Berkeley
THE UNIVERSITY OF CALIFORNIA · LIBRARY OF THE UNIVERSITY OF CALIFORNIA · LIBRARY OF THE UNIVERSITY OF CALIFORNIA

Principia Special Edition

Translated by
ANDREW MOTTE.
with
THE METHOD of FLUXIONS
Translated by
John Colson

By Sir Isaac Newton

with commentary and supplements by
J. M. F. WRIGHT
A HISTORY OF THE CONCEPTIONS OF
LIMITS AND FLUXIONS IN GREAT BRITAIN
FROM
NEWTON TO WOODHOUSE
FLORIAN CAJORI
Introductory Note by
John Clark

Introduction to the Principles of Grammar

During the evolution of a mind, it is not, nor can be, rational, i.e. cannot function by, or produce, functional grammar systems…

One of the earliest sources which point out the fact humanity will be functionally illiterate is today known, cumulatively, as the Judeo-Christian Scripture. This is possibly the only book in our history that repeatedly tells the reader humanity will not even be able to read that book until a certain time in human history, i.e., a stage in human evolution. Language does separate mankind from the other animals, however, mankind is not yet functionally literate and cannot take advantage of that distinction to the point he will survive.

Proving this is rather simple, unless one is hopelessly illiterate. Ask your computer: *How many parts of speech are possible*? Every moment that a computer is functioning, it produces all the things it can do by simple binary recursion. The Bible put this fact into a metaphor about life:—

Adam and Eve are a Conjugate Binary Pair that, *by Complete Induction and Deduction*, *produce the human race*.

As written, if one cannot read the book, then literacy is the key to opening it, just like any other book. The Key to the House of David, a simple puzzle, is simply literacy, something most of mankind believe that they are capable of, but by example prove to be the opposite.

If the metaphor is too esoteric then one has the *Dialogs* of Plato where Plato called binary recursion *Dialectic*, speech as a binary function. Or one can study Euclid's *Elements*. Or one can refer to the definition of any thing as a relative constrained by correlatives, which again, Plato called nouns and verbs. One can even examine their own body, for any particular sense can abstract either the form, limit, of a thing, or it can abstract the material difference which is again a simple binary distinction.

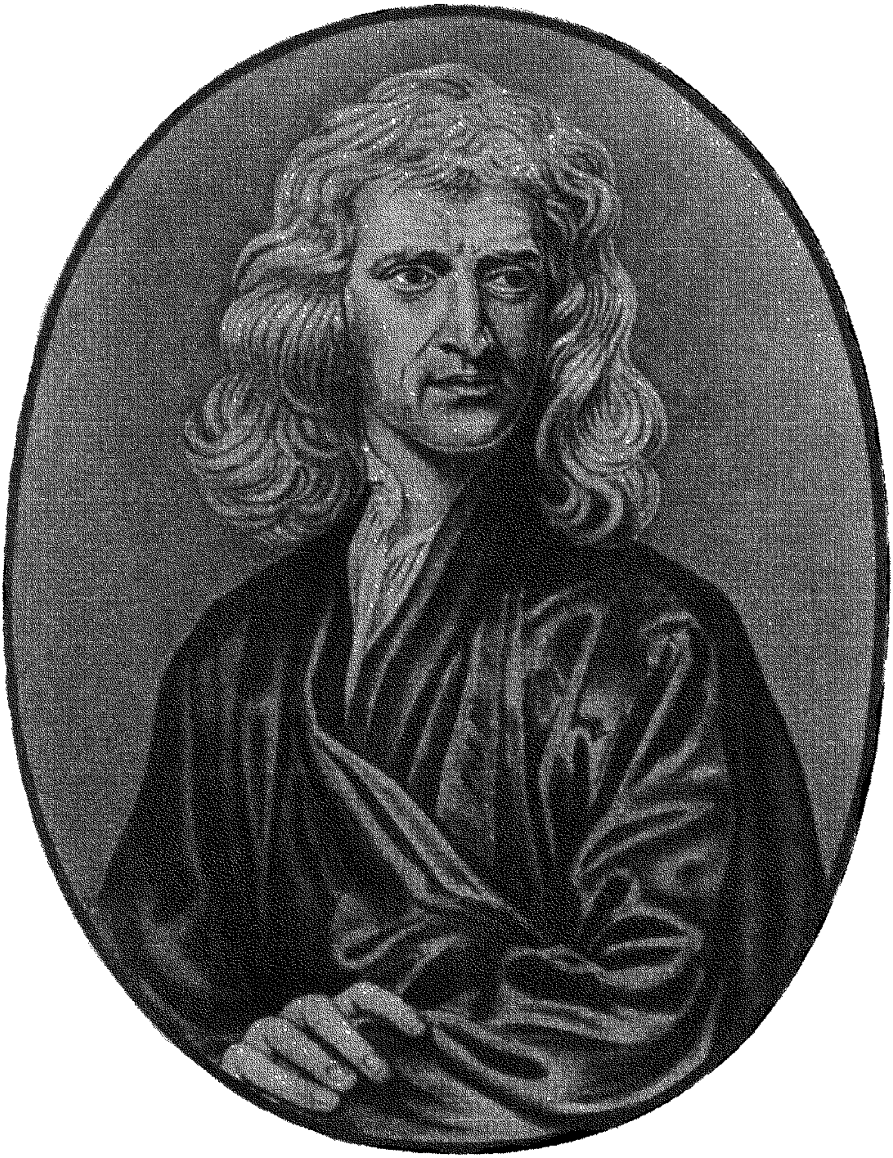
Binary recursion, when applied to the construction of grammar systems, produce methods of naming, or one can say counting, in order to manage memory. Binary recursion, used for the construction of grammar systems, produce four, and only four, categories of grammar, or four distinct methods of binary recursion: Common Grammar, Arithmetic, Algebra and Geometry. The functionally illiterate cannot comprehend or take advantage of this, of being literate.

These works give a study some material by which to study illiteracy and literacy alike.

Mankind actually believes that one grammar can contradict another, a stance championed by the Einsteins of the world, however, binary recursion is not different from itself, and binary recursion can only produce a binary result. In the words of a famous prophet, “Let your yea’s be yea’s and your nay’s be nay’s.” Which, only the functionally literate can manage.

Your computer is a machine that contradicts you every day, but long before that, your thoughts and actions, have contradicted simple reason.

Language is Universal and Intelligible, i.e. it requires intelligence to grasp it. It cannot be taught, spoken, written, or seen. While Grammar is Particular and Perceptible, which, by the simple binary recursion exampled by Language, produces exactly four methods of binary recursion for human survival: Common Grammar, Arithmetic, Algebra and Geometry. These four are also called Standards of Mental Behavior that produce Standards of Social Behavior. Psychology is commensurate with the principles of Language functionally resident in the mind of man as simple grammar systems.



SIR ISAAC NEWTON

PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore *J. S. NEWTON*, *Trin. Coll. Cantab. Soc.* Matheſeos Profeſſore *Lucaſiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Juſſu *Societatis Regiæ* ac Typis *Joſephi Streater*. Proſtat apud plures Bibliopolas. *Anno* MDCLXXXVII.

The Ode Dedicated to Newton by Edmund Halley

THIS ODE PREFIXED TO THE PRINCIPIA OF NEWTON
IS HERE TRANSLATED BY LEON J. RICHARDSON
PROFESSOR OF LATIN
IN THE UNIVERSITY OF CALIFORNIA
FROM THE VERSION AS GIVEN IN THE FIRST EDITION

TO THE ILLUSTRIOUS MAN

ISAAC NEWTON

AND THIS HIS WORK
DONE IN FIELDS OF THE MATHEMATICS AND PHYSICS
A SIGNAL DISTINCTION OF OUR TIME AND RACE

Lo, for your gaze, the pattern of the skies!
What balance of the mass, what reckonings
Divine! Here ponder too the Laws which God,
Framing the universe, set not aside
But made the fixed foundations of his work.

The inmost places of the heavens, now gained,
Break into view, nor longer hidden is
The force that turns the farthest orb. The sun
Exalted on his throne bids all things tend
Toward him by inclination and descent,
Nor suffers that the courses of the stars
Be straight, as through the boundless void they move,

But with himself as centre speeds them on
In motionless ellipses. Now we know
The sharply veering ways of comets, once
A source of dread, nor longer do we quail
Beneath appearances of bearded stars.

At last we learn wherefore the silver moon
Once seemed to travel with unequal steps,
As if she scorned to suit her pace to numbers—
Till now made clear to no astronomer;
Why, though the Seasons go and then return,
The Hours move ever forward on their way;
Explained too are the forces of the deep,
How roaming Cynthia bestirs the tides,
Whereby the surf, deserting now the kelp
Along the shore, exposes shoals of sand
Suspected by the sailors, now in turn
Driving its billows high upon the beach.

Matters that vexed the minds of ancient seers,
And for our learned doctors often led
To loud and vain contention, now are seen
In reason's light, the clouds of ignorance
Dispelled at last by science. Those on whom
Delusion cast its gloomy pall of doubt,
Upborne now on the wings that genius lends,
May penetrate the mansions of the gods
And scale the heights of heaven. O mortal men,
Arise! And, casting off your earthly cares,

Learn ye the potency of heaven-born mind,
Its thought and life far from the herd withdrawn!

The man who through the tables of the laws
Once banished theft and murder, who suppressed
Adultery and crimes of broken faith,
And put the roving peoples into cities
Girt round with walls, was founder of the state,
While he who blessed the race with Ceres' gift,
Who pressed from grapes an anodyne to care,
Or showed how on the tissue made from reeds
Growing beside the Nile one may inscribe
Symbols of sound and so present the voice
For sight to grasp, did lighten human lot,
Offsetting thus the miseries of life
With some felicity. But now, behold,
Admitted to the banquets of the gods,
We contemplate the polities of heaven;
And spelling out the secrets of the earth,
Discern the changeless order of the world
And all the aeons of its history.

Then ye who now on heavenly nectar fare,
Come celebrate with me in song the name
Of Newton, to the Muses dear; for he
Unlocked the hidden treasuries of Truth:
So richly through his mind had Phoebus cast
The radiance of his own divinity.
Nearer the gods no mortal may approach.

Newton's Preface to the First Edition

SINCE THE ANCIENTS (as we are told by *Pappus*) esteemed the science of mechanics of greatest importance in the investigation of natural things, and the moderns, rejecting substantial forms and occult qualities, have endeavored to subject the phenomena of nature to the laws of mathematics, I have in this treatise cultivated mathematics as far as it relates to philosophy. The ancients considered mechanics in a twofold respect; as rational, which proceeds accurately by demonstration, and practical. To practical mechanics all the manual arts belong, from which mechanics took its name. But as artificers do not work with perfect accuracy, it comes to pass that mechanics is so distinguished from geometry that what is perfectly accurate is called geometrical; what is less so, is called mechanical. However, the errors are not in the art, but in the artificers. He that works with less accuracy is an imperfect mechanic; and if any could work with perfect accuracy, he would be the most perfect mechanic of all, for the description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn, for it requires that the learner should first be taught to describe these accurately before he enters upon geometry, then it shows how by these operations problems may be solved. To describe right lines and circles are problems, but not geometrical problems. The solution of these problems is required from mechanics, and by geometry the use of them, when so solved, is shown; and it is the glory of geometry that from those few principles, brought from without, it is able to produce so many things. Therefore geometry is founded in mechanical practice, and is nothing but that part of universal mechanics which accurately proposes and demonstrates the art of measuring. But since the manual arts are chiefly employed in the moving of bodies, it happens that geometry is commonly referred to their magnitude, and mechanics to their motion. In this sense rational mechanics will be the science of motions resulting from any forces whatsoever, and of the forces required to produce any motions, accurately proposed and demonstrated. This part of mechanics, as far as it extended to the five powers which relate to manual arts, was cultivated by the ancients, who considered gravity (it not being a manual power) no otherwise than in moving weights by those powers. But I consider philosophy rather than arts and write not concerning manual but natural powers, and consider chiefly those things which relate to gravity, levity, elastic force, the resistance of fluids, and the like forces, whether attractive or impulsive; and therefore I offer this work as the mathematical principles of philosophy, for the whole burden of philosophy seems to consist in this—from the phenomena of motions to investigate the forces of nature, and then from these forces to dem-

onstrate the other phenomena; and to this end the general propositions in the first and second Books are directed. In the third Book I give an example of this in the explication of the System of the World; for by the propositions mathematically demonstrated in the former Books, in the third I derive from the celestial phenomena the forces of gravity with which bodies tend to the sun and the several planets. Then from these forces, by other propositions which are also mathematical, I deduce the motions of the planets, the comets, the moon, and the sea. I wish we could derive the rest of the phenomena of Nature by the same kind of reasoning from mechanical principles, for I am induced by many reasons to suspect that they may all depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards one another, and cohere in regular figures, or are repelled and recede from one another. These forces being unknown, philosophers have hitherto attempted the search of Nature in vain; but I hope the principles here laid down will afford some light either to this or some truer method of philosophy.

In the publication of this work the most acute and universally learned Mr. *Edmund Halley* not only assisted me in correcting the errors of the press and preparing the geometrical figures, but it was through his solicitations that it came to be published; for when he had obtained of me my demonstrations of the figure of the celestial orbits, he continually pressed me to communicate the same to the *Royal Society,* who afterwards, by their kind encouragement and entreaties, engaged me to think of publishing them. But after I had begun to consider the inequalities of the lunar motions, and had entered upon some other things relating to the laws and measures of gravity and other forces; and the figures that would be described by bodies attracted according to given laws; and the motion of several bodies moving among themselves; the motion of bodies in resisting mediums; the forces, densities, and motions, of mediums; the orbits of the comets, and such like, I deferred that publication till I had made a search into those matters, and could put forth the whole together. What relates to the lunar motions (being imperfect), I have put all together in the corollaries of Prop. LXVI, to avoid being obliged to propose and distinctly demonstrate the several things there contained in a method more prolix than the subject deserved and interrupt the series of the other propositions. Some things, found out after the rest, I chose to insert in places less suitable, rather than change the number of the propositions and the citations. I heartily beg that what I have here done may be read with forbearance; and that my labors in a subject so difficult may be examined, not so much with the view to censure, as to remedy their defects.

Is. NEWTON

Cambridge, Trinity College, May 8, 1686.[1]

Newton's Preface to the Second Edition

IN THIS SECOND EDITION of the *Principia* there are many emendations and some additions.[1] In the second section of the first Book, the determination of forces, by which bodies may be made to revolve in given orbits, is illustrated and enlarged. In the seventh section of the second Book the theory of the resistances of fluids was more accurately investigated, and confirmed by new experiments. In the third Book the lunar theory and the precession of the equinoxes were more fully deduced from their principles; and the theory of the comets was confirmed by more examples of the calculation of their orbits, done also with greater accuracy.

Is. NEWTON

London, March 28, 1713.

Cotes's Preface to the Second Edition[1]

WE HEREBY PRESENT to the benevolent reader the long-awaited new edition of *Newton's* Philosophy, now greatly amended and increased. The principal contents of this celebrated work may be gathered from the adjoining Table. What has been added or modified is indicated in the author's Preface. There remains for us to add something relating to the method of this philosophy.

Those who have treated of natural philosophy may be reduced to about three classes. Of these some have attributed to the several species of things, specific and occult qualities, according to which the phenomena of particular bodies are supposed to proceed in some unknown manner. The sum of the doctrine of the Schools derived from *Aristotle* and the *Peripatetics* is founded on this principle. They affirm that the several effects of bodies arise from the particular natures of those bodies. But whence it is that bodies derive those natures they don't tell us; and therefore they tell us nothing. And being entirely employed in giving names to things, and not in searching into things themselves, they have invented, we may say, a philosophical way of speaking, but they have not made known to us true philosophy.

Others have endeavored to apply their labors to greater advantage by rejecting that useless medley of words. They assume that all matter is homogeneous, and that the variety of forms which is seen in bodies arises from some very plain and simple relations of the component particles. And by going on from simple things to those which are more compounded they certainly proceed right, if they attribute to those primary relations no other relations than those which Nature has given. But when they take a liberty of imagining at pleasure unknown figures and magnitudes, and uncertain situations and motions of the parts, and moreover of supposing occult fluids, freely pervading the pores of bodies, endued with an all-performing subtilty, and agitated with occult motions, they run out into dreams and chimeras, and neglect the true constitution of things, which certainly is not to be derived from fallacious conjectures, when we can scarce reach it by the most certain observations. Those who assume hypotheses as first principles of their speculations, although they afterwards proceed with the greatest accuracy from those principles, may indeed form an ingenious romance, but a romance it will still be.

There is left then the third class, which possess experimental philosophy. These indeed derive the causes of all things from the most simple principles possible; but then they assume nothing as a principle, that is not proved by phenomena. They frame no hypotheses, nor receive them into philosophy otherwise than as questions whose truth may be disputed. They proceed therefore in a twofold

method, synthetical and analytical. From some select phenomena they deduce by analysis the forces of Nature and the more simple laws of forces; and from thence by synthesis show the constitution of the rest. This is that incomparably best way of philosophizing, which our renowned author most justly embraced in preference to the rest, and thought alone worthy to be cultivated and adorned by his excellent labors. Of this he has given us a most illustrious example, by the explication of the System of the World, most happily deduced from the Theory of Gravity. That the attribute of gravity was found in all bodies,[1] others suspected, or imagined before him, but he was the only and the first philosopher that could demonstrate it from appearances, and make it a solid foundation to the most noble speculations.

I know indeed that some persons, and those of great name, too much prepossessed with certain prejudices, are unwilling to assent to this new principle, and are ready to prefer uncertain notions to certain. It is not my intention to detract from the reputation of these eminent men; I shall only lay before the reader such considerations as will enable him to pass an equitable judgment in this dispute.

Therefore, that we may begin our reasoning from what is most simple and nearest to us, let us consider a little what is the nature of gravity in earthly bodies, that we may proceed the more safely when we come to consider it in the heavenly bodies that lie at the remotest distance from us. It is now agreed by all philosophers that all circumterrestrial bodies gravitate towards the earth. That no bodies having no weight are to be found, is now confirmed by manifold experience. That which is relative levity is not true levity, but apparent only, and arises from the preponderating gravity of the contiguous bodies.

Moreover, as all bodies gravitate towards the earth, so does the earth gravitate again towards all bodies. That the action of gravity is mutual and equal on both sides, is thus proved. Let the mass of the earth be divided into any two parts whatever, either equal or unequal; now if the weights of the parts towards each other were not mutually equal, the lesser weight would give way to the greater, and the two parts would move on together indefinitely in a right line towards that point to which the greater weight tends, which is altogether contrary to experience. Therefore we must say that the weights with which the parts tend to each other are equal; that is, that the action of gravity is mutual and equal in contrary directions.

The weights of bodies at equal distances from the centre of the earth are as the quantities of matter in the bodies. This is inferred from the equal acceleration of all bodies that fall from a state of rest by their weights; for the forces by which unequal bodies are equally accelerated must be proportional to the quantities of the matter to be moved. Now, that all falling bodies are equally accelerated, appears from this, that when the resistance of the air is taken away, as it is under

an exhausted receiver of Mr. *Boyle,* they describe equal spaces in equal times; but this is yet more accurately proved by the experiments with pendulums.

The attractive forces of bodies at equal distances are as the quantities of matter in the bodies. For since bodies gravitate towards the earth, and the earth again towards bodies with equal moments, the weight of the earth towards each body, or the force with which the body attracts the earth, will be equal to the weight of the same body towards the earth. But this weight was shown to be as the quantity of matter in the body; and therefore the force with which each body attracts the earth, or the absolute force of the body, will be as the same quantity of matter.

Therefore the attractive force of the entire bodies arises from and is composed of the attractive forces of the parts, because, as was just shown, if the bulk of the matter be augmented or diminished, its power is proportionately augmented or diminished. We must therefore conclude that the action of the earth is composed of the united actions of its parts, and therefore that all terrestrial bodies must attract one another mutually, with absolute forces that are as the matter attracting. This is the nature of gravity upon earth; let us now see what it is in the heavens.

That every body continues in its state either of rest or of moving uniformly in a right line, unless so far as it is compelled to change that state by external force, is a law of Nature universally received by all philosophers. But it follows from this that bodies which move in curved lines, and are therefore continually bent from the right lines that are tangents to their orbits, are retained in their curvilinear paths by some force continually acting. Since, then, the planets move in curvilinear orbits, there must be some force operating, by the incessant actions of which they are continually made to deflect from the tangents.

Now it is evident from mathematical reasoning, and rigorously demonstrated, that all bodies that move in any curved line described in a plane, and which, by a radius drawn to any point, whether at rest or moved in any manner, describe areas about that point proportional to the times, are urged by forces directed towards that point. This must therefore be granted. Since, then, all astronomers agree that the primary planets describe about the sun, and the secondary about the primary, areas proportional to the times, it follows that the forces by which they are continually turned aside from the rectilinear tangents, and made to revolve in curvilinear orbits, are directed towards the bodies that are placed in the centres of the orbits. This force may therefore not improperly be called centripetal in respect of the revolving body, and in respect of the central body attractive, from whatever cause it may be imagined to arise.

Moreover, it must be granted, as being mathematically demonstrated, that, if several bodies revolve with an equable motion in concentric circles, and the squares of the periodic times are as the cubes of the distances from the common centre, the centripetal forces will be inversely as the squares of the distances. Or,

if bodies revolve in orbits that are very nearly circular and the apsides of the orbits are at rest, the centripetal forces of the revolving bodies will be inversely as the squares of the distances. That both these facts hold for all the planets, all astronomers agree. Therefore the centripetal forces of all the planets are inversely as the squares of the distances from the centres of their orbits. If any should object, that the apsides of the planets, and especially of the moon, are not perfectly at rest, but are carried progressively with a slow kind of motion, one may give this answer, that, though we should grant that this very slow motion arises from a slight deviation of the centripetal force from the law of the square of the distance, yet we are able to compute mathematically the quantity of that aberration, and find it perfectly insensible. For even the ratio of the lunar centripetal force itself, which is the most irregular of them all, will vary inversely as a power a little greater than the square of the distance, but will be well-nigh sixty times nearer to the square than to the cube of the distance. But we may give a truer answer, by saying that this progression of the apsides arises not from a deviation from the law of inverse squares of the distance, but from a quite different cause, as is most admirably shown in this work. It is certain then that the centripetal forces with which the primary planets tend to the sun, and the secondary planets to their primary, are accurately as the inverse squares of the distances.

From what has been hitherto said, it is plain that the planets are retained in their orbits by some force continually acting upon them; it is plain that this force is always directed towards the centres of their orbits; it is plain that its intensity is increased in its approach and is decreased in its recession from the centre, and that it is increased in the same ratio in which the square of the distance is diminished, and decreased in the same ratio in which the square of the distance is augmented. Let us now see whether, by making a comparison between the centripetal forces of the planets and the force of gravity, we may not by chance find them to be of the same kind. Now, they will be of the same kind if we find on both sides the same laws and the same attributes. Let us then first consider the centripetal force of the moon, which is nearest to us.

The rectilinear spaces which bodies let fall from rest describe in a given time at the very beginning of the motion, when the bodies are urged by any forces whatsoever, are proportional to the forces. This appears from mathematical reasoning. Therefore the centripetal force of the moon revolving in its orbit is to the force of gravity at the surface of the earth, as the space which in a very small interval of time the moon, deprived of all its circular force and descending by its centripetal force towards the earth, would describe, is to the space which a heavy body would describe, when falling by the force of its gravity near to the earth, in the same small interval of time. The first of these spaces is equal to the versed sine of the arc described by the moon in the same time, because that versed sine

measures the translation of the moon from the tangent, produced by the centripetal force, and therefore may be computed, if the periodic time of the moon and its distance from the centre of the earth are given. The last space is found by experiments with pendulums, as Mr. *Huygens* has shown. Therefore by making a calculation we shall find that the first space is to the latter, or the centripetal force of the moon revolving in its orbit will be to the force of gravity at the surface of the earth, as the square of the semidiameter of the earth to the square of the semidiameter of the orbit. But by what was shown before, the very same ratio holds between the centripetal force of the moon revolving in its orbit, and the centripetal force of the moon near the surface of the earth. Therefore the centripetal force near the surface of the earth is equal to the force of gravity. Therefore these are not two different forces, but one and the same; for if they were different, these forces united would cause bodies to descend to the earth with twice the velocity they would fall with by the force of gravity alone. Therefore it is plain that the centripetal force, by which the moon is continually either impelled or attracted out of the tangent and retained in its orbit, is the very force of terrestrial gravity reaching up to the moon. And it is very reasonable to believe that this force should extend itself to vast distances, since upon the tops of the highest mountains we find no sensible diminution of it. Therefore the moon gravitates towards the earth; but on the other hand, the earth by a mutual action equally gravitates towards the moon, which is also abundantly confirmed in this philosophy, where the tides in the sea and the precession of the equinoxes are treated of, which arise from the action both of the moon and of the sun upon the earth. Hence lastly, we discover by what law the force of gravity decreases at great distances from the earth. For since gravity is noways different from the moon's centripetal force, and this is inversely proportional to the square of the distance, it follows that it is in that very ratio that the force of gravity decreases.

Let us now go on to the other planets. Because the revolutions of the primary planets about the sun and of the secondary about Jupiter and Saturn are phenomena of the same kind with the revolution of the moon about the earth, and because it has been moreover demonstrated that the centripetal forces of the primary planets are directed towards the centre of the sun and those of the secondary towards the centres of Jupiter and Saturn, in the same manner as the centripetal force of the moon is directed towards the centre of the earth, and since, besides, all these forces are inversely as the squares of the distances from the centres, in the same manner as the centripetal force of the moon is as the square of the distance from the earth, we must of course conclude that the nature of all is the same. Therefore as the moon gravitates towards the earth and the earth again towards the moon, so also all the secondary planets will gravitate towards their

primary, and the primary planets again towards their secondary, and so all the primary towards the sun, and the sun again towards the primary.

Therefore the sun gravitates towards all the planets, and all the planets towards the sun. For the secondary planets, while they accompany the primary, revolve the meanwhile with the primary about the sun. Therefore, by the same argument, the planets of both kinds gravitate towards the sun and the sun towards them. That the secondary planets gravitate towards the sun is moreover abundantly clear from the inequalities of the moon, a most accurate theory of which, laid open with a most admirable sagacity, we find explained in the third Book of this work.

That the attractive force of the sun is propagated on all sides to prodigious distances and is diffused to every part of the wide space that surrounds it, is most evidently shown by the motion of the comets, which, coming from places immensely distant from the sun, approach very near to it, and sometimes so near that in their perihelia they almost touch its body. The theory of these bodies was altogether unknown to astronomers till in our own times our excellent author most happily discovered it and demonstrated the truth of it by most certain observations. So that it is now apparent that the comets move in conic sections having their foci in the sun's centre, and by radii drawn to the sun describe areas proportional to the times. But from these phenomena it is manifest and mathematically demonstrated, that those forces by which the comets are retained in their orbits are directed towards the sun and are inversely proportional to the squares of the distances from its centre. Therefore the comets gravitate towards the sun, and therefore the attractive force of the sun not only acts on the bodies of the planets, placed at given distances and very nearly in the same plane, but reaches also the comets in the most different parts of the heavens, and at the most different distances. This therefore is the nature of gravitating bodies, to exert their force at all distances to all other gravitating bodies. But from thence it follows that all the planets and comets attract one another mutually, and gravitate towards one another, which is also confirmed by the perturbation of Jupiter and Saturn, observed by astronomers, and arising from the mutual actions of these two planets upon each other, as also from that very slow motion of the apsides, above taken notice of, which arises from a like cause.

We have now proceeded so far, that it must be acknowledged that the sun, and the earth, and all the heavenly bodies attending the sun, attract one another mutually. Therefore all the least particles of matter in every one must have their several attractive forces proportional to their quantities of matter, as was shown above of the terrestrial bodies. At different distances these forces will be also inversely as the squares of their distances; for it is mathematically demonstrated, that globes attracting according to this law are composed of particles attracting according to the same law.

The foregoing conclusions are grounded on this axiom which is received by all philosophers, namely, that effects of the same kind, whose known properties are the same, take their rise from the same causes and have the same unknown properties also. For if gravity be the cause of the descent of a stone in *Europe,* who doubts that it is also the cause of the same descent in *America?* If there is a mutual gravitation between a stone and the earth in *Europe,* who will deny the same to be mutual in *America?* If in *Europe* the attractive force of a stone and the earth is composed of the attractive forces of the parts, who will deny the like composition in *America?* If in *Europe* the attraction of the earth be propagated to all kinds of bodies and to all distances, why may we not say that it is propagated in like manner in *America?* All philosophy is founded on this rule; for if that be taken away, we can affirm nothing as a general truth. The constitution of particular things is known by observations and experiments; and when that is done, no general conclusion of the nature of things can thence be drawn, except by this rule.

Since, then, all bodies, whether upon earth or in the heavens, are heavy, so far as we can make any experiments or observations concerning them, we must certainly allow that gravity is found in all bodies universally. And in like manner as we ought not to suppose that any bodies can be otherwise than extended, movable, or impenetrable, so we ought not to conceive that any bodies can be otherwise than heavy. The extension, mobility, and impenetrability of bodies become known to us only by experiments; and in the very same manner their gravity becomes known to us. All bodies upon which we can make any observations, are extended, movable, and impenetrable; and thence we conclude all bodies, and those concerning which we have no observations, are extended and movable and impenetrable. So all bodies on which we can make observations, we find to be heavy; and thence we conclude all bodies, and those we have no observations of, to be heavy also. If anyone should say that the bodies of the fixed stars are not heavy because their gravity is not yet observed, they may say for the same reason that they are neither extended nor movable nor impenetrable, because these properties of the fixed stars are not yet observed. In short, either gravity must have a place among the primary qualities of all bodies, or extension, mobility, and impenetrability must not. And if the nature of things is not rightly explained by the gravity of bodies, it will not be rightly explained by their extension, mobility, and impenetrability.

Some I know disapprove this conclusion, and mutter something about occult qualities. They continually are cavilling with us, that gravity is an occult property, and occult causes are to be quite banished from philosophy. But to this the answer is easy: that those are indeed occult causes whose existence is occult, and imagined but not proved; but not those whose real existence is clearly demonstrated by

observations. Therefore gravity can by no means be called an occult cause of the celestial motions, because it is plain from the phenomena that such a power does really exist. Those rather have recourse to occult causes, who set imaginary vortices of a matter entirely fictitious and imperceptible by our senses, to direct those motions.

But shall gravity be therefore called an occult cause, and thrown out of philosophy, because the cause of gravity is occult and not yet discovered? Those who affirm this, should be careful not to fall into an absurdity that may overturn the foundations of all philosophy. For causes usually proceed in a continued chain from those that are more compounded to those that are more simple; when we are arrived at the most simple cause we can go no farther. Therefore no mechanical account or explanation of the most simple cause is to be expected or given; for if it could be given, the cause were not the most simple. These most simple causes will you then call occult, and reject them? Then you must reject those that immediately depend upon them, and those which depend upon these last, till philosophy is quite cleared and disencumbered of all causes.

Some there are who say that gravity is preternatural, and call it a perpetual miracle. Therefore they would have it rejected, because preternatural causes have no place in physics. It is hardly worth while to spend time in answering this ridiculous objection which overturns all philosophy. For either they will deny gravity to be in bodies, which cannot be said, or else, they will therefore call it preternatural because it is not produced by the other properties of bodies, and therefore not by mechanical causes. But certainly there are primary properties of bodies; and these, because they are primary, have no dependence on the others. Let them consider whether all these are not in like manner preternatural, and in like manner to be rejected; and then what kind of philosophy we are like to have.

Some there are who dislike this celestial physics because it contradicts the opinions of *Descartes,* and seems hardly to be reconciled with them. Let these enjoy their own opinion, but let them act fairly, and not deny the same liberty to us which they demand for themselves. Since the *Newtonian* Philosophy appears true to us, let us have the liberty to embrace and retain it, and to follow causes proved by phenomena, rather than causes only imagined and not yet proved. The business of true philosophy is to derive the natures of things from causes truly existent, and to inquire after those laws on which the Great Creator actually chose to found this most beautiful Frame of the World, not those by which he might have done the same, had he so pleased. It is reasonable enough to suppose that from several causes, somewhat differing from one another, the same effect may arise; but the true cause will be that from which it truly and actually does arise; the others have no place in true philosophy. The same motion of the hour-

hand in a clock may be occasioned either by a weight hung, or a spring shut up within. But if a certain clock should be really moved with a weight, we should laugh at a man that would suppose it moved by a spring, and from that principle, suddenly taken up without further examination, should go about to explain the motion of the index; for certainly the way he ought to have taken would have been actually to look into the inward parts of the machine, that he might find the true principle of the proposed motion. The like judgment ought to be made of those philosophers who will have the heavens to be filled with a most subtile matter which is continually carried round in vortices. For if they could explain the phenomena ever so accurately by their hypotheses, we could not yet say that they have discovered true philosophy and the true causes of the celestial motions, unless they could either demonstrate that those causes do actually exist, or at least that no others do exist. Therefore if it be made clear that the attraction of all bodies is a property actually existing *in rerum natura,* and if it be also shown how the motions of the celestial bodies may be solved by that property, it would be very impertinent for anyone to object that these motions ought to be accounted for by vortices; even though we should allow such an explication of those motions to be possible. But we allow no such thing; for the phenomena can by no means be accounted for by vortices, as our author has abundantly proved from the clearest reasons. So that men must be strangely fond of chimeras, who can spend their time so idly as in patching up a ridiculous figment and setting it off with new comments of their own.

If the bodies of the planets and comets are carried round the sun in vortices, the bodies so carried, and the parts of the vortices next surrounding them, must be carried with the same velocity and the same direction, and have the same density, and the same inertia, answering to the bulk of the matter. But it is certain, the planets and comets, when in the very same parts of the heavens, are carried with various velocities and various directions. Therefore it necessarily follows that those parts of the celestial fluid, which are at the same distances from the sun, must revolve at the same time with different velocities in different directions; for one kind of velocity and direction is required for the motion of the planets, and another for that of the comets. But since this cannot be accounted for, we must either say that all celestial bodies are not carried about by vortices, or else that their motions are derived, not from one and the same vortex, but from several distinct ones, which fill and pervade the spaces round about the sun.

But if several vortices are contained in the same space, and are supposed to penetrate one another, and to revolve with different motions, then because these motions must agree with those of the bodies carried about by them, which are perfectly regular, and performed in conic sections which are sometimes very eccentric, and sometimes nearly circles, one may very reasonably ask how it comes

to pass that these vortices remain entire, and have suffered no manner of perturbation in so many ages from the actions of the conflicting matter. Certainly if these fictitious motions are more compounded and harder to be accounted for than the true motions of the planets and comets, it seems to no purpose to admit them into philosophy, since every cause ought to be more simple than its effect. Allowing men to indulge their own fancies, suppose any man should affirm that the planets and comets are surrounded with atmospheres like our earth, which hypothesis seems more reasonable than that of vortices; let him then affirm that these atmospheres by their own nature move about the sun and describe conic sections, which motion is much more easily conceived than that of the vortices penetrating one another; lastly, that the planets and comets are carried about the sun by these atmospheres of theirs: and then applaud his own sagacity in discovering the causes of the celestial motions. He that rejects this fable must also reject the other; for two drops of water are not more like than this hypothesis of atmospheres, and that of vortices.

Galileo has shown that when a stone projected moves in a parabola, its deflection into that curve from its rectilinear path is occasioned by the gravity of the stone towards the earth, that is, by an occult quality. But now somebody, more cunning than he, may come to explain the cause after this manner. He will suppose a certain subtile matter, not discernible by our sight, our touch, or any other of our senses, which fills the spaces which are near and contiguous to the surface of the earth, and that this matter is carried with different directions, and various, and often contrary, motions, describing parabolic curves. Then see how easily he may account for the deflection of the stone above spoken of. The stone, says he, floats in this subtile fluid, and following its motion, can't choose but describe the same figure. But the fluid moves in parabolic curves, and therefore the stone must move in a parabola, of course. Would not the acuteness of this philosopher be thought very extraordinary, who could deduce the appearances of Nature from mechanical causes, matter and motion, so clearly that the meanest man may understand it? Or indeed should not we smile to see this new *Galileo* taking so much mathematical pains to introduce occult qualities into philosophy, from whence they have been so happily excluded? But I am ashamed to dwell so long upon trifles.

The sum of the matter is this: the number of the comets is certainly very great; their motions are perfectly regular and observe the same laws with those of the planets. The orbits in which they move are conic sections, and those very eccentric. They move every way towards all parts of the heavens, and pass through the planetary regions with all possible freedom, and their motion is often contrary to the order of the signs. These phenomena are most evidently confirmed by astronomical observations, and cannot be accounted for by vortices. Nay, indeed,

they are utterly irreconcilable with the vortices of the planets. There can be no room for the motions of the comets, unless the celestial spaces be entirely cleared of that fictitious matter.

For if the planets are carried about the sun in vortices, the parts of the vortices which immediately surround every planet must be of the same density with the planet, as was shown above. Therefore all the matter contiguous to the perimeter of the earth's orbit[1] must be of the same density as the earth. But this great orb and the orb of Saturn must have either an equal or a greater density. For to make the constitution of the vortex permanent, the parts of less density must lie near the centre, and those of greater density must go farther from it. For since the periodic times of the planets vary as the $\frac{3}{2}$th powers of their distances from the sun, the periods of the parts of the vortices must also preserve the same ratio. Thence it will follow that the centrifugal forces of the parts of the vortex must be inversely as the squares of their distances. Those parts therefore which are more remote from the centre endeavor to recede from it with less force; whence, if their density be deficient, they must yield to the greater force with which the parts that lie nearer the centre endeavor to ascend. Therefore the denser parts will ascend, and those of less density will descend, and there will be a mutual change of places, till all the fluid matter in the whole vortex be so adjusted and disposed, that being reduced to an equilibrium its parts become quiescent. If two fluids of different density be contained in the same vessel, it will certainly come to pass that the fluid of greater density will sink the lower; and by a like reasoning it follows that the denser parts of the vortex by their greater centrifugal force will ascend to the higher places. Therefore all that far greater part of the vortex which lies without the earth's orb, will have a density, and by consequence an inertia, answering to the bulk of the matter, which cannot be less than the density and inertia of the earth. But from hence will arise a mighty resistance to the passage of the comets, such as must be very sensible, not to say enough to put a stop to and absorb their motions entirely. But it appears from the perfectly regular motion of the comets, that they suffer no resistance that is in the least sensible, and therefore that they do not meet with matter of any kind that has any resisting force or, by consequence, any density or inertia. For the resistance of mediums arises either from the inertia of the matter of the fluid, or from its want of lubricity. That which arises from the want of lubricity is very small, and is scarcely observable in the fluids commonly known, unless they be very tenacious like oil and honey. The resistance we find in air, water, quicksilver, and the like fluids that are not tenacious, is almost all of the first kind, and cannot be diminished by a greater degree of subtilty, if the density and inertia, to which this resistance is proportional, remains, as is most evidently demonstrated by our author in his noble theory of resistances in the second Book.

Bodies in going on through a fluid communicate their motion to the ambient fluid by little and little, and by that communication lose their own motion, and by losing it are retarded. Therefore the retardation is proportional to the motion communicated, and the communicated motion, when the velocity of the moving body is given, is as the density of the fluid; and therefore the retardation or resistance will be as the same density of the fluid; nor can it be taken away, unless the fluid, coming about to the hinder parts of the body, restore the motion lost. Now this cannot be done unless the impression of the fluid on the hinder parts of the body be equal to the impression of the fore parts of the body on the fluid; that is, unless the relative velocity with which the fluid pushes the body behind is equal to the velocity with which the body pushes the fluid; that is, unless the absolute velocity of the recurring fluid be twice as great as the absolute velocity with which the fluid is driven forwards by the body, which is impossible. Therefore the resistance of fluids arising from their inertia can by no means be taken away. So that we must conclude that the celestial fluid has no inertia, because it has no resisting force; that it has no force to communicate motion with, because it has no inertia; that it has no force to produce any change in one or more bodies, because it has no force wherewith to communicate motion; that it has no manner of efficacy, because it has no faculty wherewith to produce any change of any kind. Therefore certainly this hypothesis may be justly called ridiculous and unworthy a philosopher, since it is altogether without foundation and does not in the least serve to explain the nature of things.[1] Those who would have the heavens filled with a fluid matter, but suppose it void of any inertia, do indeed in words deny a vacuum, but allow it in fact. For since a fluid matter of that kind can noways be distinguished from empty space, the dispute is now about the names and not the natures of things. If any are so fond of matter that they will by no means admit of a space void of body, let us consider where they must come at last.

For either they will say that this constitution of a world everywhere full was made so by the will of God to this end, that the operations of Nature might be assisted everywhere by a subtile ether pervading and filling all things; which cannot be said, however, since we have shown from the phenomena of the comets, that this ether is of no efficacy at all; or they will say, that it became so by the same will of God for some unknown end, which ought not be said, because for the same reason a different constitution may be as well supposed; or lastly, they will not say that it was caused by the will of God, but by some necessity of its nature. Therefore they will at last sink into the mire of that infamous herd who dream that all things are governed by fate and not by providence, and that matter exists by the necessity of its nature always and everywhere, being infinite and eternal. But supposing these things, it must be also everywhere uniform; for variety of forms is entirely inconsistent with necessity. It must be also unmoved;

for if it be necessarily moved in any determinate direction, with any determinate velocity, it will by a like necessity be moved in a different direction with a different velocity; but it can never move in different directions with different velocities; therefore it must be unmoved. Without all doubt this world, so diversified with that variety of forms and motions we find in it, could arise from nothing but the perfectly free will of God directing and presiding over all.

From this fountain it is that those laws, which we call the laws of Nature, have flowed, in which there appear many traces indeed of the most wise contrivance, but not the least shadow of necessity. These therefore we must not seek from uncertain conjectures, but learn them from observations and experiments. He who is presumptuous enough to think that he can find the true principles of physics and the laws of natural things by the force alone of his own mind, and the internal light of his reason, must either suppose that the world exists by necessity, and by the same necessity follows the laws proposed; or if the order of Nature was established by the will of God, that himself, a miserable reptile, can tell what was fittest to be done. All sound and true philosophy is founded on the appearances of things; and if these phenomena inevitably draw us, against our wills, to such principles as most clearly manifest to us the most excellent counsel and supreme dominion of the All-wise and Almighty Being, they are not therefore to be laid aside because some men may perhaps dislike them. These men may call them miracles or occult qualities, but names maliciously given ought not to be a disadvantage to the things themselves, unless these men will say at last that all philosophy ought to be founded in atheism. Philosophy must not be corrupted in compliance with these men, for the order of things will not be changed.

Fair and equal judges will therefore give sentence in favor of this most excellent method of philosophy, which is founded on experiments and observations. And it can hardly be said or imagined, what light, what splendor, hath accrued to that method from this admirable work of our illustrious author, whose happy and sublime genius, resolving the most difficult problems, and reaching to discoveries of which the mind of man was thought incapable before, is deservedly admired by all those who are somewhat more than superficially versed in these matters. The gates are now set open, and by the passage he has revealed we may freely enter into the knowledge of the hidden secrets and wonders of natural things. He has so clearly laid open and set before our eyes the most beautiful frame of the System of the World, that if King *Alphonso* were now alive, he would not complain for want of the graces either of simplicity or of harmony in it. Therefore we may now more nearly behold the beauties of Nature, and entertain ourselves with the delightful contemplation; and, which is the best and most valuable fruit of philosophy, be thence incited the more profoundly to reverence and adore the great Maker and Lord of all. He must be blind who from

the most wise and excellent contrivances of things cannot see the infinite Wisdom and Goodness of their Almighty Creator, and he must be mad and senseless who refuses to acknowledge them.

Newton's distinguished work will be the safest protection against the attacks of atheists, and nowhere more surely than from this quiver can one draw forth missiles against the band of godless men. This was felt long ago and first surprisingly demonstrated in learned English and Latin discourses by *Richard Bentley,* who, excelling in learning and distinguished as a patron of the highest arts, is a great ornament of his century and of our academy, the most worthy and upright Master of our *Trinity College.* To him in many ways I must express my indebtedness. And you too, benevolent reader, will not withhold the esteem due him. For many years an intimate friend of the celebrated author (since he aimed not only that the author should be esteemed by those who come after, but also that these uncommon writings should enjoy distinction among the literati of the world), he cared both for the reputation of his friend and for the advancement of the sciences. Since copies of the previous edition were very scarce and held at high prices, he persuaded by frequent entreaties and almost by chidings, the splendid man, distinguished alike for modesty and for erudition, to grant him permission for the appearance of this new edition, perfected throughout and enriched by new parts, at his expense and under his supervision. He assigned to me, as he had a right, the not unwelcome task of looking after the corrections as best I could.

ROGER COTES
Fellow of Trinity College,
Plumian Professor of Astronomy
and Experimental Philosophy.

Cambridge, May 12, 1713.

Newton's Preface to the Third Edition

IN THIS THIRD EDITION, prepared with much care by *Henry Pemberton,* M.D., a man of the greatest skill in these matters, some things in the second Book on the resistance of mediums are somewhat more comprehensively handled than before, and new experiments on the resistance of heavy bodies falling in air are added. In the third Book, the argument to prove that the moon is retained in its orbit by the force of gravity is more fully stated; and there are added new observations made by Mr. *Pound,* concerning the ratio of the diameters of Jupiter to one another. Some observations are also added on the comet which appeared in the year 1680, made in *Germany* in the month of *November* by Mr. *Kirk;* which have lately come to my hands. By the help of these it becomes apparent how nearly parabolic orbits represent the motions of comets. The orbit of that comet is determined somewhat more accurately than before, by the computation of Dr. *Halley,* in an ellipse. And it is shown that, in this elliptic orbit, the comet took its course through the nine signs of the heavens, with as much accuracy as the planets move in the elliptic orbits given in astronomy. The orbit of the comet which appeared in the year 1723 is also added, computed by Mr. *Bradley,* Professor of Astronomy at *Oxford.*[1]

Is. NEWTON

London, Jan. 12, 1725–6.

NEWTON'S PRINCIPIA.

THE

MATHEMATICAL PRINCIPLES

OF

NATURAL PHILOSOPHY,

BY SIR ISAAC NEWTON;

TRANSLATED INTO ENGLISH BY ANDREW MOTTE.

TO WHICH IS ADDED

NEWTON'S SYSTEM OF THE WORLD;

With a Portrait taken from the Bust in the Royal Observatory at Greenwich.

FIRST AMERICAN EDITION, CAREFULLY REVISED AND CORRECTED,

WITH A LIFE OF THE AUTHOR, BY N. W. CHITTENDEN, M. A., &c.

NEW-YORK:
PUBLISHED BY DANIEL ADEE, 45 LIBERTY STREET.

Entered according to Act of Congress, in the year 1846, by
DANIEL ADEE.
In the Clerk's Office of the Southern District Court of New-York.

Turney & Lockwood's Stereo.
16 Spruce St. N. Y.

DEDICATION.

TO THE

TEACHERS OF THE NORMAL SCHOOL

OF THE STATE OF NEW-YORK.

GENTLEMEN:—

A stirring freshness in the air, and ruddy streaks upon the horizon of the moral world betoken the grateful dawning of a new era. The days of a drivelling instruction are departing. With us is the opening promise of a better time, wherein genuine manhood doing its noblest work shall have adequate reward. TEACHER is the highest and most responsible office man can fill. Its dignity is, and will yet be held commensurate with its duty—a duty boundless as man's intellectual capacity, and great as his moral need—a duty from the performance of which shall emanate an influence not limited to the *now* and the *here*, but which surely will, as time flows into eternity and space into infinity, roll up, a measureless curse or a measureless blessing, in inconceivable swellings along the infinite curve. It is an office that should be esteemed of even sacred import in this country. Ere long a hundred millions, extending from the Atlantic to the Pacific, from Baffin's Bay to that of Panama, shall call themselves American citizens. What a field for those two master-passions of the human soul—the love of Rule, and the love of Gain! How shall our liberties continue to be preserved from the graspings of Ambition and the corruptions of Gold? Not by Bills of Rights

Constitutions, and Statute Books; but alone by the rightly cultivated hearts and heads of the PEOPLE. They must themselves guard the Ark. It is yours to fit them for the consecrated charge. Look well to it: for you appear clothed in the majesty of great power! It is yours to fashion, and to inform, to save, and to perpetuate. You are the Educators of the People: you are the prime Conservators of the public weal. Betray your trust, and the sacred fires would go out, and the altars crumble into dust: knowledge become lost in tradition, and Christian nobleness a fable! As you, therefore, are multiplied in number, elevated in consideration, increased in means, and fulfill, well and faithfully, all the requirements of true Teachers, so shall our favoured land lift up her head among the nations of the earth, and call herself blessed.

In conclusion, Gentlemen, to you, as the conspicuous leaders in the vast and honourable labour of Educational Reform, and Popular Teaching, the First American Edition of the PRINCIPIA of Newton—the greatest work of the greatest Teacher—is most respectfully dedicated.

N. W. CHITTENDEN.

INTRODUCTION TO THE AMERICAN EDITION.

THAT the PRINCIPIA of Newton should have remained so generally unknown in this country to the present day is a somewhat remarkable fact; because the name of the author, learned with the very elements of science, is revered at every hearth-stone where knowledge and virtue are of chief esteem, while, abroad, in all the high places of the land, the character which that name recalls is held up as the noblest illustration of what MAN may be, and may do, in the possession and manifestation of pre-eminent intellectual and moral worth; because the work is celebrated, not only in the history of one career and one mind, but in the history of all achievement and human reason itself; because of the spirit of inquiry, which has been aroused, and which, in pursuing its searchings, is not always satisfied with stopping short of the fountain-head of any given truth; and, finally, because of the earnest endeavour that has been and is constantly going on, in many sections of the Republic, to elevate the popular standard of education and give to scientific and other efforts a higher and a better aim.

True, the PRINCIPIA has been hitherto inaccessible to popular use. A few copies in Latin, and occasionally one in English may be found in some of our larger libraries, or in the possession of some ardent disciple of the great Master. But a dead language in the one case, and an enormous price in both, particularly in that of the English edition, have thus far opposed very sufficient obstacles to the wide circulation of the work. It is now, however, placed within the reach of all. And in performing this labour, the utmost care has been taken, by collation, revision, and otherwise, to render the First American Edition the most accurate and beautiful in our language. "Le plus beau monument que l'on puisse élever à la gloire de Newton, c'est une bonne édition de ses ouvrages:" and a monument like unto that we would here

set up. The PRINCIPIA, above all, glows with the immortality of a transcendant mind. Marble and brass dissolve and pass away; but the true creations of genius endure, in time and beyond time, forever: high upon the adamant of the indestructible, they send forth afar and near, over the troublous waters of life, a pure, unwavering, quenchless light whereby the myriad myriads of barques, richly laden with reason, intelligence and various faculty, are guided through the night and the storm, by the beetling shore and the hidden rock, the breaker and the shoal, safely into havens calm and secure.

To the teacher and the taught, the scholar and the student, the devotee of Science and the worshipper of Truth, the PRINCIPIA must ever continue to be of inestimable value. If to educate means, not so much to store the memory with symbols and facts, as to bring forth the faculties of the soul and develope them to the full by healthy nurture and a hardy discipline, then, what so effective to the accomplishment of that end as the study of Geometrical Synthesis? The Calculus, in some shape or other, is, indeed, necessary to the successful prosecution of researches in the higher branches of philosophy. But has not the Analytical encroached upon the Synthetical, and Algorithmic Formulæ been employed when not requisite, either for the evolution of truth, or even its apter illustration? To each method belongs, undoubtedly, an appropriate use. Newton, himself the inventor of Fluxions, censured the handling of Geometrical subjects by Algebraical calculations; and the maturest opinions which he expressed were additionally in favour of the Geometrical Method. His preference, so strongly marked, is not to be reckoned a mere matter of taste; and his authority should bear with preponderating weight upon the decision of every instructor in adopting what may be deemed the best plan to insure the completest mental development. Geometry, the vigorous product of remote time; blended with the earliest aspirations of Science and the earliest applications of Art; as well in the measures of music as in the movement of spheres; as wholly in the structure of the atom as in that of the world; directing MOTION and shaping APPEARANCE; in a word, at the moulding of the created all, is, in comprehensive

view, the outward form of that Inner Harmony of which and in which all things are. Plainly, therefore, this noble study has other and infinitely higher uses than to increase the power of abstraction. A more general and thorough cultivation of it should be strenuously insisted on. Passing from the pages of Euclid or Legendre, might not the student be led, at the suitable time, to those of the PRINCIPIA wherein Geometry may be found in varied use from the familiar to the sublime? The profoundest and the happiest results, it is believed, would attend upon this enlargement of our Educational System.

Let the PRINCIPIA, then, be gladly welcomed into every Hall where a TRUE TEACHER presides. And they who are guided to the diligent study of this incomparable work, who become strengthened by its reason, assured by its evidence, and enlightened by its truths, and who rise into loving communion with the great and pure spirit of its author, will go forth from the scenes of their pupilage, and take their places in the world as strong-minded, right-hearted men—such men as the Theory of our Government contemplates and its practical operation absolutely demands.

LIFE OF

SIR ISAAC NEWTON.

Nec fas est proprius mortali attingere Divos.—HALLEY.

FROM the thick darkness of the middle ages man's struggling spirit emerged as in new birth; breaking out of the iron control of that period; growing strong and confident in the tug and din of succeeding conflict and revolution, it bounded forwards and upwards with resistless vigour to the investigation of physical and moral truth; ascending height after height; sweeping afar over the earth, penetrating afar up into the heavens; increasing in endeavour, enlarging in endowment; every where boldly, earnestly out-stretching, till, in the AUTHOR of the PRINCIPIA, one arose, who, grasping the master-key of the universe and treading its celestial paths, opened up to the human intellect the stupendous realities of the material world, and, in the unrolling of its harmonies, gave to the human heart a new song to the goodness, wisdom, and majesty of the all-creating, all-sustaining, all-perfect God.

Sir Isaac Newton, in whom the rising intellect seemed to attain, as it were, to its culminating point, was born on the 25th of December, O. S. 1642—Christmas day—at Woolsthorpe, in the parish of Colsterworth, in Lincolnshire. His father, John Newton, died at the age of thirty-six, and only a few months after his marriage to Harriet Ayscough, daughter of James Ayscough, of Rutlandshire. Mrs. Newton, probably wrought upon by the early loss of her husband, gave premature birth to her only and posthumous child, of which, too, from its extreme diminutiveness, she appeared likely to be soon bereft. Happily, it was otherwise decreed! The tiny infant, on whose little lips the breath of life

so doubtingly hovered, lived;—lived to a vigorous maturity, to a hale old age;—lived to become the boast of his country, the wonder of his time, and the "ornament of his species."

Beyond the grandfather, Robert Newton, the descent of Sir Isaac cannot with certainty be traced. Two traditions were held in the family: one, that they were of Scotch extraction; the other, that they came originally from Newton, in Lancashire, dwelling, for a time, however, at Westby, county of Lincoln, before the removal to and purchase of Woolsthorpe—about a hundred years before this memorable birth.

The widow Newton was left with the simple means of a comfortable subsistence. The Woolsthorpe estate together with small one which she possessed at Sewstern, in Leicestershire, yielded her an income of some eighty pounds; and upon this limited sum, she had to rely chiefly for the support of herself, and the education of her child. She continued his nurture for three years, when, marrying again, she confided the tender charge to the care of her own mother.

Great genius is seldom marked by precocious development; and young Isaac, sent, at the usual age, to two day schools at Skillington and Stoke, exhibited no unusual traits of character. In his twelfth year, he was placed at the public school at Grantham, and boarded at the house of Mr. Clark, an apothecary. But even in this excellent seminary, his mental acquisitions continued for a while unpromising enough: study apparently had no charms for him; he was very inattentive, and ranked low in the school. One day, however, the boy immediately above our seemingly dull student gave him a severe kick in the stomach; Isaac, deeply affected, but with no outburst of passion, betook himself, with quiet, incessant toil, to his books; he quickly passed above the offending classmate; yet there he stopped not; the strong spirit was, for once and forever, awakened, and, yielding to its noble impulse, he speedily took up his position at the head of all.

His peculiar character began now rapidly to unfold itself. Close application grew to be habitual. Observation alternated with reflection. "A sober, silent, thinking lad," yet, the wisest and the kindliest, the indisputable leader of his fellows. Gener-

osity, modesty, and a love of truth distinguished him then as ever afterwards. He did not often join his classmates in play; but he would contrive for them various amusements of a scientific kind. Paper kites he introduced; carefully determining their best form and proportions, and the position and number of points whereby to attach the string. He also invented paper lanterns; these served ordinarily to guide the way to school in winter mornings, but occasionally for quite another purpose; they were attached to the tails of kites in a dark night, to the dismay of the country people dreading portentous comets, and to the immeasureable delight of his companions. To him, however, young as he was, life seemed to have become an earnest thing. When not occupied with his studies, his mind would be engrossed with mechanical contrivances; now imitating, now inventing. He became singularly skilful in the use of his little saws, hatchets, hammers, and other tools. A windmill was erected near Grantham; during the operations of the workmen, he was frequently present; in a short time, he had completed a perfect working model of it, which elicited general admiration. Not content, however, with this exact imitation, he conceived the idea of employing, in the place of sails, animal power, and, adapting the construction of his mill accordingly, he enclosed in it a mouse, called the miller, and which by acting on a sort of treadwheel, gave motion to the machine. He invented, too, a mechanical carriage—having four wheels, and put in motion with a handle worked by the person sitting inside. The measurement of time early drew his attention. He first constructed a water clock, in proportions somewhat like an old-fashioned house clock. The index of the dial plate was turned by a piece of wood acted upon by dropping water. This instrument, though long used by himself, and by Mr. Clark's family, did not satisfy his inquiring mind. His thoughts rose to the sun; and, by careful and oft-repeated observations of the solar movements, he subsequently formed many dials. One of these, named *Isaac's dial*, was the accurate result of years' labour, and was frequently referred to for the hour of the day by the country people.

May we not discern in these continual efforts—the diligent research, the patient meditation, the aspiring glance, and the energy

of discovery—the stirring elements of that wondrous spirit, which, clear, calm, and great, moved, in after years, through deep onward through deep of Nature's mysteries, unlocking her strongholds, dispelling darkness, educing order—everywhere silently conquering.

Newton had an early and decided taste for drawing. Pictures, taken sometimes from copies, but often from life, and drawn, coloured and framed by himself, ornamented his apartment. He was skilled also, in poetical composition, "excelled in making verses;" some of these were borne in remembrance and repeated, seventy years afterward, by Mrs. Vincent, for whom, in early youth, as Miss Storey, he formed an ardent attachment. She was the sister of a physician resident near Woolsthorpe; but Newton's intimate acquaintance with her began at Grantham, where they were both numbered among the inmates of the same house. Two or three years younger than himself, of great personal beauty, and unusual talent, her society afforded him the greatest pleasure; and their youthful friendship, it is believed, gradually rose to a higher passion; but inadequacy of fortune prevented their union. Miss Storey was afterwards twice married; Newton, never; his esteem for her continued unabated during life, accompanied by numerous acts of attention and kindness.

In 1656, Newton's mother was again left a widow, and took up her abode once more at Woolsthorpe. He was now fifteen years of age, and had made great progress in his studies; but she, desirous of his help, and from motives of economy, recalled him from school. Business occupations, however, and the management of the farm, proved utterly distasteful to him. When sent to Grantham Market on Saturdays, he would betake himself to his former lodgings in the apothecary's garret, where some of Mr. Clark's old books employed his thoughts till the aged and trustworthy servant had executed the family commissions and announced the necessity of return: or, at other times, our young philosopher would seat himself under a hedge, by the wayside, and continue his studies till the same faithful personage—proceeding alone to the town and completing the day's business—stopped as he re-

turned. The more immediate affairs of the farm received no better attention. In fact, his passion for study grew daily more absorbing, and his dislike for every other occupation more intense. His mother, therefore, wisely resolved to give him all the advantages which an education could confer. He was sent back to Grantham school, where he remained for some months in busy preparation for his academical studies. At the recommendation of one of his uncles, who had himself studied at Trinity College, Cambridge, Newton proceeded thither, and was duly admitted, on the 5th day of June 1660, in the eighteenth year of his age.

The eager student had now entered upon a new and wider field; and we find him devoting himself to the pursuit of knowledge with amazing ardour and perseverance. Among other subjects, his attention was soon drawn to that of Judicial Astrology He exposed the folly of this pseudo-science by erecting a figure with the aid of one or two of the problems of Euclid;—and thus began his study of the Mathematics. His researches into this science were prosecuted with unparallelled vigour and success. Regarding the propositions contained in Euclid as self-evident truths, he passed rapidly over this ancient system—a step which he afterward much regretted—and mastered, without further preparatory study, the Analytical Geometry of Descartes. Wallis's Arithmetic of Infinites, Saunderson's Logic, and the Optics of Kepler, he also studied with great care; writing upon them many comments; and, in these notes on Wallis's work was undoubtedly the germ of his fluxionary calculus. His progress was so great that he found himself more profoundly versed than his tutor in many branches of learning. Yet his acquisitions were not gotten with the rapidity of intuition; but they were thoroughly made and firmly secured. Quickness of apprehension, or intel lectual nimbleness did not belong to him. He saw too far: his insight was too deep. He dwelt fully, cautiously upon the least subject; while to the consideration of the greatest, he brought a massive strength joined with a matchless clearness, that, regardless of the merely trivial or unimportant, bore with unerring sagacity upon the prominences of the subject, and, grappling with its difficulties, rarely failed to surmount them.

His early and fast friend, Dr. Barrow—in compass of invention only inferior to Newton—who had been elected Professor of Greek in the University, in 1660, was made Lucasian Professor of Mathematics in 1663, and soon afterward delivered his Optical Lectures: the manuscripts of these were revised by Newton, and several oversights corrected, and many important suggestions made by him; but they were not published till 1669.

In the year 1665, he received the degree of Bachelor of Arts; and, in 1666, he entered upon those brilliant and imposing discoveries which have conferred inappreciable benefits upon science, and immortality upon his own name.

Newton, himself, states that he was in possession of his Method of Fluxions, "in the year 1666, or before." Infinite quantities had long been a subject of profound investigation; among the ancients by Archimedes, and Pappus of Alexandria; among the moderns by Kepler, Cavaleri, Roberval, Fermat and Wallis. With consummate ability Dr. Wallis had improved upon the labours of his predecessors: with a higher power, Newton moved forwards from where Wallis stopped. Our author first invented his celebrated BINOMIAL THEOREM. And then, applying this Theorem to the rectification of curves, and to the determination of the surfaces and contents of solids, and the position of their centres of gravity, he discovered the general principle of deducing the areas of curves from the ordinate, by considering the area as a nascent quantity, increasing by continual fluxion in the proportion of the length of the ordinate, and supposing the abscissa to increase uniformly in proportion to the time. Regarding lines as generated by the motion of points, surfaces by the motion of lines, and solids by the motion of surfaces, and considering that the ordinates, abscissae, &c., of curves thus formed, vary according to a regular law depending on the equation of the curve, he deduced from this equation the velocities with which these quantities are generated, and obtained by the rules of infinite series, the ultimate value required. To the velocities with which every line or quantity is generated, he gave the name of FLUXIONS, and to the lines or quantities themselves, that of FLUENTS. A discovery that successively baffled the acutest and strongest

intellects:—that, variously modified, has proved of incalculable service in aiding to develope the most abstruse and the highest truths in Mathematics and Astronomy: and that was of itself enough to render any name illustrious in the crowded Annals of Science.

At this period, the most distinguished philosophers were directing all their energies to the subject of light and the improvement of the refracting telescope. Newton, having applied himself to the grinding of "optic glasses of other figures than spherical," experienced the impracticability of executing such lenses; and conjectured that their defects, and consequently those of refracting telescopes, might arise from some other cause than the imperfect convergency of rays to a single point. He accordingly "procured a triangular glass prism to try therewith the celebrated phenomena of colours." His experiments, entered upon with zeal, and conducted with that industry, accuracy, and patient thought, for which he was so remarkable, resulted in the grand conclusion, that LIGHT WAS NOT HOMOGENEOUS, BUT CONSISTED OF RAYS, SOME OF WHICH WERE MORE REFRANGIBLE THAN OTHERS. This profound and beautiful discovery opened up a new era in the History of Optics. As bearing, however, directly upon the construction of telescopes, he saw that a lens refracting exactly like a prism would necessarily bring the different rays to different foci, at different distances from the glass, confusing and rendering the vision indistinct. Taking for granted that all bodies produced spectra of equal length, he dismissed all further consideration of the refracting instrument, and took up the principle of reflection. Rays of all colours, he found, were reflected regularly, so that the angle of reflection was equal to the angle of incidence, and hence he concluded that *optical instruments might be brought to any degree of perfection imaginable*, provided reflecting specula of the requisite figure and finish could be obtained. At this stage of his optical researches, he was forced to leave Cambridge on account of the plague which was then desolating England.

He retired to Woolsthorpe. The old manor-house, in which he was born, was situated in a beautiful little valley, on the west side of the river Witham; and here in the quiet home of his boyhood,

he passed his days in serene contemplation, while the stalking pestilence was hurrying its tens of thousands into undistinguishable graves.

Towards the close of a pleasant day in the early autumn of 1666, he was seated alone beneath a tree, in his garden, absorbed in meditation. He was a slight young man; in the twenty-fourth year of his age; his countenance mild and full of thought. For a century previous, the science of Astronomy had advanced with rapid strides. The human mind had risen from the gloom and bondage of the middle ages, in unparalleled vigour, to unfold the system, to investigate the phenomena, and to establish the laws of the heavenly bodies. Copernicus, Tycho Brahe, Kepler, Galileo, and others had prepared and lighted the way for him who was to give to their labour its just value, and to their genius its true lustre. At his bidding isolated facts were to take order as parts of one harmonious whole, and sagacious conjectures grow luminous in the certain splendour of demonstrated truth. And this ablest man had come—was here. His mind, familiar with the knowledge of past effort, and its unequalled faculties developed in transcendant strength, was now moving on to the very threshold of its grandest achievement. Step by step the untrodden path was measured, till, at length, the entrance seemed disclosed, and the tireless explorer to stand amid the first opening wonders of the universe.

The nature of gravity—that mysterious power which causes all bodies to descend towards the centre of the earth—had, indeed, dawned upon him. And reason busily united link to link of that chain which was yet to be traced joining the least to the vastest, the most remote to the nearest, in one harmonious bond. From the bottoms of the deepest caverns to the summits of the highest mountains, this power suffers no sensible change: may not its action, then, extend to the moon? Undoubtedly: and further reflection convinced him that such a power might be sufficient for retaining that luminary in her orbit round the earth. But, though this power suffers no sensible variation, in the little change of distance from the earth's centre, at which we may place ourselves, yet, at the distance of the moon, may not its force undergo

more or less diminution? The conjecture appeared most probable: and, in order to estimate what the degree of diminution might be, he considered that if the moon be retained in her orbit by the force of gravity, the primary planets must also be carried round the sun by the like power; and, by comparing the periods of the several planets with their distances from the sun, he found that, if they were held in their courses by any power like gravity, its strength must decrease in the duplicate proportion of the increase of distance. In forming this conclusion, he supposed the planets to move in perfect circles, concentric to the sun. Now was this the law of the moon's motion? Was such a force, emanating from the earth and directed to the moon, sufficient, when diminished as the square of the distance, to retain her in her orbit? To ascertain this master-fact, he compared the space through which heavy bodies fall, in a second of time, at a given distance from the centre of the earth, namely, at its surface, with the space through which the moon falls, as it were, to the earth, in the same time, while revolving in a circular orbit. He was absent from books; and, therefore, adopted, in computing the earth's diameter, the common estimate of sixty miles to a degree of latitude as then in use among geographers and navigators. The result of his calculations did not, of course, answer his expectations; hence, he concluded that some other cause, beyond the reach of observation—analogous, perhaps, to the vortices of Descartes—joined its action to that of the power of gravity upon the moon. Though by no means satisfied, he yet abandoned awhile further inquiry, and remained totally silent upon the subject.

These rapid marches in the career of discovery, combined with the youth of Newton, seem to evince a penetration the most lively, and an invention the most exuberant. But in him there was a conjunction of influences as extraordinary as fortunate. Study, unbroken, persevering and profound carried on its informing and disciplining work upon a genius, natively the greatest, and rendered freest in its movements, and clearest in its vision, through the untrammelling and enlightening power of religion. And, in this happy concurrence, are to be sought the elements of those amazing abilities, which, grasping, with equal facility, the

minute and the stupendous, brought these successively to light, and caused science to make them her own.

In 1667, Newton was made a Junior Fellow; and, in the year following, he took his degree of Master of Arts, and was appointed to a Senior Fellowship.

On his return to Cambridge, in 1668, he resumed his optical labours. Having thought of a delicate method of polishing metal, he proceeded to the construction of his newly projected reflecting telescope; a small specimen of which he actually made with his own hands. It was six inches long; and magnified about forty times;—a power greater than a refracting instrument of six feet tube could exert with distinctness. Jupiter, with his four satellites, and the horns, or moon-like phases of Venus were plainly visible through it. THIS WAS THE FIRST REFLECTING TELESCOPE EVER EXECUTED AND DIRECTED TO THE HEAVENS. He gave an account of it, in a letter to a friend, dated February 23d, 1668-9—a letter which is also remarkable for containing the first allusion to his discoveries "concerning the nature of light." Encouraged by the success of his first experiment, he again executed with his own hands, not long afterward, a second and superior instrument of the same kind. The existence of this having come to the knowledge of the Royal Society of London, in 1671, they requested it of Newton for examination. He accordingly sent it to them. It excited great admiration; it was shown to the king: a drawing and description of it was sent to Paris; and the telescope itself was carefully preserved in the Library of the Society. Newton lived to see his invention in public use, and of eminent service in the cause of science.

In the spring of 1669, he wrote to his friend Francis Aston, Esq., then about setting out on his travels, a letter of advice and directions, it was dated May 18th, and is interesting as exhibiting some of the prominent features in Newton's character. Thus:—

"Since in your letter you give me so much liberty of spending my judgment about what may be to your advantage in travelling, I shall do it more freely than perhaps otherwise would have been decent. First, then, I will lay down some general rules, most of

which, I believe, you have considered already; but if any of them be new to you, they may excuse the rest; if none at all, yet is my punishment more in writing than yours in reading.

"When you come into any fresh company. 1. Observe their humours. 2. Suit your own carriage thereto, by which insinuation you will make their converse more free and open. 3. Let your discourse be more in queries and doubtings than peremptory assertions or disputings, it being the design of travellers to learn, not to teach. Besides, it will persuade your acquaintance that you have the greater esteem of them, and so make them more ready to communicate what they know to you; whereas nothing sooner occasions disrespect and quarrels than peremptoriness. You will find little or no advantage in seeming wiser or much more ignorant than your company. 4. Seldom discommend anything though never so bad, or do it but moderately, lest you be unexpectedly forced to an unhandsome retraction. It is safer to commend any thing more than it deserves, than to discommend a thing so much as it deserves; for commendations meet not so often with oppositions, or, at least, are not usually so ill resented by men that think otherwise, as discommendations; and you will insinuate into men's favour by nothing sooner than seeming to approve and commend what they like; but beware o doing it by comparison. 5. If you be affronted, it is better, in a foreign country, to pass it by in silence, and with a jest, though with some dishonour, than to endeavour revenge; for, in the first case, your credit's ne'er the worse when you return into England, or come into other company that have not heard of the quarrel. But, in the second case, you may bear the marks of the quarrel while you live, if you outlive it at all. But, if you find yourself unavoidably engaged, 'tis best, I think, if you can command your passion and language, to keep them pretty evenly at some certain moderate pitch, not much heightening them to exasperate your adversary, or provoke his friends, nor letting them grow overmuch dejected to make him insult. In a word, if you can keep reason above passion, that and watchfulness will be your best defendants. To which purpose you may consider, that, though such excuses as this—He provok't me so much I could not forbear—may pass

among friends, yet amongst strangers they are insignificant, and only argue a traveller's weakness.

"To these I may add some general heads for inquiries or observations, such as at present I can think on. As, 1. To observe the policies, wealth, and state affairs of nations, so far as a solitary traveller may conveniently do. 2. Their impositions upon all sorts of people, trades, or commodities, that are remarkable. 3. Their laws and customs, how far they differ from ours. 4. Their trades and arts wherein they excel or come short of us in England. 5. Such fortifications as you shall meet with, their fashion, strength, and advantages for defence, and other such military affairs as are considerable. 6. The power and respect belonging to their degrees of nobility or magistracy. 7. It will not be time misspent to make a catalogue of the names and excellencies of those men that are most wise, learned, or esteemed in any nation. 8. Observe the mechanism and manner of guiding ships. 9. Observe the products of Nature in several places, especially in mines, with the circumstances of mining and of extracting metals or minerals out of their ore, and of refining them; and if you meet with any transmutations out of their own species into another (as out of iron into copper, out of any metal into quicksilver, out of one salt into another, or into an insipid body, &c.), those, above all, will be worth your noting, being the most luciferous, and many times lucriferous experiments, too, in philosophy. 10. The prices of diet and other things. 11. And the staple commodities of places.

"These generals (such as at present I could think of), if they will serve for nothing else, yet they may assist you in drawing up a model to regulate your travels by. As for particulars, these that follow are all that I can now think of, viz.; whether at Schemnitium, in Hungary (where there are mines of gold, copper, iron, vitriol, antimony, &c.). they change iron into copper by dissolving it in a vitriolate water, which they find in cavities of rocks in the mines, and then melting the slimy solution in a strong fire, which in the cooling proves copper. The like is said to be done in other places, which I cannot now remember; perhaps, too, it may be done in Italy. For about twenty or thirty years agone there was

a certain vitriol came from thence (called Roman vitriol), but of a nobler virtue than that which is now called by that name; which vitriol is not now to be gotten, because, perhaps, they make a greater gain by some such trick as turning iron into copper with it than by selling it. 2. Whether, in Hungary, Sclavonia, Bohemia, near the town Eila, or at the mountains of Bohemia near Silesia, there be rivers whose waters are impregnated with gold; perhaps, the gold being dissolved by some corrosive water like *aqua regis*, and the solution carried along with the stream, that runs through the mines. And whether the practice of laying mercury in the rivers, till it be tinged with gold, and then straining the mercury through leather, that the gold may stay behind, be a secret yet, or openly practised. 3. There is newly contrived, in Holland, a mill to grind glasses plane withal, and I think polishing them too; perhaps it will be worth the while to see it. 4. There is in Holland one —— Borry, who some years since was imprisoned by the Pope, to have extorted from him secrets (as I am told) of great worth, both as to medicine and profit, but he escaped into Holland, where they have granted him a guard. I think he usually goes clothed in green. Pray inquire what you can of him, and whether his ingenuity be any profit to the Dutch. You may inform yourself whether the Dutch have any tricks to keep their ships from being all worm-eaten in their voyages to the Indies. Whether pendulum clocks do any service in finding out the longitude, &c.

"I am very weary, and shall not stay to part with a long compliment, only I wish you a good journey, and God be with you."

It was not till the month of June, 1669, that our author made known his Method of Fluxions. He then communicated the work which he had composed upon the subject, and entitled, ANALYSIS PER EQUATIONES NUMERO TERMINORUM INFINITAS, to his friend Dr. Barrow. The latter, in a letter dated 20th of the same month, mentioned it to Mr. Collins, and transmitted it to him, on the 31st of July thereafter. Mr. Collins greatly approved of the work; took a copy of it; and sent the original back to Dr. Barrow. During the same and the two following years, Mr

Collins, by his extensive correspondence, spread the knowledge of this discovery among the mathematicians in England, Scotland, France, Holland and Italy.

Dr. Barrow, having resolved to devote himself to Theology, resigned the Lucasian Professorship of Mathematics, in 1669, in favour of Newton, who accordingly received the appointment to the vacant chair.

During the years 1669, 1670, and 1671, our author, as such Professor, delivered a course of Optical Lectures. Though these contained his principal discoveries relative to the different refrangibility of light, yet the discoveries themselves did not become publicly known, it seems, till he communicated them to the Royal Society, a few weeks after being elected a member thereof, in the spring of 1671-2. He now rose rapidly in reputation, and was soon regarded as foremost among the philosophers of the age. His paper on light excited the deepest interest in the Royal Society, who manifested an anxious solicitude to secure the author from the "arrogations of others," and proposed to publish his discourse in the monthly numbers in which the Transactions were given to the world. Newton, gratefully sensible of these expressions of esteem, willingly accepted of the proposal for publication. He gave them also, at this time, the results of some further experiments in the decomposition and re-composition of light:—that the same degree of refrangibility always belonged to the same colour, and the same colour to the same degree of refrangibility: that the seven different colours of the spectrum were original, or simple, and that *whiteness*, or white light was a compound of all these seven colours.

The publication of his new doctrines on light soon called forth violent opposition as to their soundness. Hooke and Huygens—men eminent for ability and learning—were the most conspicuous of the assailants. And though Newton effectually silenced all his adversaries, yet he felt the triumph of little gain in comparison with the loss his tranquillity had sustained. He subsequently remarked in allusion to this controversy—and to one with whom he was destined to have a longer and a bitterer conflict—"I was so persecuted with discussions arising from the publication of my

theory of light, that I blamed my own imprudence for parting with so substantial a blessing as my quiet to run after a shadow."

In a communication to Mr. Oldenburg, Secretary of the Royal Society, in 1672, our author stated many valuable suggestions relative to the construction of REFLECTING MICROSCOPES which he considered even more capable of improvement than telescopes. He also contemplated, about the same time, an edition of Kinckhuysen's Algebra, with notes and additions; partially arranging, as an introduction to the work, a treatise, entitled, A Method of Fluxions; but he finally abandoned the design. This treatise, however, he resolved, or rather consented, at a late period of his life, to put forth separately; and the plan would probably have been carried into execution had not his death intervened. It was translated into English, and published in 1736 by John Colson, Professor of Mathematics in Cambridge.

Newton, it is thought, made his discoveries concerning the INFLECTION and DIFFRACTION of light before 1674. The phenomena of the inflection of light had been first discovered more than ten years before by Grimaldi. And Newton began by repeating one of the experiments of the learned Jesuit—admitting a beam of the sun's light through a small pin hole into a dark chamber: the light diverged from the aperture in the form of a cone, and the shadows of all bodies placed in this light were larger than might have been expected, and surrounded with three coloured fringes, the nearest being widest, and the most remote the narrowest. Newton, advancing upon this experiment, took exact measures of the diameter of the shadow of a human hair, and of the breadth of the fringes, at different distances behind it, and discovered that these diameters and breadths were not proportional to the distances at which they were measured. He hence supposed that the rays which passed by the edge of the hair were deflected or turned aside from it, as if by a repulsive force, the nearest rays suffering the greatest, the more remote a less degree of deflection. In explanation of the coloured fringes, he queried: whether the rays which differ in refrangibility do not differ also in flexibility, and whether they are not, by these different inflections, separated from one another, so as after separa-

tion to make the colours in the three fringes above described? Also, whether the rays, in passing by the edges and sides of bodies, are not bent several times backwards and forwards with an eel-like motion—the three fringes arising from three such bendings? His inquiries on this subject were here interrupted and never renewed.

His Theory of the COLOURS of NATURAL BODIES was communicated to the Royal Society, in February, 1675. This is justly regarded as one of the profoundest of his speculations. The fundamental principles of the Theory in brief, are:—That bodies possessing the greatest refractive powers reflect the greatest quantity of light; and that, at the confines of equally refracting media, there is no reflection. That the minutest particles of almost all natural bodies are in some degree transparent. That between the particles of bodies there are pores, or spaces, either empty or filled with media of a less density than the particles themselves. That these particles, and pores or spaces, have some definite size. Hence he deduced the Transparency, Opacity, and colours of natural bodies. Transparency arises from the particles and their pores being too small to cause reflection at their common surfaces—the light all passing through; Opacity from the opposite cause of the particles and their pores being sufficiently large to reflect the light which is "stopped or stifled" by the multitude of reflections; and colours from the particles, according to their several sizes, reflecting rays of one colour and transmitting those of another—or in other words, the colour that meets the eye is the colour reflected, while all the other rays are transmitted or absorbed.

Analogous in origin to the colours of natural bodies, he considered the COLOURS OF THIN PLATES. This subject was interesting and important, and had attracted considerable investigation. He, however, was the first to determine the law of the production of these colours, and, during the same year made known the results of his researches herein to the Royal Society. His mode of procedure in these experiments was simple and curious. He placed a double convex lens of a large known radius of curvature, upon the flat surface of a plano-convex object glass. Thus, from

their point of contact at the centre, to the circumference of the lens, he obtained plates of air, or spaces varying from the extremest possible thinness, by slow degrees, to a considerable thickness. Letting the light fall, every different thickness of this plate of air gave different colours—the point of contact of the lens and glass forming the centre of numerous concentric colored rings. Now the radius of curvature of the lens being known, the thickness of the plate of air, at any given point, or where any particular colour appeared, could be exactly determined. Carefully noting, therefore, the order in which the different colours appeared, he measured, with the nicest accuracy, the different thicknesses at which the most luminous parts of the rings were produced, whether the medium were air, water, or mica—all these substances giving the same colours at different thicknesses;—the ratio of which he also ascertained. From the phenomena observed in these experiments, Newton deduced his Theory of Fits of EASY REFLECTION AND TRANSMISSION of light. It consists in supposing that every particle of light, from its first discharge from a luminous body, possesses, at equally distant intervals, dispositions to be reflected from, or transmitted through the surfaces of bodies upon which it may fall. For instance, if the rays are in a Fit of Easy Reflection, they are on reaching the surface, repelled, thrown off, or reflected from it; if, in a Fit of Easy Transmission, they are attracted, drawn in, or transmitted through it. By this Theory of Fits, our author likewise explained the colours of thick plates.

He regarded light as consisting of small material particles emitted from shining substances. He thought that these particles could be re-combined into solid matter, so that "gross bodies and light were convertible into one another;" that the particles of light and the particles of solid bodies acted mutually upon each other; those of light agitating and heating those of solid bodies, and the latter attracting and repelling the former. Newton was the first to suggest the idea of the POLARIZATION of light.

In the paper entitled *An Hypothesis Explaining Properties of Light*, December, 1675, our author first introduced his opinions respecting Ether—opinions which he afterward abandoned and again

permanently resumed—"A most subtle spirit which pervades" all bodies, and is expanded through all the heavens. It is electric, and almost, if not quite immeasurably elastic and rare. "By the force and action of which spirit the particles of bodies mutually attract one another, at near distances, and cohere, if contiguous; and electric bodies operate at greater distances, as well repelling as attracting the neighbouring corpuscles; and light is emitted, reflected, refracted, inflected and heats bodies; and all sensation is excited, and the members of animal bodies move at the command of the will, namely, by the vibrations of this spirit, mutually propagated along the solid filaments of the nerves, from the outward organs of sense to the brain, and from the brain into the muscles." This "spirit" was no *anima mundi;* nothing further from the thought of Newton; but was it not, on his part, a partial recognition of, or attempt to reach an ultimate material force, or primary element, by means of which, "in the roaring loom of time," this material universe, God's visible garment, may be woven for us?

The Royal Society were greatly interested in the results of some experiments, which our author had, at the same time, communicated to them relative to the excitation of electricity in glass; and they, after several attempts and further direction from him, succeeded in re-producing the same phenomena.

One of the most curious of Newton's minor inquiries related to the connexion between the refractive powers and chemical composition of bodies. He found on comparing the refractive powers and the densities of many different substances, that the former were very nearly proportional to the latter, in the same bodies. Unctuous and sulphureous bodies were noticed as remarkable exceptions—as well as the *diamond*—their refractive powers being two or three times greater in respect of their densities than in the case of other substances, while, as among themselves, the one was generally proportional to the other. He hence inferred as to the diamond a great degree of combustibility;—a conjecture which the experiments of modern chemistry have shown to be true.

The chemical researches of our author were probably pursued with more or less diligence from the time of his witnessing some

of the practical operations in that science at the Apothecary's at Grantham. De Natura Acidorum is a short chemical paper, on various topics, and published in Dr. Horsley's Edition of his works. Tabula Quantitatum et Graduum Coloris was inserted in the Philosophical Transactions; it contains a comparative scale of temperature from that of melting ice to that of a small kitchen coal-fire. He regarded fire as a body heated so hot as to emit light copiously; and flame as a vapour, fume, or exhalation heated so hot as to shine. To elective attraction, by the operation of which the small particles of bodies, as he conceived, act upon one another, at distances so minute as to escape observation, he ascribed all the various chemical phenomena of precipitation, combination, solution, and crystallization, and the mechanical phenomena of cohesion and capillary attraction. Newton's chemical views were illustrated and confirmed, in part, at least, in his own life-time. As to the structure of bodies, he was of opinion "that the smallest particles of matter may cohere by the strongest attractions, and compose bigger particles of weaker virtue; and many of these may cohere and compose bigger particles whose virtue is still weaker; and so on for divers successions, until the progression end in the biggest particles, on which the operations in chemistry and the colours of natural bodies depend, and which by adhering, compose bodies of sensible magnitude."

There is good reason to suppose that our author was a diligent student of the writings of Jacob Behmen; and that in conjunction with a relative, Dr. Newton, he was busily engaged, for several months in the earlier part of life, in quest of the philosopher's tincture. "Great Alchymist," however, very imperfectly describes the character of Behmen, whose researches into things material and things spiritual, things human and things divine, afford the strongest evidence of a great and original mind.

More appropriately here, perhaps, than elsewhere, may be given Newton's account of some curious experiments, made in his own person, on the action of light upon the retina. Locke, who was an intimate friend of our author, wrote to him for his opinion on a certain fact stated in Boyle's Book of Colours. Newton, in

his reply, dated June 30th, 1691, narrates the following circumstances, which probably took place in the course of his optical researches. Thus:—

"The observation you mention in Mr. Boyle's Book of Colours I once tried upon myself with the hazard of my eyes. The manner was this; I looked a very little while upon the sun in the looking-glass with my right eye, and then turned my eyes into a dark corner of my chamber, and winked, to observe the impression made, and the circles of colours which encompassed it, and how they decayed by degrees, and at last vanished. This I repeated a second and a third time. At the third time, when the phantasm of light and colours about it were almost vanished, intending my fancy upon them to see their last appearance, I found, to my amazement, that they began to return, and by little and little to become as lively and vivid as when I had newly looked upon the sun. But when I ceased to intend my fancy upon them, they vanished again. After this, I found, that as often as I went into the dark, and intended my mind upon them, as when a man looks earnestly to see anything which is difficult to be seen, I could make the phantasm return without looking any more upon the sun; and the oftener I made it return, the more easily I could make it return again. And, at length, by repeating this, without looking any more upon the sun, I made such an impression on my eye, that, if I looked upon the clouds, or a book, or any bright object, I saw upon it a round bright spot of light like the sun, and, which is still stranger, though I looked upon the sun with my right eye only, and not with my left, yet my fancy began to make an impression upon my left eye, as well us upon my right. For if I shut my right eye, or looked upon a book, or the clouds, with my left eye, I could see the spectrum of the sun almost as plain as with my right eye, if I did but intend my fancy a little while upon it; for at first, if I shut my right eye, and looked with my left, the spectrum of the sun did not appear till I intended my fancy upon it; but by repeating, this appeared every time more easily. And now, in a few hours' time, I had brought my eyes to such a pass, that I could look upon no bright object with either eye, but I saw the sun before me, so that I durst neither write

nor read; but to recover the use of my eyes, shut myself up in my chamber made dark, for three days together, and used all means to divert my imagination from the sun. For if I thought upon him, I presently saw his picture, though I was in the dark. But by keeping in the dark, and employing my mind about other things, I began in three or four days to have some use of my eyes again; and by forbearing to look upon bright objects, recovered them pretty well, though not so well but that, for some months after, the spectrum of the sun began to return as often as I began to meditate upon the phenomena, even though I lay in bed at midnight with my curtains drawn. But now I have been very well for many years, though I am apt to think, if I durst venture my eyes, I could still make the phantasm return by the power of my fancy. This story I tell you, to let you understand, that in the observation related by Mr. Boyle, the man's fancy probably concurred with the impression made by the sun's light to produce that phantasm of the sun which he constantly saw in bright objects. And so your question about the cause of phantasm involves another about the power of fancy, which I must confess is too hard a knot for me to untie. To place this effect in a constant motion is hard, because the sun ought then to appear perpetually. It seems rather to consist in a disposition of the sensorium to move the imagination strongly, and to be easily moved, both by the imagination and by the light, as often as bright objects are looked upon."

Though Newton had continued silent, yet his thoughts were by no means inactive upon the vast subject of the planetary motions. The idea of Universal Gravitation, first caught sight of, so to speak, in the garden at Woolsthorpe, years ago, had gradually expanded upon him. We find him, in a letter to Dr. Hooke, Secretary of the Royal Society, dated in November, 1679, proposing to verify the motion of the earth by direct experiment, namely, by the observation of the path pursued by a body falling from a considerable height. He had concluded that the path would be spiral; but Dr. Hooke maintained that it would be an eccentric ellipse in vacuo, and an ellipti-spiral in a resisting medium. Our author, aided by this correction of his error, and by

the discovery that a projectile would move in an elliptical orbit when under the influence of a force varying inversely as the square of the distance, was led to discover "the theorem by which he afterwards examined the ellipsis;" and to demonstrate the celebrated proposition that a planet acted upon by an attractive force varying inversely as the squares of the distances will describe an elliptical orbit, in one of whose foci the attractive force resides.

When he was attending a meeting of the Royal Society, in June 1682, the conversation fell upon the subject of the measurement of a degree of the meridian, executed by M. Picard, a French Astronomer, in 1679. Newton took a memorandum of the result; and afterward, at the earliest opportunity, computed from it the diameter of the earth: furnished with these new data, he resumed his calculation of 1666. As he proceeded therein, he saw that his early expectations were now likely to be realized; the thick rushing, stupendous results overpowered him; he became unable to carry on the process of calculation, and intrusted its completion to one of his friends. The discoverer had, indeed, grasped the master-fact. The law of falling bodies at the earth's surface was at length identified with that which guided the moon in her orbit. And so his GREAT THOUGHT, that had for sixteen years loomed up in dim, gigantic outline, amid the first dawn of a plausible hypothesis, now stood forth, radiant and not less grand, in the mid-day light of demonstrated truth.

It were difficult, nay impossible to imagine, even, the influence of a result like this upon a mind like Newton's. It was as if the keystone had been fitted to the glorious arch by which his spirit should ascend to the outskirts of infinite space—spanning the immeasurable—weighing the imponderable—computing the incalculable —mapping out the marchings of the planets, and the far-wanderings of the comets, and catching, bring back to earth some clearer notes of that higher melody which, as a sounding voice, bears perpetual witness to the design and omnipotence of a creating Deity.

Newton, extending the law thus obtained, composed a series of about twelve propositions on the motion of the primary planets

about the sun. These were sent to London, and communicated to the Royal Society about the end of 1683. At or near this period, other philosophers, as Sir Christopher Wren, Dr. Halley, and Dr. Hooke, were engaged in investigating the same subject; but with no definite or satisfactory results. Dr. Halley, having seen, it is presumed, our author's propositions, went in August, 1684, to Cambridge to consult with him upon the subject. Newton assured him that he had brought the demonstration to perfection. In November, Dr. Halley received a copy of the work; and, in the following month, announced it to the Royal Society, with the author's promise to have it entered upon their Register. Newton, subsequently reminded by the Society of his promise, proceeded in the diligent preparation of the work, and, though suffering an interruption of six weeks, transmitted the manuscript of the first book to London before the end of April. The work was entitled PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA, dedicated to the Royal Society, and presented thereto on the 28th of April, 1685-6. The highest encomiums were passed upon it; and the council resolved, on the 19th of May, to print it at the expense of the Society, and under the direction of Dr. Halley. The latter, a few days afterward, communicated these steps to Newton, who, in a reply, dated the 20th of June, holds the following language:—"The proof you sent me I like very well. I designed the whole to consist of three books; the second was finished last summer, being short, and only wants transcribing, and drawing the cuts fairly. Some new propositions I have since thought on, which I can as well let alone. The third wants the theory of comets. In autumn last, I spent two months in calculation to no purpose for want of a good method. which made me afterward return to the first book, and enlarge it with diverse propositions, some relating to comets, others to other things found out last winter. The third I now design to suppress. Philosophy is such an impertinently litigious lady, that a man had as good be engaged in law-suits as have to do with her. I found it so formerly, and now I can no sooner come near her again, but she gives me warning. The first two books without the third will not so well bear the title of *Philosophiæ Naturalis*

Principia Mathematicia; and thereupon I had altered it to this, *De Motu Corporum Libri duo.* But after second thought I retain the former title. It will help the sale of the book, which I ought not to diminish now 'tis yours."

This "warning" arose from some pretensions put forth by Dr. Hooke. And though Newton gave a minute and positive refutations of such claims, yet, to reconcile all differences, he generously added to Prop. IV. Cor. 6, Book I., a Scholium, in which Wren, Hooke and Halley are acknowledged to have independently deduced the law of gravity from the second law of Kepler.

The suppression of the third book Dr. Halley could not endure to see. "I must again beg you" says he, "not to let your resentments run so high as to deprive us of your third book, wherein your applications of your mathematical doctrine to the theory of comets, and several curious experiments, which, as I guess by what you write ought to compose it, will undoubtedly render it acceptable to those who will call themselves philosophers without mathematics, which are much the greater number." To these solicitations Newton yielded. There were no "resentments," however, as we conceive, in his "design to suppress." He sought peace; for he loved and valued it above all applause. But, in spite of his efforts for tranquillity's sake, his course of discovery was all along molested by ignorance or presumptuous rivalry.

The publication of the great work now went rapidly forwards. The second book was sent to the Society, and presented on the 2d March; the third, on the 6th April; and the whole was completed and published in the month of May, 1686-7. In the second Lemma of the second book, the fundamental principle of his fluxionary calculus was, for the first time, given to the world; but its algorithm or notation did not appear till published in the second volume of Dr. Wallis's works, in 1693.

And thus was ushered into existence The PRINCIPIA—a work to which pre-eminence above all the productions of the human intellect has been awarded—a work that must be esteemed of priceless worth so long as Science has a votary, or a single worshipper be left to kneel at the altar of Truth.

The entire work bears the general title of THE MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY. It consists of three books: the first two, entitled, OF THE MOTION OF BODIES, are occupied with the laws and conditions of motions and forces, and are illustrated with many scholia treating of some of the most general and best established points in philosophy, such as the density and resistance of bodies, spaces void of matter, and the motion of sound and light. From these principles, there is deduced, in the third book, drawn up in as popular a style as possible and entitled, OF THE SYSTEM OF THE WORLD, the constitution of the system of the world. In regard to this book, the author says—"I had, indeed, composed the third Book in a popular method, that it might be read by many; but afterwards, considering that such as had not sufficently entered into the principles could not easily discover the strength of the consequences, nor lay aside the prejudices to which they had been many years accustomed, therefore, to prevent disputes which might be raised upon such accounts, I chose to reduce the substance of this Book into the form of Propositions (in the mathematical way), which should be read by those only who had first made themselves masters of the principles established in the preceding Books: not that I would advise any one to the previous study of every Proposition of those Books."—"It is enough if one carefully reads the Definitions, the Laws of Motion, and the three first Sections of the first Book. He may then pass on to this Book, and consult such of the remaining Propositions of the first two Books, as the references in this, and his occasions shall require." So that "The System of the World" is composed both "in a popular method," and in the form of mathematical Propositions.

The principle of Universal Gravitation, namely, *that every particle of matter is attracted by, or gravitates to, every other particle of matter, with a force inversely proportional to the squares of their distances*—is the discovery which characterizes The PRINCIPIA. This principle the author deduced from the motion of the moon, and the three laws of Kepler—laws, which Newton, in turn, by his greater law, demonstrated to be true.

From the first law of Kepler, namely, the proportionality of

the areas to the times of their description, our author inferred that the force which retained the planet in its orbit was always directed to the sun; and from the second, namely, that every planet moves in an ellipse with the sun in one of its foci, he drew the more general inference that the force by which the planet moves round that focus varies inversely as the square of its distance therefrom: and he demonstrated that a planet acted upon by such a force could not move in any other curve than a conic section; showing when the moving body would describe a circular, an elliptical, a parabolic, or hyperbolic orbit. He demonstrated, too, that this force, or attracting, gravitating power resided in every, the least particle; but that, in spherical masses, it operated as if confined to their centres; so that, one sphere or body will act upon another sphere or body, with a force directly proportional to the quantity of matter, and inversely as the square of the distance between their centres; and that their velocities of mutual approach will be in the inverse ratio of their quantities of matter. Thus he grandly outlined the Universal Law. Verifying its truth by the motions of terrestrial bodies, then by those of the moon and other secondary orbs, he finally embraced, in one mighty generalization, the entire Solar System—all the movements of all its bodies—planets, satellites and comets—explaining and harmonizing the many diverse and theretofore inexplicable phenomena.

Guided by the genius of Newton, we see sphere bound to sphere, body to body, particle to particle, atom to mass, the minutest part to the stupendous whole—each to each, each to all, and all to each—in the mysterious bonds of a ceaseless, reciprocal influence. An influence whose workings are shown to be alike present in the globular dew-drop, or oblate-spheroidal earth; in the falling shower, or vast heaving ocean tides; in the flying thistle-down, or fixed, ponderous rock; in the swinging pendulum, or time-measuring sun; in the varying and unequal moon, or earth's slowly retrograding poles; in the uncertain meteor, or blazing comet wheeling swiftly away on its remote, yet determined round. An influence, in fine, that may link system to system through all the star-glowing firmament; then firmament to firma-

ment; aye, firmament to firmament, again and again, till, converging home, it may be, to some ineffable centre, where more presently dwells He who inhabiteth immensity, and where infinitudes meet and eternities have their conflux, and where around move, in softest, swiftest measure, all the countless hosts that crowd heaven's fathomless deeps.

And yet Newton, amid the loveliness and magnitude of Omnipotence, lost not sight of the Almighty One. A secondary, however universal, was not taken for the First Cause. An impressed force, however diffused and powerful, assumed not the functions of the creating, giving Energy. Material beauties, splendours, and sublimities, however rich in glory, and endless in extent, concealed not the attributes of an intelligent Supreme. From the depths of his own soul, through reason and the WORD, he had risen, *à priori*, to God: from the heights of Omnipotence, through the design and law of the builded universe, he proved *à posteriori*, a Deity. "I had," says he, "an eye upon such principles as might work, with considering men, for the belief of a Deity," in writing the PRINCIPIA; at the conclusion whereof, he teaches that—"this most beautiful system of the sun, planets and comets, could only proceed from the counsel and dominion of an intelligent and powerful Being. And if the fixed stars are the centres of other like systems, these, being formed by the like wise counsels, must be all subject to the dominion of One; especially since the light of the fixed stars is of the same nature with the light of the sun, and from every system light passes into all other systems: and lest the systems of the fixed stars should, by their gravity, fall on each other mutually, he hath placed those systems at immense distances one from another.

"This Being governs all things, not as the soul of the world, but as Lord over all; and on account of his dominion he is wont to be called Lord God παντοκρατωρ or Universal Ruler; for God is a relative word, and has a respect to servants; and Deity is the dominion of God, not over his own body, as those imagine who fancy God to be the soul of the world, but over servants. The Supreme God is a Being eternal, infinite, absolutely perfect; but a being, however perfect, without dominion, cannot be said to

be Lord God; for we say, my God, your God, the God of Israel, the God of Gods, and Lord of Lords; but we do not say, my Eternal, your Eternal, the Eternal of Israel, the Eternal of Gods; we do not say my Infinite, or my Perfect: these are titles which have no respect to servants. The word God usually signifies Lord; but every Lord is not God. It is the dominion of a spiritual Being which constitutes a God; a true, supreme, or imaginary dominion makes a true, supreme, or imaginary God. And from his true dominion it follows that the true God is a living, intelligent and powerful Being; and from his other perfections, that he is supreme or most perfect. He is eternal and infinite, omnipotent and omniscient; that is, his duration reaches from eternity to eternity; his presence from infinity to infinity; he governs all things and knows all things, that are or can be done. He is not eternity or infinity, but eternal and infinite; he is not duration and space, but he endures and is present. He endures forever and is everywhere present; and by existing always and everywhere, he constitutes duration and space. Since every particle of space is *always*, and every indivisible moment of duration is *everywhere*, certainly the Maker and Lord of things cannot be *never* and *nowhere*. Every soul that has perception is, though in different times and different organs of sense and motion, still the same indivisible person. There are given successive parts in duration, co-existent parts in space, but neither the one nor the other in the person of a man, or his thinking principle; and much less can they be found in the thinking substance of God. Every man, so far as he is a thing that has perception, is one and the same man during his whole life, in all and each of his organs of sense. God is one and the same God, always and everywhere. He is omnipresent, not *virtually* only, but also *substantially;* for virtue cannot subsist without substance. In him are all things contained and moved; yet neither affects the other; God suffers nothing from the motion of bodies; bodies find no resistance from the omnipresence of God. It is allowed by all that the Supreme God exists necessarily; and by the same necessity he exists *always* and *everywhere*. Whence also he is all similar, all eye, all ear, all brain, all arm, all power

to perceive, to understand, and to act; but in a manner not at all human, in a manner not at all corporeal, in a manner utterly unknown to us. As a blind man has no idea of colours, so have we no idea of the manner by which the all-wise God perceives and understands all things. He is utterly void of all body, and bodily figure, and can therefore neither be seen, nor heard, nor touched; nor ought he to be worshipped under the representation of any corporeal thing. We have ideas of his attributes, but what the real substance of anything is we know not. In bodies we see only their figures and colours, we hear only the sounds, we touch only their outward surfaces, we smell only the smells, and taste only the savours; but their inward substances are not to be known, either by our senses, or by any reflex act of our minds: much less, then, have we any idea of the substance of God. We know him only by his most wise and excellent contrivances of things, and final causes; we admire him for his perfections; but we rev erence and adore him on account of his dominion; for we adore him as his servants; and a god without dominion, providence, and final causes, is nothing else but Fate and Nature. Blind metaphysical necessity, which is certainly the same always and everywhere, could produce no variety of things. All that diversity of natural things which we find suited to different times and places could arise from nothing but the ideas and will of a Being necessarily existing."

Thus, the diligent student of science, the earnest seeker of truth, led, as through the courts of a sacred Temple, wherein, at each step, new wonders meet the eye, till, as a crowning grace, they stand before a Holy of Holies, and learn that all science and all truth are one which hath its beginning and its end in the knowledge of Him whose glory the heavens declare, and whose handiwork the firmament showeth forth.

The introduction of the pure and lofty doctrines of the PRINCIPIA was perseveringly resisted. Descartes, with his system of vortices, had sown plausibly to the imagination, and error had struck down deeply, and shot up luxuriantly, not only in the popular, but in the scientific mind. Besides the idea—in itself so simple and so grand—that the great masses of the planets were

suspended in empty space, and retained in their orbits by an invisible influence residing in the sun—was to the ignorant a thing inconceivable, and to the learned a revival of the occult qualities of the ancient physics. This remark applies particularly to the continent. Leibnitz misapprehended; Huygens in part rejected; John Bernouilli opposed; and Fontenelle never received the doctrines of the PRINCIPIA. So that, the saying of Voltaire is probably true, that though Newton survived the publication of his great work more than forty years, yet, at the time of his death, he had not above twenty followers out of England.

But in England, the reception of our author's philosophy was rapid and triumphant. His own labours, while Lucasian Professor; those of his successors in that Chair—Whiston and Saunderson; those of Dr. Samuel Clarke, Dr. Laughton, Roger Cotes, and Dr. Bentley; the experimental lectures of Dr. Keill and Desaguliers; the early and powerful exertions of David Gregory at Edinburgh, and of his brother James Gregory at St. Andrew's, tended to diffuse widely in England and Scotland a knowledge of, and taste for the truths of the PRINCIPIA. Indeed, its mathematical doctrines constituted, from the first, a regular part of academical instruction; while its physical truths, given to the public in popular lectures, illustrated by experiments, had, before the lapse of twenty years, become familiar to, and adopted by the general mind. Pemberton's popular "View of Sir Isaac Newton's Philosophy" was published, in 1728; and the year afterward, an English translation of the PRINCIPIA, and System of the World, by Andrew Motte. And since that period, the labours of Le Seur and Jacquier, of Thorpe, of Jebb, of Wright and others have greatly contributed to display the most hidden treasures of the PRINCIPIA.

About the time of the publication of the Principia, James II., bent on re-establishing the Romish Faith, had, among other illegal acts, ordered by mandamus, the University of Cambridge to confer the degree of Master of Arts upon an ignorant monk. Obedience to this mandate was resolutely refused. Newton was one of the nine delegates chosen to defend the independence of the University. They appeared before the High Court;—and

successfully: the king abandoned his design. The prominent part which our author took in these proceedings, and his eminence in the scientific world, induced his proposal as one of the parliamentary representatives of the University. He was elected, in 1688, and sat in the Convention Parliament till its dissolution. After the first year, however, he seems to have given little or no attention to his parliamentary duties, being seldom absent from the University till his appointment in the Mint, in 1695.

Newton began his theological researches sometime previous to 1691; in the prime of his years, and in the matured vigour of his intellectual powers. From his youth, as we have seen, he had devoted himself with an activity the most unceasing, and an energy almost superhuman to the discovery of physical truth;—giving to Philosophy a new foundation, and to Science a new temple. To pass on, then, from the consideration of the material, more directly to that of the spiritual, was a natural, nay, with so large and devout a soul, a necessary advance. The Bible was to him of inestimable worth. In the elastic freedom, which a pure and unswerving faith in Him of Nazareth gives, his mighty faculties enjoyed the only completest scope for development. His original endowment, however great, combined with a studious application, however profound, would never, without this liberation from the dominion of passion and sense, have enabled him to attain to that wondrous concentration and grasp of intellect, for which Fame has as yet assigned him no equal. Gratefully he owned, therefore, the same Author in the Book of Nature and the Book of Revelation. These were to him as drops of the same unfathomable ocean;—as outrayings of the same inner splendour;—as tones of the same ineffable voice;—as segments of the same infinite curve. With great joy he had found himself enabled to proclaim, as an interpreter, from the hieroglyphs of Creation, the existence of a God: and now, with greater joy, and in the fulness of his knowledge, and in the fulness of his strength, he laboured to make clear, from the utterances of the inspired Word, the far mightier confirmations of a Supreme Good, in all its glorious amplitude of Being and of Attribute; and to bring the infallible workings thereof plainly home to the understandings and the

affections of his fellow-men; and finally to add the weight of his own testimony in favour of that Religion, whose truth is now, indeed, "girded with the iron and the rock of a ponderous and colossal demonstration."

His work, entitled, OBSERVATIONS UPON THE PROPHECIES OF HOLY WRIT, PARTICULARLY THE PROPHECIES OF DANIEL AND THE APOCALYPSE OF ST. JOHN, first published in London, in 1733 4to. consists of two parts: the one devoted to the Prophecies of Daniel. and the other to the Apocalypse of St. John. In the first part, he treats concerning the compilers of the books of the Old Testament;—of the prophetic language;—of the vision of the four beasts;—of the kingdoms represented by the feet of the image composed of iron and clay;—of the ten kingdoms represented by the ten horns of the beast;—of the eleventh horn of Daniel's fourth beast; of the power which should change times and laws;—of the kingdoms represented in Daniel by the ram and he-goat;—of the prophecy of the seventy weeks;—of the times of the birth and passion of Christ;—of the prophecy of the Scripture of Truth;—of the king who doeth according to his will, and magnified himself above every god, and honoured Mahuzzims, and regarded not the desire of women;—of the Mahuzzim, honoured by the king who doeth according to his will. In the second part, he treats of the time when the Apocalypse was written, of the scene of the vision, and the relation which the Apocalypse has to the book of the law of Moses, and to the worship of God in the temple;—of the relation which the Apocalypse has to the prophecies of Daniel, and of the subject of the prophecy itself. Newton regards the prophecies as given, not for the gratification of man's curiosity, by enabling him to foreknow; but for his conviction that the world is governed by Providence, by witnessing their fulfilment. Enough of prophecy, he thinks, has already been fulfilled to afford the diligent seeker abundant evidence of God's providence. The whole work is marked by profound erudition, sagacity and argument.

And not less learning, penetration and masterly reasoning are conspicuous in his HISTORICAL ACCOUNT OF TWO NOTABLE CORRUPTIONS OF SCRIPTURES IN A LETTER TO A FRIEND. This

Treatise, first accurately published in Dr. Horsley's edition of his works, relates to two texts: the one, 1 Epistle of St. John v. 7; the other, 1 Epistle of St. Paul to Timothy iii. 16. As this work had the effect to deprive the advocates of the doctrine of the Trinity of two leading texts, Newton has been looked upon as an Arian; but there is absolutely nothing in his writings to warrant such a conclusion.

His remaining theological works consist of the LEXICON PROPHETICUM, which was left incomplete; a Latin Dissertation on the sacred cubit of the Jews, which was translated into English, and published, in 1737, among the Miscellaneous Works of John Greaves; and FOUR LETTERS *addressed to Dr. Bentley, containing some arguments in proof of a Deity.* These Letters were dated respectively: 10th December, 1692; 17th January, 1693; 25th February, 1693; and 11th February, 1693—the fourth bearing an earlier date than the third. The best faculties and the profoundest acquirements of our author are convincingly manifest in these lucid and powerful compositions. They were published in 1756, and reviewed by Dr. Samuel Johnson.

Newton's religious writings are distinguished by their absolute freedom from prejudice. Everywhere, throughout them, there glows the genuine nobleness of soul. To his whole life, indeed, we may here fitly extend the same observation. He was most richly imbued with the very spirit of the Scriptures which he so delighted to study and to meditate upon. His was a piety, so fervent, so sincere and practical, that it rose up like a holy incense from every thought and act. His a benevolence that not only willed, but endeavoured the best for all. His a philanthropy that held in the embracings of its love every brother-man. His a toleration of the largest and the truest; condemning persecution in every, even its mildest form; and kindly encouraging each striving after excellence:—a toleration that came not of indifference—for the immoral and the impious met with their quick rebuke—but a toleration that came of the wise humbleness and the Christian charity, which see, in the nothingness of self and the almightiness of TRUTH, no praise for the ablest, and no blame for the feeblest in their strugglings upward to light and life.

In the winter of 1691-2, on returning from chapel, one morning, Newton found that a favourite little dog, called Diamond, had overturned a lighted taper on his desk, and that several papers containing the results of certain optical experiments, were nearly consumed. His only exclamation, on perceiving his loss, was, "Oh Diamond, Diamond, little knowest thou the mischief thou hast done." Dr. Brewster, in his life of our author, gives the following extract from the manuscript Diary of Mr. Abraham De La Pryme, a student in the University at the time of this occurrence.

"1692. February, 3.—What I heard to-day I must relate. There is one Mr. Newton (whom I have very oft seen), Fellow of Trinity College, that is mighty famous for his learning, being a most excellent mathematician, philosopher, divine, &c. He has been Fellow of the Royal Society these many years; and among other very learned books and tracts, he's written one upon the mathematical principles of philosophy, which has given him a mighty name, he having received, especially from Scotland, abundance of congratulatory letters for the same; but of all the books he ever wrote, there was one of colours and light, established upon thousands of experiments which he had been twenty years of making, and which had cost him many hundreds of pounds. This book which he valued so much, and which was so much talked of, had the ill luck to perish, and be utterly lost just when the learned author was almost at putting a conclusion at the same, after this manner: In a winter's morning, leaving it among his other papers on his study table while he went to chapel, the candle, which he had unfortunately left burning there, too, catched hold by some means of other papers, and they fired the aforesaid book, and utterly consumed it and several other valuable writings; and which is most wonderful did no further mischief. But when Mr. Newton came from chapel, and had seen what was done, every one thought he would have run mad, he was so troubled thereat that he was not himself for a month after. A long account of this his system of colours you may find in the Transactions of the Royal Society, which he had sent up to them long before this sad mischance happened unto him."

It will be borne in mind that all of Newton's theological writings, with the exception of the Letters to Dr. Bentley, were composed before this event which, we must conclude, from Pryme's words, produced a serious impression upon our author for about a month. But M. Biot, in his Life of Newton, relying on a memorandum contained in a small manuscript Journal of Huygens, declares this occurrence to have caused a derangement of Newton's intellect. M. Biot's opinions and deductions, however, as well as those of La Place, upon this subject, were based upon erroneous data, and have been overthrown by the clearest proof. There is not, in fact, the least evidence that Newton's reason was, for a single moment, dethroned; on the contrary, the testimony is conclusive that he was, at all times, perfectly capable of carrying on his mathematical, metaphysical and astronomical inquiries. Loss of sleep, loss of appetite, and irritated nerves will disturb somewhat the equanimity of the most serene; and an act done, or language employed, under such temporary discomposure, is not a just criterion of the general tone and strength of a man's mind. As to the accident itself, we may suppose, whatever might have been its precise nature, that it greatly distressed him, and, still further, that its shock may have originated the train of nervous derangements, which afflicted him, more or less, for two years afterward. Yet, during this very period of ill health, we find him putting forth his highest powers. In 1692, he prepared for, and transmitted to Dr. Wallis the first proposition of the Treatise on Quadratures, with examples of it in first, second and third fluxions. He investigated, in the same year, the subject of haloes; making and recording numerous and important observations relative thereto. Those profound and beautiful Letters to Dr. Bentley were written at the close of this and the beginning of the next year. In October, 1693, Locke, who was then about publishing a second edition of his work on the Human Understanding, requested Newton to reconsider his opinions on innate ideas. And in 1694, he was zealously occupied in perfecting his lunar theory; visiting Flamstead, at the Royal Observatory of Greenwich, in September, and obtaining a series of lunar observations; and

commencing, in October, a correspondence with that distinguished practical Astronomer, which continued till 1698.

We now arrive at the period when Newton permanently withdrew from the seclusion of a collegiate, and entered upon a more active and public life. He was appointed Warden of the Mint, in 1695, through the influence of Charles Montague, Chancellor of the Exchequer, and afterward Earl of Halifax. The current coin of the nation had been adulterated and debased, and Montague undertook a re-coinage. Our author's mathematical and chemical knowledge proved eminently useful in accomplishing this difficult and most salutary reform. In 1699, he was promoted to the Mastership of the Mint—an office worth twelve or fifteen hundred pounds per annum, and which he held during the remainder of his life. He wrote, in this capacity, an official Report on the Coinage, which has been published; he also prepared a Table of Assays of Foreign Coins, which was printed at the end of Dr. Arbuthnot's Tables of Ancient Coins, Weights, and Measures, in 1727.

Newton retained his Professorship at Cambridge till 1703. But he had, on receiving the appointment of Master of the Mint, in 1699, made Mr. Whiston his deputy, with all the emoluments of the office; and, on finally resigning, procured his nomination to the vacant Chair.

In January 1697, John Bernouilli proposed to the most distinguished mathematicians of Europe two problems for solution. Leibnitz, admiring the beauty of one of them, requested the time for solving it to be extended to twelve months—twice the period originally named. The delay was readily granted. Newton, however, sent in, the day after he received the problems, a solution of them to the President of the Royal Society. Bernouilli obtained solutions from Newton, Leibinitz and the Marquis De L'Hopital; but Newton's though anonymous, he immediately recognised "*tanquam ungue leonem*," as the lion is known by his claw. We may mention here the famous problem of the trajectories proposed by Leibnitz, in 1716, for the purpose of "feeling the pulse of the English Analysts." Newton received the problem about five o'clock in the afternoon, as he was returning from the

Mint; and though it was extremely difficult and he himself much fatigued, yet he completed its solution, the same evening before he went to bed.

The history of these problems affords, by direct comparison, a striking illustration of Newton's vast superiority of mind. That amazing concentration and grasp of intellect, of which we have spoken, enabled him to master speedily, and, as it were, by a single effort, those things, for the achievement of which, the many would essay utterly in vain, and the very, very few attain only after long and renewed striving. And yet, with a modesty as unparalleled as his power, he attributed his successes, not to any extraordinary sagacity, but solely to industry and patient thought. He kept the subject of consideration constantly before him, and waited till the first dawning opened gradually into a full and clear light; never quitting, if possible, the mental process till the object of it were wholly gained. He never allowed this habit of meditation to appear in his intercourse with society; but in the privacy of his own chamber, or in the midst of his own family, he gave himself up to the deepest abstraction. Occupied with some interesting investigation, he would often sit down on his bedside, after he rose, and remain there, for hours, partially dressed. Meal-time would frequently come and pass unheeded; so that, unless urgently reminded, he would neglect to take the requisite quantity of nourishment. But notwithstanding his anxiety to be left undisturbed, he would, when occasion required, turn aside his thoughts, though bent upon the most intricate research, and then, when leisure served, again direct them to the very point where they ceased to act: and this he seemed to accomplish not so much by the force of his memory, as by the force of his inventive faculty, before the vigorous intensity of which, no subject, however abstruse, remained long unexplored.

He was elected a member of the Royal Academy of Sciences at Paris, in 1699, when that distinguished Body were empowered, by a new charter, to admit a small number of foreign associates. In 1700, he communicated to Dr. Halley a description of his reflecting instrument for observing the moon's distance from the fixed stars. This description was published in the Philosophical

Transactions, in 1742. The instrument was the same as that produced by Mr. Hadley, in 1731, and which, under the name of Hadley's Quadrant, has been of so great use in navigation. On the assembling of the new Parliament, in 1701, Newton was re-elected one of the members for the University of Cambridge. In 1703, he was chosen President of the Royal Society of London, to which office he was annually re-elected till the period of his decease—about twenty-five years afterward.

Our author unquestionably devoted more labour to, and, in many respects, took a greater pride in his Optical, than his other discoveries. This science he had placed on a new and indestructible basis; and he wished not only to build, but to perfect the costly and glowing structure. He had communicated, before the publication of the PRINCIPIA, his most important researches on light to the Royal Society, in detached papers which were inserted in successive numbers of the Transactions; but he did not publish a connected view of these labours till 1704, when they appeared under the title of OPTICS: OR, A TREATISE ON THE REFLEXIONS, REFRACTIONS, INFLEXIONS AND COLOURS OF LIGHT. To this, but to no subsequent edition, were added two Mathematical Treatises, entitled, TRACTATUS DUO DE SPECIEBUS ET MAGNITUDINE FIGURARUM CURVILINEARUM; the one bearing the title TRACTATUS DE QUADRATURA CURVARUM; and the other, that of ENUMERATIO LINEARUM TERTII ORDINIS. The publication of these Mathematical Treatises was made necessary in consequence of plagiarisms from the manuscripts of them loaned by the author to his friends. Dr. Samuel Clarke published a Latin translation of the Optics, in in 1706; whereupon he was presented by Newton, as a mark of his grateful approbation, with five hundred pounds, or one hundred pounds for each of his children. The work was afterward translated into French. It had a remarkably wide circulation, and appeared, in several successive editions, both in England and on the Continent. There is displayed, particularly on this Optical Treatise, the author's talent for simplifying and communicating the profoundest speculations. It is a faculty rarely united to that of the highest invention. Newton possessed both; and thus that mental perfectness which enabled him to create, to combine,

and to teach, and so render himself, not the "ornament" only, but inconceivably more, the pre-eminent benefactor of his species.

The honour of knighthood was conferred on our author in 1705. Soon afterward, he was a candidate again for the Representation of the University, but was defeated by a large majority. It is thought that a more pliant man was preferred by both ministers and electors. Newton was always remarkable for simplicity of dress, and his only known departure from it was on this occasion, when he is said to have appeared in a suit of laced clothes.

The Algebraical Lectures which he had, during nine years, delivered at Cambridge, were published by Whiston, in 1707, under the title of ARITHMETICA UNIVERSALIS, SINE DE COMPOSITIONE ET RESOLUTIONE ARITHMETICA LIBER. This publication is said to have been a breach of confidence on Whiston's part. Mr. Ralphson, not long afterward, translated the work into English; and a second edition of it, with improvements by the author, was issued at London, 1712, by Dr. Machin. Subsequent editions, both in English and Latin, with commentaries, have been published.

In June, 1709, Newton intrusted the superintendence of a second edition of the PRINCIPIA to Roger Cotes, Plumian Professor of Astronomy at Cambridge. The first edition had been sold off for some time. Copies of the work had become very rare, and could only be obtained at several times their original cost. A great number of letters passed between the author and Mr. Cotes during the preparation of the edition, which finally appeared in May, 1713. It had many alterations and improvements, and was accompanied by an admirable Preface from the pen of Cotes.

Our author's early Treatise, entitled, ANALYSIS PER EQUATIONES NUMERO TERMINORUM INFINITAS, as well as a small Tract, bearing the title of METHODUS DIFFERENTIALIS, was published, with his consent, in 1711. The former of these, and the Treatise De Quadratura Curvarum, translated into English, with a large commentary, appeared in 1745. His work, entitled, ARTIS ANALYTICÆ SPECIMINA, VEL GEOMETRIA ANALYTICA, was first given to the world in the edition of Dr. Horsley, 1779.

It is a notable fact, in Newton's history, that he never voluntarily published any one of his purely mathematical writings. The cause of this unwillingness in some, and, in other instances, of his indifference, or, at least, want of solicitude to put forth his works may be confidently sought for in his repugnance to everything like contest or dispute. But, going deeper than this aversion, we find, underlying his whole character and running parallel with all his discoveries, that extraordinary humility which always preserved him in a position so relatively just to the behests of time and eternity, that the infinite value of truth, and the utter worthlessness of fame, were alike constantly present to him. Judging of his course, however, in its more temporary aspect, as bearing upon his immediate quiet, it seemed the most unfortunate. For an early publication, especially in the case of his Method of Fluxions, would have anticipated all rivalry, and secured him from the contentious claims of Leibnitz. Still each one will solve the problem of his existence in his own way, and, with a man like Newton, his own, as we conceive, could be no other than the best way. The conduct of Leibnitz in this affair is quite irreconcilable with the stature and strength of the man; giant-like, and doing nobly, in many ways, a giant's work, yet cringing himself into the dimensions and performances of a common calumniator. Opening in 1699, the discussion in question continued till the close of Leibnitz's life, in 1716. We give the summary of the case as contained in the Report of the Committee of the Royal Society, the deliberately weighed opinion of which has been adopted as an authoritative decision in all countries.

"We have consulted the letters and letter books in the custody of the Royal Society, and those found among the papers of Mr. John Collins, dated between the years 1669 and 1677, inclusive; and showed them to such as knew and avouched the hands of Mr. Barrow, Mr. Collins, Mr. Oldenburg, and Mr. Leibnitz; and compared those of Mr. Gregory with one another, and with copies of some of them taken in the hand of Mr. Collins; and have extracted from them what relates to the matter referred to us; all which extracts, herewith delivered to you, we believe to be genuine and authentic. And by these letters and papers we find:—

"I. Mr. Leibnitz was in London in the beginning of the year 1673; and went thence in or about March, to Paris, where he kept a correspondence with Mr. Collins, by means of Mr. Oldenburg, till about September, 1676, and then returned, by London and Amsterdam, to Hanover: and that Mr. Collins was very free in communicating to able mathematicians what he had received from Mr. Newton and Mr. Gregory.

"II. That when Mr. Leibnitz was the first time in London, he contended for the invention of another differential method, properly so called; and, notwithstanding he was shown by Dr. Pell that it was Newton's method, persisted in maintaining it to be his own invention, by reason that he had found it by himself without knowing what Newton had done before, and had much improved it. And we find no mention of his having any other differential method than Newton's before his letter of the 21st of June, 1677, which was a year after a copy of Mr. Newton's letter of the 10th of December, 1672, had been sent to Paris to be communicated to him; and above four years after Mr. Collins began to communicate that letter to his correspondents; in which letter the method of fluxions was sufficiently described to any intelligent person.

"III. That by Mr. Newton's letter, of the 13th of June, 1676 it appears that he had the method of fluxions above five years before the writing of that letter. And by his Analysis per Æquationes numero Terminorum Infinitas, communicated by Dr. Barrow to Mr. Collins, in July, 1669, we find that he had invented the method before that time.

"IV. That the differential method is one and the same with the method of fluxions, excepting the name and mode of notation; Mr. Leibnitz calling those quantities differences which Mr. Newton calls moments, or fluxions; and marking them with a letter *d*—a mark not used by Mr. Newton.

"And, therefore, we take the proper question to be, not who invented this or that method, but, who was the first inventor of the method? And we believe that those who have reputed Mr. Leibnitz the first inventor knew little or nothing of his correspondence with Mr. Collins and Mr. Oldenburg long before, nor of Mr.

Newton's having that method above fifteen years before Mr Leibnitz began to publish it in the Acta Eruditorum of Leipsic.

"For which reason we reckon Mr. Newton the first inventor; and are of opinion that Mr. Keill, in asserting the same, has been no ways injurious to Mr. Leibnitz. And we submit to the judgment of the Society, whether the extract and papers, now presented to you, together with what is extant, to the same purpose, in Dr. Wallis's third volume, may not deserve to be made public."

This Report, with the collection of letters and manuscripts, under the title of COMMERCIUM EPISTOLICUM D. JOHANNIS COLLINS ET ALIORUM DE ANALYSI PROMOTA JUSSU SOCIETATIS REGIÆ EDITUM, appeared accordingly in the early part of 1713. Its publication seemed to infuse additional bitterness into the feelings of Leibnitz, who descended to unfounded charges and empty threats. He had been privy counsellor to the Elector of Hanover, before that prince was elevated to the British throne; and in his correspondence, in 1715 and 1716, with the Abbé Conti, then at the court of George I., and with Caroline, Princess of Wales, he attacked the doctrines of the PRINCIPIA, and indirectly its author, in a manner very discreditable to himself, both as a learned and as an honourable man. His assaults, however, were triumphantly met; and, to the complete overthrow of his rival pretensions, Newton was induced to give the finishing blow. The verdict is universal and irreversible that the English preceded the German philosopher, by at least ten years, in the invention of fluxions. Newton could not have borrowed from Leibnitz; but Leibnitz might have borrowed from Newton. A new edition of the Commercium Epistolicum was published in 1722-5 (?); but neither in this, nor in the former edition, did our author take any part. The disciples, enthusiastic, capable and ready, effectually shielded, with the buckler of Truth, the character of the Master, whose own conduct throughout was replete with delicacy, dignity and justice. He kept aloof from the controversy—in which Dr. Keill stood forth as the chief representative of the Newtonian side—till the very last, when, for the satisfaction of the King, George I., rather than for his own, he consented to put forth his

hand and firmly secure his rights upon a certain and impregnable basis.

A petition to have inventions for promoting the discovery of the longitude at sea, suitably rewarded, was presented to the House of Commons, in 1714. A committee, having been appointed to investigate the subject, called upon Newton and others for their opinions. That of our author was given in writing. A report, favourable to the desired measure, was then taken up, and a bill for its adoption subsequently passed.

On the ascension of George I., in 1714, Newton became an object of profound interest at court. His position under government, his surpassing fame, his spotless character, and, above all, his deep and consistent piety, attracted the reverent regard of the Princess of Wales, afterward queen-consort to George II. She was a woman of a highly cultivated mind, and derived the greatest pleasure from conversing with Newton and corresponding with Leibnitz. One day, in conversation with her, our author mentioned and explained a new system of chronology, which he had composed at Cambridge, where he had been in the habit "of refreshing himself with history and chronology, when he was weary with other studies." Subsequently, in the year 1718, she requested a copy of this interesting and ingenious work. Newton, accordingly, drew up an abstract of the system from the separate papers in which it existed, and gave it to her on condition that it should not be communicated to any other person. Sometime afterward she requested that the Abbé Conti might be allowed to have a copy of it. The author consented: and the abbé received a copy of the manuscript, under the like injunction and promise of secrecy. This manuscript bore the title of "A short Chronicle, from the First Memory of Things in Europe, to the Conquest of Persia, by Alexander the Great."

After Newton took up his residence in London, he lived in a style suited to his elevated position and rank. He kept his carriage, with an establishment of three male and three female servants. But to everything like vain show and luxury he was utterly averse. His household affairs, for the last twenty years of his life, were under the charge of his niece, Mrs. Catherine Barton,

wife and widow of Colonel Barton—a woman of great beauty and accomplishment—and subsequently married to John Conduit, Esq. At home Newton was distinguished by that dignified and gentle hospitality which springs alone from true nobleness. On all proper occasions, he gave splendid entertainments, though without ostentation. In society, whether of the palace or the cottage, his manner was self-possessed and urbane; his look benign and affable; his speech candid and modest; his whole air undisturbedly serene. He had none of what are usually called the singularities of genius; suiting himself easily to every company—except that of the vicious and wicked; and speaking of himself and others, naturally, so as never even to be suspected of vanity. There was in him, if we may be allowed the expression, a WHOLENESS of nature, which did not admit of such imperfections and weakness—the circle was too perfect, the law too constant, and the disturbing forces too slight to suffer scarcely any of those eccentricities which so interrupt and mar the movements of many bright spirits, rendering their course through the world more like that of the blazing meteor than that of the light and life-imparting sun. In brief, the words GREATNESS and GOODNESS could not, humanly speaking, be more fitly employed than when applied as the pre-eminent characteristics of this pure, meek and venerable sage.

In the eightieth year of his age, Newton was seized with symptoms of stone in the bladder. His disease was pronounced incurable. He succeeded, however, by means of a strict regimen, and other precautions, in alleviating his complaint, and procuring long intervals of ease. His diet, always frugal, was now extremely temperate, consisting chiefly of broth, vegetables, and fruit, with, now and then, a little butcher meat. He gave up the use of his carriage, and employed, in its stead, when he went out, a chair. All invitations to dinner were declined; and only small parties were received, occasionally, at his own house.

In 1724 he wrote to the Lord Provost of Edinburgh, offering to contribute twenty pounds yearly toward the salary of Mr. Maclaurin, provided he accepted the assistant Professorship of Mathematics in the University of that place. Not only in the

cause of ingenuity and learning, but in that of religion—in relieving the poor and assisting his relations, Newton annually expended large sums. He was generous and charitable almost to a fault. Those, he would often remark, who gave away nothing till they died, never gave at all. His wealth had become considerable by a prudent economy; but he regarded money in no other light than as one of the means wherewith he had been intrusted to do good, and he faithfully employed it accordingly.

He experienced, in spite of all his precautionary measures, a return of his complaint in the month of August, of the same year, 1724, when he passed a stone the size of pea; it came from him in two pieces, the one at the distance of two days from the other. Tolerable good health then followed for some months. In January, 1725, however, he was taken with a violent cough and inflammation of the lungs. In consequence of this attack, he was prevailed upon to remove to Kensington, where his health greatly improved. In February following, he was attacked in both feet with the gout, of the approach of which he had received, a few years before, a slight warning, and the presence of which now produced a very beneficial change in his general health. Mr. Conduit, his nephew, has recorded a curious conversation which took place, at or near this time, between himself and Sir Isaac.

"I was, on Sunday night, the 7th March, 1724-5, at Kensington, with Sir Isaac Newton, in his lodgings, just after he was out of a fit of the gout, which he had had in both of his feet, for the first time, in the eighty-third year of his age. He was better after it, and his head clearer and memory stronger than I had known them for some time. He then repeated to me, by way of discourse, very distinctly, though rather in answer to my queries, than in one continued narration, what he had often hinted to me before, viz.: that it was his conjecture (he would affirm nothing) that there was a sort of revolution in the heavenly bodies; that the vapours and light, emitted by the sun, which had their sediment, as water and other matter, had gathered themselves, by degrees, into a body, and attracted more matter from the planets, and at last made a secondary planet (viz.: one of those that go round another planet), and then, by gathering to them, and

attracting more matter, became a primary planet; and then, by increasing still, became a comet, which, after certain revolutions, by coming nearer and nearer to the sun, had all its volatile parts condensed, and became a matter fit to recruit and replenish the sun (which must waste by the constant heat and light it emitted), as a faggot would this fire if put into it (we were sitting by a wood fire), and that that would probably be the effect of the comet of 1680, sooner or later; for, by the observations made upon it, it appeared, before it came near the sun, with a tail only two or three degrees long; but, by the heat it contracted, in going so near the sun, it seemed to have a tail of thirty or forty degrees when it went from it; that he could not say when this comet would drop into the sun; it might perhaps have five or six revolutions more first, but whenever it did it would so much increase the heat of the sun that this earth would be burned, and no animals in it could live. That he took the three phenomena, seen by Hipparchus, Tycho Brahe, and Kepler's disciples, to have been of this kind, for he could not otherwise account for an extraordinary light, as those were, appearing, all at once, among the the fixed stars (all which he took to be suns, enlightening other planets, as our sun does ours), as big as Mercury or Venus seems to us, and gradually diminishing, for sixteen months, and then sinking into nothing. He seemed to doubt whether there were not intelligent beings, superior to us, who superintended these revolutions of the heavenly bodies, by the direction of the Supreme Being. He appeared also to be very clearly of opinion that the inhabitants of this world were of short date, and alledged, as one reason for that opinion, that all arts, as letters, ships, printing, needle, &c., were discovered within the memory of history, which could not have happened if the world had been eternal; and that there were visible marks of ruin upon it which could not be effected by flood only. When I asked him how this earth could have been repeopled if ever it had undergone the same fate it was threatened with hereafter, by the comet of 1680, he answered, that required the power of a Creator. He said he took all the planets to be composed of the same matter with this earth, viz.: earth, water, stones, &c., but variously concocted. I

asked him why he would not publish his conjectures, as conjectures, and instanced that Kepler had communicated his; and though he had not gone near so far as Kepler, yet Kepler's guesses were so just and happy that they had been proved and demonstrated by him. His answer was, "I do not deal in conjectures." But, on my talking to him about the four observations that had been made of the comet of 1680, at 574 years' distance, and asking him the particular times, he opened his *Principia*, which laid on the table, and showed me the particular periods, viz.: 1st. The Julium Sidus, in the time of Justinian, in 1106, in 1680.

"And I, observing that he said there of that comet, 'incidet in corpus solis,' and in the next paragraph adds, 'stellæ fixæ refici possunt,' told him I thought he owned there what we had been talking about, viz.: that the comet would drop into the sun, and that fixed stars were recruited and replenished by comets when they dropped into them; and, consequently, that the sun would be recruited too; and asked him why he would not own as fully what he thought of the sun as well as what he thought of the fixed stars. He said, 'that concerned us more;' and, laughing, added, that he had said enough for people to know his meaning."

In the summer of 1725, a French translation of the chronological MS., of which the Abbé Conti had been permitted, some time previous, to have a copy, was published at Paris, in violation of all good faith. The *Punic* Abbé had continued true to his promise of secrecy while he remained in England; but no sooner did he reach Paris than he placed the manuscript into the hands of M. Freret, a learned antiquarian, who translated the work, and accompanied it with an attempted refutation of the leading points of the system. In November, of the same year, Newton received a presentation copy of this publication, which bore the title of ABREGE DE CHRONOLOGIE DE M. LE CHEVALIER NEWTON, FAIT PAR LUI-MEME, ET TRADUIT SUR LE MANUSCRIPT ANGLAIS. Soon afterward a paper entitled, REMARKS ON THE OBERVATIONS MADE ON A CHRONOLOGICAL INDEX OF SIR ISAAC NEWTON, TRANSLATED INTO FRENCH BY THE OBSERVATOR, AND PUBLISHED AT PARIS,

was drawn up by our author, and printed in the Philosophical Transactions for 1725. It contained a history of the whole matter, and a triumphant reply to the objections of M. Freret. This answer called into the field a fresh antagonist, Father Soueiet, whose five dissertations on this subject were chiefly remarkable for the want of knowledge and want of decorum, which they displayed. In consequence of these discussions, Newton was induced to prepare his larger work for the press, and had nearly completed it at the time of his death. It was published in 1728, under the title of THE CHRONOLOGY OF THE ANCIENT KINGDOMS AMENDED, TO WHICH IS PREFIXED A SHORT CHRONICLE FROM THE FIRST MEMORY OF THINGS IN EUROPE TO THE CONQUEST OF PERSIA BY ALEXANDER THE GREAT. It consists of six chapters: 1. On the Chronology of the Greeks; according to Whiston, our author wrote out eighteen copies of this chapter with his own hand, differing little from one another. 2. Of the Empire of Egypt; 3. Of the Assyrian Empire; 4. Of the two contemporary Empires of the Babylonians and Medes; 5. A Description of the Temple of Solomon; 6. Of the Empire of the Persians; this chapter was not found copied with the other five, but as it was discovered among his papers, and appeared to be a continuation of the same work, the Editor thought proper to add it thereto. Newton's LETTER TO A PERSON OF DISTINCTION WHO HAD DESIRED HIS OPINION OF THE LEARNED BISHOP LLOYD'S HYPOTHESIS CONCERNING THE FORM OF THE MOST ANCIENT YEAR, closes this enumeration of his Chronological Writings.

A third edition of the PRINCIPIA appeared in 1726, with many changes and additions. About four years were consumed in its preparation and publication, which were under the superintendance of Dr. Henry Pemberton, an accomplished mathematician, and the author of "A VIEW OF SIR ISAAC NEWTON'S PHILOSOPHY." 1728. This gentleman enjoyed numerous opportunities of conversing with the aged and illustrious author. "I found," says Pemberton, "he had read fewer of the modern mathematicians than one could have expected; but his own prodigious invention readily supplied him with what he might have an occasion for in the pursuit of any subject he undertook. I have often

heard him censure the handling geometrical subjects by algebraic calculations; and his book of Algebra he called by the name of Universal Arithmetic, in opposition to the injudicious title of Geometry, which Descartes had given to the treatise, wherein he shows how the geometer may assist his invention by such kind of computations. He thought Huygens the most elegant of any mathematical writer of modern times, and the most just imitator of the ancients. Of their taste and form of demonstration, Sir Isaac always professed himself a great admirer. I have heard him even censure himself for not following them yet more closely than he did; and speak with regret of his mistake at the beginning of his mathematical studies, in applying himself to the works of Descartes and other algebraic writers, before he had considered the elements of Euclid with that attention which so excellent a writer deserves."

"Though his memory was much decayed," continues Dr. Pemberton, "he perfectly understood his own writings." And even this failure of memory, we would suggest, might have been more apparent than real, or, in medical terms, more the result of functional weakness than organic decay. Newton seems never to have confided largely to his memory: and as this faculty manifests the most susceptibility to cultivation; so, in the neglect of due exercise, it more readily and plainly shows a diminution of its powers.

Equanimity and temperance had, indeed, preserved Newton singularly free from all mental and bodily ailment. His hair was, to the last, quite thick, though as white as silver. He never made use of spectacles, and lost but one tooth to the day of his death. He was of middle stature, well-knit, and, in the latter part of his life, somewhat inclined to be corpulent. Mr. Conduit says, "he had a very lively and piercing eye, a comely and gracious aspect, with a fine head of hair, white as silver, without any baldness, and when his peruke was off was a venerable sight." According to Bishop Atterbury, "in the whole air of his face and make there was nothing of that penetrating sagacity which appears in his compositions. He had something rather languid in his look and manner which did not raise any great expectation

in those who did not know him." Hearne remarks, "Sir Isaac was a man of no very promising aspect. He was a short, well-set man. He was full of thought, and spoke very little in company, so that his conversation was not agreeable. When he rode in his coach, one arm would be out of his coach on one side and the other on the other." These different accounts we deem easily reconcilable. In the rooms of the Royal Society, in the street, or in mixed assemblages, Newton's demeanour—always courteous, unassuming and kindly—still had in it the overawings of a profound repose and reticency, out of which the communicative spirit, and the "lively and piercing eye" would only gleam in the quiet and unrestrained freedom of his own fire-side.

"But this I immediately discovered in him," adds Pemberton, still further, "which at once both surprised and charmed me. Neither his extreme great age, nor his universal reputation had rendered him stiff in opinion, or in any degree elated. Of this I had occasion to have almost daily experience. The remarks I continually sent him by letters on his Principia, were received with the utmost goodness. These were so far from being anyways displeasing to him, that, on the contrary, it occasioned him to speak many kind things of me to my friends, and to honour me with a public testimony of his good opinion." A modesty, openness, and generosity, peculiar to the noble and comprehensive spirit of Newton. "Full of wisdom and perfect in beauty," yet not lifted up by pride nor corrupted by ambition. None, however, knew so well as himself the stupendousness of his discoveries in comparison with all that had been previously achieved; and none realized so thoroughly as himself the littleness thereof in comparison with the vast region still unexplored. A short time before his death he uttered this memorable sentiment:—"I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, while the great ocean of truth lay all undiscovered before me." How few ever reach the shore even, much less find "a smoother pebble or a prettier shell!"

Newton had now resided about two years at Kensington; and

the air which he enjoyed there, and the state of absolute rest, proved of great benefit to him. Nevertheless he would occasionally go to town. And on Tuesday, the 28th of February, 1727, he proceeded to London, for the purpose of presiding at a meeting of the Royal Society. At this time his health was considered, by Mr. Conduit, better than it had been for many years. But the unusual fatigue he was obliged to suffer, in attending the meeting, and in paying and receiving visits, speedily produced a violent return of the affection in the bladder. He returned to Kensington on Saturday, the 4th of March. Dr. Mead and Dr. Cheselden attended him; they pronounced his disease to be the stone, and held out no hopes of recovery. On Wednesday, the 15th of March, he seemed a little better; and slight, though groundless, encouragement was felt that he might survive the attack. From the very first of it, his sufferings had been intense. Paroxysm followed paroxysm, in quick succession: large drops of sweat rolled down his face; but not a groan, not a complaint, not the least mark of peevishness or impatience escaped him: and during the short intervals of relief, he even smiled and conversed with his usual composure and cheerfulness. The flesh quivered, but the heart quaked not; the impenetrable gloom was settling down: the Destroyer near; the portals of the tomb opening, still, amid this utter wreck and dissolution of the mortal, the immortal remained serene, unconquerable: the radiant light broke through the gathering darkness; and Death yielded up its sting, and the grave its victory. On Saturday morning, 18th, he read the newspapers, and carried on a pretty long conversation with Dr. Mead. His senses and faculties were then strong and vigorous; but at six o'clock, the same evening, he became insensible; and in this state he continued during the whole of Sunday, and till Monday, the 20th, when he expired, between one and two o'clock in the morning, in the eighty-fifth year of his age.

And these were the last days of Isaac Newton. Thus closed the career of one of earth's greatest and best men. His mission was fulfilled. Unto the Giver, in many-fold addition, the talents were returned. While it was yet day he had worked; and for the night that quickly cometh he was not unprepared. Full of

years, and full of honours, the heaven-sent was recalled; and, in the confidence of a "certain hope," peacefully he passed away into the silent depths of Eternity.

His body was placed in Westminster Abbey, with the state and ceremonial that usually attended the interment of the most distinguished. In 1731, his relatives, the inheritors of his personal estate, erected a monument to his memory in the most conspicuous part of the Abbey, which had often been refused by the dean and chapter to the greatest of England's nobility. During the same year a medal was struck at the Tower in his honour; and, in 1755, a full-length statue of him, in white marble, admirably executed, by Roubiliac, at the expense of Dr. Robert Smith, was erected in the ante-chamber of Trinity College, Cambridge. There is a painting executed in the glass of one of the windows of the same college, made pursuant to the will of Dr. Smith, who left five hundred pounds for that purpose.

Newton left a personal estate of about thirty-two thousand pounds. It was divided among his four nephews and four nieces of the half blood, the grand-children of his mother, by the Reverend Mr. Smith. The family estates of Woolsthorpe and Sustern fell to John Newton, the heir-at-law, whose great grand-father was Sir Isaac's uncle. Before his death he made an equitable distribution of his two other estates: the one in Berkshire to the sons and daughter of a brother of Mrs. Conduit; and the other, at Kensington, to Catharine, the only daughter of Mr. Conduit, and who afterward became Viscountess Lymington. Mr. Conduit succeeded to the offices of the Mint, the duties of which he had discharged during the last two years of Sir Isaac's life.

Our author's works are found in the collection of Castilion, Berlin, 1744, 4to. 8 tom.; in Bishop Horsley's Edition, London, 1779, 4to. 5 vol.; in the Biographia Brittannica, &c. Newton also published *Bern. Varenii* Geographia, &c., 1681, 8vo. There are, however, numerous manuscripts, letters, and other papers, which have never been given to the world: these are preserved, in various collections, namely, in the library of Trinity College, Cambridge; in the library of Corpus Christi College, Oxford; in the library of Lord Macclesfield; and, lastly and

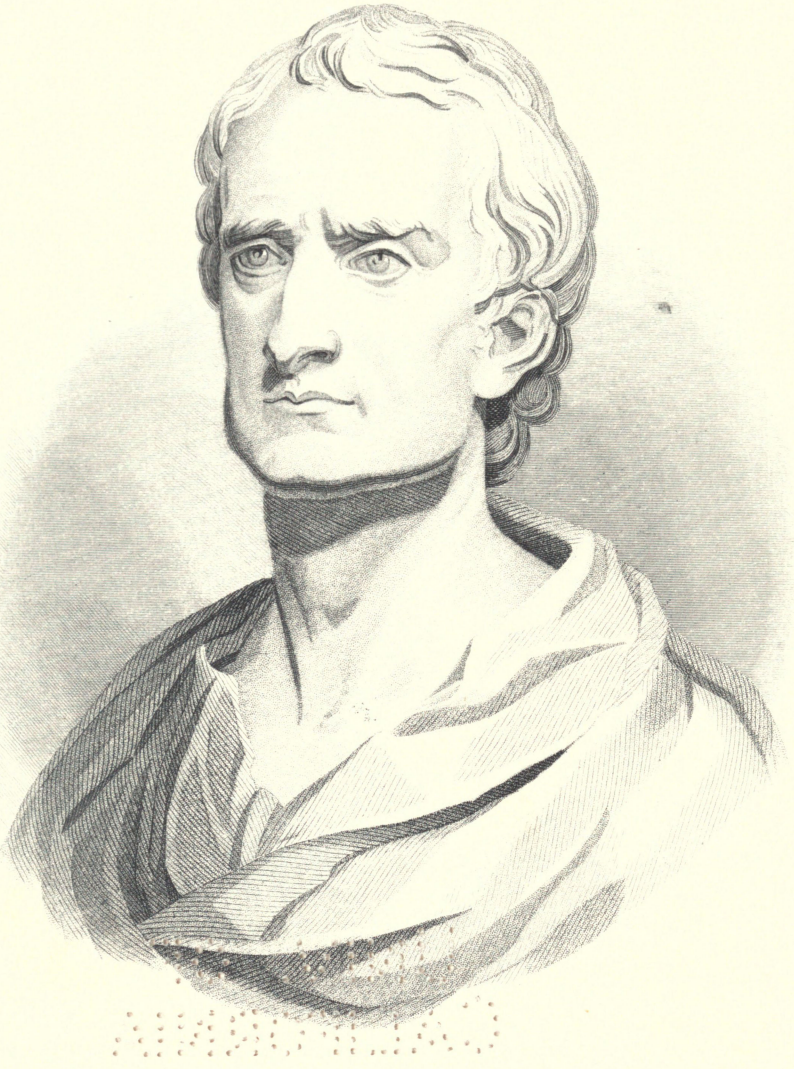
chiefly, in the possession of the family of the Earl of Portsmouth, through the Viscountess Lymington.

Everything appertaining to Newton has been kept and cherished with peculiar veneration. Different memorials of him are preserved in Trinity College, Cambridge; in the rooms of the Royal Society, of London; and in the Museum of the Royal Society of Edinburgh.

The manor-house, at Woolsthorpe, was visited by Dr. Stukeley, in October, 1721, who, in a letter to Dr. Mead, written in 1727, gave the following description of it:—"'Tis built of stone, as is the way of the country hereabouts, and a reasonably good one. They led me up stairs and showed me Sir Isaac's study, where I supposed he studied, when in the country, in his younger days, or perhaps when he visited his mother from the University. I observed the shelves were of his own making, being pieces of deal boxes, which probably he sent his books and clothes down in on those occasions. There were, some years ago, two or three hundred books in it of his father-in-law, Mr. Smith, which Sir Isaac gave to Dr. Newton, of our town." The celebrated apple-tree, the fall of one of the apples of which is said to have turned the attention of Newton to the subject of gravity, was destroyed by the wind about twenty years ago; but it has been preserved in the form of a chair. The house itself has been protected with religious care. It was repaired in 1798, and a tablet of white marble put up in the room where our author was born, with the following inscription:—

"Sir Isaac Newton, son of John Newton, Lord of the Manor of Woolsthorpe, was born in this room, on the 25th of December, 1642."

Nature and Nature's Laws were hid in night,
God said, "Let Newton be," and all was light.



SIR ISAAC NEWTON.

THE PRINCIPIA.

THE AUTHOR'S PREFACE

SINCE the ancients (as we are told by *Pappus*), made great account of the science of mechanics in the investigation of natural things; and the moderns, laying aside substantial forms and occult qualities, have endeavoured to subject the phænomena of nature to the laws of mathematics, I have in this treatise cultivated mathematics so far as it regards philosophy. The ancients considered mechanics in a twofold respect; as rational, which proceeds accurately by demonstration; and practical. To practical mechanics all the manual arts belong, from which mechanics took its name. But as artificers do not work with perfect accuracy, it comes to pass that mechanics is so distinguished from geometry, that what is perfectly accurate is called geometrical; what is less so, is called mechanical. But the errors are not in the art, but in the artificers. He that works with less accuracy is an imperfect mechanic; and if any could work with perfect accuracy, he would be the most perfect mechanic of all; for the description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn; for it requires that the learner should first be taught to describe these accurately, before he enters upon geometry; then it shows how by these operations problems may be solved. To describe right lines and circles are problems, but not geometrical problems. The solution of these problems is required from mechanics; and by geometry the use of them, when so solved, is shown; and it is the glory of geometry that from those few principles, brought from without, it is able to produce so many things. Therefore geometry is founded in mechanical practice, and is nothing but that part of universal mechanics which accurately proposes and demonstrates the art of measuring. But since the manual arts are chiefly conversant in the moving of bodies, it comes to pass that geometry is commonly referred to their magnitudes, and mechanics to their motion. In this sense rational mechanics will be the science of motions resulting from any forces whatsoever, and of the forces required to produce any motions, accurately proposed and demonstrated. This part of mechanics was

cultivated by the ancients in the five powers which relate to manual arts who considered gravity (it not being a manual power), no otherwise than as it moved weights by those powers. Our design not respecting arts, but philosophy, and our subject not manual but natural powers, we consider chiefly those things which relate to gravity, levity, elastic force, the resistance of fluids, and the like forces, whether attractive or impulsive; and therefore we offer this work as the mathematical principles of philosophy; for all the difficulty of philosophy seems to consist in this—from the phænomena of motions to investigate the forces of nature, and then from these forces to demonstrate the other phænomena; and to this end the general propositions in the first and second book are directed. In the third book we give an example of this in the explication of the System of the World; for by the propositions mathematically demonstrated in the former books, we in the third derive from the celestial phænomena the forces of gravity with which bodies tend to the sun and the several planets. Then from these forces, by other propositions which are also mathematical, we deduce the motions of the planets, the comets, the moon, and the sea. I wish we could derive the rest of the phænomena of nature by the same kind of reasoning from mechanical principles; for I am induced by many reasons to suspect that they may all depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards each other, and cohere in regular figures, or are repelled and recede from each other; which forces being unknown, philosophers have hitherto attempted the search of nature in vain; but I hope the principles here laid down will afford some light either to this or some truer method of philosophy.

In the publication of this work the most acute and universally learned Mr. Edmund Halley not only assisted me with his pains in correcting the press and taking care of the schemes, but it was to his solicitations that its becoming public is owing; for when he had obtained of me my demonstrations of the figure of the celestial orbits, he continually pressed me to communicate the same to the *Royal Society*, who afterwards, by their kind encouragement and entreaties, engaged me to think of publishing them. But after I had begun to consider the inequalities of the lunar motions, and had entered upon some other things relating to the laws and measures of gravity, and other forces; and the figures that would be described by bodies attracted according to given laws; and the motion of several bodies moving among themselves; the motion of bodies in resisting mediums; the forces, densities, and motions, of mediums; the orbits of the comets, and such like;

deferred that publication till I had made a search into those matters, and could put forth the whole together. What relates to the lunar motions (being imperfect), I have put all together in the corollaries of Prop. 66, to avoid being obliged to propose and distinctly demonstrate the several things there contained in a method more prolix than the subject deserved, and interrupt the series of the several propositions. Some things, found out after the rest, I chose to insert in places less suitable, rather than change the number of the propositions and the citations. I heartily beg that what I have here done may be read with candour; and that the defects in a subject so difficult be not so much reprehended as kindly supplied, and investigated by new endeavours of my readers.

ISAAC NEWTON.

Cambridge, Trinity College May 8, 1686.

In the second edition the second section of the first book was enlarged. In the seventh section of the second book the theory of the resistances of fluids was more accurately investigated, and confirmed by new experiments. In the third book the moon's theory and the præcession of the equinoxes were more fully deduced from their principles; and the theory of the comets was confirmed by more examples of the calculation of their orbits, done also with greater accuracy.

In this third edition the resistance of mediums is somewhat more largely handled than before; and new experiments of the resistance of heavy bodies falling in air are added. In the third book, the argument to prove that the moon is retained in its orbit by the force of gravity is enlarged on; and there are added new observations of Mr. Pound's of the proportion of the diameters of *Jupiter* to each other: there are, besides, added Mr. Kirk's observations of the comet in 1680; the orbit of that comet computed in an ellipsis by Dr. Halley; and the orbit of the comet in 1723 computed by Mr. Bradley.

BOOK I

THE

MATHEMATICAL PRINCIPLES

OF

NATURAL PHILOSOPHY.

DEFINITIONS.

DEFINITION I.

The quantity of matter is the measure of the same, arising from its density and bulk conjunctly.

Thus air of a double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction; and of all bodies that are by any causes whatever differently condensed. I have no regard in this place to a medium, if any such there is, that freely pervades the interstices between the parts of bodies. It is this quantity that I mean hereafter everywhere under the name of body or mass. And the same is known by the weight of each body; for it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which shall be shewn hereafter.

DEFINITION II.

The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjunctly.

The motion of the whole is the sum of the motions of all the parts; and therefore in a body double in quantity, with equal velocity, the motion is double; with twice the velocity, it is quadruple.

DEFINITION III.

The vis insita, *or innate force of matter, is a power of resisting, by which every body, as much as in it lies, endeavours to persevere in its present state, whether it be of rest, or of moving uniformly forward in a right line.*

This force is ever proportional to the body whose force it is; and differs nothing from the inactivity of the mass, but in our manner of conceiving

it. A body, from the inactivity of matter, is not without difficulty put out of its state of rest or motion. Upon which account, this *vis insita*, may, by a most significant name, be called *vis inertiæ*, or force of inactivity. But a body exerts this force only, when another force, impressed upon it, endeavours to change its condition; and the exercise of this force may be considered both as resistance and impulse; it is resistance, in so far as the body, for maintaining its present state, withstands the force impressed; it is impulse, in so far as the body, by not easily giving way to the impressed force of another, endeavours to change the state of that other. Resistance is usually ascribed to bodies at rest, and impulse to those in motion; but motion and rest, as commonly conceived, are only relatively distinguished; nor are those bodies always truly at rest, which commonly are taken to be so.

DEFINITION IV.

An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of moving uniformly forward in a right line.

This force consists in the action only; and remains no longer in the body, when the action is over. For a body maintains every new state it acquires, by its *vis inertiæ* only. Impressed forces are of different origins: as from percussion, from pressure, from centripetal force.

DEFINITION V.

A centripetal force is that by which bodies are drawn or impelled, or any way tend, towards a point as to a centre.

Of this sort is gravity, by which bodies tend to the centre of the earth magnetism, by which iron tends to the loadstone; and that force, whatever it is, by which the planets are perpetually drawn aside from the rectilinear motions, which otherwise they would pursue, and made to revolve in curvilinear orbits. A stone, whirled about in a sling, endeavours to recede from the hand that turns it; and by that endeavour, distends the sling, and that with so much the greater force, as it is revolved with the greater velocity, and as soon as ever it is let go, flies away. That force which opposes itself to this endeavour, and by which the sling perpetually draws back the stone towards the hand, and retains it in its orbit, because it is directed to the hand as the centre of the orbit, I call the centripetal force. And the same thing is to be understood of all bodies, revolved in any orbits. They all endeavour to recede from the centres of their orbits; and were it not for the opposition of a contrary force which restrains them to, and detains them in their orbits, which I therefore call centripetal, would fly off in right lines, with an uniform motion. A projectile, if it was not for the force of gravity, would not deviate towards the earth, but would

go off from it in a right line, and that with an uniform motion, if the resistance of the air was taken away. It is by its gravity that it is drawn aside perpetually from its rectilinear course, and made to deviate towards the earth, more or less, according to the force of its gravity, and the velocity of its motion. The less its gravity is, for the quantity of its matter, or the greater the velocity with which it is projected, the less will it deviate from a rectilinear course, and the farther it will go. If a leaden ball, projected from the top of a mountain by the force of gunpowder with a given velocity, and in a direction parallel to the horizon, is carried in a curve line to the distance of two miles before it falls to the ground; the same, if the resistance of the air were taken away, with a double or decuple velocity, would fly twice or ten times as far. And by increasing the velocity, we may at pleasure increase the distance to which it might be projected, and diminish the curvature of the line, which it might describe, till at last it should fall at the distance of 10, 30, or 90 degrees, or even might go quite round the whole earth before it falls; or lastly, so that it might never fall to the earth, but go forward into the celestial spaces, and proceed in its motion *in infinitum*. And after the same manner that a projectile, by the force of gravity, may be made to revolve in an orbit, and go round the whole earth, the moon also, either by the force of gravity, if it is endued with gravity, or by any other force, that impels it towards the earth, may be perpetually drawn aside towards the earth, out of the rectilinear way, which by its innate force it would pursue; and would be made to revolve in the orbit which it now describes; nor could the moon without some such force, be retained in its orbit. If this force was too small, it would not sufficiently turn the moon out of a rectilinear course: if it was too great, it would turn it too much, and draw down the moon from its orbit towards the earth. It is necessary, that the force be of a just quantity, and it belongs to the mathematicians to find the force, that may serve exactly to retain a body in a given orbit, with a given velocity; and *vice versa*, to determine the curvilinear way, into which a body projected from a given place, with a given velocity, may be made to deviate from its natural rectilinear way, by means of a given force.

The quantity of any centripetal force may be considered as of three kinds; absolute, accelerative, and motive.

DEFINITION VI.

The absolute quantity of a centripetal force is the measure of the same proportional to the efficacy of the cause that propagates it from the centre, through the spaces round about.

Thus the magnetic force is greater in one load-stone and less in another according to their sizes and strength of intensity.

DEFINITION VII.

The accelerative quantity of a centripetal force is the measure of the same, proportional to the velocity which it generates in a given time.

Thus the force of the same load-stone is greater at a less distance, and less at a greater: also the force of gravity is greater in valleys, less on tops of exceeding high mountains; and yet less (as shall hereafter be shown), at greater distances from the body of the earth; but at equal distances, it is the same everywhere; because (taking away, or allowing for, the resistance of the air), it equally accelerates all falling bodies, whether heavy or light, great or small.

DEFINITION VIII.

The motive quantity of a centripetal force, is the measure of the same, proportional to the motion which it generates in a given time.

Thus the weight is greater in a greater body, less in a less body; and, in the same body, it is greater near to the earth, and less at remoter distances. This sort of quantity is the centripetency, or propension of the whole body towards the centre, or, as I may say, its weight; and it is always known by the quantity of an equal and contrary force just sufficient to hinder the descent of the body.

These quantities of forces, we may, for brevity's sake, call by the names of motive, accelerative, and absolute forces; and, for distinction's sake, consider them, with respect to the bodies that tend to the centre; to the places of those bodies; and to the centre of force towards which they tend; that is to say, I refer the motive force to the body as an endeavour and propensity of the whole towards a centre, arising from the propensities of the several parts taken together; the accelerative force to the place of the body, as a certain power or energy diffused from the centre to all places around to move the bodies that are in them; and the absolute force to the centre, as endued with some cause, without which those motive forces would not be propagated through the spaces round about; whether that cause be some central body (such as is the load-stone, in the centre of the magnetic force, or the earth in the centre of the gravitating force), or anything else that does not yet appear. For I here design only to give a mathematical notion of those forces, without considering their physical causes and seats.

Wherefore the accelerative force will stand in the same relation to the motive, as celerity does to motion. For the quantity of motion arises from the celerity drawn into the quantity of matter; and the motive force arises from the accelerative force drawn into the same quantity of matter. For the sum of the actions of the accelerative force, upon the several particles of the body, is the motive force of the whole. Hence it is, that near the

surface of the earth, where the accelerative gravity, or force productive of gravity, in all bodies is the same, the motive gravity or the weight is as the body: but if we should ascend to higher regions, where the accelerative gravity is less, the weight would be equally diminished, and would always be as the product of the body, by the accelerative gravity. So in those regions, where the accelerative gravity is diminished into one half, the weight of a body two or three times less, will be four or six times less.

I likewise call attractions and impulses, in the same sense, accelerative, and motive; and use the words attraction, impulse or propensity of any sort towards a centre, promiscuously, and indifferently, one for another; considering those forces not physically, but mathematically: wherefore, the reader is not to imagine, that by those words, I anywhere take upon me to define the kind, or the manner of any action, the causes or the physical reason thereof, or that I attribute forces, in a true and physical sense, to certain centres (which are only mathematical points); when at any time I happen to speak of centres as attracting, or as endued with attractive powers.

SCHOLIUM.

Hitherto I have laid down the definitions of such words as are less known, and explained the sense in which I would have them to be understood in the following discourse. I do not define time, space, place and motion, as being well known to all. Only I must observe, that the vulgar conceive those quantities under no other notions but from the relation they bear to sensible objects. And thence arise certain prejudices, for the removing of which, it will be convenient to distinguish them into absolute and relative, true and apparent, mathematical and common.

I. Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.

II. Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is vulgarly taken for immovable space; such is the dimension of a subterraneous, an æreal, or celestial space, determined by its position in respect of the earth. Absolute and relative space, are the same in figure and magnitude; but they do not remain always numerically the same. For if the earth, for instance, moves, a space of our air, which relatively and in respect of the earth remains always the same, will at one time be one part of the absolute space into which

the air passes; at another time it will be another part of the same, and so, absolutely understood, it will be perpetually mutable.

III. Place is a part of space which a body takes up, and is according to the space, either absolute or relative. I say, a part of space; not the situation, nor the external surface of the body. For the places of equal solids are always equal; but their superfices, by reason of their dissimilar figures, are often unequal. Positions properly have no quantity, nor are they so much the places themselves, as the properties of places. The motion of the whole is the same thing with the sum of the motions of the parts; that is, the translation of the whole, out of its place, is the same thing with the sum of the translations of the parts out of their places; and therefore the place of the whole is the same thing with the sum of the places of the parts, and for that reason, it is internal, and in the whole body.

IV. Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another. Thus in a ship under sail, the relative place of a body is that part of the ship which the body possesses; or that part of its cavity which the body fills, and which therefore moves together with the ship: and relative rest is the continuance of the body in the same part of the ship, or of its cavity. But real, absolute rest, is the continuance of the body in the same part of that immovable space, in which the ship itself, its cavity, and all that it contains, is moved. Wherefore, if the earth is really at rest, the body, which relatively rests in the ship, will really and absolutely move with the same velocity which the ship has on the earth. But if the earth also moves, the true and absolute motion of the body will arise, partly from the true motion of the earth, in immovable space; partly from the relative motion of the ship on the earth; and if the body moves also relatively in the ship; its true motion will arise, partly from the true motion of the earth, in immovable space, and partly from the relative motions as well of the ship on the earth, as of the body in the ship; and from these relative motions will arise the relative motion of the body on the earth. As if that part of the earth, where the ship is, was truly moved toward the east, with a velocity of 10010 parts; while the ship itself, with a fresh gale, and full sails, is carried towards the west, with a velocity expressed by 10 of those parts; but a sailor walks in the ship towards the east, with 1 part of the said velocity; then the sailor will be moved truly in immovable space towards the east, with a velocity of 10001 parts, and relatively on the earth towards the west, with a velocity of 9 of those parts.

Absolute time, in astronomy, is distinguished from relative, by the equation or correction of the vulgar time. For the natural days are truly unequal, though they are commonly considered as equal, and used for a measure of time; astronomers correct this inequality for their more accurate deducing of the celestial motions. It may be, that there is no such thing as an equable motion, whereby time may be accurately measured. All mo

tions may be accelerated and retarded, but the true, or equable, progress of absolute time is liable to no change. The duration or perseverance of the existence of things remains the same, whether the motions are swift or slow, or none at all: and therefore it ought to be distinguished from what are only sensible measures thereof; and out of which we collect it, by means of the astronomical equation. The necessity of which equation, for determining the times of a phænomenon, is evinced as well from the experiments of the pendulum clock, as by eclipses of the satellites of *Jupiter*.

As the order of the parts of time is immutable, so also is the order of the parts of space. Suppose those parts to be moved out of their places, and they will be moved (if the expression may be allowed) out of themselves. For times and spaces are, as it were, the places as well of themselves as of all other things. All things are placed in time as to order of succession; and in space as to order of situation. It is from their essence or nature that they are places; and that the primary places of things should be moveable, is absurd. These are therefore the absolute places; and translations out of those places, are the only absolute motions.

But because the parts of space cannot be seen, or distinguished from one another by our senses, therefore in their stead we use sensible measures of them. For from the positions and distances of things from any body considered as immovable, we define all places; and then with respect to such places, we estimate all motions, considering bodies as transferred from some of those places into others. And so, instead of absolute places and motions, we use relative ones; and that without any inconvenience in common affairs; but in philosophical disquisitions, we ought to abstract from our senses, and consider things themselves, distinct from what are only sensible measures of them. For it may be that there is no body really at rest, to which the places and motions of others may be referred.

But we may distinguish rest and motion, absolute and relative, one from the other by their properties, causes and effects. It is a property of rest, that bodies really at rest do rest in respect to one another. And therefore as it is possible, that in the remote regions of the fixed stars, or perhaps far beyond them, there may be some body absolutely at rest; but impossible to know, from the position of bodies to one another in our regions whether any of these do keep the same position to that remote body; it follows that absolute rest cannot be determined from the position of bodies in our regions.

It is a property of motion, that the parts, which retain given positions to their wholes, do partake of the motions of those wholes. For all the parts of revolving bodies endeavour to recede from the axis of motion; and the impetus of bodies moving forward, arises from the joint impetus of all the parts. Therefore, if surrounding bodies are moved, those that are relatively at rest within them, will partake of their motion. Upon which account, the true and absolute motion of a body cannot be deter-

mined by the translation of it from those which only seem to rest; for the external bodies ought not only to appear at rest, but to be really at rest. For otherwise, all included bodies, beside their translation from near the surrounding ones, partake likewise of their true motions; and though that translation were not made they would not be really at rest, but only seem to be so. For the surrounding bodies stand in the like relation to the surrounded as the exterior part of a whole does to the interior, or as the shell does to the kernel; but, if the shell moves, the kernel will also move, as being part of the whole, without any removal from near the shell.

A property, near akin to the preceding, is this, that if a place is moved, whatever is placed therein moves along with it; and therefore a body, which is moved from a place in motion, partakes also of the motion of its place. Upon which account, all motions, from places in motion, are no other than parts of entire and absolute motions; and every entire motion is composed of the motion of the body out of its first place, and the motion of this place out of its place; and so on, until we come to some immovable place, as in the before-mentioned example of the sailor. Wherefore, entire and absolute motions can be no otherwise determined than by immovable places; and for that reason I did before refer those absolute motions to immovable places, but relative ones to movable places. Now no other places are immovable but those that, from infinity to infinity, do all retain the same given position one to another; and upon this account must ever remain unmoved; and do thereby constitute immovable space.

The causes by which true and relative motions are distinguished, one from the other, are the forces impressed upon bodies to generate motion. True motion is neither generated nor altered, but by some force impressed upon the body moved; but relative motion may be generated or altered without any force impressed upon the body. For it is sufficient only to impress some force on other bodies with which the former is compared, that by their giving way, that relation may be changed, in which the relative rest or motion of this other body did consist. Again, true motion suffers always some change from any force impressed upon the moving body; but relative motion does not necessarily undergo any change by such forces. For if the same forces are likewise impressed on those other bodies, with which the comparison is made, that the relative position may be preserved, then that condition will be preserved in which the relative motion consists. And therefore any relative motion may be changed when the true motion remains unaltered, and the relative may be preserved when the true suffers some change. Upon which accounts, true motion does by no means consist in such relations.

The effects which distinguish absolute from relative motion are, the forces of receding from the axis of circular motion. For there are no such forces in a circular motion purely relative, but in a true and absolute circular motion, they are greater or less, according to the quantity of the

motion. If a vessel, hung by a long cord, is so often turned about that the cord is strongly twisted, then filled with water, and held at rest together with the water; after, by the sudden action of another force, it is whirled about the contrary way, and while the cord is untwisting itself, the vessel continues for some time in this motion; the surface of the water will at first be plain, as before the vessel began to move: but the vessel, by gradually communicating its motion to the water, will make it begin sensibly to revolve, and recede by little and little from the middle, and ascend to the sides of the vessel, forming itself into a concave figure (as I have experienced), and the swifter the motion becomes, the higher will the water rise, till at last, performing its revolutions in the same times with the vessel, it becomes relatively at rest in it. This ascent of the water shows its endeavour to recede from the axis of its motion; and the true and absolute circular motion of the water, which is here directly contrary to the relative, discovers itself, and may be measured by this endeavour. At first, when the relative motion of the water in the vessel was greatest, it produced no endeavour to recede from the axis; the water showed no tendency to the circumference, nor any ascent towards the sides of the vessel, but remained of a plain surface, and therefore its true circular motion had not yet begun. But afterwards, when the relative motion of the water had decreased, the ascent thereof towards the sides of the vessel proved its endeavour to recede from the axis; and this endeavour showed the real circular motion of the water perpetually increasing, till it had acquired its greatest quantity, when the water rested relatively in the vessel. And therefore this endeavour does not depend upon any translation of the water in respect of the ambient bodies, nor can true circular motion be defined by such translation. There is only one real circular motion of any one revolving body, corresponding to only one power of endeavouring to recede from its axis of motion, as its proper and adequate effect; but relative motions, in one and the same body, are innumerable, according to the various relations it bears to external bodies, and like other relations, are altogether destitute of any real effect, any otherwise than they may perhaps partake of that one only true motion. And therefore in their system who suppose that our heavens, revolving below the sphere of the fixed stars, carry the planets along with them; the several parts of those heavens, and the planets, which are indeed relatively at rest in their heavens, do yet really move. For they change their position one to another (which never happens to bodies truly at rest), and being carried together with their heavens, partake of their motions, and as parts of revolving wholes, endeavour to recede from the axis of their motions.

Wherefore relative quantities are not the quantities themselves, whose names they bear, but those sensible measures of them (either accurate or inaccurate), which are commonly used instead of the measured quantities themselves. And if the meaning of words is to be determined by their

use, then by the names time, space, place and motion, their measures are properly to be understood; and the expression will be unusual, and purely mathematical, if the measured quantities themselves are meant. Upon which account, they do strain the sacred writings, who there interpret those words for the measured quantities. Nor do those less defile the purity of mathematical and philosophical truths, who confound real quantities themselves with their relations and vulgar measures.

It is indeed a matter of great difficulty to discover, and effectually to distinguish, the true motions of particular bodies from the apparent; because the parts of that immovable space, in which those motions are performed, do by no means come under the observation of our senses. Yet the thing is not altogether desperate; for we have some arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the causes and effects of the true motions. For instance, if two globes, kept at a given distance one from the other by means of a cord that connects them, were revolved about their common centre of gravity, we might, from the tension of the cord, discover the endeavour of the globes to recede from the axis of their motion, and from thence we might compute the quantity of their circular motions. And then if any equal forces should be impressed at once on the alternate faces of the globes to augment or diminish their circular motions, from the increase or decrease of the tension of the cord, we might infer the increment or decrement of their motions; and thence would be found on what faces those forces ought to be impressed, that the motions of the globes might be most augmented; that is, we might discover their hindermost faces, or those which, in the circular motion, do follow. But the faces which follow being known, and consequently the opposite ones that precede, we should likewise know the determination of their motions. And thus we might find both the quantity and the determination of this circular motion, even in an immense vacuum, where there was nothing external or sensible with which the globes could be compared. But now, if in that space some remote bodies were placed that kept always a given position one to another, as the fixed stars do in our regions, we could not indeed determine from the relative translation of the globes among those bodies, whether the motion did belong to the globes or to the bodies. But if we observed the cord, and found that its tension was that very tension which the motions of the globes required, we might conclude the motion to be in the globes, and the bodies to be at rest; and then, lastly, from the translation of the globes among the bodies, we should find the determination of their motions. But how we are to collect the true motions from their causes, effects, and apparent differences; and, *vice versa*, how from the motions, either true or apparent, we may come to the knowledge of their causes and effects, shall be explained more at large in the following tract. For to this end it was that I composed it.

AXIOMS, OR LAWS OF MOTION.

LAW I.

Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.

Projectiles persevere in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity A top, whose parts by their cohesion are perpetually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in more free spaces, preserve their motions both progressive and circular for a much longer time.

LAW II.

The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subducted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

LAW III.

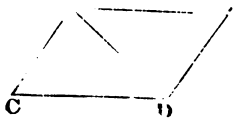
To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone: for the distended rope, by the same endeavour to relax or unbend itself, will draw the horse as much towards the stone, as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other.

If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are reciprocally proportional to the bodies. This law takes place also in attractions, as will be proved in the next scholium.

COROLLARY I.

A body by two forces conjoined will describe the diagonal of a parallelogram, in the same time that it would describe the sides, by those forces apart.

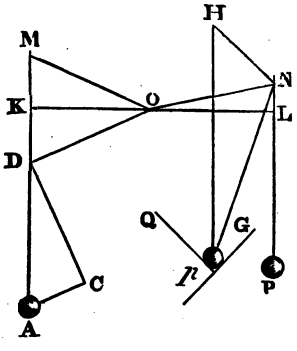


If a body in a given time, by the force M impressed apart in the place A, should with an uniform motion be carried from A to B; and by the force N impressed apart in the same place, should be carried from A to C; complete the parallelogram ABCD, and, by both forces acting together, it will in the same time be carried in the diagonal from A to D. For since the force N acts in the direction of the line AC, parallel to BD, this force (by the second law) will not at all alter the velocity generated by the other force M, by which the body is carried towards the line BD. The body therefore will arrive at the line BD in the same time, whether the force N be impressed or not; and therefore at the end of that time it will be found somewhere in the line BD. By the same argument, at the end of the same time it will be found somewhere in the line CD. Therefore it will be found in the point D, where both lines meet. But it will move in a right line from A to D, by Law I.

COROLLARY II.

And hence is explained the composition of any one direct force AD, out of any two oblique forces AC and CD; and, on the contrary, the resolution of any one direct force AD into two oblique forces AC and CD: which composition and resolution are abundantly confirmed from mechanics.

As if the unequal radii OM and ON drawn from the centre O of any wheel, should sustain the weights A and P by the cords MA and NP; and the forces of those weights to move the wheel were required. Through the centre O draw the right line KOL, meeting the cords perpendicularly in K and L; and from the centre O, with OL the greater of the distances



OK and OL, describe a circle, meeting the cord MA in D: and drawing OD, make AC parallel and DC perpendicular thereto. Now, it being indifferent whether the points K, L, D, of the cords be fixed to the plane of the wheel or not, the weights will have the same effect whether they are suspended from the points K and L, or from D and L. Let the whole force of the weight A be represented by the line AD, and let it be resolved into the forces AC and CD; of which the force AC, drawing the radius OD directly from the centre, will have no effect to move the wheel: but the other force DC, drawing the radius DO perpendicularly, will have the same effect as if it drew perpendicularly the radius OL equal to OD; that is, it will have the same effect as the weight P, if that weight is to the weight A as the force DC is to the force DA; that is (because of the similar triangles ADC, DOK), as OK to OD or OL. Therefore the weights A and P, which are reciprocally as the radii OK and OL that lie in the same right line, will be equipollent, and so remain in equilibrio; which is the well known property of the balance, the lever, and the wheel. If either weight is greater than in this ratio, its force to move the wheel will be so much greater.

If the weight p, equal to the weight P, is partly suspended by the cord Np, partly sustained by the oblique plane pG; draw pH, NH, the former perpendicular to the horizon, the latter to the plane pG; and if the force of the weight p tending downwards is represented by the line pH, it may be resolved into the forces pN, HN. If there was any plane pQ, perpendicular to the cord pN, cutting the other plane pG in a line parallel to the horizon, and the weight p was supported only by those planes pQ, pG, it would press those planes perpendicularly with the forces pN, HN; to wit, the plane pQ with the force pN, and the plane pG with the force HN. And therefore if the plane pQ was taken away, so that the weight might stretch the cord, because the cord, now sustaining the weight, supplies the place of the plane that was removed, it will be strained by the same force pN which pressed upon the plane before. Therefore, the tension of this oblique cord pN will be to that of the other perpendicular cord PN as pN to pH. And therefore if the weight p is to the weight A in a ratio compounded of the reciprocal ratio of the least distances of the cords PN, AM, from the centre of the wheel, and of the direct ratio of pH to pN, the weights will have the same effect towards moving the wheel, and will therefore sustain each other; as any one may find by experiment.

But the weight p pressing upon those two oblique planes, may be considered as a wedge between the two internal surfaces of a body split by it; and hence the forces of the wedge and the mallet may be determined; for

because the force with which the weight p presses the plane pQ is to the force with which the same, whether by its own gravity, or by the blow of a mallet, is impelled in the direction of the line pH towards both the planes, as pN to pH; and to the force with which it presses the other plane pG, as pN to NH. And thus the force of the screw may be deduced from a like resolution of forces; it being no other than a wedge impelled with the force of a lever. Therefore the use of this Corollary spreads far and wide, and by that diffusive extent the truth thereof is farther confirmed. For on what has been said depends the whole doctrine of mechanics variously demonstrated by different authors. For from hence are easily deduced the forces of machines, which are compounded of wheels, pullics, levers, cords, and weights, ascending directly or obliquely, and other mechanical powers; as also the force of the tendons to move the bones of animals.

COROLLARY III.

The quantity of motion, which is collected by taking the sum of the motions directed towards the same parts, and the difference of those that are directed to contrary parts, suffers no change from the action of bodies among themselves.

For action and its opposite re-action are equal, by Law III, and therefore, by Law II, they produce in the motions equal changes towards opposite parts. Therefore if the motions are directed towards the same parts, whatever is added to the motion of the preceding body will be subducted from the motion of that which follows; so that the sum will be the same as before. If the bodies meet, with contrary motions, there will be an equal deduction from the motions of both; and therefore the difference of the motions directed towards opposite parts will remain the same.

Thus if a spherical body A with two parts of velocity is triple of a spherical body B which follows in the same right line with ten parts of velocity, the motion of A will be to that of B as 6 to 10. Suppose, then, their motions to be of 6 parts and of 10 parts, and the sum will be 16 parts. Therefore, upon the meeting of the bodies, if A acquire 3, 4, or 5 parts of motion, B will lose as many; and therefore after reflexion A will proceed with 9, 10, or 11 parts, and B with 7, 6, or 5 parts; the sum remaining always of 16 parts as before. If the body A acquire 9, 10, 11, or 12 parts of motion, and therefore after meeting proceed with 15, 16, 17, or 18 parts, the body B, losing so many parts as A has got, will either proceed with 1 part, having lost 9, or stop and remain at rest, as having lost its whole progressive motion of 10 parts; or it will go back with 1 part, having not only lost its whole motion, but (if I may so say) one part more; or it will go back with 2 parts, because a progressive motion of 12 parts is taken off. And so the sums of the conspiring motions 15 + 1, or 16 + 0, and the differences of the contrary motions 17 — 1 and

18—2, will always be equal to 16 parts, as they were before the meeting and reflexion of the bodies. But, the motions being known with which the bodies proceed after reflexion, the velocity of either will be also known, by taking the velocity after to the velocity before reflexion, as the motion after is to the motion before. As in the last case, where the motion of the body A was of 6 parts before reflexion and of 18 parts after, and the velocity was of 2 parts before reflexion, the velocity thereof after reflexion will be found to be of 6 parts; by saying, as the 6 parts of motion before to 18 parts after, so are 2 parts of velocity before reflexion to 6 parts after.

But if the bodies are either not spherical, or, moving in different right lines, impinge obliquely one upon the other, and their motions after reflexion are required, in those cases we are first to determine the position of the plane that touches the concurring bodies in the point of concourse, then the motion of each body (by Corol. II) is to be resolved into two, one perpendicular to that plane, and the other parallel to it. This done, because the bodies act upon each other in the direction of a line perpendicular to this plane, the parallel motions are to be retained the same after reflexion as before; and to the perpendicular motions we are to assign equal changes towards the contrary parts; in such manner that the sum of the conspiring and the difference of the contrary motions may remain the same as before. From such kind of reflexions also sometimes arise the circular motions of bodies about their own centres. But these are cases which I do not consider in what follows; and it would be too tedious to demonstrate every particular that relates to this subject.

COROLLARY IV.

The common centre of gravity of two or more bodies does not alter its state of motion or rest by the actions of the bodies among themselves; and therefore the common centre of gravity of all bodies acting upon each other (excluding outward actions and impediments) is either at rest, or moves uniformly in a right line.

For if two points proceed with an uniform motion in right lines, and their distance be divided in a given ratio, the dividing point will be either at rest, or proceed uniformly in a right line. This is demonstrated hereafter in Lem. XXIII and its Corol., when the points are moved in the same plane; and by a like way of arguing, it may be demonstrated when the points are not moved in the same plane. Therefore if any number of bodies move uniformly in right lines, the common centre of gravity of any two of them is either at rest, or proceeds uniformly in a right line; because the line which connects the centres of those two bodies so moving is divided at that common centre in a given ratio. In like manner the common centre of those two and that of a third body will be either at rest or moving uniformly in a right line because at that centre the distance between the

common centre of the two bodies, and the centre of this last, is divided in a given ratio. In like manner the common centre of these three, and of a fourth body, is either at rest, or moves uniformly in a right line; because the distance between the common centre of the three bodies, and the centre of the fourth is there also divided in a given ratio, and so on *in infinitum*. Therefore, in a system of bodies where there is neither any mutual action among themselves, nor any foreign force impressed upon them from without, and which consequently move uniformly in right lines, the common centre of gravity of them all is either at rest or moves uniformly forward in a right line.

Moreover, in a system of two bodies mutually acting upon each other, since the distances between their centres and the common centre of gravity of both are reciprocally as the bodies, the relative motions of those bodies, whether of approaching to or of receding from that centre, will be equal among themselves. Therefore since the changes which happen to motions are equal and directed to contrary parts, the common centre of those bodies, by their mutual action between themselves, is neither promoted nor retarded, nor suffers any change as to its state of motion or rest. But in a system of several bodies, because the common centre of gravity of any two acting mutually upon each other suffers no change in its state by that action; and much less the common centre of gravity of the others with which that action does not intervene: but the distance between those two centres is divided by the common centre of gravity of all the bodies into parts reciprocally proportional to the total sums of those bodies whose centres they are; and therefore while those two centres retain their state of motion or rest, the common centre of all does also retain its state: it is manifest that the common centre of all never suffers any change in the state of its motion or rest from the actions of any two bodies between themselves. But in such a system all the actions of the bodies among themselves either happen between two bodies, or are composed of actions interchanged between some two bodies; and therefore they do never produce any alteration in the common centre of all as to its state of motion or rest. Wherefore since that centre, when the bodies do not act mutually one upon another, either is at rest or moves uniformly forward in some right line, it will, notwithstanding the mutual actions of the bodies among themselves, always persevere in its state, either of rest, or of proceeding uniformly in a right line, unless it is forced out of this state by the action of some power impressed from without upon the whole system. And therefore the same law takes place in a system consisting of many bodies as in one single body, with regard to their persevering in their state of motion or of rest. For the progressive motion, whether of one single body, or of a whole system of bodies, is always to be estimated from the motion of the centre of gravity.

COROLLARY V.

The motions of bodies included in a given space are the same among

themselves, whether that space is at rest, or moves uniformly forwards in a right line without any circular motion.

For the differences of the motions tending towards the same parts, and the sums of those that tend towards contrary parts, are, at first (by supposition), in both cases the same; and it is from those sums and differences that the collisions and impulses do arise with which the bodies mutually impinge one upon another. Wherefore (by Law II), the effects of those collisions will be equal in both cases; and therefore the mutual motions of the bodies among themselves in the one case will remain equal to the mutual motions of the bodies among themselves in the other. A clear proof of which we have from the experiment of a ship; where all motions happen after the same manner, whether the ship is at rest, or is carried uniformly forwards in a right line.

COROLLARY VI.

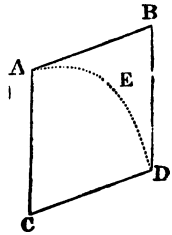
If bodies, any how moved among themselves, are urged in the direction of parallel lines by equal accelerative forces, they will all continue to move among themselves, after the same manner as if they had been urged by no such forces.

For these forces acting equally (with respect to the quantities of the bodies to be moved), and in the direction of parallel lines, will (by Law II) move all the bodies equally (as to velocity), and therefore will never produce any change in the positions or motions of the bodies among themselves.

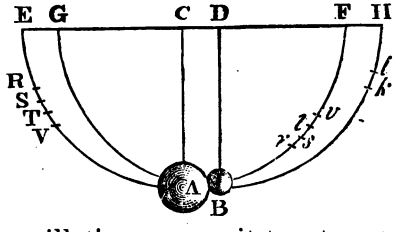
SCHOLIUM.

Hitherto I have laid down such principles as have been received by mathematicians, and are confirmed by abundance of experiments. By the first two Laws and the first two Corollaries, Galileo discovered that the descent of bodies observed the duplicate ratio of the time, and that the motion of projectiles was in the curve of a parabola; experience agreeing with both, unless so far as these motions are a little retarded by the resistance of the air. When a body is falling, the uniform force of its gravity acting equally, impresses, in equal particles of time, equal forces upon that body, and therefore generates equal velocities; and in the whole time impresses a whole force, and generates a whole velocity proportional to the time. And the spaces described in proportional times are as the velocities and the times conjunctly; that is, in a duplicate ratio of the times. And when a body is thrown upwards, its uniform gravity impresses forces and takes off velocities proportional to the times; and the times of ascending to the greatest heights are as the velocities to be taken off, and those heights are as the velocities and the times conjunctly, or in the duplicate ratio of the velocities. And if a body be projected in any direction, the motion arising from its projection is compounded with the

motion arising from its gravity. As if the body A by its motion of projection alone could describe in a given time the right line AB, and with its motion of falling alone could describe in the same time the altitude AC; complete the paralellogram ABDC, and the body by that compounded motion will at the end of the time be found in the place D; and the curve line AED, which that body describes, will be a parabola, to which the right line AB will be a tangent in A; and whose ordinate BD will be as the square of the line AB.



On the same Laws and Corollaries depend those things which have been demonstrated concerning the times of the vibration of pendulums, and are confirmed by the daily experiments of pendulum clocks. By the same, together with the third Law, Sir Christ. Wren, Dr. Wallis, and Mr. Huygens, the greatest geometers of our times, did severally determine the rules of the congress and reflexion of hard bodies, and much about the same time communicated their discoveries to the Royal Society, exactly agreeing among themselves as to those rules. Dr. Wallis, indeed, was something more early in the publication; then followed Sir Christopher Wren, and, lastly, Mr. Huygens. But Sir Christopher Wren confirmed the truth of the thing before the Royal Society by the experiment of pendulums, which Mr. Mariotte soon after thought fit to explain in a treatise entirely upon that subject. But to bring this experiment to an accurate agreement with the theory, we are to have a due regard as well to the resistance of the air as to the elastic force of the concurring bodies. Let the spherical bodies A, B be suspended by the parallel and equal strings AC, BD, from the centres C, D. About these centres, with those intervals, describe the semicircles EAF, GBH, bisected by the radii CA, DB. Bring the body A to any point R of the arc EAF, and (withdrawing the body B) let it go from thence, and after one oscillation suppose it to return to the point V: then RV will be the retardation arising from the resistance of the air. Of this RV let ST be a fourth part, situated in the middle, to wit, so as RS and TV may be equal, and RS may be to ST as 3 to 2 then will ST represent very nearly the retardation during the descent from S to A. Restore the body B to its place: and, supposing the body A to be let fall from the point S, the velocity thereof in the place of reflexion A, without sensible error, will be the same as if it had descended *in vacuo* from the point T. Upon which account this velocity may be represented by the chord of the arc TA. For it is a proposition well known to geometers, that the velocity of a pendulous body in the lowest point is as the chord of the arc which it has described in its descent. After



reflexion, suppose the body A comes to the place *s*, and the body B to the place *k*. Withdraw the body B, and find the place *v*, from which if the body A, being let go, should after one oscillation return to the place *r*, *st* may be a fourth part of *rv*, so placed in the middle thereof as to leave *rs* equal to *tv*, and let the chord of the arc *t*A represent the velocity which the body A had in the place A immediately after reflexion. For *t* will be the true and correct place to which the body A should have ascended, if the resistance of the air had been taken off. In the same way we are to correct the place *k* to which the body B ascends, by finding the place *l* to which it should have ascended *in vacuo*. And thus everything may be subjected to experiment, in the same manner as if we were really placed *in vacuo*. These things being done, we are to take the product (if I may so say) of the body A, by the chord of the arc TA (which represents its velocity), that we may have its motion in the place A immediately before reflexion; and then by the chord of the arc *t*A, that we may have its motion in the place A immediately after reflexion. And so we are to take the product of the body B by the chord of the arc B*l*, that we may have the motion of the same immediately after reflexion. And in like manner, when two bodies are let go together from different places, we are to find the motion of each, as well before as after reflexion; and then we may compare the motions between themselves, and collect the effects of the reflexion. Thus trying the thing with pendulums of ten feet, in unequal as well as equal bodies, and making the bodies to concur after a descent through large spaces, as of 8, 12, or 16 feet, I found always, without an error of 3 inches, that when the bodies concurred together directly, equal changes towards the contrary parts were produced in their motions, and, of consequence, that the action and reaction were always equal. As if the body A impinged upon the body B at rest with 9 parts of motion, and losing 7, proceeded after reflexion with 2, the body B was carried backwards with those 7 parts. If the bodies concurred with contrary motions, A with twelve parts of motion, and B with six, then if A receded with 2, B receded with 8; to wit, with a deduction of 14 parts of motion on each side. For from the motion of A subducting twelve parts, nothing will remain; but subducting 2 parts more, a motion will be generated of 2 parts towards the contrary way; and so, from the motion of the body B of 6 parts, subducting 14 parts, a motion is generated of 8 parts towards the contrary way. But if the bodies were made both to move towards the same way, A, the swifter, with 14 parts of motion, B, the slower, with 5, and after reflexion A went on with 5, B likewise went on with 14 parts; 9 parts being transferred from A to B. And so in other cases. By the congress and collision of bodies, the quantity of motion, collected from the sum of the motions directed towards the same way, or from the difference of those that were directed towards contrary ways, was never changed. For the error of an inch or two in measures may be easily ascribed to the

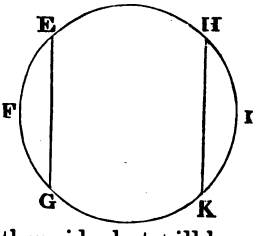
difficulty of executing everything with accuracy. It was not easy to let go the two pendulums so exactly together that the bodies should impinge one upon the other in the lowermost place AB; nor to mark the places *s*, and *k*, to which the bodies ascended after congress. Nay, and some errors, too, might have happened from the unequal density of the parts of the pendulous bodies themselves, and from the irregularity of the texture proceeding from other causes.

But to prevent an objection that may perhaps be alledged against the rule, for the proof of which this experiment was made, as if this rule did suppose that the bodies were either absolutely hard, or at least perfectly elastic (whereas no such bodies are to be found in nature), I must add, that the experiments we have been describing, by no means depending upon that quality of hardness, do succeed as well in soft as in hard bodies. For if the rule is to be tried in bodies not perfectly hard, we are only to diminish the reflexion in such a certain proportion as the quantity of the elastic force requires. By the theory of Wren and Huygens, bodies absolutely hard return one from another with the same velocity with which they meet. But this may be affirmed with more certainty of bodies perfectly elastic. In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force (except when the parts of bodies are bruised by their congress, or suffer some such extension as happens under the strokes of a hammer) is (as far as I can perceive) certain and determined, and makes the bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met. This I tried in balls of wool, made up tightly, and strongly compressed. For, first, by letting go the pendulous bodies, and measuring their reflexion, I determined the quantity of their elastic force; and then, according to this force, estimated the reflexions that ought to happen in other cases of congress. And with this computation other experiments made afterwards did accordingly agree; the balls always receding one from the other with a relative velocity, which was to the relative velocity with which they met as about 5 to 9. Balls of steel returned with almost the same velocity: those of cork with a velocity something less; but in balls of glass the proportion was as about 15 to 16. And thus the third Law, so far as it regards percussions and reflexions, is proved by a theory exactly agreeing with experience.

In attractions, I briefly demonstrate the thing after this manner. Suppose an obstacle is interposed to hinder the congress of any two bodies A, B, mutually attracting one the other: then if either body, as A, is more attracted towards the other body B, than that other body B is towards the first body A, the obstacle will be more strongly urged by the pressure of the body A than by the pressure of the body B, and therefore will not remain in equilibrio: but the stronger pressure will prevail, and will make the system of the two bodies, together with the obstacle, to move directly

towards the parts on which B lies; and in free spaces, to go forward *in infinitum* with a motion perpetually accelerated; which is absurd and contrary to the first Law. For, by the first Law, the system ought to persevere in its state of rest, or of moving uniformly forward in a right line; and therefore the bodies must equally press the obstacle, and be equally attracted one by the other. I made the experiment on the loadstone and iron. If these, placed apart in proper vessels, are made to float by one another in standing water, neither of them will propel the other; but, by being equally attracted, they will sustain each other's pressure, and rest at last in an equilibrium.

So the gravitation betwixt the earth and its parts is mutual. Let the earth FI be cut by any plane EG into two parts EGF and EGI, and their weights one towards the other will be mutually equal. For if by another plane HK, parallel to the former EG, the greater part EGI is cut into two parts EGKH and HKI, whereof HKI is equal to the part EFG, first cut off, it is evident that the middle part EGKH, will have no propension by its proper weight towards either side, but will hang as it were, and rest in an equilibrium betwixt both. But the one extreme part HKI will with its whole weight bear upon and press the middle part towards the other extreme part EGF; and therefore the force with which EGI, the sum of the parts HKI and EGKH, tends towards the third part EGF, is equal to the weight of the part HKI, that is, to the weight of the third part EGF. And therefore the weights of the two parts EGI and EGF, one towards the other, are equal, as I was to prove. And indeed if those weights were not equal, the whole earth floating in the non-resisting æther would give way to the greater weight, and, retiring from it, would be carried off *in infinitum*.



And as those bodies are equipollent in the congress and reflexion, whose velocities are reciprocally as their innate forces, so in the use of mechanic instruments those agents are equipollent, and mutually sustain each the contrary pressure of the other, whose velocities, estimated according to the determination of the forces, are reciprocally as the forces.

So those weights are of equal force to move the arms of a balance; which during the play of the balance are reciprocally as their velocities upwards and downwards; that is, if the ascent or descent is direct, those weights are of equal force, which are reciprocally as the distances of the points at which they are suspended from the axis of the balance; but if they are turned aside by the interposition of oblique planes, or other obstacles, and made to ascend or descend obliquely, those bodies will be equipollent, which are reciprocally as the heights of their ascent and descent taken according to the perpendicular; and that on account of the determination of gravity downwards.

And in like manner in the pully, or in a combination of pullies, the force of a hand drawing the rope directly, which is to the weight, whether ascending directly or obliquely, as the velocity of the perpendicular ascent of the weight to the velocity of the hand that draws the rope, will sustain the weight.

In clocks and such like instruments, made up from a combination of wheels, the contrary forces that promote and impede the motion of the wheels, if they are reciprocally as the velocities of the parts of the wheel on which they are impressed, will mutually sustain the one the other.

The force of the screw to press a body is to the force of the hand that turns the handles by which it is moved as the circular velocity of the handle in that part where it is impelled by the hand is to the progressive velocity of the screw towards the pressed body.

The forces by which the wedge presses or drives the two parts of the wood it cleaves are to the force of the mallet upon the wedge as the progress of the wedge in the direction of the force impressed upon it by the mallet is to the velocity with which the parts of the wood yield to the wedge, in the direction of lines perpendicular to the sides of the wedge. And the like account is to be given of all machines.

The power and use of machines consist only in this, that by diminishing the velocity we may augment the force, and the contrary: from whence in all sorts of proper machines, we have the solution of this problem; *To move a given weight with a given power*, or with a given force to overcome any other given resistance. For if machines are so contrived that the velocities of the agent and resistant are reciprocally as their forces, the agent will just sustain the resistant, but with a greater disparity of velocity will overcome it. So that if the disparity of velocities is so great as to overcome all that resistance which commonly arises either from the attrition of contiguous bodies as they slide by one another, or from the cohesion of continuous bodies that are to be separated, or from the weights of bodies to be raised, the excess of the force remaining, after all those resistances are overcome, will produce an acceleration of motion proportional thereto, as well in the parts of the machine as in the resisting body. But to treat of mechanics is not my present business. I was only willing to show by those examples the great extent and certainty of the third Law of motion. For if we estimate the action of the agent from its force and velocity conjunctly, and likewise the reaction of the impediment conjunctly from the velocities of its several parts, and from the forces of resistance arising from the attrition, cohesion, weight, and acceleration of those parts, the action and reaction in the use of all sorts of machines will be found always equal to one another. And so far as the action is propagated by the intervening instruments, and at last impressed upon the resisting body, the ultimate determination of the action will be always contrary to the determination of the reaction.

BOOK I.

OF THE MOTION OF BODIES.

SECTION I.

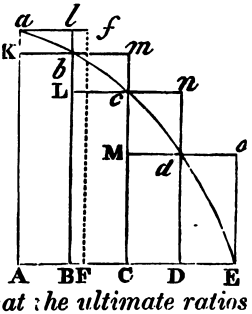
Of the method of first and last ratios of quantities, by the help whereof we demonstrate the propositions that follow.

LEMMA I.

Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal.

If you deny it, suppose them to be ultimately unequal, and let D be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference D; which is against the supposition.

LEMMA II.



If in any figure AacE, *terminated by the right lines* Aa, AE, *and the curve* acE, *there be inscribed any number of parallelograms* Ab, Bc, Cd, *&c., comprehended under equal bases* AB, BC, CD, *&c., and the sides,* Bb, Cc, Dd, *&c., parallel to one side* Aa *of the figure; and the parallelograms* aKbl, bLcm, cMdn, *&c., are completed. Then if the breadth of those parallelograms be supposed to be diminished, and their number to be augmented* in infinitum; *I say, that the ultimate ratios which the inscribed figure* AKbLcMdD, *the circumscribed figure* AalbmcndoE, *and curvilinear figure* AabcdE, *will have to one another, are ratios of equality.*

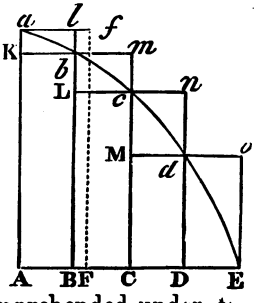
For the difference of the inscribed and circumscribed figures is the sum of the parallelograms K*l*, L*m*, M*n*, D*o*, that is (from the equality of all their bases), the rectangle under one of their bases K*b* and the sum of their altitudes A*a*, that is, the rectangle AB*la*. But this rectangle, because

its breadth AB is supposed diminished *in infinitum*, becomes less than any given space. And therefore (by Lem. I) the figures inscribed and circumscribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. Q.E.D.

LEMMA III.

The same ultimate ratios are also ratios of equality, when the breadths AB, BC, DC, &c., *of the parallelograms are unequal, and are all diminished* in infinitum.

For suppose AF equal to the greatest breadth, and complete the parallelogram FA*af*. This parallelogram will be greater than the difference of the inscribed and circumscribed figures; but, because its breadth AF is diminished *in infinitum*, it will become less than any given rectangle. Q.E.D.



Cor. 1. Hence the ultimate sum of those evanescent parallelograms will in all parts coincide with the curvilinear figure.

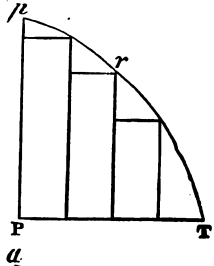
Cor. 2. Much more will the rectilinear figure comprehended under the chords of the evanescent arcs *ab*, *bc*, *cd*, &c., ultimately coincide with the curvilinear figure.

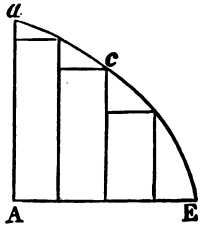
Cor. 3. And also the circumscribed rectilinear figure comprehended under the tangents of the same arcs.

Cor. 4 And therefore these ultimate figures (as to their perimeters *acE*) are not rectilinear, but curvilinear limits of rectilinear figures.

LEMMA IV.

If in two figures AacE, PprT, *you inscribe (as before) two ranks of parallelograms, an equal number in each rank, and, when their breadths are diminished* in infinitum, *the ultimate ratios of the parallelograms in one figure to those in the other, each to each respectively, are the same; I say, that those two figures* AacE, PprT, *are to one another in that same ratio.*





For as the parallelograms in the one are severally to the parallelograms in the other, so (by composition) is the sum of all in the one to the sum of all in the other; and so is the one figure to the other; because (by Lem. III) the former figure to the former sum, and the latter figure to the latter sum, are both in the ratio of equality. Q.E.D.

Cor. Hence if two quantities of any kind are any how divided into an equal number of parts, and those

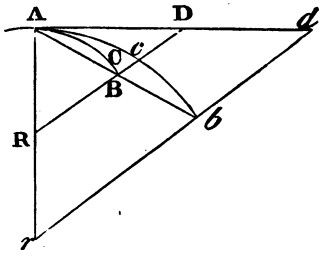
parts, when their number is augmented, and their magnitude diminished *in infinitum*, have a given ratio one to the other, the first to the first, the second to the second, and so on in order, the whole quantities will be one to the other in that same given ratio. For if, in the figures of this Lemma, the parallelograms are taken one to the other in the ratio of the parts, the sum of the parts will always be as the sum of the parallelograms; and therefore supposing the number of the parallelograms and parts to be augmented, and their magnitudes diminished *in infinitum*, those sums will be in the ultimate ratio of the parallelogram in the one figure to the correspondent parallelogram in the other; that is (by the supposition), in the ultimate ratio of any part of the one quantity to the correspondent part of the other.

LEMMA V.

In similar figures, all sorts of homologous sides, whether curvilinear or rectilinear, are proportional; and the areas are in the duplicate ratio of the homologous sides.

LEMMA VI.

If any arc ACB, *given in position is subtended by its chord* AB, *and in any point* A, *in the middle of the continued curvature, is touched by a right line* AD, *produced both ways; then if the points* A *and* B *approach one another and meet, I say, the angle* BAD, *contained between the chord and the tangent, will be diminished* in infinitum, *and ultimately will vanish.*



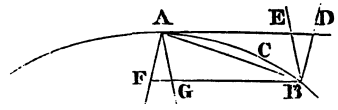
For if that angle does not vanish, the arc ACB will contain with the tangent AD an angle equal to a rectilinear angle; and therefore the curvature at the point A will not be continued, which is against the supposition.

LEMMA VII.

The same things being supposed, I say that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality.

For while the point B approaches towards the point A, consider always AB and AD as produced to the remote points *b* and *d*, and parallel to the secant BD draw *bd*: and let the arc A*cb* be always similar to the arc ACB. Then, supposing the points A and B to coincide, the angle *dAb* will vanish, by the preceding Lemma; and therefore the right lines A*b*, A*d* (which are always finite), and the intermediate arc A*cb*, will coincide, and become equal among themselves. Wherefore, the right lines AB, AD,

and the intermediate arc ACB (which are always proportional to the former), will vanish, and ultimately acquire the ratio of equality. Q.E.D.



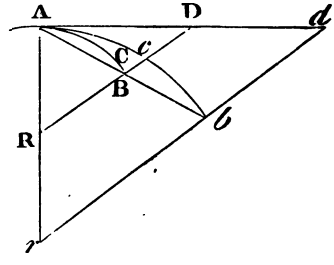
COR. 1. Whence if through B we draw BF parallel to the tangent, always cutting any right line AF passing through A in F, this line BF will be ultimately in the ratio of equality with the evanescent arc ACB; because, completing the parallelogram AFBD, it is always in a ratio of equality with AD.

COR. 2. And if through B and A more right lines are drawn, as BE, BD, AF, AG, cutting the tangent AD and its parallel BF; the ultimate ratio of all the abscissas AD, AE, BF, BG, and of the chord and arc AB, any one to any other, will be the ratio of equality.

COR. 3. And therefore in all our reasoning about ultimate ratios, we may freely use any one of those lines for any other.

LEMMA VIII.

If the right lines AR, BR, *with the arc* ACB, *the chord* AB, *and the tangent* AD, *constitute three triangles* RAB, RACB, RAD, *and the points* A *and* B *approach and meet: I say, that the ultimate form of these evanescent triangles is that of similitude, and their ultimate ratio that of equality.*

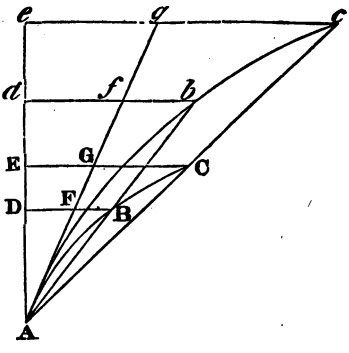


For while the point B approaches towards the point A, consider always AB, AD, AR, as produced to the remote points *b*, *d*, and *r*, and *rbd* as drawn parallel to RD, and let the arc A*cb* be always similar to the arc ACB. Then supposing the points A and B to coincide. the angle *b*A*d* will vanish; and therefore the three triangles *r*A*b*, *r*A*cb*, *r*A*d* (which are always finite), will coincide, and on that account become both similar and equal. And therefore the triangles RAB, RACB, RAD which are always similar and proportional to these, will ultimately become both similar and equal among themselves. Q.E.D.

COR. And hence in all reasonings about ultimate ratios, we may indifferently use any one of those triangles for any other.

LEMMA IX.

If a right line AE. *and a curve line* ABC, *both given by position, cut each other in a given angle,* A; *and to that right line, in another given angle,* BD, CE *are ordinately applied, meeting the curve in* B, C; *and the points* B *and* C *together approach towards and meet in the point* A: *I say, that the areas of the triangles* ABD, ACE, *will ultimately be one to the other in the duplicate ratio of the sides.*



For while the points B, C, approach towards the point A, suppose always AD to be produced to the remote points *d* and *e*, so as *Ad*, *Ae* may be proportional to AD, AE; and the ordinates *db*, *ec*, to be drawn parallel to the ordinates DB and EC, and meeting AB and AC produced in *b* and *c*. Let the curve *Abc* be similar to the curve ABC, and draw the right line *Ag* so as to touch both curves in A, and cut the ordinates DB, EC, *db ec*, in F, G, *f*, *g*. Then, supposing the length *Ae* to remain the same, let the points B and C meet in the point A; and the angle *cAg* vanishing, the curvilinear areas *Abd*, *Ace* will coincide with the rectilinear areas *Afd*, *Age*; and therefore (by Lem. V) will be one to the other in the duplicate ratio of the sides *Ad*, *Ae*. But the areas ABD, ACE are always proportional to these areas; and so the sides AD, AE are to these sides. And therefore the areas ABD, ACE are ultimately one to the other in the duplicate ratio of the sides AD, AE. Q.E.D.

LEMMA X.

The spaces which a body describes by any finite force urging it, whether that force is determined and immutable, or is continually augmented or continually diminished, are in the very beginning of the motion one to the other in the duplicate ratio of the times.

Let the times be represented by the lines AD, AE, and the velocities generated in those times by the ordinates DB, EC. The spaces described with these velocities will be as the areas ABD, ACE, described by those ordinates, that is, at the very beginning of the motion (by Lem. IX), in the duplicate ratio of the times AD, AE. Q.E.D.

COR. 1. And hence one may easily infer, that the errors of bodies describing similar parts of similar figures in proportional times, are nearly as the squares of the times in which they are generated; if so be these errors are generated by any equal forces similarly applied to the bodies, and measured by the distances of the bodies from those places of the similar figures, at which, without the action of those forces, the bodies would have arrived in those proportional times.

COR. 2. But the errors that are generated by proportional forces, similarly applied to the bodies at similar parts of the similar figures, are as the forces and the squares of the times conjunctly.

COR. 3. The same thing is to be understood of any spaces whatsoever described by bodies urged with different forces; all which, in the very beginning of the motion, are as the forces and the squares of the times conjunctly.

COR. 4. And therefore the forces are as the spaces described in the very beginning of the motion directly, and the squares of the times inversely.

COR. 5. And the squares of the times are as the spaces described directly, and the forces inversely.

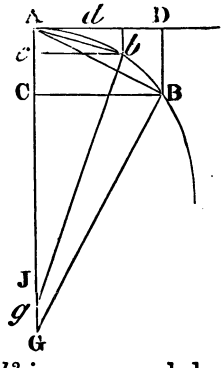
SCHOLIUM.

If in comparing indetermined quantities of different sorts one with another, any one is said to be as any other directly or inversely, the meaning is, that the former is augmented or diminished in the same ratio with the latter, or with its reciprocal. And if any one is said to be as any other two or more directly or inversely, the meaning is, that the first is augmented or diminished in the ratio compounded of the ratios in which the others, or the reciprocals of the others, are augmented or diminished. As if A is said to be as B directly, and C directly, and D inversely, the meaning is, that A is augmented or diminished in the same ratio with $B \times C \times \frac{1}{D}$, that is to say, that A and $\frac{BC}{D}$ are one to the other in a given ratio.

LEMMA XI.

The evanescent subtense of the angle of contact, in all curves which at the point of contact have a finite curvature, is ultimately in the duplicate ratio of the subtense of the conterminate arc.

CASE 1. Let AB be that arc, AD its tangent, BD the subtense of the angle of contact perpendicular on the tangent, AB the subtense of the arc. Draw BG perpendicular to the subtense AB, and AG to the tangent AD, meeting in G; then let the points D, B, and G, approach to the points *d*, *b*, and *g*, and suppose J to be the ultimate intersection of the lines BG, AG, when the points D, B, have come to A. It is evident that the distance GJ may be less than any assignable. But (from the nature of the circles passing through the points A, B, G, A, *b*, *g*,) $AB^2 = AG \times BD$, and $Ab^2 = Ag \times bd$; and therefore the ratio of AB^2 to Ab^2 is compounded of the ratios of AG to A*g*, and of B*d* to *bd*. But because GJ may be assumed of less length than any assignable, the ratio of AG to A*g* may be such as to differ from the ratio of equality by less than any assignable difference; and therefore the ratio of AB^2 to Ab^2 may be such as to differ from the ratio of BD to *bd* by less than any assignable difference. Therefore, by Lem. I, the ultimate ratio of AB^2 to Ab^2 is the same with the ultimate ratio of BD to *bd*. Q.E.D.



CASE 2. Now let BD be inclined to AD in any given angle, and the ultimate ratio of BD to *bd* will always be the same as before, and therefore the same with the ratio of AB^2 to Ab^2. Q.E.D.

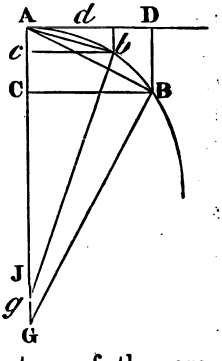
CASE 3. And if we suppose the angle D not to be given, but that the right line BD converges to a given point, or is determined by any other condition whatever; nevertheless the angles D, *d*, being determined by the same law, will always draw nearer to equality, and approach nearer to each other than by any assigned difference, and therefore, by Lem. I, will at last be equal; and therefore the lines BD, *bd* arc in the same ratio to each other as before. Q.E.D.

COR. 1. Therefore since the tangents AD, A*d*, the arcs AB, A*b*, and their sines, BC, *bc*, become ultimately equal to the chords AB, A*b*, their squares will ultimately become as the subtenses BD, *bd*.

COR. 2. Their squares are also ultimately as the versed sines of the arcs, bisecting the chords, and converging to a given point. For those versed sines are as the subtenses BD, *bd*.

COR. 3. And therefore the versed sine is in the duplicate ratio of the time in which a body will describe the arc with a given velocity.

COR. 4. The rectilinear triangles ADB, A*db* are ultimately in the triplicate ratio of the sides AD, A*d*, and in a sesquiplicate ratio of the sides DB, *db*; as being in the ratio compounded of the sides AD to DB, and of A*d* to *db*. So also the triangles ABC, A*bc* are ultimately in the triplicate ratio of the sides BC, *bc*. What I call the sesquiplicate ratio is the subduplicate of the triplicate, as being compounded of the simple and subduplicate ratio.



COR. 5. And because DB, *db* are ultimately parallel and in the duplicate ratio of the lines AD, A*d*, the ultimate curvilinear areas ADB, A*db* will be (by the nature of the parabola) two thirds of the rectilinear triangles ADB, A*db* and the segments AB, A*b* will be one third of the same triangles. And thence those areas and those segments will be in the triplicate ratio as well of the tangents AD, A*d*, as of the chords and arcs AB, AB.

SCHOLIUM.

But we have all along supposed the angle of contact to be neither infinitely greater nor infinitely less than the angles of contact made by circles and their tangents; that is, that the curvature at the point A is neither infinitely small nor infinitely great, or that the interval AJ is of a finite magnitude. For DB may be taken as AD^3: in which case no circle can be drawn through the point A, between the tangent AD and the curve AB, and therefore the angle of contact will be infinitely less than those of circles. And by a like reasoning, if DB be made successfully as AD^4, AD^5, AD^6, AD^7, &c., we shall have a series of angles of contact, proceeding *in infinitum*, wherein every succeeding term is infinitely less than the pre-

ceding. And if DB be made successively as AD^2, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$ $AD^{\frac{7}{6}}$, &c., we shall have another infinite series of angles of contact, the first of which is of the same sort with those of circles, the second infinitely greater, and every succeeding one infinitely greater than the preceding. But between any two of these angles another series of intermediate angles of contact may be interposed, proceeding both ways *in infinitum*, wherein every succeeding angle shall be infinitely greater or infinitely less than the preceding. As if between the terms AD^2 and AD^3 there were interposed the series $AD^{\frac{13}{5}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$, &c. And again, between any two angles of this series, a new series of intermediate angles may be interposed, differing from one another by infinite intervals. Nor is nature confined to any bounds.

Those things which have been demonstrated of curve lines, and the superfices which they comprehend, may be easily applied to the curve superfices and contents of solids. These Lemmas are premised to avoid the tediousness of deducing perplexed demonstrations *ad absurdum*, according to the method of the ancient geometers. For demonstrations are more contracted by the method of indivisibles: but because the hypothesis of indivisibles seems somewhat harsh, and therefore that method is reckoned less geometrical, I chose rather to reduce the demonstrations of the following propositions to the first and last sums and ratios of nascent and evanescent quantities, that is, to the limits of those sums and ratios; and so to premise, as short as I could, the demonstrations of those limits. For hereby the same thing is performed as by the method of indivisibles; and now those principles being demonstrated, we may use them with more safety. Therefore if hereafter I should happen to consider quantities as made up of particles, or should use little curve lines for right ones, I would not be understood to mean indivisibles, but evanescent divisible quantities; not the sums and ratios of determinate parts, but always the limits of sums and ratios; and that the force of such demonstrations always depends on the method laid down in the foregoing Lemmas.

Perhaps it may be objected, that there is no ultimate proportion, of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But by the same argument, it may be alledged, that a body arriving at a certain place, and there stopping, has no ultimate velocity: because the velocity, before the body comes to the place, is not its ultimate velocity; when it has arrived, is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases. And in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities

not before they vanish, nor afterwards, but with which they vanish. In like manner the first ratio of nascent quantities is that with which they begin to be. And the first or last sum is that with which they begin and cease to be (or to be augmented or diminished). There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. But whatever is geometrical we may be allowed to use in determining and demonstrating any other thing that is likewise geometrical.

It may also be objected, that if the ultimate ratios of evanescent quantities are given, their ultimate magnitudes will be also given: and so all quantities will consist of indivisibles, which is contrary to what *Euclid* has demonstrated concerning incommensurables, in the 10th Book of his Elements. But this objection is founded on a false supposition. For those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*. This thing will appear more evident in quantities infinitely great. If two quantities, whose difference is given, be augmented *in infinitum*, the ultimate ratio of these quantities will be given, to wit, the ratio of equality; but it does not from thence follow, that the ultimate or greatest quantities themselves, whose ratio that is, will be given. Therefore if in what follows, for the sake of being more easily understood, I should happen to mention quantities as least, or evanescent, or ultimate, you are not to suppose that quantities of any determinate magnitude are meant, but such as are conceived to be always diminished without end.

SECTION II.

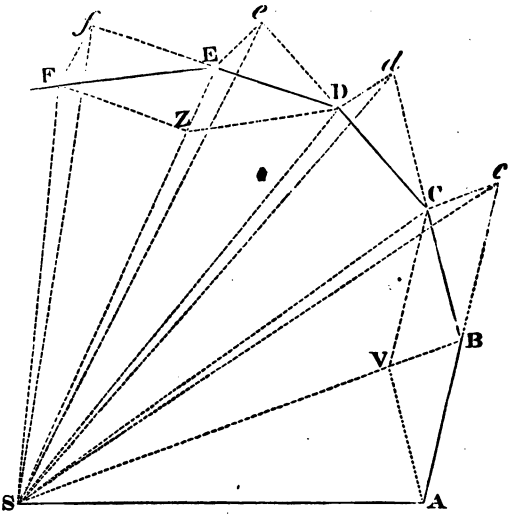
Of the Invention of Centripetal Forces.

PROPOSITION I. THEOREM I.

The areas, which revolving bodies describe by radii drawn to an immovable centre of force do lie in the same immovable planes, and are proportional to the times in which they are described.

For suppose the time to be divided into equal parts, and in the first part of that time let the body by its innate force describe the right line AB. In the second part of that time, the same would (by Law I.), if not hindered, proceed directly to *c*, along the line Bc equal to AB; so that by the radii AS, BS, *c*S, drawn to the centre, the equal areas ASB, BSc, would be de-

scribed. But when the body is arrived at B, suppose that a centripetal force acts at once with a great impulse, and, turning aside the body from the right line Bc, compels it afterwards to continue its motion along the right line BC. Draw cC parallel to BS meeting BC in C; and at the end of the second part of the time, the body (by Cor. I. of the Laws) will be found in C, in the same plane with the triangle ASB. Join SC, and, because SB and Cc are parallel, the triangle SBC will be equal to the triangle SBc, and therefore also to the triangle SAB. By the like argument, if the centripetal force acts successively in C, D, E, &c., and makes the body, in each single particle of time, to describe the right lines CD, DE, EF, &c., they will all lie in the same plane; and the triangle SCD will be equal to the triangle SBC, and SDE to SCD, and SEF to SDE. And therefore, in equal times, equal areas are described in one immovable plane: and, by composition, any sums SADS, SAFS, of those areas, are one to the other as the times in which they are described. Now let the number of those triangles be augmented, and their breadth diminished *in infinitum*; and (by Cor. 4, Lem. III.) their ultimate perimeter ADF will be a curve line: and therefore the centripetal force, by which the body is perpetually drawn back from the tangent of this curve, will act continually; and any described areas SADS, SAFS, which are always proportional to the times of description, will, in this case also, be proportional to those times. Q.E.D.



Cor. 1. The velocity of a body attracted towards an immovable centre, in spaces void of resistance, is reciprocally as the perpendicular let fall from that centre on the right line that touches the orbit. For the velocities in those places A, B, C, D, E, are as the bases AB, BC, CD, DE, EF, of equal triangles; and these bases are reciprocally as the perpendiculars let fall upon them.

Cor. 2. If the chords AB, BC of two arcs, successively described in equal times by the same body, in spaces void of resistance, are completed into a parallelogram ABCV, and the diagonal BV of this parallelogram, in the position which it ultimately acquires when those arcs are diminished *in infinitum*, is produced both ways, it will pass through the centre of force.

Cor. 3. If the chords AB, BC, and DE, EF, of arcs described in equal

times, in spaces void of resistance, are completed into the parallelograms ABCV, DEFZ; the forces in B and E are one to the other in the ultimate ratio of the diagonals BV, EZ, when those arcs are diminished *in infinitum*. For the motions BC and EF of the body (by Cor. 1 of the Laws) are compounded of the motions Bc, BV, and E*f*, EZ: but BV and EZ, which are equal to C*c* and F*f*, in the demonstration of this Proposition, were generated by the impulses of the centripetal force in B and E, and are therefore proportional to those impulses.

COR. 4. The forces by which bodies, in spaces void of resistance, are drawn back from rectilinear motions, and turned into curvilinear orbits, are one to another as the versed sines of arcs described in equal times; which versed sines tend to the centre of force, and bisect the chords when those arcs are diminished to infinity. For such versed sines are the halves of the diagonals mentioned in Cor. 3.

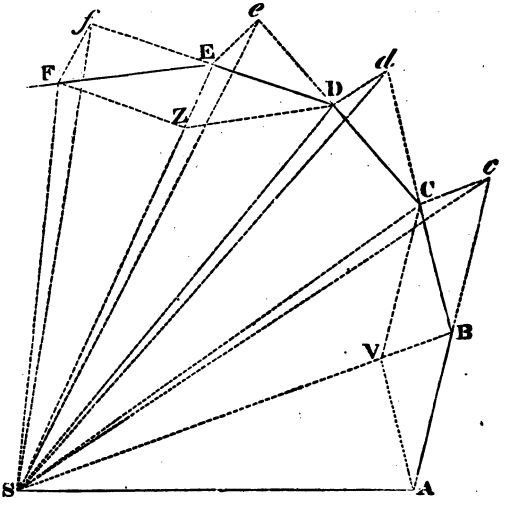
COR. 5. And therefore those forces are to the force of gravity as the said versed sines to the versed sines perpendicular to the horizon of those parabolic arcs which projectiles describe in the same time.

COR. 6. And the same things do all hold good (by Cor. 5 of the Laws), when the planes in which the bodies are moved, together with the centres of force which are placed in those planes, are not at rest, but move uniformly forward in right lines.

PROPOSITION II. THEOREM II.

Every body that moves in any curve line described in a plane, and by a radius, drawn to a point either immovable, or moving forward with an uniform rectilinear motion, describes about that point areas proportional to the times, is urged by a centripetal force directed to that point

CASE. 1. For every body that moves in a curve line, is (by Law 1) turned aside from its rectilinear course by the action of some force that impels it. And that force by which the body is turned off from its rectilinear course, and is made to describe, in equal times, the equal least triangles SAB, SBC, SCD, &c., about the immovable point S (by Prop. XL. Book 1, Elem. and Law II), acts in the place B, according to the direction of a line par-



allel to *cC*, that is, in the direction of the line BS; and in the place C, according to the direction of a line parallel to *dD*, that is, in the direction of the line CS, &c.; and therefore acts always in the direction of lines tending to the immovable point S. Q.E.D.

Case. 2. And (by Cor. 5 of the Laws) it is indifferent whether the superfices in which a body describes a curvilinear figure be quiescent, or moves together with the body, the figure described, and its point S, uniformly forward in right lines.

Cor. 1. In non-resisting spaces or mediums, if the areas are not proportional to the times, the forces are not directed to the point in which the radii meet; but deviate therefrom *in consequentia*, or towards the parts to which the motion is directed, if the description of the areas is accelerated; but *in antecedentia*, if retarded.

Cor. 2. And even in resisting mediums, if the description of the areas is accelerated, the directions of the forces deviate from the point in which the radii meet, towards the parts to which the motion tends.

SCHOLIUM.

A body may be urged by a centripetal force compounded of several forces; in which case the meaning of the Proposition is, that the force which results out of all tends to the point S. But if any force acts perpetually in the direction of lines perpendicular to the described surface, this force will make the body to deviate from the plane of its motion: but will neither augment nor diminish the quantity of the described surface, and is therefore to be neglected in the composition of forces.

PROPOSITION III. THEOREM III.

Every body, that by a radius drawn to the centre of another body, howsoever moved, describes areas about that centre proportional to the times, is urged by a force compounded out of the centripetal force tending to that other body, and of all the accelerative force by which that other body is impelled.

Let L represent the one, and T the other body; and (by Cor. 6 of the Laws) if both bodies are urged in the direction of parallel lines, by a new force equal and contrary to that by which the second body T is urged, the first body L will go on to describe about the other body T the same areas as before: but the force by which that other body T was urged will be now destroyed by an equal and contrary force; and therefore (by Law I.) that other body T, now left to itself, will either rest, or move uniformly forward in a right line: and the first body L impelled by the difference of the forces, that is, by the force remaining, will go on to describe about the other body T areas proportional to the times. And therefore (by Theor. II.) the difference of the forces is directed to the other body T as its centre. Q.E.D

COR. 1. Hence if the one body L, by a radius drawn to the other body T, describes areas proportional to the times; and from the whole force, by which the first body L is urged (whether that force is simple, or, according to Cor. 2 of the Laws, compounded out of several forces), we subduct (by the same Cor.) that whole accelerative force by which the other body is urged; the whole remaining force by which the first body is urged will tend to the other body T, as its centre.

COR. 2. And, if these areas are proportional to the times nearly, the remaining force will tend to the other body T nearly.

COR. 3. And *vice versa*, if the remaining force tends nearly to the other body T, those areas will be nearly proportional to the times.

COR. 4. If the body L, by a radius drawn to the other body T, describes areas, which, compared with the times, are very unequal; and that other body T be either at rest, or moves uniformly forward in a right line: the action of the centripetal force tending to that other body T is either none at all, or it is mixed and compounded with very powerful actions of other forces: and the whole force compounded of them all, if they are many, is directed to another (immovable or moveable) centre. The same thing obtains, when the other body is moved by any motion whatsoever; provided that centripetal force is taken, which remains after subducting that whole force acting upon that other body T.

SCHOLIUM.

Because the equable description of areas indicates that a centre is respected by that force with which the body is most affected, and by which it is drawn back from its rectilinear motion, and retained in its orbit; why may we not be allowed, in the following discourse, to use the equable description of areas as an indication of a centre, about which all circular motion is performed in free spaces?

PROPOSITION IV. THEOREM IV.

The centripetal forces of bodies, which by equable motions describe different circles, tend to the centres of the same circles; and are one to the other as the squares of the arcs described in equal times applied to the radii of the circles.

These forces tend to the centres of the circles (by Prop. II., and Cor. 2, Prop. I.), and are one to another as the versed sines of the least arcs described in equal times (by Cor. 4, Prop. I.); that is, as the squares of the same arcs applied to the diameters of the circles (by Lem. VII.); and therefore since those arcs are as arcs described in any equal times, and the diameters are as the radii, the forces will be as the squares of any arcs described in the same time applied to the radii of the circles. Q.E.D.

COR. 1. Therefore, since those arcs are as the velocities of the bodies

the centripetal forces are in a ratio compounded of the duplicate ratio of the velocities directly, and of the simple ratio of the radii inversely.

COR. 2. And since the periodic times are in a ratio compounded of the ratio of the radii directly, and the ratio of the velocities inversely, the centripetal forces, are in a ratio compounded of the ratio of the radii directly, and the duplicate ratio of the periodic times inversely.

COR. 3. Whence if the periodic times are equal, and the velocities therefore as the radii, the centripetal forces will be also as the radii; and the contrary.

COR. 4. If the periodic times and the velocities are both in the subduplicate ratio of the radii, the centripetal forces will be equal among themselves; and the contrary.

COR. 5. If the periodic times are as the radii, and therefore the velocities equal, the centripetal forces will be reciprocally as the radii; and the contrary.

COR. 6. If the periodic times are in the sesquiplicate ratio of the radii, and therefore the velocities reciprocally in the subduplicate ratio of the radii, the centripetal forces will be in the duplicate ratio of the radii inversely; and the contrary.

COR. 7. And universally, if the periodic time is as any power R^n of the radius R, and therefore the velocity reciprocally as the power R^{n-1} of the radius, the centripetal force will be reciprocally as the power R^{2n-1} of the radius; and the contrary.

COR. 8. The same things all hold concerning the times, the velocities, and forces by which bodies describe the similar parts of any similar figures that have their centres in a similar position with those figures; as appears by applying the demonstration of the preceding cases to those. And the application is easy, by only substituting the equable description of areas in the place of equable motion, and using the distances of the bodies from the centres instead of the radii.

COR. 9. From the same demonstration it likewise follows, that the arc which a body, uniformly revolving in a circle by means of a given centripetal force, describes in any time, is a mean proportional between the diameter of the circle, and the space which the same body falling by the same given force would descend through in the same given time.

SCHOLIUM.

The case of the 6th Corollary obtains in the celestial bodies (as Sir Christopher Wren, Dr. Hooke, and Dr. Halley have severally observed); and therefore in what follows, I intend to treat more at large of those things which relate to centripetal force decreasing in a duplicate ratio of the distances from the centres.

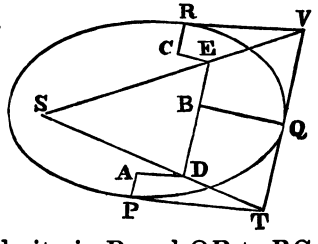
Moreover, by means of the preceding Proposition and its Corollaries, we

may discover the proportion of a centripetal force to any other known force, such as that of gravity. For if a body by means of its gravity revolves in a circle concentric to the earth, this gravity is the centripetal force of that body. But from the descent of heavy bodies, the time of one entire revolution, as well as the arc described in any given time, is given (by Cor. 9 of this Prop.). And by such propositions, Mr. Huygens, in his excellent book *De Horologio Oscillatorio*, has compared the force of gravity with the centrifugal forces of revolving bodies.

The preceding Proposition may be likewise demonstrated after this manner. In any circle suppose a polygon to be inscribed of any number of sides. And if a body, moved with a given velocity along the sides of the polygon, is reflected from the circle at the several angular points, the force, with which at every reflection it strikes the circle, will be as its velocity: and therefore the sum of the forces, in a given time, will be as that velocity and the number of reflections conjunctly; that is (if the species of the polygon be given), as the length described in that given time, and increased or diminished in the ratio of the same length to the radius of the circle; that is, as the square of that length applied to the radius; and therefore the polygon, by having its sides diminished *in infinitum*, coincides with the circle, as the square of the arc described in a given time applied to the radius. This is the centrifugal force, with which the body impels the circle; and to which the contrary force, wherewith the circle continually repels the body towards the centre, is equal.

PROPOSITION V. PROBLEM I.

There being given, in any places, the velocity with which a body describes a given figure, by means of forces directed to some common centre: to find that centre.



Let the three right lines PT, TQV, VR touch the figure described in as many points, P, Q, R, and meet in T and V. On the tangents erect the perpendiculars PA, QB, RC, reciprocally proportional to the velocities of the body in the points P, Q, R, from which the perpendiculars were raised; that is, so that PA may be to QB as the velocity in Q to the velocity in P, and QB to RC as the velocity in R to the velocity in Q. Through the ends A, B, C, of the perpendiculars draw AD, DBE, EC, at right angles, meeting in D and E: and the right lines TD, VE produced, will meet in S, the centre required.

For the perpendiculars let fall from the centre S on the tangents PT, QT, are reciprocally as the velocities of the bodies in the points P and Q

(by Cor. 1, Prop. I.), and therefore, by construction, as the perpendiculars AP, BQ directly; that is, as the perpendiculars let fall from the point D on the tangents. Whence it is easy to infer that the points S, D, T, are in one right line. And by the like argument the points S, E, V are also in one right line; and therefore the centre S is in the point where the right lines TD, VE meet. Q.E.D.

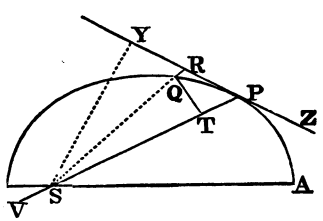
PROPOSITION VI. THEOREM V.

In a space void of resistance, if a body revolves in any orbit about an immovable centre, and in the least time describes any arc just then nascent; and the versed sine of that arc is supposed to be drawn bisecting the chord, and produced passing through the centre of force: the centripetal force in the middle of the arc will be as the versed sine directly and the square of the time inversely.

For the versed sine in a given time is as the force (by Cor. 4, Prop. 1); and augmenting the time in any ratio, because the arc will be augmented in the same ratio, the versed sine will be augmented in the duplicate of that ratio (by Cor. 2 and 3, Lem. XI.), and therefore is as the force and the square of the time. Subduct on both sides the duplicate ratio of the time, and the force will be as the versed sine directly, and the square of the time inversely. Q.E.D.

And the same thing may also be easily demonstrated by Corol. 4, Lem. X.

Cor. 1. If a body P revolving about the centre S describes a curve line APQ, which a right line ZPR touches in any point P; and from any other point Q of the curve, QR is drawn parallel to the distance SP, meeting the tangent in R; and QT is drawn perpendicular to the distance SP; the centripetal force will be reciprocally as the solid $\frac{SP^2 \times QT^2}{QR}$, if the solid be taken of that magnitude which it ultimately acquires when the points P and Q coincide. For QR is equal to the versed sine of double the arc QP, whose middle is P: and double the triangle SQP, or SP × QT is proportional to the time in which that double arc is described; and therefore may be used for the exponent of the time.



Cor. 2. By a like reasoning, the centripetal force is reciprocally as the solid $\frac{SY^2 \times QP^2}{QR}$; if SY is a perpendicular from the centre of force on PR the tangent of the orbit. For the rectangles SY × QP and SP × QT are equal.

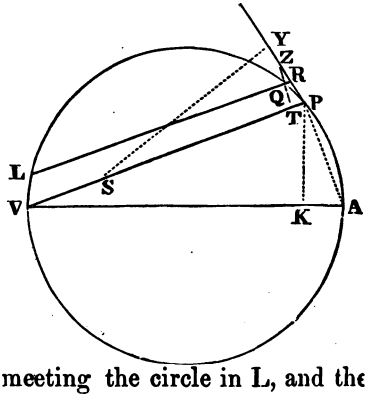
COR. 3. If the orbit is either a circle, or touches or cuts a circle concentrically, that is, contains with a circle the least angle of contact or section, having the same curvature and the same radius of curvature at the point P; and if PV be a chord of this circle, drawn from the body through the centre of force; the centripetal force will be reciprocally as the solid $SY^2 \times PV$. For PV is $\frac{QP^2}{QR}$.

COR. 4. The same things being supposed, the centripetal force is as the square of the velocity directly, and that chord inversely. For the velocity is reciprocally as the perpendicular SY, by Cor. 1. Prop. I.

COR. 5. Hence if any curvilinear figure APQ is given, and therein a point S is also given, to which a centripetal force is perpetually directed, that law of centripetal force may be found, by which the body P will be continually drawn back from a rectilinear course, and, being detained in the perimeter of that figure, will describe the same by a perpetual revolution. That is, we are to find, by computation, either the solid $\frac{SP^2 \times QT^2}{QR}$ or the solid $SY^2 \times PV$, reciprocally proportional to this force. Example of this we shall give in the following Problems.

PROPOSITION VII. PROBLEM II.

If a body revolves in the circumference of a circle; it is proposed to find the law of centripetal force directed to any given point.



Let VQPA be the circumference of the circle; S the given point to which as to a centre the force tends; P the body moving in the circumference; Q the next place into which it is to move; and PRZ the tangent of the circle at the preceding place. Through the point S draw the chord PV, and the diameter VA of the circle: join AP, and draw QT perpendicular to SP, which produced, may meet the tangent PR in Z; and lastly, through the point Q, draw LR parallel to SP, meeting the circle in L, and the tangent PZ in R. And, because of the similar triangles ZQR, ZTP, VPA, we shall have RP^2, that is, QRL to QT^2 as AV^2 to PV^2. And therefore $\frac{QRL \times PV^2}{AV^2}$ is equal to QT^2. Multiply those equals by $\frac{SP^2}{QR}$, and the points P and Q coinciding, for RL write PV; then we shall have $\frac{SP^2 \times PV^3}{AV^2} = \frac{SP^2 \times QT^2}{QR}$. And therefore (by Cor 1 and 5, Prop. VI.)

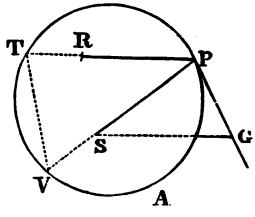
the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$; that is (because AV^2 is given), reciprocally as the square of the distance or altitude SP, and the cube of the chord PV conjunctly. Q.E.I.

The same otherwise.

On the tangent PR produced let fall the perpendicular SY; and (because of the similar triangles SYP, VPA), we shall have AV to PV as SP to SY, and therefore $\frac{SP \times PV}{AV} = SY$, and $\frac{SP^2 \times PV^3}{AV^2} = SY^2 \times PV$. And therefore (by Corol. 3 and 5, Prop. VI), the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$; that is (because AV is given), reciprocally as $SP^2 \times PV^3$. Q.E.I.

COR. 1. Hence if the given point S, to which the centripetal force always tends, is placed in the circumference of the circle, as at V, the centripetal force will be reciprocally as the quadrato-cube (or fifth power) of the altitude SP.

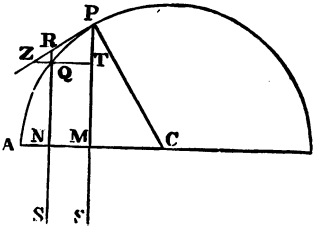
COR. 2. The force by which the body P in the circle APTV revolves about the centre of force S is to the force by which the same body P may revolve in the same circle, and in the same periodic time, about any other centre of force R, as $RP^2 \times SP$ to the cube of the right line SG, which from the first centre of force S is drawn parallel to the distance PR of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For by the construction of this Proposition, the former force is to the latter as $RP^2 \times PT^3$ to $SP^2 \times PV^3$; that is, as $SP \times RP^2$ to $\frac{SP^3 \times PV^3}{PT^3}$; or (because of the similar triangles PSG, TPV) to SG^3.



COR. 3. The force by which the body P in any orbit revolves about the centre of force S, is to the force by which the same body may revolve in the same orbit, and the same periodic time, about any other centre of force R, as the solid $SP \times RP^2$, contained under the distance of the body from the first centre of force S, and the square of its distance from the second centre of force R, to the cube of the right line SG, drawn from the first centre of the force S, parallel to the distance RP of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For the force in this orbit at any point P is the same as in a circle of the same curvature.

PROPOSITION VIII. PROBLEM III.

If a body moves in the semi-circumference PQA; *it is proposed to find the law of the centripetal force tending to a point* S, *so remote, that all the lines* PS, RS *drawn thereto, may be taken for parallels.*



From C, the centre of the semi-circle, let the semi-diameter CA be drawn, cutting the parallels at right angles in M and N, and join CP. Because of the similar triangles CPM, PZT, and RZQ, we shall have CP^2 to PM^2 as PR^2 to QT^2; and, from the nature of the circle, PR^2 is equal to the rectangle $QR \times \overline{RN + QN}$, or, the points P, Q coinciding, to the rectangle $QR \times 2PM$. Therefore CP^2 is to PM^2 as $QR \times 2PM$ to QT^2; and $\frac{QT^2}{QR} = \frac{2PM^3}{CP^2}$, and $\frac{QT^2 \times SP^2}{QR} = \frac{2PM^3 \times SP^2}{CP^2}$. And therefore (by Corol. 1 and 5, Prop. VI.), the centripetal force is reciprocally as $\frac{2PM^3 \times SP^2}{CP^2}$; that is (neglecting the given ratio $\frac{2SP^2}{CP^2}$), reciprocally as PM^3. Q.E.I.

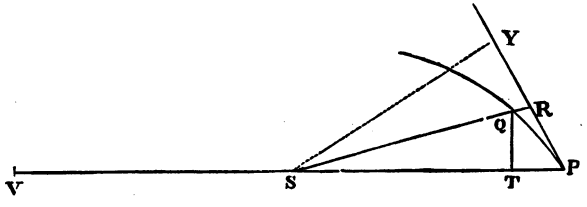
And the same thing is likewise easily inferred from the preceding Proposition.

SCHOLIUM.

And by a like reasoning, a body will be moved in an ellipsis, or even in an hyperbola, or parabola, by a centripetal force which is reciprocally as the cube of the ordinate directed to an infinitely remote centre of force.

PROPOSITION IX. PROBLEM IV.

If a body revolves in a spiral PQS, *cutting all the radii* SP, SQ, *&c., in a given angle; it is proposed to find the law of the centripetal force tending to the centre of that spiral.*



Suppose the indefinitely small angle PSQ to be given; because, then, all the angles are given, the figure SPRQT will be given in specie. Therefore the ratio $\frac{QT}{QR}$ is also given, and $\frac{QT^2}{QR}$ is as QT, that is (because the figure is given in specie), as SP. But if the angle PSQ is any way changed, the right line QR, subtending the angle of contact QPR

(by Lemma XI) will be changed in the duplicate ratio of PR or QT. Therefore the ratio $\frac{QT^2}{QR}$ remains the same as before, that is, as SP. And $\frac{QT^2 \times SP^2}{QR}$ is as SP^3, and therefore (by Corol. 1 and 5, Prop. VI) the centripetal force is reciprocally as the cube of the distance SP. Q.E.I.

The same otherwise.

The perpendicular SY let fall upon the tangent, and the chord PV of the circle concentrically cutting the spiral, are in given ratios to the height SP; and therefore SP^3 is as $SY^2 \times PV$, that is (by Corol. 3 and 5, Prop. VI) reciprocally as the centripetal force.

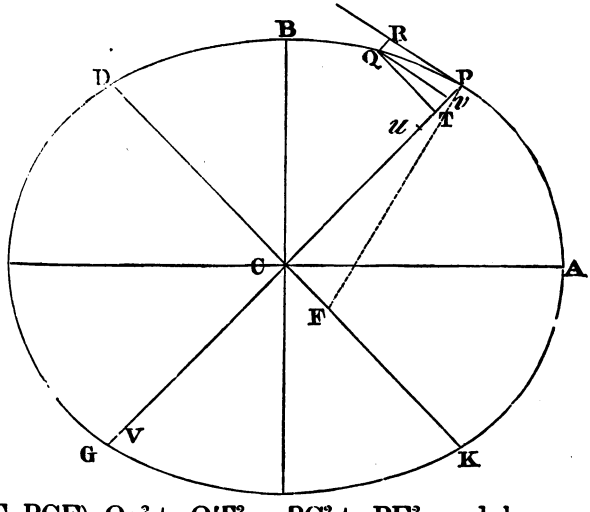
LEMMA XII.

All parallelograms circumscribed about any conjugate diameters of a given ellipsis or hyperbola are equal among themselves.

This is demonstrated by the writers on the conic sections.

PROPOSITION X. PROBLEM V.

If a body revolves in an ellipsis; it is proposed to find the law of the centripetal force tending to the centre of the ellipsis.



Suppose CA, CB to be semi-axes of the ellipsis; GP, DK, conjugate diameters; PF, QT perpendiculars to those diameters; Qv an ordinate to the diameter GP; and if the parallelogram $QvPR$ be completed, then (by the properties of the conic sections) the rectangle PvG will be to Qv^2 as PC^2 to CD^2; and (because of the similar triangles QvT, PCF), Qv^2 to QT^2 as PC^2 to PF^2; and, by composition, the ratio of PvG to QT^2 is compounded of the ratio of PC^2 to CD^2, and of the ratio of PC^2 to PF^2, that is, vG to $\frac{QT^2}{Pv}$ as PC^2 to $\frac{CD^2 \times PF^2}{PC^2}$. Put QR for Pv, and (by Lem. XII) BC × CA for CD × PF; also (the points P and Q coinciding) 2PC for vG; and multiply-

ing the extremes and means together, we shall have $\frac{QT^2 \times PC^2}{QR}$ equal to $\frac{2BC^2 \times CA^2}{PC}$. Therefore (by Cor. 5, Prop. VI), the centripetal force is reciprocally as $\frac{2BC^2 \times CA^2}{PC}$; that is (because $2BC^2 \times CA^2$ is given), reciprocally as $\frac{1}{PC}$; that is, directly as the distance PC. QEI.

The same otherwise.

In the right line PG on the other side of the point T, take the point u so that Tu may be equal to Tv; then take uV, such as shall be to vG as DC^2 to PC^2. And because Qv^2 is to PvG as DC^2 to PC^2 (by the conic sections), we shall have $Qv^2 = Pv \times uV$. Add the rectangle uPv to both sides, and the square of the chord of the arc PQ will be equal to the rectangle VPv; and therefore a circle which touches the conic section in P, and passes through the point Q, will pass also through the point V. Now let the points P and Q meet, and the ratio of uV to vG, which is the same with the ratio of DC^2 to PC^2, will become the ratio of PV to PG, or PV to 2PC; and therefore PV will be equal to $\frac{2DC^2}{PC}$. And therefore the force by which the body P revolves in the ellipsis will be reciprocally as $\frac{2DC^2}{PC} \times PF^2$ (by Cor. 3, Prop. VI); that is (because $2DC^2 \times PF^2$ is given) directly as PC. Q.E.I.

COR. 1. And therefore the force is as the distance of the body from the centre of the ellipsis; and, *vice versa*, if the force is as the distance, the body will move in an ellipsis whose centre coincides with the centre of force, or perhaps in a circle into which the ellipsis may degenerate.

COR. 2. And the periodic times of the revolutions made in all ellipses whatsoever about the same centre will be equal. For those times in similar ellipses will be equal (by Corol. 3 and S, Prop. IV); but in ellipses that have their greater axis common, they are one to another as the whole areas of the ellipses directly, and the parts of the areas described in the same time inversely; that is, as the lesser axes directly, and the velocities of the bodies in their principal vertices inversely; that is, as those lesser axes directly, and the ordinates to the same point of the common axes inversely; and therefore (because of the equality of the direct and inverse ratios) in the ratio of equality.

SCHOLIUM.

If the ellipsis, by having its centre removed to an infinite distance, degenerates into a parabola, the body will move in this parabola; and the

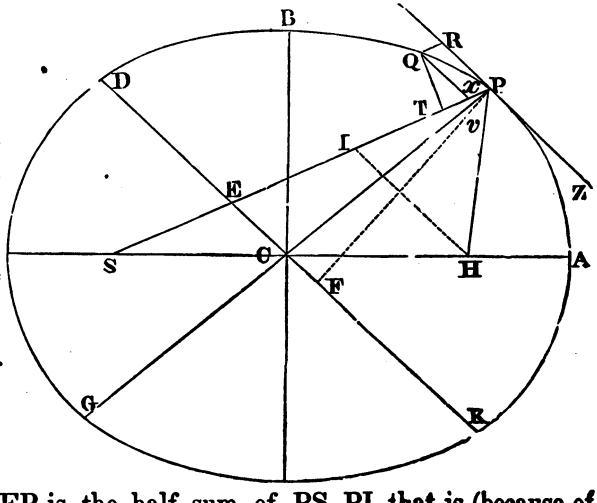
force, now tending to a centre infinitely remote, will become equable. Which is *Galileo's* theorem. And if the parabolic section of the cone (by changing the inclination of the cutting plane to the cone) degenerates into an hyperbola, the body will move in the perimeter of this hyperbola, having its centripetal force changed into a centrifugal force. And in like manner as in the circle, or in the ellipsis, if the forces are directed to the centre of the figure placed in the abscissa, those forces by increasing or diminishing the ordinates in any given ratio, or even by changing the angle of the inclination of the ordinates to the abscissa, are always augmented or diminished in the ratio of the distances from the centre; provided the periodic times remain equal; so also in all figures whatsoever, if the ordinates are augmented or diminished in any given ratio, or their inclination is any way changed, the periodic time remaining the same, the forces directed to any centre placed in the abscissa are in the several ordinates augmented or diminished in the ratio of the distances from the centre

SECTION III.

Of the motion of bodies in eccentric conic sections.

PROPOSITION XI. PROBLEM VI.

If a body revolves in an ellipsis; it is required to find the law of the centripetal force tending to the focus of the ellipsis.



Let S be the focus of the ellipsis. Draw SP cutting the diameter DK of the ellipsis in E, and the ordinate Q*v* in *x*; and complete the parallelogram Q*x*PR. It is evident that EP is equal to the greater semi-axis AC: for drawing HI from the other focus H of the ellipsis parallel to EC, because CS, CH are equal, ES, EI will be also equal; so that EP is the half sum of PS, PI, that is (because of the parallels HI, PR, and the equal angles IPR, HPZ), of PS, PH, which taken together are equal to the whole axis 2AC. Draw QT perpendicular to SP, and putting L for the princi al latus rectum of the ellipsis (or for

$\frac{2BC^2}{AC}$), we shall have L × QR to L × Pv as QR to Pv, that is, as PE or AC to PC; and L × Pv to GvP as L to Gv; and GvP to Qv^2 as PC^2 to CD^2; and by (Corol. 2, Lem. VII) the points Q and P coinciding, Qv^2 is to Qx^2 in the ratio of equality; and Qx^2 or Qv^2 is to QT^2 as EP^2 to PF^2, that is, as CA^2 to PF^2, or (by Lem. XII) as CD^2 to CB^2. And compounding all those ratios together, we shall have L × QR to QT^2 as AC × L × PC^2 × CD^2, or 2CB^2 × PC^2 × CD^2 to PC × Gv × CD^2 × CB^2, or as 2PC to Gv. But the points Q and P coinciding, 2PC and Gr are equal. And therefore the quantities L × QR and QT^2, proportional to these, will be also equal. Let those equals be drawn into $\frac{SP^2}{QR}$, and L × SP^2 will become equal to $\frac{SP^2 \times QT^2}{QR}$. And therefore (by Corol. 1 and 5, Prop. VI) the centripetal force is reciprocally as L × SP^2, that is, reciprocally in the duplicate ratio of the distance SP. Q.E.I.

The same otherwise.

Since the force tending to the centre of the ellipsis, by which the body P may revolve in that ellipsis, is (by Corol. 1, Prop. X.) as the distance CP of the body from the centre C of the ellipsis; let CE be drawn parallel to the tangent PR of the ellipsis; and the force by which the same body P may revolve about any other point S of the ellipsis, if CE and PS intersect in E, will be as $\frac{PE^3}{SP^2}$ (by Cor. 3, Prop. VII.); that is, if the point S is the focus of the ellipsis, and therefore PE be given as SP^2 reciprocally. Q.E.I.

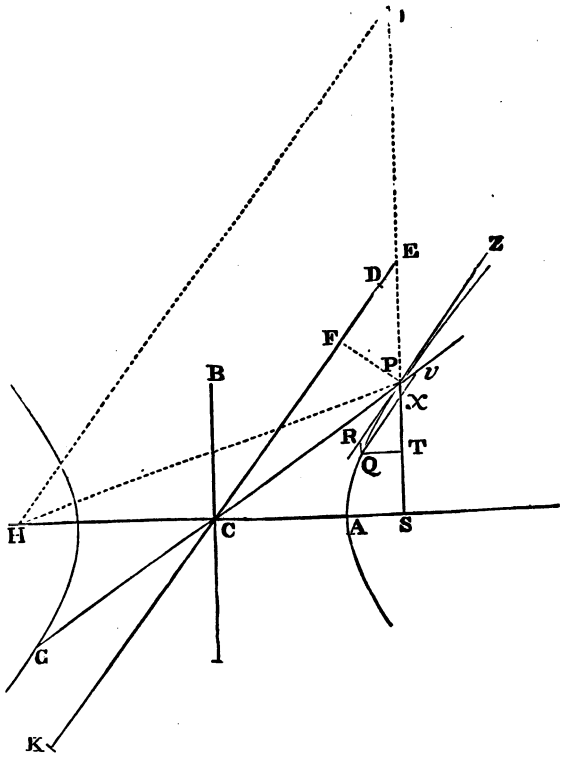
With the same brevity with which we reduced the fifth Problem to the parabola, and hyperbola, we might do the like here: but because of the dignity of the Problem and its use in what follows, I shall confirm the other cases by particular demonstrations.

PROPOSITION XII. PROBLEM VII.

Suppose a body to move in an hyperbola; it is required to find the law of the centripetal force tending to the focus of that figure.

Let CA, CB be the semi-axes of the hyperbola; PG, KD other conjugate diameters; PF a perpendicular to the diameter KD; and Qv an ordinate to the diameter GP. Draw SP cutting the diameter DK in E, and the ordinate Qv in x, and complete the parallelogram QRPx. It is evident that EP is equal to the semi-transverse axis AC; for drawing HI, from the other focus H of the hyperbola, parallel to EC, because CS, CH are equal, ES EI will be also equal; so that EP is the half difference

of PS, PI; that is (because of the parallels IH, PR, and the equal angles IPR, HPZ), of PS, PH, the difference of which is equal to the whole axis 2AC. Draw QT perpendicular to SP; and putting L for the principal latus rectum of the hyperbola (that is, for $\frac{2BC^2}{AC}$), we shall have L × QR to L × Pv as QR to Pv, or Px to Pv, that is (because of the similar triangles Pxv, PEC), as PE to PC, or AC to PC. And L × Pv will be to Gv × Pv as L to Gv; and (by the properties of the conic sections) the rectangle GvP is to Qv^2 as PC² to CD²; and by (Cor. 2, Lem. VII.), Qv^2 to Qx^2, the points Q and P coinciding, becomes a ratio of equality; and Qx^2 or Qv^2 is to QT² as EP² to PF², that is, as CA² to PF², or (by Lem. XII.) as CD² to CB²: and, compounding all those ratios together, we shall have L × QR to QT² as AC × L × PC² × CD², or 2CB² × PC² × CD² to PC × Gv × CD² × CB², or as 2PC to Gv. But the points P and Q coinciding, 2PC and Gv are equal. And therefore the quantities L × QR and QT², proportional to them, will be also equal. Let those equals be drawn into $\frac{SP^2}{QR}$, and we shall have L × SP² equal to $\frac{SP^2 \times QT^2}{QR}$. And therefore (by Cor. 1 and 5, Prop. VI.) the centripetal force is reciprocally as L × SP², that is, reciprocally in the duplicate ratio of the distance SP. Q.E.I.



The same otherwise.

Find out the force tending from the centre C of the hyperbola. This will be proportional to the distance CP. But from thence (by Cor. 3, Prop. VII.) the force tending to the focus S will be as $\frac{PE^3}{SP^2}$, that is, because PE is given reciprocally as SP². Q.E.I.

And the same way may it be demonstrated, that the body having its centripetal changed into a centrifugal force, will move in the conjugate hyperbola.

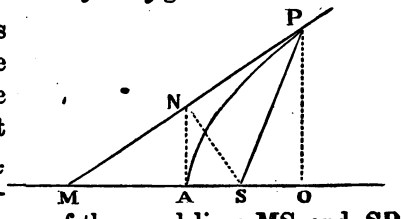
LEMMA XIII.

The latus rectum of a parabola belonging to any vertex is quadruple the distance of that vertex from the focus of the figure.

This is demonstrated by the writers on the conic sections.

LEMMA XIV.

The perpendicular, let fall from the focus of a parabola on its tangent, is a mean proportional between the distances of the focus from the point of contact, and from the principal vertex of the figure.



For, let AP be the parabola, S its focus, A its principal vertex, P the point of contact, PO an ordinate to the principal diameter, PM the tangent meeting the principal diameter in M, and SN the perpendicular from the focus on the tangent: join AN, and because of the equal lines MS and SP, MN and NP, MA and AO, the right lines AN, OP, will be parallel; and thence the triangle SAN will be right-angled at A, and similar to the equal triangles SNM, SNP; therefore PS is to SN as SN to SA. Q.E.D.

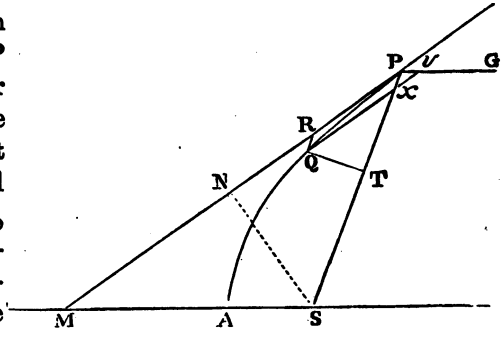
COR. 1. PS^2 is to SN^2 as PS to SA.

COR. 2. And because SA is given, SN^2 will be as PS.

COR. 3. And the concourse of any tangent PM, with the right line SN, drawn from the focus perpendicular on the tangent, falls in the right line AN that touches the parabola in the principal vertex.

PROPOSITION XIII. PROBLEM VIII.

If a body moves in the perimeter of a parabola; it is required to find the law of the centripetal force tending to the focus of that figure.



Retaining the construction of the preceding Lemma, let P be the body in the perimeter of the parabola; and from the place Q, into which it is next to succeed, draw QR parallel and QT perpendicular to SP, as also Q*v* parallel to the tangent, and meeting the diameter PG in *v*, and the distance

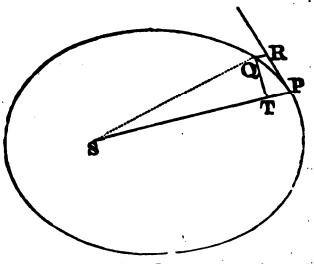
SP in x. Now, because of the similar triangles Pxv, SPM, and of the equal sides SP, SM of the one, the sides Px or QR and Pv of the other will be also equal. But (by the conic sections) the square of the ordinate Qv is equal to the rectangle under the latus rectum and the segment Pv of the diameter; that is (by Lem. XIII.), to the rectangle $4PS \times Pv$, or $4PS \times QR$; and the points P and Q coinciding, the ratio of Qv to Qx (by Cor. 2, Lem. VII.) becomes a ratio of equality. And therefore Qx^2, in this case, becomes equal to the rectangle $4PS \times QR$. But (because of the similar triangles QxT, SPN), Qx^2 is to QT^2 as PS^2 to SN^2, that is (by Cor. 1, Lem. XIV.), as PS to SA; that is, as $4PS \times QR$ to $4SA \times QR$, and therefore (by Prop. IX. Lib. V., Elem.) QT^2 and $4SA \times QR$ are equal. Multiply these equals by $\frac{SP^2}{QR}$, and $\frac{SP^2 \times QT^2}{QR}$ will become equal to $SP^2 \times 4SA$: and therefore (by Cor. 1 and 5, Prop. VI.), the centripetal force is reciprocally as $SP^2 \times 4SA$; that is, because 4SA is given, reciprocally in the duplicate ratio of the distance SP. Q.E.I.

COR. 1. From the three last Propositions it follows, that if any body P goes from the place P with any velocity in the direction of any right line PR, and at the same time is urged by the action of a centripetal force that is reciprocally proportional to the square of the distance of the places from the centre, the body will move in one of the conic sections, having its focus in the centre of force; and the contrary. For the focus, the point of contact, and the position of the tangent, being given, a conic section may be described, which at that point shall have a given curvature. But the curvature is given from the centripetal force and velocity of the body being given; and two orbits, mutually touching one the other, cannot be described by the same centripetal force and the same velocity.

COR. 2. If the velocity with which the body goes from its place P is such, that in any infinitely small moment of time the lineola PR may be thereby described; and the centripetal force such as in the same time to move the same body through the space QR; the body will move in one of the conic sections, whose principal latus rectum is the quantity $\frac{QT^2}{QR}$ in its ultimate state, when the lineolæ PR, QR are diminished *in infinitum*. In these Corollaries I consider the circle as an ellipsis; and I except the case where the body descends to the centre in a right line.

PROPOSITION XIV. THEOREM VI.

If several bodies revolve about one common centre, and the centripetal force is reciprocally in the duplicate ratio of the distance of places from the centre; I say, that the principal latera recta of their orbits are in the duplicate ratio of the areas, which the bodies by radii drawn to the centre describe in the same time.



For (by Cor 2, Prop. XIII) the latus rectum L is equal to the quantity $\frac{QT^2}{QR}$ in its ultimate state when the points P and Q coincide. But the lineola QR in a given time is as the generating centripetal force; that is (by supposition), reciprocally as SP^2. And therefore $\frac{QT^2}{QR}$ is as $QT^2 \times SP^2$; that is, the latus rectum L is in the duplicate ratio of the area $QT \times SP$. Q.E.D.

COR. Hence the whole area of the ellipsis, and the rectangle under the axes, which is proportional to it, is in the ratio compounded of the subduplicate ratio of the latus rectum, and the ratio of the periodic time. For the whole area is as the area $QT \times SP$, described in a given time, multiplied by the periodic time.

PROPOSITION XV. THEOREM VII.

The same things being supposed, I say, that the periodic times in ellipses are in the sesquiplicate ratio of their greater axes.

For the lesser axis is a mean proportional between the greater axis and the latus rectum; and, therefore, the rectangle under the axes is in the ratio compounded of the subduplicate ratio of the latus rectum and the sesquiplicate ratio of the greater axis. But this rectangle (by Cor. 3. Prop. XIV) is in a ratio compounded of the subduplicate ratio of the latus rectum, and the ratio of the periodic time. Subduct from both sides the subduplicate ratio of the latus rectum, and there will remain the sesquiplicate ratio of the greater axis, equal to the ratio of the periodic time. Q.E.D.

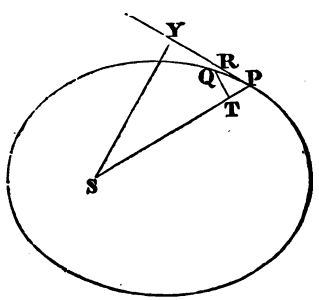
COR. Therefore the periodic times in ellipses are the same as in circles whose diameters are equal to the greater axes of the ellipses.

PROPOSITION XVI. THEOREM VIII.

The same things being supposed, and right lines being drawn to the bodies that shall touch the orbits, and perpendiculars being let fall on those tangents from the common focus; I say, that the velocities of the bodies are in a ratio compounded of the ratio of the perpendiculars inversely, and the subduplicate ratio of the principal latera recta directly.

From the focus S draw SY perpendicular to the tangent PR, and the velocity of the body P will be reciprocally in the subduplicate ratio of the quantity $\frac{SY^2}{L}$. For that velocity is as the infinitely small arc PQ de-

scribed in a given moment of time, that is (by Lem. VII), as the tangent PR; that is (because of the proportionals PR to QT, and SP to SY), as $\frac{SP \times QT}{SY}$; or as SY reciprocally, and SP × QT directly; but SP × QT is as the area described in the given time, that is (by Prop. XIV), in the subduplicate ratio of the latus rectum. Q.E.D.



Cor. 1. The principal latera recta are in a ratio compounded of the duplicate ratio of the perpendiculars and the duplicate ratio of the velocities.

Cor. 2. The velocities of bodies, in their greatest and least distances from the common focus, are in the ratio compounded of the ratio of the distances inversely, and the subduplicate ratio of the principal latera recta directly. For those perpendiculars are now the distances.

Cor. 3. And therefore the velocity in a conic section, at its greatest or least distance from the focus, is to the velocity in a circle, at the same distance from the centre, in the subduplicate ratio of the principal latus rectum to the double of that distance.

Cor. 4. The velocities of the bodies revolving in ellipses, at their mean distances from the common focus, are the same as those of bodies revolving in circles, at the same distances; that is (by Cor. 6, Prop. IV), reciprocally in the subduplicate ratio of the distances. For the perpendiculars are now the lesser semi-axes, and these are as mean proportionals between the distances and the latera recta. Let this ratio inversely be compounded with the subduplicate ratio of the latera recta directly, and we shall have the subduplicate ratio of the distance inversely.

Cor. 5. In the same figure, or even in different figures, whose principal latera recta are equal, the velocity of a body is reciprocally as the perpendicular let fall from the focus on the tangent.

Cor. 6. In a parabola, the velocity is reciprocally in the subduplicate ratio of the distance of the body from the focus of the figure; it is more variable in the ellipsis, and less in the hyperbola, than according to this ratio. For (by Cor. 2, Lem. XIV) the perpendicular let fall from the focus on the tangent of a parabola is in the subduplicate ratio of the distance. In the hyperbola the perpendicular is less variable; in the ellipsis more.

Cor. 7. In a parabola, the velocity of a body at any distance from the focus is to the velocity of a body revolving in a circle, at the same distance from the centre, in the subduplicate ratio of the number 2 to 1; in the ellipsis it is less, and in the hyperbola greater, than according to this ratio. For (by Cor. 2 of this Prop.) the velocity at the vertex of a parabola is in

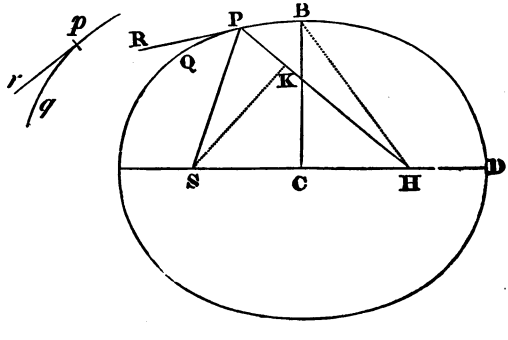
this ratio, and (by Cor. 6 of this Prop. and Prop. IV) the same proportion holds in all distances. And hence, also, in a parabola, the velocity is everywhere equal to the velocity of a body revolving in a circle at half the distance; in the ellipsis it is less, and in the hyperbola greater.

COR. 8. The velocity of a body revolving in any conic section is to the velocity of a body revolving in a circle, at the distance of half the principal latus rectum of the section, as that distance to the perpendicular let fall from the focus on the tangent of the section. This appears from Cor. 5.

COR. 9. Wherefore since (by Cor. 6, Prop. IV), the velocity of a body revolving in this circle is to the velocity of another body revolving in any other circle reciprocally in the subduplicate ratio of the distances; therefore, *ex æquo*, the velocity of a body revolving in a conic section will be to the velocity of a body revolving in a circle at the same distance as a mean proportional between that common distance, and half the principal latus rectum of the section, to the perpendicular let fall from the common focus upon the tangent of the section.

PROPOSITION XVII. PROBLEM IX.

Supposing the centripetal force to be reciprocally proportional to the squares of the distances of places from the centre, and that the absolute quantity of that force is known; it is required to determine the line which a body will describe that is let go from a given place with a given velocity in the direction of a given right line.



Let the centripetal force tending to the point S be such as will make the body *p* revolve in any given orbit *pq*; and suppose the velocity of this body in the place *p* is known. Then from the place P suppose the body P to be let go with a given velocity in the direction of the line PR; but by virtue of a centripetal force to be immediately turned aside from that right line into the conic section PQ. This, the right line PR will therefore touch in P. Suppose likewise that the right line *pr* touches the orbit *pq* in *p*; and if from S you suppose perpendiculars let fall on those tangents, the principal latus rectum of the conic section (by Cor. 1, Prop. XVI) will be to the principal latus rectum of that orbit in a ratio compounded of the duplicate ratio of the perpendiculars, and the duplicate ratio of the velocities; and is therefore given. Let this latus rectum be L; the focus S of the conic

section is also given. Let the angle RPH be the complement of the angle RPS to two right; and the line PH, in which the other focus H is placed, is given by position. Let fall SK perpendicular on PH, and erect the conjugate semi-axis BC; this done, we shall have $SP^2 - 2KPH + PH^2 = SH^2 = 4CH^2 = 4BH^2 - 4BC^2 = \overline{SP + PH}^2 - L \times \overline{SP + PH} = SP^2 + 2SPH + PH^2 - L \times \overline{SP + PH}$. Add on both sides $2KPH - SP^2 - PH^2 + L \times \overline{SP + PH}$, and we shall have $L \times \overline{SP + PH} = 2SPH + 2KPH$, or SP + PH to PH, as 2SP + 2KP to L. Whence PH is given both in length and position. That is, if the velocity of the body in P is such that the latus rectum L is less than 2SP + 2KP, PH will lie on the same side of the tangent PR with the line SP; and therefore the figure will be an ellipsis, which from the given foci S, H, and the principal axis SP + PH, is given also. But if the velocity of the body is so great, that the latus rectum L becomes equal to 2SP + 2KP, the length PH will be infinite; and therefore, the figure will be a parabola, which has its axis SH parallel to the line PK, and is thence given. But if the body goes from its place P with a yet greater velocity, the length PH is to be taken on the other side the tangent; and so the tangent passing between the foci, the figure will be an hyperbola having its principal axis equal to the difference of the lines SP and PH, and thence is given. For if the body, in these cases, revolves in a conic section so found, it is demonstrated in Prop. XI, XII, and XIII, that the centripetal force will be reciprocally as the square of the distance of the body from the centre of force S; and therefore we have rightly determined the line PQ, which a body let go from a given place P with a given velocity, and in the direction of the right line PR given by position, would describe with such a force. Q.E.F.

Cor. 1. Hence in every conic section, from the principal vertex D, the latus rectum L, and the focus S given, the other focus H is given, by taking DH to DS as the latus rectum to the difference between the latus rectum and 4DS. For the proportion, SP + PH to PH as 2SP + 2KP to L, becomes, in the case of this Corollary, DS + DH to DH as 4DS to L, and by division DS to DH as 4DS — L to L.

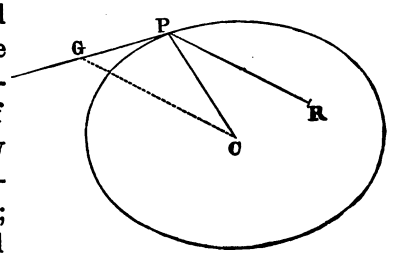
Cor. 2. Whence if the velocity of a body in the principal vertex D is given, the orbit may be readily found; to wit, by taking its latus rectum to twice the distance DS, in the duplicate ratio of this given velocity to the velocity of a body revolving in a circle at the distance DS (by Cor. 3, Prop. XVI.), and then taking DH to DS as the latus rectum to the difference between the latus rectum and 4DS.

Cor. 3. Hence also if a body move in any conic section, and is forced out of its orbit by any impulse, you may discover the orbit in which it will afterwards pursue its course. For by compounding the proper motion of

the body with that motion, which the impulse alone would generate, you will have the motion with which the body will go off from a given place of impulse in the direction of a right line given in position.

COR. 4. And if that body is continually disturbed by the action of some foreign force, we may nearly know its course, by collecting the changes which that force introduces in some points, and estimating the continual changes it will undergo in the intermediate places, from the analogy that appears in the progress of the series.

SCHOLIUM.



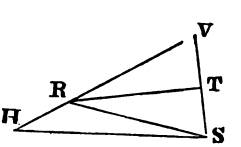
If a body P, by means of a centripetal force tending to any given point R, move in the perimeter of any given conic section whose centre is C; and the law of the centripetal force is required: draw CG parallel to the radius RP, and meeting the tangent PG of the orbit in G; and the force required (by Cor. 1, and Schol. Prop. X., and Cor. 3, Prop. VII.) will be as $\frac{CG^3}{RP^2}$.

SECTION IV.

Of the finding of elliptic, parabolic, and hyperbolic orbits, from the focus given.

LEMMA XV.

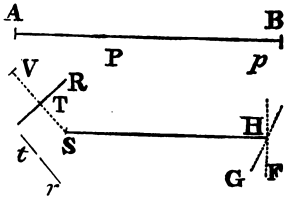
If from the two foci S, H, *of any ellipsis or hyberbola, we draw to any third point* V *the right lines* SV, HV, *whereof one* HV *is equal to the principal axis of the figure, that is, to the axis in which the foci are situated, the other,* SV, *is bisected in* T *by the perpendicular* TR *let fall upon it; that perpendicular* TR *will somewhere touch the conic section: and,* vice versa, *if it does touch it,* HV *will be equal to the principal axis of the figure.*



For, let the perpendicular TR cut the right line HV, produced, if need be, in R; and join SR. Because TS, TV are equal, therefore the right lines SR, VR, as well as the angles TRS, TRV, will be also equal. Whence the point R will be in the conic section, and the perpendicular TR will touch the same; and the contrary. Q.E.D.

PROPOSITION XVIII. PROBLEM X.

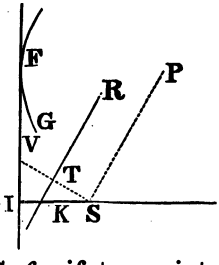
From a focus and the principal axes given, to describe elliptic and hyperbolic trajectories, which shall pass through given points, and touch right lines given by position.



Let S be the common focus of the figures; AB the length of the principal axis of any trajectory; P a point through which the trajectory should pass; and TR a right line which it should touch. About the centre P, with the interval AB — SP, if the orbit is an ellipsis, or AB + SP, if the orbit is an hyperbola, describe the circle HG. On the tangent TR let fall the perpendicular ST, and produce the same to V, so that TV may be equal to ST; and about V as a centre with the interval AB describe the circle FH. In this manner, whether two points P, *p*, are given, or two tangents TR, *tr*, or a point P and a tangent TR, we are to describe two circles. Let H be their common intersection, and from the foci S, H, with the given axis describe the trajectory: I say, the thing is done. For (because PH + SP in the ellipsis, and PH — SP in the hyperbola, is equal to the axis) the described trajectory will pass through the point P, and (by the preceding Lemma) will touch the right line TR. And by the same argument it will either pass through the two points P, *p*, or touch the two right lines TR, *tr*. Q.E.F.

PROPOSITION XIX. PROBLEM XI.

About a given focus, to describe a parabolic trajectory, which shall pass through given points, and touch right lines given by position.



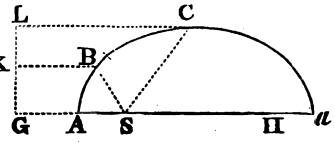
Let S be the focus, P a point, and TR a tangent of the trajectory to be described. About P as a centre, with the interval PS, describe the circle FG. From the focus let fall ST perpendicular on the tangent, and produce the same to V, so as TV may be equal to ST. After the same manner another circle *fg* is to be described, if another point *p* is given; or another point *v* is to be found, if another tangent *tr* is given; then draw the right line IF, which shall touch the two circles FG, *fg*, if two points P, *p* are given; or pass through the two points V, *v*, if two tangents TR, *tr*, are given: or touch the circle FG, and pass through the point V, if the point P and the tangent TR are given. On FI let fall the perpendicular SI, and bisect the same in K; and with the axis SK and principal vertex K describe a parabola: I say the thing is done. For this parabola (because SK is equal to IK, and SP to FP) will pass through the point P; and

(by Cor. 3, Lem. XIV) because ST is equal to TV, and STR a right angle, it will touch the right line TR. Q.E.F.

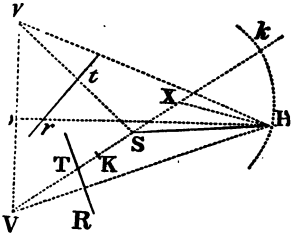
PROPOSITION XX. PROBLEM XII.

About a given focus to describe any trajectory given in specie which shall pass through given points, and touch right lines given by position.

CASE 1. About the focus S it is required to describe a trajectory ABC, passing through two points B, C. Because the trajectory is given in specie, the ratio of the principal axis to the distance of the foci will be given. In that ratio take KB to BS, and LC to CS. About the centres B, C, with the intervals BK, CL, describe two circles; and on the right line KL, that touches the same in K and L, let fall the perpendicular SG; which cut in A and *a*, so that GA may be to AS, and G*a* to *a*S, as KB to BS; and with the axis A*a*, and vertices A, *a*, describe a trajectory: I say the thing is done. For let H be the other focus of the described figure, and seeing GA is to AS as G*a* to *a*S, then by division we shall have G*a*—GA, or A*a* to *a*S—AS, or SH in the same ratio, and therefore in the ratio which the principal axis of the figure to be described has to the distance of its foci; and therefore the described figure is of the same species with the figure which was to be described. And since KB to BS, and LC to CS, are in the same ratio, this figure will pass through the points B, C, as is manifest from the conic sections.

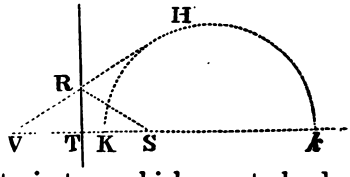


CASE 2. About the focus S it is required to describe a trajectory which shall somewhere touch two right lines TR, *tr*. From the focus on those tangents let fall the perpendiculars ST, S*t*, which produce to V, *v*, so that TV, *tv* may be equal to TS, *t*S. Bisect V*v* in O, and erect the indefinite perpendicular OH, and cut the right line VS infinitely produced in K and *k*, so that VK be to KS, and V*k* to *k*S, as the principal axis of the trajectory to be described is to the distance of its foci. On the diameter K*k* describe a circle cutting OH in H; and with the foci S, H, and principal axis equal to VH, describe a trajectory: I say, the thing is done. For bisecting K*k* in X, and joining HX, HS, HV, H*v*, because VK is to KS as V*k* to *k*S; and by composition, as VK + V*k* to KS + *k*S; and by division, as V*k* — VK to *k*S — KS, that is, as 2VX to 2KX, and 2KX to 2SX, and therefore as VX to HX and HX to SX, the triangles VXH, HXS will be similar; therefore VH will be to SH as VX to XH; and therefore as VK to KS. Wherefore VH, the principal axis of the described trajectory, has the same ratio to SH, the distance of the foci, as



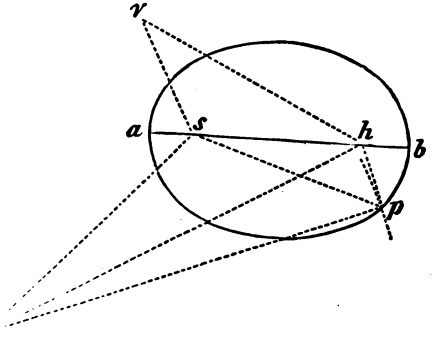
the principal axis of the trajectory which was to be described has to the distance of its foci; and is therefore of the same species. And seeing VH, *v*H are equal to the principal axis, and VS, *v*S are perpendicularly bisected by the right lines TR, *tr*, it is evident (by Lem. XV) that those right lines touch the described trajectory. Q.E.F.

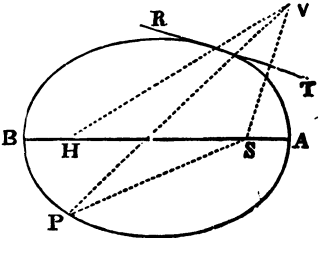
Case. 3. About the focus S it is required to describe a trajectory, which shall touch a right line TR in a given Point R. On the right line TR let fall the perpendicular S'T, which produce to V, so that TV may be equal to ST; join VR, and cut the right line VS indefinitely produced in K and *k*, so that VK may be to SK, and V*k* to S*k*, as the principal axis of the ellipsis to be described to the distance of its foci; and on the diameter K*k* describing a circle, cut the right line VR produced in H; then with the foci S, H, and principal axis equal to VH, describe a trajectory: I say, the thing is done. For VH is to SH as VK to SK,



and therefore as the principal axis of the trajectory which was to be described to the distance of its foci (as appears from what we have demonstrated in Case 2); and therefore the described trajectory is of the same species with that which was to be described; but that the right line TR, by which the angle VRS is bisected, touches the trajectory in the point R, is certain from the properties of the conic sections. Q.E.F.

Case 4. About the focus S it is required to describe a trajectory APB that shall touch a right line TR, and pass through any given point P without the tangent, and shall be similar to the figure *apb*, described with the principal axis *ab*, and foci *s*, *h*. On the tangent TR let fall the perpendicular ST, which produce to V, so that TV may be equal to ST; and making the angles *hsq*, *shq*, equal to the angles VSP, SVP, about *q* as a centre, and with an interval which shall be to *ab* as SP to VS, describe a circle cutting the figure *apb* in *p*: join *sp*, and draw SH such that it may be to *sh* as SP is to *sp*, and may make the angle PSH equal to the angle *psh*, and the angle VSH equal to the angle *psq*. Then with the foci S, H, and principal axis AB, equal to the distance VH, describe a conic section: I say, the thing is done; for if *sv* is drawn so that it shall be to



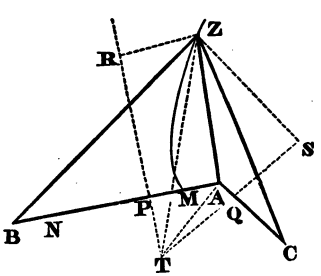


sp as *sh* is to *sq*, and shall make the angle *vsp* equal to the angle *hsq*, and the angle *vsh* equal to the angle *psq*, the triangles *svh*, *spq*, will be similar, and therefore *vh* will be to *pq* as *sh* is to *sq*; that is (because of the similar triangles VSP, *hsq*), as VS is to SP, or as *ab* to *pq*. Wherefore *vh* and *ab* are equal. But, because of the similar triangles VSH, *vsh*, VH is to SH as *vh* to *sh*; that is, the axis of the conic section now described is to the distance of its foci as the axis *ab* to the distance of the foci *sh*; and therefore the figure now described is similar to the figure *aph*. But, because the triangle PSH is similar to the triangle *psh*, this figure passes through the point P; and because VH is equal to its axis, and VS is perpendicularly bisected by the right line TR, the said figure touches the right line TR. Q.E.F.

LEMMA XVI.

From three given points to draw to a fourth point that is not given three right lines whose differences shall be either given, or none at all.

CASE 1. Let the given points be A, B, C, and Z the fourth point which we are to find; because of the given difference of the lines AZ, BZ, the locus of the point Z will be an hyperbola whose foci are A and B, and whose principal axis is the given difference. Let that axis be MN. Taking PM to MA as MN is to AB, erect PR perpendicular to AB, and let fall ZR perpendicular to PR; then from the nature of the hyperbola, ZR will be to AZ as MN is to AB. And by the like argument, the locus of the point Z will be another hyperbola, whose foci are A, C, and whose principal axis is the difference between AZ and CZ; and QS a perpendicular on AC may be drawn, to which (QS) if from any point Z of this hyperbola a perpendicular ZS is let fall (this ZS), shall be to AZ as the difference between AZ and CZ is to AC. Wherefore the ratios of ZR and ZS to AZ are given, and consequently the ratio of ZR to ZS one to the other; and therefore if the right lines RP, SQ, meet in T, and TZ and TA are drawn, the figure TRZS will be given in specie, and the right line TZ, in which the point Z is somewhere placed, will be given in position. There will be given also the right line TA, and the angle ATZ; and because the ratios of AZ and TZ to ZS are given, their ratio to each other is given also; and thence will be given likewise the triangle ATZ, whose vertex is the point Z. Q.E.I.



CASE 2. If two of the three lines, for example AZ and BZ, are equal, draw the right line TZ so as to bisect the right line AB; then find the triangle ATZ as above. Q.E.I.

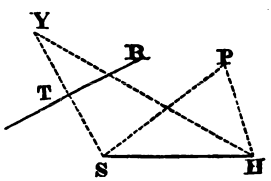
Case 3. If all the three are equal, the point Z will be placed in the centre of a circle that passes through the points A, B, C. Q.E.I.

This problematic Lemma is likewise solved in Apollonius's Book of Tactions restored by Vieta.

PROPOSITION XXI. PROBLEM XIII.

About a given focus to describe a trajectory that shall pass through given points and touch right lines given by position.

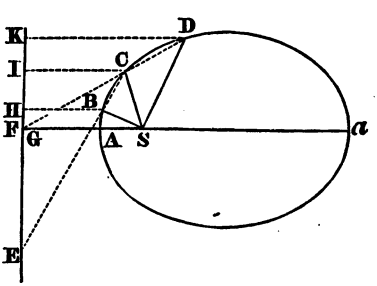
Let the focus S, the point P, and the tangent TR be given, and suppose that the other focus H is to be found. On the tangent let fall the perpendicular ST, which produce to Y, so that TY may be equal to ST, and YH will be equal to the principal axis. Join SP, HP, and SP will be the difference between HP and the principal axis. After this manner, if more tangents TR are given, or more points P, we shall always determine as many lines YH, or PH, drawn from the said points Y or P, to the focus H, which either shall be equal to the axes, or differ from the axes by given lengths SP; and therefore which shall either be equal among themselves, or shall have given differences; from whence (by the preceding Lemma), that other focus H is given. But having the foci and the length of the axis (which is either YH, or, if the trajectory be an ellipsis, PH + SP; or PH — SP, if it be an hyperbola), the trajectory is given. Q.E.I.



SCHOLIUM.

When the trajectory is an hyperbola, I do not comprehend its conjugate hyperbola under the name of this trajectory. For a body going on with a continued motion can never pass out of one hyperbola into its conjugate hyperbola.

The case when three points are given is more readily solved thus. Let B, C, D, be the given points. Join BC, CD, and produce them to E, F, so as EB may be to EC as SB to SC; and FC to FD as SC to SD. On EF drawn and produced let fall the perpendiculars SG, BH, and in GS produced indefinitely take GA to AS, and G*a* to *a*S, as HB is to BS; then A will be the vertex, and A*a* the principal axis of the trajectory; which, according as GA is greater than, equal to, or less than



AS, will be either an ellipsis, a parabola, or an hyperbola; the point a in the first case falling on the same side of the line GF as the point A; in the second, going off to an infinite distance; in the third, falling on the other side of the line GF. For if on GF the perpendiculars CI, DK are let fall, IC will be to HB as EC to EB; that is, as SC to SB; and by permutation, IC to SC as HB to SB, or as GA to SA. And, by the like argument, we may prove that KD is to SD in the same ratio. Wherefore the points B, C, D lie in a conic section described about the focus S, in such manner that all the right lines drawn from the focus S to the several points of the section, and the perpendiculars let fall from the same points on the right line GF, are in that given ratio.

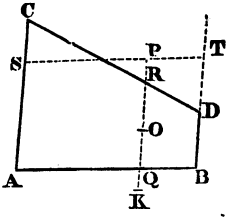
That excellent geometer M. De la Hire has solved this Problem much after the same way, in his Conics, Prop. XXV., Lib. VIII.

SECTION V.

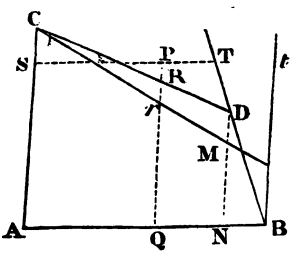
How the orbits are to be found when neither focus is given.

LEMMA XVII.

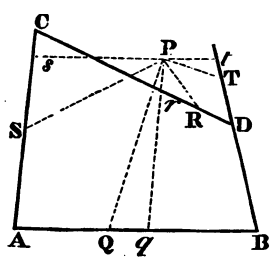
If from any point P *of a given conic section, to the four produced sides* AB, CD, AC, DB, *of any trapezium* ABDC *inscribed in that section, as many right lines* PQ, PR, PS, PT *are drawn in given angles, each line to each side; the rectangle* PQ × PR *of those on the opposite sides* AB, CD, *will be to the rectangle* PS × PT *of those on the other two opposite sides* AC, BD, *in a given ratio.*



CASE 1. Let us suppose, first, that the lines drawn to one pair of opposite sides are parallel to either of the other sides; as PQ and PR to the side AC, and PS and PT to the side AB. And farther, that one pair of the opposite sides, as AC and BD, are parallel betwixt themselves; then the right line which bisects those parallel sides will be one of the diameters of the conic section, and will likewise bisect RQ. Let O be the point in which RQ is bisected, and PO will be an ordinate to that diameter. Produce PO to K, so that OK may be equal to PO, and OK will be an ordinate on the other side of that diameter. Since, therefore, the points A, B, P, and K are placed in the conic section, and PK cuts AB in a given angle, the rectangle PQK (by Prop. XVII., XIX., XXI. and XXIII., Book III., of Apollonius's Conics) will be to the rectangle AQB in a given ratio. But QK and PR are equal, as being the differences of the equal lines OK, OP, and OQ, OR; whence the rectangles PQK and PQ × PR are equal; and therefore the rectangle PQ × PR is to the rectangle AQB, that is, to the rectangle PS × PT in a given ratio. Q.E.D



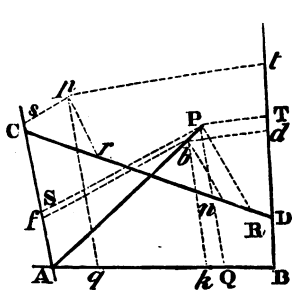
Case 2. Let us next suppose that the opposite sides AC and BD of the trapezium are not parallel. Draw B*d* parallel to AC, and meeting as well the right line ST in *t*, as the conic section in *d*. Join C*d* cutting PQ in *r*, and draw DM parallel to PQ, cutting C*d* in M, and AB in N. Then (because of the similar triangles BT*t*, DBN), B*t* or PQ is to T*t* as DN to NB. And so R*r* is to AQ or PS as DM to AN. Wherefore, by multiplying the antecedents by the antecedents, and the consequents by the consequents, as the rectangle PQ × R*r* is to the rectangle PS × T*t*, so will the rectangle NDM be to the rectangle ANB; and (by Case 1) so is the rectangle PQ × P*r* to the rectangle PS × P*t*; and by division, so is the rectangle PQ × PR to the rectangle PS × PT. Q.E.D.



Case 3. Let us suppose, lastly, the four lines PQ, PR, PS, PT, not to be parallel to the sides AC, AB, but any way inclined to them. In their place draw P*q*, P*r*, parallel to AC; and P*s*, P*t* parallel to AB; and because the angles of the triangles PQ*q*, PR*r*, PS*s*, PT*t* are given, the ratios of PQ to P*q*, PR to P*r*, PS to P*s*, PT to P*t* will be also given; and therefore the compounded ratios PQ × PR to P*q* × P*r*, and PS × PT to P*s* × P*t* are given. But from what we have demonstrated before, the ratio of P*q* × P*r* to P*s* × P*t* is given; and therefore also the ratio of PQ × PR to PS × PT. Q.E.D.

LEMMA XVIII.

The same things supposed, if the rectangle PQ × PR *of the lines drawn to the two opposite sides of the trapezium is to the rectangle* PS × PT *of those drawn to the other two sides in a given ratio, the point* P, *from whence those lines are drawn, will be placed in a conic section described about the trapezium.*



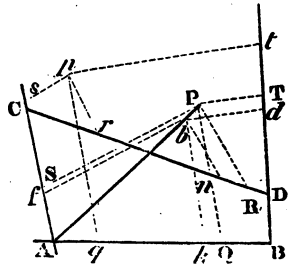
Conceive a conic section to be described passing through the points A, B, C, D, and any one of the infinite number of points P, as for example *p*; I say, the point P will be always placed in this section. If you deny the thing, join AP cutting this conic section somewhere else, if possible, than in P, as in *b*. Therefore if from those points *p* and *b*, in the given angles to the sides of the trapezium, we draw the right lines *pq*, *pr*, *ps*, *pt*, and *bk*, *bn*, *bf*, *bd*, we shall have, as *bk* × *bn* to *bf* × *bd*,

so (by Lem. XVII) $pq \times pr$ to $ps \times pt$; and so (by supposition) PQ × PR to PS × PT. And because of the similar trapezia *bk*A*f*, PQAS, as *bk* to *bf*, so PQ to PS. Wherefore by dividing the terms of the preceding proportion by the correspondent terms of this, we shall have *bn* to *bd* as PR to PT. And therefore the equiangular trapezia D*nbd*, DRPT, are similar, and consequently their diagonals D*b*, DP do coincide. Wherefore *b* falls in the intersection of the right lines AP, DP, and consequently coincides with the point P. And therefore the point P, wherever it is taken, falls to be in the assigned conic section. Q.E.D.

COR. Hence if three right lines PQ, PR, PS, are drawn from a common point P, to as many other right lines given in position, AB, CD, AC, each to each, in as many angles respectively given, and the rectangle PQ × PR under any two of the lines drawn be to the square of the third PS in a given ratio; the point P, from which the right lines are drawn, will be placed in a conic section that touches the lines AB, CD in A and C; and the contrary. For the position of the three right lines AB, CD, AC remaining the same, let the line BD approach to and coincide with the line AC; then let the line PT come likewise to coincide with the line PS; and the rectangle PS × PT will become PS^2, and the right lines AB, CD, which before did cut the curve in the points A and B, C and D, can no longer cut, but only touch, the curve in those coinciding points.

SCHOLIUM.

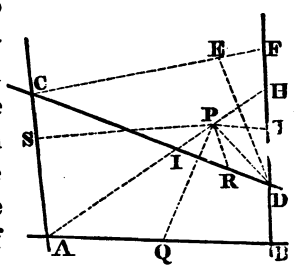
In this Lemma, the name of conic section is to be understood in a large sense, comprehending as well the rectilinear section through the vertex of the cone, as the circular one parallel to the base. For if the point *p* happens to be in a right line, by which the points A and D, or C and B are joined, the conic section will be changed into two right lines, one of which is that right line upon which the point *p* falls, and the other is a right line that joins the other two of the four points. If the two opposite angles of the trapezium taken together are equal to two right angles, and if the four lines PQ, PR, PS, PT, are drawn to the sides thereof at right angles, or any other equal angles, and the rectangle PQ × PR under two of the lines drawn PQ and PR, is equal to the rectangle PS × PT under the other two PS and PT, the conic section will become a circle. And the same thing will happen if the four lines are drawn in any angles, and the rectangle PQ × PR, under one pair of the lines drawn, is to the rectangle PS × PT under the other pair as the rectangle under the sines of the angles S, T, in which the two last lines PS, PT are drawn to the rectangle under the sines of the angles Q, R, in which the first two



PQ, PR are drawn. In all other cases the locus of the point P will be one of the three figures which pass commonly by the name of the conic sections. But in room of the trapezium ABCD, we may substitute a quadrilateral figure whose two opposite sides cross one another like diagonals. And one or two of the four points A, B, C, D may be supposed to be removed to an infinite distance, by which means the sides of the figure which converge to those points, will become parallel; and in this case the conic section will pass through the other points, and will go the same way as the parallels *in infinitum*.

LEMMA XIX.

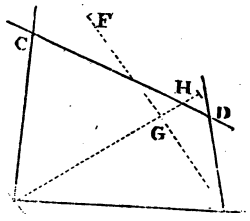
To find a point P *from which if four right lines* PQ, PR, PS, PT *are drawn to as many other right lines* AB, CD, AC, BD, *given by position, each to each, at given angles, the rectangle* PQ × PR, *under any two of the lines drawn, shall be to the rectangle* PS × PT, *under the other two, in a given ratio.*



Suppose the lines AB, CD, to which the two right lines PQ, PR, containing one of the rectangles, are drawn to meet two other lines, given by position, in the points A, B, C, D. From one of those, as A, draw any right line AH, in which you would find the point P. Let this cut the opposite lines BD, CD, in H and I; and, because all the angles of the figure are given, the ratio of PQ to PA, and PA to PS, and therefore of PQ to PS, will be also given. Subducting this ratio from the given ratio of PQ × PR to PS × PT, the ratio of PR to PT will be given; and adding the given ratios of PI to PR, and PT to PH, the ratio of PI to PH, and therefore the point P will be given. Q.E.I.

Cor. 1. Hence also a tangent may be drawn to any point D of the locus of all the points P. For the chord PD, where the points P and D meet, that is, where AH is drawn through the point D, becomes a tangent. In which case the ultimate ratio of the evanescent lines IP and PH will be found as above. Therefore draw CF parallel to AD, meeting BD in F, and cut it in E in the same ultimate ratio, then DE will be the tangent; because CF and the evanescent IH are parallel, and similarly cut in E and P.

Cor. 2. Hence also the locus of all the points P may be determined. Through any of the points A, B, C, D, as A, draw AE touching the locus, and through any other point B parallel to the tangent, draw BF meeting the locus in F; and find the point F by this Lemma. Bisect BF in G, and, drawing the indefinite line AG, this will be the position of the diameter to which BG and FG are ordinates. Let this AG meet the locus

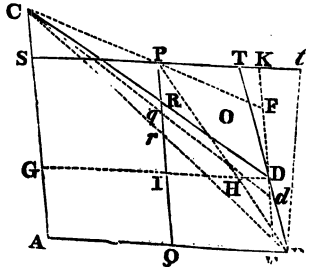


in H, and AH will be its diameter or latus transversum, to which the latus rectum will be as BG^2 to $AG \times GH$. If AG nowhere meets the locus, the line AH being infinite, the locus will be a parabola; and its latus rectum corresponding to the diameter AG will be $\frac{BG^2}{AG}$. But if it does meet it anywhere, the locus will be an hyperbola, when the points A and H are placed on the same side the point G; and an ellipsis, if the point G falls between the points A and H; unless, perhaps, the angle AGB is a right angle, and at the same time BG^2 equal to the rectangle AGH, in which case the locus will be a circle.

And so we have given in this Corollary a solution of that famous Problem of the ancients concerning four lines, begun by Euclid, and carried on by Apollonius; and this not an analytical calculus, but a geometrical composition, such as the ancients required.

LEMMA XX.

If the two opposite angular points A *and* P *of any parallelogram* ASPQ *touch any conic section in the points* A *and* P; *and the sides* AQ, AS *of one of those angles, indefinitely produced, meet the same conic section in* B *and* C; *and from the points of concourse* B *and* C *to any fifth point* D *of the conic section, two right lines* BD, CD *are drawn meeting the two other sides* PS, PQ *of the parallelogram, indefinitely produced in* T *and* R; *the parts* PR *and* PT, *cut off from the sides, will always be one to the other in a given ratio. And* vice versa, *if those parts cut off are one to the other in a given ratio, the locus of the point* D *will be a conic section passing through the four points* A, B, C, P



CASE 1. Join BP, CP, and from the point D draw the two right lines DG, DE, of which the first DG shall be parallel to AB, and meet PB, PQ, CA in H, I, G; and the other DE shall be parallel to AC, and meet PC, PS, AB, in F, K, E; and (by Lem. XVII) the rectangle $DE \times DF$ will be to the rectangle $DG \times DH$ in a given ratio. But PQ is to DE (or IQ) as PB to HB, and consequently as PT to DH; and by permutation PQ is to PT as DE to DH. Likewise PR is to DF as RC to DC, and therefore as (IG or) PS to DG; and by permutation PR is to PS as DF to DG; and, by compounding those ratios, the rectangle $PQ \times PR$ will be to the rectangle $PS \times PT$ as the rectangle $DE \times DF$ is to the rectangle $DG \times DH$, and consequently in a given ratio. But PQ and PS are given, and therefore the ratio of PR to PT is given. Q.E.D.

CASE 2. But if PR and PT are supposed to be in a given ratio one to the other, then by going back again, by a like reasoning, it will follow that the rectangle DE × DF is to the rectangle DG × DH in a given ratio; and so the point D (by Lem. XVIII) will lie in a conic section passing through the points A, B, C, P, as its locus. Q.E.D.

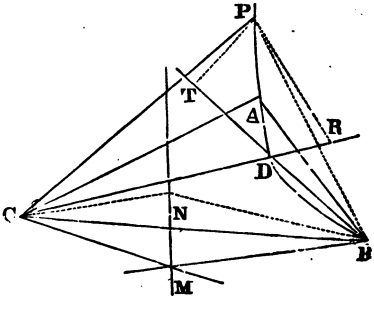
COR. 1. Hence if we draw BC cutting PQ in r and in PT take Pt to Pr in the same ratio which PT has to PR; then Bt will touch the conic section in the point B. For suppose the point D to coalesce with the point B, so that the chord BD vanishing, BT shall become a tangent, and CD and BT will coincide with CB and Bt.

COR. 2. And, vice versa, if Bt is a tangent, and the lines BD, CD meet in any point D of a conic section, PR will be to PT as Pr to Pt. And, on the contrary, if PR is to PT as Pr to Pt, then BD and CD will meet in some point D of a conic section.

COR. 3. One conic section cannot cut another conic section in more than four points. For, if it is possible, let two conic sections pass through the five points A, B, C, P, O; and let the right line BD cut them in the points D, d, and the right line Cd cut the right line PQ in q. Therefore PR is to PT as Pq to PT: whence PR and Pq are equal one to the other, against the supposition.

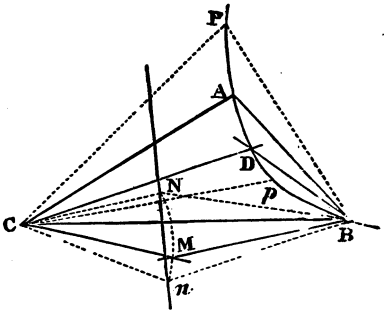
LEMMA XXI.

If two moveable and indefinite right lines BM, CM *drawn through given points* B, C, *as poles, do by their point of concourse* M *describe a third right line* MN *given by position; and other two indefinite right lines* BD, CD *are drawn, making with the former two at those given points* B, C, *given angles,* MBD, MCD: *I say, that those two right lines* BD, CD *will by their point of concourse* D *describe a conic section passing through the points* B, C. *And,* vice versa, *if the right lines* BD, CD *do by their point of concourse* D *describe a conic section passing through the given points* B, C, A, *and the angle* DBM *is always equal to the given angle* ABC, *as well as the angle* DCM *always equal to the given angle* ACB, *the point* M *will lie in a right line given by position, as its locus.*



For in the right line MN let a point N be given, and when the moveable point M falls on the immoveable point N, let the moveable point D fall on an immovable point P. Join CN, BN, CP, BP, and from the point P draw the right lines PT, PR meeting BD, CD in T and R, and making the angle BPT equal to the given angle BNM, and the angle CPR

equal to the given angle CNM. Wherefore since (by supposition) the angles MBD, NBP are equal, as also the angles MCD, NCP, take away the angles NBD and NCD that are common, and there will remain the angles NBM and PBT, NCM and PCR equal; and therefore the triangles NBM, PBT are similar, as also the triangles NCM, PCR. Wherefore PT is to NM as PB to NB; and PR to NM as PC to NC. But the points, B, C, N, P are immovable: wherefore PT and PR have a given ratio to NM, and consequently a given ratio between themselves; and therefore, (by Lemma XX) the point D wherein the moveable right lines BT and CR perpetually concur, will be placed in a conic section passing through the points B, C, P. Q.E.D.

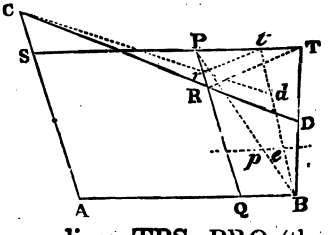


And, *vice versa*, if the moveable point D lies in a conic section passing through the given points B, C, A; and the angle DBM is always equal to the given angle ABC, and the angle DCM always equal to the given angle ACB, and when the point D falls successively on any two immovable points *p*, P, of the conic section, the moveable point M falls successively on two immovable points *n*, N. Through these points *n*, N, draw the right line *n*N: this line *n*N will be the perpetual locus of that moveable point M. For, if possible, let the point M be placed in any curve line. Therefore the point D will be placed in a conic section passing through the five points B, C, A, *p*, P, when the point M is perpetually placed in a curve line. But from what was demonstrated before, the point D will be also placed in a conic section passing through the same five points B, C, A, *p*, P, when the point M is perpetually placed in a right line. Wherefore the two conic sections will both pass through the same five points, against Corol. 3, Lem. XX. It is therefore absurd to suppose that the point M is placed in a curve line. Q.E.D.

PROPOSITION XXII. PROBLEM XIV.

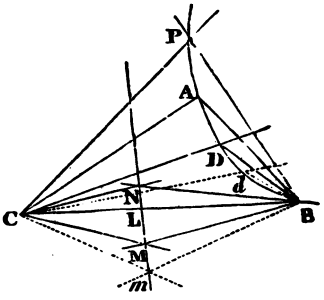
To describe a trajectory that shall pass through five given points.

Let the five given points be A, B, C, P, D. From any one of them, as A, to any other two as B, C, which may be called the poles, draw the right lines AB, AC, and parallel to those the lines TPS, PRQ through the fourth point P. Then from the two poles B, C, draw through the fifth point D two indefinite lines BDT, CRD, meeting with the last drawn lines TPS, PRQ (the



former with the former, and the latter with the latter) in T and R. Then drawing the right line *tr* parallel to TR, cutting off from the right lines PT, PR, any segments P*t*, P*r*, proportional to PT, PR; and if through their extremities, *t*, *r*, and the poles B, C, the right lines B*t*, C*r* are drawn, meeting in *d*, that point *d* will be placed in the trajectory required. For (by Lem. XX) that point *d* is placed in a conic section passing through the four points A, B, C, P; and the lines R*r*, T*t* vanishing, the point *d* comes to coincide with the point D. Wherefore the conic section passes through the five points A, B, C, P, D. Q.E.D.

The same otherwise.

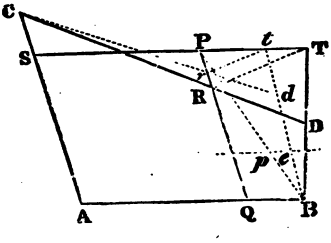


Of the given points join any three, as A, B, C; and about two of them B, C, as poles, making the angles ABC, ACB of a given magnitude to revolve, apply the legs BA, CA, first to the point D, then to the point P, and mark the points M, N, in which the other legs BL, CL intersect each other in both cases. Draw the indefinite right line MN, and let those moveable angles revolve about their poles B, C, in such manner that the intersection, which is now supposed to be *m*, of the legs BL, CL, or BM, CM, may always fall in that indefinite right line MN; and the intersection, which is now supposed to be *d*, of the legs BA, CA, or BD, CD, will describe the trajectory required, PAD*d*B. For (by Lem. XXI) the point *d* will be placed in a conic section passing through the points B, C; and when the point *m* comes to coincide with the points L, M, N, the point *d* will (by construction) come to coincide with the points A, D, P. Wherefore a conic section will be described that shall pass through the five points A, B, C, P, D. Q.E.F.

COR. 1. Hence a right line may be readily drawn which shall be a tangent to the trajectory in any given point B. Let the point *d* come to coincide with the point B, and the right line B*d* will become the tangent required.

COR. 2. Hence also may be found the centres, diameters, and latera recta of the trajectories, as in Cor. 2, Lem. XIX.

SCHOLIUM.

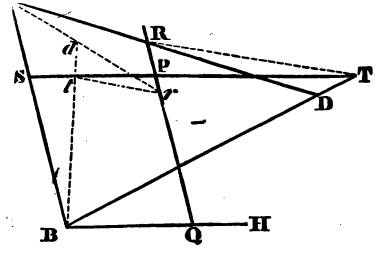


The former of these constructions will become something more simple by joining , and in that line, produced, if need be, taking B*p* to BP as PR is to PT; and through *p* draw the indefinite right line *pe* parallel to S PT, and in that line *pe* taking always *pe* equal to P*r*, and draw the right lines B*e*, C*r*

to meet in *d*. For since P*r* to P*t*, PR to PT, *p*B to PB, *pe* to P*t*, are all in the same ratio, *pe* and P*r* will be always equal. After this manner the points of the trajectory are most readily found, unless you would rather describe the curve mechanically, as in the second construction.

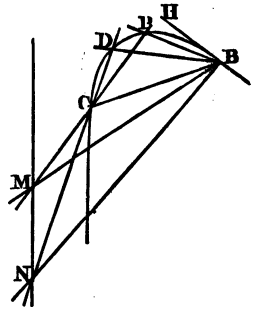
PROPOSITION XXIII. PROBLEM XV.

To describe a trajectory that shall pass through four given points, and touch a right line given by position.



CASE 1. Suppose that HB is the given tangent, B the point of contact, and C, D, P, the three other given points. Join BC, and draw PS parallel to BH, and PQ parallel to BC; complete the parallelogram BSPQ. Draw BD cutting SP in T, and CD cutting PQ in R. Lastly, draw any line *tr* parallel to TR, cutting off from PQ, PS, the segments P*r*, P*t* proportional to PR, PT respectively; and draw C*r*, B*t* their point of concourse *d* will (by Lem. XX) always fall on the trajectory to be described.

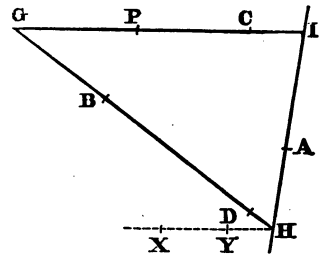
The same otherwise.



Let the angle CBH of a given magnitude revolve about the pole B, as also the rectilinear radius DC, both ways produced, about the pole C. Mark the points M, N, on which the leg BC of the angle cuts that radius when BH, the other leg thereof, meets the same radius in the points P and D. Then drawing the indefinite line MN, let that radius CP or CD and the leg BC of the angle perpetually meet in this line; and the point of concourse of the other leg BH with the radius will delineate the trajectory required.

For if in the constructions of the preceding Problem the point A comes to a coincidence with the point B, the lines CA and CB will coincide, and the line AB, in its last situation, will become the tangent BH; and therefore the constructions there set down will become the same with the constructions here described. Wherefore the concourse of the leg BH with the radius will describe a conic section passing through the points C, D, P, and touching the line BH in the point B. Q.E.F.

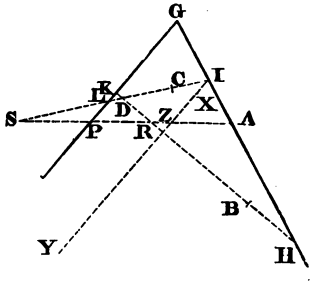
CASE 2. Suppose the four points B, C, D, P, given, being situated without the tangent HI. Join each two by the lines BD, CP meeting in G, and cutting the tangent in H and I. Cut the tangent in A in such manner



that HA may be to IA as the rectangle under a mean proportional between CG and GP, and a mean proportional between BH and HD is to a rectangle under a mean proportional between GD and GB, and a mean proportional betweeen PI and IC, and A will be the point of contact. For if HX, a parallel to the right line PI, cuts the trajectory in any points X and Y, the point A (by the properties of the conic sections) will come to be so placed, that HA^2 will become to AI^2 in a ratio that is compounded out of the ratio of the rectangle XHY to the rectangle BHD, or of the rectangle CGP to the rectangle DGB; and the ratio of the rectangle BHD to the rectangle PIC. But after the point of contact A is found, the trajectory will be described as in the first Case. Q.E.F. But the point A may be taken either between or without the points H and I, upon which account a twofold trajectory may be described.

PROPOSITION XXIV. PROBLEM XVI.

To describe a trajectory that shall pass through three given points, and touch two right lines given by position.



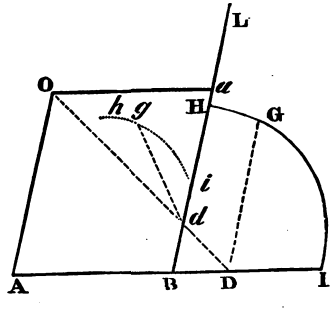
Suppose HI, KL to be the given tangents and B, C, D, the given points. Through any two of those points, as B, D, draw the indefinite right line BD meeting the tangents in the points H, K. Then likewise through any other two of these points, as C, D, draw the indefinite right line CD meeting the tangents in the points I, L. Cut the lines drawn in R and S, so that HR may be to KR as the mean proportional between BH and HD is to the mean proportional between BK and KD; and IS to LS as the mean proportional between CI and ID is to the mean proportional between CL and LD. But you may cut, at pleasure, either within or between the points K and H, I and L, or without them; then draw RS cutting the tangents in A and P, and A and P will be the points of contact. For if A and P are supposed to be the points of contact, situated anywhere else in the tangents, and through any of the points H, I, K, L, as I, situated in either tangent HI, a right line IY is drawn parallel to the other tangent KL, and meeting the curve in X and Y, and in that right line there be taken IZ equal to a mean proportional between IX and IY, the rectangle XIY or IZ^2, will (by the properties of the conic sections) be to LP^2 as the rectangle CID is to the rectangle CLD, that is (by the construction), as SI is to SL^2, and therefore

IZ is to LP as SI to SL. Wherefore the points S, P, Z, are in one right line. Moreover, since the tangents meet in G, the rectangle XIY or IZ^2 will (by the properties of the conic sections) be to IA^2 as GP^2 is to GA^2, and consequently IZ will be to IA as GP to GA. Wherefore the points P, Z, A, lie in one right line, and therefore the points S, P, and A are in one right line. And the same argument will prove that the points R, P, and A are in one right line. Wherefore the points of contact A and P lie in the right line RS. But after these points are found, the trajectory may be described, as in the first Case of the preceding Problem. Q.E.F.

In this Proposition, and Case 2 of the foregoing, the constructions are the same, whether the right line XY cut the trajectory in X and Y, or not; neither do they depend upon that section. But the constructions being demonstrated where that right line does cut the trajectory, the constructions where it does not are also known; and therefore, for brevity's sake, I omit any farther demonstration of them.

LEMMA XXII.

To transform figures into other figures of the same kind.



Suppose that any figure HGI is to be transformed. Draw, at pleasure, two parallel lines AO, BL, cutting any third line AB, given by position, in A and B, and from any point G of the figure, draw out any right line GD, parallel to OA, till it meet the right line AB. Then from any given point O in the line OA, draw to the point D the right line OD, meeting BL in *d*; and from the point of concourse raise the right line *dg* containing any given angle with the right line BL, and having such ratio to O*d* as DG has to OD; and *g* will be the point in the new figure *hgi*, corresponding to the point G. And in like manner the several points of the first figure will give as many correspondent points of the new figure. If we therefore conceive the point G to be carried along by a continual motion through all the points of the first figure, the point *g* will be likewise carried along by a continual motion through all the points of the new figure, and describe the same. For distinction's sake, let us call DG the first ordinate, *dg* the new ordinate, AD the first abscissa, *ad* the new abscissa; O the pole, OD the abscinding radius, OA the first ordinate radius, and O*a* (by which the parallelogram OAB*a* is completed) the new ordinate radius.

I say, then, that if the point G is placed in a right line given by position, the point *g* will be also placed in a right line given by position. If the point G is placed in a conic section, the point *g* will be likewise placed

in a conic section. And here I understand the circle as one of the conic sections. But farther, if the point G is placed in a line of the third analytical order, the point g will also be placed in a line of the third order, and so on in curve lines of higher orders. The two lines in which the points G, g, are placed, will be always of the same analytical order. For as ad is to OA, so are Od to OD, dg to DG, and AB to AD; and therefore AD is equal to $\frac{OA \times AB}{ad}$, and DG equal to $\frac{OA \times dg}{ad}$. Now if the point G is placed in a right line, and therefore, in any equation by which the relation between the abscissa AD and the ordinate GD is expressed, those indetermined lines AD and DG rise no higher than to one dimension, by writing this equation $\frac{OA \times AB}{ad}$ in place of AD, and $\frac{OA \times dg}{ad}$ in place of DG, a new equation will be produced, in which the new abscissa ad and new ordinate dg rise only to one dimension; and which therefore must denote a right line. But if AD and DG (or either of them) had risen to two dimensions in the first equation, ad and dg would likewise have risen to two dimensions in the second equation. And so on in three or more dimensions. The indetermined lines, ad, dg in the second equation, and AD, DG, in the first, will always rise to the same number of dimensions; and therefore the lines in which the points G, g, are placed are of the same analytical order.

I say farther, that if any right line touches the curve line in the first figure, the same right line transferred the same way with the curve into the new figure will touch that curve line in the new figure, and *vice versa*. For if any two points of the curve in the first figure are supposed to approach one the other till they come to coincide, the same points transferred will approach one the other till they come to coincide in the new figure; and therefore the right lines with which those points are joined will become together tangents of the curves in both figures. I might have given demonstrations of these assertions in a more geometrical form; but I study to be brief.

Wherefore if one rectilinear figure is to be transformed into another, we need only transfer the intersections of the right lines of which the first figure consists, and through the transferred intersections to draw right lines in the new figure. But if a curvilinear figure is to be transformed, we must transfer the points, the tangents, and other right lines, by means of which the curve line is defined. This Lemma is of use in the solution of the more difficult Problems; for thereby we may transform the proposed figures, if they are intricate, into others that are more simple. Thus any right lines converging to a point are transformed into parallels, by taking for the first ordinate radius any right line that passes through the point of concourse of the converging lines, and that because their point of con-

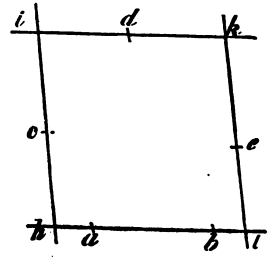
course is by this means made to go off *in infinitum;* and parallel lines are such as tend to a point infinitely remote. And after the problem is solved in the new figure, if by the inverse operations we transform the new into the first figure, we shall have the solution required.

This Lemma is also of use in the solution of solid problems. For as often as two conic sections occur, by the intersection of which a problem may be solved, any one of them may be transformed, if it is an hyperbola or a parabola, into an ellipsis, and then this ellipsis may be easily changed into a circle. So also a right line and a conic section, in the construction of plane problems, may be transformed into a right line and a circle

PROPOSITION XXV. PROBLEM XVII.

To describe a trajectory that shall pass through two given points, and touch three right lines given by position.

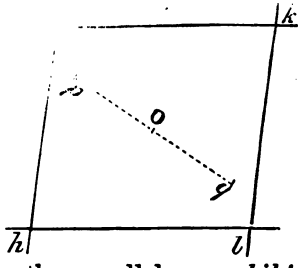
Through the concourse of any two of the tangents one with the other, and the concourse of the third tangent with the right line which passes through the two given points, draw an indefinite right line; and, taking this line for the first ordinate radius, transform the figure by the preceding Lemma into a new figure. In this figure those two tangents will become parallel to each other, and the third tangent will be parallel to the right line that passes through the two given points. Suppose *hi, kl* to be those two parallel tangents, *ik* the third tangent, and *hl* a right line parallel thereto, passing through those points *a, b,* through which the conic section ought to pass in this new figure; and completing the parallelogram *hikl,* let the right lines *hi, ik, kl* be so cut in *c, d, e,* that *hc* may be to the square root of the rectangle *ahb, ic,* to *id,* and *ke* to *kd,* as the sum of the right lines *hi* and *kl* is to the sum of the three lines, the first whereof is the right line *ik,* and the other two are the square roots of the rectangles *ahb* and *alb;* and *c, d, e,* will be the points of contact. For by the properties of the conic sections, hc^2 to the rectangle *ahb,* and ic^2 to id^2, and ke^2 to kd^2, and el^2 to the rectangle *alb,* are all in the same ratio; and therefore *hc* to the square root of *ahb, ic* to *id, ke* to *kd,* and *el* to the square root of *alb,* are in the subduplicate of that ratio; and by composition, in the given ratio of the sum of all the antecedents $hi + kl$, to the sum of all the consequents $\sqrt{ahb} + ik + \sqrt{alb}$. Wherefore from that given ratio we have the points of contact *c, d, e,* in the new figure. By the inverted operations of the last Lemma, let those points be transferred into the first figure, and the trajectory will be there described by Prob. XIV. Q.E.F. But according as the points *a, b,* fall between the points *h, l,* or without them, the points *c, d, e,* must be taken



either between the points, *h*, *i*, *k*, *l*, or without them. If one of the points *a*, *b*, falls between the points *h*, *i*, and the other without the points *h*, *l*, the Problem is impossible.

PROPOSITION XXVI. PROBLEM XVIII.

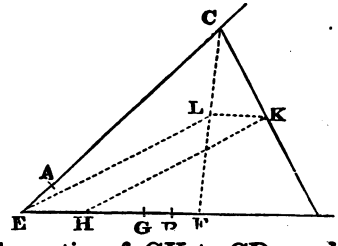
To describe a trajectory that shall pass through a given point, and touch four right lines given by position.



From the common intersections, of any two of the tangents to the common intersection of the other two, draw an indefinite right line; and taking this line for the first ordinate radius, transform the figure (by Lem. XXII) into a new figure, and the two pairs of tangents, each of which before concurred in the first ordinate radius, will now become parallel. Let *hi* and *kl*, *ik* and *hl*, be those pairs of parallels completing the parallelogram *hikl*. And let *p* be the point in this new figure corresponding to the given point in the first figure. Through O the centre of the figure draw *pq*: and O*q* being equal to O*p*, *q* will be the other point through which the conic section must pass in this new figure. Let this point be transferred, by the inverse operation of Lem. XXII into the first figure, and there we shall have the two points through which the trajectory is to be described. But through those points that trajectory may be described by Prop. XVII.

LEMMA XXIII.

If two right lines, as AC, BD *given by position, and terminating in given points* A, B, *are in a given ratio one to the other, and the right line* CD, *by which the indetermined points* C, D *are joined is cut in* K *in a given ratio; I say, that the point* K *will be placed in a right line given by position.*



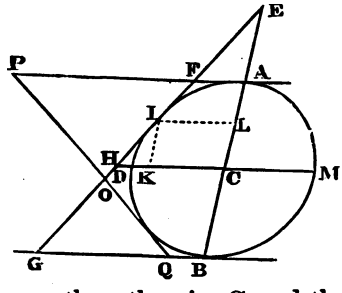
For let the right lines AC, BD meet in E, and in BE take BG to AE as BD is to AC, and let FD be always equal to the given line EG; and, by construction, EC will be to GD, that is, to EF, as AC to BD, and therefore in a given ratio; and therefore the triangle EFC will be given in kind. Let CF be cut in L so as CL may be to CF in the ratio of CK to CD; and because that is a given ratio, the triangle EFL will be given in kind, and therefore the point L will be placed in the right line EL given by position. Join LK, and the triangles CLK, CFD will be similar; and because FD is a given line, and LK is to FD in a given ratio, LK will be also given

To this let EH be taken equal, and ELKH will be always a parallelogram. And therefore the point K is always placed in the side HK (given by position) of that parallelogram. Q.E.D.

COR. Because the figure EFLC is given in kind, the three right lines EF, EL, and EC, that is, GD, HK, and EC, will have given ratios to each other.

LEMMA XXIV.

If three right lines, two whereof are parallel, and given by position, touch any conic section; I say, that the semi-diameter of the section which is parallel to those two is a mean proportional between the segments of those two that are intercepted between the points of contact and the third tangent.



Let AF, GB be the two parallels touching the conic section ADB in A and B; EF the third right line touching the conic section in I, and meeting the two former tangents in F and G, and let CD be the semi-diameter of the figure parallel to those tangents; I say, that AF, CD, BG are continually proportional.

For if the conjugate diameters AB, DM meet the tangent FG in E and H, and cut one the other in C, and the parallelogram IKCL be completed; from the nature of the conic sections, EC will be to CA as CA to CL; and so by division, EC — CA to CA — CL, or EA to AL; and by composition, EA to EA + AL or EL, as EC to EC+CA or EB; and therefore (because of the similitude of the triangles EAF, ELI, ECH, EBG) AF is to LI as CH to BG. Likewise, from the nature of the conic sections, LI (or CK) is to CD as CD to CH; and therefore (*ex æquo perturbatè*) AF is to CD as CD to BG. Q.E.D.

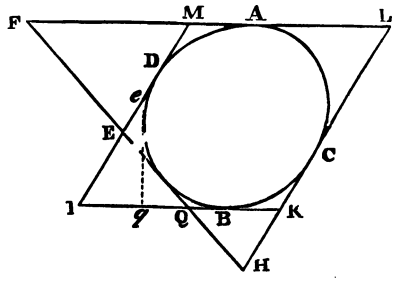
COR. 1. Hence if two tangents FG, PQ meet two parallel tangents AF, BG in F and G, P and Q, and cut one the other in O; AF (*ex æquo perturbatè*) will be to BQ as AP to BG, and by division, as FP to GQ, and therefore as FO to OG.

COR. 2. Whence also the two right lines PG, FQ drawn through the points P and G, F and Q, will meet in the right line ACB passing through the centre of the figure and the points of contact A, B.

LEMMA XXV.

If four sides of a parallelogram indefinitely produced touch any conic section, and are cut by a fifth tangent; I say, that, taking those segments of any two conterminous sides that terminate in opposite angles

of the parallelogram, either segment is to the side from which it is cut off as that part of the other conterminous side which is intercepted between the point of contact and the third side is to the other segment.



Let the four sides ML, IK, KL, MI, of the parallelogram MLIK touch the conic section in A, B, C, D; and let the fifth tangent FQ cut those sides in F, Q, H, and E; and taking the segments ME, KQ of the sides MI, KI, or the segments KH, MF of the sides KL, ML, I say, that ME is to MI as BK to KQ; and KH to KL as AM to MF. For, by Cor. 1 of the preceding Lemma, ME is to EI as (AM or) BK to BQ; and, by composition, ME is to MI as BK to KQ. Q.E.D. Also KH is to HL as (BK or) AM to AF; and by division, KH to KL as AM to MF. Q.E.D.

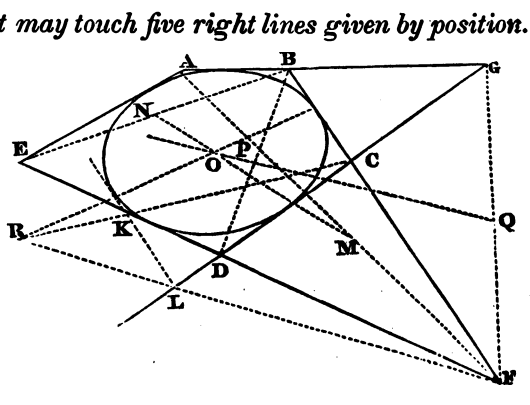
Cor. 1. Hence if a parallelogram IKLM described about a given conic section is given, the rectangle KQ × ME, as also the rectangle KH × MF equal thereto, will be given. For, by reason of the similar triangles KQH MFE, those rectangles are equal.

Cor. 2. And if a sixth tangent *eq* is drawn meeting the tangents KI, MI in *q* and *e*, the rectangle KQ × ME will be equal to the rectangle K*q* × M*e*, and KQ will be to M*e* as K*q* to ME, and by division as Q*q* to E*e*.

Cor. 3. Hence, also, if E*q*, *e*Q, are joined and bisected, and a right line is drawn through the points of bisection, this right line will pass through the centre of the conic section. For since Q*q* is to E*e* as KQ to M*e*, the same right line will pass through the middle of all the lines E*q*, *e*Q, MK (by Lem. XXIII), and the middle point of the right line MK is the centre of the section.

PROPOSITION XXVII. PROBLEM XIX.

To describe a trajectory that may touch five right lines given by position.

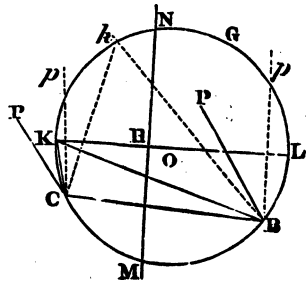


Supposing ABG, BCF, GCD, FDE, EA to be the tangents given by position. Bisect in M and N, AF, BE, the diagonals of the quadrilateral figure ABFE contained under any four of them; and (by Cor. 3, Lem. XXV) the right line MN drawn through the points of

bisection will pass through the centre of the trajectory. Again, bisect in P and Q the diagonals (if I may so call them) BD, GF of the quadrilateral figure BGDF contained under any other four tangents, and the right line PQ drawn through the points of bisection will pass through the centre of the trajectory; and therefore the centre will be given in the concourse of the bisecting lines. Suppose it to be O. Parallel to any tangent BC draw KL at such distance that the centre O may be placed in the middle between the parallels; this KL will touch the trajectory to be described. Let this cut any other two tangents GCD, FDE, in L and K. Through the points C and K, F and L, where the tangents not parallel, CL, FK meet the parallel tangents CF, KL, draw CK, FL meeting in R; and the right line OR drawn and produced, will cut the parallel tangents CF, KL, in the points of contact. This appears from Cor. 2, Lem. XXIV. And by the same method the other points of contact may be found, and then the trajectory may be described by Prob. XIV. Q.E.F.

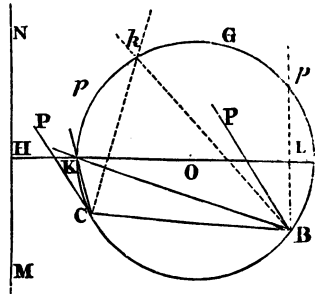
SCHOLIUM.

Under the preceding Propositions are comprehended those Problems wherein either the centres or asymptotes of the trajectories are given. For when points and tangents and the centre are given, as many other points and as many other tangents are given at an equal distance on the other side of the centre. And an asymptote is to be considered as a tangent, and its infinitely remote extremity (if we may say so) is a point of contact. Conceive the point of contact of any tangent removed *in infinitum*, and the tangent will degenerate into an asymptote, and the constructions of the preceding Problems will be changed into the constructions of those Problems wherein the asymptote is given.



After the trajectory is described, we may find its axes and foci in this manner. In the construction and figure of Lem. XXI, let those legs BP, CP, of the moveable angles PBN, PCN, by the concourse of which the trajectory was described, be made parallel one to the other; and retaining that position, let them revolve about their poles P, C, in that figure. In the mean while let the other legs CN, BN, of those angles, by their concourse K or *k*, describe the circle BKGC. Let O be the centre of this circle; and from this centre upon the ruler MN, wherein those legs CN, BN did concur while the trajectory was described, let fall the perpendicular OH meeting the circle in K and L. And when those other legs CK, BK meet in the point K that is nearest to the ruler, the first legs CP, BP will be parallel to the greater axis, and perpendicular on the lesser; and the con-

trary will happen if those legs meet in the remotest point L. Whence if the centre of the trajectory is given, the axes will be given; and those being given, the foci will be readily found.

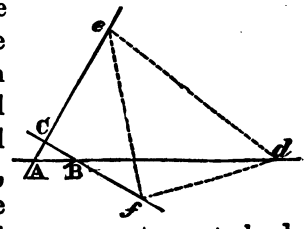


But the squares of the axes are one to the other as KH to LH, and thence it is easy to describe a trajectory given in kind through four given points. For if two of the given points are made the poles C, B, the third will give the moveable angles PCK, PBK; but those being given, the circle BGKC may be described. Then, because the trajectory is given in kind, the ratio of OH to OK, and and therefore OH itself, will be given. About the centre O, with the interval OH, describe another circle, and the right line that touches this circle, and passes through the concourse of the legs CK, BK, when the first legs CP, BP meet in the fourth given point, will be the ruler MN, by means of which the trajectory may be described Whence also on the other hand a trapezium given in kind (excepting a few cases that are impossible) may be inscribed in a given conic section.

There are also other Lemmas, by the help of which trajectories given in kind may be described through given points, and touching given lines. Of such a sort is this, that if a right line is drawn through any point given by position, that may cut a given conic section in two points, and the distance of the intersections is bisected, the point of bisection will touch another conic section of the same kind with the former, and having its axes parallel to the axes of the former. But I hasten to things of greater use.

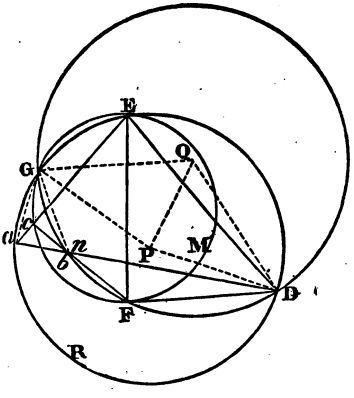
LEMMA XXVI.

To place the three angles of a triangle, given both in kind and magnitude, in respect of as many right lines given by position, provided they are not all parallel among themselves, in such manner that the several angles may touch the several lines.



Three indefinite right lines AB, AC, BC, are given by position, and it is required so to place the triangle DEF that its angle D may touch the line AB, its angle E the line AC, and its angle F the line BC. Upon DE, DF, and EF, describe three segments of circles DRE, DGF, EMF, capable of angles equal to the angles BAC, ABC, ACB respectively. But those segments are to be described towards such sides of the lines DE, DF, EF, that the letters

DRED may turn round about in the same order with the letters BACB; the letters DGFD in the same order with the letters ABCA; and the letters EMFE in the same order with the letters ACBA; then, completing those segments into entire circles let the two former circles cut one the other in G, and suppose P and Q to be their centres. Then joining GP, PQ, take G*a* to AB as GP is to PQ; and about the centre G, with the interval G*a*, describe a circle that may cut the first circle DGE in *a*. Join *a*D cutting the second circle DFG in *b*, as well as *a*E cutting the third circle EMF in *c*. Complete the figure ABC*def* similar and equal to the figure *abc*DEF: I say, the thing is done.



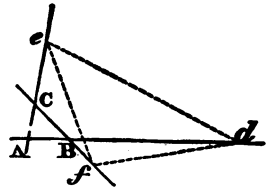
For drawing F*c* meeting *a*D in *n*, and joining *a*G, *b*G, QG, QD, PD, by construction the angle E*a*D is equal to the angle CAB, and the angle *ac*F equal to the angle ACB; and therefore the triangle *anc* equiangular to the triangle ABC. Wherefore the angle *anc* or F*n*D is equal to the angle ABC, and consequently to the angle F*b*D; and therefore the point *n* falls on the point *b*. Moreover the angle GPQ, which is half the angle GPD at the centre, is equal to the angle G*a*D at the circumference; and the angle GQP, which is half the angle GQD at the centre, is equal to the complement to two right angles of the angle G*b*D at the circumference, and therefore equal to the angle G*ba*. Upon which account the triangles GPQ, G*ab*, are similar, and G*a* is to *ab* as GP to PQ; that is (by construction), as G*a* to AB. Wherefore *ab* and AB are equal; and consequently the triangles *abc*, ABC, which we have now proved to be similar, are also equal. And therefore since the angles D, E, F, of the triangle DEF do respectively touch the sides *ab*, *ac*, *bc* of the triangle *abc*, the figure ABC*def* may be completed similar and equal to the figure *abc*DEF, and by completing it the Problem will be solved. Q.E.F.

COR. Hence a right line may be drawn whose parts given in length may be intercepted between three right lines given by position. Suppose the triangle DEF, by the access of its point D to the side EF, and by having the sides DE, DF placed *in directum* to be changed into a right line whose given part DE is to be interposed between the right lines AB, AC given by position; and its given part DF is to be interposed between the right lines AB, BC, given by position; then, by applying the preceding construction to this case, the Problem will be solved.

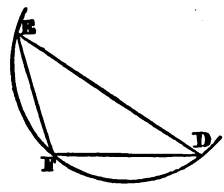
PROPOSITION XXVIII. PROBLEM XX.

To describe a trajectory given both in kind and magnitude, given parts of which shall be interposed between three right lines given by position.

Suppose a trajectory is to be described that may be similar and equal to the curve line DEF, and may be cut by three right lines AB, AC, BC, given by position, into parts DE and EF, similar and equal to the given parts of this curve line.



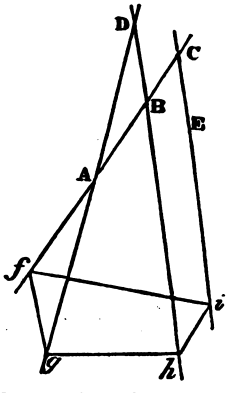
Draw the right lines DE, EF, DF: and place the angles D, E, F, of this triangle DEF, so as to touch those right lines given by position (by Lem. XXVI). Then about the triangle describe the trajectory, similar and equal to the curve DEF. Q.E.F.

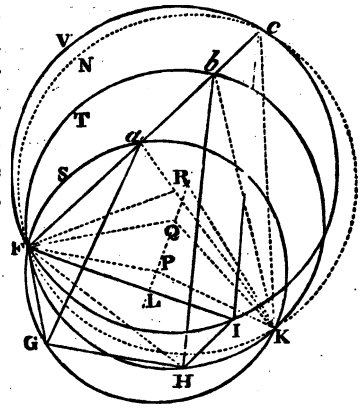


LEMMA XXVII.

To describe a trapezium given in kind, the angles whereof may be so placed, in respect of four right lines given by position, that are neither all parallel among themselves, nor converge to one common point, that the several angles may touch the several lines.

Let the four right lines ABC, AD, BD, CE, be given by position; the first cutting the second in A, the third in B, and the fourth in C; and suppose a trapezium *fghi* is to be described that may be similar to the trapezium FGHI, and whose angle *f*, equal to the given angle F, may touch the right line ABC; and the other angles *g*, *h*, *i*, equal to the other given angles, G, H, I, may touch the other lines AD, BD, CE, respectively. Join FH, and upon FG, FH, FI describe as many segments of circles FSG, FTH, FVI, the first of which FSG may be capable of an angle equal to the angle BAD; the second FTH capable of an angle equal to the angle CBD; and the third FVI of an angle equal to the angle ACE. But the segments are to be described towards those sides of the lines FG, FH, FI, that the circular order of the letters FSGF may be the same as of the letters BADB, and that the letters FTHF may turn about in the same order as the letters CBDC and the letters FVIF in the same order as the letters ACEA. Complete the segments into entire circles, and let P be the centre of the first circle FSG, Q the centre of the second FTH. Join and produce both ways the line PQ, and in it take QR in the same ratio to PQ as BC has to AB. But QR is to be taken towards that side of the point Q, that the order of the letters P, Q, R

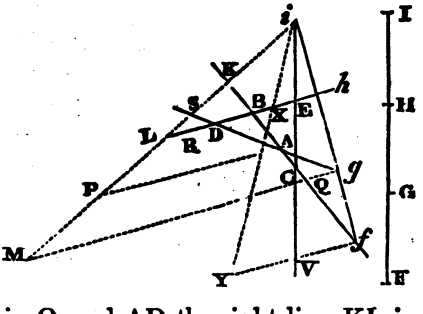




may be the same as of the letters A, B, C; and about the centre R with the interval RF describe a fourth circle FN*c* cutting the third circle FVI in *c*. Join F*c* cutting the first circle in *a*, and the second in *b*. Draw *a*G, *b*H, *c*I, and let the figure ABC*fghi* be made similar to the figure *abc*FGHI; and the trapezium *fghi* will be that which was required to be described.

For let the two first circles FSG, FTH cut one the other in K; join PK, QK, RK, *a*K, *b*K, *c*K, and produce QP to L. The angles F*a*K, F*b*K, F*c*K at the circumferences are the halves of the angles FPK, FQK, FRK, at the centres, and therefore equal to LPK, LQK, LRK, the halves of those angles. Wherefore the figure PQRK is equiangular and similar to the figure *abc*K, and consequently *ab* is to *bc* as PQ to QR, that is, as AB to BC. But by construction, the angles *f*Ag, *f*B*h*, *f*C*i*, are equal to the angles F*a*G, F*b*H, F*c*I. And therefore the figure ABC*fghi* may be completed similar to the figure *abc*FGHI. Which done a trapezium *fghi* will be constructed similar to the trapezium FGHI, and which by its angles *f*, *g*, *h*, *i* will touch the right lines ABC, AD, BD, CE. Q.E.F.

COR. Hence a right line may be drawn whose parts intercepted in a given order, between four right lines given by position, shall have a given proportion among themselves. Let the angles FGH, GHI, be so far increased that the right lines FG, GH, HI, may lie *in directum*; and by constructing the Problem in this case, a right line *fghi* will be drawn, whose parts *fg*, *gh*, *hi*, intercepted between the four right lines given by position, AB and AD, AD and BD, BD and CE, will be one to another as the lines FG, GH, HI, and will observe the same order among themselves. But the same thing may be more readily done in this manner.



Produce AB to K and BD to L, so as BK may be to AB as HI to GH; and DL to BD as GI to FG; and join KL meeting the right line CE in *i*. Produce *i*L to M, so as LM may be to *i*L as GH to HI; then draw MQ parallel to LB, and meeting the right line AD in *g*, and join *gi* cutting AB, BD in *f*, *h*; I say, the thing is done.

For let M*g* cut the right line AB in Q, and AD the right line KL in

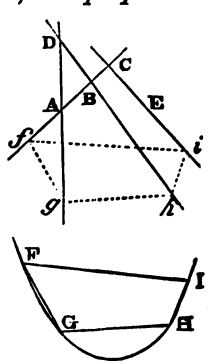
S, and draw AP parallel to BD, and meeting *i*L in P, and *g*M to L*h* (*g*S to *hi*, M*i* to L*i*, GI to HI, AK to BK) and AP to BL, will be in the same ratio. Cut DL in R, so as DL to RL may be in that same ratio; and because *g*S to *g*M, AS to AP, and DS to DL are proportional; therefore (*ex æquo*) as *g*S to L*h*, so will AS be to BL, and DS to RL; and mixtly, BL — RL to L*h* — BL, as AS — DS to *g*S — AS. That is, BR is to B*h* as AD is to A*g*, and therefore as BD to *g*Q. And alternately BR is to BD as B*h* to *g*Q, or as *fh* to *fg*. But by construction the line BL was cut in D and R in the same ratio as the line FI in G and H; and therefore BR is to BD as FH to FG. Wherefore *fh* is to *fg* as FH to FG. Since, therefore, *gi* to *hi* likewise is as M*i* to L*i*, that is, as GI to HI, it is manifest that the lines FI, *fi*, are similarly cut in G and H, *g* and *h*. Q.E.F.

In the construction of this Corollary, after the line LK is drawn cutting CE in *i*, we may produce *i*E to V, so as EV may be to E*i* as FH to HI, and then draw V*f* parallel to BD. It will come to the same, if about the centre *i* with an interval IH, we describe a circle cutting BD in X, and produce *i*X to Y so as *i*Y may be equal to IF, and then draw Y*f* parallel to BD.

Sir Christopher Wren and Dr. Wallis have long ago given other solutions of this Problem.

PROPOSITION XXIX. PROBLEM XXI.

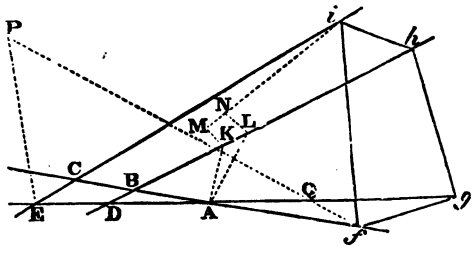
To describe a trajectory given in kind, that may be cut by four right lines given by position, into parts given in order, kind, and proportion.



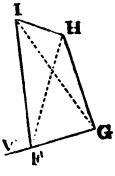
Suppose a trajectory is to be described that may be similar to the curve line FGHI, and whose parts, similar and proportional to the parts FG, GH, HI of the other, may be intercepted between the right lines AB and AD, AD, and BD, BD and CE given by position, viz., the first between the first pair of those lines, the second between the second, and the third between the third. Draw the right lines FG, GH, HI, FI; and (by Lem. XXVII) describe a trapezium *fghi* that may be similar to the trapezium FGHI, and whose angles *f*, *g*, *h*, *i*, may touch the right lines given by position AB, AD, BD, CE, severally according to their order. And then about this trapezium describe a trajectory, that trajectory will be similar to the curve line FGHI.

SCHOLIUM.

This problem may be likewise constructed in the following manner. Joining FG, GH, HI, FI, produce GF to V, and join FH, IG, and make



the angles CAK, DAL equal to the angles FGH, VFH. Let AK, AL meet the right line BD in K and L, and thence draw KM, LN, of which let KM make the angle AKM equal to the angle GHI, and be itself to AK as HI is to GH; and let LN make the angle ALN equal to the angle FHI, and be itself to AL as HI to FH. But AK, KM, AL, LN are to be drawn towards those sides of the lines AD, AK, AL, that the letters CAKMC, ALKA, DALND may be carried round in the same order as the letters FGHIF; and draw MN meeting the right line CE in *i*. Make the angle *i*EP equal to the angle IGF, and let PE be to E*i* as FG to GI; and through P draw PQ*f* that may with the right line ADE contain an angle PQE equal to the angle FIG, and may meet the right line AB in *f*, and join *fi*. But PE and PQ are to be drawn towards those sides of the lines CE, PE, that the circular order of the letters PE*i*P and PEQP may be the same as of the letters FGHIF; and if upon the line *fi*, in the same order of letters, and similar to the trapezium FGHI, a trapezium *fghi* is constructed, and a trajectory given in kind is circumscribed about it, the Problem will be solved.



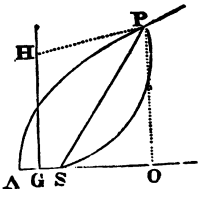
So far concerning the finding of the orbits. It remains that we determine the motions of bodies in the orbits so found.

SECTION VI.

How the motions are to be found in given orbits.

PROPOSITION XXX. PROBLEM XXII.

To find at any assigned time the place of a body moving in a given parabolic trajectory.



Let S be the focus, and A the principal vertex of the parabola; and suppose 4AS × M equal to the parabolic area to be cut off APS, which either was described by the radius SP, since the body's departure from the vertex, or is to be described thereby before its arrival there. Now the quantity of that area to be cut off is known from the time which is proportional to it. Bisect AS in G, and erect the perpendicular GH equal to 3M, and a circle described about the centre H, with the interval HS, will cut the parabola in the place P required. For letting fall PO perpendicular on the axis, and drawing PH, there will be $AG^2 + GH^2 (= HP^2 = \overline{AO - AG}^2 + \overline{PO - GH}^2) = AO^2 + PO^2 - 2GA \cdot O - 2GH + PO +$

$AG^2 + GH^2$. Whence $2GH \times PO (= AO^2 + PO^2 - 2GAO) = AO^2 + \frac{3}{4} PO^2$. For AO^2 write $AO \times \frac{PO^2}{4AS}$; then dividing all the terms by $2PO$, and multiplying them by $2AS$, we shall have $\frac{1}{3}GH \times AS (= \frac{1}{6}AO \times PO + \frac{1}{2}AS \times PO = \frac{AO + 3AS}{6} \times PO = \frac{4AO - 3SO}{6} \times PO =$ to the area $\overline{APO - SPO}|$ = to the area APS. But GH was 3M, and therefore $\frac{1}{3}GH \times AS$ is $4AS \times M$. Wherefore the area cut off APS is equal to the area that was to be cut off $4AS \times M$. Q.E.D.

Cor. 1. Hence GH is to AS as the time in which the body described the arc AP to the time in which the body described the arc between the vertex A and the perpendicular erected from the focus S upon the axis.

Cor. 2. And supposing a circle ASP perpetually to pass through the moving body P, the velocity of the point H is to the velocity which the body had in the vertex A as 3 to 8; and therefore in the same ratio is the line GH to the right line which the body, in the time of its moving from A to P, would describe with that velocity which it had in the vertex A.

Cor. 3. Hence, also, on the other hand, the time may be found in which the body has described any assigned arc AP. Join AP, and on its middle point erect a perpendicular meeting the right line GH in H.

LEMMA XXVIII.

There is no oval figure whose area, cut off by right lines at pleasure, can be universally found by means of equations of any number of finite terms and dimensions.

Suppose that within the oval any point is given, about which as a pole a right line is perpetually revolving with an uniform motion, while in that right line a moveable point going out from the pole moves always forward with a velocity proportional to the square of that right line within the oval. By this motion that point will describe a spiral with infinite circumgyrations. Now if a portion of the area of the oval cut off by that right line could be found by a finite equation, the distance of the point from the pole, which is proportional to this area, might be found by the same equation, and therefore all the points of the spiral might be found by a finite equation also; and therefore the intersection of a right line given in position with the spiral might also be found by a finite equation. But every right line infinitely produced cuts a spiral in an infinite number of points; and the equation by which any one intersection of two lines is found at the same time exhibits all their intersections by as many roots, and therefore rises to as many dimensions as there are intersections. Because two circles mutually cut one another in two points, one of those in-

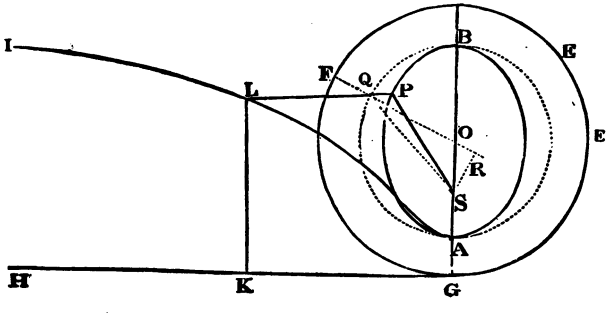
tersections is not to be found but by an equation of two dimensions, by which the other intersection may be also found. Because there may be four intersections of two conic sections, any one of them is not to be found universally, but by an equation of four dimensions, by which they may be all found together. For if those intersections are severally sought, because the law and condition of all is the same, the calculus will be the same in every case, and therefore the conclusion always the same, which must therefore comprehend all those intersections at once within itself, and exhibit them all indifferently. Hence it is that the intersections of the conic sections with the curves of the third order, because they may amount to six, come out together by equations of six dimensions; and the intersections of two curves of the third order, because they may amount to nine, come out together by equations of nine dimensions. If this did not necessarily happen, we might reduce all solid to plane Problems, and those higher than solid to solid Problems. But here I speak of curves irreducible in power. For if the equation by which the curve is defined may be reduced to a lower power, the curve will not be one single curve, but composed of two, or more, whose intersections may be severally found by different calculusses. After the same manner the two intersections of right lines with the conic sections come out always by equations of two dimensions; the three intersections of right lines with the irreducible curves of the third order by equations of three dimensions; the four intersections of right lines with the irreducible curves of the fourth order, by equations of four dimensions; and so on *in infinitum*. Wherefore the innumerable intersections of a right line with a spiral, since this is but one simple curve and not reducible to more curves, require equations infinite in number of dimensions and roots, by which they may be all exhibited together. For the law and calculus of all is the same. For if a perpendicular is let fall from the pole upon that intersecting right line, and that perpendicular together with the intersecting line revolves about the pole, the intersections of the spiral will mutually pass the one into the other; and that which was first or nearest, after one revolution, will be the second; after two, the third; and so on: nor will the equation in the mean time be changed but as the magnitudes of those quantities are changed, by which the position of the intersecting line is determined. Wherefore since those quantities after every revolution return to their first magnitudes, the equation will return to its first form; and consequently one and the same equation will exhibit all the intersections, and will therefore have an infinite number of roots, by which they may be all exhibited. And therefore the intersection of a right line with a spiral cannot be universally found by any finite equation; and of consequence there is no oval figure whose area, cut off by right lines at pleasure, can be universally exhibited by any such equation.

By the same argument, if the interval of the pole and point by which the spiral is described is taken proportional to that part of the perimeter of the oval which is cut off, it may be proved that the length of the perimeter cannot be universally exhibited by any finite equation. But here I speak of ovals that are not touched by conjugate figures running out *in infinitum*.

Cor. Hence the area of an ellipsis, described by a radius drawn from the focus to the moving body, is not to be found from the time given by a finite equation; and therefore cannot be determined by the description of curves geometrically rational. Those curves I call geometrically rational, all the points whereof may be determined by lengths that are definable by equations; that is, by the complicated ratios of lengths. Other curves (such as spirals, quadratrixes, and cycloids) I call geometrically irrational. For the lengths which are or are not as number to number (according to the tenth Book of Elements) are arithmetically rational or irrational. And therefore I cut off an area of an ellipsis proportional to the time in which it is described by a curve geometrically irrational, in the following manner.

PROPOSITION XXXI. PROBLEM XXIII.

To find the place of a body moving in a given elliptic trajectory at any assigned time.

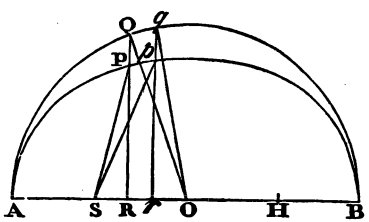


Suppose A to be the principal vertex, S the focus, and O the centre of the ellipsis APB; and let P be the place of the body to be found. Produce OA to G so as OG may be to OA as OA to OS. Erect the perpendicular GH; and about the centre O, with the interval OG, describe the circle GEF; and on the ruler GH, as a base, suppose the wheel GEF to move forwards, revolving about its axis, and in the mean time by its point A describing the cycloid ALI. Which done, take GK to the perimeter GEFG of the wheel, in the ratio of the time in which the body proceeding from A described the arc AP, to the time of a whole revolution in the ellipsis. Erect the perpendicular KL meeting the cycloid in L; then LP drawn parallel to KG will meet the ellipsis in P, the required place of the body.

For about the centre O with the interval OA describe the semi-circle AQB, and let LP, produced, if need be, meet the arc AQ in Q, and join

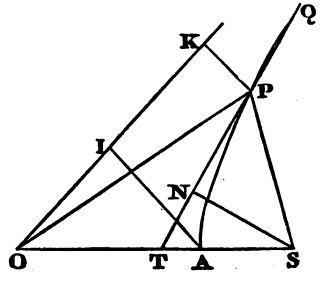
SQ, OQ. Let OQ meet the arc EFG in F, and upon OQ let fall the perpendicular SR. The area APS is as the area AQS, that is, as the difference between the sector OQA and the triangle OQS, or as the difference of the rectangles $\frac{1}{2}$OQ × AQ, and $\frac{1}{2}$OQ × SR, that is, because $\frac{1}{2}$OQ is given, as the difference between the arc AQ and the right line SR; and therefore (because of the equality of the given ratios SR to the sine of the arc AQ, OS to OA, OA to OG, AQ to GF; and by division, AQ—SR to GF — sine of the arc AQ) as GK, the difference between the arc GF and the sine of the arc AQ. Q.E.D.

SCHOLIUM.

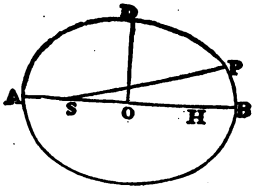


But since the description of this curve is difficult, a solution by approximation will be preferable. First, then, let there be found a certain angle B which may be to an angle of 57,29578 degrees, which an arc equal to the radius subtends, as SH, the distance of the foci, to AB, the diameter of the ellipsis. Secondly, a certain length L, which may be to the radius in the same ratio inversely. And these being found, the Problem may be solved by the following analysis. By any construction (or even by conjecture), suppose we know P the place of the body near its true place *p*. Then letting fall on the axis of the ellipsis the ordinate PR from the proportion of the diameters of the ellipsis, the ordinate RQ of the circumscribed circle AQB will be given; which ordinate is the sine of the angle AOQ, supposing AO to be the radius, and also cuts the ellipsis in P. It will be sufficient if that angle is found by a rude calculus in numbers near the truth. Suppose we also know the angle proportional to the time, that is, which is to four right angles as the time in which the body described the arc A*p*, to the time of one revolution in the ellipsis. Let this angle be N. Then take an angle D, which may be to the angle B as the sine of the angle AOQ to the radius; and an angle E which may be to the angle N — AOQ + D as the length L to the same length L diminished by the cosine of the angle AOQ, when that angle is less than a right angle, or increased thereby when greater. In the next place, take an angle F that may be to the angle B as the sine of the angle AOQ + E to the radius, and an angle G, that may be to the angle N — AOQ — E + F as the length L to the same length L diminished by the cosine of the angle AOQ + E, when that angle is less than a right angle, or increased thereby when greater. For the third time take an angle H, that may be to the angle B as the sine of the angle AOQ + E + G to the radius; and an angle I to the angle N — AOQ — E — G + H, as the

length L is to the same length L diminished by the cosine of the angle AOQ + E + G, when that angle is less than a right angle, or increased thereby when greater. And so we may proceed *in infinitum*. Lastly, take the angle AO*q* equal to the angle AOQ + E + G + I +, &c. and from its cosine O*r* and the ordinate *pr*, which is to its sine *qr* as the lesser axis of the ellipsis to the greater, we shall have *p* the correct place of the body. When the angle N — AOQ + D happens to be negative, the sign + of the angle E must be every where changed into —, and the sign — into +. And the same thing is to be understood of the signs of the angles G and I, when the angles N — AOQ — E + F, and N — AOQ — E — G + H come out negative. But the infinite series AOQ + E + G + I +, &c. converges so very fast, that it will be scarcely ever needful to proceed beyond the second term E. And the calculus is founded upon this Theorem, that the area APS is as the difference between the arc AQ and the right line let fall from the focus S perpendicularly upon the radius OQ.



And by a calculus not unlike, the Problem is solved in the hyperbola. Let its centre be O, its vertex A, its focus S, and asymptote OK; and suppose the quantity of the area to be cut off is known, as being proportional to the time. Let that be A, and by conjecture suppose we know the position of a right line SP, that cuts off an area APS near the truth. Join OP, and from A and P to the asymptote draw AI, PK parallel to the other asymptote; and by the table of logarithms the area AIKP will be given, and equal thereto the area OPA, which subducted from the triangle OPS, will leave the area cut off APS. And by applying 2APS — 2A, or 2A — 2APS, the double difference of the area A that was to be cut off, and the area APS that is cut off, to the line SN that is let fall from the focus S, perpendicular upon the tangent TP, we shall have the length of the chord PQ. Which chord PQ is to be inscribed between A and P, if the area APS that is cut off be greater than the area A that was to be cut off, but towards the contrary side of the point P, if otherwise: and the point Q will be the place of the body more accurately. And by repeating the computation the place may be found perpetually to greater and greater accuracy.



And by such computations we have a general analytical resolution of the Problem. But the particular calculus that follows is better fitted for astronomical purposes. Supposing AO, OB, OD, to be the semi-axis of the ellipsis, and L its latus rectum, and D the difference betwixt the lesser semi-

axis OD, and $\frac{1}{2}$L the half of the latus rectum: let an angle Y be found, whose sine may be to the radius as the rectangle under that difference D, and AO + OD the half sum of the axes to the square of the greater axis AB. Find also an angle Z, whose sine may be to the radius as the double rectangle under the distance of the foci SH and that difference D to triple the square of half the greater semi-axis AO. Those angles being once found, the place of the body may be thus determined. Take the angle T proportional to the time in which the arc BP was described, or equal to what is called the mean motion; and an angle V the first equation of the mean motion to the angle Y, the greatest first equation, as the sine of double the angle T is to the radius; and an angle X, the second equation, to the angle Z, the second greatest equation, as the cube of the sine of the angle T is to the cube of the radius. Then take the angle BHP the mean motion equated equal to T + X + V, the sum of the angles T, V, X, if the angle T is less than a right angle; or equal to T + X — V, the difference of the same, if that angle T is greater than one and less than two right angles; and if HP meets the ellipsis in P, draw SP, and it will cut off the area BSP nearly proportional to the time.

This practice seems to be expeditious enough, because the angles V and X, taken in second minutes, if you please, being very small, it will be sufficient to find two or three of their first figures. But it is likewise sufficiently accurate to answer to the theory of the planet's motions. For even in the orbit of Mars, where the greatest equation of the centre amounts to ten degrees, the error will scarcely exceed one second. But when the angle of the mean motion equated BHP is found, the angle of the true motion BSP, and the distance SP, are readily had by the known methods.

And so far concerning the motion of bodies in curve lines. But it may also come to pass that a moving body shall ascend or descend in a right line; and I shall now go on to explain what belongs to such kind of motions.

SECTION VII.

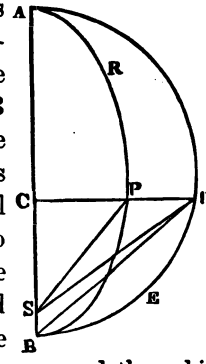
Concerning the rectilinear ascent and descent of bodies.

PROPOSITION XXXII. PROBLEM XXIV.

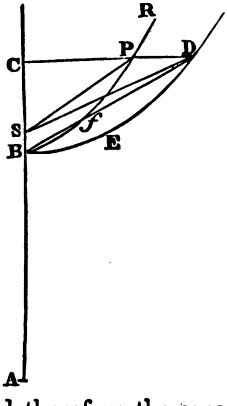
Supposing that the centripetal force is reciprocally proportional to the square of the distance of the places from the centre; it is required to define the spaces which a body, falling directly, describes in given times.

CASE 1. If the body does not fall perpendicularly, it will (by Cor. 1

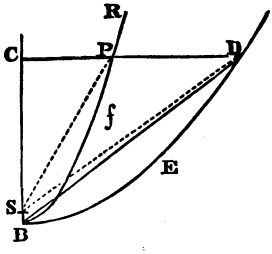
Prop. XIII) describe some conic section whose focus is placed in the centre of force. Suppose that conic section to be ARPB and its focus S. And, first, if the figure be an ellipsis, upon the greater axis thereof AB describe the semi-circle ADB, and let the right line DPC pass through the falling body, making right angles with the axis; and drawing DS, PS, the area ASD will be proportional to the area ASP, and therefore also to the time. The axis AB still remaining the same, let the breadth of the ellipsis be perpetually diminished, and the area ASD will always remain proportional to the time. Suppose that breadth to be diminished *in infinitum;* and the orbit APB in that case coinciding with the axis AB, and the focus S with the extreme point of the axis B, the body will descend in the right line AC, and the area ABD will become proportional to the time. Wherefore the space AC will be given which the body describes in a given time by its perpendicular fall from the place A, if the area ABD is taken proportional to the time, and from the point D the right line DC is let fall perpendicularly on the right line AB. Q.E.I.



Case 2. If the figure RPB is an hyperbola, on the same principal diameter AB describe the rectangular hyperbola BED; and because the areas CSP, CB*f*P, SP*f*B, are severally to the several areas CSD, CBED, SDEB, in the given ratio of the heights CP, CD, and the area SP*f*B is proportional to the time in which the body P will move through the arc P*f*B, the area SDEB will be also proportional to that time. Let the latus rectum of the hyperbola RPB be diminished *in infinitum*, the latus transversum remaining the same; and the arc PB will come to coincide with the right line CB, and the focus S, with the vertex B, and the right line SD with the right line BD. And therefore the area BDEB will be proportional to the time in which the body C, by its perpendicular descent, describes the line CB. Q.E.I.

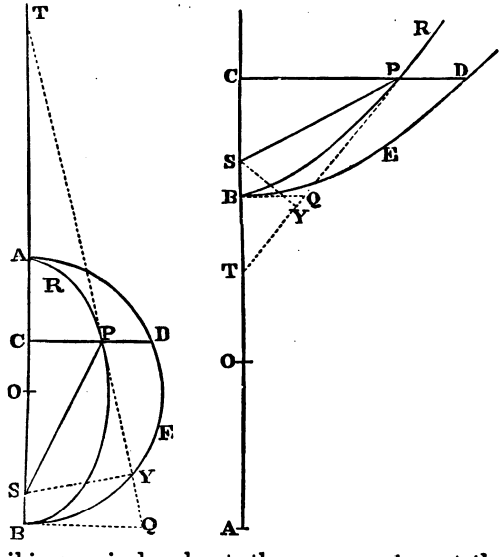


Case 3. And by the like argument, if the figure RPB is a parabola, and to the same principal vertex B another parabola BED is described, that may always remain given while the former parabola in whose perimeter the body P moves, by having its latus rectum diminished and reduced to nothing, comes to coincide with the line CB, the parabolic segment BDEB will be proportional to the time in which that body P or C will descend to the centre S or B Q.E.I



PROPOSITION XXXIII. THEOREM IX.

The things above found being supposed, I say, that the velocity of a falling body in any place C *is to the velocity of a body, describing a circle about the centre* B *at the distance* BC, *in the subduplicate ratio of* AC, *the distance of the body from the remoter vertex* A *of the circle or rectangular hyperbola, to* ½AB, *the principal semi-diameter of the figure.*



Let AB, the common diameter of both figures RPB, DEB, be bisected in O; and draw the right line PT that may touch the figure RPB in P, and likewise cut that common diameter AB (produced, if need be) in T; and let SY be perpendicular to this line, and BQ to this diameter, and suppose the latus rectum of the figure RPB to be L. From Cor. 9, Prop. XVI, it is manifest that the velocity of a body, moving in the line RPB about the centre S, in any place P, is to the velocity of a body describing a circle about the same centre, at the distance SP, in the subduplicate ratio of the rectangle ½L × SP to SY^2. For by the properties of the conic sections ACB is to CP^2 as 2AO to L, and therefore $\frac{2CP^2 \times AO}{ACB}$ is equal to L. Therefore those velocities are to each other in the subduplicate ratio of $\frac{CP^2 \times AO \times SP}{ACB}$ to SY^2. Moreover, by the properties of the conic sections, CO is to BO as BO to TO, and (by composition or division) as CB to BT. Whence (by division or composition) BO — or + CO will be to BO as CT to BT, that is, AC will be to AO as CP to BQ; and therefore $\frac{CP^2 \times AO \times SP}{ACB}$ is equal to $\frac{BQ^2 \times AC \times SP}{AO \times BC}$. Now suppose CP, the breadth of the figure RPB, to be diminished *in infinitum*, so as the point P may come to coincide with the point C, and the point S with the point B, and the line SP with the line BC, and the line SY with the line BQ; and the velocity of the body now descending perpendicularly in the line CB will be to the velocity of

a body describing a circle about the centre B, at the distance BC, in the subduplicate ratio of $\frac{BQ^2 \times AC \times SP}{AO \times BC}$ to SY^2, that is (neglecting the ratios of equality of SP to BC, and BQ^2 to SY^2), in the subduplicate ratio of AC to AO, or ½AB. Q.E.D.

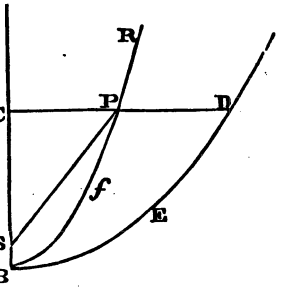
Cor. 1. When the points B and S come to coincide, TC will become to TS as AC to AO.

Cor. 2. A body revolving in any circle at a given distance from the centre, by its motion converted upwards, will ascend to double its distance from the centre.

PROPOSITION XXXIV. THEOREM X.

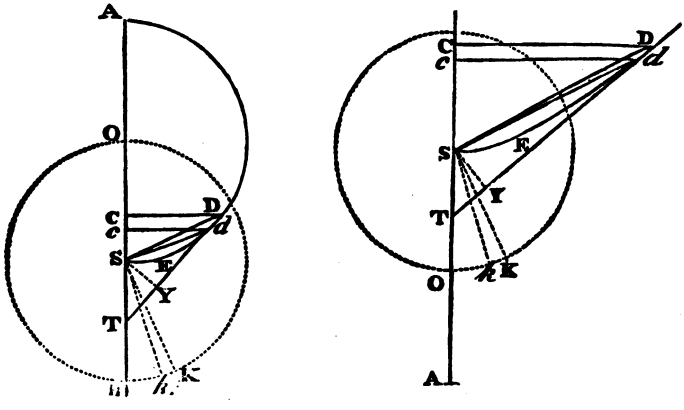
If the figure BED *is a parabola, I say, that the velocity of a falling body in any place* C *is equal to the velocity by which a body may uniformly describe a circle about the centre* B *at half the interval* BC.

For (by Cor. 7, Prop. XVI) the velocity of a body describing a parabola RPB about the centre S, in any place P, is equal to the velocity of a body uniformly describing a circle about the same centre S at half the interval SP. Let the breadth CP of the parabola be diminished *in infinitum*, so as the parabolic arc P*f*B may come to coincide with the right line CB, the centre S with the vertex B, and the interval SP with the interval BC, and the proposition will be manifest. Q.E.D.



PROPOSITION XXXV. THEOREM XI.

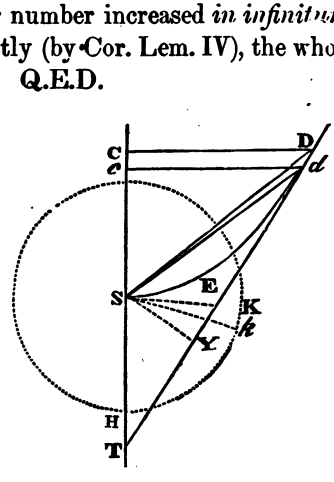
The same things supposed, I say, that the area of the figure DES, *described by the indefinite radius* SD, *is equal to the area which a body with a radius equal to half the latus rectum of the figure* DES, *by uniformly revolving about the centre* S, *may describe in the same time.*



For suppose a body C in the smallest moment of time describes in falling the infinitely little line C*c*, while another body K, uniformly revolving about the centre S in the circle OK*k*, describes the arc K*k*. Erect the perpendiculars CD, *cd*, meeting the figure DES in D, *d*. Join SD, S*d*, SK, S*k*, and draw D*d* meeting the axis AS in T, and thereon let fall the perpendicular SY.

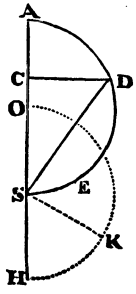
CASE 1. If the figure DES is a circle, or a rectangular hyperbola, bisect its transverse diameter AS in O, and SO will be half the latus rectum. And because TC is to TD as C*c* to D*d*, and TD to TS as CD to SY; *ex æquo* TC will be to TS as CD × C*c* to SY × D*d*. But (by Cor. 1, Prop. XXXIII) TC is to TS as AC to AO; to wit, if in the coalescence of the points D, *d*, the ultimate ratios of the lines are taken. Wherefore AC is to AO or SK as CD × C*c* to SY × D*d*. Farther, the velocity of the descending body in C is to the velocity of a body describing a circle about the centre S, at the interval SC, in the subduplicate ratio of AC to AO or SK (by Prop. XXXIII); and this velocity is to the velocity of a body describing the circle OK*k* in the subduplicate ratio of SK to SC (by Cor. 6, Prop IV); and, *ex æquo*, the first velocity to the last, that is, the little line C*c* to the arc K*k*, in the subduplicate ratio of AC to SC, that is, in the ratio of AC to CD. Wherefore CD × C*c* is equal to AC × K*k*, and consequently AC to SK as AC × K*k* to SY × D*d*, and thence SK × K*k* equal to SY × D*d*, and ½SK × K*k* equal to ½SY × D*d*, that is, the area KS*k* equal to the area SD*d*. Therefore in every moment of time two equal particles, KS*k* and SD*d*, of areas are generated, which, if their magnitude is diminished, and their number increased *in infinitum*, obtain the ratio of equality, and consequently (by Cor. Lem. IV), the whole areas together generated are always equal. Q.E.D.

CASE 2. But if the figure DES is a parabola, we shall find, as above, CD × C*c* to SY × D*d* as TC to TS, that is, as 2 to 1; and that therefore ½CD × C*c* is equal to ½ SY × D*d*. But the velocity of the falling body in C is equal to the velocity with which a circle may be uniformly described at the interval ½SC (by Prop. XXXIV). And this velocity to the velocity with which a circle may be described with the radius SK, that is, the little line C*c* to the arc K*k*, is (by Cor. 6, Prop. IV) in the subduplicate ratio of SK to ½SC; that is, in the ratio of SK to ½CD. Wherefore ½SK × K*k* is equal to ½CD × C*c*, and therefore equal to ½SY × D*d*; that is, the area KS*k* is equal to the area SD*d*, as above. Q.E.D.



PROPOSITION XXXVI. PROBLEM XXV.

To determine the times of the descent of a body falling from a given place A.

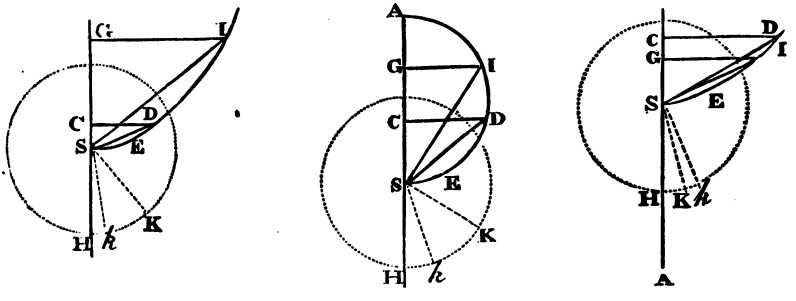


Upon the diameter AS, the distance of the body from the centre at the beginning, describe the semi-circle ADS, as likewise the semi-circle OKH equal thereto, about the centre S. From any place C of the body erect the ordinate CD. Join SD, and make the sector OSK equal to the area ASD. It is evident (by Prop. XXXV) that the body in falling will describe the space AC in the same time in which another body, uniformly revolving about the centre S, may describe the arc OK. Q.E.F.

PROPOSITION XXXVII. PROBLEM XXVI.

To define the times of the ascent or descent of a body projected upwards or downwards from a given place.

Suppose the body to go off from the given place G, in the direction of the line GS, with any velocity. In the duplicate ratio of this velocity to the uniform velocity in a circle, with which the body may revolve about



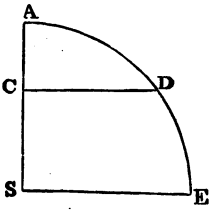
the centre S at the given interval SG, take GA to ½AS. If that ratio is the same as of the number 2 to 1, the point A is infinitely remote; in which case a parabola is to be described with any latus rectum to the vertex S, and axis SG; as appears by Prop. XXXIV. But if that ratio is less or greater than the ratio of 2 to 1, in the former case a circle, in the latter a rectangular hyperbola, is to be described on the diameter SA; as appears by Prop. XXXIII. Then about the centre S, with an interval equal to half the latus rectum, describe the circle H*k*K; and at the place G of the ascending or descending body, and at any other place C, erect the perpendiculars GI, CD, meeting the conic section or circle in I and D. Then joining SI, SD, let the sectors HSK, HS*k* be made equal to the segments SEIS, SEDS, and (by Prop. XXXV) the body G will describe

the space GC in the same time in which the body K may describe the arc K*k*. Q.E.F.

PROPOSITION XXXVIII. THEOREM XII.

Supposing that the centripetal force is proportional to the altitude or distance of places from the centre, I say, that the times and velocities of falling bodies, and the spaces which they describe, are respectively proportional to the arcs, and the right and versed sines of the arcs.

Suppose the body to fall from any place A in the right line AS; and about the centre of force S, with the interval AS, describe the quadrant of a circle AE; and let CD be the right sine of any arc AD; and the body A will in the time AD in falling describe the space AC, and in the place C will acquire the velocity CD.



This is demonstrated the same way from Prop. X, as Prop. XXXII was demonstrated from Prop. XI.

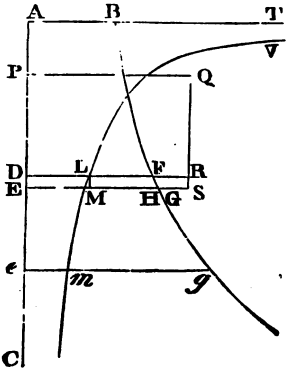
COR. 1. Hence the times are equal in which one body falling from the place A arrives at the centre S, and another body revolving describes the quadrantal arc ADE.

COR. 2. Wherefore all the times are equal in which bodies falling from whatsoever places arrive at the centre. For all the periodic times of revolving bodies are equal (by Cor. 3, Prop. IV).

PROPOSITION XXXIX. PROBLEM XXVII.

Supposing a centripetal force of any kind, and granting the quadratures of curvilinear figures; it is required to find the velocity of a body, ascending or descending in a right line, in the several places through which it passes; as also the time in which it will arrive at any place: and vice versa.

Suppose the body E to fall from any place A in the right line ADEC; and from its place E imagine a perpendicular EG always erected proportional to the centripetal force in that place tending to the centre C; and let BFG be a curve line, the locus of the point G. And in the beginning of the motion suppose EG to coincide with the perpendicular AB; and the velocity of the body in any place E will be as a right line whose square is equal to the curvilinear area ABGE. Q.E.I.



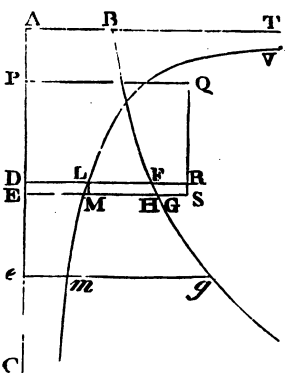
In EG take EM reciprocally proportional to

a right line whose square is equal to the area ABGE, and let VLM be a curve line wherein the point M is always placed, and to which the right line AB produced is an asymptote; and the time in which the body in falling describes the line AE, will be as the curvilinear area ABTVME. Q.E.I.

For in the right line AE let there be taken the very small line DE of a given length, and let DLF be the place of the line EMG, when the body was in D; and if the centripetal force be such, that a right line, whose square is equal to the area ABGE, is as the velocity of the descending body, the area itself will be as the square of that velocity; that is, if for the velocities in D and E we write V and V + I, the area ABFD will be as VV, and the area ABGE as VV + 2VI + II; and by division, the area DFGE as 2VI + II, and therefore $\frac{\text{DFGE}}{\text{DE}}$ will be as $\frac{2\text{VI} + \text{II}}{\text{DE}}$; that is, if we take the first ratios of those quantities when just nascent, the length DF is as the quantity $\frac{2\text{VI}}{\text{DE}}$, and therefore also as half that quantity $\frac{\text{I} \times \text{V}}{\text{DE}}$. But the time in which the body in falling describes the very small line DE, is as that line directly and the velocity V inversely; and the force will be as the increment I of the velocity directly and the time inversely; and therefore if we take the first ratios when those quantities are just nascent, as $\frac{\text{I} \times \text{V}}{\text{DE}}$, that is, as the length DF. Therefore a force proportional to DF or EG will cause the body to descend with a velocity that is as the right line whose square is equal to the area ABGE. Q.E.D.

Moreover, since the time in which a very small line DE of a given length may be described is as the velocity inversely, and therefore also inversely as a right line whose square is equal to the area ABFD; and since the line DL, and by consequence the nascent area DLME, will be as the same right line inversely, the time will be as the area DLME, and the sum of all the times will be as the sum of all the areas; that is (by Cor. Lem. IV), the whole time in which the line AE is described will be as the whole area ATVME. Q.E.D.

COR. 1. Let P be the place from whence a body ought to fall, so as that, when urged by any known uniform centripetal force (such as gravity is vulgarly supposed to be), it may acquire in the place D a velocity equal to the velocity which another body, falling by any force whatever, hath acquired in that place D. In the perpendicular DF let there be taken DR, which may be to DF as that uniform force to the other force in the place D. Complete the rectangle PDRQ, and cut off the area ABFD equal to that rectangle. Then A will be the place



from whence the other body fell. For completing the rectangle DRSE, since the area ABFD is to the area DFGE as VV to 2VI, and therefore as ½V to I, that is, as half the whole velocity to the increment of the velocity of the body falling by the unequable force; and in like manner the area PQRD to the area DRSE as half the whole velocity to the increment of the velocity of the body falling by the uniform force; and since those increments (by reason of the equality of the nascent times) are as the generating forces, that is, as the ordinates DF, DR, and consequently as the nascent areas DFGE, DRSE; therefore, *ex æquo*, the whole areas ABFD, PQRD will be to one another as the halves of the whole velocities; and therefore, because the velocities are equal, they become equal also.

COR. 2. Whence if any body be projected either upwards or downwards with a given velocity from any place D, and there be given the law of centripetal force acting on it, its velocity will be found in any other place, as *e*, by erecting the ordinate *eg*, and taking that velocity to the velocity in the place D as a right line whose square is equal to the rectangle PQRD, either increased by the curvilinear area DF*ge*, if the place *e* is below the place D, or diminished by the same area DF*ge*, if it be higher, is to the right line whose square is equal to the rectangle PQRD alone.

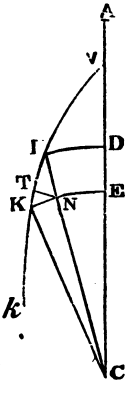
COR. 3. The time is also known by erecting the ordinate *em* reciprocally proportional to the square root of PQRD + or — DF*ge*, and taking the time in which the body has described the line D*e* to the time in which another body has fallen with an uniform force from P, and in falling arrived at D in the proportion of the curvilinear area DL*me* to the rectangle 2PD × DL. For the time in which a body falling with an uniform force hath described the line PD, is to the time in which the same body has described the line PE in the subduplicate ratio of PD to PE; that is (the very small line DE being just nascent), in the ratio of PD to PD + ½DE, or 2PD to 2PD + DE, and, by division, to the time in which the body hath described the small line DE, as 2PD to DE, and therefore as the rectangle 2PD × DL to the area DLME; and the time in which both the bodies described the very small line DE is to the time in which the body moving unequably hath described the line D*e* as the area DLME to the area DL*me*; and, *ex æquo*, the first mentioned of these times is to the last as the rectangle 2PD × DL to the area DL*me*.

SECTION VIII.

Of the invention of orbits wherein bodies will revolve, being acted upon by any sort of centripetal force.

PROPOSITION XL. THEOREM XIII.

If a body, acted upon by any centripetal force, is any how moved, and another body ascends or descends in a right line, and their velocities be equal in any one case of equal altitudes, their velocities will be also equal at all equal altitudes.



Let a body descend from A through D and E, to the centre C; and let another body move from V in the curve line VIK*k*. From the centre C, with any distances, describe the concentric circles DI, EK, meeting the right line AC in D and E, and the curve VIK in I and K. Draw IC meeting KE in N, and on IK let fall the perpendicular NT; and let the interval DE or IN between the circumferences of the circles be very small; and imagine the bodies in D and I to have equal velocities. Then because the distances CD and CI are equal, the centripetal forces in D and I will be also equal. Let those forces be expressed by the equal lineolæ DE and IN; and let the force IN (by Cor. 2 of the Laws of Motion) be resolved into two others, NT and IT. Then the force NT acting in the direction of the line NT perpendicular to the path ITK of the body will not at all affect or change the velocity of the body in that path, but only draw it aside from a rectilinear course, and make it deflect perpetually from the tangent of the orbit, and proceed in the curvilinear path ITK*k*. That whole force, therefore, will be spent in producing this effect; but the other force IT, acting in the direction of the course of the body, will be all employed in accelerating it, and in the least given time will produce an acceleration proportional to itself. Therefore the accelerations of the bodies in D and I, produced in equal times, are as the lines DE, IT (if we take the first ratios of the nascent lines DE, IN, IK, IT, NT); and in unequal times as those lines and the times conjunctly. But the times in which DE and IK are described, are, by reason of the equal velocities (in D and I) as the spaces described DE and IK, and therefore the accelerations in the course of the bodies through the lines DE and IK are as DE and IT, and DE and IK conjunctly; that is, as the square of DE to the rectangle IT into IK. But the rectangle IT × IK is equal to the square of IN, that is, equal to the square of DE; and therefore the accelerations generated in the passage of the bodies from D and I to E and K are equal. Therefore the velocities of the bodies in E and K are also equal. and by the same reasoning they will always be found equal in any subsequent equal distances. Q.E.D.

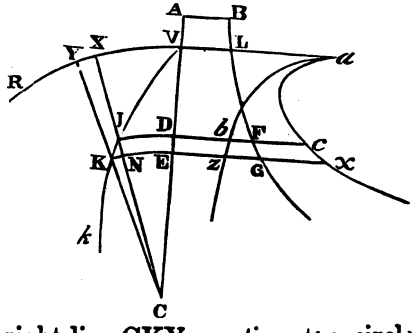
By the same reasoning, bodies of equal velocities and equal distances from the centre will be equally retarded in their ascent to equal distances. Q.E.D.

COR. 1. Therefore if a body either oscillates by hanging to a string, or by any polished and perfectly smooth impediment is forced to move in a curve line; and another body ascends or descends in a right line, and their velocities be equal at any one equal altitude, their velocities will be also equal at all other equal altitudes. For by the string of the pendulous body, or by the impediment of a vessel perfectly smooth, the same thing will be effected as by the transverse force N'T. The body is neither accelerated nor retarded by it, but only is obliged to leave its rectilinear course.

COR. 2. Suppose the quantity P to be the greatest distance from the centre to which a body can ascend, whether it be oscillating, or revolving in a trajectory, and so the same projected upwards from any point of a trajectory with the velocity it has in that point. Let the quantity A be the distance of the body from the centre in any other point of the orbit; and let the centripetal force be always as the power A^{n-1}, of the quantity A, the index of which power $n-1$ is any number n diminished by unity. Then the velocity in every altitude A will be as $\sqrt{P^n - A^n}$, and therefore will be given. For by Prop. XXXIX, the velocity of a body ascending and descending in a right line is in that very ratio.

PROPOSITION XLI. PROBLEM XXVIII.

Supposing a centripetal force of any kind, and granting the quadratures of curvilinear figures, it is required to find as well the trajectories in which bodies will move, as the times of their motions in the trajectories found.



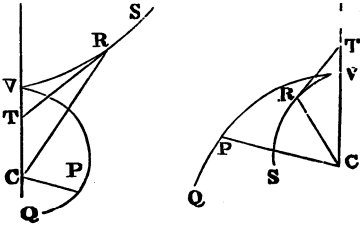
Let any centripetal force tend to the centre C, and let it be required to find the trajectory VIK*k*. Let there be given the circle VR, described from the centre C with any interval CV; and from the same centre describe any other circles ID, KE cutting the trajectory in I and K, and the right line CV in D and E. Then draw the right line CNIX cutting the circles KE, VR in N and X, and the right line CKY meeting the circle VR in Y. Let the points I and K be indefinitely near; and let the body go on from V through I and K to *k*; and let the point A be the place from whence another body is to fall, so as in the place D to acquire a velocity equal to the velocity of the first body in I. And things remaining as in Prop. XXXIX, the lineola IK, described in the least given time

will be as the velocity, and therefore as the right line whose square is equal to the area ABFD, and the triangle ICK proportional to the time will be given, and therefore KN will be reciprocally as the altitude IC: that is (if there be given any quantity Q, and the altitude IC be called A), as $\frac{Q}{A}$. This quantity $\frac{Q}{A}$ call Z, and suppose the magnitude of Q to be such that in some case $\surd$ABFD may be to Z as IK to KN, and then in all cases $\surd$ ABFD will be to Z as IK to KN, and ABFD to ZZ as IK^2 to KN^2, and by division ABFD — ZZ to ZZ as IN^2 to KN^2, and therefore $\surd\overline{ABFD - ZZ}$ to Z, or $\frac{Q}{A}$ as IN to KN; and therefore A × KN will be equal to $\frac{Q \times IN}{\surd\overline{ABFD - ZZ}}$. Therefore since YX × XC is to A × KN as CX^2, to AA, the rectangle XY × XC will be equal to $\frac{Q \times IN \times CX^2}{AA\surd\overline{ABFD - ZZ}}$. Therefore in the perpendicular DF let there be taken continually D*b*, D*c* equal to $\frac{Q}{2\surd\overline{ABFD - ZZ}}$, $\frac{Q \times CX^2}{2AA\surd\overline{ABFD - ZZ}}$ respectively, and let the curve lines *ab*, *ac*, the foci of the points *b* and *c*, be described: and from the point V let the perpendicular V*a* be erected to the line AC, cutting off the curvilinear areas VD*ba*, VD*ca*, and let the ordinates E*z*, E*x*, be erected also. Then because the rectangle D*b* × IN or D*bz*E is equal to half the rectangle A × KN, or to the triangle ICK; and the rectangle D*c* × IN or D*cx*E is equal to half the rectangle YX × XC, or to the triangle XCY; that is, because the nascent particles D*bz*E, ICK of the areas VD*ba*, VIC are always equal; and the nascent particles D*cx*E, XCY of the areas VD*ca*, VCX are always equal: therefore the generated area VD*ba* will be equal to the generated area VIC, and therefore proportional to the time; and the generated area VD*ca* is equal to the generated sector VCX. If, therefore, any time be given during which the body has been moving from V, there will be also given the area proportional to it VD*ba*; and thence will be given the altitude of the body CD or CI; and the area VD*ca*, and the sector VCX equal thereto, together with its angle VCI. But the angle VCI, and the altitude CI being given, there is also given the place I, in which the body will be found at the end of that time. Q.E.I.

Cor. 1. Hence the greatest and least altitudes of the bodies, that is, the apsides of the trajectories, may be found very readily. For the apsides are those points in which a right line IC drawn through the centre falls perpendicularly upon the trajectory VIK; which comes to pass when the right lines IK and NK become equal; that is, when the area ABFD is equal to ZZ.

COR. 2. So also the angle KIN, in which the trajectory at any place cuts the line IC, may be readily found by the given altitude IC of the body: to wit, by making the sine of that angle to radius as KN to IK that is, as Z to the square root of the area ABFD.

COR. 3. If to the centre C, and the principal vertex V, there be described a conic section VRS; and from any point thereof, as R, there be drawn the tangent RT meeting the axis CV indefinitely produced in the point T; and then joining CR there be drawn the right line CP,

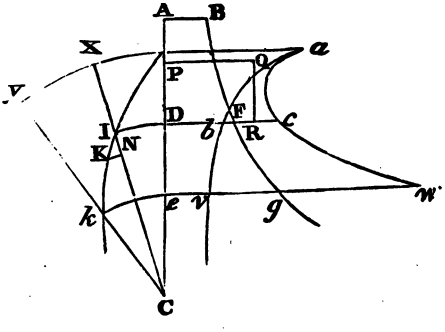


equal to the abscissa CT, making an angle VCP proportional to the sector VCR; and if a centripetal force, reciprocally proportional to the cubes of the distances of the places from the centre, tends to the centre C; and from the place V there sets out a body with a just velocity in the direction of a line perpendicular to the right line CV; that body will proceed in a trajectory VPQ, which the point P will always touch; and therefore if the conic section VRS be an hyberbola, the body will descend to the centre; but if it be an ellipsis, it will ascend perpetually, and go farther and farther off *in infinitum*. And, on the contrary, if a body endued with any velocity goes off from the place V, and according as it begins either to descend obliquely to the centre, or ascends obliquely from it, the figure VRS be either an hyperbola or an ellipsis, the trajectory may be found by increasing or diminishing the angle VCP in a given ratio. And the centripetal force becoming centrifugal, the body will ascend obliquely in the trajectory VPQ, which is found by taking the angle VCP proportional to the elliptic sector VRC, and the length CP equal to the length CT, as before. All these things follow from the foregoing Proposition, by the quadrature of a certain curve, the invention of which, as being easy enough, for brevity's sake I omit.

PROPOSITION XLII. PROBLEM XXIX.

The law of centripetal force being given, it is required to find the motion of a body setting out from a given place, with a given velocity, in the direction of a given right line.

Suppose the same things as in the three preceding propositions; and let the body go off from the place I in the direction of the little line, IK, with the same velocity as another body, by falling with an uniform centripetal force from the place P, may acquire in D; and let this uniform force be to the force with which the body



is at first urged in I, as DR to DF. Let the body go on towards *k*; and about the centre C, with the interval C*k*, describe the circle *ke*, meeting the right line PD in *e*, and let there be erected the lines *eg*, *ev*, *ew*, ordinately applied to the curves BF*g*, *abv*, *acw*. From the given rectangle PDRQ and the given law of centripetal force, by which the first body is acted on, the curve line BF*g* is also given, by the construction of Prop. XXVII, and its Cor. 1. Then from the given angle CIK is given the proportion of the nascent lines IK, KN; and thence, by the construction of Prop. XXVIII, there is given the quantity Q, with the curve lines *abv*, *acw*; and therefore, at the end of any time D*bve*, there is given both the altitude of the body C*e* or C*k*, and the area D*cwe*, with the sector equal to it XC*y*, the angle IC*k*, and the place *k*, in which the body will then be found. Q.E.I.

We suppose in these Propositions the centripetal force to vary in its recess from the centre according to some law, which any one may imagine at pleasure; but at equal distances from the centre to be everywhere the same.

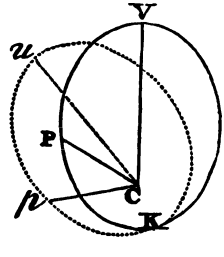
I have hitherto considered the motions of bodies in immovable orbits. It remains now to add something concerning their motions in orbits which revolve round the centres of force.

SECTION IX.

Of the motion of bodies in moveable orbits; and of the motion of the apsides.

PROPOSITION XLIII. PROBLEM XXX.

It is required to make a body move in a trajectory that revolves about the centre of force in the same manner as another body in the same trajectory at rest.

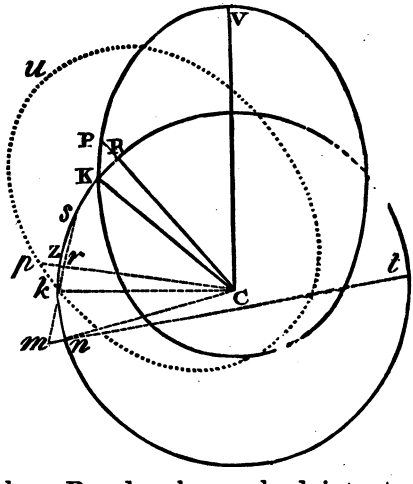


In the orbit VPK, given by position, let the body P revolve, proceeding from V towards K. From the centre C let there be continually drawn C*p*, equal to CP, making the angle V*Cp* proportional to the angle VCP; and the area which the line C*p* describes will be to the area VCP, which the line CP describes at the same time, as the velocity of the describing line C*p* to the velocity of the describing line CP; that is, as the angle V*Cp* to the angle VCP, therefore in a given ratio, and therefore proportional to the time. Since, then, the area described by the line C*p* in an immovable plane is proportional to the time, it is manifest that a body, being acted upon by a just quantity of centripetal force may

revolve with the point p in the curve line which the same point p, by the method just now explained, may be made to describe an immovable plane. Make the angle VCu equal to the angle PCp, and the line Cu equal to CV, and the figure uCp equal to the figure VCP, and the body being always in the point p, will move in the perimeter of the revolving figure uCp, and will describe its (revolving) arc up in the same time that the other body P describes the similar and equal arc VP in the quiescent figure VPK. Find, then, by Cor. 5, Prop. VI., the centripetal force by which the body may be made to revolve in the curve line which the point p describes in an immovable plane, and the Problem will be solved. Q.E.F.

PROPOSITION XLIV. THEOREM XIV.

The difference of the forces, by which two bodies may be made to move equally, one in a quiescent, the other in the same orbit revolving, is in a triplicate ratio of their common altitudes inversely.



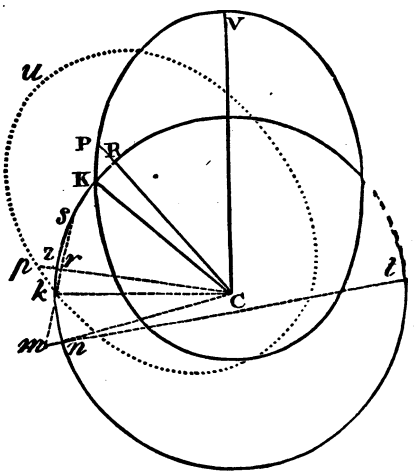
Let the parts of the quiescent orbit VP, PK be similar and equal to the parts of the revolving orbit up, pk; and let the distance of the points P and K be supposed of the utmost smallness Let fall a perpendicular kr from the point k to the right line pC, and produce it to m, so that mr may be to kr as the angle VCp to the angle VCP. Because the altitudes of the bodies PC and pC, KC and kC, are always equal, it is manifest that the increments or decrements of the lines PC and pC are always equal; and therefore if each of the several motions of the bodies in the places P and p be resolved into two (by Cor. 2 of the Laws of Motion), one of which is directed towards the centre, or according to the lines PC, pC, and the other, transverse to the former, hath a direction perpendicular to the lines PC and pC; the motions towards the centre will be equal, and the transverse motion of the body p will be to the transverse motion of the body P as the angular motion of the line pC to the angular motion of the line PC; that is, as the angle VCp to the angle VCP. Therefore, at the same time that the body P, by both its motions, comes to the point K, the body p, having an equal motion towards the centre, will be equally moved from p towards C; and therefore that time being expired, it will be found somewhere in the line mkr, which, passing through the point k, is perpendicular to the line pC; and by its transverse motion will acquire a distance from the line

vC, that will be to the distance which the other body P acquires from the line PC as the transverse motion of the body p to the transverse motion of the other body P. Therefore since kr is equal to the distance which the body P acquires from the line PC, and mr is to kr as the angle VCp to the angle VCP, that is, as the transverse motion of the body p to the transverse motion of the body P, it is manifest that the body p, at the expiration of that time, will be found in the place m. These things will be so, if the bodies p and P are equally moved in the directions of the lines pC and PC, and are therefore urged with equal forces in those directions. But if we take an angle pCn that is to the angle pCk as the angle VCp to the angle VCP, and nC be equal to kC, in that case the body p at the expiration of the time will really be in n; and is therefore urged with a greater force than the body P, if the angle nCp is greater than the angle kCp, that is, if the orbit npk, move either *in consequentia*, or *in antecedentia*, with a celerity greater than the double of that with which the line CP moves *in consequentia*; and with a less force if the orbit moves slower *in antecedentia*. And the difference of the forces will be as the interval mn of the places through which the body would be carried by the action of that difference in that given space of time. About the centre C with the interval Cn or Ck suppose a circle described cutting the lines mr, nu produced in s and t, and the rectangle $mn \times mt$ will be equal to the rectangle $mk \times ms$, and therefore mn will be equal to $\frac{mk \times ms}{mt}$. But since the triangles pCk, pCn, in a given time, are of a given magnitude, kr and mr, and their difference mk, and their sum ms, are reciprocally as the altitude pC, and therefore the rectangle $mk \times ms$ is reciprocally as the square of the altitude pC. But, moreover, mt is directly as $\frac{1}{2}mt$, that is, as the altitude pC. These are the first ratios of the nascent lines; and hence $\frac{mk \times ms}{mt}$, that is, the nascent lineola mn, and the difference of the forces proportional thereto, are reciprocally as the cube of the altitude pC. Q.E.D.

COR. 1. Hence the difference of the forces in the places P and p, or K and k, is to the force with which a body may revolve with a circular motion from R to K, in the same time that the body P in an immovable orb describes the arc PK, as the nascent line mn to the versed sine of the nascent arc RK, that is, as $\frac{mk \times ms}{mt}$ to $\frac{rk^2}{2kC}$, or as $mk \times ms$ to the square of rk; that is, if we take given quantities F and G in the same ratio to one another as the angle VCP bears to the angle VCp, as GG — FF to FF. And, therefore, if from the centre C, with any distance CP or Cp, there be described a circular sector equal to the whole area VPC, which the body

revolving in an immovable orbit has by a radius drawn to the centre described in any certain time, the difference of the forces, with which the body P revolves in an immovable orbit, and the body p in a movable orbit, will be to the centripetal force, with which another body by a radius drawn to the centre can uniformly describe that sector in the same time as the area VPC is described, as GG — FF to FF. For that sector and the area pCk are to one another as the times in which they are described.

COR. 2. If the orbit VPK be an ellipsis, having its focus C, and its highest apsis V, and we suppose the the ellipsis upk similar and equal to it, so that pC may be always equal to PC, and the angle VCp be to the angle VCP in the given ratio of G to F; and for the altitude PC or pC we put A, and 2R for the latus rectum of the ellipsis, the force with which a body may be made to revolve in a movable ellipsis will be as $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$, and *vice versa*.



Let the force with which a body may revolve in an immovable ellipsis be expressed by the quantity $\frac{FF}{AA}$, and the force in V will be $\frac{FF}{CV^2}$. But the force with which a body may revolve in a circle at the distance CV, with the same velocity as a body revolving in an ellipsis has in V, is to the force with which a body revolving in an ellipsis is acted upon in the apsis V, as half the latus rectum of the ellipsis to the semi-diameter CV of the circle, and therefore is as $\frac{RFF}{CV^3}$; and the force which is to this, as GG — FF to FF, is as $\frac{RGG - RFF}{CV^3}$: and this force (by Cor. 1 of this Prop.) is the difference of the forces in V, with which the body P revolves in the immovable ellipsis VPK, and the body p in the movable ellipsis upk. Therefore since by this Prop. that difference at any other altitude A is to itself at the altitude CV as $\frac{1}{A^3}$ to $\frac{1}{CV^3}$, the same difference in every altitude A will be as $\frac{RGG - RFF}{A^3}$. Therefore to the force $\frac{FF}{AA}$, by which the body may revolve in an immovable ellipsis VPK

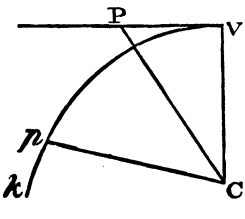
add the excess $\frac{RGG-RFF}{A^3}$, and the sum will be the whole force $\frac{FF}{AA} + \frac{RGG-RFF}{A^3}$ by which a body may revolve in the same time in the movable ellipsis *upk*.

Cor. 3. In the same manner it will be found, that, if the immovable orbit VPK be an ellipsis having its centre in the centre of the forces C, and there be supposed a movable ellipsis *upk*, similar, equal, and concentrical to it; and 2R be the principal latus rectum of that ellipsis, and 2T the latus transversum, or greater axis; and the angle V*Cp* be continually to the angle VCP as G to F; the forces with which bodies may revolve in the immovable and movable ellipsis, in equal times, will be as $\frac{FFA}{T^3}$ and $\frac{FFA}{T^3} + \frac{RGG-RFF}{A^3}$ respectively.

Cor. 4. And universally, if the greatest altitude CV of the body be called T, and the radius of the curvature which the orbit VPK has in V, that is, the radius of a circle equally curve, be called R, and the centripetal force with which a body may revolve in any immovable trajectory VPK at the place V be called $\frac{VFF}{TT}$, and in other places P be indefinitely styled X; and the altitude CP be called A, and G be taken to F in the given ratio of the angle V*Cp* to the angle VCP; the centripetal force with which the same body will perform the same motions in the same time, in the same trajectory *upk* revolving with a circular motion, will be as the sum of the forces $X + \frac{VRGG-VRFF}{A^3}$.

Cor. 5. Therefore the motion of a body in an immovable orbit being given, its angular motion round the centre of the forces may be increased or diminished in a given ratio; and thence new immovable orbits may be found in which bodies may revolve with new centripetal forces.

Cor. 6. Therefore if there be erected the line VP of an indeterminate length, perpendicular to the line CV given by position, and CP be drawn, and C*p* equal to it, making the angle V*Cp* having a given ratio to the angle VCP, the force with which a body may revolve in the curve line V*pk*, which the point *p* is continually describing, will be reciprocally as the cube of the altitude C*p*. For the body P, by its *vis inertiæ* alone, no other force impelling it, will proceed uniformly in the right line VP. Add, then, a force tending to the centre C reciprocally as the cube of the altitude CP or C*p*, and (by what was just demonstrated) the



body will deflect from the rectilinear motion into the curve line V*pk*. But this curve V*pk* is the same with the curve VPQ found in Cor. 3, Prop XLI, in which, I said, bodies attracted with such forces would ascend obliquely.

PROPOSITION XLV. PROBLEM XXXI.

To find the motion of the apsides in orbits approaching very near to circles.

This problem is solved arithmetically by reducing the orbit, which a body revolving in a movable ellipsis (as in Cor. 2 and 3 of the above Prop.) describes in an immovable plane, to the figure of the orbit whose apsides are required; and then seeking the apsides of the orbit which that body describes in an immovable plane. But orbits acquire the same figure, if the centripetal forces with which they are described, compared between themselves, are made proportional at equal altitudes. Let the point V be the highest apsis, and write T for the greatest altitude CV, A for any other altitude CP or C*p*, and X for the difference of the altitudes CV — CP; and the force with which a body moves in an ellipsis revolving about its focus C (as in Cor. 2), and which in Cor. 2 was as $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$, that is as, $\frac{FFA + RGG - RFF}{A^3}$, by substituting T — X for A, will become as $\frac{RGG - RFF + TFF - FFX}{A^3}$. In like manner any other centripetal force is to be reduced to a fraction whose denominator is A^3, and the numerators are to be made analogous by collating together the homologous terms. This will be made plainer by Examples.

EXAMPLE 1. Let us suppose the centripetal force to be uniform, and therefore as $\frac{A^3}{A^3}$ or, writing T — X for A in the numerator, as $\frac{T^3 - 3TTX + 3TXX - X^3}{A^3}$. Then collating together the correspondent terms of the numerators, that is, those that consist of given quantities, with those of given quantities, and those of quantities not given with those of quantities not given, it will become RGG — RFF + TFF to T^3 as — FFX to 3TTX + 3TXX — X^3, or as —FF to —3TT + 3TX — XX. Now since the orbit is supposed extremely near to a circle, let it coincide with a circle; and because in that case R and T become equal, and X is infinitely diminished, the last ratios will be, as RGG to T^2, so — FF to — 3TT, or as GG to TT, so FF to 3TT; and again, as GG to FF, so TT to 3TT, that is, as 1 to 3; and therefore G is to F, that is, the angle V*Cp* to the angle VCP, as 1 to $\sqrt{3}$. Therefore since the body, in an immovable

ellipsis, in descending from the upper to the lower apsis, describes an angle, if I may so speak, of 180 deg., the other body in a movable ellipsis, and therefore in the immovable orbit we are treating of, will in its descent from the upper to the lower apsis, describe an angle VC*p* of $\frac{180}{\sqrt{3}}$ deg. And this comes to pass by reason of the likeness of this orbit which a body acted upon by an uniform centripetal force describes, and of that orbit which a body performing its circuits in a revolving ellipsis will describe in a quiescent plane. By this collation of the terms, these orbits are made similar; not universally, indeed, but then only when they approach very near to a circular figure. A body, therefore revolving with an uniform centripetal force in an orbit nearly circular, will always describe an angle of $\frac{180}{\sqrt{3}}$ deg., or 103 deg., 55 m., 23 sec., at the centre; moving from the upper apsis to the lower apsis when it has once described that angle, and thence returning to the upper apsis when it has described that angle again; and so on *in infinitum*.

EXAM. 2. Suppose the centripetal force to be as any power of the altitude A, as, for example, A^{n-3}, or $\frac{A^n}{A^3}$; where $n - 3$ and n signify any indices of powers whatever, whether integers or fractions, rational or surd, affirmative or negative. That numerator A^n or $\overline{T - X}|^n$ being reduced to an indeterminate series by my method of converging series, will become $T^n - nXT^{n-1} + \frac{nn-n}{2}XXT^{n-2}$, &c. And conferring these terms with the terms of the other numerator $RGG - RFF + TFF - FFX$, it becomes as $RGG - RFF + TFF$ to T^n, so $-FF$ to $-nT^{n-1} + \frac{nn-n}{2}XT^{n-2}$, &c. And taking the last ratios where the orbits approach to circles, it becomes as RGG to T^n, so $-FF$ to $-nT^{n-1}$, or as GG to T^{n-1}, so FF to nT^n —; and again, GG to FF, so T^{n-1} to nT^{n-1}, that is, as 1 to n; and therefore G is to F, that is the angle VC*p* to the angle VCP, as 1 to $\sqrt{n}$. Therefore since the angle VCP, described in the descent of the body from the upper apsis to the lower apsis in an ellipsis, is of 180 deg., the angle VC*p*, described in the descent of the body from the upper apsis to the lower apsis in an orbit nearly circular which a body describes with a centripetal force proportional to the power A^{n-3}, will be equal to an angle of $\frac{180}{\sqrt{n}}$ deg., and this angle being repeated, the body will return from the lower to the upper apsis, and so on *in infinitum*. As if the centripetal force be as the distance of the body from the centre, that is, as A, or $\frac{A^4}{A^3}$, n will be equal to 4, and $\sqrt{n}$ equal to 2; and therefore the angle

between the upper and the lower apsis will be equal to $\frac{180}{2}$ deg., or 90 deg. Therefore the body having performed a fourth part of one revolution, will arrive at the lower apsis, and having performed another fourth part, will arrive at the upper apsis, and so on by turns *in infinitum*. This appears also from Prop. X. For a body acted on by this centripetal force will revolve in an immovable ellipsis, whose centre is the centre of force. If the centripetal force is reciprocally as the distance, that is, directly as $\frac{1}{A}$ or $\frac{A^2}{A^3}$, n will be equal to 2; and therefore the angle between the upper and lower apsis will be $\frac{180}{\sqrt{2}}$ deg., or 127 deg., 16 min., 45 sec.; and therefore a body revolving with such a force, will by a perpetual repetition of this angle, move alternately from the upper to the lower and from the lower to the upper apsis for ever. So, also, if the centripetal force be reciprocally as the biquadrate root of the eleventh power of the altitude, that is, reciprocally as $A^{\frac{11}{4}}$, and, therefore, directly as $\frac{1}{A^{\frac{11}{4}}}$, or as $\frac{A^{\frac{1}{4}}}{A^3}$, n will be equal to $\frac{1}{4}$, and $\frac{180}{\sqrt{n}}$ deg. will be equal to 360 deg.; and therefore the body parting from the upper apsis, and from thence perpetually descending, will arrive at the lower apsis when it has completed one entire revolution; and thence ascending perpetually, when it has completed another entire revolution, it will arrive again at the upper apsis; and so alternately for ever.

EXAM. 3. Taking m and n for any indices of the powers of the altitude, and b and c for any given numbers, suppose the centripetal force to be as $\frac{bA^m + cA^n}{A^3}$, that is, as $\frac{b \text{ into } \overline{T - X}^m + c \text{ into } \overline{T - X}^n}{A^3}$ or (by the method of converging series above-mentioned) as $\frac{bT^m + cT^n - mbXT^{m-1} \quad ncXT^{n-1} + \frac{mm-m}{2} bXXT^{m-2} + \frac{nn-n}{2}}{A^3}$ $\frac{cXXT^{n-2}, \&c.}{A^3}$ and comparing the terms of the numerators, there will arise RGG — RFF + TFF to $bT^m + cT^n$ as — FF to $-mbT^{m-1} - ncT^{n-1} + \frac{mm-m}{2} bXT^{m-2} + \frac{nn-n}{2} cXT^{n-2}$, &c. And taking the last ratios that arise when the orbits come to a circular form, there will come forth GG to $bT^{m-1} + cT^{n-1}$ as FF to $mbT^{m-1} + ncT^{n-1}$; and again, GG to FF as $bT^{m-1} + cT^{n-1}$ to $mbT^{n-1} + ncT^{n-1}$. This proportion, by expressing the greatest altitude CV or T arithmetically by unity, becomes, GG to FF as $b + c$ to $mb + nc$, and therefore as 1

to $\frac{mb + nc}{b + c}$. Whence G becomes to F, that is, the angle VCp to the angle VCP, as 1 to $\sqrt{\frac{mb + nc}{b + c}}$. And therefore since the angle VCP between the upper and the lower apsis, in an immovable ellipsis, is of 180 deg., the angle VCp between the same apsides in an orbit which a body describes with a centripetal force, that is, as $\frac{bA^m + cA^n}{A^3}$, will be equal to an angle of $180\sqrt{\frac{b + c}{mb + nc}}$ deg. And by the same reasoning, if the centripetal force be as $\frac{bA^m - cA^n}{A^3}$, the angle between the apsides will be found equal to $180\sqrt{\frac{b - c}{mb - nc}}$ deg. After the same manner the Problem is solved in more difficult cases. The quantity to which the centripetal force is proportional must always be resolved into a converging series whose denominator is A^3. Then the given part of the numerator arising from that operation is to be supposed in the same ratio to that part of it which is not given, as the given part of this numerator RGG — RFF + TFF — FFX is to that part of the same numerator which is not given. And taking away the superfluous quantities, and writing unity for T, the proportion of G to F is obtained.

Cor. 1. Hence if the centripetal force be as any power of the altitude, that power may be found from the motion of the apsides; and so contrariwise. That is, if the whole angular motion, with which the body returns to the same apsis, be to the angular motion of one revolution, or 360 deg., as any number as m to another as n, and the altitude called A; the force will be as the power $A^{\frac{nn}{mm} - 3}$ of the altitude A; the index of which power is $\frac{nn}{mm} - 3$. This appears by the second example. Hence it is plain that the force in its recess from the centre cannot decrease in a greater than a triplicate ratio of the altitude. A body revolving with such a force, and parting from the apsis, if it once begins to descend, can never arrive at the lower apsis or least altitude, but will descend to the centre, describing the curve line treated of in Cor. 3, Prop. XLI. But if it should, at its parting from the lower apsis, begin to ascend never so little, it will ascend *in infinitum*, and never come to the upper apsis; but will describe the curve line spoken of in the same Cor., and Cor. 6, Prop. XLIV. So that where the force in its recess from the centre decreases in a greater than a triplicate ratio of the altitude, the body at its parting from the apsis, will either descend to the centre, or ascend *in infinitum*, according as it descends or ascends at the beginning of its motion. But if the force in its recess from

the centre either decreases in a less than a triplicate ratio of the altitude, or increases in any ratio of the altitude whatsoever, the body will never descend to the centre, but will at some time arrive at the lower apsis; and, on the contrary, if the body alternately ascending and descending from one apsis to another never comes to the centre, then either the force increases in the recess from the centre, or it decreases in a less than a triplicate ratio of the altitude; and the sooner the body returns from one apsis to another, the farther is the ratio of the forces from the triplicate ratio. As if the body should return to and from the upper apsis by an alternate descent and ascent in 8 revolutions, or in 4, or 2, or $1\frac{1}{2}$; that is, if m should be to n as 8, or 4, or 2, or $1\frac{1}{2}$ to 1, and therefore $\frac{nn}{mm} - 3$, be $\frac{1}{64} - 3$, or $\frac{1}{16} - 3$, or $\frac{1}{4} - 3$, or $\frac{4}{9} - 3$; then the force will be as $A^{\frac{1}{64}-3}$, or $A^{\frac{1}{16}-3}$, or $A^{\frac{1}{4}-3}$, or $A^{\frac{4}{9}-3}$; that is, it will be reciprocally as $A^{3-\frac{1}{64}}$, or $A^{3-\frac{1}{16}}$, or $A^{3-\frac{1}{4}}$, or $A^{3-\frac{4}{9}}$. If the body after each revolution returns to the same apsis, and the apsis remains unmoved, then m will be to n as 1 to 1, and therefore $A^{\frac{nn}{mm}-3}$ will be equal to A^{-2}, or $\frac{1}{AA}$; and therefore the decrease of the forces will be in a duplicate ratio of the altitude; as was demonstrated above. If the body in three fourth parts, or two thirds, or one third, or one fourth part of an entire revolution, return to the same apsis; m will be to n as $\frac{3}{4}$ or $\frac{2}{3}$ or $\frac{1}{3}$ or $\frac{1}{4}$ to 1, and therefore $A^{\frac{nn}{mm}-3}$ is equal to $A^{\frac{16}{9}-3}$, or $A^{\frac{9}{4}-3}$ or A^{9-3}, or A^{16-3}; and therefore the force is either reciprocally as $A^{\frac{11}{9}}$ or $A^{\frac{3}{4}}$, or directly as A^{6} or A^{13}. Lastly if the body in its progress from the upper apsis to the same upper apsis again, goes over one entire revolution and three deg. more, and therefore that apsis in each revolution of the body moves three deg. *in consequentia;* then m will be to n as 363 deg. to 360 deg. or as 121 to 120, and therefore $A^{\frac{nn}{mm}-3}$ will be equal to $A^{-\frac{295231}{146641}}$, and therefore the centripetal force will be reciprocally as $A^{\frac{295231}{146641}}$, or reciprocally as $A^{2\frac{4}{243}}$ very nearly. Therefore the centripetal force decreases in a ratio something greater than the duplicate; but approaching $59\frac{3}{4}$ times nearer to the duplicate than the triplicate.

COR. 2. Hence also if a body, urged by a centripetal force which is reciprocally as the square of the altitude, revolves in an ellipsis whose focus is in the centre of the forces; and a new and foreign force should be added to or subducted from this centripetal force, the motion of the apsides arising from that foreign force may (by the third Example) be known; and so on the contrary. As if the force with which the body revolves in the ellipsis

be as $\frac{1}{AA}$; and the foreign force subducted as cA, and therefore the remaining force as $\frac{A - cA^4}{A^3}$; then (by the third Example) b will be equal to 1, m equal to 1, and n equal to 4; and therefore the angle of revolution between the apsides is equal to $180\sqrt{\frac{1-c}{1-4c}}$ deg. Suppose that foreign force to be 357.45 parts less than the other force with which the body revolves in the ellipsis; that is, c to be $\frac{1}{357.45}$; A or T being equal to 1; and then $180\sqrt{\frac{1-c}{1-4c}}$ will be $180\sqrt{\frac{356.45}{357.45}}$ or 180.7623, that is, 180 deg., 45 min., 44 sec. Therefore the body, parting from the upper apsis, will arrive at the lower apsis with an angular motion of 180 deg., 45 min., 44 sec, and this angular motion being repeated, will return to the upper apsis; and therefore the upper apsis in each revolution will go forward 1 deg., 31 min., 28 sec. The apsis of the moon is about twice as swift

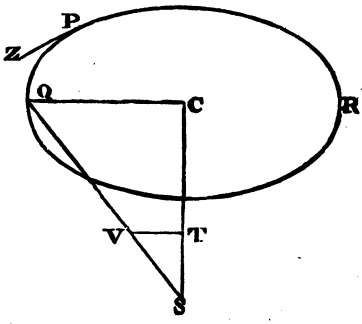
So much for the motion of bodies in orbits whose planes pass through the centre of force. It now remains to determine those motions in eccentrical planes. For those authors who treat of the motion of heavy bodies used to consider the ascent and descent of such bodies, not only in a perpendicular direction, but at all degrees of obliquity upon any given planes; and for the same reason we are to consider in this place the motions of bodies tending to centres by means of any forces whatsoever, when those bodies move in eccentrical planes. These planes are supposed to be perfectly smooth and polished, so as not to retard the motion of the bodies in the least. Moreover, in these demonstrations, instead of the planes upon which those bodies roll or slide, and which are therefore tangent planes to the bodies, I shall use planes parallel to them, in which the centres of the bodies move, and by that motion describe orbits. And by the same method I afterwards determine the motions of bodies performed in curve superficies.

SECTION X.

Of the motion of bodies in given superficies, and of the reciprocal motion of funependulous bodies.

PROPOSITION XLVI. PROBLEM XXXII.

Any kind of centripetal force being supposed, and the centre of force, and any plane whatsoever in which the body revolves, being given, and the quadratures of curvilinear figures being allowed; it is required to determine the motion of a body going off from a given place, with a given velocity, in the direction of a given right line in that plane.



Let S be the centre of force, SC the least distance of that centre from the given plane, P a body issuing from the place P in the direction of the right line PZ, Q the same body revolving in its trajectory, and PQR the trajectory itself which is required to be found, described in that given plane. Join CQ, QS, and if in QS we take SV proportional to the centripetal force with which the body is attracted towards the centre S, and draw VT parallel to CQ, and meeting SC in T; then will the force SV be resolved into two (by Cor. 2, of the Laws of Motion), the force ST, and the force TV; of which ST attracting the body in the direction of a line perpendicular to that plane, does not at all change its motion in that plane. But the action of the other force TV, coinciding with the position of the plane itself, attracts the body directly towards the given point C in that plane; and therefore causes the body to move in this plane in the same manner as if the force ST were taken away, and the body were to revolve in free space about the centre C by means of the force TV alone. But there being given the centripetal force TV with which the body Q revolves in free space about the given centre C, there is given (by Prop. XLII) the trajectory PQR which the body describes; the place Q, in which the body will be found at any given time; and, lastly, the velocity of the body in that place Q. And so è contra. Q.E.I.

PROPOSITION XLVII. THEOREM XV.

Supposing the centripetal force to be proportional to the distance of the body from the centre; all bodies revolving in any planes whatsoever will describe ellipses, and complete their revolutions in equal times; and those which move in right lines, running backwards and forwards alternately, will complete their several periods of going and returning in the same times.

For letting all things stand as in the foregoing Proposition, the force SV, with which the body Q revolving in any plane PQR is attracted towards the centre S, is as the distance SQ; and therefore because SV and SQ, TV and CQ are proportional, the force TV with which the body is attracted towards the given point C in the plane of the orbit is as the distance CQ. Therefore the forces with which bodies found in the plane PQR are attracted towards the point C, are in proportion to the distances equal to the forces with which the same bodies are attracted every way towards the centre S; and therefore the bodies will move in the same times, and in the same figures, in any plane PQR about the point C, as they

would do in free spaces about the centre S; and therefore (by Cor. 2, Prop. X, and Cor. 2, Prop. XXXVIII.) they will in equal times either describe ellipses in that plane about the centre C, or move to and fro in right lines passing through the centre C in that plane; completing the same periods of time in all cases. Q.E.D.

SCHOLIUM.

The ascent and descent of bodies in curve superficies has a near relation to these motions we have been speaking of. Imagine curve lines to be described on any plane, and to revolve about any given axes passing through the centre of force, and by that revolution to describe curve superficies; and that the bodies move in such sort that their centres may be always found in those superficies. If those bodies reciprocate to and fro with an oblique ascent and descent, their motions will be performed in planes passing through the axis, and therefore in the curve lines, by whose revolution those curve superficies were generated. In those cases, therefore, it will be sufficient to consider the motion in those curve lines.

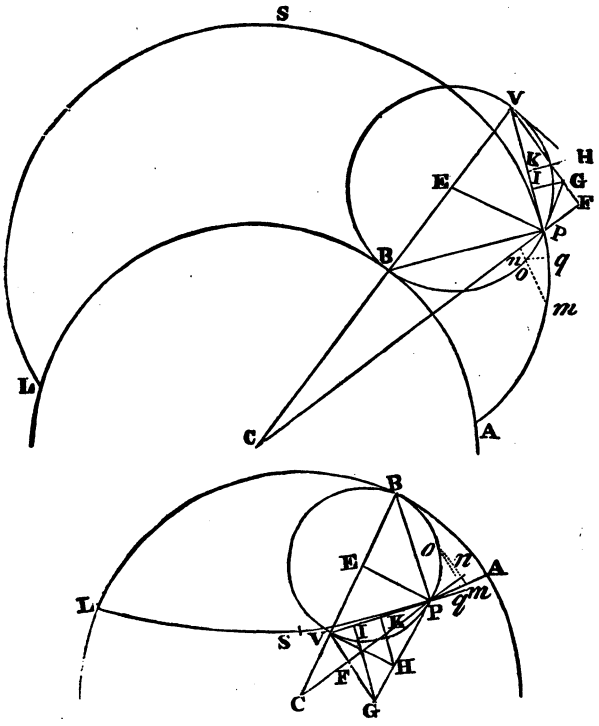
PROPOSITION XLVIII. THEOREM XVI.

If a wheel stands upon the outside of a globe at right angles thereto, and revolving about its own axis goes forward in a great circle, the length of the curvilinear path which any point, given in the perimeter of the wheel, hath described since the time that it touched the globe (which curvilinear path we may call the cycloid or epicycloid), will be to double the versed sine of half the arc which since that time has touched the globe in passing over it, as the sum of the diameters of the globe and the wheel to the semi-diameter of the globe.

PROPOSITION XLIX. THEOREM XVII.

If a wheel stand upon the inside of a concave globe at right angles thereto, and revolving about its own axis go forward in one of the great circles of the globe, the length of the curvilinear path which any point, given in the perimeter of the wheel, hath described since it touched the globe, will be to the double of the versed sine of half the arc which in all that time has touched the globe in passing over it, as the difference of the diameters of the globe and the wheel to the semi-diameter of the globe.

Let ABL be the globe, C its centre, BPV the wheel insisting thereon, E the centre of the wheel, B the point of contact, and P the given point in the perimeter of the wheel. Imagine this wheel to proceed in the great circle ABL from A through B towards L, and in its progress to revolve in such a manner that the arcs AB, PB may be always equal one to the other, and the given point P in the perimeter of the wheel may describe in the



mean time the curvilinear path AP. Let AP be the whole curvilinear path described since the wheel touched the globe in A, and the length of this path AP will be to twice the versed sine of the arc $\frac{1}{2}$PB as 2CE to CB. For let the right line CE (produced if need be) meet the wheel in V, and join CP, BP, EP, VP; produce CP, and let fall thereon the perpendicular VF. Let PH, VH, meeting in H, touch the circle in P and V, and let PH cut VF in G, and to VP let fall the perpendiculars GI, HK. From the centre C with any interval let there be described the circle *nom*, cutting the right line CP in *n*, the perimeter of the wheel BP in *o*, and the curvilinear path AP in *m*; and from the centre V with the interval V*o* let there be described a circle cutting VP produced in *q*.

Because the wheel in its progress always revolves about the point of contact B, it is manifest that the right line BP is perpendicular to that curve line AP which the point P of the wheel describes, and therefore that the right line VP will touch this curve in the point P. Let the radius of the circle *nom* be gradually increased or diminished so that at last it become equal to the distance CP; and by reason of the similitude of the evanescent figure P*nomq*, and the figure PFGVI, the ultimate ratio of the evanescent lineolæ P*m*, P*n*, P*o*, P*q*, that is, the ratio of the momentary mutations of the curve AP, the right line CP, the circular arc BP, and the right line VP, will be

the same as of the lines PV, PF, PG, PI, respectively. But since VF is perpendicular to CF, and VH to CV, and therefore the angles HVG, VCF equal; and the angle VHG (because the angles of the quadrilateral figure HVEP are right in V and P) is equal to the angle CEP, the triangles VHG, CEP will be similar; and thence it will come to pass that as EP is to CE so is HG to HV or HP, and so KI to KP, and by composition or division as CB to CE so is PI to PK, and doubling the consequents as CB to 2CE so PI to PV, and so is P*q* to P*m*. Therefore the decrement of the line VP, that is, the increment of the line BV—VP to the increment of the curve line AP is in a given ratio of CB to 2CE, and therefore (by Cor. Lem. IV) the lengths BV—VP and AP, generated by those increments, are in the same ratio. But if BV be radius, VP is the cosine of the angle BVP or $\frac{1}{2}$BEP, and therefore BV—VP is the versed sine of the same angle, and therefore in this wheel, whose radius is $\frac{1}{2}$BV, BV—VP will be double the versed sine of the arc $\frac{1}{2}$BP. Therefore AP is to double the versed sine of the arc $\frac{1}{2}$BP as 2CE to CB. Q.E.D.

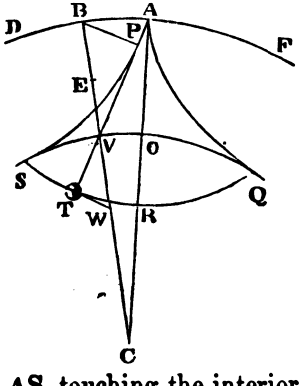
The line AP in the former of these Propositions we shall name the cycloid without the globe, the other in the latter Proposition the cycloid within the globe, for distinction sake.

Cor. 1. Hence if there be described the entire cycloid ASL, and the same be bisected in S, the length of the part PS will be to the length PV (which is the double of the sine of the angle VBP, when EB is radius) as 2CE to CB, and therefore in a given ratio.

Cor. 2. And the length of the semi-perimeter of the cycloid AS will be equal to a right line which is to the diameter of the wheel BV as 2CE to CB.

PROPOSITION L. PROBLEM XXXIII.

To cause a pendulous body to oscillate in a given cycloid.



Let there be given within the globe QVS described with the centre C, the cycloid QRS, bisected in R, and meeting the superficies of the globe with its extreme points Q and S on either hand. Let there be drawn CR bisecting the arc QS in O, and let it be produced to A in such sort that CA may be to CO as CO to CR. About the centre C, with the interval CA, let there be described an exterior globe DAF; and within this globe, by a wheel whose diameter is AO, let there be described two semi-cycloids AQ, AS, touching the interior globe in Q and S, and meeting the exterior globe in A. From that point A, with a thread APT in length equal to the line AR, let the body T depend, and oscillate in such manner between the two

semi-cycloids AQ, AS, that, as often as the pendulum parts from the perpendicular AR, the upper part of the thread AP may be applied to that semi-cycloid APS towards which the motion tends, and fold itself round that curve line, as if it were some solid obstacle, the remaining part of the same thread PT which has not yet touched the semi-cycloid continuing straight. Then will the weight T oscillate in the given cycloid QRS. Q.E.F.

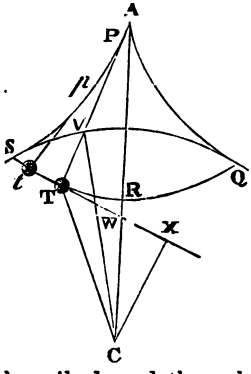
For let the thread PT meet the cycloid QRS in T, and the circle QOS in V, and let CV be drawn; and to the rectilinear part of the thread PT from the extreme points P and T let there be erected the perpendiculars BP, TW, meeting the right line CV in B and W. It is evident, from the construction and generation of the similar figures AS, SR, that those perpendiculars PB, TW, cut off from CV the lengths VB, VW equal the diameters of the wheels OA, OR. Therefore TP is to VP (which is double the sine of the angle VBP when $\frac{1}{2}$BV is radius) as BW to BV, or AO + OR to AO, that is (since CA and CO, CO and CR, and by division AO and OR are proportional), as CA + CO to CA, or, if BV be bisected in E, as 2CE to CB. Therefore (by Cor. 1, Prop. XLIX), the length of the rectilinear part of the thread PT is always equal to the arc of the cycloid PS, and the whole thread APT is always equal to the half of the cycloid APS, that is (by Cor. 2, Prop. XLIX), to the length AR. And therefore contrariwise, if the string remain always equal to the length AR, the point T will always move in the given cycloid QRS. Q.E.D.

COR. The string AR is equal to the semi-cycloid AS, and therefore has the same ratio to AC the semi-diameter of the exterior globe as the like semi-cycloid SR has to CO the semi-diameter of the interior globe.

PROPOSITION LI. THEOREM XVIII.

If a centripetal force tending on all sides to the centre C *of a globe, be in all places as the distance of the place from the centre, and by this force alone acting upon it, the body* T *oscillate (in the manner above described) in the perimeter of the cycloid* QRS; *I say, that all the oscillations, how unequal soever in themselves, will be performed in equal times.*

For upon the tangent TW infinitely produced let fall the perpendicular CX, and join CT. Because the centripetal force with which the body T is impelled towards C is as the distance CT, let this (by Cor. 2, of the Laws) be resolved into the parts CX, TX, of which CX impelling the body directly from P stretches the thread PT, and by the resistance the thread makes to it is totally employed, producing no other effect; but the other part TX, impelling the body transversely or towards X, directly accelerates the motion in the cycloid. Then it is plain that the acceleration of the body, proportional to this accelerating force, will be every

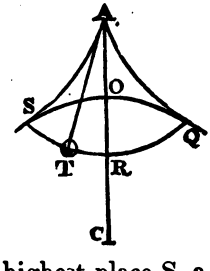


moment as the length TX, that is (because CV, WV, and TX, TW proportional to them are given), as the length TW, that is (by Cor. 1, Prop. XLIX) as the length of the arc of the cycloid TR. If therefore two pendulums APT, A*pt*, be unequally drawn aside from the perpendicular AR, and let fall together, their accelerations will be always as the arcs to be described TR, *t*R. But the parts described at the beginning of the motion are as the accelerations, that is, as the wholes that are to be described at the beginning, and therefore the parts which remain to be described, and the subsequent accelerations proportional to those parts, are also as the wholes, and so on. Therefore the accelerations, and consequently the velocities generated, and the parts described with those velocities, and the parts to be described, are always as the wholes; and therefore the parts to be described preserving a given ratio to each other will vanish together, that is, the two bodies oscillating will arrive together at the perpendicular AR. And since on the other hand the ascent of the pendulums from the lowest place R through the same cycloidal arcs with a retrograde motion, is retarded in the several places they pass through by the same forces by which their descent was accelerated; it is plain that the velocities of their ascent and descent through the same arcs are equal, and consequently performed in equal times; and, therefore, since the two parts of the cycloid RS and RQ lying on either side of the perpendicular are similar and equal, the two pendulums will perform as well the wholes as the halves of their oscillations in the same times. Q.E.D.

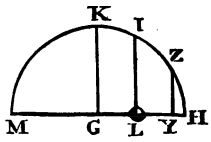
COR. The force with which the body T is accelerated or retarded in any place T of the cycloid, is to the whole weight of the same body in the highest place S or Q as the arc of the cycloid TR is to the arc SR or QR

PROPOSITION LII. PROBLEM XXXIV.

To define the velocities of the pendulums in the several places, and the times in which both the entire oscillations, and the several parts of them are performed.



About any centre G, with the interval GH equal to the arc of the cycloid RS, describe a semi-circle HKM bisected by the semi-diameter GK. And if a centripetal force proportional to the distance of the places from the centre tend to the centre G, and it be in the perimeter HIK equal to the centripetal force in the perimeter of the globe QOS tending towards its centre, and at the same time that the pendulum T is let fall from the highest place S, a body, as L, is let fall from H to G; then because the



forces which act upon the bodies are equal at the beginning, and always proportional to the spaces to be described TR, LG, and therefore if TR and LG are equal, are also equal in the places T and L, it is plain that those bodies describe at the beginning equal spaces ST, HL, and therefore are still acted upon equally, and continue to describe equal spaces. Therefore by Prop. XXXVIII, the time in which the body describes the arc ST is to the time of one oscillation, as the arc HI the time in which the body H arrives at L, to the semi-periphery HKM, the time in which the body H will come to M. And the velocity of the pendulous body in the place T is to its velocity in the lowest place R, that is, the velocity of the body H in the place L to its velocity in the place G, or the momentary increment of the line HL to the momentary increment of the line HG (the arcs HI, HK increasing with an equable flux) as the ordinate LI to the radius GK, or as $\sqrt{SR^2 - TR^2}$ to SR. Hence, since in unequal oscillations there are described in equal time arcs proportional to the entire arcs of the oscillations, there are obtained from the times given, both the velocities and the arcs described in all the oscillations universally. Which was first required.

Let now any pendulous bodies oscillate in different cycloids described within different globes, whose absolute forces are also different; and if the absolute force of any globe QOS be called V, the accelerative force with which the pendulum is acted on in the circumference of this globe, when it begins to move directly towards its centre, will be as the distance of the pendulous body from that centre and the absolute force of the globe conjunctly, that is, as $CO \times V$. Therefore the lineola HY, which is as this accelerated force $CO \times V$, will be described in a given time; and if there be erected the perpendicular YZ meeting the circumference in Z, the nascent arc HZ will denote that given time. But that nascent arc HZ is in the subduplicate ratio of the rectangle GHY, and therefore as $\sqrt{GH \times CO \times V}$ Whence the time of an entire oscillation in the cycloid QRS (it being as the semi-periphery HKM, which denotes that entire oscillation, directly; and as the arc HZ which in like manner denotes a given time inversely) will be as GH directly and $\sqrt{GH \times CO \times V}$ inversely; that is, because GH and SR are equal, as $\sqrt{\frac{SR}{CO \times V}}$ or (by Cor. Prop. L,) as $\sqrt{\frac{AR}{AC \times V}}$. Therefore the oscillations in all globes and cycloids, performed with what absolute forces soever, are in a ratio compounded of the subduplicate ratio of the length of the string directly, and the subduplicate ratio of the distance between the point of suspension and the centre of the globe inversely, and the subduplicate ratio of the absolute force of the globe inversely also Q.E.I.

Cor. 1. Hence also the times of oscillating, falling, and revolving bodies may be compared among themselves. For if the diameter of the wheel with which the cycloid is described within the globe is supposed equal to the semi-diameter of the globe, the cycloid will become a right line passing through the centre of the globe, and the oscillation will be changed into a descent and subsequent ascent in that right line. Whence there is given both the time of the descent from any place to the centre, and the time equal to it in which the body revolving uniformly about the centre of the globe at any distance describes an arc of a quadrant For this time (by Case 2) is to the time of half the oscillation in any cycloid QRS as 1 to $\sqrt{\frac{AR}{AC}}$.

Cor. 2. Hence also follow what Sir *Christopher Wren* and M. *Huygens* have discovered concerning the vulgar cycloid. For if the diameter of the globe be infinitely increased, its sphærical superficies will be changed into a plane, and the centripetal force will act uniformly in the direction of lines perpendicular to that plane, and this cycloid of our's will become the same with the common cycloid. But in that case the length of the arc of the cycloid between that plane and the describing point will become equal to four times the versed sine of half the arc of the wheel between the same plane and the describing point, as was discovered by Sir *Christopher Wren*. And a pendulum between two such cycloids will oscillate in a similar and equal cycloid in equal times, as M. *Huygens* demonstrated. The descent of heavy bodies also in the time of one oscillation will be the same as M. *Huygens* exhibited.

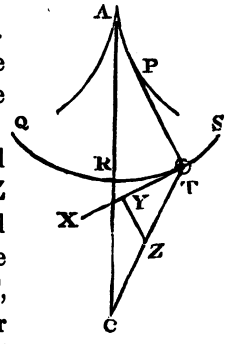
The propositions here demonstrated are adapted to the true constitution of the Earth, in so far as wheels moving in any of its great circles will describe, by the motions of nails fixed in their perimeters, cycloids without the globe; and pendulums, in mines and deep caverns of the Earth, must oscillate in cycloids within the globe, that those oscillations may be performed in equal times. For gravity (as will be shewn in the third book) decreases in its progress from the superficies of the Earth; upwards in a duplicate ratio of the distances from the centre of the Earth; downwards in a simple ratio of the same.

PROPOSITION LIII. PROBLEM XXXV.

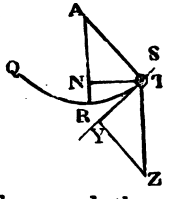
Granting the quadratures of curvilinear figures, it is required to find the forces with which bodies moving in given curve lines may always perform their oscillations in equal times.

Let the body T oscillate in any curve line STRQ, whose axis is AR passing through the centre of force C. Draw TX touching that curve in any place of the body T, and in that tangent TX take TY equal to the arc TR. The length of that arc is known from the common methods used

for the quadratures of figures. From the point Y draw the right line YZ perpendicular to the tangent. Draw CT meeting that perpendicular in Z, and the centripetal force will be proportional to the right line TZ. Q.E.I.



For if the force with which the body is attracted from T towards C be expressed by the right line TZ taken proportional to it, that force will be resolved into two forces TY, YZ, of which YZ drawing the body in the direction of the length of the thread PT, does not at all change its motion; whereas the other force TY directly accelerates or retards its motion in the curve STRQ. Wherefore since that force is as the space to be described TR, the accelerations or retardations of the body in describing two proportional parts (a greater and a less) of two oscillations, will be always as those parts, and therefore will cause those parts to be described together. But bodies which continually describe together parts proportional to the wholes, will describe the wholes together also. Q.E.D.



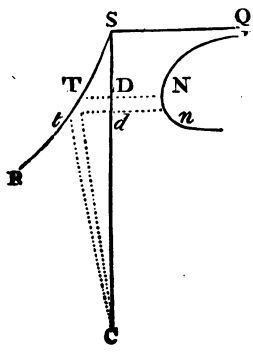
COR. 1. Hence if the body T, hanging by a rectilinear thread AT from the centre A, describe the circular arc STRQ, and in the mean time be acted on by any force tending downwards with parallel directions, which is to the uniform force of gravity as the arc TR to its sine TN, the times of the several oscillations will be equal. For because TZ, AR are parallel, the triangles ATN, ZTY are similar; and therefore TZ will be to AT as TY to TN; that is, if the uniform force of gravity be expressed by the given length AT, the force TZ, by which the oscillations become isochronous, will be to the force of gravity AT, as the arc TR equal to TY is to TN the sine of that arc.

COR. 2. And therefore in clocks, if forces were impressed by some machine upon the pendulum which preserves the motion, and so compounded with the force of gravity that the whole force tending downwards should be always as a line produced by applying the rectangle under the arc TR and the radius AR to the sine TN, all the oscillations will become isochronous.

PROPOSITION LIV. PROBLEM XXXVI.

Granting the quadratures of curvilinear figures, it is required to find the times in which bodies by means of any centripetal force will descend or ascend in any curve lines described in a plane passing through the centre of force.

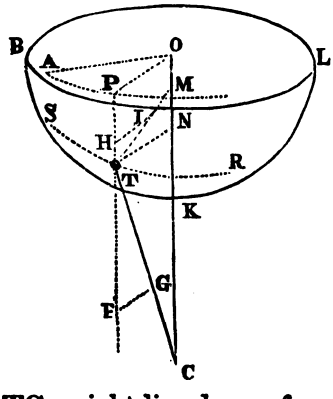
Let the body descend from any place S, and move in any curve S'T*t*R given in a plane passing through the centre of force C. Join CS, and let



it be divided into innumerable equal parts, and let D*d* be one of those parts. From the centre C, with the intervals CD, C*d*, let the circles DT, *dt* be described, meeting the curve line ST*t*R in T and *t*. And because the law of centripetal force is given, and also the altitude CS from which the body at first fell, there will be given the velocity of the body in any other altitude CT (by Prop. XXXIX). But the time in which the body describes the lineola *Tt* is as the length of that lineola, that is, as the secant of the angle *t*TC directly, and the velocity inversely. Let the ordinate DN, proportional to this time, be made perpendicular to the right line CS at the point D, and because D*d* is given, the rectangle D*d* × DN, that is, the area DN*nd*, will be proportional to the same time. Therefore if PN*n* be a curve line in which the point N is perpetually found, and its asymptote be the right line SQ standing upon the line CS at right angles, the area SQPND will be proportional to the time in which the body in its descent hath described the line ST; and therefore that area being found, the time is also given. Q.E.I.

PROPOSITION LV. THEOREM XIX.

If a body move in any curve superficies, whose axis passes through the centre of force, and from the body a perpendicular be let fall upon the axis; and a line parallel and equal thereto be drawn from any given point of the axis; I say, that this parallel line will describe an area proportional to the time.



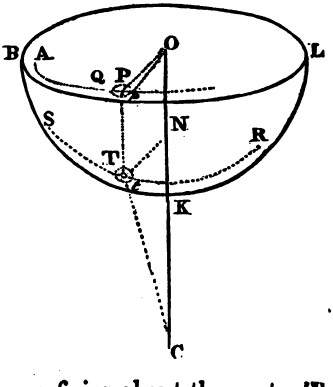
Let BKL be a curve superficies, T a body revolving in it, STR a trajectory which the body describes in the same, S the beginning of the trajectory, OMK the axis of the curve superficies, TN a right line let fall perpendicularly from the body to the axis; OP a line parallel and equal thereto drawn from the given point O in the axis; AP the orthographic projection of the trajectory described by the point P in the plane AOP in which the revolving line OP is found; A the beginning of that projection, answering to the point S; TC a right line drawn from the body to the centre; TG a part thereof proportional to the centripetal force with which the body tends towards the centre C; TM a right line perpendicular to the curve superficies; TI a part thereof proportional to the force of pressure with which the body urges

the superficies, and therefore with which it is again repelled by the superficies towards M; PTF a right line parallel to the axis and passing through the body, and GF, IH right lines let fall perpendicularly from the points G and I upon that parallel PHTF. I say, now, that the area AOP, described by the radius OP from the beginning of the motion, is proportional to the time. For the force TG (by Cor. 2, of the Laws of Motion) is resolved into the forces TF, FG; and the force TI into the forces TH, HI; but the forces TF, TH, acting in the direction of the line PF perpendicular to the plane AOP, introduce no change in the motion of the body but in a direction perpendicular to that plane. Therefore its motion, so far as it has the same direction with the position of the plane, that is, the motion of the point P, by which the projection AP of the trajectory is described in that plane, is the same as if the forces TF, TH were taken away, and the body were acted on by the forces FG, HI alone; that is, the same as if the body were to describe in the plane AOP the curve AP by means of a centripetal force tending to the centre O, and equal to the sum of the forces FG and HI. But with such a force as that (by Prop. 1) the area AOP will be described proportional to the time. Q.E.D.

COR. By the same reasoning, if a body, acted on by forces tending to two or more centres in any the same right line CO, should describe in a free space any curve line ST, the area AOP would be always proportional to the time.

PROPOSITION LVI. PROBLEM XXXVII.

Granting the quadratures of curvilinear figures, and supposing that there are given both the law of centripetal force tending to a given centre, and the curve superficies whose axis passes through that centre; it is required to find the trajectory which a body will describe in that superficies, when going off from a given place with a given velocity, and in a given direction in that superficies.



The last construction remaining, let the body T go from the given place S, in the direction of a line given by position, and turn into the trajectory sought STR, whose orthographic projection in the plane BDO is AP. And from the given velocity of the body in the altitude SC, its velocity in any other altitude TC will be also given. With that velocity, in a given moment of time, let the body describe the particle T*t* of its trajectory, and let P*p* be the projection of that particle described in the plane AOP. Join O*p*, and a little circle being described upon the curve superficies about the centre T

with the interval T*t* let the projection of that little circle in the plane AOP be the ellipsis *p*Q. And because the magnitude of that little circle T*t*, and TN or PO its distance from the axis CO is also given, the ellipsis *p*Q will be given both in kind and magnitude, as also its position to the right line PO. And since the area PO*p* is proportional to the time, and therefore given because the time is given, the angle PO*p* will be given. And thence will be given *p* the common intersection of the ellipsis and the right line O*p*, together with the angle OP*p*, in which the projection AP*p* of the trajectory cuts the line OP. But from thence (by conferring Prop. XLI, with its 2d Cor.) the manner of determining the curve AP*p* easily appears. Then from the several points P of that projection erecting to the plane AOP, the perpendiculars PT meeting the curve superficies in T, there will be given the several points T of the trajectory. Q.E.I.

SECTION XI.

Of the motions of bodies tending to each other with centripetal forces.

I have hitherto been treating of the attractions of bodies towards an immovable centre; though very probably there is no such thing existent in nature. For attractions are made towards bodies, and the actions of the bodies attracted and attracting are always reciprocal and equal, by Law III; so that if there are two bodies, neither the attracted nor the attracting body is truly at rest, but both (by Cor. 4, of the Laws of Motion), being as it were mutually attracted, revolve about a common centre of gravity. And if there be more bodies, which are either attracted by one single one which is attracted by them again, or which all of them, attract each other mutually, these bodies will be so moved among themselves, as that their common centre of gravity will either be at rest, or move uniformly forward in a right line. I shall therefore at present go on to treat of the motion of bodies mutually attracting each other; considering the centripetal forces as attractions; though perhaps in a physical strictness they may more truly be called impulses. But these propositions are to be considered as purely mathematical; and therefore, laying aside all physical considerations, I make use of a familiar way of speaking, to make myself the more easily understood by a mathematical reader.

PROPOSITION LVII. THEOREM XX.

Two bodies attracting each other mutually describe similar figures about their common centre of gravity, and about each other mutually.

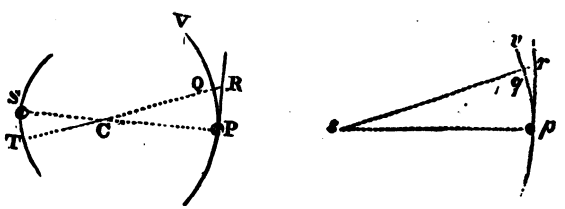
For the distances of the bodies from their common centre of gravity are reciprocally as the bodies; and therefore in a given ratio to each other: and thence, by composition of ratios, in a given ratio to the whole distance

between the bodies. Now these distances revolve about their common term with an equable angular motion, because lying in the same right line they never change their inclination to each other mutually But right lines that are in a given ratio to each other, and revolve about their terms with an equal angular motion, describe upon planes, which either rest with those terms, or move with any motion not angular, figures entirely similar round those terms. Therefore the figures described by the revolution of these distances are similar. Q.E.D.

PROPOSITION LVIII. THEOREM XXI.

If two bodies attract each other mutually with forces of any kind, and in the mean time revolve about the common centre of gravity; I say, that, by the same forces, there may be described round either body unmoved a figure similar and equal to the figures which the bodies so moving describe round each other mutually.

Let the bodies S and P revolve about their common centre of gravity C, proceeding from S to T, and from P to Q. From the given point *s* let



there be continually drawn *sp*, *sq*, equal and parallel to SP, TQ; and the curve *pqv*, which the point *p* describes in its revolution round the immovable point *s*, will be similar and equal to the curves which the bodies S and P describe about each other mutually; and therefore, by Theor. XX, similar to the curves ST and PQV which the same bodies describe about their common centre of gravity C; and that because the proportions of the lines SC, CP, and SP or *sp*, to each other, are given.

CASE 1. The common centre of gravity C (by Cor. 4, of the Laws of Motion) is either at rest, or moves uniformly in a right line. Let us first suppose it at rest, and in *s* and *p* let there be placed two bodies, one immovable in *s*, the other movable in *p*, similar and equal to the bodies S and P. Then let the right lines PR and *pr* touch the curves PQ and *pq* in P and *p*, and produce CQ and *sq* to R and *r*. And because the figures CPRQ, *sprq* are similar, RQ will be to *rq* as CP to *sp*, and therefore in a given ratio. Hence if the force with which the body P is attracted towards the body S, and by consequence towards the intermediate point the centre C, were to the force with which the body *p* is attracted towards the centre *s*. in the same given ratio, these forces would in equal times attract

the bodies from the tangents PR, *pr* to the arcs PQ, *pq*, through the intervals proportional to them RQ, *rq*; and therefore this last force (tending to *s*) would make the body *p* revolve in the curve *pqv*, which would become similar to the curve PQV, in which the first force obliges the body P to revolve; and their revolutions would be completed in the same times. But because those forces are not to each other in the ratio of CP to *sp*, but (by reason of the similarity and equality of the bodies S and *s*, P and *p*, and the equality of the distances SP, *sp*) mutually equal, the bodies in equal times will be equally drawn from the tangents; and therefore that the body *p* may be attracted through the greater interval *rq*, there is required a greater time, which will be in the subduplicate ratio of the intervals; because, by Lemma X, the spaces described at the very beginning of the motion are in a duplicate ratio of the times. Suppose, then the velocity of the body *p* to be to the velocity of the body P in a subduplicate ratio of the distance *sp* to the distance CP, so that the arcs *pq*, PQ, which are in a simple proportion to each other, may be described in times that are in a subduplicate ratio of the distances; and the bodies P, *p*, always attracted by equal forces, will describe round the quiescent centres C and *s* similar figures PQV, *pqv*, the latter of which *pqv* is similar and equal to the figure which the body P describes round the movable body S. Q.E.D.

CASE 2. Suppose now that the common centre of gravity, together with the space in which the bodies are moved among themselves, proceeds uniformly in a right line; and (by Cor. 6, of the Laws of Motion) all the motions in this space will be performed in the same manner as before; and therefore the bodies will describe mutually about each other the same figures as before, which will be therefore similar and equal to the figure *pqv*. Q.E.D.

COR. 1. Hence two bodies attracting each other with forces proportional to their distance, describe (by Prop. X) both round their common centre of gravity, and round each other mutually concentrical ellipses; and, *vice versa*, if such figures are described, the forces are proportional to the distances.

COR. 2. And two bodies, whose forces are reciprocally proportional to the square of their distance, describe (by Prop. XI, XII, XIII), both round their common centre of gravity, and round each other mutually, conic sections having their focus in the centre about which the figures are described. And, *vice versa*, if such figures are described, the centripetal forces are reciprocally proportional to the squares of the distance.

COR. 3. Any two bodies revolving round their common centre of gravity describe areas proportional to the times, by radii drawn both to that centre and to each other mutually.

PROPOSITION LIX. THEOREM XXII.

The periodic time of two bodies S *and* P *revolving round their common centre of gravity* C, *is to the periodic time of one of the bodies* P *revolving round the other* S *remaining unmoved, and describing a figure similar and equal to those which the bodies describe about each other mutually, in a subduplicate ratio of the other body* S *to the sum of the bodies* S + P.

For, by the demonstration of the last Proposition, the times in which any similar arcs PQ and *pq* are described are in a subduplicate ratio of the distances CP and SP, or *sp*, that is, in a subduplicate ratio of the ody S to the sum of the bodies S + P. And by composition of ratios, the sums of the times in which all the similar arcs PQ and *pq* are described, that is, the whole times in which the whole similar figures are described are in the same subduplicate ratio. Q.E.D.

PROPOSITION LX. THEOREM XXIII.

If two bodies S *and* P, *attracting each other with forces reciprocally proportional to the squares of their distance, revolve about their common centre of gravity; I say, that the principal axis of the ellipsis which either of the bodies, as* P, *describes by this motion about the other* S, *will be to the principal axis of the ellipsis, which the same body* P *may describe in the same periodical time about the other body* S *quiescent, as the sum of the two bodies* S + P *to the first of two mean proportionals between that sum and the other body* S.

For if the ellipses described were equal to each other, their periodic times by the last Theorem would be in a subduplicate ratio of the body S to the sum of the bodies S + P. Let the periodic time in the latter ellipsis be diminished in that ratio, and the periodic times will become equal; but, by Prop. XV, the principal axis of the ellipsis will be diminished in a ratio sesquiplicate to the former ratio; that is, in a ratio to which the ratio of S to S + P is triplicate; and therefore that axis will be to the principal axis of the other ellipsis as the first of two mean proportionals between S + P and S to S + P. And inversely the principal axis of the ellipsis described about the movable body will be to the principal axis of that described round the immovable as S + P to the first of two mean proportionals between S + P and S. Q.E.D.

PROPOSITION LXI. THEOREM XXIV.

If two bodies attracting each other with any kind of forces, and not otherwise agitated or obstructed, are moved in any manner whatsoever, those motions will be the same as if they did not at all attract each other mutually, but were both attracted with the same forces by a third body placed in their common centre of gravity; and the law of the

attracting forces will be the same in respect of the distance of the bodies from the common centre, as in respect of the distance between the two bodies.

For those forces with which the bodies attract each other mutually, by tending to the bodies, tend also to the common centre of gravity lying directly between them; and therefore are the same as if they proceeded from an intermediate body. Q.E.D.

And because there is given the ratio of the distance of either body from that common centre to the distance between the two bodies, there is given, of course, the ratio of any power of one distance to the same power of the other distance; and also the ratio of any quantity derived in any manner from one of the distances compounded any how with given quantities, to another quantity derived in like manner from the other distance, and as many given quantities having that given ratio of the distances to the first. Therefore if the force with which one body is attracted by another be directly or inversely as the distance of the bodies from each other, or as any power of that distance; or, lastly, as any quantity derived after any manner from that distance compounded with given quantities; then will the same force with which the same body is attracted to the common centre of gravity be in like manner directly or inversely as the distance of the attracted body from the common centre, or as any power of that distance; or, lastly, as a quantity derived in like sort from that distance compounded with analogous given quantities. That is, the law of attracting force will be the same with respect to both distances. Q.E.D.

PROPOSITION LXII. PROBLEM XXXVIII.

To determine the motions of two bodies which attract each other with forces reciprocally proportional to the squares of the distance between them, and are let fall from given places.

The bodies, by the last Theorem, will be moved in the same manner as if they were attracted by a third placed in the common centre of their gravity; and by the hypothesis that centre will be quiescent at the beginning of their motion, and therefore (by Cor. 4, of the Laws of Motion) will be always quiescent. The motions of the bodies are therefore to be determined (by Prob. XXV) in the same manner as if they were impelled by forces tending to that centre; and then we shall have the motions of the bodies attracting each other mutually. Q.E.I.

PROPOSITION LXIII. PROBLEM XXXIX.

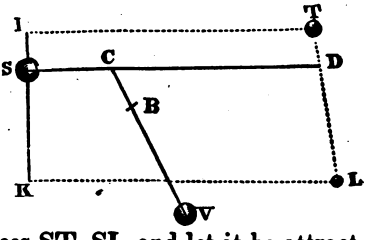
To determine the motions of two bodies attracting each other with forces reciprocally proportional to the squares of their distance, and going off from given places in given directions with given velocities.

The motions of the bodies at the beginning being given, there is given

also the uniform motion of the common centre of gravity, and the motion of the space which moves along with this centre uniformly in a right line, and also the very first, or beginning motions of the bodies in respect of this space. Then (by Cor. 5, of the Laws, and the last Theorem) the subsequent motions will be performed in the same manner in that space, as if that space together with the common centre of gravity were at rest, and as if the bodies did not attract each other, but were attracted by a third body placed in that centre. The motion therefore in this movable space of each body going off from a given place, in a given direction, with a given velocity, and acted upon by a centripetal force tending to that centre, is to be determined by Prob. IX and XXVI, and at the same time will be obtained the motion of the other round the same centre. With this motion compound the uniform progressive motion of the entire system of the space and the bodies revolving in it, and there will be obtained the absolute motion of the bodies in immovable space. Q.E.I.

PROPOSITION LXIV. PROBLEM XL.

Supposing forces with which bodies mutually attract each other to increase in a simple ratio of their distances from the centres; it is required to find the motions of several bodies among themselves.



Suppose the first two bodies T and L to have their common centre of gravity in D. These, by Cor. 1, Theor. XXI, will describe ellipses having their centres in D, the magnitudes of which ellipses are known by Prob. V.

Let now a third body S attract the two former T and L with the accelerative forces ST, SL, and let it be attracted again by them. The force ST (by Cor. 2, of the Laws of Motion) is resolved into the forces SD, DT; and the force SL into the forces SD and DL. Now the forces DT, DL, which are as their sum TL, and therefore as the accelerative forces with which the bodies T and L attract each other mutually, added to the forces of the bodies T and L, the first to the first, and the last to the last, compose forces proportional to the distances DT and DL as before, but only greater than those former forces; and therefore (by Cor. 1, Prop. X, and Cor. 1, and 8, Prop. IV) they will cause those bodies to describe ellipses as before, but with a swifter motion. The remaining accelerative forces SD and DL, by the motive forces SD × T and SD × L, which are as the bodies attracting those bodies equally and in the direction of the lines TI, LK parallel to DS, do not at all change their situations with respect to one another, but cause them equally to approach to the line IK; which must be imagined drawn through the middle of the body S, and perpendicular to the line DS. But that approach to the line

IK will be hindered by causing the system of the bodies T and L on one side, and the body S on the other, with proper velocities, to revolve round the common centre of gravity C. With such a motion the body S, because the sum of the motive forces SD × T and SD × L is proportional to the distance CS, tends to the centre C, will describe an ellipsis round the same centre C; and the point D, because the lines CS and CD are proportional, will describe a like ellipsis over against it. But the bodies T and L, attracted by the motive forces SD × T and SD × L, the first by the first, and the last by the last, equally and in the direction of the parallel lines TI and LK, as was said before, will (by Cor. 5 and 6, of the Laws of Motion) continue to describe their ellipses round the movable centre D, as before. Q.E.I.

Let there be added a fourth body V, and, by the like reasoning, it will be demonstrated that this body and the point C will describe ellipses about the common centre of gravity B; the motions of the bodies T, L, and S round the centres D and C remaining the same as before; but accelerated. And by the same method one may add yet more bodies at pleasure. Q.E.I.

This would be the case, though the bodies T and L attract each other mutually with accelerative forces either greater or less than those with which they attract the other bodies in proportion to their distance. Let all the mutual accelerative attractions be to each other as the distances multiplied into the attracting bodies; and from what has gone before it will easily be concluded that all the bodies will describe different ellipses with equal periodical times about their common centre of gravity B, in an immovable plane. Q.E.I.

PROPOSITION LXV. THEOREM XXV.

Bodies, whose forces decrease in a duplicate ratio of their distances from their centres, may move among themselves in ellipses; and by radii drawn to the foci may describe areas proportional to the times very nearly.

In the last Proposition we demonstrated that case in which the motions will be performed exactly in ellipses. The more distant the law of the forces is from the law in that case, the more will the bodies disturb each other's motions; neither is it possible that bodies attracting each other mutually according to the law supposed in this Proposition should move exactly in ellipses, unless by keeping a certain proportion of distances from each other. However, in the following cases the orbits will not much differ from ellipses.

CASE 1. Imagine several lesser bodies to revolve about some very great one at different distances from it, and suppose absolute forces tending to every one of the bodies proportional to each. And because (by Cor. 4, of the Laws) the common centre of gravity of them all is either at rest, or

moves uniformly forward in a right line, suppose the lesser bodies so small that the great body may be never at a sensible distance from that centre; and then the great body will, without any sensible error, be either at rest, or move uniformly forward in a right line; and the lesser will revolve about that great one in ellipses, and by radii drawn thereto will describe areas proportional to the times; if we except the errors that may be introduced by the receding of the great body from the common centre of gravity, or by the mutual actions of the lesser bodies upon each other. But the lesser bodies may be so far diminished, as that this recess and the mutual actions of the bodies on each other may become less than any assignable; and therefore so as that the orbits may become ellipses, and the areas answer to the times, without any error that is not less than any assignable. Q.E.O.

CASE 2. Let us imagine a system of lesser bodies revolving about a very great one in the manner just described, or any other system of two bodies revolving about each other to be moving uniformly forward in a right line, and in the mean time to be impelled sideways by the force of another vastly greater body situate at a great distance. And because the equal accelerative forces with which the bodies are impelled in parallel directions do not change the situation of the bodies with respect to each other, but only oblige the whole system to change its place while the parts still retain their motions among themselves, it is manifest that no change in those motions of the attracted bodies can arise from their attractions towards the greater, unless by the inequality of the accelerative attractions, or by the inclinations of the lines towards each other, in whose directions the attractions are made. Suppose, therefore, all the accelerative attractions made towards the great body to be among themselves as the squares of the distances reciprocally; and then, by increasing the distance of the great body till the differences of the right lines drawn from that to the others in respect of their length, and the inclinations of those lines to each other, be less than any given, the motions of the parts of the system will continue without errors that are not less than any given. And because, by the small distance of those parts from each other, the whole system is attracted as if it were but one body, it will therefore be moved by this attraction as if it were one body; that is, its centre of gravity will describe about the great body one of the conic sections (that is, a parabola or hyperbola when the attraction is but languid and an ellipsis when it is more vigorous); and by radii drawn thereto, it will describe areas proportional to the times, without any errors but those which arise from the distances of the parts, which are by the supposition exceedingly small, and may be diminished at pleasure. Q.E.O.

By a like reasoning one may proceed to more compounded cases *in infinitum*.

COR 1. In the second Case, the nearer the very great body approaches to

the system of two or more revolving bodies, the greater will the perturbation be of the motions of the parts of the system among themselves; because the inclinations of the lines drawn from that great body to those parts become greater; and the inequality of the proportion is also greater.

Cor. 2. But the perturbation will be greatest of all, if we suppose the accelerative attractions of the parts of the system towards the greatest body of all are not to each other reciprocally as the squares of the distances from that great body; especially if the inequality of this proportion be greater than the inequality of the proportion of the distances from the great body. For if the accelerative force, acting in parallel directions and equally, causes no perturbation in the motions of the parts of the system, it must of course, when it acts unequally, cause a perturbation somewhere, which will be greater or less as the inequality is greater or less. The excess of the greater impulses acting upon some bodies, and not acting upon others, must necessarily change their situation among themselves. And this perturbation, added to the perturbation arising from the inequality and inclination of the lines, makes the whole perturbation greater.

Cor. 3. Hence if the parts of this system move in ellipses or circles without any remarkable perturbation, it is manifest that, if they are at all impelled by accelerative forces tending to any other bodies, the impulse is very weak, or else is impressed very near equally and in parallel directions upon all of them.

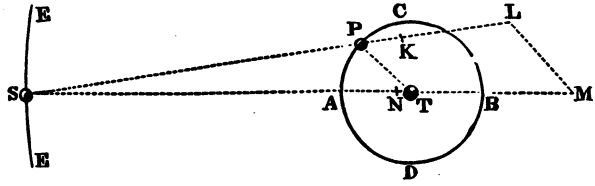
PROPOSITION LXVI. THEOREM XXVI.

If three bodies whose forces decrease in a duplicate ratio of the distances attract each other mutually; and the accelerative attractions of any two towards the third be between themselves reciprocally as the squares of the distances; and the two least revolve about the greatest; I say, that the interior of the two revolving bodies will, by radii drawn to the innermost and greatest, describe round that body areas more proportional to the times, and a figure more approaching to that of an ellipsis having its focus in the point of concourse of the radii, if that great body be agitated by those attractions, than it would do if that great body were not attracted at all by the lesser, but remained at rest; or than it would if that great body were very much more or very much less attracted, or very much more or very much less agitated, by the attractions.

This appears plainly enough from the demonstration of the second Corollary of the foregoing Proposition; but it may be made out after this manner by a way of reasoning more distinct and more universally convincing.

Case 1. Let the lesser bodies P and S revolve in the same plane about the greatest body T, the body P describing the interior orbit PAB, and S

the exterior orbit ESE. Let SK be the mean distance of the bodies P and S; and let the accelerative attraction of the body P towards S, at that mean distance, be expressed by that line SK. Make SL to SK as the



square of SK to the square of SP, and SL will be the accelerative attraction of the body P towards S at any distance SP. Join PT, and draw LM parallel to it meeting ST in M; and the attraction SL will be resolved (by Cor. 2, of the Laws of Motion) into the attractions SM, LM. And so the body P will be urged with a threefold accelerative force. One of these forces tends towards T, and arises from the mutual attraction of the bodies T and P. By this force alone the body P would describe round the body T, by the radius PT, areas proportional to the times, and an ellipsis whose focus is in the centre of the body T; and this it would do whether the body T remained unmoved, or whether it were agitated by that attraction. This appears from Prop. XI, and Cor. 2 and 3 of Theor. XXI. The other force is that of the attraction LM, which, because it tends from P to T, will be superadded to and coincide with the former force; and cause the areas to be still proportional to the times, by Cor. 3, Theor. XXI. But because it is not reciprocally proportional to the square of the distance PT, it will compose, when added to the former, a force varying from that proportion; which variation will be the greater by how much the proportion of this force to the former is greater, *cæteris paribus*. Therefore, since by Prop. XI, and by Cor. 2, Theor. XXI, the force with which the ellipsis is described about the focus T ought to be directed to that focus, and to be reciprocally proportional to the square of the distance PT, that compounded force varying from that proportion will make the orbit PAB vary from the figure of an ellipsis that has its focus in the point T; and so much the more by how much the variation from that proportion is greater; and by consequence by how much the proportion of the second force LM to the first force is greater, *cæteris paribus*. But now the third force SM, attracting the body P in a direction parallel to ST, composes with the other forces a new force which is no longer directed from P to T; and which varies so much more from this direction by how much the proportion of this third force to the other forces is greater, *cæteris paribus;* and therefore causes the body P to describe, by the radius TP, areas no longer proportional to the times; and therefore makes the variation from that proportionality so much greater by how much the proportion of this force to the others is greater. But this third force will increase the variation of the orbit PAB from the

elliptical figure before-mentioned upon two accounts; first because that force is not directed from P to T; and, secondly, because it is not reciprocally proportional to the square of the distance PT. These things being premised, it is manifest that the areas are then most nearly proportional to the times, when that third force is the least possible, the rest preserving their former quantity; and that the orbit PAB does then approach nearest to the elliptical figure above-mentioned, when both the second and third, but especially the third force, is the least possible; the first force remaining in its former quantity.

Let the accelerative attraction of the body T towards S be expressed by the line SN; then if the accelerative attractions SM and SN were equal, these, attracting the bodies T and P equally and in parallel directions would not at all change their situation with respect to each other. The motions of the bodies between themselves would be the same in that case as if those attractions did not act at all, by Cor. 6, of the Laws of Motion. And, by a like reasoning, if the attraction SN is less than the attraction SM, it will take away out of the attraction SM the part SN, so that there will remain only the part (of the attraction) MN to disturb the proportionality of the areas and times, and the elliptical figure of the orbit. And in like manner if the attraction SN be greater than the attraction SM, the perturbation of the orbit and proportion will be produced by the difference MN alone. After this manner the attraction SN reduces always the attraction SM to the attraction MN, the first and second attractions remaining perfectly unchanged; and therefore the areas and times come then nearest to proportionality, and the orbit PAB to the above-mentioned elliptical figure, when the attraction MN is either none, or the least that is possible; that is, when the accelerative attractions of the bodies P and T approach as near as possible to equality; that is, when the attraction SN is neither none at all, nor less than the least of all the attractions SM, but is, as it were, a mean between the greatest and least of all those attractions SM, that is, not much greater nor much less than the attraction SK. Q.E.D.

CASE 2. Let now the lesser bodies P, S, revolve about a greater T in different planes; and the force LM, acting in the direction of the line PT situate in the plane of the orbit PAB, will have the same effect as before; neither will it draw the body P from the plane of its orbit. But the other force NM acting in the direction of a line parallel to ST (and which, therefore, when the body S is without the line of the nodes is inclined to the plane of the orbit PAB), besides the perturbation of the motion just now spoken of as to longitude, introduces another perturbation also as to latitude, attracting the body P out of the plane of its orbit. And this perturbation, in any given situation of the bodies P and T to each other, will be as the generating force MN; and therefore becomes least when the force MN is least, that is (as was just now shewn), where the attraction SN is not much greater nor much less than the attraction SK. Q.E.D.

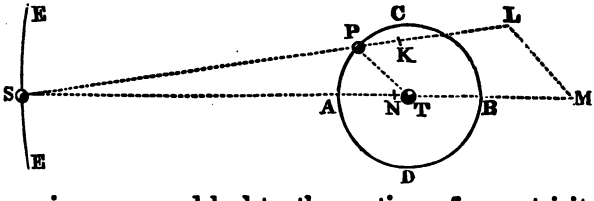
COR. 1. Hence it may be easily collected, that if several less bodies P S, R, &c., revolve about a very great body T, the motion of the innermost revolving body P will be least disturbed by the attractions of the others, when the great body is as well attracted and agitated by the rest (according to the ratio of the accelerative forces) as the rest are by each other mutually.

COR. 2. In a system of three bodies, T, P, S, if the accelerative attractions of any two of them towards a third be to each other reciprocally as the squares of the distances, the body P, by the radius PT, will describe its area about the body T swifter near the conjunction A and the opposition B than it will near the quadratures C and D. For every force with which the body P is acted on and the body T is not, and which does not act in the direction of the line PT, does either accelerate or retard the description of the area, according as it is directed, whether *in consequentia* or *in antecedentia*. Such is the force NM. This force in the passage of the body P from C to A is directed *in consequentia* to its motion, and therefore accelerates it; then as far as D *in antecedentia*, and retards the motion; then *in consequentia* as far as B; and lastly *in antecedentia* as it moves from B to C.

COR. 3. And from the same reasoning it appears that the body P *cæteris paribus*, moves more swiftly in the conjunction and opposition than in the quadratures.

COR. 4. The orbit of the body P, *cæteris paribus*, is more curve at the quadratures than at the conjunction and opposition. For the swifter bodies move, the less they deflect from a rectilinear path. And besides the force KL, or NM, at the conjunction and opposition, is contrary to the force with which the body T attracts the body P, and therefore diminishes that force; but the body P will deflect the less from a rectilinear path the less it is impelled towards the body T.

COR. 5. Hence the body P, *cæteris paribus*, goes farther from the body T at the quadratures than at the conjunction and opposition. This is said,



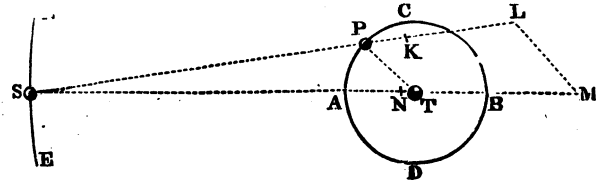
however, supposing no regard had to the motion of eccentricity. For if the orbit of the body P be eccentrical, its eccentricity (as will be shewn presently by Cor. 9) will be greatest when the apsides are in the syzygies; and thence it may sometimes come to pass that the body P, in its near approach to the farther apsis, may go farther from the body T at the syzygies than at the quadratures.

COR. 6. Because the centripetal force of the central body T, by which

the body P is retained in its orbit, is increased at the quadratures by the addition caused by the force LM, and diminished at the syzygies by the subduction caused by the force KL, and, because the force KL is greater than LM, it is more diminished than increased; and, moreover, since that centripetal force (by Cor. 2, Prop. IV) is in a ratio compounded of the simple ratio of the radius TP directly, and the duplicate ratio of the periodical time inversely; it is plain that this compounded ratio is diminished by the action of the force KL; and therefore that the periodical time, supposing the radius of the orbit PT to remain the same, will be increased, and that in the subduplicate of that ratio in which the centripetal force is diminished; and, therefore, supposing this radius increased or diminished, the periodical time will be increased more or diminished less than in the sesquiplicate ratio of this radius, by Cor. 6, Prop. IV. If that force of the central body should gradually decay, the body P being less and less attracted would go farther and farther from the centre T; and, on the contrary, if it were increased, it would draw nearer to it. Therefore if the action of the distant body S, by which that force is diminished, were to increase and decrease by turns, the radius TP will be also increased and diminshed by turns; and the periodical time will be increased and diminished in a ratio compounded of the sesquiplicate ratio of the radius, and of the subduplicate of that ratio in which the centripetal force of the central body T is diminished or increased, by the increase or decrease of the action of the distant body S.

Cor. 7. It also follows, from what was before laid down, that the axis of the ellipsis described by the body P, or the line of the apsides, does as to its angular motion go forwards and backwards by turns, but more forwards than backwards, and by the excess of its direct motion is in the whole carried forwards. For the force with which the body P is urged to the body T at the quadratures, where the force MN vanishes, is compounded of the force LM and the centripetal force with which the body T attracts the body P. The first force LM, if the distance PT be increased, is increased in nearly the same proportion with that distance, and the other force decreases in the duplicate ratio of the distance; and therefore the sum of these two forces decreases in a less than the duplicate ratio of the distance PT; and therefore, by Cor. 1, Prop. XLV, will make the line of the apsides, or, which is the same thing, the upper apsis, to go backward. But at the conjunction and opposition the force with which the body P is urged towards the body T is the difference of the force KL, and of the force with which the body T attracts the body P; and that difference, because the force KL is very nearly increased in the ratio of the distance PT, decreases in more than the duplicate ratio of the distance PT; and therefore, by Cor. 1, Prop. XLV, causes the line of the apsides to go forwards. In the places between the syzygies and the quadratures, the motion

of the line of the apsides depends upon both of these causes conjunctly, so that it either goes forwards or backwards in proportion to the excess of one of these causes above the other. Therefore since the force KL in the syzygies is almost twice as great as the force LM in the quadratures, the excess will be on the side of the force KL, and by consequence the line of the apsides will be carried forwards. The truth of this and the foregoing



Corollary will be more easily understood by conceiving the system of the two bodies T and P to be surrounded on every side by several bodies S, S, S, &c., disposed about the orbit ESE. For by the actions of these bodies the action of the body T will be diminished on every side, and decrease in more than a duplicate ratio of the distance.

COR. 8. But since the progress or regress of the apsides depends upon the decrease of the centripetal force, that is, upon its being in a greater or less ratio than the duplicate ratio of the distance TP, in the passage of the body from the lower apsis to the upper; and upon a like increase in its return to the lower apsis again; and therefore becomes greatest where the proportion of the force at the upper apsis to the force at the lower apsis recedes farthest from the duplicate ratio of the distances inversely; it is plain, that, when the apsides are in the syzygies, they will, by reason of the subducting force KL or NM — LM, go forward more swiftly; and in the quadratures by the additional force LM go backward more slowly. Because the velocity of the progress or slowness of the regress is continued for a long time; this inequality becomes exceedingly great.

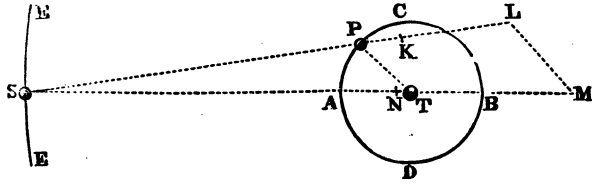
COR. 9. If a body is obliged, by a force reciprocally proportional to the square of its distance from any centre, to revolve in an ellipsis round that centre; and afterwards in its descent from the upper apsis to the lower apsis, that force by a perpetual accession of new force is increased in more than a duplicate ratio of the diminished distance; it is manifest that the body, being impelled always towards the centre by the perpetual accession of this new force, will incline more towards that centre than if it were urged by that force alone which decreases in a duplicate ratio of the diminished distance, and therefore will describe an orbit interior to that elliptical orbit, and at the lower apsis approaching nearer to the centre than before. Therefore the orbit by the accession of this new force will become more eccentrical. If now, while the body is returning from the lower to the upper apsis, it should decrease by the same degrees by which it increases before the body would return to its first distance; and there-

fore if the force decreases in a yet greater ratio, the body, being now less attracted than before, will ascend to a still greater distance, and so the eccentricity of the orbit will be increased still more. Therefore if the ratio of the increase and decrease of the centripetal force be augmented each revolution, the eccentricity will be augmented also; and, on the contrary, if that ratio decrease, it will be diminished.

Now, therefore, in the system of the bodies T, P, S, when the apsides of the orbit PAB are in the quadratures, the ratio of that increase and decrease is least of all, and becomes greatest when the apsides are in the syzygies. If the apsides are placed in the quadratures, the ratio near the apsides is less, and near the syzygies greater, than the duplicate ratio of the distances; and from that greater ratio arises a direct motion of the line of the apsides, as was just now said. But if we consider the ratio of the whole increase or decrease in the progress between the apsides, this is less than the duplicate ratio of the distances. The force in the lower is to the force in the upper apsis in less than a duplicate ratio of the distance of the upper apsis from the focus of the ellipsis to the distance of the lower apsis from the same focus; and, contrariwise, when the apsides are placed in the syzygies, the force in the lower apsis is to the force in the upper apsis in a greater than a duplicate ratio of the distances. For the forces LM in the quadratures added to the forces of the body T compose forces in a less ratio; and the forces KL in the syzygies subducted from the forces of the body T, leave the forces in a greater ratio. Therefore the ratio of the whole increase and decrease in the passage between the apsides is least at the quadratures and greatest at the syzygies; and therefore in the passage of the apsides from the quadratures to the syzygies it is continually augmented, and increases the eccentricity of the ellipsis; and in the passage from the syzygies to the quadratures it is perpetually decreasing, and diminishes the eccentricity.

COR. 10. That we may give an account of the errors as to latitude, let us suppose the plane of the orbit EST to remain immovable; and from the cause of the errors above explained, it is manifest, that, of the two forces NM, ML, which are the only and entire cause of them, the force ML acting always in the plane of the orbit PAB never disturbs the motions as to latitude; and that the force NM, when the nodes are in the syzygies, acting also in the same plane of the orbit, does not at that time affect those motions. But when the nodes are in the quadratures, it disturbs them very much, and, attracting the body P perpetually out of the plane of its orbit, it diminishes the inclination of the plane in the passage of the body from the quadratures to the syzygies, and again increases the same in the passage from the syzygies to the quadratures. Hence it comes to pass that when the body is in the syzygies, the inclination is then least of all, and returns to the first magnitude nearly, when the body

arrives at the next node. But if the nodes are situate at the octants after the quadratures, that is, between C and A, D and B, it will appear, from



what was just now shewn, that in the passage of the body P from either node to the ninetieth degree from thence, the inclination of the plane is perpetually diminished; then, in the passage through the next 45 degrees to the next quadrature, the inclination is increased; and afterwards, again, in its passage through another 45 degrees to the next node, it is diminished. Therefore the inclination is more diminished than increased, and is therefore always less in the subsequent node than in the preceding one. And, by a like reasoning, the inclination is more increased than diminished when the nodes are in the other octants between A and D, B and C. The inclination, therefore, is the greatest of all when the nodes are in the syzygies In their passage from the syzygies to the quadratures the inclination is diminished at each appulse of the body to the nodes; and becomes least of all when the nodes are in the quadratures, and the body in the syzygies; then it increases by the same degrees by which it decreased before; and, when the nodes come to the next syzygies, returns to its former magnitude.

COR. 11. Because when the nodes are in the quadratures the body P is perpetually attracted from the plane of its orbit; and because this attraction is made towards S in its passage from the node C through the conjunction A to the node D; and to the contrary part in its passage from the node D through the opposition B to the node C; it is manifest that, in its motion from the node C, the body recedes continually from the former plane CD of its orbit till it comes to the next node; and therefore at that node, being now at its greatest distance from the first plane CD, it will pass through the plane of the orbit EST not in D, the other node of that plane, but in a point that lies nearer to the body S, which therefore becomes a new place of the node *in antecedentia* to its former place. And, by a like reasoning, the nodes will continue to recede in their passage from this node to the next. The nodes, therefore, when situate in the quadratures, recede perpetually; and at the syzygies, where no perturbation can be produced in the motion as to latitude, are quiescent: in the intermediate places they partake of both conditions, and recede more slowly; and, therefore, being always either retrograde or stationary, they will be carried backwards, or *in antecedentia*, each revolution.

COR. 12. All the errors described in these corrollaries are a little greater

at the conjunction of the bodies P, S, than at their opposition; because the generating forces NM and ML are greater.

Cor. 13. And since the causes and proportions of the errors and variations mentioned in these Corollaries do not depend upon the magnitude of the body S, it follows that all things before demonstrated will happen, if the magnitude of the body S be imagined so great as that the system of the two bodies P and T may revolve about it. And from this increase of the body S, and the consequent increase of its centripetal force, from which the errors of the body P arise, it will follow that all these errors, at equal distances, will be greater in this case, than in the other where the body S revolves about the system of the bodies P and T.

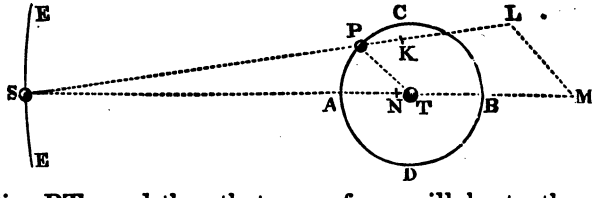
Cor. 14. But since the forces NM, ML, when the body S is exceedingly distant, are very nearly as the force SK and the ratio PT to ST conjunctly; that is, if both the distance PT, and the absolute force of the body S be given, as ST^3 reciprocally; and since those forces NM, ML are the causes of all the errors and effects treated of in the foregoing Corollaries; it is manifest that all those effects, if the system of bodies T and P continue as before, and only the distance ST and the absolute force of the body S be changed, will be very nearly in a ratio compounded of the direct ratio of the absolute force of the body S, and the triplicate inverse ratio of the distance ST. Hence if the system of bodies T and P revolve about a distant body S, those forces NM, ML, and their effects, will be (by Cor. 2 and 6, Prop IV) reciprocally in a duplicate ratio of the periodical time. And thence, also, if the magnitude of the body S be proportional to its absolute force, those forces NM, ML, and their effects, will be directly as the cube of the apparent diameter of the distant body S viewed from T, and so *vice versa*. For these ratios are the same as the compounded ratio above mentioned.

Cor. 15. And because if the orbits ESE and PAB, retaining their figure, proportions, and inclination to each other, should alter their magnitude; and the forces of the bodies S and T should either remain, or be changed in any given ratio; these forces (that is, the force of the body T, which obliges the body P to deflect from a rectilinear course into the orbit PAB, and the force of the body S, which causes the body P to deviate from that orbit) would act always in the same manner, and in the same proportion; it follows, that all the effects will be similar and proportional, and the times of those effects proportional also; that is, that all the linear errors will be as the diameters of the orbits, the angular errors the same as before; and the times of similar linear errors, or equal angular errors, as the periodical times of the orbits.

Cor. 16. Therefore if the figures of the orbits and their inclination to each other be given, and the magnitudes, forces, and distances of the bodies be any how changed, we may, from the errors and times of those errors in

one case, collect very nearly the errors and times of the errors in any other case. But this may be done more expeditiously by the following method. The forces NM, ML, other things remaining unaltered, are as the radius TP; and their periodical effects (by Cor. 2, Lem. X) are as the forces and the square of the periodical time of the body P conjunctly. These are the linear errors of the body P; and hence the angular errors as they appear from the centre T (that is, the motion of the apsides and of the nodes, and all the apparent errors as to longitude and latitude) are in each revolution of the body P as the square of the time of the revolution, very nearly. Let these ratios be compounded with the ratios in Cor. 14, and in any system of bodies T, P, S, where P revolves about T very near to it, and T revolves about S at a great distance, the angular errors of the body P, observed from the centre T, will be in each revolution of the body P as the square of the periodical time of the body P directly, and the square of the periodical time of the body T inversely. And therefore the mean motion of the line of the apsides will be in a given ratio to the mean motion of the nodes; and both those motions will be as the periodical time of the body P directly, and the square of the periodical time of the body T inversely. The increase or diminution of the eccentricity and inclination of the orbit PAB makes no sensible variation in the motions of the apsides and nodes, unless that increase or diminution be very great indeed.

COR. 17. Since the line LM becomes sometimes greater and sometimes less than the radius PT, let the mean quantity of the force LM be expressed



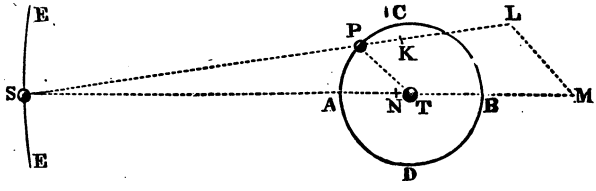
by that radius PT; and then that mean force will be to the mean force SK or SN (which may be also expressed by ST) as the length PT to the length ST. But the mean force SN or ST, by which the body T is retained in the orbit it describes about S, is to the force with which the body P is retained in its orbit about T in a ratio compounded of the ratio of the radius ST to the radius PT, and the duplicate ratio of the periodical time of the body P about T to the periodical time of the body T about S. And, *ex æquo*, the mean force LM is to the force by which the body P is retained in its orbit about T (or by which the same body P might revolve at the distance PT in the same periodical time about any immovable point T) in the same duplicate ratio of the periodical times. The periodical times therefore being given, together with the distance PT, the mean force LM is also given; and that force being given, there is given also the force MN, very nearly, by the analogy of the lines PT and MN.

COR. 18. By the same laws by which the body P revolves about the body T, let us suppose many fluid bodies to move round T at equal distances from it; and to be so numerous, that they may all become contiguous to each other, so as to form a fluid annulus, or ring, of a round figure, and concentrical to the body T; and the several parts of this annulus, performing their motions by the same law as the body P, will draw nearer to the body T, and move swifter in the conjunction and opposition of themselves and the body S, than in the quadratures. And the nodes of this annulus, or its intersections with the plane of the orbit of the body S or T, will rest at the syzygies; but out of the syzygies they will be carried backward, or *in antecedentia*; with the greatest swiftness in the quadratures, and more slowly in other places. The inclination of this annulus also will vary, and its axis will oscillate each revolution, and when the revolution is completed will return to its former situation, except only that it will be carried round a little by the præcession of the nodes.

COR. 19. Suppose now the sphærical body T, consisting of some matter not fluid, to be enlarged, and to extend itself on every side as far as that annulus, and that a channel were cut all round its circumference containing water; and that this sphere revolves uniformly about its own axis in the same periodical time. This water being accelerated and retarded by turns (as in the last Corollary), will be swifter at the syzygies, and slower at the quadratures, than the surface of the globe, and so will ebb and flow in its channel after the manner of the sea. If the attraction of the body S were taken away, the water would acquire no motion of flux and reflux by revolving round the quiescent centre of the globe. The case is the same of a globe moving uniformly forwards in a right line, and in the mean time revolving about its centre (by Cor. 5 of the Laws of Motion), and of a globe uniformly attracted from its rectilinear course (by Cor. 6, of the same Laws). But let the body S come to act upon it, and by its unequable attraction the water will receive this new motion; for there will be a stronger attraction upon that part of the water that is nearest to the body, and a weaker upon that part which is more remote. And the force LM will attract the water downwards at the quadratures, and depress it as far as the syzygies; and the force KL will attract it upwards in the syzygies, and withhold its descent, and make it rise as far as the quadratures; except only in so far as the motion of flux and reflux may be directed by the channel of the water, and be a little retarded by friction.

COR. 20. If, now, the annulus becomes hard, and the globe is diminished, the motion of flux and reflux will cease; but the oscillating motion of the inclination and the præcession of the nodes will remain. Let the globe have the same axis with the annulus, and perform its revolutions in the same times, and at its surface touch the annulus within, and adhere to it; then the globe partaking of the motion of the annulus, this whole compages

will oscillate, and the nodes will go backward, for the globe, as we shall shew presently, is perfectly indifferent to the receiving of all impressions. The greatest angle of the inclination of the annulus single is when the nodes are in the syzygies. Thence in the progress of the nodes to the quadratures, it endeavours to diminish its inclination, and by that endeavour impresses a motion upon the whole globe. The globe retains this motion impressed, till the annulus by a contrary endeavour destroys that motion, and impresses a new motion in a contrary direction. And by this means the greatest motion of the decreasing inclination happens when the nodes are in the quadratures, and the least angle of inclination in the octants



after the quadratures; and, again, the greatest motion of reclination happens when the nodes are in the syzygies; and the greatest angle of reclination in the octants following. And the case is the same of a globe without this annulus, if it be a little higher or a little denser in the equatorial than in the polar regions; for the excess of that matter in the regions near the equator supplies the place of the annulus. And though we should suppose the centripetal force of this globe to be any how increased, so that all its parts were to tend downwards, as the parts of our earth gravitate to the centre, yet the phænomena of this and the preceding Corollary would scarce be altered; except that the places of the greatest and least height of the water will be different; for the water is now no longer sustained and kept in its orbit by its centrifugal force, but by the channel in which it flows. And, besides, the force LM attracts the water downwards most in the quadratures, and the force KL or NM — LM attracts it upwards most in the syzygies. And these forces conjoined cease to attract the water downwards, and begin to attract it upwards in the octants before the syzygies; and cease to attract the water upwards, and begin to attract the water downwards in the octants after the syzygies. And thence the greatest height of the water may happen about the octants after the syzygies; and the least height about the octants after the quadratures; excepting only so far as the motion of ascent or descent impressed by these forces may by the *vis insita* of the water continue a little longer, or be stopped a little sooner by impediments in its channel.

COR. 21. For the same reason that redundant matter in the equatorial regions of a globe causes the nodes to go backwards, and therefore by the increase of that matter that retrogradation is increased, by the diminution is diminished, and by the removal quite ceases; it follows, that, if more than

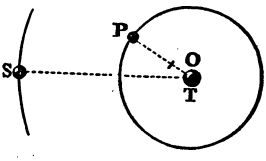
that redundant matter be taken away, that is, if the globe be either more depressed, or of a more rare consistence near the equator than near the poles, there will arise a motion of the nodes *in consequentia.*

Cor. 22. And thence from the motion of the nodes is known the constitution of the globe. That is, if the globe retains unalterably the same poles, and the motion (of the nodes) be *in antecedentia*, there is a redundance of the matter near the equator; but if *in consequentia*, a deficiency. Suppose a uniform and exactly sphærical globe to be first at rest in a free space; then by some impulse made obliquely upon its superficies to be driven from its place, and to receive a motion partly circular and partly right forward. Because this globe is perfectly indifferent to all the axes that pass through its centre, nor has a greater propensity to one axis or to one situation of the axis than to any other, it is manifest that by its own force it will never change its axis, or the inclination of it. Let now this globe be impelled obliquely by a new impulse in the same part of its superficies as before, and since the effect of an impulse is not at all changed by its coming sooner or later, it is manifest that these two impulses, successively impressed, will produce the same motion as if they were impressed at the same time; that is, the same motion as if the globe had been impelled by a simple force compounded of them both (by Cor. 2, of the Laws), that is, a simple motion about an axis of a given inclination. And the case is the same if the second impulse were made upon any other place of the equator of the first motion; and also if the first impulse were made upon any place in the equator of the motion which would be generated by the second impulse alone; and therefore, also, when both impulses are made in any places whatsoever; for these impulses will generate the same circular motion as if they were impressed together, and at once, in the place of the intersections of the equators of those motions, which would be generated by each of them separately. Therefore, a homogeneous and perfect globe will not retain several distinct motions, but will unite all those that are impressed on it, and reduce them into one; revolving, as far as in it lies, always with a simple and uniform motion about one single given axis, with an inclination perpetually invariable. And the inclination of the axis, or the velocity of the rotation, will not be changed by centripetal force. For if the globe be supposed to be divided into two hemispheres, by any plane whatsoever passing through its own centre, and the centre to which the force is directed, that force will always urge each hemisphere equally; and therefore will not incline the globe any way as to its motion round its own axis. But let there be added any where between the pole and the equator a heap of new matter like a mountain, and this, by its perpetual endeavour to recede from the centre of its motion, will disturb the motion of the globe, and cause its poles to wander about its superficies, describing circles about themselves and their opposite points. Neither can this enormous evagation

of the poles be corrected, unless by placing that mountain either in one of the poles; in which case, by Cor. 21, the nodes of the equator will go forwards; or in the equatorial regions, in which case, by Cor. 20, the nodes will go backwards; or, lastly, by adding on the other side of the axis a new quantity of matter, by which the mountain may be balanced in its motion; and then the nodes will either go forwards or backwards, as the mountain and this newly added matter happen to be nearer to the pole or to the equator.

PROPOSITION LXVII. THEOREM XXVII.

The same laws of attraction being supposed, I say, that the exterior body S *does, by radii drawn to the point* O, *the common centre of gravity of the interior bodies* P *and* T, *describe round that centre areas more proportional to the times, and an orbit more approaching to the form of an ellipsis having its focus in that centre, than it can describe round the innermost and greatest body* T *by radii drawn to that body.*



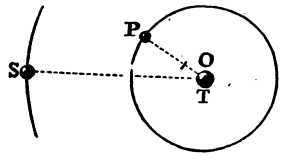
For the attractions of the body S towards T and P compose its absolute attraction, which is more directed towards O, the common centre of gravity of the bodies T and P, than it is to the greatest body T; and which is more in a reciprocal proportion to the square of the distance SO, than it is to the square of the distance ST; as will easily appear by a little consideration.

PROPOSITION LXVIII. THEOREM XXVIII.

The same laws of attraction supposed, I say, that the exterior body S *will, by radii drawn to* O, *the common centre of gravity of the interior bodies P and T, describe round that centre areas more proportional to the times, and an orbit more approaching to the form of an ellipsis having its focus in that centre, if the innermost and greatest body be agitated by these attractions as well as the rest, than it would do if that body were either at rest as not attracted, or were much more or much less attracted, or much more or much less agitated.*

This may be demonstrated after the same manner as Prop. LXVI, but by a more prolix reasoning, which I therefore pass over. It will be sufficient to consider it after this manner. From the demonstration of the last Proposition it is plain, that the centre, towards which the body S is urged by the two forces conjunctly, is very near to the common centre of gravity of those two other bodies. If this centre were to coincide with that common centre, and moreover the common centre of gravity of all the three bodies were at rest, the body S on one side, and the common centre of gravity of the other two bodies on the other side, would describe true ellip-

ses about that quiescent common centre. This appears from Cor. 2, Prop LVIII, compared with what was demonstrated in Prop. LXIV, and LXV Now this accurate elliptical motion will be disturbed a little by the distance of the centre of the two bodies from the centre towards which the third body S is attracted. Let there be added, moreover, a motion to the common centre of the three, and the perturbation will be increased yet more. Therefore the perturbation is least when the common centre of the three bodies is at rest; that is, when the innermost and greatest body T is attracted according to the same law as the rest are; and is always greatest when the common centre of the three, by the diminution of the motion of the body T, begins to be moved, and is more and more agitated.



Cor. And hence if more lesser bodies revolve about the great one, it may easily be inferred that the orbits described will approach nearer to ellipses; and the descriptions of areas will be more nearly equable, if all the bodies mutually attract and agitate each other with accelerative forces that are as their absolute forces directly, and the squares of the distances inversely; and if the focus of each orbit be placed in the common centre of gravity of all the interior bodies (that is, if the focus of the first and innermost orbit be placed in the centre of gravity of the greatest and innermost body; the focus of the second orbit in the common centre of gravity of the two innermost bodies; the focus of the third orbit in the common centre of gravity of the three innermost; and so on), than if the innermost body were at rest, and was made the common focus of all the orbits.

PROPOSITION LXIX. THEOREM XXIX.

In a system of several bodies A, B, C, D, *&c., if any one of those bodies, as* A, *attract all the rest,* B, C, D, *&c., with accelerative forces that are reciprocally as the squares of the distances from the attracting body; and another body, as* B, *attracts also the rest,* A, C, D, *&c., with forces that are reciprocally as the squares of the distances from the attracting body; the absolute forces of the attracting bodies* A *and* B *will be to each other as those very bodies* A *and* B *to which those forces belong.*

For the accelerative attractions of all the bodies B, C, D, towards A, are by the supposition equal to each other at equal distances; and in like manner the accelerative attractions of all the bodies towards B are also equal to each other at equal distances. But the absolute attractive force of the body A is to the absolute attractive force of the body B as the accelerative attraction of all the bodies towards A to the accelerative attraction of all the bodies towards B at equal distances; and so is also the accelerative attraction of the body B towards A to the accelerative attraction

of the body A towards B. But the accelerative attraction of the body B towards A is to the accelerative attraction of the body A towards B as the mass of the body A to the mass of the body B; because the motive forces which (by the 2d, 7th, and 8th Definition) are as the accelerative forces and the bodies attracted conjunctly are here equal to one another by the third Law. Therefore the absolute attractive force of the body A is to the absolute attractive force of the body B as the mass of the body A to the mass of the body B. Q.E.D.

COR. 1. Therefore if each of the bodies of the system A, B, C, D, &c. does singly attract all the rest with accelerative forces that are reciprocally as the squares of the distances from the attracting body, the absolute forces of all those bodies will be to each other as the bodies themselves.

COR. 2. By a like reasoning, if each of the bodies of the system A, B, C, D, &c., do singly attract all the rest with accelerative forces, which are either reciprocally or directly in the ratio of any power whatever of the distances from the attracting body; or which are defined by the distances from each of the attracting bodies according to any common law; it is plain that the absolute forces of those bodies are as the bodies themselves.

COR. 3. In a system of bodies whose forces decrease in the duplicate ratio of the distances, if the lesser revolve about one very great one in ellipses, having their common focus in the centre of that great body, and of a figure exceedingly accurate; and moreover by radii drawn to that great body describe areas proportional to the times exactly; the absolute forces of those bodies to each other will be either accurately or very nearly in the ratio of the bodies. And so on the contrary. This appears from Cor. of Prop. XLVIII, compared with the first Corollary of this Prop.

SCHOLIUM.

These Propositions naturally lead us to the analogy there is between centripetal forces, and the central bodies to which those forces used to be directed; for it is reasonable to suppose that forces which are directed to bodies should depend upon the nature and quantity of those bodies, as we see they do in magnetical experiments. And when such cases occur, we are to compute the attractions of the bodies by assigning to each of their particles its proper force, and then collecting the sum of them all. I here use the word attraction in general for any endeavour, of what kind soever, made by bodies to approach to each other; whether that endeavour arise from the action of the bodies themselves, as tending mutually to or agitating each other by spirits emitted; or whether it arises from the action of the æther or of the air, or of any medium whatsoever, whether corporeal or incorporeal, any how impelling bodies placed therein towards each other. In the same general sense I use the word impulse, not defining in this treatise the species or physical qualities of forces, but investigating the quantities

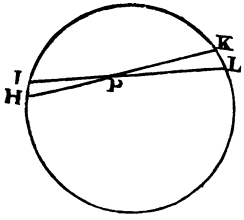
and mathematical proportions of them; as I observed before in the Definitions. In mathematics we are to investigate the quantities of forces with their proportions consequent upon any conditions supposed; then, when we enter upon physics, we compare those proportions with the phænomena of Nature, that we may know what conditions of those forces answer to the several kinds of attractive bodies. And this preparation being made, we argue more safely concerning the physical species, causes, and proportions of the forces. Let us see, then, with what forces sphærical bodies consisting of particles endued with attractive powers in the manner above spoken of must act mutually upon one another; and what kind of motions will follow from thence.

SECTION XII.

Of the attractive forces of sphærical bodies.

PROPOSITION LXX. THEOREM XXX.

If to every point of a sphærical surface there tend equal centripetal forces decreasing in the duplicate ratio of the distances from those points; I say, that a corpuscle placed within that superficies will not be attracted by those forces any way.



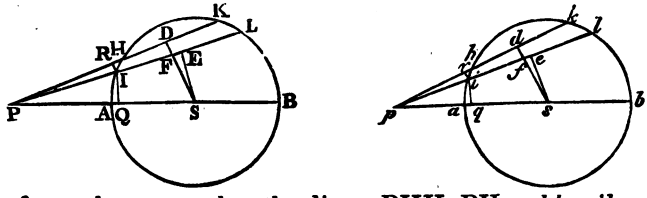
Let HIKL, be that sphærical superficies, and P a corpuscle placed within. Through P let there be drawn to this superficies to two lines HK, IL, intercepting very small arcs HI, KL; and because (by Cor. 3, Lem. VII) the triangles HPI, LPK are alike, those arcs will be proportional to the distances HP LP; and any particles at HI and KL of the sphærical superficies, terminated by right lines passing through P, will be in the duplicate ratio of those distances. Therefore the forces of these particles exerted upon the body P are equal between themselves. For the forces are as the particles directly, and the squares of the distances inversely. And these two ratios compose the ratio of equality. The attractions therefore, being made equally towards contrary parts, destroy each other. And by a like reasoning all the attractions through the whole sphærical superficies are destroyed by contrary attractions. Therefore the body P will not be any way impelled by those attractions. Q.E.D.

PROPOSITION LXXI. THEOREM XXXI.

The same things supposed as above, I say, that a corpuscle placed without the sphærical superficies is attracted towards the centre of the sphere with a force reciprocally proportional to the square of its distance from that centre.

Let AHKB, *ahkb*, be two equal sphærical superficies described about

the centre S, *s*; their diameters AB, *ab*; and let P and *p* be two corpuscles situate without the spheres in those diameters produced. Let there



be drawn from the corpuscles the lines PHK, PIL, *phk*, *pil*, cutting off from the great circles AHB, *ahb*, the equal arcs HK, *hk*, IL, *il*; and to those lines let fall the perpendiculars SD, *sd*, SE, *se*, IR, *ir*; of which let SD, *sd*, cut PL, *pl*, in F and *f*. Let fall also to the diameters the perpendiculars IQ, *iq*. Let now the angles DPE, *dpe*, vanish; and because DS and *ds*, ES and *es* are equal, the lines PE, PF, and *pe*, *pf*, and the lineolæ DF, *df* may be taken for equal; because their last ratio, when the angles DPE, *dpe* vanish together, is the ratio of equality. These things then supposed, it will be, as PI to PF so is RI to DF, and as *pf* to *pi* so is *df* or DF to *ri*; and, *ex æquo*, as PI × *pf* to PF × *pi* so is RI to *ri*, that is (by Cor. 3, Lem VII), so is the arc IH to the arc *ih*. Again, PI is to PS as IQ to SE, and *ps* to *pi* as *se* or SE to *iq*; and, *ex æquo*, PI × *ps* to PS × *pi* as IQ to *iq*. And compounding the ratios $PI^2 \times pf \times ps$ is to $pi^2 \times PF \times PS$, as IH × IQ to *ih* × *iq*; that is, as the circular superficies which is described by the arc IH, as the semi-circle AKB revolves about the diameter AB, is to the circular superficies described by the arc *ih* as the semi-circle *akb* revolves about the diameter *ab*. And the forces with which these superficies attract the corpuscles P and *p* in the direction of lines tending to those superficies are by the hypothesis as the superficies themselves directly, and the squares of the distances of the superficies from those corpuscles inversely; that is, as *pf* × *ps* to PF × PS. And these forces again are to the oblique parts of them which (by the resolution of forces as in Cor. 2, of the Laws) tend to the centres in the directions of the lines PS, *ps*, as PI to PQ, and *pi* to *pq*; that is (because of the like triangles PIQ and PSF, *piq* and *psf*), as PS to PF and *ps* to *pf*. Thence *ex æquo*, the attraction of the corpuscle P towards S is to the attraction of the corpuscle *p* towards *s* as $\frac{PF \times pf \times ps}{PS}$ is to $\frac{pf \times PF \times PS}{ps}$, that is, as ps^2 to PS^2. And, by a like reasoning, the forces with which the superficies described by the revolution of the arcs KL, *kl* attract those corpuscles, will be as ps^2 to PS^2. And in the same ratio will be the forces of all the circular superficies into which each of the sphærical superficies may be divided by taking *sd* always equal to SD, and *se* equal to SE. And therefore, by composition, the forces of the entire sphærical superficies exerted upon those corpuscles will be in the same ratio. Q.E.D

PROPOSITION LXXII. THEOREM XXXII.

If to the several points of a sphere there tend equal centripetal forces decreasing in a duplicate ratio of the distances from those points; and there be given both the density of the sphere and the ratio of the diameter of the sphere to the distance of the corpuscle from its centre; I say, that the force with which the corpuscle is attracted is proportional to the semi-diameter of the sphere.

For conceive two corpuscles to be severally attracted by two spheres, one by one, the other by the other, and their distances from the centres of the spheres to be proportional to the diameters of the spheres respectively, and the spheres to be resolved into like particles, disposed in a like situation to the corpuscles. Then the attractions of one corpuscle towards the several particles of one sphere will be to the attractions of the other towards as many analogous particles of the other sphere in a ratio compounded of the ratio of the particles directly, and the duplicate ratio of the distances inversely. But the particles are as the spheres, that is, in a triplicate ratio of the diameters, and the distances are as the diameters; and the first ratio directly with the last ratio taken twice inversely, becomes the ratio of diameter to diameter. Q.E.D.

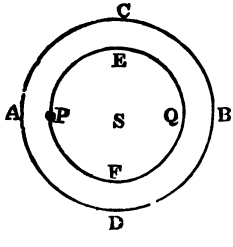
Cor. 1. Hence if corpuscles revolve in circles about spheres composed of matter equally attracting, and the distances from the centres of the spheres be proportional to their diameters, the periodic times will be equal.

Cor. 2. And, *vice versa*, if the periodic times are equal, the distances will be proportional to the diameters. These two Corollaries appear from Cor. 3, Prop. IV.

Cor. 3. If to the several points of any two solids whatever, of like figure and equal density, there tend equal centripetal forces decreasing in a duplicate ratio of the distances from those points, the forces, with which corpuscles placed in a like situation to those two solids will be attracted by them, will be to each other as the diameters of the solids.

PROPOSITION LXXIII. THEOREM XXXIII.

If to the several points of a given sphere there tend equal centripetal forces decreasing in a duplicate ratio of the distances from the points; I say, that a corpuscle placed within the sphere is attracted by a force proportional to its distance from the centre.



In the sphere ABCD, described about the centre S, let there be placed the corpuscle P; and about the same centre S, with the interval SP, conceive described an interior sphere PEQF. It is plain (by Prop. LXX) that the concentric sphærical superficies, of which the difference AEBF of the spheres is composed, have no effect at all upon the body P, their at-

tractions being destroyed by contrary attractions. There remains, therefore, only the attraction of the interior sphere PEQF. And (by Prop. LXXII) this is as the distance PS. Q.E.D.

SCHOLIUM.

By the superficies of which I here imagine the solids composed, I do not mean superficies purely mathematical, but orbs so extremely thin, that their thickness is as nothing; that is, the evanescent orbs of which the sphere will at last consist, when the number of the orbs is increased, and their thickness diminished without end. In like manner, by the points of which lines, surfaces, and solids are said to be composed, are to be understood equal particles, whose magnitude is perfectly inconsiderable.

PROPOSITION LXXIV. THEOREM XXXIV.

The same things supposed, I say, that a corpuscle situate without the sphere is attracted with a force reciprocally proportional to the square of its distance from the centre.

For suppose the sphere to be divided into innumerable concentric sphærical superficies, and the attractions of the corpuscle arising from the several superficies will be reciprocally proportional to the square of the distance of the corpuscle from the centre of the sphere (by Prop. LXXI). And, by composition, the sum of those attractions, that is, the attraction of the corpuscle towards the entire sphere, will be in the same ratio. Q.E.D.

COR. 1. Hence the attractions of homogeneous spheres at equal distances from the centres will be as the spheres themselves. For (by Prop. LXXII) if the distances be proportional to the diameters of the spheres, the forces will be as the diameters. Let the greater distance be diminished in that ratio; and the distances now being equal, the attraction will be increased in the duplicate of that ratio; and therefore will be to the other attraction in the triplicate of that ratio; that is, in the ratio of the spheres.

COR. 2. At any distances whatever the attractions are as the spheres applied to the squares of the distances.

COR. 3. If a corpuscle placed without an homogeneous sphere is attracted by a force reciprocally proportional to the square of its distance from the centre, and the sphere consists of attractive particles, the force of every particle will decrease in a duplicate ratio of the distance from each particle.

PROPOSITION LXXV. THEOREM XXXV.

If to the several points of a given sphere there tend equal centripetal forces decreasing in a duplicate ratio of the distances from the points; I say, that another similar sphere will be attracted by it with a force reciprocally proportional to the square of the distance of the centres.

For the attraction of every particle is reciprocally as the square of its

distance from the centre of the attracting sphere (by Prop. LXXIV), and is therefore the same as if that whole attracting force issued from one single corpuscle placed in the centre of this sphere. But this attraction is as great as on the other hand the attraction of the same corpuscle would be, if that were itself attracted by the several particles of the attracted sphere with the same force with which they are attracted by it. But that attraction of the corpuscle would be (by Prop. LXXIV) reciprocally proportional to the square of its distance from the centre of the sphere; therefore the attraction of the sphere, equal thereto, is also in the same ratio. Q.E.D.

COR. 1. The attractions of spheres towards other homogeneous spheres are as the attracting spheres applied to the squares of the distances of their centres from the centres of those which they attract.

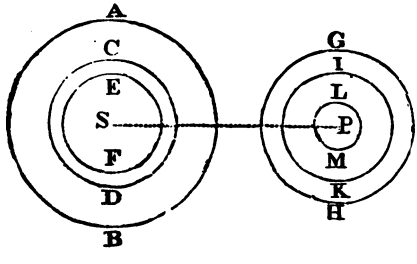
COR. 2. The case is the same when the attracted sphere does also attract. For the several points of the one attract the several points of the other with the same force with which they themselves are attracted by the others again; and therefore since in all attractions (by Law III) the attracted and attracting point are both equally acted on, the force will be doubled by their mutual attractions, the proportions remaining.

COR. 3. Those several truths demonstrated above concerning the motion of bodies about the focus of the conic sections will take place when an attracting sphere is placed in the focus, and the bodies move without the sphere.

COR. 4. Those things which were demonstrated before of the motion of bodies about the centre of the conic sections take place when the motions are performed within the sphere.

PROPOSITION LXXVI. THEOREM XXXVI.

If spheres be however dissimilar (as to density of matter and attractive force) in the same ratio onward from the centre to the circumference; but every where similar, at every given distance from the centre, on all sides round about; and the attractive force of every point decreases in the duplicate ratio of the distance of the body attracted; I say, that the whole force with which one of these spheres attracts the other will be reciprocally proportional to the square of the distance of the centres.



Imagine several concentric similar spheres, AB, CD, EF, &c., the innermost of which added to the outermost may compose a matter more dense towards the centre, or subducted from them may leave the same more lax and rare. Then, by Prop. LXXV, these spheres will attract other similar con-

centric spheres GH, IK, LM, &c., each the other, with forces reciprocally proportional to the square of the distance SP. And, by composition or division, the sum of all those forces, or the excess of any of them above the others; that is, the entire force with which the whole sphere AB (composed of any concentric spheres or of their differences) will attract the whole sphere GH (composed of any concentric spheres or their differences) in the same ratio. Let the number of the concentric spheres be increased *in infinitum*, so that the density of the matter together with the attractive force may, in the progress from the circumference to the centre, increase or decrease according to any given law; and by the addition of matter not attractive, let the deficient density be supplied, that so the spheres may acquire any form desired; and the force with which one of these attracts the other will be still, by the former reasoning, in the same ratio of the square of the distance inversely. Q.E.D.

COR. 1. Hence if many spheres of this kind, similar in all respects, attract each other mutually, the accelerative attractions of each to each, at any equal distances of the centres, will be as the attracting spheres.

COR. 2. And at any unequal distances, as the attracting spheres applied to the squares of the distances between the centres.

COR. 3. The motive attractions, or the weights of the spheres towards one another, will be at equal distances of the centres as the attracting and attracted spheres conjunctly; that is, as the products arising from multiplying the spheres into each other.

COR. 4. And at unequal distances, as those products directly, and the squares of the distances between the centres inversely.

COR. 5. These proportions take place also when the attraction arises from the attractive virtue of both spheres mutually exerted upon each other. For the attraction is only doubled by the conjunction of the forces, the proportions remaining as before.

COR. 6. If spheres of this kind revolve about others at rest, each about each; and the distances between the centres of the quiescent and revolving bodies are proportional to the diameters of the quiescent bodies; the periodic times will be equal.

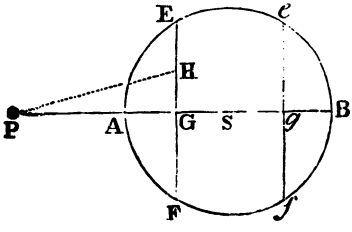
COR. 7. And, again, if the periodic times are equal, the distances will be proportional to the diameters.

COR. 8. All those truths above demonstrated, relating to the motions of bodies about the foci of conic sections, will take place when an attracting sphere, of any form and condition like that above described, is placed in the focus.

COR. 9. And also when the revolving bodies are also attracting spheres of any condition like that above described.

PROPOSITION LXXVII. THEOREM XXXVII.

If to the several points of spheres there tend centripetal forces proportional to the distances of the points from the attracted bodies; I say, that the compounded force with which two spheres attract each other mutually is as the distance between the centres of the spheres.



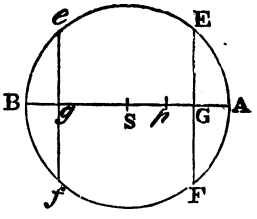
CASE 1. Let AEBF be a sphere; S its centre. P a corpuscle attracted; PASB the axis of the sphere passing through the centre of the corpuscle; EF, *ef* two planes cutting the sphere, and perpendicular to the axis, and equi-distant, one on one side, the other on the other, from the centre of the sphere; G and *g* the intersections of the planes and the axis; and H any point in the plane EF. The centripetal force of the point H upon the corpuscle P, exerted in the direction of the line PH, is as the distance PH; and (by Cor. 2, of the Laws) the same exerted in the direction of the line PG, or towards the centre S, is as the length PG. Therefore the force of all the points in the plane EF (that is, of that whole plane) by which the corpuscle P is attracted towards the centre S is as the distance PG multiplied by the number of those points, that is, as the solid contained under that plane EF and the distance PG. And in like manner the force of the plane *ef*, by which the corpuscle P is attracted towards the centre S, is as that plane drawn into its distance P*g*, or as the equal plane EF drawn into that distance P*g*; and the sum of the forces of both planes as the plane EF drawn into the sum of the distances PG + P*g*, that is, as that plane drawn into twice the distance PS of the centre and the corpuscle; that is, as twice the plane EF drawn into the distance PS, or as the sum of the equal planes EF + *ef* drawn into the same distance. And, by a like reasoning, the forces of all the planes in the whole sphere, equi-distant on each side from the centre of the sphere, are as the sum of those planes drawn into the distance PS, that is, as the whole sphere and the distance PS conjunctly. Q.E.D.

CASE 2. Let now the corpuscle P attract the sphere AEBF. And, by the same reasoning, it will appear that the force with which the sphere is attracted is as the distance PS. Q.E.D.

CASE 3. Imagine another sphere composed of innumerable corpuscles P; and because the force with which every corpuscle is attracted is as the distance of the corpuscle from the centre of the first sphere, and as the same sphere conjunctly, and is therefore the same as if it all proceeded from a single corpuscle situate in the centre of the sphere, the entire force with which all the corpuscles in the second sphere are attracted, that is, with which that whole sphere is attracted, will be the same as if that sphere

were attracted by a force issuing from a single corpuscle in the centre of the first sphere; and is therefore proportional to the distance between the centres of the spheres. Q.E.D.

CASE 4. Let the spheres attract each other mutually, and the force will be doubled, but the proportion will remain. Q.E.D.



CASE 5. Let the corpuscle p be placed within the sphere AEBF; and because the force of the plane ef upon the corpuscle is as the solid contained under that plane and the distance pg; and the contrary force of the plane EF as the solid contained under that plane and the distance pG; the force compounded of both will be as the difference of the solids, that is, as the sum of the equal planes drawn into half the difference of the distances; that is, as that sum drawn into pS, the distance of the corpuscle from the centre of the sphere. And, by a like reasoning, the attraction of all the planes EF, ef, throughout the whole sphere, that is, the attraction of the whole sphere, is conjunctly as the sum of all the planes, or as the whole sphere, and as pS, the distance of the corpuscle from the centre of the sphere. Q.E.D.

CASE 6. And if there be composed a new sphere out of innumerable corpuscles such as p, situate within the first sphere AEBF, it may be proved, as before, that the attraction, whether single of one sphere towards the other, or mutual of both towards each other, will be as the distance pS of the centres. Q.E.D.

PROPOSITION LXXVIII. THEOREM XXXVIII.

If spheres in the progress from the centre to the circumference be however dissimilar and unequable, but similar on every side round about at all given distances from the centre; and the attractive force of every point be as the distance of the attracted body; I say, that the entire force with which two spheres of this kind attract each other mutually is proportional to the distance between the centres of the spheres.

This is demonstrated from the foregoing Proposition, in the same manner as Proposition LXXVI was demonstrated from Proposition LXXV.

COR. Those things that were above demonstrated in Prop. X and LXIV, of the motion of bodies round the centres of conic sections, take place when all the attractions are made by the force of sphærical bodies of the condition above described, and the attracted bodies are spheres of the same kind.

SCHOLIUM.

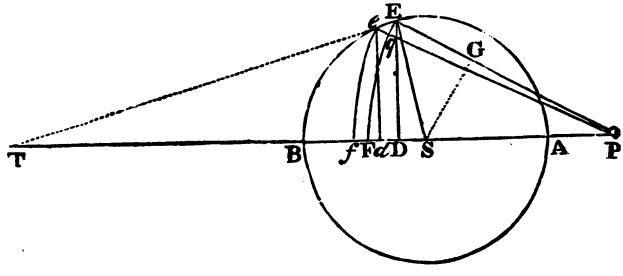
I have now explained the two principal cases of attractions; to wit, when the centripetal forces decrease in a duplicate ratio of the distances, or increase in a simple ratio of the distances, causing the bodies in both

cases to revolve in conic sections, and composing sphærical bodies whose centripetal forces observe the same law of increase or decrease in the recess from the centre as the forces of the particles themselves do; which is very remarkable. It would be tedious to run over the other cases, whose conclusions are less elegant and important, so particularly as I have done these. I choose rather to comprehend and determine them all by one general method as follows.

LEMMA XXIX.

If about the centre S *there be described any circle as* AEB, *and about the centre* P *there be also described two circles* EF, ef, *cutting the first in* E *and* e, *and the line* PS *in* F *and* f; *and there be let fall to* PS *the perpendiculars* ED, ed; *I say, that if the distance of the arcs* EF, ef *be supposed to be infinitely diminished, the last ratio of the evanescent line* Dd *to the evanescent line* Ff *is the same as that of the line* PE *to the line* PS.

For if the line Pe cut the arc EF in *q*; and the right line E*e*, which

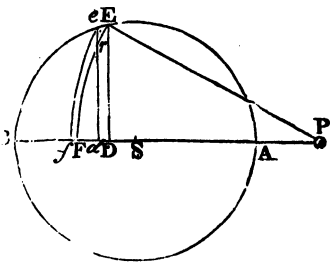


coincides with the evanescent arc E*e*, be produced, and meet the right line PS in T; and there be let fall from S to PE the perpendicular SG; then, because of the like triangles DTE, *d*T*e*, DES, it will be as D*d* to E*e* so DT to TE, or DE to ES; and because the triangles, E*eq*, ESG (by Lem. VIII, and Cor. 3, Lem. VII) are similar, it will be as E*e* to *eq* or F*f* so ES to SG; and, *ex æquo*, as D*d* to F*f* so DE to SG; that is (because of the similar triangles PDE, PGS), so is PE to PS. Q.E.D.

PROPOSITION LXXIX. THEOREM XXXIX.

Suppose a superficies as EFfe *to have its breadth infinitely diminished, and to be just vanishing; and that the same superficies by its revolution round the axis* PS *describes a sphærical concavo-convex solid, to the several equal particles of which there tend equal centripetal forces; I say, that the force with which that solid attracts a corpuscle situate in* P *is in a ratio compounded of the ratio of the solid* $DE^2 \times Ff$ *and the ratio of the force with which the given particle in the place* Ff *would attract the same corpuscle.*

For if we consider, first, the force of the sphærical superficies FE which

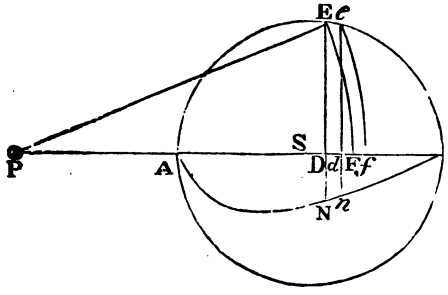


is generated by the revolution of the arc FE, and is cut any where, as in r, by the line de, the annular part of the superficies generated by the revolution of the arc rE will be as the lineola Dd, the radius of the sphere PE remaining the same; as *Archimedes* has demonstrated in his Book of the Sphere and Cylinder. And the force of this superficies exerted in the direction of the lines PE or Pr situate all round in the conical superficies, will be as this annular superficies itself; that is as the lineola Dd, or, which is the same, as the rectangle under the given radius PE of the sphere and the lineola Dd; but that force, exerted in the direction of the line PS tending to the centre S, will be less in the ratio PD to PE, and therefore will be as $PD \times Dd$. Suppose now the line DF to be divided into innumerable little equal particles, each of which call Dd, and then the superficies FE will be divided into so many equal annuli, whose forces will be as the sum of all the rectangles $PD \times Dd$, that is, as $\frac{1}{2}PF^2 - \frac{1}{2}PD^2$, and therefore as DE^2. Let now the superficies FE be drawn into the altitude Ff; and the force of the solid $EFfe$ exerted upon the corpuscle P will be as $DE^2 \times Ff$; that is, if the force be given which any given particle as Ff exerts upon the corpuscle P at the distance PF. But if that force be not given, the force of the solid $EFfe$ will be as the solid $DE^2 \times Ff$ and that force not given, conjunctly. Q.E.D.

PROPOSITION LXXX. THEOREM XL.

If to the several equal parts of a sphere ABE *described about the centre* S *there tend equal centripetal forces; and from the several points* D *in the axis of the sphere* AB *in which a corpuscle, as* P, *is placed, there be erected the perpendiculars* DE *meeting the sphere in* E, *and if in those perpendiculars the lengths* DN *be taken as the quantity* $\frac{DE^2 \times PS}{PE}$, *and as the force which a particle of the sphere situate in the axis exerts at the distance* PE *upon the corpuscle* P *conjunctly; I say, that the whole force with which the corpuscle P is attracted towards the sphere is as the area* ANB, *comprehended under the axis of the sphere* AB, *and the curve line* ANB, *the locus of the point* N.

For supposing the construction in the last Lemma and Theorem to stand, conceive the axis of the sphere AB to be divided into innumerable equal particles Dd, and the whole sphere to be divided into so many sphærical concavo-convex laminæ $EFfe$; and erect the perpendicular dn. By the last Theorem, the force with which the laminæ $EFfe$ attracts the corpuscle P is as $DE^2 \times Ff$ and the force of one particle exerted at the



distance PE or PF, conjunctly. But (by the last Lemma) D*d* is to F*f* as PE to PS, and therefore F*f* is equal to $\frac{PS \times Dd}{PE}$; and $DE^2 \times Ff$ is equal to $Dd \times \frac{DE^2 \times PS}{PE}$; and therefore the force of the lamina EF*fe* is as $Dd \times \frac{DE^2 \times PS}{PE}$ and the force of a particle exerted at the distance PF conjunctly; that is, by the supposition, as DN × D*d*, or as the evanescent area DN*nd*. Therefore the forces of all the laminæ exerted upon the corpuscle P are as all the areas DN*nd*, that is, the whole force of the sphere will be as the whole area ANB. Q.E.D.

Cor. 1. Hence if the centripetal force tending to the several particles remain always the same at all distances, and DN be made as $\frac{DE^2 \times PS}{PE}$, the whole force with which the corpuscle is attracted by the sphere is as the area ANB.

Cor. 2. If the centripetal force of the particles be reciprocally as the distance of the corpuscle attracted by it, and DN be made as $\frac{DE^2 \times PS}{PE^2}$, the force with which the corpuscle P is attracted by the whole sphere will be as the area ANB.

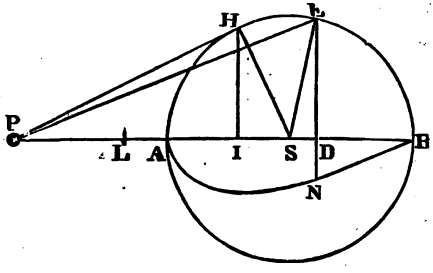
Cor. 3. If the centripetal force of the particles be reciprocally as the cube of the distance of the corpuscle attracted by it, and DN be made as $\frac{DE^2 \times PS}{PE^4}$, the force with which the corpuscle is attracted by the whole sphere will be as the area ANB.

Cor. 4. And universally if the centripetal force tending to the several particles of the sphere be supposed to be reciprocally as the quantity V; and DN be made as $\frac{DE^2 \times PS}{PE \times V}$; the force with which a corpuscle is attracted by the whole sphere will be as the area ANB.

PROPOSITION LXXXI. PROBLEM XLI.

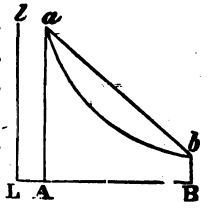
The things remaining as above, it is required to measure the area ANB.

From the point P let there be drawn the right line PH touching the sphere in H; and to the axis PAB, letting fall the perpendicular HI, bisect PI in L; and (by Prop. XII, Book II, Elem.) PE^2 is equal to



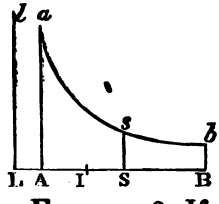
$PS^2 + SE^2 + 2PSD$. But because the triangles SPH, SHI are alike, SE^2 or SH^2 is equal to the rectangle PSI. Therefore PE^2 is equal to the rectangle contained under PS and $PS + SI + 2SD$; that is, under PS and $2LS + 2SD$; that is, under PS and 2LD. Moreover DE^2 is equal to $SE^2 - SD^2$, or $SE^2 - LS^2 + 2SLD - LD^2$, that is, $2SLD - LD^2 - ALB$. For $LS^2 - SE^2$ or $LS^2 - SA^2$ (by Prop. VI, Book II, Elem.) is equal to the rectangle ALB. Therefore if instead of DE^2 we write $2SLD - LD^2 - ALB$, the quantity $\frac{DE^2 \times PS}{PE \times V}$, which (by Cor. 4 of the foregoing Prop.) is as the length of the ordinate DN, will now resolve itself into three parts $\frac{2SLD \times PS}{PE \times V} - \frac{LD^2 \times PS}{PE \times V} - \frac{ALB \times PS}{PE \times V}$; where if instead of V we write the inverse ratio of the centripetal force, and instead of PE the mean proportional between PS and 2LD, those three parts will become ordinates to so many curve lines, whose areas are discovered by the common methods. Q.E.D.

EXAMPLE 1. If the centripetal force tending to the several particles of the sphere be reciprocally as the distance; instead of V write PE the distance, then $2PS \times LD$ for PE^2; and DN will become as $SL - \frac{1}{2} LD - \frac{ALB}{2LD}$. Suppose DN equal to its double $2SL - LD - \frac{ALB}{LD}$; and 2SL the given part of the ordinate drawn into the length AB will describe the rectangular area $2SL \times AB$; and the indefinite part LD, drawn perpendicularly into the same length with a continued motion, in such sort as in its motion one way or another it may either by increasing or decreasing remain always equal to the length LD, will describe the area $\frac{LB^2 - LA^2}{2}$, that is, the area $SL \times AB$; which taken from the former area $2SL \times AB$, leaves the area $SL \times AE$. But the third part $\frac{ALB}{LD}$, drawn after the same manner with a continued motion perpendicularly into the same length, will describe the area of an hyperbola, which subducted from the area $SL \times AB$ will leave ANB the area sought. Whence arises this construction of the Problem. At the points, L, A, B, erect the perpendiculars L*l*, A*a*, B*b*; making A*a* equal to LB, and B*b* equal to LA. Making L*l* and LB asymptotes, describe through the points *a*, *b*,

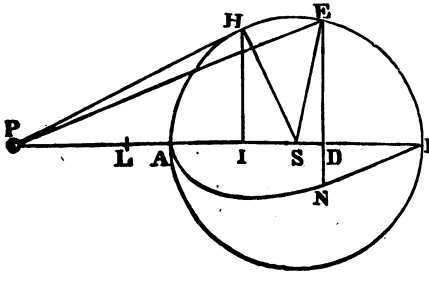


the hyperbolic curve *ab*. And the chord *ba* being drawn, will inclose the area *aba* equal to the area sought ANB.

EXAMPLE 2. If the centripetal force tending to the several particles of the sphere be reciprocally as the cube of the distance, or (which is the same thing) as that cube applied to any given plane; write $\frac{PE^3}{2AS^2}$ for V, and $2PS \times LD$ for PE^2; and DN will become as $\frac{SL \times AS^2}{PS \times LD} \quad \frac{AS^2}{2PS}$ $\frac{ALB \times AS^2}{2PS \times LD^2}$ that is (because PS, AS, SI are continually proportional), as $\frac{LSI}{LD} - \frac{1}{2}SI - \frac{ALB \times SI}{2LD^2}$. If we draw then these three parts into th length AB, the first $\frac{LSI}{LD}$ will generate the area of an hyperbola; the second $\frac{1}{2}SI$ the area $\frac{1}{2}AB \times SI$; the third $\frac{ALB \times SI}{2LD^2}$ the area $\frac{ALB \times SI}{2LA}$ $\frac{ALB \times SI}{2LB}$, that is, $\frac{1}{2}AB \times SI$. From the first subduct the sum of the second and third, and there will remain ANB, the area sought. Whence arises this construction of the problem. At the points L, A, S, B, erect the perpendiculars L*l* A*a* S*s*, B*b*, of which suppose S*s* equal to SI; and through the point *s*, to the asymptotes L*l*, LB, describe the hyperbola *asb* meeting the perpendiculars A*a*, B*b*, in *a* and *b*; and the rectangle 2ASI, subducted from the hyberbolic area A*asb*B, will leave ANB the area sought.



EXAMPLE 3. If the centripetal force tending to the several particles of the spheres decrease in a quadruplicate ratio of the distance from the particles; write $\frac{PE^4}{2AS^3}$ for V, then $\sqrt{2PS + LD}$ for PE, and DN will become as $\frac{SI^2 \times SL}{\sqrt{2SI}} \times \frac{1}{\sqrt{LD^3}} - \frac{SI^2}{2\sqrt{2SI}} \times \frac{1}{\sqrt{LD}} - \frac{SI^2 \times ALB}{2\sqrt{2SI}} \times \frac{1}{\sqrt{LD^5}}$. These three parts drawn into the length AB, produce so many areas, viz. $\frac{2SI^2 \times SL}{\sqrt{2SI}}$ into $\frac{1}{\sqrt{LA}} - \frac{1}{\sqrt{LB}}$; $\frac{SI^2}{\sqrt{2SI}}$ into $\sqrt{LB} - \sqrt{LA}$; and $\frac{SI^2 \times ALB}{3\sqrt{2SI}}$ into $\frac{1}{\sqrt{LA^3}} - \frac{1}{\sqrt{LB^3}}$. And these after due reduction come forth $\frac{2SI^2 \times SL}{LI}$, SI^2, and $SI^2 +$

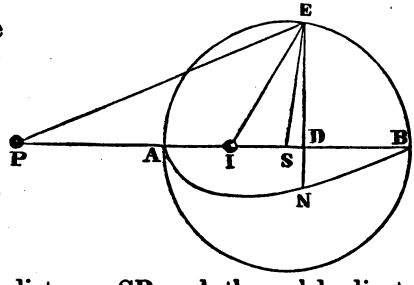


$\frac{2SI^3}{3LI}$. And these by subducting the last from the first, become $\frac{4SI^3}{3LI}$. Therefore the entire force with which the corpuscle P is attracted towards the centre of the sphere is as $\frac{SI^3}{PI}$, that is, reciprocally as $PS^3 \times PI$ Q.E.I.

By the same method one may determine the attraction of a corpuscle situate within the sphere, but more expeditiously by the following Theorem.

PROPOSITION LXXXII. THEOREM XLI.

In a sphere described about the centre S *with the interval* SA, *if there be taken* SI, SA, SP *continually proportional; I say, that the attraction of a corpuscle within the sphere in any place* I *is to its attraction without the sphere in the place* P *in a ratio compounded of the subduplicate ratio of* IS, PS, *the distances from the centre, and the subduplicate ratio of the centripetal forces tending to the centre in those places* P *and* I.



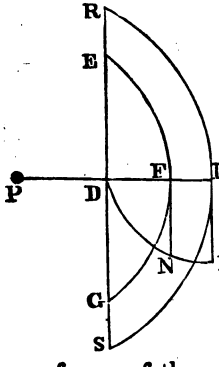
As if the centripetal forces of the particles of the sphere be reciprocally as the distances of the corpuscle attracted by them; the force with which the corpuscle situate in I is attracted by the entire sphere will be to the force with which it is attracted in P in a ratio compounded of the subduplicate ratio of the distance SI to the distance SP, and the subduplicate ratio of the centripetal force in the place I arising from any particle in the centre to the centripetal force in the place P arising from the same particle in the centre; that is, in the subduplicate ratio of the distances SI, SP to each other reciprocally. These two subduplicate ratios compose the ratio of equality, and therefore the attractions in I and P produced by the whole sphere are equal. By the like calculation, if the forces of the particles of the sphere are reciprocally in a duplicate ratio of the distances, it will be found that the attraction in I is to the attraction in P as the distance SP to the semi-diameter SA of the sphere. If those forces are reciprocally in a triplicate ratio of the distances, the attractions in I and P will be to each other as SP^2 to SA^2; if in a quadruplicate ratio, as SP^3 to SA^3. Therefore since the attraction in P was found in this last case to be reciprocally as $PS^3 \times PI$, the attraction in I will be reciprocally as $SA^3 \times PI$, that is, because SA^3 is given reciprocally as PI. And the progression is the same *in infinitum*. The demonstration of this Theorem is as follows:

The things remaining as above constructed, and a corpuscle being in any

place P, the ordinate DN was found to be as $\frac{DE^2 \times PS}{PE \times V}$. Therefore if IE be drawn, that ordinate for any other place of the corpuscle, as I, will become (*mutatis mutandis*) as $\frac{DE^2 \times IS}{IE \times V}$. Suppose the centripetal forces flowing from any point of the sphere, as E, to be to each other at the distances IE and PE as PE^n to IE^n (where the number n denotes the index of the powers of PE and IE), and those ordinates will become as $\frac{DE^2 \times PS}{PE \times PE^n}$ and $\frac{DE^2 \times IS}{IE \times IE^n}$ whose ratio to each other is as $PS \times IE \times IE^n$ to $IS \times PE \times PE^n$. Because SI, SE, SP are in continued proportion, the triangles SPE, SEI are alike; and thence IE is to PE as IS to SE or SA. For the ratio of IE to PE write the ratio of IS to SA; and the ratio of the ordinates becomes that of $PS \times IE^n$ to $SA \times PE^n$. But the ratio of PS to SA is subduplicate of that of the distances PS, SI; and the ratio of IE^n to PE^n (because IE is to PE as IS to SA) is subduplicate of that of the forces at the distances PS, IS. Therefore the ordinates, and consequently the areas which the ordinates describe, and the attractions proportional to them, are in a ratio compounded of those subduplicate ratios. Q.E.D.

PROPOSITION LXXXIII. PROBLEM XLII.

To find the force with which a corpuscle placed in the centre of a sphere is attracted towards any segment of that sphere whatsoever.



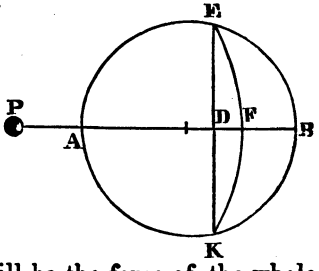
Let P be a body in the centre of that sphere and RBSD a segment thereof contained under the plane RDS, and the sphærical superficies RBS. Let DB be cut in F by a sphærical superficies EFG described from the centre P, and let the segment be divided into the parts BREFGS, FEDG. Let us suppose that segment to be not a purely mathematical but a physical superficies, having some, but a perfectly inconsiderable thickness. Let that thickness be called O, and (by what *Archimedes* has demonstrated) that superficies will be as $PF \times DF \times O$. Let us suppose besides the attractive forces of the particles of the sphere to be reciprocally as that power of the distances, of which n is index; and the force with which the superficies EFG attracts the body P will be (by Prop. LXXIX) as $\frac{DE^2 \times O}{PF^n}$, that is, as $\frac{2DF \times O}{PF^{n-1}} - \frac{DF^2 \times O}{PF^{n+1}}$. Let the perpendicular FN drawn into

O be proportional to this quantity; and the curvilinear area BDI, which the ordinate FN, drawn through the length DB with a continued motion will describe, will be as the whole force with which the whole segment RBSD attracts the body P. Q.E.I.

PROPOSITION LXXXIV. PROBLEM XLIII.

To find the force with which a corpuscle, placed without the centre of a sphere in the axis of any segment, is attracted by that segment.

Let the body P placed in the axis ADB of the segment EBK be attracted by that segment. About the centre P, with the interval PE, let the sphærical superficies EFK be described; and let it divide the segment into two parts EBKFE and EFKDE. Find the force of the first of those parts by Prop. LXXXI, and the force of the latter part by Prop. LXXXIII, and the sum of the forces will be the force of the whole segment EBKDE. Q.E.I.



SCHOLIUM.

The attractions of sphærical bodies being now explained, it comes next in order to treat of the laws of attraction in other bodies consisting in like manner of attractive particles; but to treat of them particularly is not necessary to my design. It will be sufficient to subjoin some general propositions relating to the forces of such bodies, and the motions thence arising, because the knowledge of these will be of some little use in philosophical inquiries.

SECTION XIII.

Of the attractive forces of bodies which are not of a sphærical figure.

PROPOSITION LXXXV. THEOREM XLII.

If a body be attracted by another, and its attraction be vastly stronger when it is contiguous to the attracting body than when they are separated from one another by a very small interval; the forces of the particles of the attracting body decrease, in the recess of the body attracted, in more than a duplicate ratio of the distance of the particles.

For if the forces decrease in a duplicate ratio of the distances from the particles, the attraction towards a sphærical body being (by Prop. LXXIV) reciprocally as the square of the distance of the attracted body from the centre of the sphere, will not be sensibly increased by the contact, and it

will be still less increased by it, if the attraction, in the recess of the body attracted, decreases in a still less proportion. The proposition, therefore, is evident concerning attractive spheres. And the case is the same of concave sphærical orbs attracting external bodies. And much more does it appear in orbs that attract bodies placed within them, because there the attractions diffused through the cavities of those orbs are (by Prop. LXX) destroyed by contrary attractions, and therefore have no effect even in the place of contact. Now if from these spheres and sphærical orbs we take away any parts remote from the place of contact, and add new parts any where at pleasure, we may change the figures of the attractive bodies at pleasure; but the parts added or taken away, being remote from the place of contact, will cause no remarkable excess of the attraction arising from the contact of the two bodies. Therefore the proposition holds good in bodies of all figures. Q.E.D.

PROPOSITION LXXXVI. THEOREM XLIII.

If the forces of the particles of which an attractive body is composed decrease, in the recess of the attractive body, in a triplicate or more than a triplicate ratio of the distance from the particles, the attraction will be vastly stronger in the point of contact than when the attracting and attracted bodies are separated from each other, though by never so small an interval.

For that the attraction is infinitely increased when the attracted corpuscle comes to touch an attracting sphere of this kind, appears, by the solution of Problem XLI, exhibited in the second and third Examples. The same will also appear (by comparing those Examples and Theorem XLI together) of attractions of bodies made towards concavo-convex orbs, whether the attracted bodies be placed without the orbs, or in the cavities within them. And by adding to or taking from those spheres and orbs any attractive matter any where without the place of contact, so that the attractive bodies may receive any assigned figure, the Proposition will hold good of all bodies universally. Q.E.D.

PROPOSITION LXXXVII. THEOREM XLIV.

If two bodies similar to each other, and consisting of matter equally attractive, attract separately two corpuscles proportional to those bodies, and in a like situation to them, the accelerative attractions of the corpuscles towards the entire bodies will be as the accelerative attractions of the corpuscles towards particles of the bodies proportional to the wholes, and alike situated in them.

For if the bodies are divided into particles proportional to the wholes, and alike situated in them, it will be, as the attraction towards any particle of one of the bodies to the attraction towards the correspondent particle

in the other body, so are the attractions towards the several particles of the first body, to the attractions towards the several correspondent particles of the other body; and, by composition, so is the attraction towards the first whole body to the attraction towards the second whole body. Q.E.D.

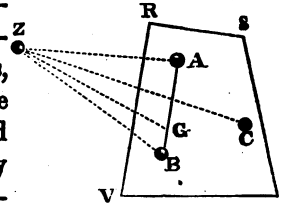
COR. 1. Therefore if, as the distances of the corpuscles attracted increase, the attractive forces of the particles decrease in the ratio of any power of the distances, the accelerative attractions towards the whole bodies will be as the bodies directly, and those powers of the distances inversely. As if the forces of the particles decrease in a duplicate ratio of the distances from the corpuscles attracted, and the bodies are as A^3 and B^3, and therefore both the cubic sides of the bodies, and the distance of the attracted corpuscles from the bodies, are as A and B; the accelerative attractions towards the bodies will be as $\frac{A^3}{A^2}$ and $\frac{B^3}{B^2}$, that is, as A and B the cubic sides of those bodies. If the forces of the particles decrease in a triplicate ratio of the distances from the attracted corpuscles, the accelerative attractions towards the whole bodies will be as $\frac{A^3}{A^3}$ and $\frac{B^3}{B^3}$, that is, equal. If the forces decrease in a quadruplicate ratio, the attractions towards the bodies will be as $\frac{A^3}{A^4}$ and $\frac{B^3}{B^4}$, that is, reciprocally as the cubic sides A and B. And so in other cases.

COR. 2. Hence, on the other hand, from the forces with which like bodies attract corpuscles similarly situated, may be collected the ratio of the decrease of the attractive forces of the particles as the attracted corpuscle recedes from them; if so be that decrease is directly or inversely in any ratio of the distances.

PROPOSITION LXXXVIII. THEOREM XLV.

If the attractive forces of the equal particles of any body be as the distance of the places from the particles, the force of the whole body will tend to its centre of gravity; and will be the same with the force of a globe, consisting of similar and equal matter, and having its centre in the centre of gravity.

Let the particles A, B, of the body RSTV attract any corpuscle Z with forces which, supposing the particles to be equal between themselves, are as the distances AZ, BZ; but, if they are supposed unequal, are as those particles and their distances AZ, BZ, conjunctly, or (if I may so speak) as those particles drawn into their distances AZ, BZ respectively. And let those forces be expressed by the



contents under A × AZ, and B × BZ. Join AB, and let it be cut in G, so that AG may be to BG as the particle B to the particle A; and G will be the common centre of gravity of the particles A and B. The force A × AZ will (by Cor. 2, of the Laws) be resolved into the forces A × GZ and A × AG; and the force B × BZ into the forces B × GZ and B × BG. Now the forces A × AG and B × BG, because A is proportional to B, and BG to AG, are equal, and therefore having contrary directions destroy one another. There remain then the forces A × GZ and B × GZ. These tend from Z towards the centre G, and compose the force $\overline{A + B}$ × GZ; that is, the same force as if the attractive particles A and B were placed in their common centre of gravity G, composing there a little globe.

By the same reasoning, if there be added a third particle C, and the force of it be compounded with the force $\overline{A + B}$ × GZ tending to the centre G, the force thence arising will tend to the common centre of gravity of that globe in G and of the particle C; that is, to the common centre of gravity of the three particles A, B, C; and will be the same as if that globe and the particle C were placed in that common centre composing a greater globe there; and so we may go on *in infinitum*. Therefore the whole force of all the particles of any body whatever RSTV is the same as if that body, without removing its centre of gravity, were to put on the form of a globe. Q.E.D.

COR. Hence the motion of the attracted body Z will be the same as if the attracting body RSTV were sphærical; and therefore if that attracting body be either at rest, or proceed uniformly in a right line, the body attracted will move in an ellipsis having its centre in the centre of gravity of the attracting body.

PROPOSITION LXXXIX. THEOREM XLVI.

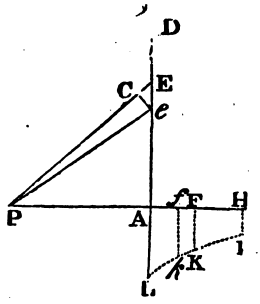
If there be several bodies consisting of equal particles whose forces are as the distances of the places from each, the force compounded of all the forces by which any corpuscle is attracted will tend to the common centre of gravity of the attracting bodies; and will be the same as if those attracting bodies, preserving their common centre of gravity, should unite there, and be formed into a globe.

This is demonstrated after the same manner as the foregoing Proposition.

COR. Therefore the motion of the attracted body will be the same as if the attracting bodies, preserving their common centre of gravity, should unite there, and be formed into a globe. And, therefore, if the common centre of gravity of the attracting bodies be either at rest, or proceed uniformly in a right line, the attracted body will move in an ellipsis having its centre in the common centre of gravity of the attracting bodies.

PROPOSITION XC. PROBLEM XLIV.

If to the several points of any circle there tend equal centripetal forces, increasing or decreasing in any ratio of the distances; it is required to find the force with which a corpuscle is attracted, that is, situate any where in a right line which stands at right angles to the plane of the circle at its centre.



Suppose a circle to be described about the centre A with any interval AD in a plane to which the right line AP is perpendicular; and let it be required to find the force with which a corpuscle P is attracted towards the same. From any point E of the circle, to the attracted corpuscle P, let there be drawn the right line PE. In the right line PA take PF equal to PE, and make a perpendicular FK, erected at F, to be as the force with which the point E attracts the corpuscle P. And let the curve line IKL be the locus of the point K. Let that curve meet the plane of the circle in L. In PA take PH equal to PD, and erect the perpendicular HI meeting that curve in I; and the attraction of the corpuscle P towards the circle will be as the area AHIL drawn into the altitude AP. Q.E.I.

For let there be taken in AE a very small line E*e*. Join P*e*, and in PE, PA take PC, P*f* equal to P*e*. And because the force, with which any point E of the annulus described about the centre A with the interval AE in the aforesaid plane attracts to itself the body P, is supposed to be as FK; and, therefore, the force with which that point attracts the body P towards A is as $\frac{AP \times FK}{PE}$; and the force with which the whole annulus attracts the body P towards A is as the annulus and $\frac{AP \times FK}{PE}$ conjunctly; and that annulus also is as the rectangle under the radius AE and the breadth E*e*, and this rectangle (because PE and AE, E*e* and CE are proportional) is equal to the rectangle PE × CE or PE × F*f*; the force with which that annulus attracts the body P towards A will be as PE × F*f* and $\frac{AP \times FK}{PE}$ conjunctly; that is, as the content under F*f* × FK × AP, or as the area FK*kf* drawn into AP. And therefore the sum of the forces with which all the annuli, in the circle described about the centre A with the interval AD, attract the body P towards A, is as the whole area AHIKL drawn into AP. Q.E.D.

COR. 1. Hence if the forces of the points decrease in the duplicate ratio

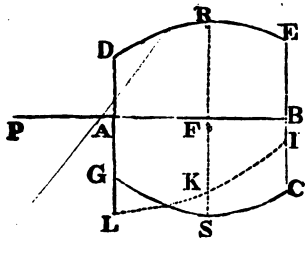
of the distances, that is, if FK be as $\frac{1}{PF^2}$, and therefore the area AHIKL as $\frac{1}{PA} - \frac{1}{PH}$; the attraction of the corpuscle P towards the circle will be as $1 - \frac{PA}{PH}$; that is, as $\frac{AH}{PH}$

Cor. 2. And universally if the forces of the points at the distances D be reciprocally as any power D^n of the distances; that is, if FK be as $\frac{1}{D^n}$, and therefore the area AHIKL as $\frac{1}{PA^{n-1}} - \frac{1}{PH^{n-1}}$; the attraction of the corpuscle P towards the circle will be as $\frac{1}{PA^{n-2}} - \frac{PA}{PH^{n-1}}$.

Cor. 3. And if the diameter of the circle be increased *in infinitum*, and the number n be greater than unity; the attraction of the corpuscle P towards the whole infinite plane will be reciprocally as PA^{n-2}, because the other term $\frac{PA}{PH^{n-1}}$ vanishes.

PROPOSITION XCI. PROBLEM XLV.

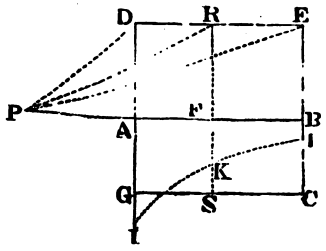
To find the attraction of a corpuscle situate in the axis of a round solid, to whose several points there tend equal centripetal forces decreasing in any ratio of the distances whatsoever.



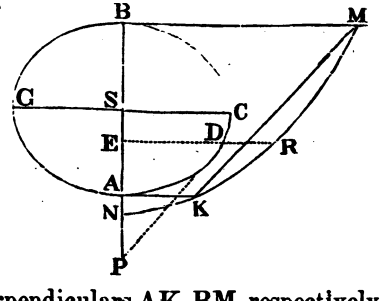
Let the corpuscle P, situate in the axis AB of the solid DECG, be attracted towards that solid. Let the solid be cut by any circle as RFS, perpendicular to the axis; and in its semi-diameter FS, in any plane PALKB passing through the axis, let there be taken (by Prop. XC) the length FK proportional to the force with which the corpuscle P is attracted towards that circle. Let the locus of the point K be the curve line LKI, meeting the planes of the outermost circles AL and BI in L and I; and the attraction of the corpuscle P towards the solid will be as the area LABI. Q.E.I.

Cor. 1. Hence if the solid be a cylinder described by the parallelogram ADEB revolved about the axis AB, and the centripetal forces tending to the several points be reciprocally as the squares of the distances from the points; the attraction of the corpuscle P towards this cylinder will be as $AB - PE + PD$. For the ordinate FK (by Cor. 1, Prop. XC) will be as $1 - \frac{PF}{PR}$. The part 1 of this quantity, drawn into the length AB, de-

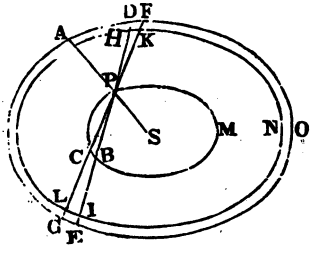
scribes the area $1 \times AB$; and the other part $\frac{PF}{PR}$, drawn into the length PB describes the area 1 into $\overline{PE - AD}$ (as may be easily shewn from the quadrature of the curve LKI); and, in like manner, the same part drawn into the length PA describes the area 1 into $\overline{PD - AD}$, and drawn into AB, the difference of PB and PA, describes 1 into $\overline{PE - PD}$, the difference of the areas. From the first content $1 \times AB$ take away the last content 1 into $\overline{PE - PD}$, and there will remain the area LABI equal to 1 into $\overline{AB - PE + PD}$. Therefore the force, being proportional to this area, is as $AB - PE + PD$.



COR. 2. Hence also is known the force by which a spheroid AGBC attracts any body P situate externally in its axis AB. Let NKRM be a conic section whose ordinate ER perpendicular to PE may be always equal to the length of the line PD, continually drawn to the point D in which that ordinate cuts the spheroid. From the vertices A, B, of the spheriod, let there be erected to its axis AB the perpendiculars AK, BM, respectively equal to AP, BP, and therefore meeting the conic section in K and M; and join KM cutting off from it the segment KMRK. Let S be the centre of the spheroid, and SC its greatest semi-diameter; and the force with which the spheroid attracts the body P will be to the force with which a sphere described with the diameter AB attracts the same body as $\frac{AS \times CS^2 - PS \times KMRK}{PS^2 + CS^2 - AS^2}$ is to $\frac{AS^3}{3PS^2}$. And by a calculation founded on the same principles may be found the forces of the segments of the spheroid.



COR. 3. If the corpuscle be placed within the spheroid and in its axis, the attraction will be as its distance from the centre. This may be easily collected from the following reasoning, whether the particle be in the axis or in any other given diameter. Let AGOF be an attracting spheroid, S its centre, and P the body attracted. Through the body P let there be drawn the semi-diameter SPA, and two right lines DE, FG meeting the spheroid in D and E, F and G; and let PCM, HLN be the superficies of



two interior spheroids similar and concentrical to the exterior, the first of which passes through the body P, and cuts the right lines DE, FG in B and C; and the latter cuts the same right lines in H and I, K and L. Let the spheroids have all one common axis, and the parts of the right lines intercepted on both sides DP and BE, FP and CG, DH and IE, FK and LG, will be mutually equal; because the right lines DE, PB, and HI, are bisected in the same point, as are also the right lines FG, PC, and KL. Conceive now DPF, EPG to represent opposite cones described with the infinitely small vertical angles DPF, EPG, and the lines DH, EI to be infinitely small also. Then the particles of the cones DHKF, GLIE, cut off by the spheroidical superficies, by reason of the equality of the lines DH and EI, will be to one another as the squares of the distances from the body P, and will therefore attract that corpuscle equally. And by a like reasoning if the spaces DPF, EGCB be divided into particles by the superficies of innumerable similar spheroids concentric to the former and having one common axis, all these particles will equally attract on both sides the body P towards contrary parts. Therefore the forces of the cone DPF, and of the conic segment EGCB, are equal, and by their contrariety destroy each other. And the case is the same of the forces of all the matter that lies without the interior spheroid PCBM. Therefore the body P is attracted by the interior spheroid PCBM alone, and therefore (by Cor. 3, Prop. LXXII) its attraction is to the force with which the body A is attracted by the whole spheroid AGOD as the distance PS to the distance AS. Q.E.D.

PROPOSITION XCII. PROBLEM XLVI.

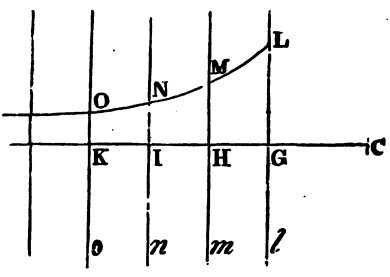
An attracting body being given, it is required to find the ratio of the decrease of the centripetal forces tending to its several points.

The body given must be formed into a sphere, a cylinder, or some regular figure, whose law of attraction answering to any ratio of decrease may be found by Prop. LXXX, LXXXI, and XCI. Then, by experiments, the force of the attractions must be found at several distances, and the law of attraction towards the whole, made known by that means, will give the ratio of the decrease of the forces of the several parts; which was to be found.

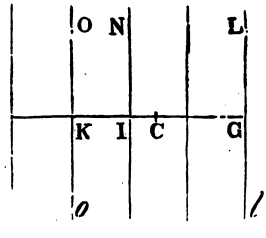
PROPOSITION XCIII. THEOREM XLVII.

If a solid be plane on one side, and infinitely extended on all other sides, and consist of equal particles equally attractive, whose forces decrease, in the recess from the solid, in the ratio of any power greater than the square of the distances; and a corpuscle placed towards either part of the plane is attracted by the force of the whole solid; I say that the attractive force of the whole solid, in the recess from its plane superfi-

cies, will decrease in the ratio of a power whose side is the distance of the corpuscle from the plane, and its index less by 3 *than the index of the power of the distances.*



CASE 1. Let LG*l* be the plane by which the solid is terminated. Let the solid lie on that hand of the plane that is towards I, and let it be resolved into innumerable planes *m*HM, *n*IN, *o*KO, &c., parallel to G*l*. And first let the attracted body C be placed without the solid. Let there be drawn CGHI perpendicular to those innumerable planes, and let the attractive forces of the points of the solid decrease in the ratio of a power of the distances whose index is the number n not less than 3. Therefore (by Cor. 3, Prop. XC) the force with which any plane *m*HM attracts the point C is reciprocally as CH^{n-2}. In the plane *m*HM take the length HM reciprocally proportional to CH^{n-2}, and that force will be as HM. In like manner in the several planes *l*GL, *n*IN, *o*KO, &c., take the lengths GL, IN, KO, &c., reciprocally proportional to CG^{n-2}, CI^{n-2}, CK^{n-2}, &c., and the forces of those planes will be as the lengths so taken, and therefore the sum of the forces as the sum of the lengths, that is, the force of the whole solid as the area GLOK produced infinitely towards OK. But that area (by the known methods of quadratures) is reciprocally as CG^{n-3}, and therefore the force of the whole solid is reciprocally as CG^{n-3}. Q.E.D.



CASE 2. Let the corpuscle C be now placed on that hand of the plane *l*GL that is within the solid, and take the distance CK equal to the distance CG. And the part of the solid LG*lo*KO terminated by the parallel planes *l*GL, *o*KO, will attract the corpuscle C, situate in the middle, neither one way nor another, the contrary actions of the opposite points destroying one another by reason of their equality. Therefore the corpuscle C is attracted by the force only of the solid situate beyond the plane OK. But this force (by Case 1) is reciprocally as CK^{n-3}, that is, (because CG, CK are equal) reciprocally as CG^{n-3}. Q.E.D.

COR. 1. Hence if the solid LGIN be terminated on each side by two infinite parallel planes LG, IN, its attractive force is known, subducting from the attractive force of the whole infinite solid LGKO the attractive force of the more distant part NIKO infinitely produced towards KO.

COR. 2. If the more distant part of this solid be rejected, because its attraction compared with the attraction of the nearer part is inconsiderable,

the attraction of that nearer part will, as the distance increases, decrease nearly in the ratio of the power CG^{n-3}.

COR. 3. And hence if any finite body, plane on one side, attract a corpuscle situate over against the middle of that plane, and the distance between the corpuscle and the plane compared with the dimensions of the attracting body be extremely small; and the attracting body consist of homogeneous particles, whose attractive forces decrease in the ratio of any power of the distances greater than the quadruplicate; the attractive force of the whole body will decrease very nearly in the ratio of a power whose side is that very small distance, and the index less by 3 than the index of the former power. This assertion does not hold good, however, of a body consisting of particles whose attractive forces decrease in the ratio of the triplicate power of the distances; because, in that case, the attraction of the remoter part of the infinite body in the second Corollary is always infinitely greater than the attraction of the nearer part.

SCHOLIUM.

If a body is attracted perpendicularly towards a given plane, and from the law of attraction given, the motion of the body be required; the Problem will be solved by seeking (by Prop. XXXIX) the motion of the body descending in a right line towards that plane, and (by Cor. 2, of the Laws) compounding that motion with an uniform motion performed in the direction of lines parallel to that plane. And, on the contrary, if there be required the law of the attraction tending towards the plane in perpendicular directions, by which the body may be caused to move in any given curve line, the Problem will be solved by working after the manner of the third Problem.

But the operations may be contracted by resolving the ordinates into converging series. As if to a base A the length B be ordinately applied in any given angle, and that length be as any power of the base $A^{\frac{m}{n}}$; and there be sought the force with which a body, either attracted towards the base or driven from it in the direction of that ordinate, may be caused to move in the curve line which that ordinate always describes with its superior extremity; I suppose the base to be increased by a very small part O, and I resolve the ordinate $\overline{A+O}|^{\frac{m}{n}}$ into an infinite series $A^{\frac{m}{n}} + \frac{m}{n} OA^{\frac{m-n}{n}} + \frac{mm-mn}{2nn} OOA^{\frac{m-2n}{n}}$ &c., and I suppose the force proportional to the term of this series in which O is of two dimensions, that is, to the term $\frac{mm-mn}{2nn} OOA^{\frac{m-2n}{n}}$. Therefore the force sought is as

$\frac{mm-mn}{nn}$ A $\frac{m-2n}{n}$, or, which is the same thing, as $\frac{mm-mn}{nn}$ B $\frac{m-2n}{m}$. As if the ordinate describe a parabola, m being $= 2$, and $n = 1$, the force will be as the given quantity $2B^{\circ}$, and therefore is given. Therefore with a given force the body will move in a parabola, as *Galileo* has demonstrated. If the ordinate describe an hyperbola, m being $= 0 - 1$, and $n = 1$, the force will be as $2A^{-3}$ or $2B^{3}$; and therefore a force which is as the cube of the ordinate will cause the body to move in an hyperbola. But leaving this kind of propositions, I shall go on to some others relating to motion which I have not yet touched upon.

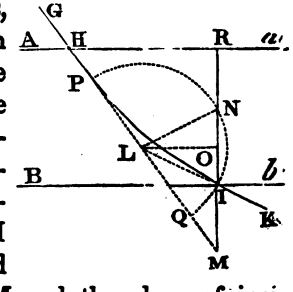
SECTION XIV.

Of the motion of very small bodies when agitated by centripetal forces tending to the several parts of any very great body.

PROPOSITION XCIV. THEOREM XLVIII.

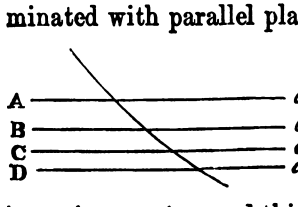
If two similar mediums be separated from each other by a space terminated on both sides by parallel planes, and a body in its passage through that space be attracted or impelled perpendicularly towards either of those mediums, and not agitated or hindered by any other force; and the attraction be every where the same at equal distances from either plane, taken towards the same hand of the plane; I say, that the sine of incidence upon either plane will be to the sine of emergence from the other plane in a given ratio.

CASE 1. Let A*a* and B*b* be two parallel planes, and let the body light upon the first plane A*a* in the direction of the line GH, and in its whole passage through the intermediate space let it be attracted or impelled towards the medium of incidence, and by that action let it be made to describe a curve line HI, and let it emerge in the direction of the line IK. Let there be erected IM perpendicular to B*b* the plane of emergence, and meeting the line of incidence GH prolonged in M, and the plane of incidence A*a* in R; and let the line of emergence KI be produced and meet HM in L. About the centre L, with the interval LI, let a circle be described cutting both HM in P and Q, and MI produced in N; and, first, if the attraction or impulse be supposed uniform, the curve HI (by what *Galileo* has demonstrated) be a parabola, whose property is that of a rec-



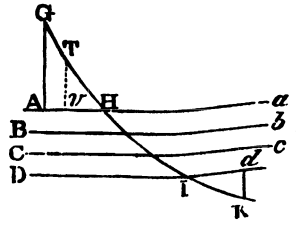
tangle under its given latus rectum and the line IM is equal to the square of HM; and moreover the line HM will be bisected in L. Whence if to MI there be let fall the perpendicular LO, MO, OR will be equal; and adding the equal lines ON, OI, the wholes MN, IR will be equal also. Therefore since IR is given, MN is also given, and the rectangle NMI is to the rectangle under the latus rectum and IM, that is, to HM^2 in a given ratio. But the rectangle NMI is equal to the rectangle PMQ, that is, to the difference of the squares ML^2, and PL^2 or LI^2; and HM^2 hath a given ratio to its fourth part ML^2; therefore the ratio of $ML^2 - LI^2$ to ML^2 is given, and by conversion the ratio of LI^2 to ML^2, and its subduplicate, the ratio of LI to ML. But in every triangle, as LMI, the sines of the angles are proportional to the opposite sides. Therefore the ratio of the sine of the angle of incidence LMR to the sine of the angle of emergence LIR is given. Q.E.D.

Case 2. Let now the body pass successively through several spaces terminated with parallel planes A*ab*B, B*bc*C, &c., and let it be acted on by a force which is uniform in each of them separately, but different in the different spaces; and by what was just demonstrated, the sine of the angle of incidence on the first plane A*a* is to the sine of emergence from the second plane B*b* in a given ratio; and this sine of incidence upon the second plane B*b* will be to the sine of emergence from the third plane C*c* in a given ratio; and this sine to the sine of emergence from the fourth plane D*d* in a given ratio; and so on *in infinitum*; and, by equality, the sine of incidence on the first plane to the sine of emergence from the last plane in a given ratio. Let now the intervals of the planes be diminished, and their number be infinitely increased, so that the action of attraction or impulse, exerted according to any assigned law, may become continual, and the ratio of the sine of incidence on the first plane to the sine of emergence from the last plane being all along given, will be given then also. Q.E.D.



PROPOSITION XCV. THEOREM XLIX.

The same things being supposed, I say, that the velocity of the body before its incidence is to its velocity after emergence as the sine of emergence to the sine of incidence.



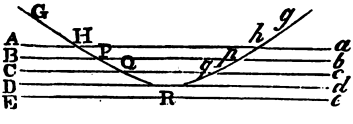
Make AH and I*d* equal, and erect the perpendiculars AG, *d*K meeting the lines of incidence and emergence GH, IK, in G and K. In GH take TH equal to IK, and to the plane A*a* let fall a perpendicular T*v*. And (by Cor. 2 of the Laws of Motion) let the motion of the body be resolved into two, one perpendicular to the planes

A*a*, B*b*, C*c*, &c, and another parallel to them. The force of attraction or impulse, acting in directions perpendicular to those planes, does not at all alter the motion in parallel directions; and therefore the body proceeding with this motion will in equal times go through those equal parallel intervals that lie between the line AG and the point H, and between the point I and the line *d*K; that is, they will describe the lines GH, IK in equal times. Therefore the velocity before incidence is to the velocity after emergence as GH to IK or TH, that is, as AH or I*d* to *v*H, that is (supposing TH or IK radius), as the sine of emergence to the sine of incidence. Q.E.D.

PROPOSITION XCVI. THEOREM L.

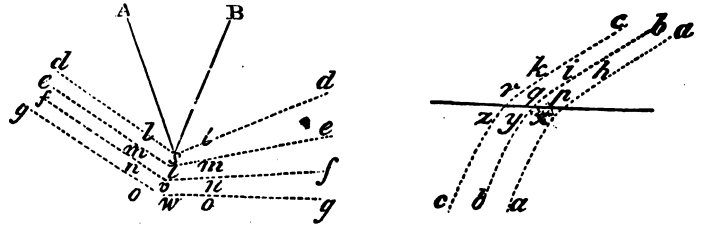
The same things being supposed, and that the motion before incidence is swifter than afterwards; I say, that if the line of incidence be inclined continually, the body will be at last reflected, and the angle of reflexion will be equal to the angle of incidence.

For conceive the body passing between the parallel planes A*a*, B*b*, C*c*, &c., to describe parabolic arcs as above; and let those arcs be HP, PQ, QR, &c. And let the obliquity of the line of incidence GH to the first plane A*a* be such that the sine of incidence may be to the radius of the circle whose sine it is, in the same ratio which the same sine of incidence hath to the sine of emergence from the plane D*d* into the space D*d*e E; and because the sine of emergence is now become equal to radius, the angle of emergence will be a right one, and therefore the line of emergence will coincide with the plane D*d*. Let the body come to this plane in the point R; and because the line of emergence coincides with that plane, it is manifest that the body can proceed no farther towards the plane E*e*. But neither can it proceed in the line of emergence R*d*; because it is perpetually attracted or impelled towards the medium of incidence. It will return, therefore, between the planes C*c*, D*d*, describing an arc of a parabola QR*q*, whose principal vertex (by what *Galileo* has demonstrated) is in R, cutting the plane C*c* in the same angle at *q*, that it did before at Q; then going on in the parabolic arcs *qp*, *ph*, &c., similar and equal to the former arcs QP, PH, &c., it will cut the rest of the planes in the same angles at *p*, *h*, &c., as it did before in P, H, &c., and will emerge at last with the same obliquity at *h* with which it first impinged on that plane at H. Conceive now the intervals of the planes A*a*, B*b*, C*c*, D*d*, E*e*, &c., to be infinitely diminished, and the number infinitely increased, so that the action of attraction or impulse, exerted according to any assigned law, may become continual; and, the angle of emergence remaining all along equal to the angle of incidence, will be equal to the same also at last. Q.E.D.



SCHOLIUM.

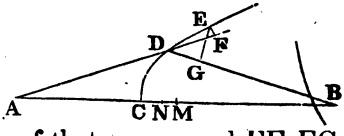
These attractions bear a great resemblance to the reflexions and refractions of light made in a given ratio of the secants, as was discovered by *Snellius*; and consequently in a given ratio of the sines, as was exhibited by *Des Cartes*. For it is now certain from the phenomena of *Jupiter's* satellites, confirmed by the observations of different astronomers, that light is propagated in succession, and requires about seven or eight minutes to travel from the sun to the earth. Moreover, the rays of light that are in our air (as lately was discovered by *Grimaldus*, by the admission of light into a dark room through a small hole, which I have also tried) in their passage near the angles of bodies, whether transparent or opaque (such as the circular and rectangular edges of gold, silver and brass coins, or of knives, or broken pieces of stone or glass), are bent or inflected round those bodies as if they were attracted to them; and those rays which in their passage come nearest to the bodies are the most inflected, as if they were most attracted; which thing I myself have also carefully observed. And those which pass at greater distances are less inflected; and those at still greater distances are a little inflected the contrary way, and form three fringes of colours. In the figure *s* represents the edge of a knife, or any



kind of wedge AsB; and *gowog*, *fnunf*, *emtme*, *dlsld*, are rays inflected towards the knife in the arcs *owo*, *nvn*, *mtm*, *lsl*; which inflection is greater or less according to their distance from the knife. Now since this inflection of the rays is performed in the air without the knife, it follows that the rays which fall upon the knife are first inflected in the air before they touch the knife. And the case is the same of the rays falling upon glass. The refraction, therefore, is made not in the point of incidence, but gradually, by a continual inflection of the rays; which is done partly in the air before they touch the glass, partly (if I mistake not) within the glass, after they have entered it; as is represented in the rays *ckzc*, *byb*, *ahxa*, falling upon *r*, *q*, *p*, and inflected between *k* and *z*, *i* and *y*, *h* and *x*. Therefore because of the analogy there is between the propagation of the rays of light and the motion of bodies, I thought it not amiss to add the following Propositions for optical uses; not at all considering the nature of the rays of light, or inquiring whether they are bodies or not; but only determining the trajectories of bodies which are extremely like the trajectories of the rays.

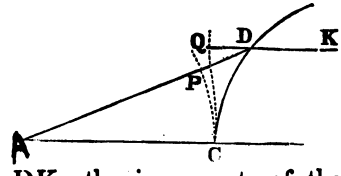
PROPOSITION XCVII. PROBLEM XLVII.

Supposing the sine of incidence upon any superficies to be in a given ratio to the sine of emergence; and that the inflection of the paths of those bodies near that superficies is performed in a very short space, which may be considered as a point; it is required to determine such a superficies as may cause all the corpuscles issuing from any one given place to converge to another given place.



Let A be the place from whence the corpuscles diverge; B the place to which they should converge; CDE the curve line which by its revolution round the axis AB describes the superficies sought; D, E, any two points of that curve; and EF, EG, perpendiculars let fall on the paths of the bodies AD, DB. Let the point D approach to and coalesce with the point E; and the ultimate ratio of the line DF by which AD is increased, to the line DG by which DB is diminished, will be the same as that of the sine of incidence to the sine of emergence Therefore the ratio of the increment of the line AD to the decrement of the line DB is given; and therefore if in the axis AB there be taken any where the point C through which the curve CDE must pass, and CM the increment of AC be taken in that given ratio to CN the decrement of BC, and from the centres A, B, with the intervals AM, BN, there be described two circles cutting each other in D; that point D will touch the curve sought CDE, and, by touching it any where at pleasure, will determine that curve. Q.E.I.

COR. 1. By causing the point A or B to go off sometimes *in infinitum*, and sometimes to move towards other parts of the point C, will be obtained all those figures which *Cartesius* has exhibited in his Optics and Geometry relating to refractions. The invention of which *Cartesius* having thought fit to conceal, is here laid open in this Proposition.

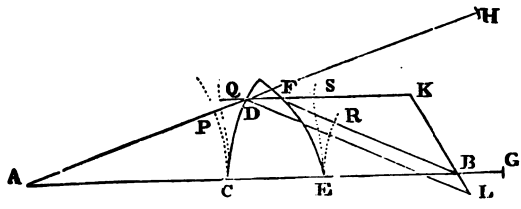


COR. 2. If a body lighting on any superficies CD in the direction of a right line AD, drawn according to any law, should emerge in the direction of another right line DK: and from the point C there be drawn curve lines CP, CQ, always perpendicular to AD, DK; the increments of the lines PD, QD, and therefore the lines themselves PD, QD, generated by those increments, will be as the sines of incidence and emergence to each other, and *è contra.*

PROPOSITION XCVIII. PROBLEM XLVIII.

The same things supposed; if round the axis AB *any attractive superficies be described as* CD, *regular or irregular, through which the bodies issuing from the given place* A *must pass; it is required to find*

a second attractive superficies EF, *which may make those bodies converge to a given place* B.



Let a line joining AB cut the first superficies in C and the second in E, the point D being taken any how at pleasure. And supposing the sine of incidence on the first superficies to the sine of emergence from the same, and the sine of emergence from the second superficies to the sine of incidence on the same, to be as any given quantity M to another given quantity N; then produce AB to G, so that BG may be to CE as M — N to N; and AD to H, so that AH may be equal to AG; and DF to K, so that DK may be to DH as N to M. Join KB, and about the centre D with the interval DH describe a circle meeting KB produced in L, and draw BF parallel to DL; and the point F will touch the line EF, which, being turned round the axis AB, will describe the superficies sought. Q.E.F.

For conceive the lines CP, CQ to be every where perpendicular to AD, DF, and the lines ER, ES to FB, FD respectively, and therefore QS to be always equal to CE; and (by Cor. 2, Prop. XCVII) PD will be to QD as M to N, and therefore as DL to DK, or FB to FK; and by division as DL — FB or PH — PD — FB to FD or FQ — QD; and by composition as PH — FB to FQ, that is (because PH and CG, QS and CE, are equal), as CE + BG — FR to CE — FS. But (because BG is to CE as M — N to N) it comes to pass also that CE + BG is to CE as M to N; and therefore, by division, FR is to FS as M to N; and therefore (by Cor. 2, Prop XCVII) the superficies EF compels a body, falling upon it in the direction DF, to go on in the line FR to the place B. Q.E.D.

SCHOLIUM.

In the same manner one may go on to three or more superficies. But of all figures the sphærical is the most proper for optical uses. If the object glasses of telescopes were made of two glasses of a sphærical figure, containing water between them, it is not unlikely that the errors of the refractions made in the extreme parts of the superficies of the glasses may be accurately enough corrected by the refractions of the water. Such object glasses are to be preferred before elliptic and hyperbolic glasses, not only because they may be formed with more ease and accuracy, but because the pencils of rays situate without the axis of the glass would be more accurately refracted by them. But the different refrangibility of different rays is the real obstacle that hinders optics from being made perfect by sphærical or any other figures. Unless the errors thence arising can be corrected, all the labour spent in correcting the others is quite thrown away.

BOOK II.

BOOK II.

OF THE MOTION OF BODIES.

SECTION I.

Of the motion of bodies that are resisted in the ratio of the velocity.

PROPOSITION I. THEOREM I.

If a body is resisted in the ratio of its velocity, the motion lost by resistance is as the space gone over in its motion.

For since the motion lost in each equal particle of time is as the velocity, that is, as the particle of space gone over, then, by composition, the motion lost in the whole time will be as the whole space gone over. Q.E.D.

Cor. Therefore if the body, destitute of all gravity, move by its innate force only in free spaces, and there be given both its whole motion at the beginning, and also the motion remaining after some part of the way is gone over, there will be given also the whole space which the body can describe in an infinite time. For that space will be to the space now described as the whole motion at the beginning is to the part lost of that motion.

LEMMA I.

Quantities proportional to their differences are continually proportional.

Let A be to A — B as B to B — C and C to C — D, &c., and, by conversion, A will be to B as B to C and C to D, &c. Q.E.D.

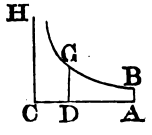
PROPOSITION II. THEOREM II.

If a body is resisted in the ratio of its velocity, and moves, by its vis insita *only, through a similar medium, and the times be taken equal, the velocities in the beginning of each of the times are in a geometrical progression, and the spaces described in each of the times are as the velocities.*

Case 1. Let the time be divided into equal particles; and if at the very beginning of each particle we suppose the resistance to act with one single impulse which is as the velocity, the decrement of the velocity in each of

the particles of time will be as the same velocity. Therefore the velocities are proportional to their differences, and therefore (by Lem. 1, Book II) continually proportional. Therefore if out of an equal number of particles there be compounded any equal portions of time, the velocities at the beginning of those times will be as terms in a continued progression, which are taken by intervals, omitting every where an equal number of intermediate terms. But the ratios of these terms are compounded of the equal ratios of the intermediate terms equally repeated, and therefore are equal Therefore the velocities, being proportional to those terms, are in geometrical progression. Let those equal particles of time be diminished, and their number increased *in infinitum*, so that the impulse of resistance may become continual; and the velocities at the beginnings of equal times, always continually proportional, will be also in this case continually proportional. Q.E.D.

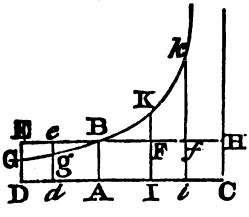
CASE 2. And, by division, the differences of the velocities, that is, the parts of the velocities lost in each of the times, are as the wholes; but the spaces described in each of the times are as the lost parts of the velocities (by Prop. 1, Book I), and therefore are also as the wholes. Q.E.D.



COROL. Hence if to the rectangular asymptotes AC, CH, the hyperbola BG is described, and AB, DG be drawn perpendicular to the asymptote AC, and both the velocity of the body, and the resistance of the medium, at the very beginning of the motion, be expressed by any given line AC, and, after some time is elapsed, by the indefinite line DC; the time may be expressed by the area ABGD, and the space described in that time by the line AD. For if that area, by the motion of the point D, be uniformly increased in the same manner as the time, the right line DC will decrease in a geometrical ratio in the same manner as the velocity; and the parts of the right line AC, described in equal times, will decrease in the same ratio.

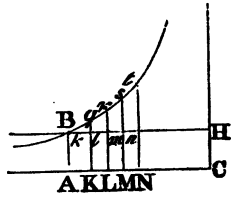
PROPOSITION III. PROBLEM I.

To define the motion of a body which, in a similar medium, ascends or descends in a right line, and is resisted in the ratio of its velocity, and acted upon by an uniform force of gravity.



The body ascending, let the gravity be expounded by any given rectangle BACH; and the resistance of the medium, at the beginning of the ascent, by the rectangle BADE, taken on the contrary side of the right line AB. Through the point B, with the rectangular asymptotes AC, CH, describe an hyperbola, cutting the perpendiculars DE, *de*, in

G, *g*; and the body ascending will in the time DG*gd* describe the space EG*ge*; in the time DGBA, the space of the whole ascent EGB; in the time ABKI, the space of descent BFK; and in the time IK*ki* the space of descent KF*fk*; and the velocities of the bodies (proportional to the resistance of the medium) in these periods of time will be ABED, AB*ed*, O, ABFI, AB*fi* respectively; and the greatest velocity which the body can acquire by descending will be BACH.



For let the rectangle BACH be resolved into innumerable rectangles A*k*, K*l*, L*m*, M*n*, &c., which shall be as the increments of the velocities produced in so many equal times; then will 0, A*k*, A*l*, A*m*, A*n*, &c., be as the whole velocities, and therefore (by supposition) as the resistances of the medium in the beginning of each of the equal times. Make AC to AK, or ABHC to AB*k*K, as the force of gravity to the resistance in the beginning of the second time; then from the force of gravity subduct the resistances, and ABHC, K*k*HC, L*l*HC, M*m*HC, &c., will be as the absolute forces with which the body is acted upon in the beginning of each of the times, and therefore (by Law I) as the increments of the velocities, that is, as the rectangles A*k*, K*l*, L*m*, M*n*, &c., and therefore (by Lem. 1, Book II) in a geometrical progression. Therefore, if the right lines K*k*, L*l* M*m*, N*n*, &c., are produced so as to meet the hyperbola in *q*, *r*, *s*, *t*, &c., the areas AB*q*K, K*qr*L, L*rs*M, M*st*N, &c., will be equal, and therefore analogous to the equal times and equal gravitating forces. But the area AB*q*K (by Corol. 3, Lem. VII and VIII, Book I) is to the area B*kq* as K*q* to $\frac{1}{2}kq$, or AC to $\frac{1}{2}$AK, that is, as the force of gravity to the resistance in the middle of the first time. And by the like reasoning, the areas *q*KL*r*, *r*LM*s*, *s*MN*t*, &c., are to the areas *qklr*, *rlms*, *smnt*, &c., as the gravitating forces to the resistances in the middle of the second, third, fourth time, and so on. Therefore since the equal areas BAK*q*, *q*KL*r*, *r*LM*s*, *s*MN*t*, &c., are analogous to the gravitating forces, the areas B*kq*, *qklr*, *rlms*, *smnt*, &c., will be analogous to the resistances in the middle of each of the times, that is (by supposition), to the velocities, and so to the spaces described. Take the sums of the analogous quantities, and the areas B*kq*, B*lr*, B*ms*, B*nt*, &c., will be analogous to the whole spaces described; and also the areas AB*q*K, AB*r*L, AB*s*M, AB*t*N, &c., to the times. Therefore the body, in descending, will in any time AB*r*L describe the space B*lr*, and in the time L*rt*N the space *rlnt*. Q.E.D. And the like demonstration holds in ascending motion.

COROL. 1. Therefore the greatest velocity that the body can acquire by falling is to the velocity acquired in any given time as the given force of gravity which perpetually acts upon it to the resisting force which opposes it at the end of that time.

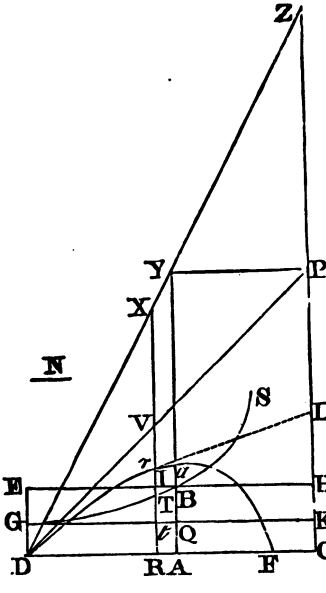
Corol. 2. But the time being augmented in an arithmetical progression, the sum of that greatest velocity and the velocity in the ascent, and also their difference in the descent, decreases in a geometrical progression.

Corol. 3. Also the differences of the spaces, which are described in equal differences of the times, decrease in the same geometrical progression.

Corol. 4. The space described by the body is the difference of two spaces, whereof one is as the time taken from the beginning of the descent, and the other as the velocity; which [spaces] also at the beginning of the descent are equal among themselves.

PROPOSITION IV. PROBLEM II.

Supposing the force of gravity in any similar medium to be uniform, and to tend perpendicularly to the plane of the horizon; to define the motion of a projectile therein, which suffers resistance proportional to its velocity.



Let the projectile go from any place D in the direction of any right line DP, and let its velocity at the beginning of the motion be expounded by the length DP. From the point P let fall the perpendicular PC on the horizontal line DC, and cut DC in A, so that DA may be to AC as the resistance of the medium arising from the motion upwards at the beginning to the force of gravity; or (which comes to the same) so that the rectangle under DA and DP may be to that under AC and CP as the whole resistance at the beginning of the motion to the force of gravity. With the asymptotes DC, CP describe any hyperbola GTBS cutting the perpendiculars DG, AB in G and B; complete the parallelogram DGKC, and let its side GK cut AB in Q. Take a line N in the same ratio to QB as DC is in to CP; and from any point R of the right line DC erect RT perpendicular to it, meeting the hyperbola in T, and the right lines EH, GK, DP in I, *t*, and V; in that perpendicular take V*r* equal to $\frac{t\mathrm{GT}}{\mathrm{N}}$, or which is the same thing, take R*r* equal to $\frac{\mathrm{GTIE}}{\mathrm{N}}$; and the projectile in the time DRTG will arrive at the point *r* describing the curve line D*ra*F, the locus of the point *r*; thence it will come to its greatest height *a* in the perpendicular AB; and afterwards

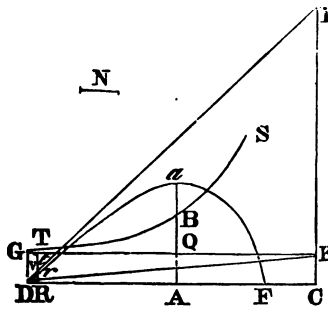
ever approach to the asymptote PC. And its velocity in any point r will be as the tangent rL to the curve. Q.E.I.

For N is to QB as DC to CP or DR to RV, and therefore RV is equal to $\frac{DR \times QB}{N}$, and Rr (that is, RV — Vr, or $\frac{DR \times QB - tGT}{N}$) is equal to $\frac{DR \times AB - RDGT}{N}$. Now let the time be expounded by the area RDGT and (by Laws, Cor. 2), distinguish the motion of the body into two others, one of ascent, the other lateral. And since the resistance is as the motion, let that also be distinguished into two parts proportional and contrary to the parts of the motion: and therefore the length described by the lateral motion will be (by Prop. II, Book II) as the line DR, and the height (by Prop. III, Book II) as the area DR × AB — RDGT, that is, as the line Rr. But in the very beginning of the motion the area RDGT is equal to the rectangle DR × AQ, and therefore that line Rr (or $\frac{DR \times AB - DR \times AQ}{N}$) will then be to DR as AB — AQ or QB to N, that is, as CP to DC; and therefore as the motion upwards to the motion lengthwise at the beginning. Since, therefore, Rr is always as the height, and DR always as the length, and Rr is to DR at the beginning as the height to the length, it follows, that Rr is always to DR as the height to the length; and therefore that the body will move in the line DraF, which is the locus of the point r. Q.E.D.

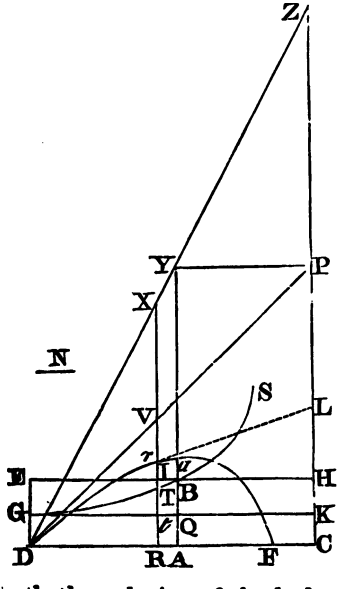
COR. 1. Therefore Rr is equal to $\frac{DR \times AB}{N} - \frac{RDGT}{N}$; and therefore if RT be produced to X so that RX may be equal to $\frac{DR \times AB}{N}$, that is, if the parallelogram ACPY be completed, and DY cutting CP in Z be drawn, and RT be produced till it meets DY in X; Xr will be equal to $\frac{RDGT}{N}$, and therefore proportional to the time.

COR. 2. Whence if innumerable lines CR, or, which is the same, innumerable lines ZX, be taken in a geometrical progression, there will be as many lines Xr in an arithmetical progression. And hence the curve DraF is easily delineated by the table of logarithms.

COR. 3. If a parabola be constructed to the vertex D, and the diameter DG produced downwards, and its latus rectum is to 2 DP as the whole resistance at the beginning of the motion to the gravitating force, the velocity with which the body ought to go from the place D, in the direction of the right line DP, so as in an uniform resisting medium to describe the curve DraF, will be the same as that with which it ought to go from the same place D in the direction of the same right line DP, so as to describe



a parabola in a non-resisting medium. For the latus rectum of this parabola, at the very beginning of the motion, is $\frac{DV^2}{Vr}$; and Vr is $\frac{tGT}{N}$ or $\frac{DR \times Tt}{2N}$. But a right line, which, if drawn, would touch the hyperbola GTS in G, is parallel to DK, and therefore Tt is $\frac{CK \times DR}{DC}$, and N is $\frac{QB \times DC}{CP}$. And therefore Vr is equal to $\frac{DR^2 \times CK \times CP}{2DC^2 \times QB}$, that is (because DR and DC, DV and DP are proportionals), to $\frac{DV^2 \times CK \times CP}{2DP^2 \times QB}$; and the latus rectum $\frac{DV^2}{Vr}$ comes out $\frac{2DP^2 \times QB}{CK \times CP}$, that is (because QB and CK, DA and AC are proportional), $\frac{2DP^2 \times DA}{AC \times CP}$, and therefore is to 2DP as DP × DA to CP × AC; that is, as the resistance to the gravity. Q.E.D.



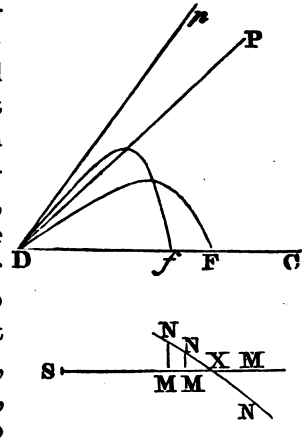
Cor. 4. Hence if a body be projected from any place D with a given velocity, in the direction of a right line DP given by position, and the resistance of the medium, at the beginning of the motion, be given, the curve D*ra*F, which that body will describe, may be found. For the velocity being given, the latus rectum of the parabola is given, as is well known. And taking 2DP to that latus rectum, as the force of gravity to the resisting force, DP is also given. Then cutting DC in A, so that CP × AC may be to DP × DA in the same ratio of the gravity to the resistance, the point A will be given. And hence the curve D*ra*F is also given.

Cor. 5. And, on the contrary, if the curve D*ra*F be given, there will be given both the velocity of the body and the resistance of the medium in each of the places r. For the ratio of CP × AC to DP × DA being given, there is given both the resistance of the medium at the beginning of the motion and the latus rectum of the parabola; and thence the velocity at the beginning of the motion is given also. Then from the length of the tangent

L there is given both the velocity proportional to it, and the resistance proportional to the velocity in any place r.

COR. 6. But since the length 2DP is to the latus rectum of the parabola as the gravity to the resistance in D; and, from the velocity augmented, the resistance is augmented in the same ratio, but the latus rectum of the parabola is augmented in the duplicate of that ratio, it is plain that the length 2DP is augmented in that simple ratio only; and is therefore always proportional to the velocity; nor will it be augmented or diminished by the change of the angle CDP, unless the velocity be also changed.

COR. 7. Hence appears the method of determining the curve DraF nearly from the phenomena, and thence collecting the resistance and velocity with which the body is projected. Let two similar and equal bodies be projected with the same velocity, from the place D, in different angles CDP, CDp; and let the places F, f, where they fall upon the horizontal plane DC, be known. Then taking any length for DP or Dp suppose the resistance in D to be to the gravity in any ratio whatsoever, and let that ratio be expounded by any length SM. Then, by computation, from that assumed length DP, find the lengths DF, Df; and from the ratio $\frac{Ff}{DF}$, found by calculation, subduct the same ratio as found by experiment; and let the difference be expounded by the perpendicular MN. Repeat the same a second and a third time, by assuming always a new ratio SM of the resistance to the gravity, and collecting a new difference MN. Draw the affirmative differences on one side of the right line SM, and the negative on the other side; and through the points N, N, N, draw a regular curve NNN, cutting the right line SMMM in X, and SX will be the true ratio of the resistance to the gravity, which was to be found. From this ratio the length DF is to be collected by calculation; and a length, which is to the assumed length DP as the length DF known by experiment to the length DF just now found, will be the true length DP. This being known, you will have both the curve line DraF which the body describes, and also the velocity and resistance of the body in each place.



SCHOLIUM.

But, yet, that the resistance of bodies is in the ratio of the velocity, is more a mathematical hypothesis than a physical one. In mediums void of all tenacity, the resistances made to bodies are in the duplicate ratio of the velocities. For by the action of a swifter body, a greater motion in propor-

tion to a greater velocity is communicated to the same quantity of the medium in a less time; and in an equal time, by reason of a greater quantity of the disturbed medium, a motion is communicated in the duplicate ratio greater; and the resistance (by Law II and III) is as the motion communicated. Let us, therefore, see what motions arise from this law of resistance.

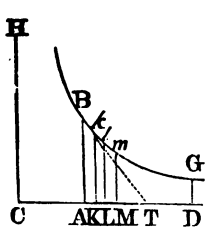
SECTION II.

Of the motion of bodies that are resisted in the duplicate ratio of their velocities.

PROPOSITION V. THEOREM III.

If a body is resisted in the duplicate ratio of its velocity, and moves by its innate force only through a similar medium; and the times be taken in a geometrical progression, proceeding from less to greater terms: I say, that the velocities at the beginning of each of the times are in the same geometrical progression inversely; and that the spaces are equal, which are described in each of the times.

For since the resistance of the medium is proportional to the square of the velocity, and the decrement of the velocity is proportional to the resistance: if the time be divided into innumerable equal particles, the squares of the velocities at the beginning of each of the times will be proportional to the differences of the same velocities. Let those particles of time be AK, KL, LM, &c., taken in the right line CD; and erect the perpendiculars AB, K*k*, L*l*, M*m*, &c., meeting the hyperbola B*klm*G, described with the centre C, and the rectangular asymptotes CD, CH, in B, *k*, *l*, *m*, &c.; then AB will be to K*k* as CK to CA, and, by division, AB — K*k* to K*k* as AK to CA, and alternately, AB — K*k* to AK as K*k* to CA; and therefore as AB × K*k* to AB × CA. Therefore since AK and AB × CA are given, AB — K*k* will be as AB × K*k*; and, lastly, when AB and K*k* coincide, as AB^2. And, by the like reasoning, K*k* − L*l*, L*l* − M*m*, &c., will be as Kk^2, Ll^2, &c. Therefore the squares of the lines AB, K*k*, L*l*, M*m*, &c., are as their differences; and, therefore, since the squares of the velocities were shewn above to be as their differences, the progression of both will be alike. This being demonstrated it follows also that the areas described by these lines are in a like progression with the spaces described by these velocities. Therefore if the velocity at the beginning of the first time AK be expounded by the line AB,



and the velocity at the beginning of the second time KL by the line K*k* and the length described in the first time by the area AK*k*B, all the following velocities will be expounded by the following lines L*l*, M*m*, &c. and the lengths described, by the areas K*l*, L*m*, &c. And, by composition, if the whole time be expounded by AM, the sum of its parts, the whole length described will be expounded by AM*m*B the sum of its parts. Now conceive the time AM to be divided into the parts AK, KL, LM, &c so that CA, CK, CL, CM, &c. may be in a geometrical progression; and those parts will be in the same progression, and the velocities AB, K*k*, L*l*, M*m*, &c., will be in the same progression inversely, and the spaces described A*k*, K*l*, L*m*, &c., will be equal. Q.E.D.

COR. 1. Hence it appears, that if the time be expounded by any part AD of the asymptote, and the velocity in the beginning of the time by the ordinate AB, the velocity at the end of the time will be expounded by the ordinate DG; and the whole space described by the adjacent hyperbolic area ABGD; and the space which any body can describe in the same time AD, with the first velocity AB, in a non-resisting medium, by the rectangle AB × AD.

COR 2. Hence the space described in a resisting medium is given, by taking it to the space described with the uniform velocity AB in a non-resisting medium, as the hyperbolic area ABGD to the rectangle AB × AD.

COR. 3. The resistance of the medium is also given, by making it equal, in the very beginning of the motion, to an uniform centripetal force, which could generate, in a body falling through a non-resisting medium, the velocity AB in the time AC. For if BT be drawn touching the hyperbola in B, and meeting the asymptote in T, the right line AT will be equal to AC, and will express the time in which the first resistance, uniformly continued, may take away the whole velocity AB.

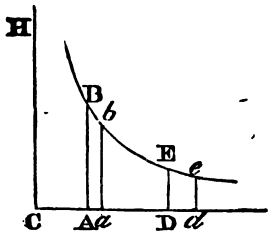
COR. 4. And thence is also given the proportion of this resistance to the force of gravity, or any other given centripetal force.

COR. 5. And, *vice versa*, if there is given the proportion of the resistance to any given centripetal force, the time AC is also given, in which a centripetal force equal to the resistance may generate any velocity as AB; and thence is given the point B, through which the hyperbola, having CH, CD for its asymptotes, is to be described: as also the space ABGD, which a body, by beginning its motion with that velocity AB, can describe in any time AD, in a similar resisting medium.

PROPOSITION VI. THEOREM IV.

Homogeneous and equal spherical bodies, opposed by resistances that are in the duplicate ratio of the velocities, and moving on by their innate force only, will, in times which are reciprocally as the velocities at the

beginning, describe equal spaces, and lose parts of their velocities proportional to the wholes.



To the rectangular asymptotes CD, CH describe any hyperbola B*b*E*e*, cutting the perpendiculars AB, *ab*, DE, *de* in B, *b*, E, *e*; let the initial velocities be expounded by the perpendiculars AB, DE, and the times by the lines A*a*, D*d*. Therefore as A*a* is to D*d*, so (by the hypothesis) is DE to AB, and so (from the nature of the hyperbola) is CA to CD; and, by composition, so is C*a* to C*d*. Therefore the areas AB*ba*, DE*ed*, that is, the spaces described, are equal among themselves, and the first velocities AB, DE are proportional to the last *ab*, *de*; and therefore, by division, proportional to the parts of the velocities lost, AB — *ab*, DE — *de*. Q.E.D.

PROPOSITION VII. THEOREM V.

If spherical bodies are resisted in the duplicate ratio of their velocities, in times which are as the first motions directly, and the first resistances inversely, they will lose parts of their motions proportional to the wholes, and will describe spaces proportional to those times and the first velocities conjunctly.

For the parts of the motions lost are as the resistances and times conjunctly. Therefore, that those parts may be proportional to the wholes, the resistance and time conjunctly ought to be as the motion. Therefore the time will be as the motion directly and the resistance inversely. Wherefore the particles of the times being taken in that ratio, the bodies will always lose parts of their motions proportional to the wholes, and therefore will retain velocities always proportional to their first velocities. And because of the given ratio of the velocities, they will always describe spaces which are as the first velocities and the times conjunctly. Q.E.D.

COR. 1. Therefore if bodies equally swift are resisted in a duplicate ratio of their diameters, homogeneous globes moving with any velocities whatsoever, by describing spaces proportional to their diameters, will lose parts of their motions proportional to the wholes. For the motion of each globe will be as its velocity and mass conjunctly, that is, as the velocity and the cube of its diameter; the resistance (by supposition) will be as the square of the diameter and the square of the velocity conjunctly; and the time (by this proposition) is in the former ratio directly, and in the latter inversely, that is, as the diameter directly and the velocity inversely; and therefore the space, which is proportional to the time and velocity is as the diameter.

COR. 2. If bodies equally swift are resisted in a sesquiplicate ratio of their diameters, homogeneous globes, moving with any velocities whatso-

ever, by describing spaces that are in a sesquiplicate ratio of the diameters, will lose parts of their motions proportional to the wholes.

COR. 3. And universally, if equally swift bodies are resisted in the ratio of any power of the diameters, the spaces, in which homogeneous globes, moving with any velocity whatsoever, will lose parts of their motions proportional to the wholes, will be as the cubes of the diameters applied to that power. Let those diameters be D and E; and if the resistances, where the velocities are supposed equal, are as D^n and E^n; the spaces in which the globes, moving with any velocities whatsoever, will lose parts of their motions proportional to the wholes, will be as D^{3-n} and E^{3-n}. And therefore homogeneous globes, in describing spaces proportional to D^{3-n} and E^{3-n}, will retain their velocities in the same ratio to one another as at the beginning.

COR. 4. Now if the globes are not homogeneous, the space described by the denser globe must be augmented in the ratio of the density. For the motion, with an equal velocity, is greater in the ratio of the density, and the time (by this Prop.) is augmented in the ratio of motion directly, and the space described in the ratio of the time.

COR. 5. And if the globes move in different mediums, the space, in a medium which, *cæteris paribus*, resists the most, must be diminished in the ratio of the greater resistance. For the time (by this Prop.) will be diminished in the ratio of the augmented resistance, and the space in the ratio of the time.

LEMMA II.

The moment of any genitum *is equal to the moments of each of the generating sides drawn into the indices of the powers of those sides, and into their co-efficients continually.*

I call any quantity a *genitum* which is not made by addition or subduction of divers parts, but is generated or produced in arithmetic by the multiplication, division, or extraction of the root of any terms whatsoever; in geometry by the invention of contents and sides, or of the extremes and means of proportionals. Quantities of this kind are products, quotients, roots, rectangles, squares, cubes, square and cubic sides, and the like. These quantities I here consider as variable and indetermined, and increasing or decreasing, as it were, by a perpetual motion or flux; and I understand their momentaneous increments or decrements by the name of moments; so that the increments may be esteemed as added or affirmative moments; and the decrements as subducted or negative ones. But take care not to look upon finite particles as such. Finite particles are not moments, but the very quantities generated by the moments. We are to conceive them as the just nascent principles of finite magnitudes. Nor do we in this Lemma regard the magnitude of the moments, but their first

proportion, as nascent. It will be the same thing, if, instead of moments, we use either the velocities of the increments and decrements (which may also be called the motions, mutations, and fluxions of quantities), or any finite quantities proportional to those velocities. The co-efficient of any generating side is the quantity which arises by applying the genitum to that side.

Wherefore the sense of the Lemma is, that if the moments of any quantities A, B, C, &c., increasing or decreasing by a perpetual flux, or the velocities of the mutations which are proportional to them, be called a, b, c, &c., the moment or mutation of the generated rectangle AB will be $a\mathrm{B} + b\mathrm{A}$; the moment of the generated content ABC will be $a\mathrm{BC} + b\mathrm{AC} + c\mathrm{AB}$; and the moments of the generated powers A^2, A^3, A^4, $\mathrm{A}^{\frac{1}{2}}$, $\mathrm{A}^{\frac{3}{2}}$, $\mathrm{A}^{\frac{1}{3}}$, $\mathrm{A}^{\frac{2}{3}}$, A^{-1}, A^{-2}, $\mathrm{A}^{-\frac{1}{2}}$ will be $2a\mathrm{A}$, $3a\mathrm{A}^2$, $4a\mathrm{A}^3$, $\frac{1}{2}a\mathrm{A}^{-\frac{1}{2}}$, $\frac{3}{2}a\mathrm{A}^{\frac{1}{2}}$, $\frac{1}{3}a\mathrm{A}^{-\frac{2}{3}}$, $\frac{2}{3}a\mathrm{A}^{-\frac{1}{3}}$, $-a\mathrm{A}^{-2}$, $-2a\mathrm{A}^{-3}$, $-\frac{1}{2}a\mathrm{A}^{-\frac{3}{2}}$ respectively; and in general, that the moment of any power $\mathrm{A}^{\frac{n}{m}}$, will be $\frac{n}{m}\,a\mathrm{A}^{\frac{n-m}{m}}$. Also, that the moment of the generated quantity $\mathrm{A}^2\mathrm{B}$ will be $2a\mathrm{AB} + b\mathrm{A}^2$; the moment of the generated quantity $\mathrm{A}^3\mathrm{B}^4\mathrm{C}^2$ will be $3a\mathrm{A}^2\mathrm{B}^4\mathrm{C}^2 + 4b\mathrm{A}^3\mathrm{B}^3\mathrm{C}^2 + 2c\mathrm{A}^3\mathrm{B}^4\mathrm{C}$; and the moment of the generated quantity $\frac{\mathrm{A}^3}{\mathrm{B}^2}$ or $\mathrm{A}^3\mathrm{B}^{-2}$ will be $3a\mathrm{A}^2\mathrm{B}^{-2} - 2b\mathrm{A}^3\mathrm{B}^{-3}$; and so on. The Lemma is thus demonstrated.

CASE 1. Any rectangle, as AB, augmented by a perpetual flux, when, as yet, there wanted of the sides A and B half their moments $\frac{1}{2}a$ and $\frac{1}{2}b$, was $\mathrm{A} - \frac{1}{2}a$ into $\mathrm{B} - \frac{1}{2}b$, or $\mathrm{AB} - \frac{1}{2}a\,\mathrm{B} - \frac{1}{2}b\,\mathrm{A} + \frac{1}{4}ab$; but as soon as the sides A and B are augmented by the other half moments, the rectangle becomes $\mathrm{A} + \frac{1}{2}a$ into $\mathrm{B} + \frac{1}{2}b$, or $\mathrm{AB} + \frac{1}{2}a\,\mathrm{B} + \frac{1}{2}b\,\mathrm{A} + \frac{1}{4}ab$. From this rectangle subduct the former rectangle, and there will remain the excess $a\mathrm{B} + b\mathrm{A}$. Therefore with the whole increments a and b of the sides, the increment $a\mathrm{B} + b\mathrm{A}$ of the rectangle is generated. Q.E.D.

CASE 2. Suppose AB always equal to G, and then the moment of the content ABC or GC (by Case 1) will be $g\mathrm{C} + c\mathrm{G}$, that is (putting AB and $a\mathrm{B} + b\mathrm{A}$ for G and g), $a\mathrm{BC} + b\mathrm{AC} + c\mathrm{AB}$. And the reasoning is the same for contents under ever so many sides. Q.E.D.

CASE 3. Suppose the sides A, B, and C, to be always equal among themselves; and the moment $a\mathrm{B} + b\mathrm{A}$, of A^2, that is, of the rectangle AB, will be $2a\mathrm{A}$; and the moment $a\mathrm{BC} + b\mathrm{AC} + c\mathrm{AB}$ of A^3, that is, of the content ABC, will be $3a\mathrm{A}^2$. And by the same reasoning the moment of any power A^n is $na\mathrm{A}^{n-1}$. Q.E.D

CASE 4. Therefore since $\frac{1}{\mathrm{A}}$ into A is 1, the moment of $\frac{1}{\mathrm{A}}$ drawn into

A, together with $\frac{1}{A}$ drawn into a, will be the moment of 1, that is, nothing. Therefore the moment of $\frac{1}{A}$, or of A^{-1}, is $\frac{-a}{A^2}$. And generally since $\frac{1}{A^n}$ into A^n is 1, the moment of $\frac{1}{A^n}$ drawn into A^n together with $\frac{1}{A^n}$ into naA^{n-1} will be nothing. And, therefore, the moment of $\frac{1}{A^n}$ or A^{-n} will be $-\frac{na}{A^{n+1}}$. Q.E.D.

CASE 5. And since $A^{\frac{1}{2}}$ into $A^{\frac{1}{2}}$ is A, the moment of $A^{\frac{1}{2}}$ drawn into $2A^{\frac{1}{2}}$ will be a (by Case 3); and, therefore, the moment of $A^{\frac{1}{2}}$ will be $\frac{a}{2A^{\frac{1}{2}}}$ or $\frac{1}{2}aA^{-\frac{1}{2}}$. And, generally, putting $A^{\frac{m}{n}}$ equal to B, then A^m will be equal to B^n, and therefore maA^{m-1} equal to nbB^{n-1}, and maA^{-1} equal to nbB^{-1}, or $nbA^{-\frac{m}{n}}$; and therefore $\frac{m}{n}aA^{\frac{m-n}{n}}$ is equal to b, that is, equal to the moment of $A^{\frac{m}{n}}$. Q.E.D.

CASE 6. Therefore the moment of any generated quantity A^mB^n is the moment of A^m drawn into B^n, together with the moment of B^n drawn into A^m, that is, $maA^{m-1}B^n + nbB^{n-1}A^m$; and that whether the indices m and n of the powers be whole numbers or fractions, affirmative or negative. And the reasoning is the same for contents under more powers. Q.E.D.

COR. 1. Hence in quantities continually proportional, if one term is given, the moments of the rest of the terms will be as the same terms multiplied by the number of intervals between them and the given term. Let A, B, C, D, E, F, be continually proportional; then if the term C is given, the moments of the rest of the terms will be among themselves as — 2A, — B, D, 2E, 3F.

COR. 2. And if in four proportionals the two means are given, the moments of the extremes will be as those extremes. The same is to be understood of the sides of any given rectangle.

COR. 3. And if the sum or difference of two squares is given, the moments of the sides will be reciprocally as the sides.

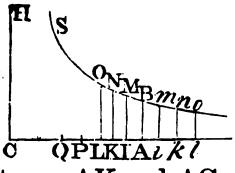
SCHOLIUM.

In a letter of mine to Mr. *J. Collins*, dated *December* 10, 1672, having described a method of tangents, which I suspected to be the same with *Slusius's* method, which at that time was not made public, I subjoined these words: *This is one particular, or rather a Corollary, of a general me*

thod, which extends itself, without any troublesome calculation, not only to the drawing of tangents to any curve lines, whether geometrical or mechanical, or any how respecting right lines or other curves, but also to the resolving other abstruser kinds of problems about the crookedness, areas, lengths, centres of gravity of curves, &c.; nor is it (*as* Hudden's *method* de Maximis & Minimis) *limited to equations which are free from surd quantities. This method I have interwoven with that other of working in equations, by reducing them to infinite series.* So far that letter. And these last words relate to a treatise I composed on that subject in the year 1671. The foundation of that general method is contained in the preceding Lemma.

PROPOSITION VIII. THEOREM VI.

If a body in an uniform medium, being uniformly acted upon by the force of gravity, ascends or descends in a right line; and the whole space described be distinguished into equal parts, and in the beginning of each of the parts (*by adding or subducting the resisting force of the medium to or from the force of gravity, when the body ascends or descends*) *you collect the absolute forces; I say, that those absolute forces are in a geometrical progression.*



For let the force of gravity be expounded by the given line AC; the force of resistance by the indefinite line AK; the absolute force in the descent of the body by the difference KC; the velocity of the body by a line AP, which shall be a mean proportional between AK and AC, and therefore in a subduplicate ratio of the resistance; the increment of the resistance made in a given particle of time by the lineola KL, and the contemporaneous increment of the velocity by the lineola PQ; and with the centre C, and rectangular asymptotes CA, CH, describe any hyperbola BNS meeting the erected perpendiculars AB, KN, LO in B, N and O. Because AK is as AP^2, the moment KL of the one will be as the moment 2APQ of the other, that is, as AP × KC; for the increment PQ of the velocity is (by Law II) proportional to the generating force KC. Let the ratio of KL be compounded with the ratio KN, and the rectangle KL × KN will become as AP × KC × KN; that is (because the rectangle KC × KN is given), as AP. But the ultimate ratio of the hyperbolic area KNOL to the rectangle KL × KN becomes, when the points K and L coincide, the ratio of equality. Therefore that hyperbolic evanescent area is as AP. Therefore the whole hyperbolic area ABOL is composed of particles KNOL which are always proportional to the velocity AP; and therefore is itself proportional to the space described with that velocity. Let that area be now divided into equal parts

as ABMI, IMNK, KNOL, &c., and the absolute forces AC, IC, KC, LC, &c., will be in a geometrical progression. Q.E.D. And by a like reasoning, in the ascent of the body, taking, on the contrary side of the point A, the equal areas AB*mi*, *imnk*, *knol*, &c., it will appear that the absolute forces AC, *i*C, *k*C, *l*C, &c., are continually proportional. Therefore if all the spaces in the ascent and descent are taken equal, all the absolute forces *l*C, *k*C, *i*C, AC, IC, KC, LC, &c., will be continually proportional. Q.E.D.

COR. 1. Hence if the space described be expounded by the hyperbolic area ABNK, the force of gravity, the velocity of the body, and the resistance of the medium, may be expounded by the lines AC, AP, and AK respectively; and *vice versa*.

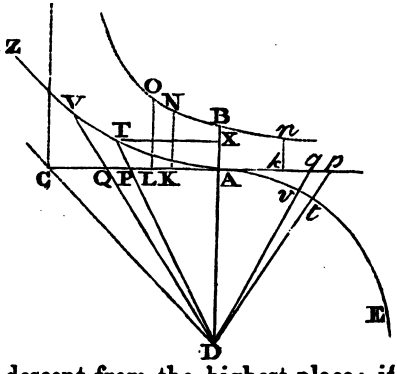
COR. 2. And the greatest velocity which the body can ever acquire in an infinite descent will be expounded by the line AC.

COR. 3. Therefore if the resistance of the medium answering to any given velocity be known, the greatest velocity will be found, by taking it to that given velocity in a ratio subduplicate of the ratio which the force of gravity bears to that known resistance of the medium.

PROPOSITION IX. THEOREM VII.

Supposing what is above demonstrated, I say, that if the tangents of the angles of the sector of a circle, and of an hyperbola, be taken proportional to the velocities, the radius being of a fit magnitude, all the time of the ascent to the highest place will be as the sector of the circle, and all the time of descending from the highest place as the sector of the hyperbola.

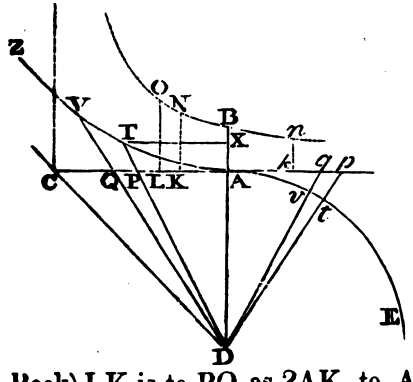
To the right line AC, which expresses the force of gravity, let AD be drawn perpendicular and equal. From the centre D with the semi-diameter AD describe as well the quadrant A*t*E of a circle, as the rectangular hyperbola AVZ, whose axis is AK, principal vertex A, and asymptote DC. Let D*p*, DP be drawn; and the circular sector A*t*D will be as all the time of the ascent to the highest place; and the hyperbolic sector ATD as all the time of descent from the highest place; if so be that the tangents A*p*, AP of those sectors be as the velocities.



CASE 1. Draw D*vq* cutting off the moments or least particles *t*D*v* and *q*D*p*, described in the same time, of the sector AD*t* and of the triangle AD*p*. Since those particles (because of the common angle D) are in a duplicate ratio of the sides, the particle *t*D*v* will be as $\frac{q\mathrm{D}p \times t\mathrm{D}^2}{p\mathrm{D}^2}$, that is

(because tD is given), as $\frac{q\mathrm{D}p}{p\mathrm{D}^2}$. But $p\mathrm{D}^2$ is $\mathrm{AD}^2 + \mathrm{A}p^2$, that is, $\mathrm{AD}^2 + \mathrm{AD} \times \mathrm{A}k$, or $\mathrm{AD} \times \mathrm{C}k$; and $q\mathrm{D}p$ is $\frac{1}{2}\,\mathrm{AD} \times pq$. Therefore $t\mathrm{D}v$, the particle of the sector, is as $\frac{pq}{\mathrm{C}k}$; that is, as the least decrement pq of the velocity directly, and the force Ck which diminishes the velocity, inversely; and therefore as the particle of time answering to the decrement of the velocity. And, by composition, the sum of all the particles $t\mathrm{D}v$ in the sector ADt will be as the sum of the particles of time answering to each of the lost particles pq of the decreasing velocity Ap, till that velocity, being diminished into nothing, vanishes; that is, the whole sector ADt is as the whole time of ascent to the highest place. Q.E.D.

Case 2. Draw DQV cutting off the least particles TDV and PDQ of the sector DAV, and of the triangle DAQ; and these particles will be to each other as DT^2 to DP^2, that is (if TX and AP are parallel), as DX^2 to DA^2 or TX^2 to AP^2; and, by division, as $\mathrm{DX}^2 - \mathrm{TX}^2$ to $\mathrm{DA}^2 - \mathrm{AP}^2$. But, from the nature of the hyperbola, $\mathrm{DX}^2 - \mathrm{TX}^2$ is AD^2; and, by the supposition, AP^2 is AD × AK. Therefore the particles are to each other as AD^2 to $\mathrm{AD}^2 - \mathrm{AD} \times \mathrm{AK}$; that is, as AD to AD — AK or AC to CK: and therefore the particle TDV of the sector is $\frac{\mathrm{PDQ} \times \mathrm{AC}}{\mathrm{CK}}$; and therefore (because AC and AD are given) as $\frac{\mathrm{PQ}}{\mathrm{CK}}$; that is, as the increment of the velocity directly, and as the force generating the increment inversely; and therefore as the particle of the time answering to the increment. And, by composition, the sum of the particles of time, in which all the particles PQ of the velocity AP are generated, will be as the sum of the particles of the sector ATD; that is, the whole time will be as the whole sector. Q.E.D.



Cor. 1. Hence if AB be equal to a fourth part of AC, the space which a body will describe by falling in any time will be to the space which the body could describe, by moving uniformly on in the same time with its greatest velocity AC, as the area ABNK, which expresses the space described in falling to the area ATD, which expresses the time. For since AC is to AP as AP to AK, then (by Cor. 1, Lem. II, of this Book) LK is to PQ as 2AK to AP, that is, as 2AP to AC, and thence LK is to $\frac{1}{2}$PQ as AP to $\frac{1}{4}$AC or AB; and KN is to AC or AD as AB to

CK; and therefore, *ex æquo*, LKNO to DPQ as AP to CK. But DPQ was to DTV as CK to AC. Therefore, *ex æquo*, LKNO is to DTV as AP to AC; that is, as the velocity of the falling body to the greatest velocity which the body by falling can acquire. Since, therefore, the moments LKNO and DTV of the areas ABNK and ATD are as the velocities, all the parts of those areas generated in the same time will be as the spaces described in the same time; and therefore the whole areas ABNK and ADT, generated from the beginning, will be as the whole spaces described from the beginning of the descent. Q.E.D.

COR. 2. The same is true also of the space described in the ascent. That is to say, that all that space is to the space described in the same time, with the uniform velocity AC, as the area AB*nk* is to the sector AD*t*.

COR. 3. The velocity of the body, falling in the time ATD, is to the velocity which it would acquire in the same time in a non-resisting space, as the triangle APD to the hyperbolic sector ATD. For the velocity in a non-resisting medium would be as the time ATD, and in a resisting medium is as AP, that is, as the triangle APD. And those velocities, at the beginning of the descent, are equal among themselves, as well as those areas ATD, APD.

COR. 4. By the same argument, the velocity in the ascent is to the velocity with which the body in the same time, in a non-resisting space, would lose all its motion of ascent, as the triangle A*p*D to the circular sector A*t*D; or as the right line A*p* to the arc A*t*.

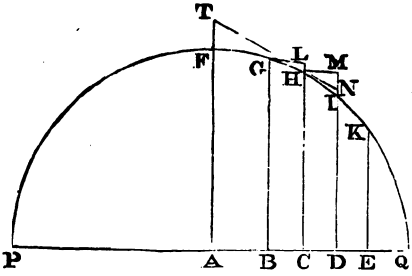
COR. 5. Therefore the time in which a body, by falling in a resisting medium, would acquire the velocity AP, is to the time in which it would acquire its greatest velocity AC, by falling in a non-resisting space, as the sector ADT to the triangle ADC: and the time in which it would lose its velocity A*p*, by ascending in a resisting medium, is to the time in which it would lose the same velocity by ascending in a non-resisting space, as the arc A*t* to its tangent A*p*.

COR. 6. Hence from the given time there is given the space described in the ascent or descent. For the greatest velocity of a body descending *in infinitum* is given (by Corol. 2 and 3, Theor. VI, of this Book); and thence the time is given in which a body would acquire that velocity by falling in a non-resisting space. And taking the sector ADT or AD*t* to the triangle ADC in the ratio of the given time to the time just now found, there will be given both the velocity AP or A*p*, and the area ABNK or AB*nk*, which is to the sector ADT, or AD*t*, as the space sought to the space which would, in the given time, be uniformly described with that greatest velocity found just before.

COR. 7. And by going backward, from the given space of ascent or descent AB*nk* or ABNK, there will be given the time AD*t* or ADT.

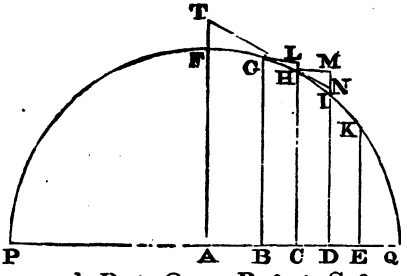
PROPOSITION X. PROBLEM III.

Suppose the uniform force of gravity to tend directly to the plane of the horizon, and the resistance to be as the density of the medium and the square of the velocity conjunctly: it is proposed to find the density of the medium in each place, which shall make the body move in any given curve line; the velocity of the body and the resistance of the medium in each place.



Let PQ be a plane perpendicular to the plane of the scheme itself; PFHQ a curve line meeting that plane in the points P and Q; G, H, I, K four places of the body going on in this curve from F to Q; and GB, HC, ID, KE four parallel ordinates let fall from these points to the horizon, and standing on the horizontal line PQ at the points B, C, D, E; and let the distances BC, CD, DE, of the ordinates be equal among themselves. From the points G and H let the right lines GL, HN, be drawn touching the curve in G and H, and meeting the ordinates CH, DI, produced upwards, in L and N: and complete the parallelogram HCDM. And the times in which the body describes the arcs GH, HI, will be in a subduplicate ratio of the altitudes LH, NI, which the bodies would describe in those times, by falling from the tangents; and the velocities will be as the lengths described GH, HI directly, and the times inversely. Let the times be expounded by T and t, and the velocities by $\frac{GH}{T}$ and $\frac{HI}{t}$; and the decrement of the velocity produced in the time t will be expounded by $\frac{GH}{T} - \frac{HI}{t}$. This decrement arises from the resistance which retards the body, and from the gravity which accelerates it. Gravity, in a falling body, which in its fall describes the space NI, produces a velocity with which it would be able to describe twice that space in the same time, as *Galileo* has demonstrated; that is, the velocity $\frac{2NI}{t}$: but if the body describes the arc HI, it augments that arc only by the length HI — HN or $\frac{MI \times NI}{HI}$; and therefore generates only the velocity $\frac{2MI \times NI}{t \times HI}$. Let this velocity be added to the beforementioned decrement, and we shall have the decrement of the velocity arising from the resistance alone, that is, $\frac{GH}{T} - \frac{HI}{t} + \frac{2MI \times NI}{t \times HI}$.

Therefore since, in the same time, the action of gravity generates, in a falling body, the velocity $\frac{2NI}{t}$, the resistance will be to the gravity as $\frac{GH}{T} - \frac{HI}{t} + \frac{2MI \times NI}{t \times HI}$ to $\frac{2NI}{t}$ or as $\frac{t \times GH}{T} - HI + \frac{2MI \times NI}{HI}$ to $2NI$.



Now for the abscissas CB, CD, CE, put $-o$, o, $2o$. For the ordinate CH put P; and for MI put any series $Qo + Ro^2 + So^3 +$, &c. And all the terms of the series after the first, that is, $Ro^2 + So^3 +$, &c., will be NI; and the ordinates DI, EK, and BG will be $P - Qo - Ro^2 - So^3 -$, &c., $P - 2Qo - 4Ro^2 - 8So^3 -$, &c., and $P + Qo - Ro^2 + So^3 -$, &c., respectively. And by squaring the differences of the ordinates BG — CH and CH — DI, and to the squares thence produced adding the squares of BC and CD themselves, you will have $oo + QQoo - 2QRo^3 +$, &c., and $oo + QQoo + 2QRo^3 +$, &c., the squares of the arcs GH, HI; whose roots $o\sqrt{1 + QQ} - \frac{QRoo}{\sqrt{1 + QQ}}$, and $o\sqrt{1 + QQ} + \frac{QRoo}{\sqrt{1 + QQ}}$ are the arcs GH and HI. Moreover, if from the ordinate CH there be subducted half the sum of the ordinates BG and DI, and from the ordinate DI there be subducted half the sum of the ordinates CH and EK, there will remain Roo and $Roo + 3So^3$, the versed sines of the arcs GI and HK. And these are proportional to the lineolæ LH and NI, and therefore in the duplicate ratio of the infinitely small times T and t: and thence the ratio $\frac{t}{T}$ is $\sqrt{\frac{R + 3So}{R}}$ or $\frac{R + \frac{3}{2}So}{R}$; and $\frac{t \times GH}{T} - HI + \frac{2MI \times NI}{HI}$, by substituting the values of $\frac{t}{T}$, GH, HI, MI and NI just found, becomes $\frac{3Soo}{2R}\sqrt{1 + QQ}$. And since $2NI$ is $2Roo$, the resistance will be now to the gravity as $\frac{3Soo}{2R}\sqrt{1 + QQ}$ to $2Roo$, that is, as $3S\sqrt{1 + QQ}$ to $4RR$.

And the velocity will be such, that a body going off therewith from any place H, in the direction of the tangent HN, would describe, in vacuo, a parabola, whose diameter is HC, and its latus rectum $\frac{HN^2}{NI}$ or $\frac{1 + QQ}{R}$.

And the resistance is as the density of the medium and the square of the velocity conjunctly; and therefore the density of the medium is as the resistance directly, and the square of the velocity inversely; that is, as

$\frac{3S\sqrt{1+QQ}}{4RR}$ directly and $\frac{1+QQ}{R}$ inversely; that is, as $\frac{S}{R\sqrt{1+QQ}}$ Q.E.I.

Cor. 1. If the tangent HN be produced both ways, so as to meet any ordinate AF in T $\frac{HT}{AC}$ will be equal to $\sqrt{1+QQ}$, and therefore in what has gone before may be put for $\sqrt{1+QQ}$. By this means the resistance will be to the gravity as $3S \times HT$ to $4RR \times AC$; the velocity will be as $\frac{HT}{AC\sqrt{R}}$, and the density of the medium will be as $\frac{S \times AC}{R \times HT}$.

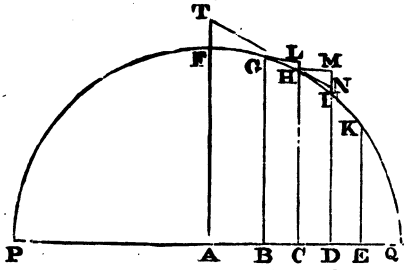
Cor. 2. And hence, if the curve line PFHQ be defined by the relation between the base or abscissa AC and the ordinate CH, as is usual, and the value of the ordinate be resolved into a converging series, the Problem will be expeditiously solved by the first terms of the series; as in the following examples.

Example 1. Let the line PFHQ be a semi-circle described upon the diameter PQ, to find the density of the medium that shall make a projectile move in that line.

Bisect the diameter PQ in A; and call AQ, n; AC, a; CH, e; and CD, o; then DI^2 or $AQ^2 - AD^2 = nn - aa - 2ao - oo$, or $ee - 2ao - oo$; and the root being extracted by our method, will give $DI = e - \frac{ao}{e} - \frac{oo}{2e} - \frac{aaoo}{2e^3} - \frac{ao^3}{2e^3} - \frac{a^3o^3}{2e^5} -$, &c. Here put nn for $ee + aa$, and DI will become $= e - \frac{ao}{e} - \frac{nnoo}{2e^3} - \frac{anno^3}{2e^5} -$, &c.

Such series I distinguish into successive terms after this manner: I call that the first term in which the infinitely small quantity o is not found; the second, in which that quantity is of one dimension only; the third, in which it arises to two dimensions; the fourth, in which it is of three; and so *ad infinitum*. And the first term, which here is e, will always denote the length of the ordinate CH, standing at the beginning of the indefinite quantity o. The second term, which here is $\frac{ao}{e}$, will denote the difference between CH and DN; that is, the lineola MN which is cut off by completing the parallelogram HCDM; and therefore always determines the position of the tangent HN; as, in this case, by taking MN to HM as $\frac{ao}{e}$ to o, or a to e. The third term, which here is $\frac{nnoo}{2e^3}$, will represent the lineola IN, which lies between the tangent and the curve; and therefore determines the angle of contact IHN, or the curvature which the curve line

has in H. If that lineola IN is of a finite magnitude, it will be expressed by the third term, together with those that follow *in infinitum*. But if that lineola be diminished *in infinitum*, the terms following become infinitely less than the third term, and therefore may be neglected. The fourth term determines the variation of the curvature; the fifth, the variation of the variation; and so on. Whence, by the way, appears no contemptible use of these series in the solution of problems that depend upon tangents, and the curvature of curves.

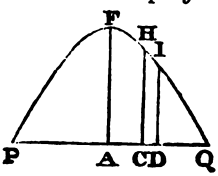


Now compare the series $e - \frac{ao}{e} - \frac{nnoo}{2e^3} - \frac{annno^3}{2e^5}$ — &c., with the series $P - Qo - Roo - So^3$ — &c., and for P, Q, R and S, put e, $\frac{a}{e}$, $\frac{nn}{2e^3}$ and $\frac{ann}{2e^5}$, and for $\sqrt{1 + QQ}$ put $\sqrt{1 + \frac{aa}{ee}}$ or $\frac{n}{e}$; and the density of the medium will come out as $\frac{a}{ne}$; that is (because n is given), as $\frac{a}{e}$ or $\frac{AC}{CH}$, that is, as that length of the tangent HT, which is terminated at the semi-diameter AF standing perpendicularly on PQ: and the resistance will be to the gravity as $3a$ to $2n$, that is, as 3AC to the diameter PQ of the circle; and the velocity will be as $\sqrt{CH}$. Therefore if the body goes from the place F, with a due velocity, in the direction of a line parallel to PQ, and the density of the medium in each of the places H is as the length of the tangent HT, and the resistance also in any place H is to the force of gravity as 3AC to PQ, that body will describe the quadrant FHQ of a circle. Q.E.I.

But if the same body should go from the place P, in the direction of a line perpendicular to PQ, and should begin to move in an arc of the semicircle PFQ, we must take AC or a on the contrary side of the centre A; and therefore its sign must be changed, and we must put $-a$ for $+a$. Then the density of the medium would come out as $-\frac{a}{e}$. But nature does not admit of a negative density, that is, a density which accelerates the motion of bodies; and therefore it cannot naturally come to pass that a body by ascending from P should describe the quadrant PF of a circle. To produce such an effect, a body ought to be accelerated by an impelling medium, and not impeded by a resisting one.

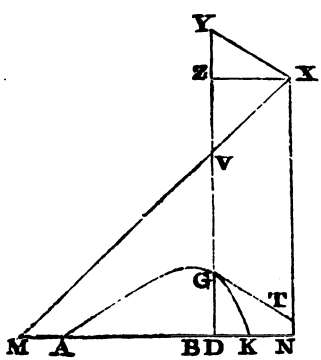
EXAMPLE 2. Let the line PFQ be a parabola, having its axis AF per-

pendicular to the horizon PQ, to find the density of the medium, which will make a projectile move in that line.



From the nature of the parabola, the rectangle PDQ is equal to the rectangle under the ordinate DI and some given right line; that is, if that right line be called b; PC, a; PQ, c; CH, e; and CD, o; the rectangle $a + o$ into $c - a - o$ or $ac - aa - 2ao + co - oo$, is equal to the rectangle b into DI, and therefore DI is equal to $\frac{ac-aa}{b} + \frac{c-2a}{b}o - \frac{oo}{b}$. Now the second term $\frac{c-2a}{b}o$ of this series is to be put for Qo, and the third term $\frac{oo}{b}$ for Roo. But since there are no more terms, the co-efficient S of the fourth term will vanish; and therefore the quantity $\frac{S}{R\sqrt{1+QQ}}$, to which the density of the medium is proportional, will be nothing. Therefore, where the medium is of no density, the projectile will move in a parabola; as *Galileo* hath heretofore demonstrated. Q.E.I.

Example 3. Let the line AGK be an hyperbola, having its asymptote NX perpendicular to the horizontal plane AK, to find the density of the medium that will make a projectile move in that line.

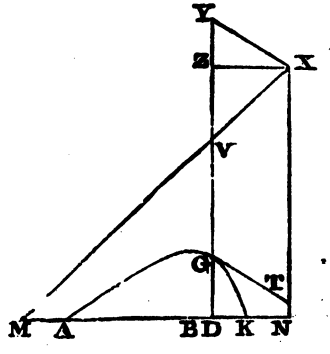


Let MX be the other asymptote, meeting the ordinate DG produced in V; and from the nature of the hyperbola, the rectangle of XV into VG will be given. There is also given the ratio of DN to VX, and therefore the rectangle of DN into VG is given. Let that be bb: and, completing the parallelogram DNXZ, let BN be called a; BD, o; NX, c; and let the given ratio of VZ to ZX or DN be $\frac{m}{n}$. Then DN will be equal to $a - o$, VG equal to $\frac{bb}{a-o}$, VZ equal to $\frac{m}{n} \times a - o$, and GD or NX $-$VZ$-$VG equal to $c - \frac{m}{n}a + \frac{m}{n}o - \frac{bb}{a-o}$. Let the term $\frac{bb}{a-o}$ be resolved into the converging series $\frac{bb}{a} + \frac{bb}{aa}o + \frac{bb}{a^3}oo + \frac{bb}{a^4}o^3$, &c., and GD will become equal to $c - \frac{m}{n}a - \frac{bb}{a} + \frac{m}{n}o - \frac{bb}{aa}o - \frac{bb}{a^3}o^2 - \frac{bb}{a^4}o^3$

&c. The second term $\frac{m}{n}o - \frac{bb}{aa}o$ of this series is to be used for Qo; the third $\frac{bb}{a^3}o^2$, with its sign changed for Ro^2; and the fourth $\frac{bb}{a^4}o^3$, with its sign changed also for So^3, and their coefficients $\frac{m}{n} - \frac{bb}{aa}$, $\frac{bb}{a^3}$ and $\frac{bb}{a^4}$ are to be put for Q, R, and S in the former rule. Which being done, the density of the medium will come out as $\frac{\frac{bb}{a^4}}{\frac{bb}{a^3}\sqrt{1 + \frac{mm}{nn} - \frac{2mbb}{naa} + \frac{b^4}{a^4}}}$ or $\frac{1}{\sqrt{aa + \frac{mm}{nn}aa - \frac{2mbb}{n} + \frac{b^4}{aa}}}$, that is, if in VZ you take VY equal to VG, as $\frac{1}{XY}$. For aa and $\frac{m^2}{n^2}a^2 - \frac{2mbb}{n} + \frac{b^4}{aa}$ are the squares of XZ and ZY. But the ratio of the resistance to gravity is found to be that of 3XY to 2YG; and the velocity is that with which the body would describe a parabola, whose vertex is G, diameter DG, latus rectum $\frac{XY^2}{VG}$. Suppose, therefore, that the densities of the medium in each of the places G are reciprocally as the distances XY, and that the resistance in any place G is to the gravity as 3XY to 2YG; and a body let go from the place A. with a due velocity, will describe that hyperbola AGK. Q.E.I.

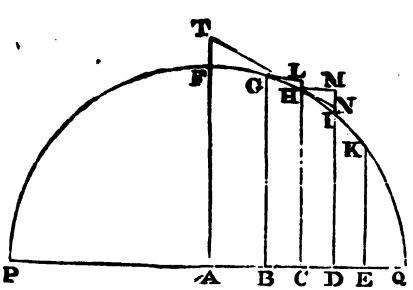
EXAMPLE 4. Suppose, indefinitely, the line AGK to be an hyperbola described with the centre X, and the asymptotes MX, NX, so that, having constructed the rectangle XZDN, whose side ZD cuts the hyperbola in G and its asymptote in V, VG may be reciprocally as any power DN^n of the line ZX or DN, whose index is the number n: to find the density of the medium in which a projected body will describe this curve.

For BN, BD, NX, put A, O, C, respectively, and let VZ be to XZ or DN as d to e, and VG be equal to $\frac{bb}{DN^n}$; then DN will be equal to $A - O$, $VG = \frac{bb}{\overline{A-O}^n}$, $VZ = \frac{d}{e}\overline{A-O}$, and GD or $NX - VZ - VG$ equal to $C - \frac{d}{e}A + \frac{d}{e}O - \frac{bb}{\overline{A-O}^n}$. Let the



term $\frac{bb}{\overline{A - O}^n}$ be resolved into an infinite series $\frac{bb}{A^n} + \frac{nbb}{A^{n+1}} \times O + \frac{nn+n}{2A^{n+2}} \times bb\,O^2 + \frac{n^3+3nn+2n}{6A^{n+3}} \times bb\,O^3$, &c., and GD will be equal to $C - \frac{d}{e}A - \frac{bb}{A^n} + \frac{d}{e}O - \frac{nbb}{A^{n+1}}O - \frac{+nn+n}{2A^{n+2}}bb\,O^2 - \frac{+n^3+3nn+2n}{6A^{n+3}}bbO^3$, &c. The second term $\frac{d}{e}O - \frac{nbb}{A^{n+1}}O$ of this series is to be used for Q*o*, the third $\frac{nn+n}{2A^{n+2}}bbO^2$ for R*oo*, the fourth $\frac{n^3+3nn+2n}{6A^{n+3}}bbO^3$ for S*o*^{3}. And thence the density of the medium $\frac{S}{R\sqrt{1+QQ}}$, in any place G, will be $\frac{n+2}{3\sqrt{A^2 + \frac{dd}{ee}A^2 - \frac{2dnbb}{eA^n}A + \frac{nnb^4}{A^{2n}}}}$, and therefore if in VZ you take VY equal to $n \times$ VG, that density is reciprocally as XY. For A^2 and $\frac{dd}{ee}A^2 - \frac{2dnbb}{eA^n}A + \frac{nnb^4}{A^{2n}}$ are the squares of XZ and ZY. But the resistance in the same place G is to the force of gravity as $3S \times \frac{XY}{A}$ to 4RR, that is, as XY to $\frac{2nn+2n}{n+2}$ VG. And the velocity there is the same wherewith the projected body would move in a parabola, whose vertex is G, diameter GD, and latus rectum $\frac{1+QQ}{R}$ or $\frac{2XY^2}{nn+n \times VG}$. Q.E.I.

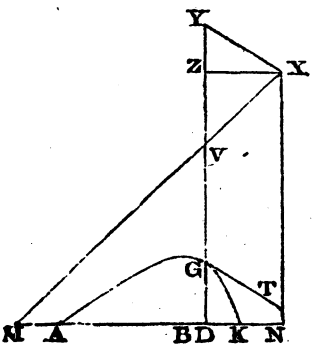
SCHOLIUM.



In the same manner that the density of the medium comes out to be as $\frac{S \times AC}{R \times HT^2}$ in Cor. 1, if the resistance is put as any power V^n of the velocity V, the density of the medium will come out to be as $\frac{S}{R^{\frac{4-n}{2}}} \times \overline{\frac{AC}{HT}}^{n-1}$.

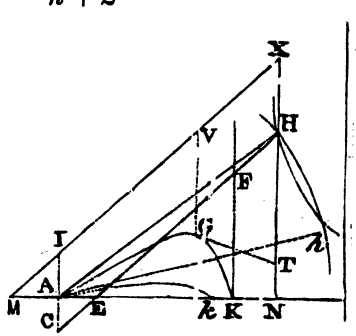
And therefore if a curve can be found, such that the ratio of $\frac{S}{R^{\frac{4-n}{2}}}$ to

$\overline{\frac{HT}{AC}}^{n-1}$, or of $\frac{S^2}{R^{4-n}}$ to $\overline{1+QQ}^{n-1}$ may be given; the body, in an uniform medium, whose resistance is as the power V^n of the velocity V, will move in this curve. But let us return to more simple curves.



Because there can be no motion in a parabola except in a non-resisting medium, but in the hyperbolas here described it is produced by a perpetual resistance; it is evident that the line which a projectile describes in an uniformly resisting medium approaches nearer to these hyperbolas than to a parabola. That line is certainly of the hyperbolic kind, but about the vertex it is more distant from the asymptotes, and in the parts remote from the vertex draws nearer to them than these hyperbolas here described. The difference, however, is not so great between the one and the other but that these latter may be commodiously enough used in practice instead of the former. And perhaps these may prove more useful than an hyperbola that is more accurate, and at the same time more compounded. They may be made use of, then, in this manner.

Complete the parallelogram XYGT, and the right line GT will touch the hyperbola in G, and therefore the density of the medium in G is reciprocally as the tangent GT, and the velocity there as $\sqrt{\frac{GT^2}{GV}}$; and the resistance is to the force of gravity as GT to $\frac{2nn+2n}{n+2} \times GV$.



Therefore if a body projected from the place A, in the direction of the right line AH, describes the hyperbola AGK and AH produced meets the asymptote NX in H, and AI drawn parallel to it meets the other asymptote MX in I; the density of the medium in A will be reciprocally as AH, and the velocity of the body as $\sqrt{\frac{AH^2}{AI}}$, and the resistance there to the force of gravity as AH to $\frac{2nn+2n}{n+2} \times AI$. Hence the following rules are deduced.

RULE 1. If the density of the medium at A, and the velocity with which the body is projected remain the same, and the angle NAH be changed, the lengths AH, AI, HX will remain. Therefore if those lengths, in any

one case, are found, the hyperbola may afterwards be easily determined from any given angle NAH.

Rule 2. If the angle NAH, and the density of the medium at A, remain the same, and the velocity with which the body is projected be changed, the length AH will continue the same; and AI will be changed in a duplicate ratio of the velocity reciprocally.

Rule 3. If the angle NAH, the velocity of the body at A, and the accelerative gravity remain the same, and the proportion of the resistance at A to the motive gravity be augmented in any ratio; the proportion of AH to AI will be augmented in the same ratio, the latus rectum of the above-mentioned parabola remaining the same, and also the length $\frac{AH^2}{AI}$ proportional to it; and therefore AH will be diminished in the same ratio, and AI will be diminished in the duplicate of that ratio. But the proportion of the resistance to the weight is augmented, when either the specific gravity is made less, the magnitude remaining equal, or when the density of the medium is made greater, or when, by diminishing the magnitude, the resistance becomes diminished in a less ratio than the weight.

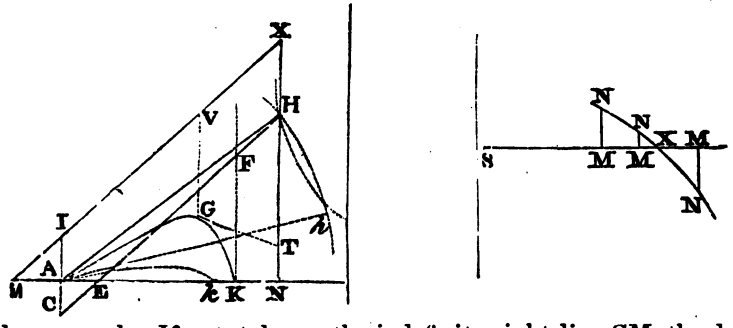
Rule 4. Because the density of the medium is greater near the vertex of the hyperbola than it is in the place A, that a mean density may be preserved, the ratio of the least of the tangents GT to the tangent AH ought to be found, and the density in A augmented in a ratio a little greater than that of half the sum of those tangents to the least of the tangents GT.

Rule 5. If the lengths AH, AI are given, and the figure AGK is to be described, produce HN to X, so that HX may be to AI as $n + 1$ to 1; and with the centre X, and the asymptotes MX, NX, describe an hyperbola through the point A, such that AI may be to any of the lines VG as XV^n to XI^n.

Rule 6. By how much the greater the number n is, so much the more accurate are these hyperbolas in the ascent of the body from A, and less accurate in its descent to K; and the contrary. The conic hyperbola keeps a mean ratio between these, and is more simple than the rest. Therefore if the hyperbola be of this kind, and you are to find the point K, where the projected body falls upon any right line AN passing through the point A, let AN produced meet the asymptotes MX, NX in M and N, and take NK equal to AM.

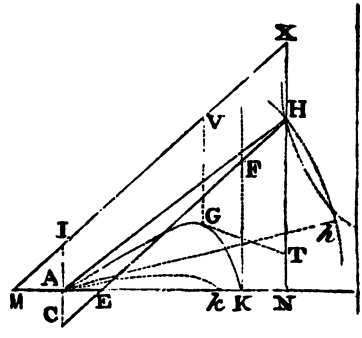
Rule 7. And hence appears an expeditious method of determining this hyperbola from the phenomena. Let two similar and equal bodies be projected with the same velocity, in different angles HAK, hAk, and let them fall upon the plane of the horizon in K and k; and note the proportion of AK to Ak. Let it be as d to e. Then erecting a perpendicular AI of any length, assume any how the length AH or Ah, and thence graphically,

or by scale and compass, collect the lengths AK, Ak (by Rule 6). If the ratio of AK to Ak be the same with that of d to e, the length of AH was

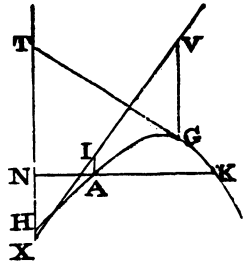


rightly assumed. If not, take on the indefinite right line SM, the length SM equal to the assumed AH; and erect a perpendicular MN equal to the difference $\frac{AK}{Ak} - \frac{d}{e}$ of the ratios drawn into any given right line. By the like method, from several assumed lengths AH, you may find several points N; and draw through them all a regular curve NNXN, cutting the right line SMMM in X. Lastly, assume AH equal to the abscissa SX, and thence find again the length AK; and the lengths, which are to the assumed length AI, and this last AH, as the length AK known by experiment, to the length AK last found, will be the true lengths AI and AH, which were to be found. But these being given, there will be given also the resisting force of the medium in the place A, it being to the force of gravity as AH to $\frac{4}{3}$AI. Let the density of the medium be increased by Rule 4, and if the resisting force just found be increased in the same ratio, it will become still more accurate.

RULE 8. The lengths AH, HX being found; let there be now required the position of the line AH, according to which a projectile thrown with that given velocity shall fall upon any point K. At the points A and K, erect the lines AC, KF perpendicular to the horizon; whereof let AC be drawn downwards, and be equal to AI or $\frac{1}{2}$HX. With the asymptotes AK, KF, describe an hyperbola, whose conjugate shall pass through the point C; and from the centre A, with the interval AH, describe a circle cutting that hyperbola in the point H; then the projectile thrown in the direction of the right line AH will fall upon the point K. Q.E.I. For the point H, because of the given length AH, must be somewhere in the circumference of the described circle. Draw CH meeting AK and KF in E and F; and because CH, MX are parallel, and AC, AI equal, AE will be equal to AM, and therefore also equal to KN. But CE is to AE as FH to KN, and therefore CE and FH are equal. Therefore the point H falls upon the hyperbolic curve described with the asymptotes AK, KF whose conjugate passes through the point C; and is therefore found in the



common intersection of this hyperbolic curve and the circumference of the described circle. Q.E.D. It is to be observed that this operation is the same, whether the right line AKN be parallel to the horizon, or inclined thereto in any angle; and that from two intersections H, *h*, there arise two angles NAH, NA*h*; and that in mechanical practice it is sufficient once to describe a circle, then to apply a ruler CH, of an indeterminate length, so to the point C, that its part FH, intercepted between the circle and the right line FK, may be equal to its part CE placed between the point C and the right line AK



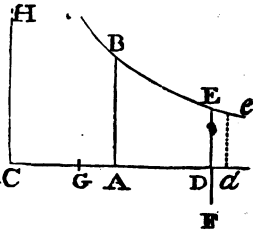
What has been said of hyperbolas may be easily applied to parabolas. For if a parabola be represented by XAGK, touched by a right line XV in the vertex X, and the ordinates IA, VG be as any powers XI^n, XV^n, of the abscissas XI, XV; draw XT, GT, AH, whereof let XT be parallel to VG, and let GT, AH touch the parabola in G and A: and a body projected from any place A, in the direction of the right line AH, with a due velocity, will describe this parabola, if the density of the medium in each of the places G be reciprocally as the tangent GT. In that case the velocity in G will be the same as would cause a body, moving in a non-resisting space, to describe a conic parabola, having G for its vertex, VG produced downwards for its diameter, and $\frac{2GT^2}{nn-n \times VG}$ for its latus rectum. And the resisting force in G will be to the force of gravity as GT to $\frac{2nn-2n}{n-2}$ VG. Therefore if NAK represent an horizontal line, and both the density of the medium at A, and the velocity with which the body is projected, remaining the same, the angle NAH be any how altered, the lengths AH, AI, HX will remain; and thence will be given the vertex X of the parabola, and the position of the right line XI; and by taking VG to IA as XV^n to XI^n, there will be given all the points G of the parabola, through which the projectile will pass.

SECTION III.

Of the motions of bodies which are resisted partly in the ratio of the velocities, and partly in the duplicate of the same ratio.

PROPOSITION XI. THEOREM VIII.

If a body be resisted partly in the ratio and partly in the duplicate ratio of its velocity, and moves in a similar medium by its innate force only; and the times be taken in arithmetical progression; then quantities reciprocally proportional to the velocities, increased by a certain given quantity, will be in geometrical progression.



With the centre C, and the rectangular asymptotes CAD*d* and CH, describe an hyperbola BE*e*, and let AB, DE, *de*, be parallel to the asymptote CH. In the asymptote CD let A, G be given points; and if the time be expounded by the hyperbolic area ABED uniformly increasing, I say, that the velocity may be expressed by the length DF, whose reciprocal GD, together with the given line CG, compose the length CD increasing in a geometrical progression.

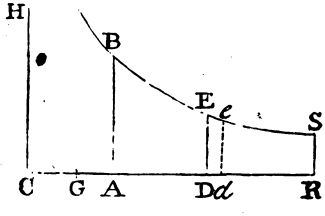
For let the areola DE*ed* be the least given increment of the time, and D*d* will be reciprocally as DE, and therefore directly as CD. Therefore the decrement of $\frac{1}{GD}$, which (by Lem. II, Book II) is $\frac{D\,d}{GD^2}$, will be also as $\frac{CD}{GD^2}$ or $\frac{CG+GD}{GD^2}$, that is, as $\frac{1}{GD}+\frac{CG}{GD^2}$. Therefore the time ABED uniformly increasing by the addition of the given particles ED*de*, it follows that $\frac{1}{GD}$ decreases in the same ratio with the velocity. For the decrement of the velocity is as the resistance, that is (by the supposition), as the sum of two quantities, whereof one is as the velocity, and the other as the square of the velocity; and the decrement of $\frac{1}{GD}$ is as the sum of the quantities $\frac{1}{GD}$ and $\frac{CG}{GD^2}$, whereof the first is $\frac{1}{GD}$ itself, and the last $\frac{CG}{GD^2}$ is as $\frac{1}{GD^2}$: therefore $\frac{1}{GD}$ is as the velocity, the decrements of both being analogous. And if the quantity GD reciprocally proportional to $\frac{1}{GD}$, be augmented by the given quantity CG; the sum CD, the time ABED uniformly increasing, will increase in a geometrical progression. Q.E.D.

Cor. 1. Therefore, if, having the points A and G given, the time be expounded by the hyperbolic area ABED, the velocity may be expounded by $\frac{1}{GD}$ the reciprocal of GD.

Cor. 2. And by taking GA to GD as the reciprocal of the velocity at the beginning to the reciprocal of the velocity at the end of any time ABED, the point G will be found. And that point being found the velocity may be found from any other time given.

PROPOSITION XII. THEOREM IX.

The same things being supposed, I say, that if the spaces described are taken in arithmetical progression, the velocities augmented by a certain given quantity will be in geometrical progression.



In the asymptote CD let there be given the point R, and, erecting the perpendicular RS meeting the hyperbola in S, let the space described be expounded by the hyperbolic area RSED; and the velocity will be as the length GD, which, together with the given line CG, composes a length CD decreasing in a geometrical progression, while the space RSED increases in an arithmetical progression.

For, because the increment ED*de* of the space is given, the lineola D*d*, which is the decrement of GD, will be reciprocally as ED, and therefore directly as CD; that is, as the sum of the same GD and the given length CG. But the decrement of the velocity, in a time reciprocally proportional thereto, in which the given particle of space D*de*E is described, is as the resistance and the time conjunctly, that is, directly as the sum of two quantities, whereof one is as the velocity, the other as the square of the velocity, and inversely as the velocity; and therefore directly as the sum of two quantities, one of which is given, the other is as the velocity. Therefore the decrement both of the velocity and the line GD is as a given quantity and a decreasing quantity conjunctly; and, because the decrements are analogous, the decreasing quantities will always be analogous; viz., the velocity, and the line GD. Q.E.D.

Cor. 1. If the velocity be expounded by the length GD, the space described will be as the hyperbolic area DESR.

Cor. 2. And if the point ... be assumed any how, the point G will be found, by taking GR to GD as the velocity at the beginning to the velocity after any space RSED is described. The point G being given, the space is given from the given velocity: and the contrary.

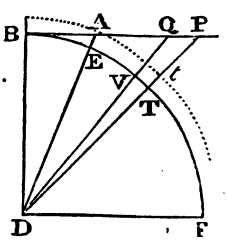
Cor. 3. Whence since (by Prop. XI) the velocity is given from the given

time, and (by this Prop.) the space is given from the given velocity; the space will be given from the given time: and the contrary.

PROPOSITION XIII. THEOREM X.

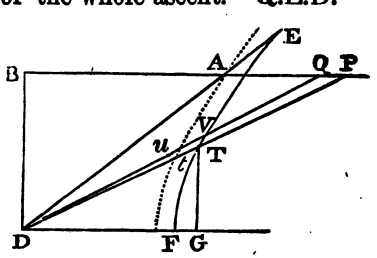
Supposing that a body attracted downwards by an uniform gravity ascends or descends in a right line; and that the same is resisted partly in the ratio of its velocity, and partly in the duplicate ratio thereof: I say, that, if right lines parallel to the diameters of a circle and an hyperbola be drawn through the ends of the conjugate diameters, and the velocities be as some segments of those parallels drawn from a given point, the times will be as the sectors of the areas cut off by right lines drawn from the centre to the ends of the segments; and the contrary.

CASE 1. Suppose first that the body is ascending, and from the centre D, with any semi-diameter DB, describe a quadrant BETF of a circle, and through the end B of the semi-diameter DB draw the indefinite line BAP, parallel to the semi-diameter DF. In that line let there be given the point A, and take the segment AP proportional to the velocity. And since one part of the resistance is as the velocity, and another part as the square of the velocity, let the whole resistance be as $AP^2 + 2BAP$. Join DA, DP, cutting the circle in E and T, and let the gravity be expounded by DA^2, so that the gravity shall be to the resistance in P as DA^2 to AP^2+2BAP; and the time of the whole ascent will be as the sector EDT of the circle.



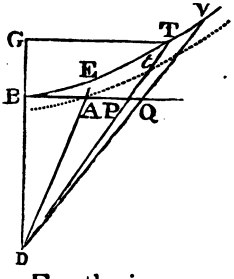
For draw DVQ, cutting off the moment PQ of the velocity AP, and the moment DTV of the sector DET answering to a given moment of time; and that decrement PQ of the velocity will be as the sum of the forces of gravity DA^2 and of resistance $AP^2 + 2BAP$, that is (by Prop. XII. Book II, Elem.), as DP^2. Then the area DPQ, which is proportional to PQ, is as DP^2, and the area DTV, which is to the area DPQ as DT^2 to DP^2, is as the given quantity DT^2. Therefore the area EDT decreases uniformly according to the rate of the future time, by subduction of given particles DTV, and is therefore proportional to the time of the whole ascent. Q.E.D.

CASE 2. If the velocity in the ascent of the body be expounded by the length AP as before, and the resistance be made as $AP^2 + 2BAP$, and if the force of gravity be less than can be expressed by DA^2; take BD of such a length, that $AB^2 - BD^2$ may be proportional to the gravity, and let DF be perpendicular and equal



to DB, and through the vertex F describe the hyperbola FTVE, whose conjugate semi-diameters are DB and DF, and which cuts DA in E, and DP, DQ in T and V; and the time of the whole ascent will be as the hyperbolic sector TDE.

For the decrement PQ of the velocity, produced in a given particle of time, is as the sum of the resistance $AP^2 + 2BAP$ and of the gravity $AB^2 - BD^2$, that is, as $BP^2 - BD^2$. But the area DTV is to the area DPQ as DT^2 to DP^2; and, therefore, if GT be drawn perpendicular to DF, as GT^2 or $GD^2 - DF^2$ to BD^2, and as GD^2 to BP^2, and, by division, as DF^2 to $BP^2 - BD^2$. Therefore since the area DPQ is as PQ, that is, as $BP^2 - BD^2$, the area DTV will be as the given quantity DF^2. Therefore the area EDT decreases uniformly in each of the equal particles of time, by the subduction of so many given particles DTV, and therefore is proportional to the time. Q.E.D.



Case 3. Let AP be the velocity in the descent of the body, and $AP^2 + 2BAP$ the force of resistance, and $BD^2 - AB^2$ the force of gravity, the angle DBA being a right one. And if with the centre D, and the principal vertex B, there be described a rectangular hyperbola BETV cutting DA, DP, and DQ produced in E, T, and V; the sector DET of this hyperbola will be as the whole time of descent.

For the increment PQ of the velocity, and the area DPQ proportional to it, is as the excess of the gravity above the resistance, that is, as $BD^2 - AB^2 - 2BAP - AP^2$ or $BD^2 - BP^2$. And the area DTV is to the area DPQ as DT^2 to DP^2; and therefore as GT^2 or $GD^2 - BD^2$ to BP^2, and as GD^2 to BD^2, and, by division, as BD^2 to $BD^2 - BP^2$. Therefore since the area DPQ is as $BD^2 - BP^2$, the area DTV will be as the given quantity BD^2. Therefore the area EDT increases uniformly in the several equal particles of time by the addition of as many given particles DTV, and therefore is proportional to the time of the descent. Q.E.D.

Cor. If with the centre D and the semi-diameter DA there be drawn through the vertex A an arc A*t* similar to the arc ET, and similarly subtending the angle ADT, the velocity AP will be to the velocity which the body in the time EDT, in a non-resisting space, can lose in its ascent, or acquire in its descent, as the area of the triangle DAP to the area of the sector DA*t*; and therefore is given from the time given. For the velocity in a non-resisting medium is proportional to the time, and therefore to this sector; in a resisting medium, it is as the triangle; and in both mediums, where it is least, it approaches to the ratio of equality, as the sector and triangle do

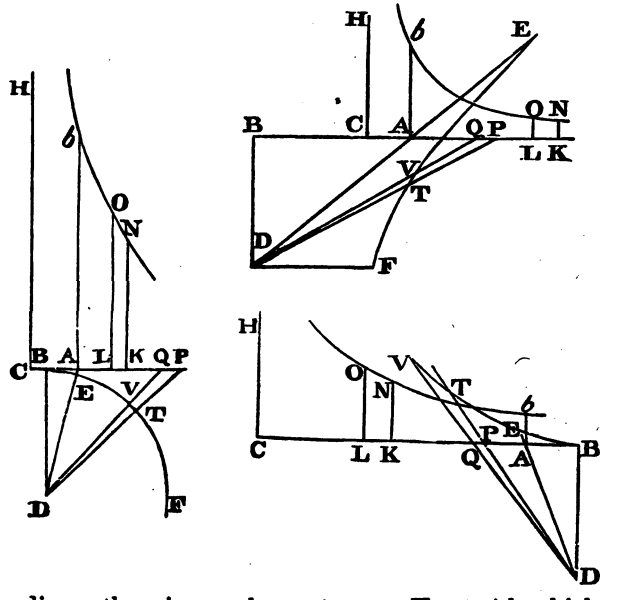
SCHOLIUM.

One may demonstrate also that case in the ascent of the body, where the force of gravity is less than can be expressed by DA^2 or $AB^2 + BD^2$, and greater than can be expressed by $AB^2 - DB^2$, and must be expressed by AB^2. But I hasten to other things.

PROPOSITION XIV. THEOREM XI.

The same things being supposed, I say, that the space described in the ascent or descent is as the difference of the area by which the time is expressed, and of some other area which is augmented or diminished in an arithmetical progression; if the forces compounded of the resistance and the gravity be taken in a geometrical progression.

Take AC (in these three figures) proportional to the gravity, and AK to the resistance; but take them on the same side of the point A, if the



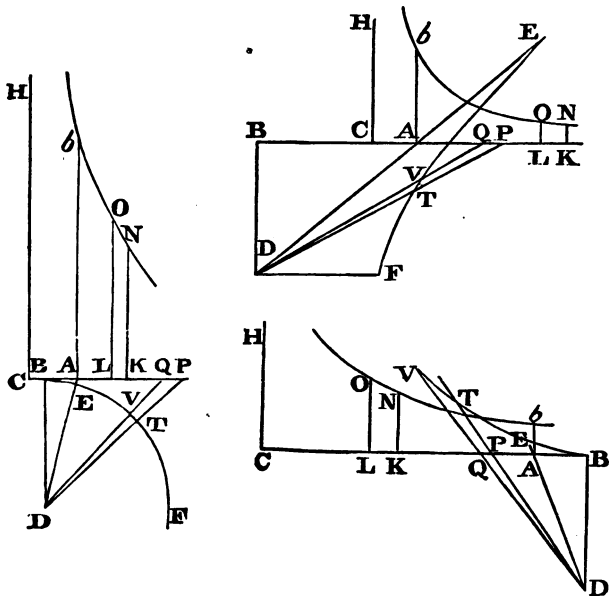
body is descending, otherwise on the contrary. Erect Ab, which make to DB as DB^2 to 4BAC: and to the rectangular asymptotes CK, CH, describe the hyperbola bN; and, erecting KN perpendicular to CK, the area $AbNK$ will be augmented or diminished in an arithmetical progression, while the forces CK are taken in a geometrical progression. I say, therefore, that the distance of the body from its greatest altitude is as the excess of the area $AbNK$ above the area DET.

For since AK is as the resistance, that is, as $AP^2 \times 2BAP$; assume any given quantity Z, and put AK equal to $\frac{AP^2 + 2BAP}{Z}$; then (by Lem.

II of this Book) the moment KL of AK will be equal to $\frac{2APQ + 2BA \times PQ}{Z}$ or $\frac{2BPQ}{Z}$, and the moment KLON of the area AbNK will be equal to $\frac{2BPQ \times LO}{Z}$ or $\frac{BPQ \times BD^3}{2Z \times CK \times AB}$.

Case 1. Now if the body ascends, and the gravity be as $AB^2 + BD^2$ BET being a circle, the line AC, which is proportional to the gravity, will be $\frac{AB^2 + BD^2}{Z}$, and DP^2 or $AP^2 + 2BAP + AB^2 + BD^2$ will be $AK \times Z + AC \times Z$ or $CK \times Z$; and therefore the area DTV will be to the area DPQ as DT^2 or DB^2 to $CK \times Z$.

Case 2. If the body ascends, and the gravity be as $AB^2 - BD^2$, the line AC will be $\frac{AB^2 - BD^2}{Z}$, and DT^2 will be to DP^2 as DF^2 or DB^2 to $BP^2 - BD^2$ or $AP^2 + 2BAP + AB^2 - BD^2$, that is, to $AK \times Z +$



$AC \times Z$ or $CK \times Z$. And therefore the area DTV will be to the area DPQ as DB^2 to $CK \times Z$.

Case 3. And by the same reasoning, if the body descends, and therefore the gravity is as $BD^2 - AB^2$, and the line AC becomes equal to $\frac{BD^2 - AB^2}{Z}$; the area DTV will be to the area DPQ as DB^2 to $CK \times Z$: as above.

Since, therefore, these areas are always in this ratio, if for the area

DTV, by which the moment of the time, always equal to itself, is expressed, there be put any determinate rectangle, as $BD \times m$, the area DPQ, that is, $\frac{1}{2}BD \times PQ$, will be to $BD \times m$ as $CK \times Z$ to BD^2. And thence $PQ \times BD^3$ becomes equal to $2BD \times m \times CK \times Z$, and the moment KLON of the area AbNK, found before, becomes $\frac{BP \times BD \times m}{AB}$. From the area DET subduct its moment DTV or $BD \times m$, and there will remain $\frac{AP \times BD \times m}{AB}$. Therefore the difference of the moments, that is, the moment of the difference of the areas, is equal to $\frac{AP \times BD \times m}{AB}$; and therefore (because of the given quantity $\frac{BD \times m}{AB}$) as the velocity AP; that is, as the moment of the space which the body describes in its ascent or descent. And therefore the difference of the areas, and that space, increasing or decreasing by proportional moments, and beginning together or vanishing together, are proportional. Q.E.D.

COR. If the length, which arises by applying the area DET to the line BD, be called M; and another length V be taken in that ratio to the length M, which the line DA has to the line DE; the space which a body, in a resisting medium, describes in its whole ascent or descent, will be to the space which a body, in a non-resisting medium, falling from rest, can describe in the same time, as the difference of the aforesaid areas to $\frac{BD \times V^2}{AB}$; and therefore is given from the time given. For the space in a non-resisting medium is in a duplicate ratio of the time, or as V^2; and, because BD and AB are given, as $\frac{BD \times V^2}{AB}$. This area is equal to the area $\frac{DA^2 \times BD \times M^2}{DE^2 \times AB}$ and the moment of M is m; and therefore the moment of this area is $\frac{DA^2 \times BD \times 2M \times m}{DE^2 \times AB}$. But this moment is to the moment of the difference of the aforesaid areas DET and AbNK, viz., to $\frac{AP \times BD \times m}{AB}$, as $\frac{DA^2 \times BD \times M}{DE^2}$ to $\frac{1}{2}BD \times AP$, or as $\frac{DA^2}{DE^2}$ into DET to DAP; and, therefore, when the areas DET and DAP are least, in the ratio of equality. Therefore the area $\frac{BD \times V^2}{AB}$ and the difference of the areas DET and AbNK, when all these areas are least, have equal moments; and are therefore equal. Therefore since the velocities, and therefore also the spaces in both mediums described together, in the beginning of the descent, or the end of the ascent, approach to equality, and therefore are then

one to another as the area $\frac{BD \times V^2}{AB}$, and the difference of the areas DET and A*b*NK; and moreover since the space, in a non-resisting medium, is perpetually as $\frac{BD \times V^2}{AB}$, and the space, in a resisting medium, is perpetually as the difference of the areas DET and A*b*NK; it necessarily follows, that the spaces, in both mediums, described in any equal times, are one to another as that area $\frac{BD \times V^2}{AB}$, and the difference of the areas DET and A*b*NK. Q.E.D.

SCHOLIUM.

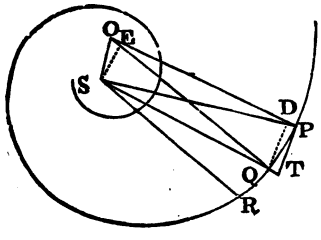
The resistance of spherical bodies in fluids arises partly from the tenacity, partly from the attrition, and partly from the density of the medium. And that part of the resistance which arises from the density of the fluid is, as I said, in a duplicate ratio of the velocity; the other part, which arises from the tenacity of the fluid, is uniform, or as the moment of the time; and, therefore, we might now proceed to the motion of bodies, which are resisted partly by an uniform force, or in the ratio of the moments of the time, and partly in the duplicate ratio of the velocity. But it is sufficient to have cleared the way to this speculation in Prop. VIII and IX foregoing, and their Corollaries. For in those Propositions, instead of the uniform resistance made to an ascending body arising from its gravity, one may substitute the uniform resistance which arises from the tenacity of the medium, when the body moves by its *vis insita* alone; and when the body ascends in a right line, add this uniform resistance to the force of gravity, and subduct it when the body descends in a right line. One might also go on to the motion of bodies which are resisted in part uniformly, in part in the ratio of the velocity, and in part in the duplicate ratio of the same velocity. And I have opened a way to this in Prop. XIII and XIV foregoing, in which the uniform resistance arising from the tenacity of the medium may be substituted for the force of gravity, or be compounded with it as before. But I hasten to other things.

SECTION IV.

Of the circular motion of bodies in resisting mediums.

LEMMA III.

Let PQR *be a spiral cutting all the radii* SP, SQ, SR, *&c., in equal angles. Draw the right line* PT *touching the spiral in any point* P, *and cutting the radius* SQ *in* T; *draw* PO, QO *perpendicular to the spiral, and meeting in* O, *and join* SO. *I say, that if the points* P *and* Q *approach and coincide, the angle* PSO *will become a right angle, and the ultimate ratio of the rectangle* TQ × 2PS *to* PQ^2 *will be the ratio of equality.*

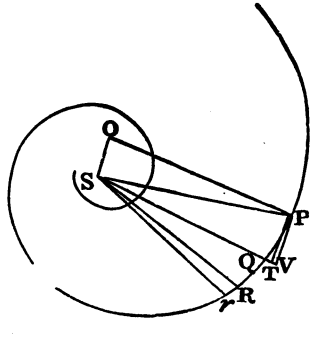


For from the right angles OPQ, OQR, subduct the equal angles SPQ, SQR, and there will remain the equal angles OPS, OQS. Therefore a circle which passes through the points OSP will pass also through the point Q. Let the points P and Q coincide, and this circle will touch the spiral in the place of coincidence PQ, and will therefore cut the right line OP perpendicularly. Therefore OP will become a diameter of this circle, and the angle OSP, being in a semi-circle, becomes a right one. Q.E.D.

Draw QD, SE perpendicular to OP, and the ultimate ratios of the lines will be as follows: TQ to PD as TS or PS to PE, or 2PO to 2PS; and PD to PQ as PQ to 2PO; and, *ex æquo perturbatè*, to TQ to PQ as PQ to 2PS. Whence PQ^2 becomes equal to TQ × 2PS. Q.E.D.

PROPOSITION XV. THEOREM XII.

If the density of a medium in each place thereof be reciprocally as the distance of the places from an immovable centre, and the centripetal force be in the duplicate ratio of the density; I say, that a body may revolve in a spiral which cuts all the radii drawn from that centre in a given angle.



Suppose every thing to be as in the foregoing Lemma, and produce SQ to V so that SV may be equal to SP. In any time let a body, in a resisting medium, describe the least arc PQ, and in double the time the least arc PR; and the decrements of those arcs arising from the resistance, or their differences from the arcs which would be described in a non-resisting medium in the same times, will be to each other as the squares of the times in which they are generated; therefore the decrement of the

arc PQ is the fourth part of the decrement of the arc PR. Whence also if the area QSr be taken equal to the area PSQ, the decrement of the arc PQ will be equal to half the lineola Rr; and therefore the force of resistance and the centripetal force are to each other as the lineola $\frac{1}{2}$Rr and TQ which they generate in the same time. Because the centripetal force with which the body is urged in P is reciprocally as SP^2, and (by Lem. X, Book I) the lineola TQ, which is generated by that force, is in a ratio compounded of the ratio of this force and the duplicate ratio of the time in which the arc PQ is described (for in this case I neglect the resistance, as being infinitely less than the centripetal force), it follows that TQ $\times$ SP^2, that is (by the last Lemma), $\frac{1}{2}PQ^2 \times SP$, will be in a duplicate ratio of the time, and therefore the time is as $PQ \times \sqrt{SP}$; and the velocity of the body, with which the arc PQ is described in that time, as $\frac{PQ}{PQ \times \sqrt{SP}}$ or $\frac{1}{\sqrt{SP}}$, that is, in the subduplicate ratio of SP reciprocally. And, by a like reasoning, the velocity with which the arc QR is described, is in the subduplicate ratio of SQ reciprocally. Now those arcs PQ and QR are as the describing velocities to each other; that is, in the subduplicate ratio of SQ to SP, or as SQ to $\sqrt{SP \times SQ}$; and, because of the equal angles SPQ, SQr, and the equal areas PSQ, QSr, the arc PQ is to the arc Qr as SQ to SP. Take the differences of the proportional consequents, and the arc PQ will be to the arc Rr as SQ to SP — $\sqrt{SP \times SQ}$, or $\frac{1}{2}$VQ. For the points P and Q coinciding, the ultimate ratio of SP — $\sqrt{SP \times SQ}$ to $\frac{1}{2}$VQ is the ratio of equality. Because the decrement of the arc PQ arising from the resistance, or its double Rr, is as the resistance and the square of the time conjunctly, the resistance will be as $\frac{Rr}{PQ^2 \times SP}$. But PQ was to R$r$ as SQ to $\frac{1}{2}$VQ, and thence $\frac{Rr}{PQ^2 \times SP}$ becomes as $\frac{\frac{1}{2}VQ}{PQ \times SP \times SQ}$, or as $\frac{\frac{1}{2}OS}{OP \times SP^2}$. For the points P and Q coinciding, SP and SQ coincide also, and the angle PVQ becomes a right one; and, because of the similar triangles PVQ, PSO, PQ becomes to $\frac{1}{2}$VQ as OP to $\frac{1}{2}$OS. Therefore $\frac{OS}{OP \times SP^2}$ is as the resistance, that is, in the ratio of the density of the medium in P and the duplicate ratio of the velocity conjunctly. Subduct the duplicate ratio of the velocity, namely, the ratio $\frac{1}{SP}$, and there will remain the density of the medium in P, as $\frac{OS}{OP \times SP}$. Let the spiral be given, and, because of the given ratio of OS to OP, the density of the medium in P will be as $\frac{1}{SP}$. Therefore in a medium whose

density is reciprocally as SP the distance from the centre, a body will revolve in this spiral. Q.E.D.

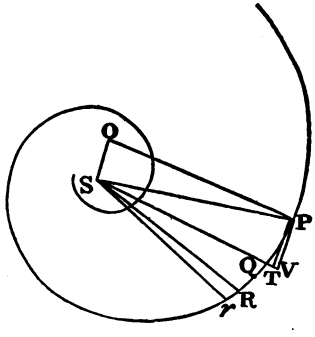
COR. 1. The velocity in any place P, is always the same wherewith a body in a non-resisting medium with the same centripetal force would revolve in a circle, at the same distance SP from the centre.

COR. 2. The density of the medium, if the distance SP be given, is as $\frac{OS}{OP^2}$, but if that distance is not given, as $\frac{OS}{OP \times SP}$. And thence a spiral may be fitted to any density of the medium.

COR. 3. The force of the resistance in any place P is to the centripetal force in the same place as $\frac{1}{2}$OS to OP. For those forces are to each other as $\frac{1}{2}Rr$ and TQ, or as $\frac{\frac{1}{4}VQ \times PQ}{SQ}$ and $\frac{\frac{1}{2}PQ^2}{SP}$, that is, as $\frac{1}{2}$VQ and PQ, or $\frac{1}{2}$OS and OP. The spiral therefore being given, there is given the proportion of the resistance to the centripetal force; and, *vice versa*, from that proportion given the spiral is given.

COR. 4. Therefore the body cannot revolve in this spiral, except where the force of resistance is less than half the centripetal force. Let the resistance be made equal to half the centripetal force, and the spiral will coincide with the right line PS, and in that right line the body will descend to the centre with a velocity that is to the velocity, with which it was proved before, in the case of the parabola (Theor. X, Book I), the descent would be made in a non-resisting medium, in the subduplicate ratio of unity to the number two. And the times of the descent will be here reciprocally as the velocities, and therefore given.

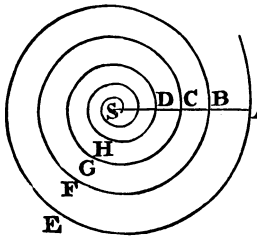
COR. 5. And because at equal distances from the centre the velocity is the same in the spiral PQR as it is in the right line SP, and the length of the spiral is to the length of the right line PS in a given ratio, namely, in the ratio of OP to OS; the time of the descent in the spiral will be to the time of the descent in the right line SP in the same given ratio, and therefore given.



COR. 6. If from the centre S, with any two given intervals, two circles are described; and these circles remaining, the angle which the spiral makes with the radius PS be any how changed; the number of revolutions which the body can complete in the space between the circumferences of those circles, going round in the spiral from one circumference to another, will be as $\frac{PS}{OS}$, or as the tangent of the angle which the spiral makes with the radius PS; and

the time of the same revolutions will be as $\frac{OP}{OS}$, that is, as the secant of the same angle, or reciprocally as the density of the medium.

Cor. 7. If a body, in a medium whose density is reciprocally as the distances of places from the centre, revolves in any curve AEB about that centre, and cuts the first radius AS in the same angle in B as it did before in A, and that with a velocity that shall be to its first velocity in A reciprocally in a subduplicate ratio of the distances from the centre (that is, as AS to a mean proportional between AS and BS) that body will continue to describe innumerable similar revolutions BFC, CGD, &c., and by its intersections will distinguish the radius AS into parts AS, BS, CS, DS, &c., that are continually proportional. But the times of the revolutions will be as the perimeters of the orbits AEB, BFC, CGD, &c., directly, and the velocities at the beginnings A, B, C of those orbits inversely; that is as $AS^{\frac{3}{2}}$, $BS^{\frac{3}{2}}$, $CS^{\frac{3}{2}}$. And the whole time in which the body will arrive at the centre, will be to the time of the first revolution as the sum of all the continued proportionals $AS^{\frac{3}{2}}$, $BS^{\frac{3}{2}}$, $CS^{\frac{3}{2}}$, going on *ad infinitum*, to the first term $AS^{\frac{3}{2}}$; that is, as the first term $AS^{\frac{3}{2}}$ to the difference of the two first $AS^{\frac{3}{2}}$ — $BS^{\frac{3}{2}}$, or as $\frac{2}{3}$ AS to AB very nearly. Whence the whole time may be easily found.

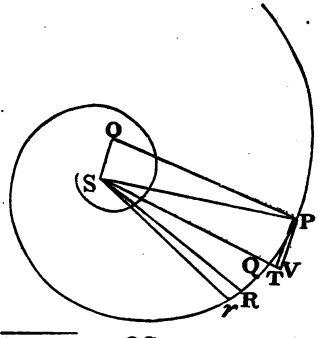


Cor. 8. From hence also may be deduced, near enough, the motions of bodies in mediums whose density is either uniform, or observes any other assigned law. From the centre S, with intervals SA, SB, SC, &c., continually proportional, describe as many circles; and suppose the time of the revolutions between the perimeters of any two of those circles, in the medium whereof we treated, to be to the time of the revolutions between the same in the medium proposed as the mean density of the proposed medium between those circles to the mean density of the medium whereof we treated, between the same circles, nearly: and that the secant of the angle in which the spiral above determined, in the medium whereof we treated, cuts the radius AS, is in the same ratio to the secant of the angle in which the new spiral, in the proposed medium, cuts the same radius: and also that the number of all the revolutions between the same two circles is nearly as the tangents of those angles. If this be done every where between every two circles. the motion will be continued through all the circles. And by this means one may without difficulty conceive at what rate and in what time bodies ought to revolve in any regular medium.

COR. 9. And although these motions becoming eccentrical should be performed in spirals approaching to an oval figure, yet, conceiving the several revolutions of those spirals to be at the same distances from each other, and to approach to the centre by the same degrees as the spiral above described, we may also understand how the motions of bodies may be performed in spirals of that kind.

PROPOSITION XVI. THEOREM XIII.

If the density of the medium in each of the places be reciprocally as the distance of the places from the immoveable centre, and the centripetal force be reciprocally as any power of the same distance, I say, that the body may revolve in a spiral intersecting all the radii drawn from that centre in a given angle.



This is demonstrated in the same manner as the foregoing Proposition. For if the centripetal force in P be reciprocally as any power SP^{n+1} of the distance SP whose index is $n+1$; it will be collected, as above, that the time in which the body describes any arc PQ, will be as $PQ \times PS^{\frac{1}{2}n}$; and the resistance in P as $\frac{Rr}{PQ^2 \times SP^n}$, or as $\frac{\overline{1-\frac{1}{2}n} \times VQ}{PQ \times SP^n \times SQ}$, and therefore as $\frac{\overline{1-\frac{1}{2}n} \times OS}{OP \times SP^{n+1}}$, that is, (because $\frac{\overline{1-\frac{1}{2}n} \times OS}{OP}$ is a given quantity), reciprocally as SP^{n+1}. And therefore, since the velocity is reciprocally as $SP^{\frac{1}{2}n}$, the density in P will be reciprocally as SP.

COR. 1. The resistance is to the centripetal force as $\overline{1-\frac{1}{2}n} \times OS$ to OP.

COR. 2. If the centripetal force be reciprocally as SP^3, $1-\frac{1}{2}n$ will be $=0$; and therefore the resistance and density of the medium will be nothing, as in Prop. IX, Book I.

COR. 3. If the centripetal force be reciprocally as any power of the radius SP, whose index is greater than the number 3, the affirmative resistance will be changed into a negative.

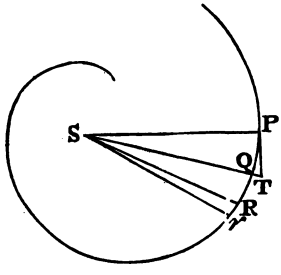
SCHOLIUM.

This Proposition and the former, which relate to mediums of unequal density, are to be understood of the motion of bodies that are so small, that the greater density of the medium on one side of the body above that on the other is not to be considered. I suppose also the resistance, *cæteris paribus*, to be proportional to its density. Whence, in mediums whose

force of resistance is not as the density, the density must be so much augmented or diminished, that either the excess of the resistance may be taken away, or the defect supplied.

PROPOSITION XVII. PROBLEM IV

To find the centripetal force and the resisting force of the medium, by which a body, the law of the velocity being given, shall revolve in a given spiral.



Let that spiral be PQR. From the velocity, with which the body goes over the very small arc PQ, the time will be given; and from the altitude TQ, which is as the centripetal force, and the square of the time, that force will be given. Then from the difference RS*r* of the areas PSQ and QSR described in equal particles of time, the retardation of the body will be given; and from the retardation will be found the resisting force and density of the medium.

PROPOSITION XVIII. PROBLEM V.

The law of centripetal force being given, to find the density of the medium in each of the places thereof, by which a body may describe a given spiral.

From the centripetal force the velocity in each place must be found; then from the retardation of the velocity the density of the medium is found, as in the foregoing Proposition.

But I have explained the method of managing these Problems in the tenth Proposition and second Lemma of this Book; and will no longer detain the reader in these perplexed disquisitions. I shall now add some things relating to the forces of progressive bodies, and to the density and resistance of those mediums in which the motions hitherto treated of, and those akin to them, are performed.

SECTION V.

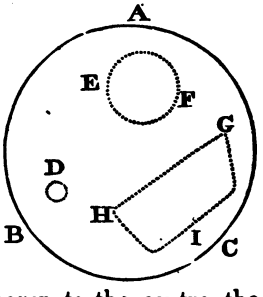
Of the density and compression of fluids; and of hydrostatics.

THE DEFINITION OF A FLUID.

A fluid is any body whose parts yield to any force impressed on it, by yielding, are easily moved among themselves.

PROPOSITION XIX. THEOREM XIV

All the parts of a homogeneous and unmoved fluid included in any unmoved vessel, and compressed on every side (setting aside the consideration of condensation, gravity, and all centripetal forces), will be equally pressed on every side, and remain in their places without any motion arising from that pressure.



CASE 1. Let a fluid be included in the spherical vessel ABC, and uniformly compressed on every side: I say, that no part of it will be moved by that pressure. For if any part, as D, be moved, all such parts at the same distance from the centre on every side must necessarily be moved at the same time by a like motion; because the pressure of them all is similar and equal; and all other motion is excluded that does not arise from that pressure. But if these parts come all of them nearer to the centre, the fluid must be condensed towards the centre, contrary to the supposition. If they recede from it, the fluid must be condensed towards the circumference; which is also contrary to the supposition. Neither can they move in any one direction retaining their distance from the centre, because for the same reason, they may move in a contrary direction; but the same part cannot be moved contrary ways at the same time. Therefore no part of the fluid will be moved from its place. Q.E.D.

CASE 2. I say now, that all the spherical parts of this fluid are equally pressed on every side. For let EF be a spherical part of the fluid; if this be not pressed equally on every side, augment the lesser pressure till it be pressed equally on every side; and its parts (by Case 1) will remain in their places. But before the increase of the pressure, they would remain in their places (by Case 1); and by the addition of a new pressure they will be moved, by the definition of a fluid, from those places. Now these two conclusions contradict each other. Therefore it was false to say that the sphere EF was not pressed equally on every side. Q.E.D.

CASE 3. I say besides, that different spherical parts have equal pressures. For the contiguous spherical parts press each other mutually and equally in the point of contact (by Law III). But (by Case 2) they are pressed on every side with the same force. Therefore any two spherical parts not

contiguous, since an intermediate spherical part can touch both, will be pressed with the same force. Q.E.D.

Case 4. I say now, that all the parts of the fluid are every where pressed equally. For any two parts may be touched by spherical parts in any points whatever; and there they will equally press those spherical parts (by Case 3), and are reciprocally equally pressed by them (by Law III). Q.E.D.

Case 5. Since, therefore, any part GHI of the fluid is inclosed by the rest of the fluid as in a vessel, and is equally pressed on every side; and also its parts equally press one another, and are at rest among themselves; it is manifest that all the parts of any fluid as GHI, which is pressed equally on every side, do press each other mutually and equally, and are at rest among themselves. Q.E.D.

Case 6. Therefore if that fluid be included in a vessel of a yielding substance, or that is not rigid, and be not equally pressed on every side, the same will give way to a stronger pressure, by the Definition of fluidity.

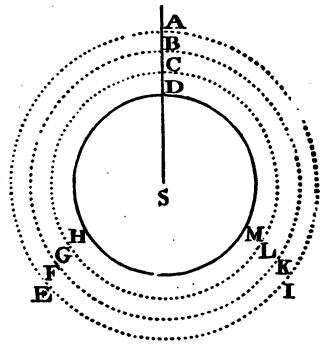
Case 7. And therefore, in an inflexible or rigid vessel, a fluid will not sustain a stronger pressure on one side than on the other, but will give way to it, and that in a moment of time; because the rigid side of the vessel does not follow the yielding liquor. But the fluid, by thus yielding, will press against the opposite side, and so the pressure will tend on every side to equality. And because the fluid, as soon as it endeavours to recede from the part that is most pressed, is withstood by the resistance of the vessel on the opposite side, the pressure will on every side be reduced to equality, in a moment of time, without any local motion: and from thence the parts of the fluid (by Case 5) will press each other mutually and equally, and be at rest among themselves. Q.E.D.

Cor. Whence neither will a motion of the parts of the fluid among themselves be changed by a pressure communicated to the external superficies, except so far as either the figure of the superficies may be somewhere altered, or that all the parts of the fluid, by pressing one another more intensely or remissly, may slide with more or less difficulty among themselves.

PROPOSITION XX. THEOREM XV.

If all the parts of a spherical fluid, homogeneous at equal distances from the centre, lying on a spherical concentric bottom, gravitate towards the centre of the whole, the bottom will sustain the weight of a cylinder, whose base is equal to the superficies of the bottom, and whose altitude is the same with that of the incumbent fluid.

Let DHM be the superficies of the bottom, and AEI the upper superficies of the fluid. Let the fluid be distinguished into concentric orbs of equal thickness, by the innumerable spherical superficies BFK, CGL: and



conceive the force of gravity to act only in the upper superficies of every orb, and the actions to be equal on the equal parts of all the superficies. Therefore the upper superficies AE is pressed by the single force of its own gravity, by which all the parts of the upper orb, and the second superficies BFK, will (by Prop. XIX), according to its measure, be equally pressed. The second superficies BFK is pressed likewise by the force of its own gravity, which, added to the former force, makes the pressure double. The third superficies CGL is, according to its measure, acted on by this pressure and the force of its own gravity besides, which makes its pressure triple. And in like manner the fourth superficies receives a quadruple pressure, the fifth superficies a quintuple, and so on. Therefore the pressure acting on every superficies is not as the solid quantity of the incumbent fluid, but as the number of the orbs reaching to the upper surface of the fluid; and is equal to the gravity of the lowest orb multiplied by the number of orbs: that is, to the gravity of a solid whose ultimate ratio to the cylinder above-mentioned (when the number of the orbs is increased and their thickness diminished, *ad infinitum*, so that the action of gravity from the lowest superficies to the uppermost may become continued) is the ratio of equality. Therefore the lowest superficies sustains the weight of the cylinder above determined. Q.E.D. And by a like reasoning the Proposition will be evident, where the gravity of the fluid decreases in any assigned ratio of the distance from the centre, and also where the fluid is more rare above and denser below. Q.E.D.

COR. 1. Therefore the bottom is not pressed by the whole weight of the incumbent fluid, but only sustains that part of it which is described in the Proposition; the rest of the weight being sustained archwise by the spherical figure of the fluid.

COR. 2. The quantity of the pressure is the same always at equal distances from the centre, whether the superficies pressed be parallel to the horizon, or perpendicular, or oblique; or whether the fluid, continued upwards from the compressed superficies, rises perpendicularly in a rectilinear direction, or creeps obliquely through crooked cavities and canals, whether those passages be regular or irregular, wide or narrow. That the pressure is not altered by any of these circumstances, may be collected by applying the demonstration of this Theorem to the several cases of fluids.

COR. 3. From the same demonstration it may also be collected (by Prop. XIX), that the parts of a heavy fluid acquire no motion among themselves by the pressure of the incumbent weight, except that motion which arises from condensation.

Cor. 4. And therefore if another body of the same specific gravity, incapable of condensation, be immersed in this fluid, it will acquire no motion by the pressure of the incumbent weight: it will neither descend nor ascend, nor change its figure. If it be spherical, it will remain so, notwithstanding the pressure; if it be square, it will remain square; and that, whether it be soft or fluid; whether it swims freely in the fluid, or lies at the bottom. For any internal part of a fluid is in the same state with the submersed body; and the case of all submersed bodies that have the same magnitude, figure, and specific gravity, is alike. If a submersed body, retaining its weight, should dissolve and put on the form of a fluid, this body, if before it would have ascended, descended, or from any pressure assume a new figure, would now likewise ascend, descend, or put on a new figure; and that, because its gravity and the other causes of its motion remain. But (by Case 5, Prop. XIX.) it would now be at rest, and retain its figure. Therefore also in the former case.

Cor. 5. Therefore a body that is specifically heavier than a fluid contiguous to it will sink; and that which is specifically lighter will ascend, and attain so much motion and change of figure as that excess or defect of gravity is able to produce. For that excess or defect is the same thing as an impulse, by which a body, otherwise *in equilibrio* with the parts of the fluid, is acted on; and may be compared with the excess or defect of a weight in one of the scales of a balance.

Cor. 6. Therefore bodies placed in fluids have a twofold gravity: the one true and absolute, the other apparent, vulgar, and comparative. Absolute gravity is the whole force with which the body tends downwards; relative and vulgar gravity is the excess of gravity with which the body tends downwards more than the ambient fluid. By the first kind of gravity the parts of all fluids and bodies gravitate in their proper places; and therefore their weights taken together compose the weight of the whole. For the whole taken together is heavy, as may be experienced in vessels full of liquor; and the weight of the whole is equal to the weights of all the parts, and is therefore composed of them. By the other kind of gravity bodies do not gravitate in their places; that is, compared with one another, they do not preponderate, but, hindering one another's endeavours to descend, remain in their proper places, as if they were not heavy. Those things which are in the air, and do not preponderate, are commonly looked on as not heavy. Those which do preponderate are commonly reckoned heavy, in as much as they are not sustained by the weight of the air. The common weights are nothing else but the excess of the true weights above the weight of the air. Hence also, vulgarly, those things are called light which are less heavy, and, by yielding to the preponderating air, mount upwards. But these are only comparatively light, and not truly so, because they descend *in vacuo*. Thus, in water, bodies which, by their greater or

less gravity, descend or ascend, are comparatively and apparently heavy or light; and their comparative and apparent gravity or levity is the excess or defect by which their true gravity either exceeds the gravity of the water or is exceeded by it. But those things which neither by preponderating descend, nor, by yielding to the preponderating fluid, ascend, although by their true weight they do increase the weight of the whole, yet comparatively, and in the sense of the vulgar, they do not gravitate in the water. For these cases are alike demonstrated.

COR. 7. These things which have been demonstrated concerning gravity take place in any other centripetal forces.

COR. 8. Therefore if the medium in which any body moves be acted on either by its own gravity, or by any other centripetal force, and the body be urged more powerfully by the same force; the difference of the forces is that very motive force, which, in the foregoing Propositions, I have considered as a centripetal force. But if the body be more lightly urged by that force, the difference of the forces becomes a centrifugal force, and is to be considered as such.

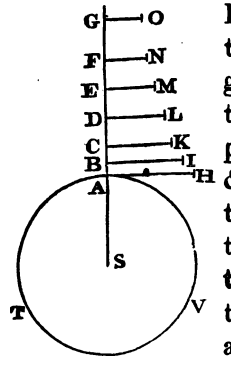
COR. 9. But since fluids by pressing the included bodies do not change their external figures, it appears also (by Cor. Prop. XIX) that they will not change the situation of their internal parts in relation to one another; and therefore if animals were immersed therein, and that all sensation did arise from the motion of their parts, the fluid will neither hurt the immersed bodies, nor excite any sensation, unless so far as those bodies may be condensed by the compression. And the case is the same of any system of bodies encompassed with a compressing fluid. All the parts of the system will be agitated with the same motions as if they were placed in a vacuum, and would only retain their comparative gravity; unless so far as the fluid may somewhat resist their motions, or be requisite to conglutinate them by compression.

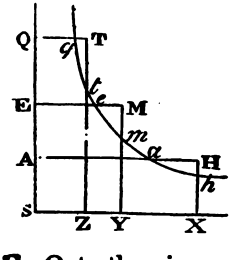
PROPOSITION XXI. THEOREM XVI.

Let the density of any fluid be proportional to the compression, and its parts be attracted downwards by a centripetal force reciprocally proportional to the distances from the centre: I say, that, if those distances be taken continually proportional, the densities of the fluid at the same distances will be also continually proportional.

Let ATV denote the spherical bottom of the fluid, S the centre, SA, SB, SC, SD, SE, SF, &c., distances continually proportional. Erect the perpendiculars AH, BI, CK, DL, EM, FN, &c., which shall be as the densities of the medium in the places A, B, C, D, E, F; and the specific gravities in those places will be as $\frac{AH}{AS}, \frac{BI}{BS}, \frac{CK}{CS}$, &c., or, which is all one, as

$\frac{AH}{AB}, \frac{BI}{BC}, \frac{CK}{CD}$, &c. Suppose, first, these gravities to be uniformly continued from A to B, from B to C, from C to D, &c., the decrements in the points B, C, D, &c., being taken by steps. And these gravities drawn into the altitudes AB, BC, CD, &c., will give the pressures AH, BI, CK, &c., by which the bottom ATV is acted on (by Theor. XV). Therefore the particle A sustains all the pressures AH, BI, CK, DL, &c., proceeding *in infinitum;* and the particle B sustains the pressures of all but the first AH; and the particle C all but the two first AH, BI; and so on: and therefore the density AH of the first particle A is to the density BI of the second particle B as the sum of all AH + BI + CK + DL, *in infinitum*, to the sum of all BI + CK + DL, &c. And BI the density of the second particle B is to CK the density of the third C, as the sum of all BI + CK + DL, &c., to the sum of all CK + DL, &c. Therefore these sums are proportional to their differences AH, BI, CK, &c., and therefore continually proportional (by Lem. 1 of this Book); and therefore the differences AH, BI, CK, &c., proportional to the sums, are also continually proportional. Wherefore since the densities in the places A, B, C, &c., are as AH, BI, CK, &c., they will also be continually proportional. Proceed intermissively, and, *ex æquo*, at the distances SA, SC, SE, continually proportional, the densities AH, CK, EM will be continually proportional. And by the same reasoning, at any distances SA, SD, SG, continually proportional, the densities AH, DL, GO, will be continually proportional. Let now the points A, B, C, D, E, &c., coincide, so that the progression of the specific gravities from the bottom A to the top of the fluid may be made continual; and at any distances SA, SD, SG, continually proportional, the densities AH, DL, GO, being all along continually proportional, will still remain continually proportional. Q.E.D.





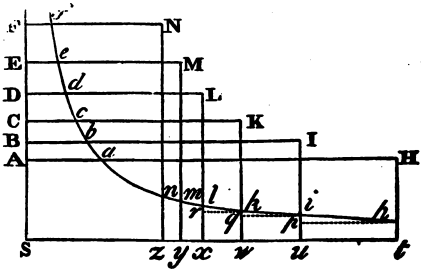
Cor. Hence if the density of the fluid in two places, as A and E, be given, its density in any other place Q may be collected. With the centre S, and the rectangular asymptotes SQ, SX, describe an hyperbola cutting the perpendiculars AH, EM, QT in *a*, *e*, and *q*, as also the perpendiculars HX, MY, TZ, let fall upon the asypmtote SX, in *h*, *m*, and *t*. Make the area Y*mt*Z to the given area Y*mh*X as the given area E*eq*Q to the given area E*ea*A; and the line Z*t* produced will cut off the line QT proportional to the density. For if the lines SA, SE, SQ are continually proportional, the areas E*eq*Q, E*ea*A will be equal, and thence

the areas $YmtZ$, $XhmY$, proportional to them, will be also equal; and the lines SX, SY, SZ, that is, AH, EM, QT continually proportional, as they ought to be. And if the lines SA, SE, SQ, obtain any other order in the series of continued proportionals, the lines AH, EM, QT, because of the proportional hyperbolic areas, will obtain the same order in another series of quantities continually proportional.

PROPOSITION XXII THEOREM XVII.

Let the density of any fluid be proportional to the compression, and its parts be attracted downwards by a gravitation reciprocally proportional to the squares of the distances from the centre: I say, that if the distances be taken in harmonic progression, the densities of the fluid at those distances will be in a geometrical progression.

Let S denote the centre, and SA, SB, SC, SD, SE, the distances in geometrical progression. Erect the perpendiculars AH, BI, CK, &c., which shall be as the densities of the fluid in the places A, B, C, D, E, &c., and the specific gravities thereof in those places will be as $\frac{AH}{SA^2}, \frac{BI}{SB^2}, \frac{CK}{SC^2}$, &c. Suppose these gravities to be uniformly continued, the first from A to B, the second from B to C, the third from C to D, &c. And these drawn into the altitudes AB, BC, CD, DE, &c., or, which is the same thing, into the distances SA, SB, SC, &c., proportional to those altitudes, will give $\frac{AH}{SA}, \frac{BI}{SB}, \frac{CK}{SC}$ &c. the exponents of the pressures. Therefore since the densities are as the sums of those pressures, the differences AH — BI, BI — CK, &c., of the densities will be as the differences of those sums $\frac{AH}{SA}, \frac{BI}{SB}, \frac{CK}{SC}$ &c. With the centre S, and the asymptotes SA, Sx, describe any hyperbola, cutting the perpendiculars AH, BI, CK, &c., in a, b, c, &c., and the perpendiculars Ht, Iu, Kw, let fall upon the asymptote Sx, in h, i, k; and the differences of the densities tu, uw, &c., will be as $\frac{AH}{SA}, \frac{BI}{SB}$, &c. And the rectangles $tu \times th$, $uw \times ui$, &c., or tp, uq, &c., as $\frac{AH \times th}{SA}, \frac{BI \times ui}{SB}$, &c., that is, as Aa, Bb, &c. For, by the nature of the hyperbola, SA is to AH or St as th to Ac, and therefore $\frac{AH \times th}{SA}$ is equal to Aa. And, by a like



reasoning, $\frac{BI \times ui}{SB}$ is equal to B*b*, &c. But A*a*, B*b*, C*c*, &c., are continually proportional, and therefore proportional to their differences A*a* — B*b*, B*b* — C*c*, &c., therefore the rectangles *tp*, *uq*, &c., are proportional to those differences; as also the sums of the rectangles $tp + uq$, or $tp + uq + wr$ to the sums of the differences A*a* — C*c* or A*a* — D*d*. Suppose several of these terms. and the sum of all the differences, as A*a* — F*f*, will be proportional to the sum of all the rectangles, as *zthn*. Increase the number of terms, and diminish the distances of the points A, B, C, &c., *in infinitum*, and those rectangles will become equal to the hyperbolic area *zthn*, and therefore the difference A*a* — F*f* is proportional to this area. Take now any distances, as SA, SD, SF, in harmonic progression, and the differences A*a* — D*d*, D*d* — F*f* will be equal; and therefore the areas *thlx*, *xlnz*, proportional to those differences will be equal among themselves, and the densities S*t*, S*x*, S*z*, that is, AH, DI., FN, continually proportional. Q.E.D.

COR. Hence if any two densities of the fluid, as AH and BI, be given, the area *thiu*, answering to their difference *tu*, will be given; and thence the density FN will be found at any height SF, by taking the area *thnz* to that given area *thiu* as the difference A*a* — F*f* to the difference A*a* — B*b*.

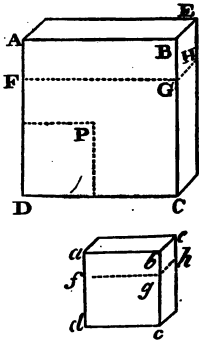
SCHOLIUM.

By a like reasoning it may be proved, that if the gravity of the particles of a fluid be diminished in a triplicate ratio of the distances from the centre; and the reciprocals of the squares of the distances SA, SB, SC, &c., (namely, $\frac{SA^3}{SA^2}, \frac{SA^3}{SB^2}, \frac{SA^3}{SC^2}$) be taken in an arithmetical progression, the densities AH, BI, CK, &c., will be in a geometrical progression. And if the gravity be diminished in a quadruplicate ratio of the distances, and the reciprocals of the cubes of the distances (as $\frac{SA^4}{SA^3}, \frac{SA^4}{SB^3}, \frac{SA^4}{SC^3}$, &c.,) be taken in arithmetical progression, the densities AH, BI, CK, &c., will be in geometrical progression. And so *in infinitum*. Again; if the gravity of the particles of the fluid be the same at all distances, and the distances be in arithmetical progression, the densities will be in a geometrical progression as Dr. *Halley* has found. If the gravity be as the distance, and the squares of the distances be in arithmetical progression, the densities will be in geometrical progression. And so *in infinitum*. These things will be so, when the density of the fluid condensed by compression is as the force of compression; or, which is the same thing, when the space possessed by the fluid is reciprocally as this force. Other laws of condensation may be supposed, as that the cube of the compressing force may be as the biquadrate of the

density; or the triplicate ratio of the force the same with the quadruplicate ratio of the density: in which case, if the gravity be reciprocally as the square of the distance from the centre, the density will be reciprocally as the cube of the distance. Suppose that the cube of the compressing force be as the quadrato-cube of the density; and if the gravity be reciprocally as the square of the distance, the density will be reciprocally in a sesquiplicate ratio of the distance. Suppose the compressing force to be in a duplicate ratio of the density, and the gravity reciprocally in a duplicate ratio of the distance, and the density will be reciprocally as the distance. To run over all the cases that might be offered would be tedious. But as to our own air, this is certain from experiment, that its density is either accurately, or very nearly at least, as the compressing force; and therefore the density of the air in the atmosphere of the earth is as the weight of the whole incumbent air, that is, as the height of the mercury in the barometer.

PROPOSITION XXIII. THEOREM XVIII.

If a fluid be composed of particles mutually flying each other, and the density be as the compression, the centrifugal forces of the particles will be reciprocally proportional to the distances of their centres. And, vice versa, *particles flying each other, with forces that are reciprocally proportional to the distances of their centres, compose an elastic fluid, whose density is as the compression.*



Let the fluid be supposed to be included in a cubic space ACE, and then to be reduced by compression into a lesser cubic space *ace;* and the distances of the particles retaining a like situation with respect to each other in both the spaces, will be as the sides AB, *ab* of the cubes; and the densities of the mediums will be reciprocally as the containing spaces AB^3, ab^3. In the plane side of the greater cube ABCD take the square DP equal to the plane side *db* of the lesser cube: and, by the supposition, the pressure with which the square DP urges the inclosed fluid will be to the pressure with which that square *db* urges the inclosed fluid as the densities of the mediums are to each other, that is, as ab^3 to AB^3. But the pressure with which the square DB urges the included fluid is to the pressure with which the square DP urges the same fluid as the square DB to the square DP, that is, as AB^2 to ab^2. Therefore, *ex æquo*, the pressure with which the square DB urges the fluid is to the pressure with which the square *db* urges the fluid as *ab* to AB. Let the planes FGH, *fgh*, be drawn through the middles of the two cubes, and divide the fluid into two parts. These parts will press each other mutually with the same forces with which they

are themselves pressed by the planes AC, *ac*, that is, in the proportion of *ab* to AB: and therefore the centrifugal forces by which these pressures are sustained are in the same ratio. The number of the particles being equal, and the situation alike, in both cubes, the forces which all the particles exert, according to the planes FGH, *fgh*, upon all, are as the forces which each exerts on each. Therefore the forces which each exerts on each, according to the plane FGH in the greater cube, are to the forces which each exerts on each, according to the plane *fgh* in the lesser cube, as *ab* to AB, that is, reciprocally as the distances of the particles from each other. Q.E.D.

And, *vice versa*, if the forces of the single particles are reciprocally as the distances, that is, reciprocally as the sides of the cubes AB, *ab*; the sums of the forces will be in the same ratio, and the pressures of the sides DB, *db* as the sums of the forces; and the pressure of the square DP to the pressure of the side DB as ab^2 to AB^2. And, *ex æquo*, the pressure of the square DP to the pressure of the side *db* as ab^3 to AB^3; that is, the force of compression in the one to the force of compression in the other as the density in the former to the density in the latter. Q.E.D.

SCHOLIUM.

By a like reasoning, if the centrifugal forces of the particles are reciprocally in the duplicate ratio of the distances between the centres, the cubes of the compressing forces will be as the biquadrates of the densities. If the centrifugal forces be reciprocally in the triplicate or quadruplicate ratio of the distances, the cubes of the compressing forces will be as the quadrato-cubes, or cubo-cubes of the densities. And universally, if D be put for the distance, and E for the density of the compressed fluid, and the centrifugal forces be reciprocally as any power D^n of the distance, whose index is the number n, the compressing forces will be as the cube roots of the power E^{n+2}, whose index is the number $n+2$; and the contrary. All these things are to be understood of particles whose centrifugal forces terminate in those particles that are next them, or are diffused not much further. We have an example of this in magnetical bodies. Their attractive virtue is terminated nearly in bodies of their own kind that are next them. The virtue of the magnet is contracted by the interposition of an iron plate, and is almost terminated at it: for bodies further off are not attracted by the magnet so much as by the iron plate. If in this manner particles repel others of their own kind that lie next them, but do not exert their virtue on the more remote, particles of this kind will compose such fluids as are treated of in this Proposition. If the virtue of any particle diffuse itself every way *in infinitum*, there will be required a greater force to produce an equal condensation of a greater quantity of the fluid. But whether

elastic fluids do really consist of particles so repelling each other, is a physical question. We have here demonstrated mathematically the property of fluids consisting of particles of this kind, that hence philosophers may take occasion to discuss that question.

SECTION VI.

Of the motion and resistance of funependulous bodies.

PROPOSITION XXIV. THEOREM XIX.

The quantities of matter in funependulous bodies, whose centres of oscillation are equally distant from the centre of suspension, are in a ratio compounded of the ratio of the weights and the duplicate ratio of the times of the oscillations in vacuo.

For the velocity which a given force can generate in a given matter in a given time is as the force and the time directly, and the matter inversely. The greater the force or the time is, or the less the matter, the greater velocity will be generated. This is manifest from the second Law of Motion. Now if pendulums are of the same length, the motive forces in places equally distant from the perpendicular are as the weights: and therefore if two bodies by oscillating describe equal arcs, and those arcs are divided into equal parts; since the times in which the bodies describe each of the correspondent parts of the arcs are as the times of the whole oscillations, the velocities in the correspondent parts of the oscillations will be to each other as the motive forces and the whole times of the oscillations directly, and the quantities of matter reciprocally: and therefore the quantities of matter are as the forces and the times of the oscillations directly and the velocities reciprocally. But the velocities reciprocally are as the times, and therefore the times directly and the velocities reciprocally are as the squares of the times; and therefore the quantities of matter are as the motive forces and the squares of the times, that is, as the weights and the squares of the times. Q.E.D.

COR. 1. Therefore if the times are equal, the quantities of matter in each of the bodies are as the weights.

COR. 2. If the weights are equal, the quantities of matter will be as the squares of the times.

COR. 3. If the quantities of matter are equal, the weights will be reciprocally as the squares of the times.

COR. 4. Whence since the squares of the times, *cæteris paribus*, are as the lengths of the pendulums, therefore if both the times and quantities of matter are equal, the weights will be as the lengths of the pendulums.

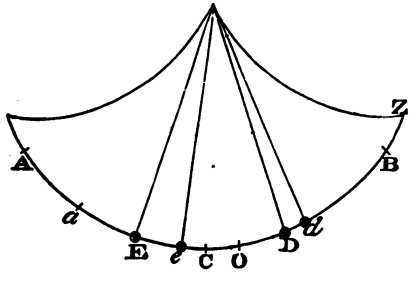
Cor. 5. And universally, the quantity of matter in the pendulous body is as the weight and the square of the time directly, and the length of the pendulum inversely.

Cor. 6. But in a non-resisting medium, the quantity of matter in the pendulous body is as the comparative weight and the square of the time directly, and the length of the pendulum inversely. For the comparative weight is the motive force of the body in any heavy medium, as was shewn above; and therefore does the same thing in such a non-resisting medium as the absolute weight does in a vacuum.

Cor. 7. And hence appears a method both of comparing bodies one among another, as to the quantity of matter in each; and of comparing the weights of the same body in different places, to know the variation of its gravity. And by experiments made with the greatest accuracy, I have always found the quantity of matter in bodies to be proportional to their weight.

PROPOSITION XXV. THEOREM XX.

Funependulous bodies that are, in any medium, resisted in the ratio of the moments of time, and funependulous bodies that move in a non-resisting medium of the same specific gravity, perform their oscillations in a cycloid in the same time, and describe proportional parts of arcs together.



Let AB be an arc of a cycloid, which a body D, by vibrating in a non-resisting medium, shall describe in any time. Bisect that arc in C, so that C may be the lowest point thereof; and the accelerative force with which the body is urged in any place D, or *d* or E, will be as the length of the arc CD, or C*d*, or CE. Let that force be expressed by that same arc; and since the resistance is as the moment of the time, and therefore given, let it be expressed by the given part CO of the cycloidal arc, and take the arc O*d* in the same ratio to the arc CD that the arc OB has to the arc CB: and the force with which the body in *d* is urged in a resisting medium, being the excess of the force C*d* above the resistance CO, will be expressed by the arc O*d*, and will therefore be to the force with which the body D is urged in a non-resisting medium in the place D, as the arc O*d* to the arc CD; and therefore also in the place B, as the arc OB to the arc CB. Therefore if two bodies D, *d* go from the place B, and are urged by these forces; since the forces at the beginning are as the arc CB and OB, the first velocities and arcs first described will be in the same ratio. Let those arcs be BD and B*d*, and the remaining arcs

CD, O*d*, will be in the same ratio. Therefore the forces, being proportional to those arcs CD, O*d*, will remain in the same ratio as at the beginning, and therefore the bodies will continue describing together arcs in the same ratio. Therefore the forces and velocities and the remaining arcs CD, O*d*, will be always as the whole arcs CB, OB, and therefore those remaining arcs will be described together. Therefore the two bodies D and *d* will arrive together at the places C and O; that which moves in the non-resisting medium, at the place C, and the other, in the resisting medium, at the place O. Now since the velocities in C and O are as the arcs CB, OB, the arcs which the bodies describe when they go farther will be in the same ratio. Let those arcs be CE and O*e*. The force with which the body D in a non-resisting medium is retarded in E is as CE, and the force with which the body *d* in the resisting medium is retarded in *e*, is as the sum of the force C*e* and the resistance CO, that is, as O*e*; and therefore the forces with which the bodies are retarded are as the arcs CB, OB, proportional to the arcs CE, O*e*; and therefore the velocities, retarded in that given ratio, remain in the same given ratio. Therefore the velocities and the arcs described with those velocities are always to each other in that given ratio of the arcs CB and OB; and therefore if the entire arcs AB, *a*B are taken in the same ratio, the bodies D and *d* will describe those arcs together, and in the places A and *a* will lose all their motion together. Therefore the whole oscillations are isochronal, or are performed in equal times; and any parts of the arcs, as BD, B*d*, or BE, B*e*, that are described together, are proportional to the whole arcs BA, B*a*. Q.E.D.

COR. Therefore the swiftest motion in a resisting medium does not fall upon the lowest point C, but is found in that point O, in which the whole arc described B*a* is bisected. And the body, proceeding from thence to *a*, is retarded at the same rate with which it was accelerated before in its descent from B to O.

PROPOSITION XXVI. THEOREM XXI.

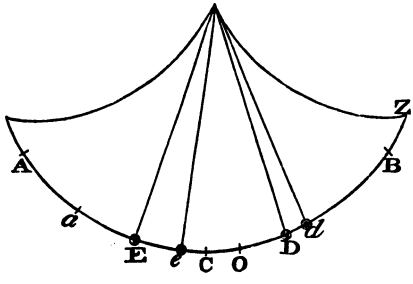
Funependulous bodies, that are resisted in the ratio of the velocity, have their oscillations in a cycloid isochronal.

For if two bodies, equally distant from their centres of suspension, describe, in oscillating, unequal arcs, and the velocities in the correspondent parts of the arcs be to each other as the whole arcs; the resistances, proportional to the velocities, will be also to each other as the same arcs. Therefore if these resistances be subducted from or added to the motive forces arising from gravity which are as the same arcs, the differences or sums will be to each other in the same ratio of the arcs; and since the increments and decrements of the velocities are as these differences or sums, the velocities will be always as the whole arcs; therefore if the velocities are in any one case as the whole arcs, they will remain always in the same

ratio. But at the beginning of the motion, when the bodies begin to descend and describe those arcs, the forces, which at that time are proportional to the arcs, will generate velocities proportional to the arcs. Therefore the velocities will be always as the whole arcs to be described, and therefore those arcs will be described in the same time. Q.E.D.

PROPOSITION XXVII. THEOREM XXII.

If funependulous bodies are resisted in the duplicate ratio of their velocities, the differences between the times of the oscillations in a resisting medium, and the times of the oscillations in a non-resisting medium of the same specific gravity, will be proportional to the arcs described in oscillating nearly.



For let equal pendulums in a resisting medium describe the unequal arcs A, B; and the resistance of the body in the arc A will be to the resistance of the body in the correspondent part of the arc B in the duplicate ratio of the velocities, that is, as AA to BB nearly. If the resistance in the arc B were to the resistance in the arc A as AB to AA, the times in the arcs A and B would be equal (by the last Prop.) Therefore the resistance AA in the arc A, or AB in the arc B, causes the excess of the time in the arc A above the time in a non-resisting medium; and the resistance BB causes the excess of the time in the arc B above the time in a non-resisting medium. But those excesses are as the efficient forces AB and BB nearly, that is, as the arcs A and B. Q.E.D.

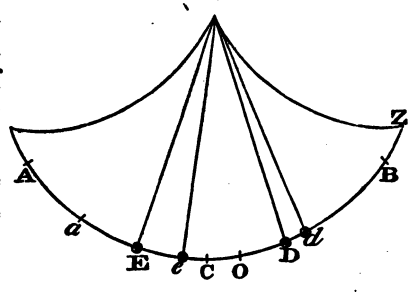
Cor. 1. Hence from the times of the oscillations in unequal arcs in a resisting medium, may be known the times of the oscillations in a non-resisting medium of the same specific gravity. For the difference of the times will be to the excess of the time in the lesser arc above the time in a non-resisting medium as the difference of the arcs to the lesser arc.

Cor. 2. The shorter oscillations are more isochronal, and very short ones are performed nearly in the same times as in a non-resisting medium. But the times of those which are performed in greater arcs are a little greater, because the resistance in the descent of the body, by which the time is prolonged, is greater, in proportion to the length described in the descent than the resistance in the subsequent ascent, by which the time is contracted. But the time of the oscillations, both short and long, seems to be prolonged in some measure by the motion of the medium. For retarded bodies are resisted somewhat less in proportion to the velocity, and accelerated bodies somewhat more than those that proceed uniformly forwards:

because the medium, by the motion it has received from the bodies, going forwards the same way with them, is more agitated in the former case, and less in the latter; and so conspires more or less with the bodies moved. Therefore it resists the pendulums in their descent more, and in their ascent less, than in proportion to the velocity; and these two causes concurring prolong the time.

PROPOSITION XXVIII. THEOREM XXIII.

If a funependulous body, oscillating in a cycloid, be resisted in the ratio of the moments of the time, its resistance will be to the force of gravity as the excess of the arc described in the whole descent above the arc described in the subsequent ascent to twice the length of the pendulum.

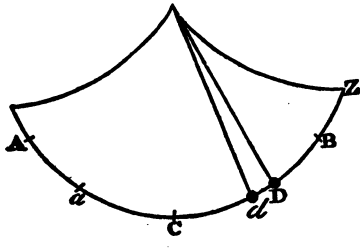


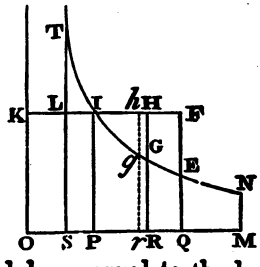
Let BC represent the arc described in the descent, C*a* the arc described in the ascent, and A*a* the difference of the arcs: and things remaining as they were constructed and demonstrated in Prop. XXV, the force with which the oscillating body is urged in any place D will be to the force of resistance as the arc CD to the arc CO, which is half of that difference A*a*. Therefore the force with which the oscillating body is urged at the beginning or the highest point of the cycloid, that is, the force of gravity, will be to the resistance as the arc of the cycloid, between that highest point and lowest point C, is to the arc CO; that is (doubling those arcs), as the whole cycloidal arc, or twice the length of the pendulum, to the arc A*a*. Q.E.D.

PROPOSITION XXIX. PROBLEM VI.

Supposing that a body oscillating in a cycloid is resisted in a duplicate ratio of the velocity: to find the resistance in each place.

Let B*a* be an arc described in one entire oscillation, C the lowest point





of the cycloid, and CZ half the whole cycloidal arc, equal to the length of the pendulum; and let it be required to find the resistance of the body in

any place D. Cut the indefinite right line OQ in the points O, S, P, Q, so that (erecting the perpendiculars OK, ST, PI, QE, and with the centre O, and the aysmptotes OK, OQ, describing the hyperbola TIGE cutting the perpendiculars ST, PI, QE in T, I, and E, and through the point I drawing KF, parallel to the asymptote OQ, meeting the asymptote OK in K, and the perpendiculars ST and QE in L and F) the hyperbolic area PIEQ may be to the hyperbolic area PITS as the arc BC, described in the descent of the body, to the arc C*a* described in the ascent; and that the area IEF may be to the area ILT as OQ to OS. Then with the perpendicular MN cut off the hyperbolic area PINM, and let that area be to the hyperbolic area PIEQ as the arc CZ to the arc BC described in the descent. And if the perpendicular RG cut off the hyperbolic area PIGR, which shall be to the area PIEQ as any arc CD to the arc BC described in the whole descent, the resistance in any place D will be to the force of gravity as the area $\frac{OR}{OQ}$ IEF — IGH to the area PINM.

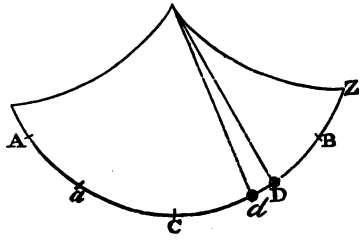
For since the forces arising from gravity with which the body is urged in the places Z, B, D, *a*, are as the arcs CZ, CB, CD, C*a* and those arcs are as the areas PINM, PIEQ, PIGR, PITS; let those areas be the exponents both of the arcs and of the forces respectively. Let D*d* be a very small space described by the body in its descent: and let it be expressed by the very small area RG*gr* comprehended between the parallels RG, *rg*; and produce *rg* to *h*, so that GH*hg* and RG*gr* may be the contemporaneous decrements of the areas IGH, PIGR. And the increment GH*hg* — $\frac{Rr}{OQ}$ IEF, or R*r* × HG — $\frac{Rr}{OQ}$ IEF, of the area $\frac{OR}{OQ}$ IEF — IGH will be to the decrement RG*gr*, or R*r* × RG, of the area PIGR, as HG — $\frac{IEF}{OQ}$ to RG; and therefore as OR × HG — $\frac{OR}{OQ}$ IEF to OR × GR or OP × PI, that is (because of the equal quantities OR × HG, OR × HR — OR × GR, ORHK — OPIK, PIHR and PIGR + IGH), as PIGR + IGH — $\frac{OR}{OQ}$ IEF to OPIK. Therefore if the area $\frac{OR}{OQ}$ IEF — IGH be called Y, and RG*gr* the decrement of the area PIGR be given, the increment of the area Y will be as PIGR — Y.

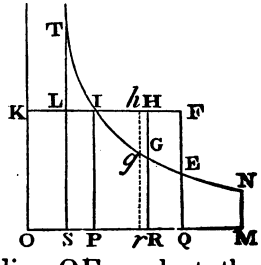
Then if V represent the force arising from the gravity, proportional to the arc CD to be described, by which the body is acted upon in D, and R be put for the resistance, V — R will be the whole force with which the body is urged in D. Therefore the increment of the velocity is as V — R and the particle of time in which it is generated conjunctly. But the velocity itself is as the contemporaneous increment of the space described di-

rectly and the same particle of time inversely. Therefore, since the resistance is, by the supposition, as the square of the velocity, the increment of the resistance will (by Lem. II) be as the velocity and the increment of the velocity conjunctly, that is, as the moment of the space and V — R conjunctly; and, therefore, if the moment of the space be given, as V — R; that is, if for the force V we put its exponent PIGR, and the resistance R be expressed by any other area Z, as PIGR — Z.

Therefore the area PIGR uniformly decreasing by the subduction of given moments, the area Y increases in proportion of PIGR — Y, and the area Z in proportion of PIGR — Z. And therefore if the areas Y and Z begin together, and at the beginning are equal, these, by the addition of equal moments, will continue to be equal; and in like manner decreasing by equal moments, will vanish together. And, *vice versa*, if they together begin and vanish, they will have equal moments and be always equal; and that, because if the resistance Z be augmented, the velocity together with the arc C*a*, described in the ascent of the body, will be diminished; and the point in which all the motion together with the resistance ceases coming nearer to the point C, the resistance vanishes sooner than the area Y. And the contrary will happen when the resistance is diminished.

Now the area Z begins and ends where the resistance is nothing, that is, at the beginning of the motion where the arc CD is equal to the arc CB,





and the right line RG falls upon the right line QE; and at the end of the motion where the arc CD is equal to the arc C*a*, and RG falls upon the right line ST. And the area Y or $\frac{OR}{OQ}$ IEF — IGH begins and ends also where the resistance is nothing, and therefore where $\frac{OR}{OQ}$ IEF and IGH are equal; that is (by the construction), where the right line RG falls successively upon the right lines QE and ST. Therefore those areas begin and vanish together, and are therefore always equal. Therefore the area $\frac{OR}{OQ}$ IEF — IGH is equal to the area Z, by which the resistance is expressed, and therefore is to the area PINM, by which the gravity is expressed, as the resistance to the gravity. Q.E.D.

Cor. 1. Therefore the resistance in the lowest place C is to the force of gravity as the area $\frac{OP}{OQ}$ IEF to the area PINM.

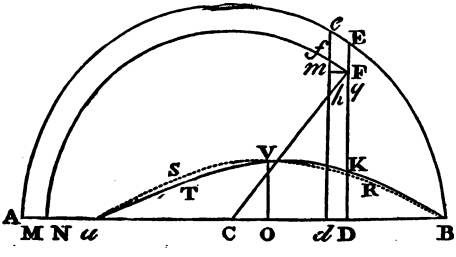
Cor. 2. But it becomes greatest where the area PIHR is to the area IEF as OR to OQ. For in that case its moment (that is, PIGR — Y) becomes nothing.

Cor. 3. Hence also may be known the velocity in each place, as being in the subduplicate ratio of the resistance, and at the beginning of the motion equal to the velocity of the body oscillating in the same cycloid without any resistance.

However, by reason of the difficulty of the calculation by which the resistance and the velocity are found by this Proposition, we have thought fit to subjoin the Proposition following.

PROPOSITION XXX. THEOREM XXIV.

If a right line aB *be equal to the arc of a cycloid which an oscillating body describes, and at each of its points* D *the perpendiculars* DK *be erected, which shall be to the length of the pendulum as the resistance of the body in the corresponding points of the arc to the force of gravity; I say, that the difference between the arc described in the whole descent and the arc described in the whole subsequent ascent drawn into half the sum of the same arcs will be equal to the area* BKa *which all those perpendiculars take up.*



Let the arc of the cycloid, described in one entire oscillation, be expressed by the right line *a*B, equal to it, and the arc which would have been described *in vacuo* by the length AB. Bisect AB in C, and the point C will represent the lowest point of the cycloid, and CD will be as the force arising from gravity, with which the body in D is urged in the direction of the tangent of the cycloid, and will have the same ratio to the length of the pendulum as the force in D has to the force of gravity. Let that force, therefore, be expressed by that length CD, and the force of gravity by the length of the pendulum; and if in DE you take DK in the same ratio to the length of the pendulum as the resistance has to the gravity, DK will be the exponent of the resistance. From the centre C with the interval CA or CB describe a semi-circle BE*e*A. Let the body describe, in the least time, the space D*d*; and, erecting the perpendiculars DE, *de*, meeting the circumference in E and *e*, they will be as the velocities which the body descending *in vacuo* from the point B would acquire in the places D and *d*. This appears by Prop. LII, Book I. Let

therefore, these velocities be expressed by those perpendiculars DE, *de*; and let DF be the velocity which it acquires in D by falling from B in the resisting medium. And if from the centre C with the interval CF we describe the circle F*f*M meeting the right lines *de* and AB in *f* and M, then M will be the place to which it would thenceforward, without farther resistance, ascend, and *df* the velocity it would acquire in *d*. Whence, also, if F*g* represent the moment of the velocity which the body D, in describing the least space D*d*, loses by the resistance of the medium; and CN be taken equal to C*g*; then will N be the place to which the body, if it met no farther resistance, would thenceforward ascend, and MN will be the decrement of the ascent arising from the loss of that velocity. Draw F*m* perpendicular to *df*, and the decrement F*g* of the velocity DF generated by the resistance DK will be to the increment *fm* of the same velocity, generated by the force CD, as the generating force DK to the generating force CD. But because of the similar triangles F*mf*, F*hg*, FDC, *fm* is to F*m* or D*d* as CD to DF; and, *ex æquo*, F*g* to D*d* as DK to DF. Also F*h* is to F*g* as DF to CF; and, *ex æquo perturbatè*, F*h* or MN to D*d* as DK to CF or CM; and therefore the sum of all the MN × CM will be equal to the sum of all the D*d* × DK. At the moveable point M suppose always a rectangular ordinate erected equal to the indeterminate CM, which by a continual motion is drawn into the whole length A*a*; and the trapezium described by that motion, or its equal, the rectangle A*a* × ½*a*B, will be equal to the sum of all the MN × CM, and therefore to the sum of all the D*d* × DK, that is, to the area BKVT*a* Q.E.D.

COR. Hence from the law of resistance, and the difference A*a* of the arcs C*a*, CB, may be collected the proportion of the resistance to the gravity nearly.

For if the resistance DK be uniform, the figure BKT*a* will be a rectangle under B*a* and DK; and thence the rectangle under ½B*a* and A*a* will be equal to the rectangle under B*a* and DK, and DK will be equal to ½A*a*. Wherefore since DK is the exponent of the resistance, and the length of the pendulum the exponent of the gravity, the resistance will be to the gravity as ½A*a* to the length of the pendulum; altogether as in Prop. XXVIII is demonstrated.

If the resistance be as the velocity, the figure BKT*a* will be nearly an ellipsis. For if a body, in a non-resisting medium, by one entire oscillation, should describe the length BA, the velocity in any place D would be as the ordinate DE of the circle described on the diameter AB. Therefore since B*a* in the resisting medium, and BA in the non-resisting one, are described nearly in the same times; and therefore the velocities in each of the points of B*a* are to the velocities in the correspondent points of the length BA nearly as B*a* is to BA, the velocity in the point D in the re-

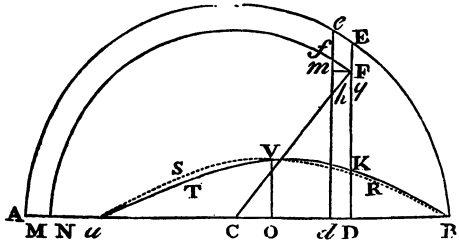
sisting medium will be as the ordinate of the circle or ellipsis described upon the diameter Ba; and therefore the figure BKVTa will be nearly an ellipsis. Since the resistance is supposed proportional to the velocity, let OV be the exponent of the resistance in the middle point O; and an ellipsis BRVSa described with the centre O, and the semi-axes OB, OV, will be nearly equal to the figure BKVTa, and to its equal the rectangle Aa × BO. Therefore Aa × BO is to OV × BO as the area of this ellipsis to OV × BO; that is, Aa is to OV as the area of the semi-circle to the square of the radius, or as 11 to 7 nearly; and, therefore, $\frac{7}{11}$Aa is to the length of the pendulum as the resistance of the oscillating body in O to its gravity.

Now if the resistance DK be in the duplicate ratio of the velocity, the figure BKVTa will be almost a parabola having V for its vertex and OV for its axis, and therefore will be nearly equal to the rectangle under $\frac{2}{3}$Ba and OV. Therefore the rectangle under $\frac{1}{2}$Ba and Aa is equal to the rectangle $\frac{2}{3}$Ba × OV, and therefore OV is equal to $\frac{3}{4}$Aa; and therefore the resistance in O made to the oscillating body is to its gravity as $\frac{3}{4}$Aa to the length of the pendulum.

And I take these conclusions to be accurate enough for practical uses. For since an ellipsis or parabola BRVSa falls in with the figure BKVTa in the middle point V, that figure, if greater towards the part BRV or VSa than the other, is less towards the contrary part, and is therefore nearly equal to it.

PROPOSITION XXXI. THEOREM XXV.

If the resistance made to an oscillating body in each of the proportional parts of the arcs described be augmented or diminished in a given ratio, the difference between the arc described in the descent and the arc described in the subsequent ascent will be augmented or diminished in the same ratio.



For that difference arises from the retardation of the pendulum by the resistance of the medium, and therefore is as the whole retardation and the retarding resistance proportional thereto. In the foregoing Proposition the rectangle under the right line $\frac{1}{2}a$B and the difference Aa of the arcs CB, Ca, was equal to the area BKTa. And that area, if the length aB remains, is augmented or diminished in the ratio of the ordinates DK; that is, in the ratio of the resistance and is therefore as the length aB and the resistance conjunctly. And therefore the rectangle under Aa and $\frac{1}{2}a$B is as aB and the resistance conjunctly, and therefore Aa is as the resistance. Q.E.D.

COR. 1. Hence if the resistance be as the velocity, the difference of the arcs in the same medium will be as the whole arc described: and the contrary.

COR. 2. If the resistance be in the duplicate ratio of the velocity, that difference will be in the duplicate ratio of the whole arc: and the contrary.

COR. 3. And universally, if the resistance be in the triplicate or any other ratio of the velocity, the difference will be in the same ratio of the whole arc: and the contrary.

COR. 4. If the resistance be partly in the simple ratio of the velocity, and partly in the duplicate ratio of the same, the difference will be partly in the ratio of the whole arc, and partly in the duplicate ratio of it: and the contrary. So that the law and ratio of the resistance will be the same for the velocity as the law and ratio of that difference for the length of the arc.

COR. 5. And therefore if a pendulum describe successively unequal arcs, and we can find the ratio of the increment or decrement of this difference for the length of the arc described, there will be had also the ratio of the increment or decrement of the resistance for a greater or less velocity.

GENERAL SCHOLIUM.

From these propositions we may find the resistance of mediums by pendulums oscillating therein. I found the resistance of the air by the following experiments. I suspended a wooden globe or ball weighing $57\frac{7}{2}\frac{7}{3}$ ounces troy, its diameter $6\frac{7}{8}$ *London* inches, by a fine thread on a firm hook, so that the distance between the hook and the centre of oscillation of the globe was $10\frac{1}{2}$ feet. I marked on the thread a point 10 feet and 1 inch distant from the centre of suspension; and even with that point I placed a ruler divided into inches, by the help whereof I observed the lengths of the arcs described by the pendulum. Then I numbered the oscillations in which the globe would lose $\frac{1}{8}$ part of its motion. If the pendulum was drawn aside from the perpendicular to the distance of 2 inches, and thence let go, so that in its whole descent it described an arc of 2 inches, and in the first whole oscillation, compounded of the descent and subsequent ascent, an arc of almost 4 inches, the same in 164 oscillations lost $\frac{1}{8}$ part of its motion, so as in its last ascent to describe an arc of $1\frac{3}{4}$ inches. If in the first descent it described an arc of 4 inches, it lost $\frac{1}{8}$ part of its motion in 121 oscillations, so as in its last ascent to describe an arc of $3\frac{1}{2}$ inches. If in the first descent it described an arc of 8, 16, 32, or 64 inches, it lost $\frac{1}{8}$ part of its motion in 69, $35\frac{1}{2}$, $18\frac{1}{2}$, $9\frac{2}{3}$ oscillations, respectively. Therefore the difference between the arcs described in the first descent and the last ascent was in the 1st, 2d, 3d, 4th, 5th, 6th cases, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, 8 inches respectively. Divide those differences by the number of oscillations in each case, and in one mean oscillation, wherein an arc of $3\frac{3}{4}$, $7\frac{1}{2}$, 15, 30

60, 120 inches was described, the difference of the arcs described in the descent and subsequent ascent will be $\frac{1}{6\,5\,6}$, $\frac{1}{2\,4\,3}$, $\frac{1}{6\,5}$, $\frac{4}{7\,1}$, $\frac{8}{3\,7}$, $\frac{2\,4}{6\,5}$ parts of an inch, respectively. But these differences in the greater oscillations are in the duplicate ratio of the arcs described nearly, but in lesser oscillations something greater than in that ratio; and therefore (by Cor. 2, Prop. XXXI of this Book) the resistance of the globe, when it moves very swift, is in the duplicate ratio of the velocity, nearly; and when it moves slowly, somewhat greater than in that ratio.

Now let V represent the greatest velocity in any oscillation, and let A, B, and C be given quantities, and let us suppose the difference of the arcs to be $AV + BV^{\frac{3}{2}} + CV^2$. Since the greatest velocities are in the cycloid as $\frac{1}{2}$ the arcs described in oscillating, and in the circle as $\frac{1}{2}$ the chords of those arcs; and therefore in equal arcs are greater in the cycloid than in the circle in the ratio of $\frac{1}{2}$ the arcs to their chords; but the times in the circle are greater than in the cycloid, in a reciprocal ratio of the velocity; it is plain that the differences of the arcs (which are as the resistance and the square of the time conjunctly) are nearly the same in both curves: for in the cycloid those differences must be on the one hand augmented, with the resistance, in about the duplicate ratio of the arc to the chord, because of the velocity augmented in the simple ratio of the same; and on the other hand diminished, with the square of the time, in the same duplicate ratio. Therefore to reduce these observations to the cycloid, we must take the same differences of the arcs as were observed in the circle, and suppose the greatest velocities analogous to the half, or the whole arcs. that is, to the numbers $\frac{1}{2}$, 1, 2, 4, 8, 16. Therefore in the 2d, 4th, and 6th cases, put 1, 4, and 16 for V; and the difference of the arcs in the 2d case will become $\frac{\frac{1}{2}}{121} = A + B + C$; in the 4th case, $\frac{2}{35\frac{1}{2}} = 4A + 8B + 16C$; in the 6th case, $\frac{8}{9\frac{2}{3}} = 16A + 64B + 256C$. These equations reduced give $A = 0{,}0000916$, $B = 0{,}0010847$, and $C = 0{,}0029558$. Therefore the difference of the arcs is as $0{,}0000916V + 0{,}0010847V^{\frac{3}{2}} + 0{,}0029558V^2$: and therefore since (by Cor. Prop. XXX, applied to this case) the resistance of the globe in the middle of the arc described in oscillating, where the velocity is V, is to its weight as $\frac{7}{11}AV + \frac{7}{10}BV^{\frac{3}{2}} + \frac{3}{4}CV^2$ to the length of the pendulum, if for A, B, and C you put the numbers found, the resistance of the globe will be to its weight as $0{,}0000583V + 0{,}0007593V^{\frac{3}{2}} + 0{,}0022169V^2$ to the length of the pendulum between the centre of suspension and the ruler, that is, to 121 inches. Therefore since V in the second case represents 1, in the 4th case 4, and in the 6th case 16, the resistance will be to the weight of the globe, in the 2d case, as 0,0030345 to 121; in the 4th, as 0,041748 to 121; in the 6th, as 0,61705 to 121.

The arc, which the point marked in the thread described in the 6th case, was of $120 - \frac{8}{9\frac{2}{3}}$, or $119\frac{5}{19}$ inches. And therefore since the radius was 121 inches, and the length of the pendulum between the point of suspension and the centre of the globe was 126 inches, the arc which the centre of the globe described was $124\frac{3}{51}$ inches. Because the greatest velocity of the oscillating body, by reason of the resistance of the air, does not fall on the lowest point of the arc described, but near the middle place of the whole arc, this velocity will be nearly the same as if the globe in its whole descent in a non-resisting medium should describe $62\frac{3}{8\frac{1}{2}}$ inches, the half of that arc, and that in a cycloid, to which we have above reduced the motion of the pendulum; and therefore that velocity will be equal to that which the globe would acquire by falling perpendicularly from a height equal to the versed sine of that arc. But that versed sine in the cycloid is to that arc $62\frac{3}{8\frac{1}{2}}$ as the same arc to twice the length of the pendulum 252, and therefore equal to 15,278 inches. Therefore the velocity of the pendulum is the same which a body would acquire by falling, and in its fall describing a space of 15,278 inches. Therefore with such a velocity the globe meets with a resistance which is to its weight as 0,61705 to 121, or (if we take that part only of the resistance which is in the duplicate ratio of the velocity) as 0,56752 to 121.

I found, by an hydrostatical experiment, that the weight of this wooden globe was to the weight of a globe of water of the same magnitude as 55 to 97: and therefore since 121 is to 213,4 in the same ratio, the resistance made to this globe of water, moving forwards with the above-mentioned velocity, will be to its weight as 0,56752 to 213,4, that is, as 1 to $376\frac{1}{5\frac{1}{0}}$. Whence since the weight of a globe of water, in the time in which the globe with a velocity uniformly continued describes a length of 30,556 inches, will generate all that velocity in the falling globe, it is manifest that the force of resistance uniformly continued in the same time will take away a velocity, which will be less than the other in the ratio of 1 to $376\frac{1}{5\frac{1}{0}}$, that is, the $\frac{1}{376\frac{1}{5\frac{1}{0}}}$ part of the whole velocity. And therefore in the time that the globe, with the same velocity uniformly continued, would describe the length of its semi-diameter, or $3\frac{7}{10}$ inches, it would lose the $\frac{1}{3\,3\,4\,2}$ part of its motion.

I also counted the oscillations in which the pendulum lost $\frac{1}{4}$ part of its motion. In the following table the upper numbers denote the length of the arc described in the first descent, expressed in inches and parts of an inch; the middle numbers denote the length of the arc described in the last ascent; and in the lowest place are the numbers of the oscillations. I give an account of this experiment, as being more accurate than that in which

only $\frac{1}{8}$ part of the motion was lost. I leave the calculation to such as are disposed to make it.

First descent . . .	2	4	8	16	32	64
Last ascent . . .	$1\frac{1}{2}$	3	6	12	24	48
Numb. of oscill. . .	374	272	$162\frac{1}{2}$	$83\frac{1}{3}$	$41\frac{2}{3}$	$22\frac{1}{3}$

I afterward suspended a leaden globe of 2 inches in diameter, weighing $26\frac{1}{4}$ ounces troy by the same thread, so that between the centre of the globe and the point of suspension there was an interval of $10\frac{1}{2}$ feet, and I counted the oscillations in which a given part of the motion was lost. The first of the following tables exhibits the number of oscillations in which $\frac{1}{8}$ part of the whole motion was lost; the second the number of oscillations in which there was lost $\frac{1}{4}$ part of the same.

First descent	1	2	4	8	16	32	64
Last ascent	$\frac{7}{8}$	$\frac{7}{4}$	$3\frac{1}{2}$	7	14	28	56
Numb. of oscill. . .	226	228	193	140	$90\frac{1}{2}$	53	30
First descent	1	2	4	8	16	32	64
Last ascent	$\frac{3}{4}$	$1\frac{1}{2}$	3	6	12	24	48
Numb. of oscill. . .	510	518	420	318	204	121	70

Selecting in the first table the 3d, 5th, and 7th observations, and expressing the greatest velocities in these observations particularly by the numbers 1, 4, 16 respectively, and generally by the quantity V as above, there will come out in the 3d observation $\frac{\frac{1}{2}}{193} = A + B + C$, in the 5th observation $\frac{2}{90\frac{1}{2}} = 4A + 8B + 16C$, in the 7th observation $\frac{8}{30} = 16A + 64B + 256C$. These equations reduced give $A = 0{,}001414$, $B = 0{,}000297$, $C = 0{,}000879$. And thence the resistance of the globe moving with the velocity V will be to its weight $26\frac{1}{4}$ ounces in the same ratio as $0{,}0009V + 0{,}000208V^{\frac{3}{2}} + 0{,}000659V^2$ to 121 inches, the length of the pendulum. And if we regard that part only of the resistance which is in the duplicate ratio of the velocity, it will be to the weight of the globe as $0{,}000659V^2$ to 121 inches. But this part of the resistance in the first experiment was to the weight of the wooden globe of $57\frac{7}{2}\frac{7}{2}$ ounces as $0{,}002217V^2$ to 121; and thence the resistance of the wooden globe is to the resistance of the leaden one (their velocities being equal) as $57\frac{7}{2}\frac{7}{2}$ into 0,002217 to $26\frac{1}{4}$ into 0,000659, that is, as $7\frac{1}{3}$ to 1. The diameters of the two globes were $6\frac{7}{8}$ and 2 inches, and the squares of these are to each other as $47\frac{1}{4}$ and 4, or $11\frac{3}{16}$ and 1, nearly. Therefore the resistances of these equally swift globes were in less than a duplicate ratio of the diameters. But we have not yet considered the resistance of the thread, which was certainly very considerable, and ought to be subducted from the resistance of the pendulums here found. I could not determine this accurately, but I found it

greater than a third part of the whole resistance of the lesser pendulum; and thence I gathered that the resistances of the globes, when the resistance of the thread is subducted, are nearly in the duplicate ratio of their diameters. For the ratio of $7\frac{1}{2} - \frac{1}{3}$ to $1 - \frac{1}{3}$, or $10\frac{1}{2}$ to 1 is not very different from the duplicate ratio of the diameters $11\frac{1}{16}$ to 1.

Since the resistance of the thread is of less moment in greater globes, I tried the experiment also with a globe whose diameter was $18\frac{3}{4}$ inches. The length of the pendulum between the point of suspension and the centre of oscillation was $122\frac{1}{2}$ inches, and between the point of suspension and the knot in the thread $109\frac{1}{2}$ inches. The arc described by the knot at the first descent of the pendulum was 32 inches. The arc described by the same knot in the last ascent after five oscillations was 28 inches. The sum of the arcs, or the whole arc described in one mean oscillation, was 60 inches. The difference of the arcs 4 inches. The $\frac{1}{16}$ part of this, or the difference between the descent and ascent in one mean oscillation, is $\frac{2}{5}$ of an inch. Then as the radius $109\frac{1}{2}$ to the radius $122\frac{1}{2}$, so is the whole arc of 60 inches described by the knot in one mean oscillation to the whole arc of $67\frac{1}{8}$ inches described by the centre of the globe in one mean oscillation; and so is the difference $\frac{2}{5}$ to a new difference 0,4475. If the length of the arc described were to remain, and the length of the pendulum should be augmented in the ratio of 126 to $122\frac{1}{2}$, the time of the oscillation would be augmented, and the velocity of the pendulum would be diminished in the subduplicate of that ratio; so that the difference 0,4475 of the arcs described in the descent and subsequent ascent would remain. Then if the arc described be augmented in the ratio of $124\frac{3}{11}$ to $67\frac{1}{8}$, that difference 0.4475 would be augmented in the duplicate of that ratio, and so would become 1,5295. These things would be so upon the supposition that the resistance of the pendulum were in the duplicate ratio of the velocity. Therefore if the pendulum describe the whole arc of $124\frac{3}{11}$ inches, and its length between the point of suspension and the centre of oscillation be 126 inches, the difference of the arcs described in the descent and subsequent ascent would be 1,5295 inches. And this difference multiplied into the weight of the pendulous globe, which was 208 ounces, produces 318,136. Again; in the pendulum above-mentioned, made of a wooden globe, when its centre of oscillation, being 126 inches from the point of suspension, described the whole arc of $124\frac{3}{11}$ inches, the difference of the arcs described in the descent and ascent was $\frac{126}{121}$ into $\frac{8}{9\frac{2}{3}}$. This multiplied into the weight of the globe, which was $57\frac{7}{12}$ ounces, produces 49,396. But I multiply these differences into the weights of the globes, in order to find their resistances. For the differences arise from the resistances, and are as the resistances directly and the weights inversely. Therefore the resistances are as the numbers 318,136 and 49,396. But that part of the resistance

of the lesser globe, which is in the duplicate ratio of the velocity, was to the whole resistance as 0,56752 to 0,61675, that is, as 45,453 to 49,396; whereas that part of the resistance of the greater globe is almost equal to its whole resistance; and so those parts are nearly as 318,136 and 45,453, that is, as 7 and 1. But the diameters of the globes are $18\frac{3}{4}$ and $6\frac{7}{8}$; and their squares $351\frac{9}{16}$ and $47\frac{1}{64}$ are as 7,438 and 1, that is, as the resistances of the globes 7 and 1, nearly. The difference of these ratios is scarce greater than may arise from the resistance of the thread. Therefore those parts of the resistances which are, when the globes are equal, as the squares of the velocities, are also, when the velocities are equal, as the squares of the diameters of the globes.

But the greatest of the globes I used in these experiments was not perfectly spherical, and therefore in this calculation I have, for brevity's sake, neglected some little niceties; being not very solicitous for an accurate calculus in an experiment that was not very accurate. So that I could wish that these experiments were tried again with other globes, of a larger size, more in number, and more accurately formed; since the demonstration of a vacuum depends thereon. If the globes be taken in a geometrical proportion, as suppose whose diameters are 4, 8, 16, 32 inches; one may collect from the progression observed in the experiments what would happen if the globes were still larger.

In order to compare the resistances of different fluids with each other, I made the following trials. I procured a wooden vessel 4 feet long, 1 foot broad, and 1 foot high. This vessel, being uncovered, I filled with spring water, and, having immersed pendulums therein, I made them oscillate in the water. And I found that a leaden globe weighing $166\frac{1}{2}$ ounces, and in diameter $3\frac{5}{8}$ inches, moved therein as it is set down in the following table; the length of the pendulum from the point of suspension to a certain point marked in the thread being 126 inches, and to the centre of oscillation $134\frac{3}{8}$ inches.

The arc described in the first descent, by a point marked in the thread was inches.	64 . 32 . 16 . 8 . 4 . 2 . 1 . $\frac{1}{2}$. $\frac{1}{4}$
The arc described in the last ascent was inches.	48 . 24 . 12 . 6 . 3 . $1\frac{1}{2}$. $\frac{3}{4}$. $\frac{3}{8}$. $\frac{3}{16}$
The difference of the arcs, proportional to the motion lost, was inches.	16 . 8 . 4 . 2 . 1 . $\frac{1}{2}$. $\frac{1}{4}$. $\frac{1}{8}$. $\frac{1}{16}$
The number of the oscillations in water.	$\frac{2}{16}$. $1\frac{1}{2}$. 3 . 7 . $11\frac{1}{4}$. $12\frac{2}{3}$. $13\frac{1}{4}$
The number of the oscillations in air.	$85\frac{1}{2}$. 287 . 535

In the experiments of the 4th column there were equal motions lost in 535 oscillations made in the air, and $1\frac{1}{5}$ in water. The oscillations in the air were indeed a little swifter than those in the water. But if the oscillations in the water were accelerated in such a ratio that the motions of the pendulums might be equally swift in both mediums, there would be still the same number $1\frac{1}{5}$ of oscillations in the water, and by these the same quantity of motion would be lost as before; because the resistance is increased, and the square of the time diminished in the same duplicate ratio. The pendulums, therefore, being of equal velocities, there were equal motions lost in 535 oscillations in the air, and $1\frac{1}{5}$ in the water; and therefore the resistance of the pendulum in the water is to its resistance in the air as 535 to $1\frac{1}{5}$. This is the proportion of the whole resistances in the case of the 4th column.

Now let $AV + CV^2$ represent the difference of the arcs described in the descent and subsequent ascent by the globe moving in air with the greatest velocity V; and since the greatest velocity is in the case of the 4th column to the greatest velocity in the case of the 1st column as 1 to 8; and that difference of the arcs in the case of the 4th column to the difference in the case of the 1st column as $\frac{2}{535}$ to $\frac{16}{85\frac{1}{2}}$, or as $85\frac{1}{2}$ to 4280; put in these cases 1 and 8 for the velocities, and $85\frac{1}{2}$ and 4280 for the differences of the arcs, and $A + C$ will be $= 85\frac{1}{2}$, and $8A + 64C = 4280$ or $A + 8C = 535$; and then by reducing these equations, there will come out $7C = 449\frac{1}{2}$ and $C = 64\frac{3}{14}$ and $A = 21\frac{2}{7}$; and therefore the resistance, which is as $\frac{7}{11}AV + \frac{3}{4}CV^2$, will become as $13\frac{6}{11}V + 48\frac{9}{56}V^2$. Therefore in the case of the 4th column, where the velocity was 1, the whole resistance is to its part proportional to the square of the velocity as $13\frac{6}{11} + 48\frac{9}{56}$ or $61\frac{2}{11}$ to $48\frac{9}{56}$; and therefore the resistance of the pendulum in water is to that part of the resistance in air, which is proportional to the square of the velocity, and which in swift motions is the only part that deserves consideration, as $61\frac{2}{11}$ to $48\frac{9}{56}$ and 535 to $1\frac{1}{5}$ conjunctly, that is, as 571 to 1. If the whole thread of the pendulum oscillating in the water had been immersed its resistance would have been still greater; so that the resistance of the pendulum oscillating in the water, that is, that part which is proportional to the square of the velocity, and which only needs to be considered in swift bodies, is to the resistance of the same whole pendulum, oscillating in air with the same velocity, as about 850 to 1, that is as, the density of water to the density of air, nearly.

In this calculation we ought also to have taken in that part of the resistance of the pendulum in the water which was as the square of the velocity; but I found (which will perhaps seem strange) that the resistance in the water was augmented in more than a duplicate ratio of the velocity. In searching after the cause, I thought upon this, that the vessel was too

narrow for the magnitude of the pendulous globe, and by its narrowness obstructed the motion of the water as it yielded to the oscillating globe. For when I immersed a pendulous globe, whose diameter was one inch only, the resistance was augmented nearly in a duplicate ratio of the velocity. I tried this by making a pendulum of two globes, of which the lesser and lower oscillated in the water, and the greater and higher was fastened to the thread just above the water, and, by oscillating in the air, assisted the motion of the pendulum, and continued it longer. The experiments made by this contrivance proved according to the following table.

Arc descr. in first descent . .	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
Arc descr. in last ascent . .	12	6	3	$1\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{3}{16}$
Diff. of arcs, proport. to motion lost	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
Number of oscillations . . .	$3\frac{3}{8}$	$6\frac{1}{2}$	$12\frac{1}{12}$	$21\frac{1}{2}$	34	53	$62\frac{1}{2}$

In comparing the resistances of the mediums with each other, I also caused iron pendulums to oscillate in quicksilver. The length of the iron wire was about 3 feet, and the diameter of the pendulous globe about $\frac{1}{3}$ of an inch. To the wire, just above the quicksilver, there was fixed another leaden globe of a bigness sufficient to continue the motion of the pendulum for some time. Then a vessel, that would hold about 3 pounds of quicksilver, was filled by turns with quicksilver and common water, that, by making the pendulum oscillate successively in these two different fluids, I might find the proportion of their resistances; and the resistance of the quicksilver proved to be to the resistance of water as about 13 or 14 to 1; that is, as the density of quicksilver to the density of water. When I made use of a pendulous globe something bigger, as of one whose diameter was about $\frac{1}{2}$ or $\frac{2}{3}$ of an inch, the resistance of the quicksilver proved to be to the resistance of the water as about 12 or 10 to 1. But the former experiment is more to be relied on, because in the latter the vessel was too narrow in proportion to the magnitude of the immersed globe; for the vessel ought to have been enlarged together with the globe. I intended to have repeated these experiments with larger vessels, and in melted metals, and other liquors both cold and hot; but I had not leisure to try all: and besides, from what is already described, it appears sufficiently that the resistance of bodies moving swiftly is nearly proportional to the densities of the fluids in which they move. I do not say accurately; for more tenacious fluids, of equal density, will undoubtedly resist more than those that are more liquid; as cold oil more than warm, warm oil more than rain-water, and water more than spirit of wine. But in liquors, which are sensibly fluid enough, as in air, in salt and fresh water, in spirit of wine, of turpentine, and salts, in oil cleared of its fæces by distillation and warmed, in oil of vitriol, and in mercury, and melted metals, and any other such like, that are fluid enough to retain for some time the motion impressed

upon them by the agitation of the vessel, and which being poured out are easily resolved into drops, I doubt not but the rule already laid down may be accurate enough, especially if the experiments be made with larger pendulous bodies and more swiftly moved.

Lastly, since it is the opinion of some that there is a certain æthereal medium extremely rare and subtile, which freely pervades the pores of all bodies; and from such a medium, so pervading the pores of bodies, some resistance must needs arise; in order to try whether the resistance, which we experience in bodies in motion, be made upon their outward superficies only, or whether their internal parts meet with any considerable resistance upon their superficies, I thought of the following experiment. I suspended a round deal box by a thread 11 feet long, on a steel hook, by means of a ring of the same metal, so as to make a pendulum of the aforesaid length. The hook had a sharp hollow edge on its upper part, so that the upper arc of the ring pressing on the edge might move the more freely; and the thread was fastened to the lower arc of the ring. The pendulum being thus prepared, I drew it aside from the perpendicular to the distance of about 6 feet, and that in a plane perpendicular to the edge of the hook, lest the ring, while the pendulum oscillated, should slide to and fro on the edge of the hook: for the point of suspension, in which the ring touches the hook, ought to remain immovable. I therefore accurately noted the place to which the pendulum was brought, and letting it go, I marked three other places, to which it returned at the end of the 1st, 2d, and 3d oscillation. This I often repeated, that I might find those places as accurately as possible. Then I filled the box with lead and other heavy metals that were near at hand. But, first, I weighed the box when empty, and that part of the thread that went round it, and half the remaining part, extended between the hook and the suspended box; for the thread so extended always acts upon the pendulum, when drawn aside from the perpendicular, with half its weight. To this weight I added the weight of the air contained in the box. And this whole weight was about $\frac{1}{778}$ of the weight of the box when filled with the metals. Then because the box when full of the metals, by extending the thread with its weight, increased the length of the pendulum, I shortened the thread so as to make the length of the pendulum, when oscillating, the same as before. Then drawing aside the pendulum to the place first marked, and letting it go, I reckoned about 77 oscillations before the box returned to the second mark, and as many afterwards before it came to the third mark, and as many after that before it came to the fourth mark. From whence I conclude that the whole resistance of the box, when full, had not a greater proportion to the resistance of the box, when empty, than 78 to 77. For if their resistances were equal, the box, when full, by reason of its *vis insita*, which was 78 times greater than the *vis insita* of the same when empty, ought to have continued its oscillating motion so

much the longer, and therefore to have returned to those marks at the end of 78 oscillations. But it returned to them at the end of 77 oscillations.

Let, therefore, A represent the resistance of the box upon its external superficies, and B the resistance of the empty box on its internal superficies; and if the resistances to the internal parts of bodies equally swift be as the matter, or the number of particles that are resisted, then 78B will be the resistance made to the internal parts of the box, when full; and therefore the whole resistance A + B of the empty box will be to the whole resistance A + 78B of the full box as 77 to 78, and, by division, A + B to 77B as 77 to 1; and thence A + B to B as 77 × 77 to 1, and, by division again, A to B as 5928 to 1. Therefore the resistance of the empty box in its internal parts will be above 5000 times less than the resistance on its external superficies. This reasoning depends upon the supposition that the greater resistance of the full box arises not from any other latent cause, but only from the action of some subtile fluid upon the included metal.

This experiment is related by memory, the paper being lost in which I had described it; so that I have been obliged to omit some fractional parts, which are slipt out of my memory; and I have no leisure to try it again. The first time I made it, the hook being weak, the full box was retarded sooner. The cause I found to be, that the hook was not strong enough to bear the weight of the box; so that, as it oscillated to and fro, the hook was bent sometimes this and sometimes that way. I therefore procured a hook of sufficient strength, so that the point of suspension might remain unmoved, and then all things happened as is above described.

SECTION VII.

Of the motion of fluids, and the resistance made to projected bodies.

PROPOSITION XXXII. THEOREM XXVI.

Suppose two similar systems of bodies consisting of an equal number of particles, and let the correspondent particles be similar and proportional, each in one system to each in the other, and have a like situation among themselves, and the same given ratio of density to each other; and let them begin to move among themselves in proportional times, and with like motions (that is, those in one system among one another, and those in the other among one another). And if the particles that are in the same system do not touch one another, except in the moments of reflexion; nor attract, nor repel each other, except with accelerative forces that are as the diameters of the correspondent particles inversely, and the squares of the velocities directly; I say, that the particles of those systems will continue to move among themselves with like motions and in proportional times.

Like bodies in like situations are said to be moved among themselves with like motions and in proportional times, when their situations at the end of those times are always found alike in respect of each other; as suppose we compare the particles in one system with the correspondent particles in the other. Hence the times will be proportional, in which similar and proportional parts of similar figures will be described by correspondent particles. Therefore if we suppose two systems of this kind, the correspondent particles, by reason of the similitude of the motions at their beginning, will continue to be moved with like motions, so long as they move without meeting one another; for if they are acted on by no forces, they will go on uniformly in right lines, by the 1st Law. But if they do agitate one another with some certain forces, and those forces are as the diameters of the correspondent particles inversely and the squares of the velocities directly, then, because the particles are in like situations, and their forces are proportional, the whole forces with which correspondent particles are agitated, and which are compounded of each of the agitating forces (by Corol. 2 of the Laws), will have like directions, and have the same effect as if they respected centres placed alike among the particles; and those whole forces will be to each other as the several forces which compose them, that is, as the diameters of the correspondent particles inversely, and the squares of the velocities directly: and therefore will cause

correspondent particles to continue to describe like figures. These things will be so (by Cor. 1 and 8, Prop. IV., Book 1), if those centres are at rest but if they are moved, yet by reason of the similitude of the translations, their situations among the particles of the system will remain similar, so that the changes introduced into the figures described by the particles will still be similar. So that the motions of correspondent and similar particles will continue similar till their first meeting with each other; and thence will arise similar collisions, and similar reflexions; which will again beget similar motions of the particles among themselves (by what was just now shown), till they mutually fall upon one another again, and so on *ad infinitum*.

COR. 1. Hence if any two bodies, which are similar and in like situations to the correspondent particles of the systems, begin to move amongst them in like manner and in proportional times, and their magnitudes and densities be to each other as the magnitudes and densities of the corresponding particles, these bodies will continue to be moved in like manner and in proportional times; for the case of the greater parts of both systems and of the particles is the very same.

COR. 2. And if all the similar and similarly situated parts of both systems be at rest among themselves; and two of them, which are greater than the rest, and mutually correspondent in both systems, begin to move in lines alike posited, with any similar motion whatsoever, they will excite similar motions in the rest of the parts of the systems, and will continue to move among those parts in like manner and in proportional times; and will therefore describe spaces proportional to their diameters.

PROPOSITION XXXIII. THEOREM XXVII.

The same things being supposed, I say, that the greater parts of the systems are resisted in a ratio compounded of the duplicate ratio of their velocities, and the duplicate ratio of their diameters, and the simple ratio of the density of the parts of the systems.

For the resistance arises partly from the centripetal or centrifugal forces with which the particles of the system mutually act on each other, partly from the collisions and reflexions of the particles and the greater parts. The resistances of the first kind are to each other as the whole motive forces from which they arise, that is, as the whole accelerative forces and the quantities of matter in corresponding parts; that is (by the supposition), as the squares of the velocities directly, and the distances of the corresponding particles inversely, and the quantities of matter in the correspondent parts directly: and therefore since the distances of the particles in one system are to the correspondent distances of the particles of the other as the diameter of one particle or part in the former system to the

diameter of the correspondent particle or part in the other, and since the quantities of matter are as the densities of the parts and the cubes of the diameters; the resistances are to each other as the squares of the velocities and the squares of the diameters and the densities of the parts of the systems. Q.E.D. The resistances of the latter sort are as the number of correspondent reflexions and the forces of those reflexions conjunctly; but the number of the reflexions are to each other as the velocities of the corresponding parts directly and the spaces between their reflexions inversely. And the forces of the reflexions are as the velocities and the magnitudes and the densities of the corresponding parts conjunctly; that is, as the velocities and the cubes of the diameters and the densities of the parts. And, joining all these ratios, the resistances of the corresponding parts are to each other as the squares of the velocities and the squares of the diameters and the densities of the parts conjunctly. Q.E.D.

COR. 1. Therefore if those systems are two elastic fluids, like our air, and their parts are at rest among themselves; and two similar bodies proportional in magnitude and density to the parts of the fluids, and similarly situated among those parts, be any how projected in the direction of lines similarly posited; and the accelerative forces with which the particles of the fluids mutually act upon each other are as the diameters of the bodies projected inversely and the squares of their velocities directly; those bodies will excite similar motions in the fluids in proportional times, and will describe similar spaces and proportional to their diameters.

COR. 2. Therefore in the same fluid a projected body that moves swiftly meets with a resistance that is, in the duplicate ratio of its velocity, nearly. For if the forces with which distant particles act mutually upon one another should be augmented in the duplicate ratio of the velocity, the projected body would be resisted in the same duplicate ratio accurately; and therefore in a medium, whose parts when at a distance do not act mutually with any force on one another, the resistance is in the duplicate ratio of the velocity accurately. Let there be, therefore, three mediums A, B, C, consisting of similar and equal parts regularly disposed at equal distances. Let the parts of the mediums A and B recede from each other with forces that are among themselves as T and V; and let the parts of the medium C be entirely destitute of any such forces. And if four equal bodies D, E, F, G, move in these mediums, the two first D and E in the two first A and B, and the other two F and G in the third C; and if the velocity of the body D be to the velocity of the body E, and the velocity of the body F to the velocity of the body G, in the subduplicate ratio of the force T to the force V; the resistance of the body D to the resistance of the body E, and the resistance of the body F to the resistance of the body G, will be in the duplicate ratio of the velocities; and therefore the resistance of the body D will be to the resistance of the body F as the re-

sistance of the body E to the resistance of the body G. Let the bodies D and F be equally swift, as also the bodies E and G; and, augmenting the velocities of the bodies D and F in any ratio, and diminishing the forces of the particles of the medium B in the duplicate of the same ratio, the medium B will approach to the form and condition of the medium C at pleasure; and therefore the resistances of the equal and equally swift bodies E and G in these mediums will perpetually approach to equality so that their difference will at last become less than any given. Therefore since the resistances of the bodies D and F are to each other as the resistances of the bodies E and G, those will also in like manner approach to the ratio of equality. Therefore the bodies D and F, when they move with very great swiftness, meet with resistances very nearly equal; and therefore since the resistance of the body F is in a duplicate ratio of the velocity, the resistance of the body D will be nearly in the same ratio.

Cor. 3. The resistance of a body moving very swift in an elastic fluid is almost the same as if the parts of the fluid were destitute of their centrifugal forces, and did not fly from each other; if so be that the elasticity of the fluid arise from the centrifugal forces of the particles, and the velocity be so great as not to allow the particles time enough to act.

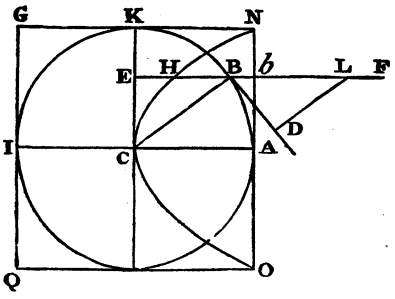
Cor. 4. Therefore, since the resistances of similar and equally swift bodies, in a medium whose distant parts do not fly from each other, are as the squares of the diameters, the resistances made to bodies moving with very great and equal velocities in an elastic fluid will be as the squares of the diameters, nearly.

Cor. 5. And since similar, equal, and equally swift bodies, moving through mediums of the same density, whose particles do not fly from each other mutually, will strike against an equal quantity of matter in equal times, whether the particles of which the medium consists be more and smaller, or fewer and greater, and therefore impress on that matter an equal quantity of motion, and in return (by the 3d Law of Motion) suffer an equal re-action from the same, that is, are equally resisted; it is manifest, also, that in elastic fluids of the same density, when the bodies move with extreme swiftness, their resistances are nearly equal, whether the fluids consist of gross parts, or of parts ever so subtile. For the resistance of projectiles moving with exceedingly great celerities is not much diminished by the subtilty of the medium.

Cor. 6. All these things are so in fluids whose elastic force takes its rise from the centrifugal forces of the particles. But if that force arise from some other cause, as from the expansion of the particles after the manner of wool, or the boughs of trees, or any other cause, by which the particles are hindered from moving freely among themselves, the resistance, by reason of the lesser fluidity of the medium, will be greater than in the Corollaries above.

PROPOSITION XXXIV. THEOREM XXVIII.

If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe, and a cylinder described on equal diameters move with equal velocities in the direction of the axis of the cylinder, the resistance of the globe will be but half so great as that of the cylinder.

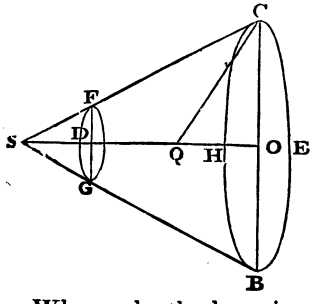


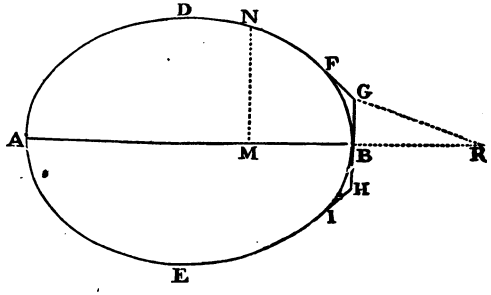
For since the action of the medium upon the body is the same (by Cor. 5 of the Laws) whether the body move in a quiescent medium, or whether the particles of the medium impinge with the same velocity upon the quiescent body, let us consider the body as if it were quiescent, and see with what force it would be impelled by the moving medium. Let, therefore, ABKI represent a spherical body described from the centre C with the semi-diameter CA, and let the particles of the medium impinge with a given velocity upon that spherical body in the directions of right lines parallel to AC; and let FB be one of those right lines. In FB take LB equal to the semi-diameter CB, and draw BD touching the sphere in B. Upon KC and BD let fall the perpendiculars BE, LD; and the force with which a particle of the medium, impinging on the globe obliquely in the direction FB, would strike the globe in B, will be to the force with which the same particle, meeting the cylinder ONGQ described about the globe with the axis ACI, would strike it perpendicularly in b, as LD to LB, or BE to BC. Again; the efficacy of this force to move the globe, according to the direction of its incidence FB or AC, is to the efficacy of the same to move the globe, according to the direction of its determination, that is, in the direction of the right line BC in which it impels the globe directly, as BE to BC. And, joining these ratios, the efficacy of a particle, falling upon the globe obliquely in the direction of the right line FB, to move the globe in the direction of its incidence, is to the efficacy of the same particle falling in the same line perpendicularly on the cylinder, to move it in the same direction, as BE^2 to BC^2. Therefore if in bE, which is perpendicular to the circular base of the cylinder NAO, and equal to the radius AC, we take bH equal to $\frac{BE^2}{CB}$; then bH will be to bE as the effect of the particle upon the globe to the effect of the particle upon the cylinder. And therefore the solid which is formed by all the right lines bH will be to the solid formed by all the right lines bE as the effect of all the particles upon the globe to the effect of all the particles upon the cylinder. But the former of these solids is a

paraboloid whose vertex is C, its axis CA, and latus rectum CA, and the latter solid is a cylinder circumscribing the paraboloid; and it is known that a paraboloid is half its circumscribed cylinder. Therefore the whole force of the medium upon the globe is half of the entire force of the same upon the cylinder. And therefore if the particles of the medium are at rest, and the cylinder and globe move with equal velocities, the resistance of the globe will be half the resistance of the cylinder. Q.E.D.

SCHOLIUM.

By the same method other figures may be compared together as to their resistance; and those may be found which are most apt to continue their motions in resisting mediums. As if upon the circular base CEBH from the centre O, with the radius OC, and the altitude OD, one would construct a frustum CBGF of a cone, which should meet with less resistance than any other frustum constructed with the same base and altitude, and going forwards towards D in the direction of its axis: bisect the altitude OD in Q, and produce OQ to S, so that QS may be equal to QC, and S will be the vertex of the cone whose frustum is sought.





Whence, by the bye, since the angle CSB is always acute, it follows, that, if the solid ADBE be generated by the convolution of an elliptical or oval figure ADBE about its axis AB, and the generating figure be touched by three right lines FG, GH, HI, in the points F, B, and I, so that GH shall be perpendicular to the axis in the point of contact B, and FG, HI may be inclined to GH in the angles FGB, BHI of 135 degrees: the solid arising from the convolution of the figure ADFGHIE about the same axis AB will be less resisted than the former solid; if so be that both move forward in the direction of their axis AB, and that the extremity B of each go foremost. Which Proposition I conceive may be of use in the building of ships.

If the figure DNFG be such a curve, that if, from any point thereof, as N, the perpendicular NM be let fall on the axis AB, and from the given point G there be drawn the right line GR parallel to a right line touching the figure in N, and cutting the axis produced in R, MN becomes to GR as GR^3 to $4BR \times GB^2$, the solid described by the revolution of this figure

about its axis AB, moving in the before-mentioned rare medium from A towards B, will be less resisted than any other circular solid whatsoever, described of the same length and breadth.

PROPOSITION XXXV. PROBLEM VII.

If a rare medium consist of very small quiescent particles of equal magnitudes, and freely disposed at equal distances from one another: to find the resistance of a globe moving uniformly forward in this medium.

CASE 1. Let a cylinder described with the same diameter and altitude be conceived to go forward with the same velocity in the direction of its axis through the same medium; and let us suppose that the particles of the medium, on which the globe or cylinder falls, fly back with as great a force of reflexion as possible. Then since the resistance of the globe (by the last Proposition) is but half the resistance of the cylinder, and since the globe is to the cylinder as 2 to 3, and since the cylinder by falling perpendicularly on the particles, and reflecting them with the utmost force, communicates to them a velocity double to its own; it follows that the cylinder, in moving forward uniformly half the length of its axis, will communicate a motion to the particles which is to the whole motion of the cylinder as the density of the medium to the density of the cylinder; and that the globe, in the time it describes one length of its diameter in moving uniformly forward, will communicate the same motion to the particles; and in the time that it describes two thirds of its diameter, will communicate a motion to the particles which is to the whole motion of the globe as the density of the medium to the density of the globe. And therefore the globe meets with a resistance, which is to the force by which its whole motion may be either taken away or generated in the time in which it describes two thirds of its diameter moving uniformly forward, as the density of the medium to the density of the globe.

CASE 2. Let us suppose that the particles of the medium incident on the globe or cylinder are not reflected; and then the cylinder falling perpendicularly on the particles will communicate its own simple velocity to them, and therefore meets a resistance but half so great as in the former case, and the globe also meets with a resistance but half so great.

CASE 3. Let us suppose the particles of the medium to fly back from the globe with a force which is neither the greatest, nor yet none at all, but with a certain mean force; then the resistance of the globe will be in the same mean ratio between the resistance in the first case and the resistance in the second. Q.E.I.

COR. 1. Hence if the globe and the particles are infinitely hard, and destitute of all elastic force, and therefore of all force of reflexion; the resistance of the globe will be to the force by which its whole motion may

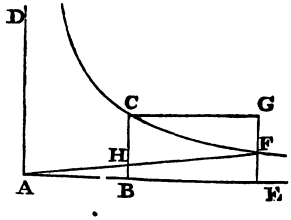
be destroyed or generated, in the time that the globe describes four third parts of its diameter, as the density of the medium to the density of the globe.

Cor. 2. The resistance of the globe, *cæteris paribus*, is in the duplicate ratio of the velocity.

Cor. 3. The resistance of the globe, *cæteris paribus*, is in the duplicate ratio of the diameter.

Cor. 4. The resistance of the globe is, *cæteris paribus*, as the density of the medium.

Cor. 5. The resistance of the globe is in a ratio compounded of the duplicate ratio of the velocity, and the duplicate ratio of the diameter, and the ratio of the density of the medium.



Cor. 6. The motion of the globe and its resistance may be thus expounded Let AB be the time in which the globe may, by its resistance uniformly continued, lose its whole motion. Erect AD, BC perpendicular to AB. Let BC be that whole motion, and through the point C, the asymptotes being AD, AB, describe the hyperbola CF. Produce AB to any point E. Erect the perpendicular EF meeting the hyperbola in F. Complete the parallelogram CBEG, and draw AF meeting BC in H. Then if the globe in any time BE, with its first motion BC uniformly continued, describes in a non-resisting medium the space CBEG expounded by the area of the parallelogram, the same in a resisting medium will describe the space CBEF expounded by the area of the hyperbola; and its motion at the end of that time will be expounded by EF, the ordinate of the hyperbola, there being lost of its motion the part FG. And its resistance at the end of the same time will be expounded by the length BH, there being lost of its resistance the part CH. All these things appear by Cor. 1 and 3, Prop. V., Book II.

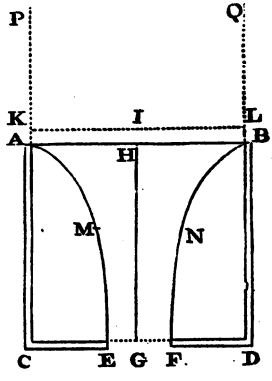
Cor. 7. Hence if the globe in the time T by the resistance R uniformly continued lose its whole motion M, the same globe in the time t in a resisting medium, wherein the resistance R decreases in a duplicate ratio of the velocity, will lose out of its motion M the part $\frac{tM}{T+t}$, the part $\frac{TM}{T+t}$ remaining; and will describe a space which is to the space described in the same time t, with the uniform motion M, as the logarithm of the number $\frac{T+t}{T}$ multiplied by the number 2,302585092994 is to the number $\frac{t}{T}$, because the hyperbolic area BCFE is to the rectangle BCGE in that proportion.

SCHOLIUM.

I have exhibited in this Proposition the resistance and retardation of spherical projectiles in mediums that are not continued, and shewn that this resistance is to the force by which the whole motion of the globe may be destroyed or produced in the time in which the globe can describe two thirds of its diameter, with a velocity uniformly continued, as the density of the medium to the density of the globe, if so be the globe and the particles of the medium be perfectly elastic, and are endued with the utmost force of reflexion; and that this force, where the globe and particles of the medium are infinitely hard and void of any reflecting force, is diminished one half. But in continued mediums, as water, hot oil, and quicksilver, the globe as it passes through them does not immediately strike against all the particles of the fluid that generate the resistance made to it, but presses only the particles that lie next to it, which press the particles beyond, which press other particles, and so on; and in these mediums the resistance is diminished one other half. A globe in these extremely fluid mediums meets with a resistance that is to the force by which its whole motion may be destroyed or generated in the time wherein it can describe, with that motion uniformly continued, eight third parts of its diameter, as the density of the medium to the density of the globe. This I shall endeavour to shew in what follows.

PROPOSITION XXXVI. PROBLEM VIII.

To define the motion of water running out of a cylindrical vessel through a hole made at the bottom.



Let ACDB be a cylindrical vessel, AB the mouth of it, CD the bottom parallel to the horizon, EF a circular hole in the middle of the bottom, G the centre of the hole, and GH the axis of the cylinder perpendicular to the horizon. And suppose a cylinder of ice APQB to be of the same breadth with the cavity of the vessel, and to have the same axis, and to descend perpetually with an uniform motion, and that its parts, as soon as they touch the superficies AB, dissolve into water, and flow down by their weight into the vessel, and in their fall compose the cataract or column of water ABNFEM, passing through the hole EF, and filling up the same exactly. Let the uniform velocity of the descending ice and of the contiguous water in the circle AB be that which the water would acquire by falling through the space IH; and let IH and HG lie in the same right line; and through

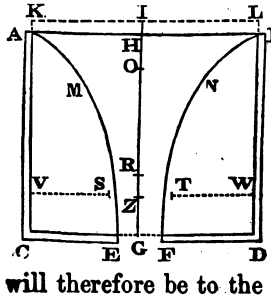
the point I let there be drawn the right line KL parallel to the horizon and meeting the ice on both the sides thereof in K and L. Then the velocity of the water running out at the hole EF will be the same that it would acquire by falling from I through the space IG. Therefore, by *Galileo's* Theorems, IG will be to IH in the duplicate ratio of the velocity of the water that runs out at the hole to the velocity of the water in the circle AB, that is, in the duplicate ratio of the circle AB to the circle EF; those circles being reciprocally as the velocities of the water which in the same time and in equal quantities passes severally through each of them, and completely fills them both. We are now considering the velocity with which the water tends to the plane of the horizon. But the motion parallel to the same, by which the parts of the falling water approach to each other, is not here taken notice of; since it is neither produced by gravity, nor at all changes the motion perpendicular to the horizon which the gravity produces. We suppose, indeed, that the parts of the water cohere a little, that by their cohesion they may in falling approach to each other with motions parallel to the horizon in order to form one single cataract, and to prevent their being divided into several: but the motion parallel to the horizon arising from this cohesion does not come under our present consideration.

Case 1. Conceive now the whole cavity in the vessel, which encompasses the falling water ABNFEM, to be full of ice, so that the water may pass through the ice as through a funnel. Then if the water pass very near to the ice only, without touching it; or, which is the same thing, if by reason of the perfect smoothness of the surface of the ice, the water, though touching it, glides over it with the utmost freedom, and without the least resistance; the water will run through the hole EF with the same velocity as before, and the whole weight of the column of water ABNFEM will be all taken up as before in forcing out the water, and the bottom of the vessel will sustain the weight of the ice encompassing that column.

Let now the ice in the vessel dissolve into water; yet will the efflux of the water remain, as to its velocity, the same as before. It will not be less, because the ice now dissolved will endeavour to descend; it will not be greater, because the ice, now become water, cannot descend without hindering the descent of other water equal to its own descent. The same force ought always to generate the same velocity in the effluent water.

But the hole at the bottom of the vessel, by reason of the oblique motions of the particles of the effluent water, must be a little greater than before. For now the particles of the water do not all of them pass through the hole perpendicularly, but, flowing down on all parts from the sides of the vessel, and converging towards the hole, pass through it with oblique motions; and in tending downwards meet in a stream whose diameter is a little smaller below the hole than at the hole itself; its diameter being to the

diameter of the hole as 5 to 6, or as $5\frac{1}{2}$ to $6\frac{1}{2}$, very nearly, if I took the measures of those diameters right. I procured a very thin flat plate, having a hole pierced in the middle, the diameter of the circular hole being $\frac{5}{8}$ parts of an inch. And that the stream of running waters might not be accelerated in falling, and by that acceleration become narrower, I fixed this plate not to the bottom, but to the side of the vessel, so as to make the water go out in the direction of a line parallel to the horizon. Then, when the vessel was full of water, I opened the hole to let it run out; and the diameter of the stream, measured with great accuracy at the distance of about half an inch from the hole, was $\frac{21}{40}$ of an inch. Therefore the diameter of this circular hole was to the diameter of the stream very nearly as 25 to 21. So that the water in passing through the hole converges on all sides, and, after it has run out of the vessel, becomes smaller by converging in that manner, and by becoming smaller is accelerated till it comes to the distance of half an inch from the hole, and at that distance flows in a smaller stream and with greater celerity than in the hole itself, and this in the ratio of 25×25 to 21×21, or 17 to 12, very nearly; that is, in about the subduplicate ratio of 2 to 1. Now it is certain from experiments, that the quantity of water running out in a given time through a circular hole made in the bottom of a vessel is equal to the quantity, which, flowing with the aforesaid velocity, would run out in the same time through another circular hole, whose diameter is to the diameter of the former as 21 to 25. And therefore that running water in passing through the hole itself has a velocity downwards equal to that which a heavy body would acquire in falling through half the height of the stagnant water in the vessel, nearly. But, then, after it has run out, it is still accelerated by converging, till it arrives at a distance from the hole that is nearly equal to its diameter, and acquires a velocity greater than the other in about the subduplicate ratio of 2 to 1; which velocity a heavy body would nearly acquire by falling through the whole height of the stagnant water in the vessel.



Therefore in what follows let the diameter of the stream be represented by that lesser hole which we called EF. And imagine another plane VW above the hole EF, and parallel to the plane thereof, to be placed at a distance equal to the diameter of the same hole, and to be pierced through with a greater hole ST, of such a magnitude that a stream which will exactly fill the lower hole EF may pass through it; the diameter of which hole will therefore be to the diameter of the lower hole as 25 to 21, nearly. By this means the water will run perpendicularly out at the lower hole; and the quantity of the water running out will be, according to the magnitude

of this last hole, the same, very nearly, which the solution of the Problem requires. The space included between the two planes and the falling stream may be considered as the bottom of the vessel. But, to make the solution more simple and mathematical, it is better to take the lower plane alone for the bottom of the vessel, and to suppose that the water which flowed through the ice as through a funnel, and ran out of the vessel through the hole EF made in the lower plane, preserves its motion continually, and that the ice continues at rest. Therefore in what follows let ST be the diameter of a circular hole described from the centre Z, and let the stream run out of the vessel through that hole, when the water in the vessel is all fluid. And let EF be the diameter of the hole, which the stream, in falling through, exactly fills up, whether the water runs out of the vessel by that upper hole ST, or flows through the middle of the ice in the vessel, as through a funnel. And let the diameter of the upper hole ST be to the diameter of the lower EF as about 25 to 21, and let the perpendicular distance between the planes of the holes be equal to the diameter of the lesser hole EF. Then the velocity of the water downwards, in running out of the vessel through the hole ST, will be in that hole the same that a body may acquire by falling from half the height IZ; and the velocity of both the falling streams will be in the hole EF, the same which a body would acquire by falling from the whole height IG.

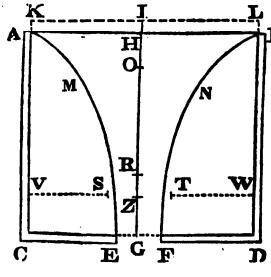
Case 2. If the hole EF be not in the middle of the bottom of the vessel, but in some other part thereof, the water will still run out with the same velocity as before, if the magnitude of the hole be the same. For though an heavy body takes a longer time in descending to the same depth, by an oblique line, than by a perpendicular line, yet in both cases it acquires in its descent the same velocity; as *Galileo* has demonstrated.

Case 3. The velocity of the water is the same when it runs out through a hole in the side of the vessel. For if the hole be small, so that the interval between the superficies AB and KL may vanish as to sense, and the stream of water horizontally issuing out may form a parabolic figure; from the latus rectum of this parabola may be collected, that the velocity of the effluent water is that which a body may acquire by falling the height IG or HG of the stagnant water in the vessel. For, by making an experiment, I found that if the height of the stagnant water above the hole were 20 inches, and the height of the hole above a plane parallel to the horizon were also 20 inches, a stream of water springing out from thence would fall upon the plane, at the distance of 37 inches, very nearly, from a perpendicular let fall upon that plane from the hole. For without resistance the stream would have fallen upon the plane at the distance of 40 inches, the latus rectum of the parabolic stream being 80 inches.

Case 4. If the effluent water tend upward, it will still issue forth with the same velocity. For the small stream of water springing upward, as-

cends with a perpendicular motion to GH or GI, the height of the stagnant water in the vessel; excepting in so far as its ascent is hindered a little by the resistance of the air; and therefore it springs out with the same velocity that it would acquire in falling from that height. Every particle of the stagnant water is equally pressed on all sides (by Prop. XIX., Book II), and, yielding to the pressure, tends always with an equal force, whether it descends through the hole in the bottom of the vessel, or gushes out in an horizontal direction through a hole in the side, or passes into a canal, and springs up from thence through a little hole made in the upper part of the canal. And it may not only be collected from reasoning, but is manifest also from the well-known experiments just mentioned, that the velocity with which the water runs out is the very same that is assigned in this Proposition.

CASE 5. The velocity of the effluent water is the same, whether the figure of the hole be circular, or square, or triangular, or any other figure equal to the circular; for the velocity of the effluent water does not depend upon the figure of the hole, but arises from its depth below the plane KL.



CASE 6. If the lower part of the vessel ABDC be immersed into stagnant water, and the height of the stagnant water above the bottom of the vessel be GR, the velocity with which the water that is in the vessel will run out at the hole EF into the stagnant water will be the same which the water would acquire by falling from the height IR; for the weight of all the water in the vessel that is below the superficies of the stagnant water will be sustained in equilibrio by the weight of the stagnant water, and therefore does not at all accelerate the motion of the descending water in the vessel. This case will also appear by experiments, measuring the times in which the water will run out.

COR. 1. Hence if CA the depth of the water be produced to K, so that AK may be to CK in the duplicate ratio of the area of a hole made in any part of the bottom to the area of the circle AB, the velocity of the effluent water will be equal to the velocity which the water would acquire by falling from the height KC.

COR. 2. And the force with which the whole motion of the effluent water may be generated is equal to the weight of a cylindric column of water, whose base is the hole EF, and its altitude 2GI or 2CK. For the effluent water, in the time it becomes equal to this column, may acquire, by falling by its own weight from the height GI, a velocity equal to that with which it runs out.

COR. 3. The weight of all the water in the vessel ABDC is to that part

of the weight which is employed in forcing out the water as the sum of the circles AB and EF to twice the circle EF. For let IO be a mean proportional between IH and IG, and the water running out at the hole EF will, in the time that a drop falling from I would describe the altitude IG, become equal to a cylinder whose base is the circle EF and its altitude 2IG, that is, to a cylinder whose base is the circle AB, and whose altitude is 2IO. For the circle EF is to the circle AB in the subduplicate ratio of the altitude IH to the altitude IG; that is, in the simple ratio of the mean proportional IO to the altitude IG. Moreover, in the time that a drop falling from I can describe the altitude IH, the water that runs out will have become equal to a cylinder whose base is the circle AB, and its altitude 2IH; and in the time that a drop falling from I through H to G describes HG, the difference of the altitudes, the effluent water, that is, the water contained within the solid ABNFEM, will be equal to the difference of the cylinders, that is, to a cylinder whose base is AB, and its altitude 2HO. And therefore all the water contained in the vessel ABDC is to the whole falling water contained in the said solid ABNFEM as HG to 2HO, that is, as HO + OG to 2HO, or IH + IO to 2IH. But the weight of all the water in the solid ABNFEM is employed in forcing out the water; and therefore the weight of all the water in the vessel is to that part of the weight that is employed in forcing out the water as IH + IO to 2IH, and therefore as the sum of the circles EF and AB to twice the circle EF.

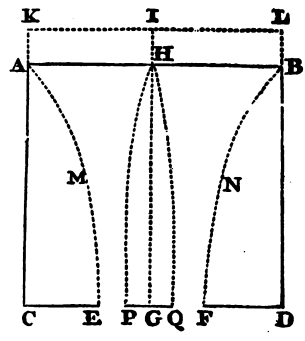
Cor. 4. And hence the weight of all the water in the vessel ABDC is to the other part of the weight which is sustained by the bottom of the vessel as the sum of the circles AB and EF to the difference of the same circles.

Cor. 5. And that part of the weight which the bottom of the vessel sustains is to the other part of the weight employed in forcing out the water as the difference of the circles AB and EF to twice the lesser circle EF, or as the area of the bottom to twice the hole.

Cor. 6. That part of the weight which presses upon the bottom is to the whole weight of the water perpendicularly incumbent thereon as the circle AB to the sum of the circles AB and EF, or as the circle AB to the excess of twice the circle AB above the area of the bottom. For that part of the weight which presses upon the bottom is to the weight of the whole water in the vessel as the difference of the circles AB and EF to the sum of the same circles (by Cor. 4); and the weight of the whole water in the vessel is to the weight of the whole water perpendicularly incumbent on the bottom as the circle AB to the difference of the circles AB and EF. Therefore, *ex æquo perturbatè*, that part of the weight which presses upon the bottom is to the weight of the whole water perpendicularly incumbent

thereon as the circle AB to the sum of the circles AB and EF. or the excess of twice the circle AB above the bottom.

COR. 7. If in the middle of the hole EF there be placed the little circle PQ described about the centre G, and parallel to the horizon, the weight of water which that little circle sustains is greater than the weight of a third part of a cylinder of water whose base is that little circle and its height GH. For let ABNFEM be the cataract or column of falling water whose axis is GH, as above, and let all the water, whose fluidity is not requisite for the ready and quick descent of the water, be supposed to be congealed, as well round about the cataract, as above the little circle. And let PHQ be the column of water congealed above the little circle, whose vertex is H, and its altitude GH. And suppose this cataract to fall with its whole weight downwards, and not in the least to lie against or to press PHQ, but to glide freely by it without any friction, unless, perhaps, just at the very vertex of the ice, where the cataract at the beginning of its fall may tend to a concave figure. And as the congealed water AMEC, BNFD, lying round the cataract, is convex in its internal superficies AME, BNF, towards the falling cataract, so this column PHQ will be convex towards the cataract also, and will therefore be greater than a cone whose base is that little circle PQ and its altitude GH; that is, greater than a third part of a cylinder described with the same base and altitude. Now that little circle sustains the weight of this column, that is, a weight greater than the weight of the cone, or a third part of the cylinder.



COR. 8. The weight of water which the circle PQ, when very small, sustains, seems to be less than the weight of two thirds of a cylinder of water whose base is that little circle, and its altitude HG. For, things standing as above supposed, imagine the half of a spheroid described whose base is that little circle, and its semi-axis or altitude HG. This figure will be equal to two thirds of that cylinder, and will comprehend within it the column of congealed water PHQ, the weight of which is sustained by that little circle. For though the motion of the water tends directly downwards, the external superficies of that column must yet meet the base PQ in an angle somewhat acute, because the water in its fall is perpetually accelerated, and by reason of that acceleration become narrower. Therefore, since that angle is less than a right one, this column in the lower parts thereof will lie within the hemi-spheroid. In the upper parts also it will be acute or pointed; because to make it otherwise, the horizontal motion of the water must be at the vertex infinitely more swift than its motion towards the horizon. And the less this circle PQ is, the more acute will

the vertex of this column be; and the circle being diminished *in infinitum*, the angle PHQ will be diminished *in infinitum*, and therefore the column will lie within the hemi-spheroid. Therefore that column is less than that hemi-spheroid, or than two-third parts of the cylinder whose base is that little circle, and its altitude GH. Now the little circle sustains a force of water equal to the weight of this column, the weight of the ambient water being employed in causing its efflux out at the hole.

COR. 9. The weight of water which the little circle PQ sustains, when it is very small, is very nearly equal to the weight of a cylinder of water whose base is that little circle, and its altitude $\frac{1}{2}$GH; for this weight is an arithmetical mean between the weights of the cone and the hemi-spheroid above mentioned. But if that little circle be not very small, but on the contrary increased till it be equal to the hole EF, it will sustain the weight of all the water lying perpendicularly above it, that is, the weight of a cylinder of water whose base is that little circle, and its altitude GH.

COR. 10. And (as far as I can judge) the weight which this little circle sustains is always to the weight of a cylinder of water whose base is that little circle, and its altitude $\frac{1}{2}$GH, as EF^2 to $EF^2 - \frac{1}{2}PQ^2$, or as the circle EF to the excess of this circle above half the little circle PQ, very nearly.

LEMMA IV.

If a cylinder move uniformly forward in the direction of its length, the resistance made thereto is not at all changed by augmenting or diminishing that length; and is therefore the same with the resistance of a circle, described with the same diameter, and moving forward with the same velocity in the direction of a right line perpendicular to its plane.

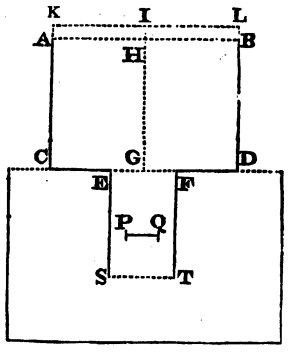
For the sides are not at all opposed to the motion; and a cylinder becomes a circle when its length is diminished *in infinitum*.

PROPOSITION XXXVII. THEOREM XXIX.

If a cylinder move uniformly forward in a compressed, infinite, and non-elastic fluid, in the direction of its length, the resistance arising from the magnitude of its transverse section is to the force by which its whole motion may be destroyed or generated, in the time that it moves four times its length, as the density of the medium to the density of the cylinder, nearly.

For let the vessel ABDC touch the surface of stagnant water with its bottom CD, and let the water run out of this vessel into the stagnant water through the cylindric canal EFTS perpendicular to the horizon; and let the little circle PQ be placed parallel to the horizon any where in the

middle of the canal; and produce CA to K, so that AK may be to CK in the duplicate of the ratio, which the excess of the orifice of the canal EF above the little circle PQ bears to the circle AB. Then it is manifest (by Case 5, Case 6, and Cor. 1, Prop. XXXVI) that the velocity of the water passing through the annular space between the little circle and the sides of the vessel will be the very same which the water would acquire by falling, and in its fall describing the altitude KC or IG.



And (by Cor. 10, Prop. XXXVI) if the breadth of the vessel be infinite, so that the lineola HI may vanish, and the altitudes IG, HG become equal; the force of the water that flows down and presses upon the circle will be to the weight of a cylinder whose base is that little circle, and the altitude $\frac{1}{2}$IG, as EF^2 to $EF^2 - \frac{1}{2}PQ^2$, very nearly. For the force of the water flowing downward uniformly through the whole canal will be the same upon the little circle PQ in whatsoever part of the canal it be placed.

Let now the orifices of the canal EF, ST be closed, and let the little circle ascend in the fluid compressed on every side, and by its ascent let it oblige the water that lies above it to descend through the annular space between the little circle and the sides of the canal. Then will the velocity of the ascending little circle be to the velocity of the descending water as the difference of the circles EF and PQ is to the circle PQ; and the velocity of the ascending little circle will be to the sum of the velocities, that is, to the relative velocity of the descending water with which it passes by the little circle in its ascent, as the difference of the circles EF and PQ to the circle EF, or as $EF^2 - PQ^2$ to EF^2. Let that relative velocity be equal to the velocity with which it was shewn above that the water would pass through the annular space, if the circle were to remain unmoved, that is, to the velocity which the water would acquire by falling, and in its fall describing the altitude IG; and the force of the water upon the ascending circle will be the same as before (by Cor. 5, of the Laws of Motion); that is, the resistance of the ascending little circle will be to the weight of a cylinder of water whose base is that little circle, and its altitude $\frac{1}{2}$IG, as EF^2 to $EF^2 - \frac{1}{2}PQ^2$, nearly. But the velocity of the little circle will be to the velocity which the water acquires by falling, and in its fall describing the altitude IG, as $EF^2 - PQ^2$ to EF^2.

Let the breadth of the canal be increased *in infinitum;* and the ratios between $EF^2 - PQ^2$ and EF^2, and between EF^2 and $EF^2 - \frac{1}{2}PQ^2$, will become at last ratios of equality. And therefore the velocity of the little circle will now be the same which the water would acquire in falling, and in its fall describing the altitude IG: and the resistance will become

equal to the weight of a cylinder whose base is that little circle, and its altitude half the altitude IG, from which the cylinder must fall to acquire the velocity of the ascending circle; and with this velocity the cylinder in the time of its fall will describe four times its length. But the resistance of the cylinder moving forward with this velocity in the direction of its length is the same with the resistance of the little circle (by Lem. IV), and is therefore nearly equal to the force by which its motion may be generated while it describes four times its length.

If the length of the cylinder be augmented or diminished, its motion, and the time in which it describes four times its length, will be augmented or diminished in the same ratio, and therefore the force by which the motion so increased or diminished, may be destroyed or generated, will continue the same; because the time is increased or diminished in the same proportion; and therefore that force remains still equal to the resistance of the cylinder, because (by Lem. IV) that resistance will also remain the same.

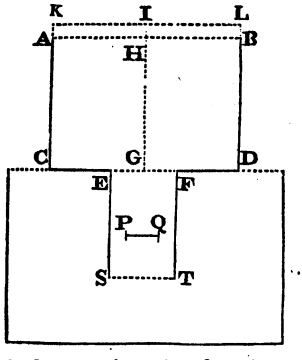
If the density of the cylinder be augmented or diminished, its motion, and the force by which its motion may be generated or destroyed in the same time, will be augmented or diminished in the same ratio. Therefore the resistance of any cylinder whatsoever will be to the force by which its whole motion may be generated or destroyed, in the time during which it moves four times its length, as the density of the medium to the density of the cylinder, nearly. Q.E.D.

A fluid must be compressed to become continued; it must be continued and non-elastic, that all the pressure arising from its compression may be propagated in an instant; and so, acting equally upon all parts of the body moved, may produce no change of the resistance. The pressure arising from the motion of the body is spent in generating a motion in the parts of the fluid, and this creates the resistance. But the pressure arising from the compression of the fluid, be it ever so forcible, if it be propagated in an instant, generates no motion in the parts of a continued fluid, produces no change at all of motion therein; and therefore neither augments nor lessens the resistance. This is certain, that the action of the fluid arising from the compression cannot be stronger on the hinder parts of the body moved than on its fore parts, and therefore cannot lessen the resistance described in this proposition. And if its propagation be infinitely swifter than the motion of the body pressed, it will not be stronger on the fore parts than on the hinder parts. But that action will be infinitely swifter, and propagated in an instant, if the fluid be continued and non-elastic.

COR. 1. The resistances, made to cylinders going uniformly forward in the direction of their lengths through continued infinite mediums are in a

ratio compounded of the duplicate ratio of the velocities and the duplicate ratio of the diameters, and the ratio of the density of the mediums.

COR. 2. If the breadth of the canal be not infinitely increased but the cylinder go forward in the direction of its length through an included quiescent medium, its axis all the while coinciding with the axis of the canal, its resistance will be to the force by which its whole motion, in the time in which it describes four times its length, may be generated or destroyed, in a ratio compounded of the ratio of EF^2 to $EF^2 - \frac{1}{2}PQ^2$ once, and the ratio of EF^2 to $EF^2 - PQ^2$ twice, and the ratio of the density of the medium to the density of the cylinder.

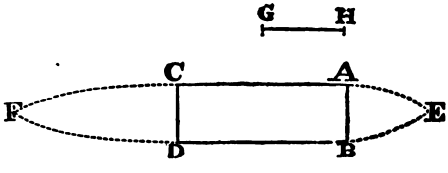


COR. 3. The same thing supposed, and that a length L is to the quadruple of the length of the cylinder in a ratio compounded of the ratio $EF^2 - \frac{1}{2}PQ^2$ to EF^2 once, and the ratio of $EF^2 - PQ^2$ to EF^2 twice; the resistance of the cylinder will be to the force by which its whole motion, in the time during which it describes the length L, may be destroyed or generated, as the density of the medium to the density of the cylinder.

SCHOLIUM.

In this proposition we have investigated that resistance alone which arises from the magnitude of the transverse section of the cylinder, neglecting that part of the same which may arise from the obliquity of the motions. For as, in Case 1, of Prop. XXXVI., the obliquity of the motions with which the parts of the water in the vessel converged on every side to the hole EF hindered the efflux of the water through the hole, so, in this Proposition, the obliquity of the motions, with which the parts of the water, pressed by the antecedent extremity of the cylinder, yield to the pressure, and diverge on all sides, retards their passage through the places that lie round that antecedent extremity, toward the hinder parts of the cylinder, and causes the fluid to be moved to a greater distance; which increases the resistance, and that in the same ratio almost in which it diminished the efflux of the water out of the vessel, that is, in the duplicate ratio of 25 to 21, nearly. And as, in Case 1, of that Proposition, we made the parts of the water pass through the hole EF perpendicularly and in the greatest plenty, by supposing all the water in the vessel lying round the cataract to be frozen, and that part of the water whose motion was oblique and useless to remain without motion, so in this Proposition, that the obliquity of the motions may be taken away, and the parts of the water may give the freest passage to the cylinder, by yielding to it with the most direct and quick motion possible, so that only so much resistance may re-

main as arises from the magnitude of the transverse section, and which is incapable of diminution, unless by diminishing the diameter of the cylinder; we must conceive those parts of the fluid whose motions are oblique and useless, and produce resistance, to be at rest among themselves at both extremities of the cylinder, and there to cohere, and be joined to the cylinder. Let ABCD be a rectangle, and let AE and BE be two parabolic arcs, described with the axis AB, and with a latus rectum that is to the space HG, which must be described by the cylinder in falling, in order to acquire the velocity with which it moves, as HG to $\frac{1}{4}$AB. Let CF and DF be two other parabolic arcs described with the axis CD, and a latus rectum quadruple of the former; and by the convolution of the figure about the axis EF let there be generated a solid, whose middle part ABDC is the cylinder we are here speaking of, and whose extreme parts ABE and CDF contain the parts of the fluid at rest among themselves, and concreted into two hard bodies, adhering to the cylinder at each end like a head and tail. Then if this solid EACFDB move in the direction of the length of its axis FE toward the parts beyond E, the resistance will be the same which we have here determined in this Proposition, nearly; that is, it will have the same ratio to the force with which the whole motion of the cylinder may be destroyed or generated, in the time that it is describing the length 4AC with that motion uniformly continued, as the density of the fluid has to the density of the cylinder, nearly. And (by Cor. 7, Prop. XXXVI) the resistance must be to this force in the ratio of 2 to 3, at the least.



LEMMA V.

If a cylinder, a sphere, and a spheroid, of equal breadths be placed successively in the middle of a cylindric canal, so that their axes may coincide with the axis of the canal, these bodies will equally hinder the passage of the water through the canal.

For the spaces lying between the sides of the canal, and the cylinder, sphere, and spheroid, through which the water passes, are equal; and the water will pass equally through equal spaces.

This is true, upon the supposition that all the water above the cylinder, sphere, or spheroid, whose fluidity is not necessary to make the passage of the water the quickest possible, is congealed, as was explained above in Cor 7, Prop. XXXVI.

LEMMA VI.

The same supposition remaining, the fore-mentioned bodies are equally acted on by the water flowing through the canal.

This appears by Lem. V and the third Law. For the water and the bodies act upon each other mutually and equally.

LEMMA VII.

If the water be at rest in the canal, and these bodies move with equal velocity and the contrary way through the canal, their resistances will be equal among themselves.

This appears from the last Lemma, for the relative motions remain the same among themselves.

SCHOLIUM.

The case is the same of all convex and round bodies, whose axes coincide with the axis of the canal. Some difference may arise from a greater or less friction; but in these *Lemmata* we suppose the bodies to be perfectly smooth, and the medium to be void of all tenacity and friction; and that those parts of the fluid which by their oblique and superfluous motions may disturb, hinder, and retard the flux of the water through the canal, are at rest among themselves; being fixed like water by frost, and adhering to the fore and hinder parts of the bodies in the manner explained in the Scholium of the last Proposition; for in what follows we consider the very least resistance that round bodies described with the greatest given transverse sections can possibly meet with.

Bodies swimming upon fluids, when they move straight forward, cause the fluid to ascend at their fore parts and subside at their hinder parts, especially if they are of an obtuse figure; and thence they meet with a little more resistance than if they were acute at the head and tail. And bodies moving in elastic fluids, if they are obtuse behind and before, condense the fluid a little more at their fore parts, and relax the same at their hinder parts; and therefore meet also with a little more resistance than if they were acute at the head and tail. But in these Lemmas and Propositions we are not treating of elastic but non-elastic fluids; not of bodies floating on the surface of the fluid, but deeply immersed therein. And when the resistance of bodies in non-elastic fluids is once known, we may then augment this resistance a little in elastic fluids, as our air; and in the surfaces of stagnating fluids, as lakes and seas.

PROPOSITION XXXVIII. THEOREM XXX.

If a globe move uniformly forward in a compressed, infinite, and non-elastic fluid, its resistance is to the force by which its whole motion

may be destroyed or generated, in the time that it describes eight third parts of its diameter, as the density of the fluid to the density of the globe, very nearly.

For the globe is to its circumscribed cylinder as two to three; and therefore the force which can destroy all the motion of the cylinder, while the same cylinder is describing the length of four of its diameters, will destroy all the motion of the globe, while the globe is describing two thirds of this length, that is, eight third parts of its own diameter. Now the resistance of the cylinder is to this force very nearly as the density of the fluid to the density of the cylinder or globe (by Prop. XXXVII), and the resistance of the globe is equal to the resistance of the cylinder (by Lem. V, VI, and VII). Q.E.D.

Cor. 1. The resistances of globes in infinite compressed mediums are in a ratio compounded of the duplicate ratio of the velocity, and the duplicate ratio of the diameter, and the ratio of the density of the mediums.

Cor. 2. The greatest velocity, with which a globe can descend by its comparative weight through a resisting fluid, is the same which it may acquire by falling with the same weight, and without any resistance, and in its fall describing a space that is, to four third parts of its diameter as the density of the globe to the density of the fluid. For the globe in the time of its fall, moving with the velocity acquired in falling, will describe a space that will be to eight third parts of its diameter as the density of the globe to the density of the fluid; and the force of its weight which generates this motion will be to the force that can generate the same motion, in the time that the globe describes eight third parts of its diameter, with the same velocity as the density of the fluid to the density of the globe; and therefore (by this Proposition) the force of weight will be equal to the force of resistance, and therefore cannot accelerate the globe.

Cor. 3. If there be given both the density of the globe and its velocity at the beginning of the motion, and the density of the compressed quiescent fluid in which the globe moves, there is given at any time both the velocity of the globe and its resistance, and the space described by it (by Cor. 7, Prop. XXXV).

Cor. 4. A globe moving in a compressed quiescent fluid of the same density with itself will lose half its motion before it can describe the length of two of its diameters (by the same Cor. 7).

PROPOSITION XXXIX. THEOREM XXXI.

If a globe move uniformly forward through a fluid inclosed and compressed in a cylindric canal, its resistance is to the force by which its whole motion may be generated or destroyed, in the time in which it describes eight third parts of its diameter, in a ratio compounded of

the ratio of the orifice of the canal to the excess of that orifice above half the greatest circle of the globe; and the duplicate ratio of the orifice of the canal to the excess of that orifice above the greatest circle of the globe; and the ratio of the density of the fluid to the density of the globe, nearly.

This appears by Cor. 2, Prop. XXXVII, and the demonstration proceeds in the same manner as in the foregoing Proposition.

SCHOLIUM.

In the last two Propositions we suppose (as was done before in Lem. V) that all the water which precedes the globe, and whose fluidity increases the resistance of the same, is congealed. Now if that water becomes fluid, it will somewhat increase the resistance. But in these Propositions that increase is so small, that it may be neglected, because the convex superficies of the globe produces the very same effect almost as the congelation of the water.

PROPOSITION XL. PROBLEM IX.

To find by phenomena the resistance of a globe moving through a perfectly fluid compressed medium.

Let A be the weight of the globe *in vacuo*, B its weight in the resisting medium, D the diameter of the globe. F a space which is to $\frac{4}{3}$D as the density of the globe to the density of the medium, that is, as A to A — B, G the time in which the globe falling with the weight B without resistance describes the space F, and H the velocity which the body acquires by that fall. Then H will be the greatest velocity with which the globe can possibly descend with the weight B in the resisting medium, by Cor. 2, Prop XXXVIII; and the resistance which the globe meets with, when descending with that velocity, will be equal to its weight B; and the resistance it meets with in any other velocity will be to the weight B in the duplicate ratio of that velocity to the greatest velocity H, by Cor. 1, Prop. XXXVIII.

This is the resistance that arises from the inactivity of the matter of the fluid. That resistance which arises from the elasticity, tenacity, and friction of its parts, may be thus investigated.

Let the globe be let fall so that it may descend in the fluid by the weight B; and let P be the time of falling, and let that time be expressed in seconds, if the time G be given in seconds. Find the absolute number N agreeing to the logarithm 0,4342944819 $\frac{2P}{G}$, and let L be the logarithm of the number $\frac{N+1}{N}$: and the velocity acquired in falling will be

$\frac{N-1}{N+1}$ H, and the height described will be $\frac{2PF}{G}$ — 1,3862943611F + 4,6051701S6LF. If the fluid be of a sufficient depth, we may neglect the term 4,6051701S6LF; and $\frac{2PF}{G}$ — 1,3862943611F will be the altitude described, nearly. These things appear by Prop. IX, Book II, and its Corollaries, and are true upon this supposition, that the globe meets with no other resistance but that which arises from the inactivity of matter. Now if it really meet with any resistance of another kind, the descent will be slower, and from the quantity of that retardation will be known the quantity of this new resistance.

That the velocity and descent of a body falling in a fluid might more easily be known, I have composed the following table; the first column of which denotes the times of descent; the second shews the velocities acquired in falling, the greatest velocity being 100000000; the third exhibits the spaces described by falling in those times, 2F being the space which the body describes in the time G with the greatest velocity; and the fourth gives the spaces described with the greatest velocity in the same times. The numbers in the fourth column are $\frac{2P}{G}$, and by subducting the number 1,3862944 — 4,6051702L, are found the numbers in the third column; and these numbers must be multiplied by the space F to obtain the spaces described in falling. A fifth column is added to all these, containing the spaces described in the same times by a body falling *in vacuo* with the force of B its comparative weight.

The Times P.	Velocities of the body falling in the fluid.	The spaces described in falling in the fluid.	The spaces described with the greatest motion.	The spaces described by falling *in vacuo*.
0,001G	99999$\frac{2}{3}\frac{9}{0}$	0,000001F	0,002F	0,000001F
0,01G	999967	0,0001F	0,02F	0,0001F
0,1G	9966799	0,0099834F	0,2F	0,01F
0,2G	19737532	0,0397361F	0,4F	0,04F
0,3G	29131261	0,0886815F	0,6F	0,09F
0,4G	37994896	0,1559070F	0,8F	0,16F
0,5G	46211716	0,2402290F	1,0F	0,25F
0,6G	53704957	0,3402706F	1,2F	0,36F
0,7G	60436778	0,4545405F	1,4F	0,49F
0,8G	66403677	0,5815071F	1,6F	0,64F
0,9G	71629787	0,7196609F	1,8F	0,81F
1G	76159416	0,8675617F	2F	1F
2G	96402758	2,6500055F	4F	4F
3G	99505475	4,6186570F	6F	9F
4G	99932930	6,6143765F	8F	16F
5G	99990920	8,6137964F	10F	25F
6G	99998771	10,6137179F	12F	36F
7G	99999834	12,6137073F	14F	49F
8G	99999980	14,6137059F	16F	64F
9G	99999997	16,6137057F	18F	81F
10G	99999999$\frac{3}{5}$	18,6137056F	20F	100F

SCHOLIUM.

In order to investigate the resistances of fluids from experiments, I procured a square wooden vessel, whose length and breadth on the inside was 9 inches *English* measure, and its depth 9 feet $\frac{1}{2}$; this I filled with rain-water: and having provided globes made up of wax, and lead included therein, I noted the times of the descents of these globes, the height through which they descended being 112 inches. A solid cubic foot of *English* measure contains 76 pounds *troy* weight of rain water; and a solid inch contains $\frac{19}{36}$ ounces *troy* weight, or $253\frac{1}{3}$ grains; and a globe of water of one inch in diameter contains 132,645 grains in air, or 132,8 grains *in vacuo;* and any other globe will be as the excess of its weight *in vacuo* above its weight in water.

EXPER. 1. A globe whose weight was $156\frac{1}{4}$ grains in air, and 77 grains in water, described the whole height of 112 inches in 4 seconds. And, upon repeating the experiment, the globe spent again the very same time of 4 seconds in falling.

The weight of this globe *in vacuo* is $156\frac{1}{3}\frac{3}{8}$ grains; and the excess of this weight above the weight of the globe in water is $79\frac{1}{3}\frac{3}{8}$ grains. Hence the diameter of the globe appears to be 0,84224 parts of an inch. Then it will be, as that excess to the weight of the globe *in vacuo*, so is the density of the water to the density of the globe; and so is $\frac{8}{3}$ parts of the diameter of the globe (viz. 2,24597 inches) to the space 2F, which will be therefore 4,4256 inches. Now a globe falling *in vacuo* with its whole weight of $156\frac{1}{3}\frac{3}{8}$ grains in one second of time will describe $193\frac{1}{3}$ inches; and falling in water in the same time with the weight of 77 grains without resistance, will describe 95,219 inches; and in the time G, which is to one second of time in the subduplicate ratio of the space F, or of 2,2128 inches to 95,219 inches, will describe 2,2128 inches, and will acquire the greatest velocity H with which it is capable of descending in water. Therefore the time G is 0″,15244. And in this time G, with that greatest velocity H, the globe will describe the space 2F, which is 4,4256 inches; and therefore in 4 seconds will describe a space of 116,1245 inches. Subduct the space 1,3862944F, or 3,0676 inches, and there will remain a space of 113,0569 inches, which the globe falling through water in a very wide vessel will describe in 4 seconds. But this space, by reason of the narrowness of the wooden vessel before mentioned, ought to be diminished in a ratio compounded of the subduplicate ratio of the orifice of the vessel to the excess of this orifice above half a great circle of the globe, and of the simple ratio of the same orifice to its excess above a great circle of the globe, that is, in a ratio of 1 to 0,9914. This done, we have a space of 112,08 inches, which a globe falling through the water in this wooden vessel in 4 seconds of time ought nearly to describe by this theory; but it described 112 inches by the experiment.

Exper. 2. Three equal globes, whose weights were severally $76\frac{1}{3}$ grains in air, and $5\frac{1}{176}$ grains in water, were let fall successively; and every one fell through the water in 15 seconds of time, describing in its fall a height of 112 inches.

By computation, the weight of each globe *in vacuo* is $76\frac{5}{12}$ grains; the excess of this weight above the weight in water is 71 grains $\frac{17}{176}$; the diameter of the globe 0,81296 of an inch; $\frac{3}{5}$ parts of this diameter 2,1678[illegible] inches; the space 2F is 2,3217 inches; the space which a globe of $5\frac{1}{176}$ grains in weight would describe in one second without resistance, 12,80[illegible] inches, and the time G0″,301056. Therefore the globe, with the greatest velocity it is capable of receiving from a weight of $5\frac{1}{176}$ grains in its descent through water, will describe in the time 0″,301056 the space of 2,3217 inches; and in 15 seconds the space 115,678 inches. Subduct the space 1,3862944F, or 1,609 inches, and there remains the space 114,069 inches, which therefore the falling globe ought to describe in the same time, if the vessel were very wide. But because our vessel was narrow, the space ought to be diminished by about 0,895 of an inch. And so the space will remain 113,174 inches, which a globe falling in this vessel ought nearly to describe in 15 seconds, by the theory. But by the experiment it described 112 inches. The difference is not sensible.

Exper. 3. Three equal globes, whose weights were severally 121 grains in air, and 1 grain in water, were successively let fall; and they fell through the water in the times 46″, 47″, and 50″, describing a height of 112 inches.

By the theory, these globes ought to have fallen in about 40″. Now whether their falling more slowly were occasioned from hence, that in slow motions the resistance arising from the force of inactivity does really bear a less proportion to the resistance arising from other causes; or whether it is to be attributed to little bubbles that might chance to stick to the globes, or to the rarefaction of the wax by the warmth of the weather, or of the hand that let them fall; or, lastly, whether it proceeded from some insensible errors in weighing the globes in the water, I am not certain. Therefore the weight of the globe in water should be of several grains, that the experiment may be certain, and to be depended on.

Exper. 4. I began the foregoing experiments to investigate the resistances of fluids, before I was acquainted with the theory laid down in the Propositions immediately preceding. Afterward, in order to examine the theory after it was discovered, I procured a wooden vessel, whose breadth on the inside was $8\frac{2}{3}$ inches, and its depth 15 feet and $\frac{1}{3}$. Then I made four globes of wax, with lead included, each of which weighed $139\frac{1}{4}$ grains in air, and $7\frac{1}{8}$ grains in water. These I let fall, measuring the times of their falling in the water with a pendulum oscillating to half seconds. The globes were cold, and had remained so some time, both when they were

weighed and when they were let fall; because warmth rarefies the wax, and by rarefying it diminishes the weight of the globe in the water; and wax, when rarefied, is not instantly reduced by cold to its former density. Before they were let fall, they were totally immersed under water, lest, by the weight of any part of them that might chance to be above the water, their descent should be accelerated in its beginning. Then, when after their immersion they were perfectly at rest, they were let go with the greatest care, that they might not receive any impulse from the hand that let them down. And they fell successively in the times of $47\frac{1}{2}$, $48\frac{1}{2}$, 50, and 51 oscillations, describing a height of 15 feet and 2 inches. But the weather was now a little colder than when the globes were weighed, and therefore I repeated the experiment another day; and then the globes fell in the times of 49, $49\frac{1}{2}$, 50, and 53; and at a third trial in the times of $49\frac{1}{2}$, 50, 51, and 53 oscillations. And by making the experiment several times over, I found that the globes fell mostly in the times of $49\frac{1}{2}$ and 50 oscillations. When they fell slower, I suspect them to have been retarded by striking against the sides of the vessel.

Now, computing from the theory, the weight of the globe *in vacuo* is $139\frac{2}{5}$ grains; the excess of this weight above the weight of the globe in water $132\frac{1}{4}\frac{1}{0}$ grains; the diameter of the globe 0,99868 of an inch; $\frac{8}{3}$ parts of the diameter 2,66315 inches; the space 2F 2,8066 inches; the space which a globe weighing $7\frac{1}{8}$ grains falling without resistance describes in a second of time 9,88164 inches; and the time G0″,376843. Therefore the globe with the greatest velocity with which it is capable of descending through the water by the force of a weight of $7\frac{1}{8}$ grains, will in the time 0″,376843 describe a space of 2,8066 inches, and in one second of time a space of 7,44766 inches, and in the time 25″, or in 50 oscillations, the space 186,1915 inches. Subduct the space 1,386294F, or 1,9454 inches, and there will remain the space 184,2461 inches which the globe will describe in that time in a very wide vessel. Because our vessel was narrow, let this space be diminished in a ratio compounded of the subduplicate ratio of the orifice of the vessel to the excess of this orifice above half a great circle of the globe, and of the simple ratio of the same orifice to its excess above a great circle of the globe; and we shall have the space of 181,86 inches, which the globe ought by the theory to describe in this vessel in the time of 50 oscillations, nearly. But it described the space of 182 inches, by experiment, in $49\frac{1}{2}$ or 50 oscillations.

EXPER. 5. Four globes weighing $154\frac{3}{4}$ grains in air, and $21\frac{1}{2}$ grains in water, being let fall several times, fell in the times of $28\frac{1}{2}$, 29, $29\frac{1}{2}$, and 30, and sometimes of 31, 32, and 33 oscillations, describing a height of 15 feet and 2 inches.

They ought by the theory to have fallen in the time of 29 oscillations, nearly.

EXPER. 6. Five globes, weighing 212⅜ grains in air, and 79½ in water, being several times let fall, fell in the times of 15, 15½, 16, 17, and 18 oscillations, describing a height of 15 feet and 2 inches.

By the theory they ought to have fallen in the time of 15 oscillations, nearly.

EXPER. 7. Four globes, weighing 293⅜ grains in air, and 35⅞ grains in water, being let fall several times, fell in the times of 29½ 30, 30½ 31, 32, and 33 oscillations, describing a height of 15 feet and 1 inch and ½.

By the theory they ought to have fallen in the time of 28 oscillations, nearly.

In searching for the cause that occasioned these globes of the same weight and magnitude to fall, some swifter and some slower, I hit upon this; that the globes, when they were first let go and began to fall, oscillated about their centres; that side which chanced to be the heavier descending first, and producing an oscillating motion. Now by oscillating thus, the globe communicates a greater motion to the water than if it descended without any oscillations; and by this communication loses part of its own motion with which it should descend; and therefore as this oscillation is greater or less, it will be more or less retarded. Besides, the globe always recedes from that side of itself which is descending in the oscillation, and by so receding comes nearer to the sides of the vessel, so as even to strike against them sometimes. And the heavier the globes are, the stronger this oscillation is; and the greater they are, the more is the water agitated by it. Therefore to diminish this oscillation of the globes, I made new ones of lead and wax, sticking the lead in one side of the globe very near its surface; and I let fall the globe in such a manner, that, as near as possible, the heavier side might be lowest at the beginning of the descent. By this means the oscillations became much less than before, and the times in which the globes fell were not so unequal: as in the following experiments.

EXPER. 8. Four globes weighing 139 grains in air, and 6½ in water, were let fall several times, and fell mostly in the time of 51 oscillations, never in more than 52, or in fewer than 50, describing a height of 182 inches.

By the theory they ought to fall in about the time of 52 oscillations

EXPER. 9. Four globes weighing 273¼ grains in air, and 140¾ in water, being several times let fall, fell in never fewer than 12, and never more than 13 oscillations, describing a height of 182 inches.

These globes by the theory ought to have fallen in the time of 11⅓ oscillations, nearly.

EXPER. 10. Four globes, weighing 384 grains in air, and 119⅓ in water, being let fall several times, fell in the times of 17¾ 18, 18½, and 19 oscillations, describing a height of 181½ inches. And when they fell in the time

of 19 oscillations, I sometimes heard them hit against the sides of the vessel before they reached the bottom.

By the theory they ought to have fallen in the time of $15\frac{5}{9}$ oscillations, nearly.

EXPER. 11. Three equal globes, weighing 48 grains in the air, and $3\frac{2}{3}\frac{9}{2}$ in water, being several times let fall, fell in the times of $43\frac{1}{2}$, 44, $44\frac{1}{2}$, 45, and 46 oscillations, and mostly in 44 and 45. describing a height of $182\frac{1}{2}$ inches, nearly.

By the theory they ought to have fallen in the time of 46 oscillations and $\frac{4}{9}$, nearly.

EXPER. 12. Three equal globes, weighing 141 grains in air, and $4\frac{3}{8}$ in water, being let fall several times, fell in the times of 61, 62, 63, 64, and 65 oscillations, describing a space of 182 inches.

And by the theory they ought to have fallen in $64\frac{1}{2}$ oscillations nearly.

From these experiments it is manifest, that when the globes fell slowly, as in the second, fourth, fifth, eighth, eleventh, and twelfth experiments, the times of falling are rightly exhibited by the theory; but when the globes fell more swiftly, as in the sixth, ninth, and tenth experiments, the resistance was somewhat greater than in the duplicate ratio of the velocity. For the globes in falling oscillate a little; and this oscillation, in those globes that are light and fall slowly, soon ceases by the weakness of the motion; but in greater and heavier globes, the motion being strong, it continues longer, and is not to be checked by the ambient water till after several oscillations Besides, the more swiftly the globes move, the less are they pressed by the fluid at their hinder parts; and if the velocity be perpetually increased, they will at last leave an empty space behind them, unless the compression of the fluid be increased at the same time. For the compression of the fluid ought to be increased (by Prop. XXXII and XXXIII) in the duplicate ratio of the velocity, in order to preserve the resistance in the same duplicate ratio. But because this is not done, the globes that move swiftly are not so much pressed at their hinder parts as the others; and by the defect of this pressure it comes to pass that their resistance is a little greater than in a duplicate ratio of their velocity.

So that the theory agrees with the phænomena of bodies falling in water It remains that we examine the phænomena of bodies falling in air.

EXPER. 13. From the top of St. *Paul's* Church in *London*, in *June* 1710, there were let fall together two glass globes, one full of quicksilver, the other of air; and in their fall they described a height of 220 *English* feet. A wooden table was suspended upon iron hinges on one side, and the other side of the same was supported by a wooden pin. The two globes lying upon this table were let fall together by pulling out the pin by means of an iron wire reaching from thence quite down to the ground; so

that, the pin being removed, the table, which had then no support but the iron hinges, fell downward, and turning round upon the hinges, gave leave to the globes to drop off from it. At the same instant, with the same pull of the iron wire that took out the pin, a pendulum oscillating to seconds was let go, and began to oscillate. The diameters and weights of the globes, and their times of falling, are exhibited in the following table.

The globes filled with mercury.			*The globes full of air.*		
Weights.	Diameters.	Times in falling.	Weights.	Diameters.	Times in falling.
908 grains	0,8 of an inch	4″ ·	510 grains	5,1 inches	8″½
983	0,8	4 —	642	5,2	8
866	0,8	4	599	5,1	8
747	0,75	4 +	515	5,0	8¼
808	0,75	4	483	5,0	8¼
784	0,75	4 +	641	5,2	8

But the times observed must be corrected; for the globes of mercury (by *Galileo's* theory), in 4 seconds of time, will describe 257 *English* feet, and 220 feet in only 3″42‴. So that the wooden table, when the pin was taken out, did not turn upon its hinges so quickly as it ought to have done; and the slowness of that revolution hindered the descent of the globes at the beginning. For the globes lay about the middle of the table, and indeed were rather nearer to the axis upon which it turned than to the pin. And hence the times of falling were prolonged about 18‴; and therefore ought to be corrected by subducting that excess, especially in the larger globes, which, by reason of the largeness of their diameters, lay longer upon the revolving table than the others. This being done, the times in which the six larger globes fell will come forth 8″ 12‴, 7″ 42‴, 7″ 42‴, 7″ 57‴, 8″ 12‴ and 7″ 42‴.

Therefore the fifth in order among the globes that were full of air being 5 inches in diameter, and 483 grains in weight, fell in 8″ 12‴, describing a space of 220 feet. The weight of a bulk of water equal to this globe is 16600 grains; and the weight of an equal bulk of air is $\frac{16600}{850}$ grains, or $19\frac{3}{10}$ grains; and therefore the weight of the globe *in vacuo* is $502\frac{3}{10}$ grains; and this weight is to the weight of a bulk of air equal to the globe as $502\frac{3}{10}$ to $19\frac{3}{10}$; and so is 2F to $\frac{8}{3}$ of the diameter of the globe, that is, to $13\frac{1}{3}$ inches. Whence 2F becomes 28 feet 11 inches. A globe, falling *in vacuo* with its whole weight of $502\frac{3}{10}$ grains, will in one second of time describe $193\frac{1}{3}$ inches as above; and with the weight of 483 grains will describe 185,905 inches; and with that weight 483 grains *in vacuo* will describe the space F, or 14 feet $5\frac{1}{2}$ inches, in the time of 57‴ 58⁗, and acquire the greatest velocity it is capable of descending with in the air. With this velocity the globe in 8″ 12‴ of time will describe 245 feet and $5\frac{1}{3}$ inches. Subduct 1,3863F, or 20 feet and $\frac{1}{2}$ an inch, and there remain 225 feet 5 inches. This space, therefore, the falling globe ought by the

theory to describe in 8″ 12‴. But by the experiment it described a space of 220 feet. The difference is insensible.

By like calculations applied to the other globes full of air, I composed the following table.

The weights of the globe	The diameters	The times of falling from a height of 220 feet	The spaces which they would describe by the theory	The excesses.
510 grains	5.1 inches	8″ 12‴	226 feet 11 inch.	6 feet 11 inch.
642	5,2	7 42	230 9	10 9
599	5,1	7 42	227 10	7 0
515	5	7 57	224 5	4 5
483	5	8 12	225 5	5 5
641	5,2	7 42	230 7	10 7

EXPER. 14. *Anno* 1719, in the month of *July*, Dr. *Desaguliers* made some experiments of this kind again, by forming hogs' bladders into spherical orbs; which was done by means of a concave wooden sphere, which the bladders, being wetted well first, were put into. After that being blown full of air, they were obliged to fill up the spherical cavity that contained them; and then, when dry, were taken out. These were let fall from the lantern on the top of the cupola of the same church, namely, from a height of 272 feet; and at the same moment of time there was let fall a leaden globe, whose weight was about 2 pounds *troy* weight. And in the mean time some persons standing in the upper part of the church where the globes were let fall observed the whole times of falling; and others standing on the ground observed the differences of the times between the fall of the leaden weight and the fall of the bladder. The times were measured by pendulums oscillating to half seconds. And one of those that stood upon the ground had a machine vibrating four times in one second; and another had another machine accurately made with a pendulum vibrating four times in a second also. One of those also who stood at the top of the church had a like machine; and these instruments were so contrived, that their motions could be stopped or renewed at pleasure. Now the leaden globe fell in about four seconds and $\frac{1}{4}$ of time; and from the addition of this time to the difference of time above spoken of, was collected the whole time in which the bladder was falling. The times which the five bladders spent in falling, after the leaden globe had reached the ground, were, the first time, $14\frac{3}{4}''$, $12\frac{3}{4}''$, $14\frac{5}{8}''$, $17\frac{3}{4}''$, and $16\frac{7}{8}''$; and the second time, $14\frac{1}{2}''$, $14\frac{1}{4}''$, $14''$, $19''$, and $16\frac{3}{4}''$. Add to these $4\frac{1}{4}''$, the time in which the leaden globe was falling, and the whole times in which the five bladders fell were, the first time, $19''$, $17''$, $18\frac{7}{8}''$, $22''$, and $21\frac{1}{8}''$; and the second time, $18\frac{3}{4}''$, $18\frac{1}{2}''$, $18\frac{1}{4}''$, $23\frac{1}{4}''$, and $21''$. The times observed at the top of the church were, the first time, $19\frac{3}{8}''$, $17\frac{1}{4}''$, $18\frac{3}{4}''$, $22\frac{1}{8}''$, and $21\frac{5}{8}''$; and the second time, $19''$, $18\frac{5}{8}''$, $18\frac{3}{8}''$, $24''$. and $21\frac{1}{4}''$. But the bladders did not always fall directly down, but sometimes fluttered a little in the air, and waved to and fro, as

they were descending. And by these motions the times of their falling were prolonged, and increased by half a second sometimes, and sometimes by a whole second. The second and fourth bladder fell most directly the first time, and the first and third the second time. The fifth bladder was wrinkled, and by its wrinkles was a little retarded. I found their diameters by their circumferences measured with a very fine thread wound about them twice. In the following table I have compared the experiments with the theory; making the density of air to be to the density of rain-water as 1 to 860, and computing the spaces which by the theory the globes ought to describe in falling.

The weight of the bladders.	The diameters	The times of falling from a height of 272 feet	The spaces which by the theory ought to have been described in those times	The difference between the theory and the experiments
128 grains	5,28 inches	19″	271 feet 11 in.	— 0 ft. 1 in.
156	5,19	17	272 0½	+ 0 0½
137½	5.3	18	272 7	+ 0 7
97½	5,26	22	277 4	+ 5 4
99⅛	5	21⅛	282 0	+ 10 0

Our theory, therefore, exhibits rightly, within a very little, all the resistance that globes moving either in air or in water meet with; which appears to be proportional to the densities of the fluids in globes of equal velocities and magnitudes.

In the Scholium subjoined to the sixth Section, we shewed, by experiments of pendulums, that the resistances of equal and equally swift globes moving in air, water, and quicksilver, are as the densities of the fluids. We here prove the same more accurately by experiments of bodies falling in air and water. For pendulums at each oscillation excite a motion in the fluid always contrary to the motion of the pendulum in its return; and the resistance arising from this motion, as also the resistance of the thread by which the pendulum is suspended, makes the whole resistance of a pendulum greater than the resistance deduced from the experiments of falling bodies. For by the experiments of pendulums described in that Scholium, a globe of the same density as water in describing the length of its semidiameter in air would lose the $\frac{1}{3342}$ part of its motion. But by the theory delivered in this seventh Section, and confirmed by experiments of falling bodies, the same globe in describing the same length would lose only a part of its motion equal to $\frac{1}{4586}$, supposing the density of water to be to the density of air as 860 to 1. Therefore the resistances were found greater by the experiments of pendulums (for the reasons just mentioned) than by the experiments of falling globes; and that in the ratio of about 4 to 3. But yet since the resistances of pendulums oscillating in air, water, and quicksilver, are alike increased by like causes, the proportion of the resistances in these mediums will be rightly enough exhibited by the

experiments of pendulums, as well as by the experiments of falling bodies. And from all this it may be concluded, that the resistances of bodies, moving in any fluids whatsoever, though of the most extreme fluidity, are, *cæteris paribus*, as the densities of the fluids.

These things being thus established, we may now determine what part of its motion any globe projected in any fluid whatsoever would nearly lose in a given time. Let D be the diameter of the globe, and V its velocity at the beginning of its motion, and T the time in which a globe with the velocity V can describe *in vacuo* a space that is, to the space $\frac{2}{3}$D as the density of the globe to the density of the fluid; and the globe projected in that fluid will, in any other time t lose the part $\frac{tV}{T+t}$, the part $\frac{TV}{T+t}$ remaining; and will describe a space, which will be to that described in the same time *in vacuo* with the uniform velocity V, as the logarithm of the number $\frac{T+t}{T}$ multiplied by the number 2,302585093 is to the number $\frac{t}{T}$, by Cor. 7, Prop. XXXV. In slow motions the resistance may be a little less, because the figure of a globe is more adapted to motion than the figure of a cylinder described with the same diameter. In swift motions the resistance may be a little greater, because the elasticity and compression of the fluid do not increase in the duplicate ratio of the velocity. But these little nicetics I take no notice of.

And though air, water, quicksilver, and the like fluids, by the division of their parts *in infinitum*, should be subtilized, and become mediums infinitely fluid, nevertheless, the resistance they would make to projected globes would be the same. For the resistance considered in the preceding Propositions arises from the inactivity of the matter; and the inactivity of matter is essential to bodies, and always proportional to the quantity of matter. By the division of the parts of the fluid the resistance arising from the tenacity and friction of the parts may be indeed diminished; but the quantity of matter will not be at all diminished by this division; and if the quantity of matter be the same, its force of inactivity will be the same; and therefore the resistance here spoken of will be the same, as being always proportional to that force. To diminish this resistance, the quantity of matter in the spaces through which the bodies move must be diminished; and therefore the celestial spaces, through which the globes of the planets and comets are perpetually passing towards all parts, with the utmost freedom, and without the least sensible diminution of their motion, must be utterly void of any corporeal fluid, excepting, perhaps, some extremely rare vapours and the rays of light.

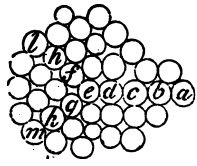
Projectiles excite a motion in fluids as they pass through them, and this motion arises from the excess of the pressure of the fluid at the fore parts of the projectile above the pressure of the same at the hinder parts; and cannot be less in mediums infinitely fluid than it is in air, water, and quicksilver, in proportion to the density of matter in each. Now this excess of pressure does, in proportion to its quantity, not only excite a motion in the fluid, but also acts upon the projectile so as to retard its motion; and therefore the resistance in every fluid is as the motion excited by the projectile in the fluid; and cannot be less in the most subtile æther in proportion to the density of that æther, than it is in air, water, and quicksilver, in proportion to the densities of those fluids.

SECTION VIII.

Of motion propagated through fluids.

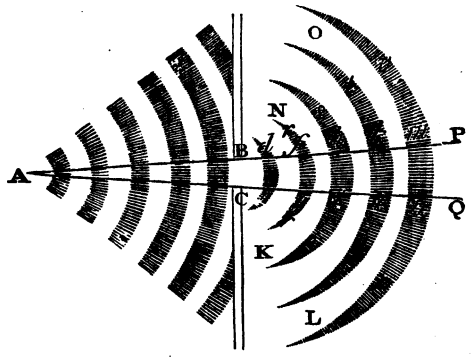
PROPOSITION XLI. THEOREM XXXII.

A pressure is not propagated through a fluid in rectilinear directions unless where the particles of the fluid lie in a right line.



If the particles *a*, *b*, *c*, *d*, *e*, lie in a right line, the pressure may be indeed directly propagated from *a* to *e*; but then the particle *e* will urge the obliquely posited particles *f* and *g* obliquely, and those particles *f* and *g* will not sustain this pressure, unless they be supported by the particles *h* and *k* lying beyond them; but the particles that support them are also pressed by them; and those particles cannot sustain that pressure, without being supported by, and pressing upon, those particles that lie still farther, as *l* and *m*, and so on *in infinitum*. Therefore the pressure, as soon as it is propagated to particles that lie out of right lines, begins to deflect towards one hand and the other, and will be propagated obliquely *in infinitum*; and after it has begun to be propagated obliquely, if it reaches more distant particles lying out of the right line, it will deflect again on each hand and this it will do as often as it lights on particles that do not lie exactly in a right line. Q.E.D.

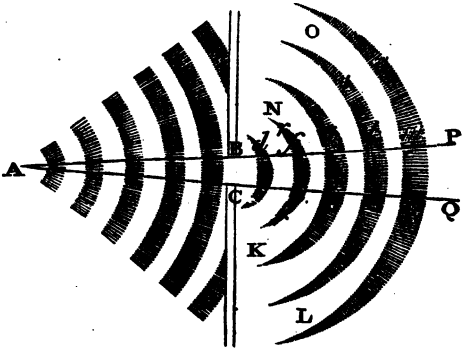
COR. If any part of a pressure, propagated through a fluid from a given point, be intercepted by any obstacle, the remaining part, which is not intercepted, will deflect into the spaces behind the obstacle. This may be demonstrated also after the following manner. Let a pressure be propagated from the point A towards any part, and, if it be possible, in rectilinear



directions; and the obstacle NBCK being perforated in BC, let all the pressure be intercepted but the coniform part APQ passing through the circular hole BC. Let the cone APQ be divided into frustums by the transverse planes, *de*, *fg*, *hi*. Then while the cone ABC, propagating the pressure, urges the conic frustum *degf* beyond it on the superficies *de*, and this frustum urges the next frustum *fgih* on the superficies *fg*, and that frustum urges a third frustum, and so *in infinitum;* it is manifest (by the third Law) that the first frustum *defg* is, by the re-action of the second frustum *fghi*, as much urged and pressed on the superficies *fg*, as it urges and presses that second frustum. Therefore the frustum *degf* is compressed on both sides, that is, between the cone *Ade* and the frustum *fhig;* and therefore (by Case 6, Prop. XIX) cannot preserve its figure, unless it be compressed with the same force on all sides. Therefore with the same force with which it is pressed on the superficies *de*, *fg*, it will endeavour to break forth at the sides *df*, *eg;* and there (being not in the least tenacious or hard, but perfectly fluid) it will run out, expanding itself, unless there be an ambient fluid opposing that endeavour. Therefore, by the effort it makes to run out, it will press the ambient fluid, at its sides *df*, *eg*, with the same force that it does the frustum *fghi;* and therefore, the pressure will be propagated as much from the sides *df*, *eg*, into the spaces NO, KL this way and that way, as it is propagated from the superficies *fg* towards PQ. Q.E.D.

PROPOSITION XLII. THEOREM XXXIII.

All motion propagated through a fluid diverges from a rectilinear progress into the unmoved spaces.



CASE 1. Let a motion be propagated from the point A through the hole BC, and, if it be possible, let it proceed in the conic space BCQP according to right lines diverging from the point A. And let us first suppose this motion to be that of waves in the surface of standing water; and let *de*, *fg*, *hi*, *kl*, &c., be the tops of the several waves, divided from each other by as any intermediate valleys or hollows. Then, because the water in the

ridges of the waves is higher than in the unmoved parts of the fluid KL, NO, it will run down from off the tops of those ridges, *e*, *g*, *i*, *l*, &c., *d*, *f*, *h*, *k*, &c., this way and that way towards KL and NO; and because the water is more depressed in the hollows of the waves than in the unmoved parts of the fluid KL, NO, it will run down into those hollows out of those unmoved parts. By the first deflux the ridges of the waves will dilate themselves this way and that way, and be propagated towards KL and NO. And because the motion of the waves from A towards PQ is carried on by a continual deflux from the ridges of the waves into the hollows next to them, and therefore cannot be swifter than in proportion to the celerity of the descent; and the descent of the water on each side towards KL and NO must be performed with the same velocity; it follows that the dilatation of the waves on each side towards KL and NO will be propagated with the same velocity as the waves themselves go forward with directly from A to PQ. And therefore the whole space this way and that way towards KL and NO will be filled by the dilated waves *rfgr*, *shis*, *tklt*, *vmnv*, &c. Q.E.D. That these things are so, any one may find by making the experiment in still water.

CASE 2. Let us suppose that *de*, *fg*, *hi*, *kl*, *mn*, represent pulses successively propagated from the point A through an elastic medium. Conceive the pulses to be propagated by successive condensations and rarefactions of the medium, so that the densest part of every pulse may occupy a spherical superficies described about the centre A, and that equal intervals intervene between the successive pulses. Let the lines *de*, *fg*, *hi*, *kl*, &c., represent the densest parts of the pulses, propagated through the hole BC; and because the medium is denser there than in the spaces on either side towards KL and NO, it will dilate itself as well towards those spaces KL, NO, on each hand, as towards the rare intervals between the pulses; and thence the medium, becoming always more rare next the intervals, and more dense next the pulses, will partake of their motion. And because the progressive motion of the pulses arises from the perpetual relaxation of the denser parts towards the antecedent rare intervals; and since the pulses will relax themselves on each hand towards the quiescent parts of the medium KL, NO, with very near the same celerity; therefore the pulses will dilate themselves on all sides into the unmoved parts KL, NO, with almost the same celerity with which they are propagated directly from the centre A; and therefore will fill up the whole space KLON. Q.E.D. And we find the same by experience also in sounds which are heard through a mountain interposed; and, if they come into a chamber through the window, dilate themselves into all the parts of the room, and are heard in every corner; and not as reflected from the opposite walls, but directly propagated from the window, as far as our sense can judge.

CASE 3 Let us suppose, lastly, that a motion of any kind is propagated

from A through the hole BC. Then since the cause of this propagation is that the parts of the medium that are near the centre A disturb and agitate those which lie farther from it; and since the parts which are urged are fluid, and therefore recede every way towards those spaces where they are less pressed, they will by consequence recede towards all the parts of the quiescent medium; as well to the parts on each hand, as KL and NO, as to those right before, as PQ; and by this means all the motion, as soon as it has passed through the hole BC, will begin to dilate itself, and from thence, as from its principle and centre, will be propagated directly every way. Q.E.D.

PROPOSITION XLIII. THEOREM XXXIV.

Every tremulous body in an elastic medium propagates the motion of the pulses on every side right forward; but in a non-elastic medium excites a circular motion.

CASE. 1. The parts of the tremulous body, alternately going and returning, do in going urge and drive before them those parts of the medium that lie nearest, and by that impulse compress and condense them; and in returning suffer those compressed parts to recede again, and expand themselves. Therefore the parts of the medium that lie nearest to the tremulous body move to and fro by turns, in like manner as the parts of the tremulous body itself do; and for the same cause that the parts of this body agitate these parts of the medium, these parts, being agitated by like tremors, will in their turn agitate others next to themselves; and these others, agitated in like manner, will agitate those that lie beyond them, and so on *in infinitum*. And in the same manner as the first parts of the medium were condensed in going, and relaxed in returning, so will the other parts be condensed every time they go, and expand themselves every time they return. And therefore they will not be all going and all returning at the same instant (for in that case they would always preserve determined distances from each other, and there could be no alternate condensation and rarefaction); but since, in the places where they are condensed, they approach to, and, in the places where they are rarefied, recede from each other, therefore some of them will be going while others are returning; and so on *in infinitum*. The parts so going, and in their going condensed, are pulses, by reason of the progressive motion with which they strike obstacles in their way; and therefore the successive pulses produced by a tremulous body will be propagated in rectilinear directions; and that at nearly equal distances from each other, because of the equal intervals of time in which the body, by its several tremors produces the several pulses. And though the parts of the tremulous body go and return in some certain and determinate direction, yet the pulses propagated from thence through the medium will dilate themselves towards the sides, by the foregoing Proposition; and

will be propagated on all sides from that tremulous body, as from a common centre, in superficies nearly spherical and concentrical. An example of this we have in waves excited by shaking a finger in water, which proceed not only forward and backward agreeably to the motion of the finger, but spread themselves in the manner of concentrical circles all round the finger, and are propagated on every side. For the gravity of the water supplies the place of elastic force.

Case 2. If the medium be not elastic, then, because its parts cannot be condensed by the pressure arising from the vibrating parts of the tremulous body, the motion will be propagated in an instant towards the parts where the medium yields most easily, that is, to the parts which the tremulous body would otherwise leave vacuous behind it. The case is the same with that of a body projected in any medium whatever. A medium yielding to projectiles does not recede *in infinitum*, but with a circular motion comes round to the spaces which the body leaves behind it. Therefore as often as a tremulous body tends to any part, the medium yielding to it comes round in a circle to the parts which the body leaves; and as often as the body returns to the first place, the medium will be driven from the place it came round to, and return to its original place. And though the tremulous body be not firm and hard, but every way flexible, yet if it continue of a given magnitude, since it cannot impel the medium by its tremors any where without yielding to it somewhere else, the medium receding from the parts of the body where it is pressed will always come round in a circle to the parts that yield to it. Q.E.D.

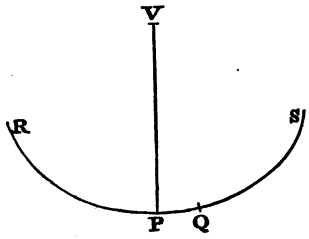
Cor. It is a mistake, therefore, to think, as some have done, that the agitation of the parts of flame conduces to the propagation of a pressure in rectilinear directions through an ambient medium. A pressure of that kind must be derived not from the agitation only of the parts of flame, but from the dilatation of the whole.

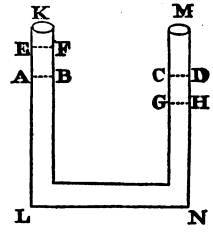
PROPOSITION XLIV. THEOREM XXXV.

If water ascend and descend alternately in the erected legs KL, MN, of a canal or pipe; and a pendulum be constructed whose length between the point of suspension and the centre of oscillation is equal to half the length of the water in the canal; I say, that the water will ascend and descend in the same times in which the pendulum oscillates.

I measure the length of the water along the axes of the canal and its legs, and make it equal to the sum of those axes; and take no notice of the resistance of the water arising from its attrition by the sides of the canal. Let, therefore, AB, CD, represent the mean height of the water in both legs; and when the water in the leg KL ascends to the height EF, the water will descend in the leg MN to the height GH. Let P be a pendulous

body, VP the thread, V the point of suspension, RPQS the cycloid which





the pendulum describes, P its lowest point, PQ an arc equal to the neight AE. The force with which the motion of the water is accelerated and retarded alternately is the excess of the weight of the water in one leg above the weight in the other; and, therefore, when the water in the leg KL ascends to EF, and in the other leg descends to GH, that force is double the weight of the water EABF, and therefore is to the weight of the whole water as AE or PQ to VP or PR. The force also with which the body P is accelerated or retarded in any place, as Q, of a cycloid, is (by Cor. Prop. LI) to its whole weight as its distance PQ from the lowest place P to the length PR of the cycloid. Therefore the motive forces of the water and pendulum, describing the equal spaces AE, PQ, are as the weights to be moved; and therefore if the water and pendulum are quiescent at first, those forces will move them in equal times, and will cause them to go and return together with a reciprocal motion. Q.E.D.

COR. 1. Therefore the reciprocations of the water in ascending and descending are all performed in equal times, whether the motion be more or less intense or remiss.

COR. 2. If the length of the whole water in the canal be of $6\frac{1}{9}$ feet of *French* measure, the water will descend in one second of time, and will ascend in another second, and so on by turns *in infinitum;* for a pendulum of $3\frac{1}{18}$ such feet in length will oscillate in one second of time.

COR. 3. But if the length of the water be increased or diminished, the time of the reciprocation will be increased or diminished in the subduplicate ratio of the length.

PROPOSITION XLV. THEOREM XXXVI.

The velocity of waves is in the subduplicate ratio of the breadths.

This follows from the construction of the following Proposition.

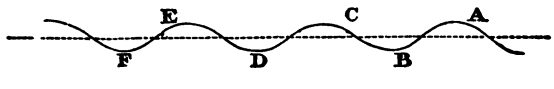
PROPOSITION XLVI. PROBLEM X.

To find the velocity of waves.

Let a pendulum be constructed, whose length between the point of suspension and the centre of oscillation is equal to the breadth of the waves

and in the time that the pendulum will perform one single oscillation the waves will advance forward nearly a space equal to their breadth.

That which I call the breadth of the waves is the transverse measure lying between the deepest part of the hollows, or the tops of the ridges. Let



ABCDEF represent the surface of stagnant water ascending and descending in successive waves; and let A, C, E, &c., be the tops of the waves; and let B, D, F, &c., be the intermediate hollows. Because the motion of the waves is carried on by the successive ascent and descent of the water, so that the parts thereof, as A, C, E, &c., which are highest at one time become lowest immediately after; and because the motive force, by which the highest parts descend and the lowest ascend, is the weight of the elevated water, that alternate ascent and descent will be analogous to the reciprocal motion of the water in the canal, and observe the same laws as to the times of its ascent and descent; and therefore (by Prop. XLIV) if the distances between the highest places of the waves A, C, E, and the lowest B, D, F, be equal to twice the length of any pendulum, the highest parts A, C, E, will become the lowest in the time of one oscillation, and in the time of another oscillation will ascend again. Therefore between the passage of each wave, the time of two oscillations will intervene; that is, the wave will describe its breadth in the time that pendulum will oscillate twice; but a pendulum of four times that length, and which therefore is equal to the breadth of the waves, will just oscillate once in that time. Q.E.I.

COR. 1. Therefore waves, whose breadth is equal to $3\frac{1}{18}$ *French* feet, will advance through a space equal to their breadth in one second of time; and therefore in one minute will go over a space of $183\frac{1}{3}$ feet; and in an hour a space of 11000 feet, nearly.

COR. 2. And the velocity of greater or less waves will be augmented or diminished in the subduplicate ratio of their breadth.

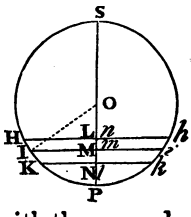
These things are true upon the supposition that the parts of water ascend or descend in a right line; but, in truth, that ascent and descent is rather performed in a circle; and therefore I propose the time defined by this Proposition as only near the truth.

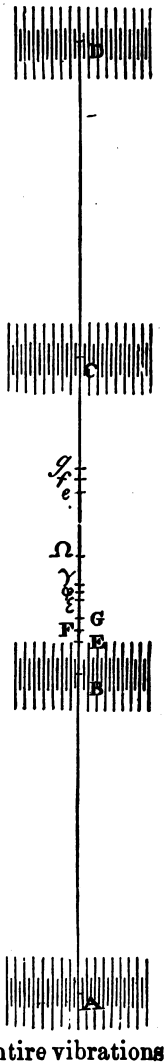
PROPOSITION XLVII. THEOREM XXXVII.

If pulses are propagated through a fluid, the several particles of the fluid, going and returning with the shortest reciprocal motion, are always accelerated or retarded according to the law of the oscillating pendulum.

Let AB, BC, CD, &c., represent equal distances of successive pulses; ABC the line of direction of the motion of the successive pulses propagated

from A to B; E, F, G three physical points of the quiescent medium situate in the right line AC at equal distances from each other; E*e*, F*f*, G*g*, equal spaces of extreme shortness, through which those points go and return with a reciprocal motion in each vibration; ε, ϕ, γ, any intermediate places of the same points; EF, FG physical lineolæ, or linear parts of the medium lying between those points, and successively transferred into the places $\varepsilon\phi$, $\phi\gamma$, and *ef*, *fg*. Let there be drawn the right line PS equal to the right line E*e*. Bisect the same in O, and from the centre O, with the interval OP, describe the circle SIP*i*. Let the whole time of one vibration; with its proportional parts, be expounded by the whole circumference of this circle and its parts, in such sort, that, when any time PH or PHS*h* is completed, if there be let fall to PS the perpendicular HL or *hl*, and there be taken Eε equal to PL or P*l*, the physical point E may be found in ε. A point, as E, moving acccording to this law with a reciprocal motion, in its going from E through ε to *e*, and returning again through ε to E, will perform its several vibrations with the same degrees of acceleration and retardation with those of an oscillating pendulum. We are now to prove that the several physical points of the medium will be agitated with such a kind of motion. Let us suppose, then, that a medium hath such a motion excited in it from any cause whatsoever, and consider what will follow from thence.





In the circumference PHS*h* let there be taken the equal arcs, HI, IK, or *hi*, *ik*, having the same ratio to the whole circumference as the equal right lines EF, FG have to BC, the whole interval of the pulses. Let fall the perpendiculars IM, KN, or *im*, *kn*; then because the points E, F, G are successively agitated with like motions, and perform their entire vibrations composed of their going and return, while the pulse is transferred from B to C; if PH or PHS*h* be the time elapsed since the beginning of the motion of the point E, then will PI or PHS*i* be the time elapsed since the beginning of the motion of the point F, and PK or PHS*k* the time elapsed since the beginning of the motion of the point G; and therefore Eε, Fϕ, Gγ, will be respectively equal to PL, PM, PN, while the points are going, and to P*l*, P*m*, P*n*, when the points are returning. Therefore $\varepsilon\gamma$ or EG + Gγ — Eε will, when the points are going, be equal to EG — LN

and in their return equal to EG + *ln*. But $\varepsilon\gamma$ is the breadth or expansion of the part EG of the medium in the place $\varepsilon\gamma$; and therefore the expansion of that part in its going is to its mean expansion as EG — LN to EG; and in its return, as EG + *ln* or EG + LN to EG. Therefore since LN is to KH as IM to the radius OP, and KH to EG as the circumference PHS*h*P to BC; that is, if we put V for the radius of a circle whose circumference is equal to BC the interval of the pulses, as OP to V; and, *ex æquo*, LN to EG as IM to V; the expansion of the part EG, or of the physical point F in the place $\varepsilon\gamma$, to the mean expansion of the same part in its first place EG, will be as V — IM to V in going, and as V + *im* to V in its return. Hence the elastic force of the point F in the place $\varepsilon\gamma$ to its mean elastic force in the place EG is as $\frac{1}{V - IM}$ to $\frac{1}{V}$ in its going, and as $\frac{1}{V + im}$ to $\frac{1}{V}$ in its return. And by the same reasoning the elastic forces of the physical points E and G in going are as $\frac{1}{V - HL}$ and $\frac{1}{V - KN}$ to $\frac{1}{V}$; and the difference of the forces to the mean elastic force of the medium as $\frac{HL - KN}{VV - V \times HL - V \times KN + HL \times KN}$ to $\frac{1}{V}$; that is, as $\frac{HL - KN}{VV}$ to $\frac{1}{V}$, or as HL — KN to V; if we suppose (by reason of the very short extent of the vibrations) HL and KN to be indefinitely less than the quantity V. Therefore since the quantity V is given, the difference of the forces is as HL — KN; that is (because HL — KN is proportional to HK, and OM to OI or OP; and because HK and OP are given) as OM; that is, if F*f* be bisected in Ω, as $\Omega\phi$. And for the same reason the difference of the elastic forces of the physical points ε and γ, in the return of the physical lineola $\varepsilon\gamma$, is as $\Omega\phi$. But that difference (that is, the excess of the elastic force of the point ε above the elastic force of the point γ) is the very force by which the intervening physical lineola $\varepsilon\gamma$ of the medium is accelerated in going, and retarded in returning; and therefore the accelerative force of the physical lineola $\varepsilon\gamma$ is as its distance from Ω, the middle place of the vibration. Therefore (by Prop. XXXVIII, Book I) the time is rightly expounded by the arc PI; and the linear part of the medium $\varepsilon\gamma$ is moved according to the law above-mentioned, that is, according to the law of a pendulum oscillating; and the case is the same of all the linear parts of which the whole medium is compounded. Q.E.D.

COR. Hence it appears that the number of the pulses propagated is the same with the number of the vibrations of the tremulous body, and is not multiplied in their progress. For the physical lineola $\varepsilon\gamma$ as soon as it returns to its first place is at rest; neither will it move again, unless it

receives a new motion either from the impulse of the tremulous body, or of the pulses propagated from that body. As soon, therefore, as the pulses cease to be propagated from the tremulous body, it will return to a state of rest, and move no more.

PROPOSITION XLVIII. THEOREM XXXVIII.

The velocities of pulses propagated in an elastic fluid are in a ratio compounded of the subduplicate ratio of the elastic force directly, and the subduplicate ratio of the density inversely; supposing the elastic force of the fluid to be proportional to its condensation

CASE 1. If the mediums be homogeneous, and the distances of the pulses in those mediums be equal amongst themselves, but the motion in one medium is more intense than in the other, the contractions and dilatations of the correspondent parts will be as those motions; not that this proportion is perfectly accurate. However, if the contractions and dilatations are not exceedingly intense, the error will not be sensible; and therefore this proportion may be considered as physically exact. Now the motive elastic forces are as the contractions and dilatations; and the velocities generated in the same time in equal parts are as the forces. Therefore equal and corresponding parts of corresponding pulses will go and return together, through spaces proportional to their contractions and dilatations, with velocities that are as those spaces; and therefore the pulses, which in the time of one going and returning advance forward a space equal to their breadth, and are always succeeding into the places of the pulses that immediately go before them, will, by reason of the equality of the distances, go forward in both mediums with equal velocity.

CASE 2. If the distances of the pulses or their lengths are greater in one medium than in another, let us suppose that the correspondent parts describe spaces, in going and returning, each time proportional to the breadths of the pulses; then will their contractions and dilatations be equal; and therefore if the mediums are homogeneous, the motive elastic forces, which agitate them with a reciprocal motion, will be equal also. Now the matter to be moved by these forces is as the breadth of the pulses; and the space through which they move every time they go and return is in the same ratio. And, moreover, the time of one going and returning is in a ratio compounded of the subduplicate ratio of the matter, and the subduplicate ratio of the space; and therefore is as the space. But the pulses advance a space equal to their breadths in the times of going once and returning once; that is, they go over spaces proportional to the times, and therefore are equally swift.

CASE 3. And therefore in mediums of equal density and elastic force all the pulses are equally swift. Now if the density or the elastic force of the medium were augmented, then, because the motive force is increased

in the ratio of the elastic force, and the matter to be moved is increased in the ratio of the density, the time which is necessary for producing the same motion as before will be increased in the subduplicate ratio of the density, and will be diminished in the subduplicate ratio of the elastic force. And therefore the velocity of the pulses will be in a ratio compounded of the subduplicate ratio of the density of the medium inversely, and the subduplicate ratio of the elastic force directly. Q.E.D.

This Proposition will be made more clear from the construction of the following Problem.

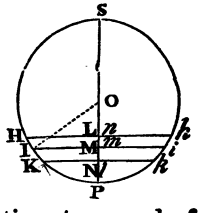
PROPOSITION XLIX. PROBLEM XI.

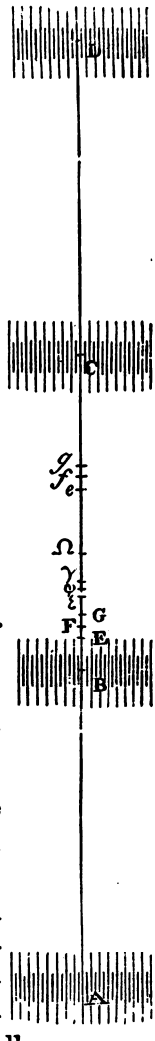
The density and elastic force of a medium being given, to find the velocity of the pulses.

Suppose the medium to be pressed by an incumbent weight after the manner of our air; and let A be the height of a homogeneous medium, whose weight is equal to the incumbent weight, and whose density is the same with the density of the compressed medium in which the pulses are propagated. Suppose a pendulum to be constructed whose length between the point of suspension and the centre of oscillation is A: and in the time in which that pendulum will perform one entire oscillation composed of its going and returning, the pulse will be propagated right onwards through a space equal to the circumference of a circle described with the radius A.

For, letting those things stand which were constructed in Prop. XLVII, if any physical line, as EF, describing the space PS in each vibration, be acted on in the extremities P and S of every going and return that it makes by an elastic force that is equal to its weight, it will perform its several vibrations in the time in which the same might oscillate in a cycloid whose whole perimeter is equal to the length PS; and that because equal forces will impel equal corpuscles through equal spaces in the same or equal times. Therefore since the times of the oscillations are in the subduplicate ratio of the lengths of the pendulums, and the length of the pendulum is equal to half the arc of the whole cycloid, the time of one vibration would be to the time of the oscillation of a pendulum whose length is A in the subduplicate ratio of the length $\frac{1}{2}$PS or PO to the length A. But the elastic force with which the physical lineola EG is urged, when it is found in its extreme places P, S, was (in the demonstration of Prop. XLVII) to its whole elastic force as HL — KN to V, that is (since the point K now falls upon P), as HK to V: and all that force, or which is the same thing, the incumbent weight by which the lineola EG is compressed, is to the weight of the lineola as the altitude A of the incumbent weight to EG the length of the lineola; and therefore, *ex æquo*, the force

with which the lineola EG is urged in the places P and S is to the weight of that lineola as HK × A to V × EG; or as PO × A to VV; because HK was to EG as PO to V. Therefore since the times in which equal bodies are impelled through equal spaces are reciprocally in the subduplicate ratio of the forces, the time of one vibration, produced by the action of that elastic force, will be to the time of a vibration, produced by the impulse of the weight in a subduplicate ratio of VV to PO × A, and therefore to the time of the oscillation of a pendulum whose length is A in the subduplicate ratio of VV to PO × A, and the subduplicate ratio of PO to A conjunctly; that is, in the entire ratio of V to A. But in the time of one vibration composed of the going and returning of the pendulum, the pulse will be propagated right onward through a space equal to its breadth BC. Therefore the time in which a pulse runs over the space BC is to the time of one oscillation composed of the going and returning of the pendulum as V to A, that is, as BC to the circumference of a circle whose radius is A. But the time in which the pulse will run over the space BC is to the time in which it will run over a length equal to that circumference in the same ratio; and therefore in the time of such an oscillation the pulse will run over a length equal to that circumference. Q.E.D.





COR. 1. The velocity of the pulses is equal to that which heavy bodies acquire by falling with an equally accelerated motion, and in their fall describing half the altitude A. For the pulse will, in the time of this fall, supposing it to move with the velocity acquired by that fall, run over a space that will be equal to the whole altitude A; and therefore in the time of one oscillation composed of one going and return, will go over a space equal to the circumference of a circle described with the radius A; for the time of the fall is to the time of oscillation as the radius of a circle to its circumference.

COR. 2. Therefore since that altitude A is as the elastic force of the fluid directly, and the density of the same inversely, the velocity of the pulses will be in a ratio compounded of the subduplicate ratio of the density inversely, and the subduplicate ratio of the elastic force directly.

PROPOSITION L. PROBLEM XII.

To find the distances of the pulses.

Let the number of the vibrations of the body, by whose tremor the pulses are produced, be found to any given time. By that number divide the space which a pulse can go over in the same time, and the part found will be the breadth of one pulse. Q.E.I.

SCHOLIUM.

The last Propositions respect the motions of light and sounds; for since light is propagated in right lines, it is certain that it cannot consist in action alone (by Prop. XLI and XLII). As to sounds, since they arise from tremulous bodies, they can be nothing else but pulses of the air propagated through it (by Prop. XLIII); and this is confirmed by the tremors which sounds, if they be loud and deep, excite in the bodies near them, as we experience in the sound of drums; for quick and short tremors are less easily excited. But it is well known that any sounds, falling upon strings in unison with the sonorous bodies, excite tremors in those strings. This is also confirmed from the velocity of sounds; for since the specific gravities of rain-water and quicksilver are to one another as about 1 to $13\frac{2}{3}$, and when the mercury in the barometer is at the height of 30 inches of our measure, the specific gravities of the air and of rain-water are to one another as about 1 to 870, therefore the specific gravity of air and quicksilver are to each other as 1 to 11890. Therefore when the height of the quicksilver is at 30 inches, a height of uniform air, whose weight would be sufficient to compress our air to the density we find it to be of, must be equal to 356700 inches, or 29725 feet of our measure; and this is that very height of the medium, which I have called A in the construction of the foregoing Proposition. A circle whose radius is 29725 feet is 186768 feet in circumference. And since a pendulum $39\frac{1}{2}$ inches in length completes one oscillation, composed of its going and return, in two seconds of time, as is commonly known, it follows that a pendulum 29725 feet, or 356700 inches in length will perform a like oscillation in $190\frac{3}{4}$ seconds. Therefore in that time a sound will go right onwards 186768 feet, and therefore in one second 979 feet.

But in this computation we have made no allowance for the crassitude of the solid particles of the air, by which the sound is propagated instantaneously. Because the weight of air is to the weight of water as 1 to 870, and because salts are almost twice as dense as water; if the particles of air are supposed to be of near the same density as those of water or salt, and the rarity of the air arises from the intervals of the particles; the diameter of one particle of air will be to the interval between the centres

of the particles as 1 to about 9 or 10, and to the interval between the particles themselves as 1 to 8 or 9. Therefore to 979 feet, which, according to the above calculation, a sound will advance forward in one second of time, we may add $\frac{979}{9}$, or about 109 feet, to compensate for the crassitude of the particles of the air: and then a sound will go forward about 1088 feet in one second of time.

Moreover, the vapours floating in the air being of another spring, and a different tone, will hardly, if at all, partake of the motion of the true air in which the sounds are propagated. Now if these vapours remain unmoved, that motion will be propagated the swifter through the true air alone, and that in the subduplicate ratio of the defect of the matter. So if the atmosphere consist of ten parts of true air and one part of vapours, the motion of sounds will be swifter in the subduplicate ratio of 11 to 10, or very nearly in the entire ratio of 21 to 20, than if it were propagated through eleven parts of true air: and therefore the motion of sounds above discovered must be increased in that ratio. By this means the sound will pass through 1142 feet in one second of time.

These things will be found true in spring and autumn, when the air is rarefied by the gentle warmth of those seasons, and by that means its elastic force becomes somewhat more intense. But in winter, when the air is condensed by the cold, and its elastic force is somewhat remitted, the motion of sounds will be slower in a subduplicate ratio of the density; and, on the other hand, swifter in the summer.

Now by experiments it actually appears that sounds do really advance in one second of time about 1142 feet of *English* measure, or 1070 feet of *French* measure.

The velocity of sounds being known, the intervals of the pulses are known also. For M. *Sauveur*, by some experiments that he made, found that an open pipe about five *Paris* feet in length gives a sound of the same tone with a viol-string that vibrates a hundred times in one second. Therefore there are near 100 pulses in a space of 1070 *Paris* feet, which a sound runs over in a second of time; and therefore one pulse fills up a space of about $10\frac{7}{10}$ *Paris* feet, that is, about twice the length of the pipe. From whence it is probable that the breadths of the pulses, in all sounds made in open pipes, are equal to twice the length of the pipes.

Moreover, from the Corollary of Prop. XLVII appears the reason why the sounds immediately cease with the motion of the sonorous body, and why they are heard no longer when we are at a great distance from the sonorous bodies than when we are very near them. And besides, from the foregoing principles, it plainly appears how it comes to pass that sounds are so mightily increased in speaking-trumpets; for all reciprocal motion uses to be increased by the generating cause at each return. And in tubes hindering the dilatation of the sounds, the motion decays more slowly, and

recurs more forcibly; and therefore is the more increased by the new motion impressed at each return. And these are the principal phænomena of sounds.

SECTION IX.

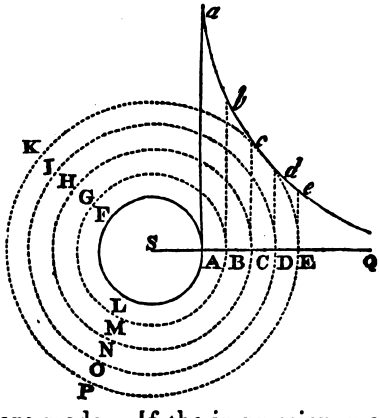
Of the circular motion of fluids.

HYPOTHESIS.

The resistance arising from the want of lubricity in the parts of a fluid, is, cæteris paribus, *proportional to the velocity with which the parts of the fluid are separated from each other.*

PROPOSITION LI. THEOREM XXXIX.

If a solid cylinder infinitely long, in an uniform and infinite fluid, revolve with an uniform motion about an axis given in position, and the fluid be forced round by only this impulse of the cylinder, and every part of the fluid persevere uniformly in its motion; I say, that the periodic times of the parts of the fluid are as their distances from the axis of the cylinder.



Let AFL be a cylinder turning uniformly about the axis S, and let the concentric circles BGM, CHN, DIO, EKP, &c., divide the fluid into innumerable concentric cylindric solid orbs of the same thickness. Then, because the fluid is homogeneous, the impressions which the contiguous orbs make upon each other mutually will be (by the Hypothesis) as their translations from each other, and as the contiguous superficies upon which the impressions are made. If the impression made upon any orb be greater or less on its concave than on its convex side, the stronger impression will prevail, and will either accelerate or retard the motion of the orb, according as it agrees with, or is contrary to, the motion of the same. Therefore, that every orb may persevere uniformly in its motion, the impressions made on both sides must be equal and their directions contrary. Therefore since the impressions are as the contiguous superficies, and as their translations from one another, the translations will be inversely as the superficies, that is, inversely as the distances of the superficies from the axis. But the differences of

the angular motions about the axis are as those translations applied to the distances, or as the translations directly and the distances inversely; that is, joining these ratios together, as the squares of the distances inversely. Therefore if there be erected the lines Aa, Bb, Cc, Dd, Ee, &c., perpendicular to the several parts of he infinite right line SABCDEQ, and reciprocally proportional to the squares of SA, SB, SC, SD, SE, &c., and through the extremities of those perpendiculars there be supposed to pass an hyperbolic curve, the sums of the differences, that is, the whole angular motions, will be as the correspondent sums of the lines Aa, Bb, Cc, Dd, Ee, that is (if to constitute a medium uniformly fluid the number of the orbs be increased and their breadth diminished *in infinitum*), as the hyperbolic areas AaQ, BbQ, CcQ, DdQ, EeQ, &c., analogous to the sums; and the times, reciprocally proportional to the angular motions, will be also reciprocally proportional to those areas. Therefore the periodic time of any particle as D, is reciprocally as the area DdQ, that is (as appears from the known methods of quadratures of curves), directly as the distance SD. Q.E.D.

COR. 1. Hence the angular motions of the particles of the fluid are reciprocally as their distances from the axis of the cylinder, and the absolute velocities are equal.

COR. 2. If a fluid be contained in a cylindric vessel of an infinite length, and contain another cylinder within, and both the cylinders revolve about one common axis, and the times of their revolutions be as their semidiameters, and every part of the fluid perseveres in its motion, the periodic times of the several parts will be as the distances from the axis of the cylinders.

COR. 3. If there be added or taken away any common quantity of angular motion from the cylinder and fluid moving in this manner; yet because this new motion will not alter the mutual attrition of the parts of the fluid, the motion of the parts among themselves will not be changed; for the translations of the parts from one another depend upon the attrition. Any part will persevere in that motion, which, by the attrition made on both sides with contrary directions, is no more accelerated than it is retarded.

COR. 4. Therefore if there be taken away from this whole system of the cylinders and the fluid all the angular motion of the outward cylinder, we shall have the motion of the fluid in a quiescent cylinder.

COR. 5. Therefore if the fluid and outward cylinder are at rest, and the inward cylinder revolve uniformly, there will be communicated a circular motion to the fluid, which will be propagated by degrees through the whole fluid; and will go on continually increasing, till such time as the several parts of the fluid acquire the motion determined in Cor. 4.

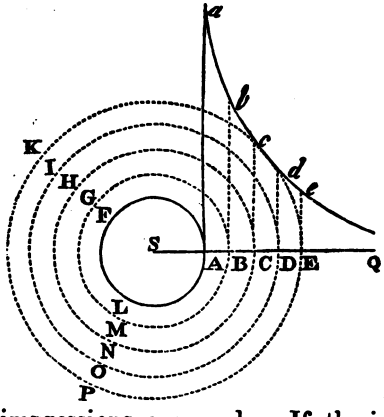
COR. 6. And because the fluid endeavours to propagate its motion still

farther, its impulse will carry the outmost cylinder also about with it, unless the cylinder be violently detained; and accelerate its motion till the periodic times of both cylinders become equal among themselves. But if the outward cylinder be violently detained, it will make an effort to retard the motion of the fluid; and unless the inward cylinder preserve that motion by means of some external force impressed thereon, it will make it cease by degrees.

All these things will be found true by making the experiment in deep standing water.

PROPOSITION LII. THEOREM XL.

If a solid sphere, in an uniform and infinite fluid, revolves about an axis given in position with an uniform motion, and the fluid be forced round by only this impulse of the sphere; and every part of the fluid perseveres uniformly in its motion; I say, that the periodic times of the parts of the fluid are as the squares of their distances from the centre of the sphere.



Case 1. Let AFL be a sphere turning uniformly about the axis S, and let the concentric circles BGM, CHN, DIO, EKP, &c., divide the fluid into innumerable concentric orbs of the same thickness. Suppose those orbs to be solid; and, because the fluid is homogeneous, the impressions which the contiguous orbs make one upon another will be (by the supposition) as their translations from one another, and the contiguous superficies upon which the impressions are made. If the impression upon any orb be greater or less upon its concave than upon its convex side, the more forcible impression will prevail, and will either accelerate or retard the velocity of the orb, according as it is directed with a conspiring or contrary motion to that of the orb. Therefore that every orb may persevere uniformly in its motion, it is necessary that the impressions made upon both sides of the orb should be equal, and have contrary directions. Therefore since the impressions are as the contiguous superficies, and as their translations from one another, the translations will be inversely as the superficies, that is, inversely as the squares of the distances of the superficies from the centre. But the differences of the angular motions about the axis are as those translations applied to the distances, or as the translations directly and the distances inversely; that is, by compounding those ratios, as the cubes of the distances inversely. Therefore if upon the several parts of the infinite right line SABCDEQ

there be erected the perpendiculars A*a*, B*b*, C*c*, D*d*, E*e*, &c., reciprocally proportional to the cubes of SA, SB, SC, SD, SE, &c., the sums of the differences, that is, the whole angular motions will be as the corresponding sums of the lines A*a*, B*b*, C*c*, D*d*, E*e*, &c., that is (if to constitute an uniformly fluid medium the number of the orbs be increased and their thickness diminished *in infinitum*), as the hyperbolic areas A*a*Q, B*b*Q, C*c*Q, D*d*Q, E*e*Q, &c., analogous to the sums; and the periodic times being reciprocally proportional to the angular motions, will be also reciprocally proportional to those areas. Therefore the periodic time of any orb DIO is reciprocally as the area D*d*Q, that is (by the known methods of quadratures), directly as the square of the distance SD. Which was first to be demonstrated.

CASE 2. From the centre of the sphere let there be drawn a great number of indefinite right lines, making given angles with the axis, exceeding one another by equal differences; and, by these lines revolving about the axis, conceive the orbs to be cut into innumerable annuli; then will every annulus have four annuli contiguous to it, that is, one on its inside, one on its outside, and two on each hand. Now each of these annuli cannot be impelled equally and with contrary directions by the attrition of the interior and exterior annuli, unless the motion be communicated according to the law which we demonstrated in Case 1. This appears from that demonstration. And therefore any series of annuli, taken in any right line extending itself *in infinitum* from the globe, will move according to the law of Case 1, except we should imagine it hindered by the attrition of the annuli on each side of it. But now in a motion, according to this law, no such is, and therefore cannot be, any obstacle to the motions persevering according to that law. If annuli at equal distances from the centre revolve either more swiftly or more slowly near the poles than near the ecliptic, they will be accelerated if slow, and retarded if swift, by their mutual attrition; and so the periodic times will continually approach to equality, according to the law of Case 1. Therefore this attrition will not at all hinder the motion from going on according to the law of Case 1, and therefore that law will take place; that is, the periodic times of the several annuli will be as the squares of their distances from the centre of the globe. Which was to be demonstrated in the second place.

CASE 3. Let now every annulus be divided by transverse sections into innumerable particles constituting a substance absolutely and uniformly fluid; and because these sections do not at all respect the law of circular motion, but only serve to produce a fluid substance, the law of circular motion will continue the same as before. All the very small annuli will either not at all change their asperity and force of mutual attrition upon account of these sections, or else they will change the same equally. Therefore the proportion of the causes remaining the same, the proportion of the effects

will remain the same also; that is, the proportion of the motions and the periodic times. Q.E.D. But now as the circular motion, and the centrifugal force thence arising, is greater at the ecliptic than at the poles, there must be some cause operating to retain the several particles in their circles; otherwise the matter that is at the ecliptic will always recede from the centre, and come round about to the poles by the outside of the vortex, and from thence return by the axis to the ecliptic with a perpetual circulation.

COR. 1. Hence the angular motions of the parts of the fluid about the axis of the globe are reciprocally as the squares of the distances from the centre of the globe, and the absolute velocities are reciprocally as the same squares applied to the distances from the axis.

COR. 2. If a globe revolve with a uniform motion about an axis of a given position in a similar and infinite quiescent fluid with an uniform motion, it will communicate a whirling motion to the fluid like that of a vortex, and that motion will by degrees be propagated onward *in infinitum*; and this motion will be increased continually in every part of the fluid, till the periodical times of the several parts become as the squares of the distances from the centre of the globe.

COR. 3. Because the inward parts of the vortex are by reason of their greater velocity continually pressing upon and driving forward the external parts, and by that action are perpetually communicating motion to them, and at the same time those exterior parts communicate the same quantity of motion to those that lie still beyond them, and by this action preserve the quantity of their motion continually unchanged, it is plain that the motion is perpetually transferred from the centre to the circumference of the vortex, till it is quite swallowed up and lost in the boundless extent of that circumference. The matter between any two spherical superficies concentrical to the vortex will never be accelerated; because that matter will be always transferring the motion it receives from the matter nearer the centre to that matter which lies nearer the circumference.

COR. 4. Therefore, in order to continue a vortex in the same state of motion, some active principle is required from which the globe may receive continually the same quantity of motion which it is always communicating to the matter of the vortex. Without such a principle it will undoubtedly come to pass that the globe and the inward parts of the vortex, being always propagating their motion to the outward parts, and not receiving any new motion, will gradually move slower and slower, and at last be carried round no longer.

COR. 5. If another globe should be swimming in the same vortex at a certain distance from its centre, and in the mean time by some force revolve constantly about an axis of a given inclination, the motion of this globe will drive the fluid round after the manner of a vortex; and at first this

new and small vortex will revolve with its globe about the centre of the other; and in the mean time its motion will creep on farther and farther, and by degrees be propagated *in infinitum*, after the manner of the first vortex. And for the same reason that the globe of the new vortex was carried about before by the motion of the other vortex, the globe of this other will be carried about by the motion of this new vortex, so that the two globes will revolve about some intermediate point, and by reason of that circular motion mutually fly from each other, unless some force restrains them. Afterward, if the constantly impressed forces, by which the globes persevere in their motions, should cease, and every thing be left to act according to the laws of mechanics, the motion of the globes will languish by degrees (for the reason assigned in Cor. 3 and 4), and the vortices at last will quite stand still.

COR. 6. If several globes in given places should constantly revolve with determined velocities about axes given in position, there would arise from them as many vortices going on *in infinitum*. For upon the same account that any one globe propagates its motion *in infinitum*, each globe apart will propagate its own motion *in infinitum* also; so that every part of the infinite fluid will be agitated with a motion resulting from the actions of all the globes. Therefore the vortices will not be confined by any certain limits, but by degrees run mutually into each other; and by the mutual actions of the vortices on each other, the globes will be perpetually moved from their places, as was shewn in the last Corollary; neither can they possibly keep any certain position among themselves, unless some force restrains them. But if those forces, which are constantly impressed upon the globes to continue these motions, should cease, the matter (for the reason assigned in Cor. 3 and 4) will gradually stop, and cease to move in vortices.

COR. 7. If a similar fluid be inclosed in a spherical vessel, and, by the uniform rotation of a globe in its centre, is driven round in a vortex; and the globe and vessel revolve the same way about the same axis, and their periodical times be as the squares of the semi-diameters; the parts of the fluid will not go on in their motions without acceleration or retardation, till their periodical times are as the squares of their distances from the centre of the vortex. No constitution of a vortex can be permanent but this.

COR. 8. If the vessel, the inclosed fluid, and the globe, retain this motion, and revolve besides with a common angular motion about any given axis, because the mutual attrition of the parts of the fluid is not changed by this motion, the motions of the parts among each other will not be changed; for the translations of the parts among themselves depend upon this attrition. Any part will persevere in that motion in which its attri-

tion on one side retards it just as much as its attrition on the other side accelerates it.

COR. 9. Therefore if the vessel be quiescent, and the motion of the globe be given, the motion of the fluid will be given. For conceive a plane to pass through the axis of the globe, and to revolve with a contrary motion; and suppose the sum of the time of this revolution and of the revolution of the globe to be to the time of the revolution of the globe as the square of the semi-diameter of the vessel to the square of the semi-diameter of the globe; and the periodic times of the parts of the fluid in respect of this plane will be as the squares of their distances from the centre of the globe.

COR. 10. Therefore if the vessel move about the same axis with the globe, or with a given velocity about a different one, the motion of the fluid will be given. For if from the whole system we take away the angular motion of the vessel, all the motions will remain the same among themselves as before, by Cor. 8, and those motions will be given by Cor. 9.

COR. 11. If the vessel and the fluid are quiescent, and the globe revolves with an uniform motion, that motion will be propagated by degrees through the whole fluid to the vessel, and the vessel will be carried round by it, unless violently detained; and the fluid and the vessel will be continually accelerated till their periodic times become equal to the periodic times of the globe. If the vessel be either withheld by some force, or revolve with any constant and uniform motion, the medium will come by little and little to the state of motion defined in Cor. 8, 9, 10, nor will it ever persevere in any other state. But if then the forces, by which the globe and vessel revolve with certain motions, should cease, and the whole system be left to act according to the mechanical laws, the vessel and globe, by means of the intervening fluid, will act upon each other, and will continue to propagate their motions through the fluid to each other, till their periodic times become equal among themselves, and the whole system revolves together like one solid body.

SCHOLIUM.

In all these reasonings I suppose the fluid to consist of matter of uniform density and fluidity; I mean, that the fluid is such, that a globe placed any where therein may propagate with the same motion of its own, at distances from itself continually equal, similar and equal motions in the fluid in the same interval of time. The matter by its circular motion endeavours to recede from the axis of the vortex, and therefore presses all the matter that lies beyond. This pressure makes the attrition greater, and the separation of the parts more difficult; and by consequence diminishes the fluidity of the matter. Again; if the parts of the fluid are in any one place denser or larger than in the others, the fluidity will be less in that place, because there are fewer superficies where the parts can be separated

from each other. In these cases I suppose the defect of the fluidity to be supplied by the smoothness or softness of the parts, or some other condition; otherwise the matter where it is less fluid will cohere more, and be more sluggish, and therefore will receive the motion more slowly, and propagate it farther than agrees with the ratio above assigned. If the vessel be not spherical, the particles will move in lines not circular, but answering to the figure of the vessel; and the periodic times will be nearly as the squares of the mean distances from the centre. In the parts between the centre and the circumference the motions will be slower where the spaces are wide, and swifter where narrow; but yet the particles will not tend to the circumference at all the more for their greater swiftness; for they then describe arcs of less curvity, and the conatus of receding from the centre is as much diminished by the diminution of this curvature as it is augmented by the increase of the velocity. As they go out of narrow into wide spaces, they recede a little farther from the centre, but in doing so are retarded; and when they come out of wide into narrow spaces, they are again accelerated; and so each particle is retarded and accelerated by turns for ever. These things will come to pass in a rigid vessel; for the state of vortices in an infinite fluid is known by Cor. 6 of this Proposition.

I have endeavoured in this Proposition to investigate the properties of vortices, that I might find whether the celestial phænomena can be explained by them; for the phænomenon is this, that the periodic times of the planets revolving about Jupiter are in the sesquiplicate ratio of their distances from Jupiter's centre; and the same rule obtains also among the planets that revolve about the sun. And these rules obtain also with the greatest accuracy, as far as has been yet discovered by astronomical observation. Therefore if those planets are carried round in vortices revolving about Jupiter and the sun, the vortices must revolve according to that law. But here we found the periodic times of the parts of the vortex to be in the duplicate ratio of the distances from the centre of motion; and this ratio cannot be diminished and reduced to the sesquiplicate, unless either the matter of the vortex be more fluid the farther it is from the centre, or the resistance arising from the want of lubricity in the parts of the fluid should, as the velocity with which the parts of the fluid are separated goes on increasing, be augmented with it in a greater ratio than that in which the velocity increases. But neither of these suppositions seem reasonable. The more gross and less fluid parts will tend to the circumference, unless they are heavy towards the centre. And though, for the sake of demonstration, I proposed, at the beginning of this Section, an Hypothesis that the resistance is proportional to the velocity, nevertheless, it is in truth probable that the resistance is in a less ratio than that of the velocity; which granted, the periodic times of the parts of the vortex will be in a greater than the duplicate ratio of the distances from its centre. If, as some think, the vortices move more swiftly near the centre, then slower

to a certain limit, then again swifter near the circumference, certainly neither the sesquiplicate, nor any other certain and determinate ratio, can obtain in them. Let philosophers then see how that phænomenon of the sesquiplicate ratio can be accounted for by vortices.

PROPOSITION LIII. THEOREM XLI.

Bodies carried about in a vortex, and returning in the same orb, are of the same density with the vortex, and are moved according to the same law with the parts of the vortex, as to velocity and direction of motion.

For if any small part of the vortex, whose particles or physical points preserve a given situation among each other, be supposed to be congealed, this particle will move according to the same law as before, since no change is made either in its density, *vis insita*, or figure. And again; if a congealed or solid part of the vortex be of the same density with the rest of the vortex, and be resolved into a fluid, this will move according to the same law as before, except in so far as its particles, now become fluid, may be moved among themselves. Neglect, therefore, the motion of the particles among themselves as not at all concerning the progressive motion of the whole, and the motion of the whole will be the same as before. But this motion will be the same with the motion of other parts of the vortex at equal distances from the centre; because the solid, now resolved into a fluid, is become perfectly like to the other parts of the vortex. Therefore a solid, if it be of the same density with the matter of the vortex, will move with the same motion as the parts thereof, being relatively at rest in the matter that surrounds it. If it be more dense, it will endeavour more than before to recede from the centre; and therefore overcoming that force of the vortex, by which, being, as it were, kept in equilibrio, it was retained in its orbit, it will recede from the centre, and in its revolution describe a spiral, returning no longer into the same orbit. And, by the same argument, if it be more rare, it will approach to the centre. Therefore it can never continually go round in the same orbit, unless it be of the same density with the fluid. But we have shewn in that case that it would revolve according to the same law with those parts of the fluid that are at the same or equal distances from the centre of the vortex.

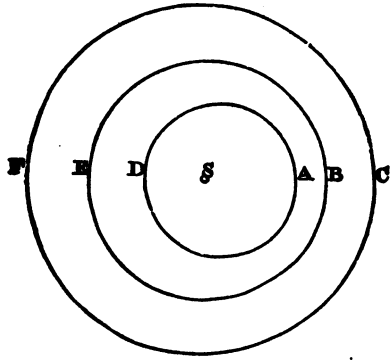
Cor. 1. Therefore a solid revolving in a vortex, and continually going round in the same orbit, is relatively quiescent in the fluid that carries it.

Cor. 2. And if the vortex be of an uniform density, the same body may revolve at any distance from the centre of the vortex.

SCHOLIUM.

Hence it is manifest that the planets are not carried round in corporeal vortices; for, according to the *Copernican* hypothesis, the planets going

round the sun revolve in ellipses, having the sun in their common focus; and by radii drawn to the sun describe areas proportional to the times. But now the parts of a vortex can never revolve with such a motion. Let AD, BE, CF, represent three orbits described about the sun S, of which let the utmost circle CF be concentric to the sun; and let the aphelia of the two innermost be A, B; and their perihelia D, E. Therefore a body revolving in the orb CF, describing, by a radius drawn to the sun, areas proportional to the times, will move with an uniform motion. And, according to the laws of astronomy, the body revolving in the orb BE will move slower in its aphelion B, and swifter in its perihelion E; whereas, according to the laws of mechanics, the matter of the vortex ought to move more swiftly in the narrow space between A and C than in the wide space between D and F; that is, more swiftly in the aphelion than in the perihelion. Now these two conclusions contradict each other. So at the beginning of the sign of Virgo, where the aphelion of Mars is at present, the distance between the orbits of Mars and Venus is to the distance between the same orbits, at the beginning of the sign of Pisces, as about 3 to 2; and therefore the matter of the vortex between those orbits ought to be swifter at the beginning of Pisces than at the beginning of Virgo in the ratio of 3 to 2; for the narrower the space is through which the same quantity of matter passes in the same time of one revolution, the greater will be the velocity with which it passes through it. Therefore if the earth being relatively at rest in this celestial matter should be carried round by it, and revolve together with it about the sun, the velocity of the earth at the beginning of Pisces would be to its velocity at the beginning of Virgo in a sesquialteral ratio. Therefore the sun's apparent diurnal motion at the beginning of Virgo ought to be above 70 minutes, and at the beginning of Pisces less than 48 minutes; whereas, on the contrary, that apparent motion of the sun is really greater at the beginning of Pisces than at the beginning of Virgo, as experience testifies; and therefore the earth is swifter at the beginning of Virgo than at the beginning of Pisces; so that the hypothesis of vortices is utterly irreconcileable with astronomical phænomena, and rather serves to perplex than explain the heavenly motions. How these motions are performed in free spaces without vortices, may be understood by the first Book; and I shall now more fully treat of it in the following Book.



BOOK III.

BOOK III.

In the preceding Books I have laid down the principles of philosophy, principles not philosophical, but mathematical: such, to wit, as we may build our reasonings upon in philosophical inquiries. These principles are the laws and conditions of certain motions, and powers or forces, which chiefly have respect to philosophy; but, lest they should have appeared of themselves dry and barren, I have illustrated them here and there with some philosophical scholiums, giving an account of such things as are of more general nature, and which philosophy seems chiefly to be founded on; such as the density and the resistance of bodies, spaces void of all bodies, and the motion of light and sounds. It remains that, from the same principles, I now demonstrate the frame of the System of the World. Upon this subject I had, indeed, composed the third Book in a popular method, that it might be read by many; but afterward, considering that such as had not sufficiently entered into the principles could not easily discern the strength of the consequences, nor lay aside the prejudices to which they had been many years accustomed, therefore, to prevent the disputes which might be raised upon such accounts, I chose to reduce the substance of this Book into the form of Propositions (in the mathematical way), which should be read by those only who had first made themselves masters of the principles established in the preceding Books: not that I would advise any one to the previous study of every Proposition of those Books; for they abound with such as might cost too much time, even to readers of good mathematical learning. It is enough if one carefully reads the Definitions, the Laws of Motion, and the first three Sections of the first Book. He may then pass on to this Book, and consult such of the remaining Propositions of the first two Books, as the references in this, and his occasions, shall require.

RULES OF REASONING IN PHILOSOPHY.

RULE I.

We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.

To this purpose the philosophers say that Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased with simplicity, and affects not the pomp of superfluous causes.

RULE II.

Therefore to the same natural effects we must, as far as possible, assign the same causes.

As to respiration in a man and in a beast; the descent of stones in *Europe* and in *America;* the light of our culinary fire and of the sun; the reflection of light in the earth, and in the planets.

RULE III.

The qualities of bodies, which admit neither intension nor remission of degrees, and which are found to belong to all bodies within the reach of our experiments, are to be esteemed the universal qualities of all bodies whatsoever.

For since the qualities of bodies are only known to us by experiments, we are to hold for universal all such as universally agree with experiments; and such as are not liable to diminution can never be quite taken away. We are certainly not to relinquish the evidence of experiments for the sake of dreams and vain fictions of our own devising; nor are we to recede from the analogy of Nature, which uses to be simple, and always consonant to itself. We no other way know the extension of bodies than by our senses, nor do these reach it in all bodies; but because we perceive extension in all that are sensible, therefore we ascribe it universally to all others also. That abundance of bodies are hard, we learn by experience; and because the hardness of the whole arises from the hardness of the parts, we therefore justly infer the hardness of the undivided particles not only of the bodies we feel but of all others. That all bodies are impenetrable, we gather not from reason, but from sensation. The bodies which we handle we find impenetrable, and thence conclude impenetrability to be an universal property of all bodies whatsoever. That all bodies are moveable, and endowed with certain powers (which we call the *vires inertiæ*) of persevering in their motion, or in their rest, we only infer from the like properties observed in the

bodies which we have seen. The extension, hardness, impenetrability, mobility, and *vis inertiæ* of the whole, result from the extension, hardness, impenetrability, mobility, and *vires inertiæ* of the parts; and thence we conclude the least particles of all bodies to be also all extended, and hard and impenetrable, and moveable, and endowed with their proper *vires inertia*. And this is the foundation of all philosophy. Moreover, that the divided but contiguous particles of bodies may be separated from one another, is matter of observation; and, in the particles that remain undivided, our minds are able to distinguish yet lesser parts, as is mathematically demonstrated. But whether the parts so distinguished, and not yet divided, may, by the powers of Nature, be actually divided and separated from one another, we cannot certainly determine. Yet, had we the proof of but one experiment that any undivided particle, in breaking a hard and solid body, suffered a division, we might by virtue of this rule conclude that the undivided as well as the divided particles may be divided and actually separated to infinity.

Lastly, if it universally appears, by experiments and astronomical observations, that all bodies about the earth gravitate towards the earth, and that in proportion to the quantity of matter which they severally contain; that the moon likewise, according to the quantity of its matter, gravitates towards the earth; that, on the other hand, our sea gravitates towards the moon; and all the planets mutually one towards another; and the comets in like manner towards the sun; we must, in consequence of this rule, universally allow that all bodies whatsoever are endowed with a principle of mutual gravitation. For the argument from the appearances concludes with more force for the universal gravitation of all bodies than for their impenetrability; of which, among those in the celestial regions, we have no experiments, nor any manner of observation. Not that I affirm gravity to be essential to bodies: by their *vis insita* I mean nothing but their *vis inertiæ*. This is immutable. Their gravity is diminished as they recede from the earth.

RULE IV.

In experimental philosophy we are to look upon propositions collected by general induction from phænomena as accurately or very nearly true, notwithstanding any contrary hypotheses that may be imagined, till such time as other phænomena occur, by which they may either be made more accurate, or liable to exceptions.

This rule we must follow, that the argument of induction may not be evaded by hypotheses.

PHÆNOMENA, OR APPEARANCES.

PHÆNOMENON I.

That the circumjovial planets, by radii drawn to Jupiter's centre, describe areas proportional to the times of description; and that their periodic times, the fixed stars being at rest, are in the sesquiplicate proportion of their distances from its centre.

This we know from astronomical observations. For the orbits of these planets differ but insensibly from circles concentric to Jupiter; and their motions in those circles are found to be uniform. And all astronomers agree that their periodic times are in the sesquiplicate proportion of the semi-diameters of their orbits; and so it manifestly appears from the following table.

The periodic times of the satellites of Jupiter.

$1^d.\ 18^h.\ 27'.\ 34''.$ $3^d.\ 13^h.\ 13'\ 42''.$ $7^d.\ 3^h.\ 42'\ 36''.$ $16^d.\ 16^h.\ 32'\ 9''.$

The distances of the satellites from Jupiter's centre.

From the observations of	1	2	3	4	
Borelli	$5\frac{2}{3}$	$8\frac{2}{3}$	14	$24\frac{2}{3}$	*semi-diameter of Jupiter.*
Townly *by the Microm.* . . .	5,52	8,78	13,47	24,72	
Cassini *by the Telescope* . . .	5	8	13	23	
Cassini *by the eclip. of the satel.* .	$5\frac{2}{3}$	9	$14\frac{2}{6}\frac{3}{0}$	$25\frac{3}{10}$	
From the periodic times	5,667	9,017	14,384	25,299	

Mr. *Pound* has determined, by the help of excellent micrometers, the diameters of Jupiter and the elongation of its satellites after the following manner. The greatest heliocentric elongation of the fourth satellite from Jupiter's centre was taken with a micrometer in a 15 feet telescope, and at the mean distance of Jupiter from the earth was found about $8'\ 16''$. The elongation of the third satellite was taken with a micrometer in a telescope of 123 feet, and at the same distance of Jupiter from the earth was found $4'\ 42''$. The greatest elongations of the other satellites, at the same distance of Jupiter from the earth, are found from the periodic times to be $2'\ 56''\ 47'''$, and $1'\ 51''\ 6'''$.

The diameter of Jupiter taken with the micrometer in a 123 feet telescope several times, and reduced to Jupiter's mean distance from the earth, proved always less than $40''$, never less than $38''$, generally $39''$. This diameter in shorter telescopes is $40''$, or $41''$; for Jupiter's light is a little dilated by the unequal refrangibility of the rays, and this dilatation bears a less ratio to the diameter of Jupiter in the longer and more perfect telescopes than in those which are shorter and less perfect. The times in

which two satellites, the first and the third, passed over Jupiter's body, were observed, from the beginning of the ingress to the beginning of the egress, and from the complete ingress to the complete egress, with the long telescope. And from the transit of the first satellite, the diameter of Jupiter at its mean distance from the earth came forth $37\frac{1}{8}''$, and from the transit of the third $37\frac{3}{8}''$. There was observed also the time in which the shadow of the first satellite passed over Jupiter's body, and thence the diameter of Jupiter at its mean distance from the earth came out about $37''$. Let us suppose its diameter to be $37\frac{1}{4}''$ very nearly, and then the greatest elongations of the first, second, third, and fourth satellite will be respectively equal to 5,965, 9,494, 15,141, and 26,63 semi-diameters of Jupiter.

PHÆNOMENON II.

That the circumsaturnal planets, by radii drawn to Saturn's centre, describe areas proportional to the times of description; and that their periodic times, the fixed stars being at rest, are in the sesquiplicate proportion of their distances from its centre.

For, as *Cassini* from his own observations has determined, their distances from Saturn's centre and their periodic times are as follow.

The periodic times of the satellites of Saturn.

$1^d.\ 21^h.\ 18'\ 27''.$ $2^d.\ 17^h.\ 41'\ 22''.$ $4^d.\ 12^h.\ 25'\ 12''.$ $15^d.\ 22^h.\ 41'\ 14''.$ $79^d.\ 7^h.\ 48'\ 00''.$

The distances of the satellites from Saturn's centre, in semi-diameters of its ring.

From observations	$1\frac{9}{10}$.	$2\frac{1}{2}$.	$3\frac{1}{2}$.	8.	24.
From the periodic times . . .	1,93.	2,47.	3,45.	8.	23,35.

The greatest elongation of the fourth satellite from Saturn's centre is commonly determined from the observations to be eight of those semi-diameters very nearly. But the greatest elongation of this satellite from Saturn's centre, when taken with an excellent micrometer in Mr. *Huygens'* telescope of 123 feet, appeared to be eight semi-diameters and $\frac{7}{10}$ of a semi-diameter. And from this observation and the periodic times the distances of the satellites from Saturn's centre in semi-diameters of the ring are 2,1. 2,69. 3,75. 8,7. and 25,35. The diameter of Saturn observed in the same telescope was found to be to the diameter of the ring as 3 to 7; and the diameter of the ring, *May* 28–29, 1719, was found to be $43''$; and thence the diameter of the ring when Saturn is at its mean distance from the earth is $42''$, and the diameter of Saturn $18''$. These things appear so in very long and excellent telescopes, because in such telescopes the apparent magnitudes of the heavenly bodies bear a greater proportion to the dilatation of light in the extremities of those bodies than in shorter telescopes.

If we, then, reject all the spurious light, the diameter of Saturn will not amount to more than 16″.

PHÆNOMENON III.

That the five primary planets, Mercury, Venus, Mars, Jupiter, and Saturn, with their several orbits, encompass the sun.

That Mercury and Venus revolve about the sun, is evident from their moon-like appearances. When they shine out with a full face, they are, in respect of us, beyond or above the sun; when they appear half full, they are about the same height on one side or other of the sun; when horned, they are below or between us and the sun; and they are sometimes, *when directly under*, seen like spots traversing the sun's disk. That Mars surrounds the sun, is as plain from its full face when near its conjunction with the sun, and from the gibbous figure which it shews in its quadratures. And the same thing is demonstrable of Jupiter and Saturn, from their appearing full in all situations; for the shadows of their satellites that appear sometimes upon their disks make it plain that the light they shine with is not their own, but borrowed from the sun.

PHÆNOMENON IV.

That the fixed stars being at rest, the periodic times of the five primary planets, and (whether of the sun about the earth, or) of the earth about the sun, are in the sesquiplicate proportion of their mean distances from the sun.

This proportion, first observed by *Kepler*, is now received by all astronomers; for the periodic times are the same, and the dimensions of the orbits are the same, whether the sun revolves about the earth, or the earth about the sun. And as to the measures of the periodic times, all astronomers are agreed about them. But for the dimensions of the orbits, *Kepler* and *Bullialdus*, above all others, have determined them from observations with the greatest accuracy; and the mean distances corresponding to the periodic times differ but insensibly from those which they have assigned, and for the most part fall in between them; as we may see from the following table.

The periodic times with respect to the fixed stars, of the planets and earth revolving *about the sun, in days and decimal parts of a day.*

♄	♃	♂	♁	♀	☿
10759,275.	4332,514.	686,9785.	365,2565.	224,6176.	87,9692.

The mean distances of the planets and of the earth from the sun.

	♄	♃	♂
According to *Kepler*	951000.	519650.	152350.
" to *Bullialdus*	954198.	522520.	152350.
" to the periodic times	954006.	520096.	152369

	♂	♀	☿
According to *Kepler*	100000.	72400.	38806
" to *Bullialdus*	100000.	72398.	38585
" to the periodic times	100000.	72333.	38710.

As to Mercury and Venus, there can be no doubt about their distances from the sun; for they are determined by the elongations of those planets from the sun; and for the distances of the superior planets, all dispute is cut off by the eclipses of the satellites of Jupiter. For by those eclipses the position of the shadow which Jupiter projects is determined; whence we have the heliocentric longitude of Jupiter. And from its heliocentric and geocentric longitudes compared together, we determine its distance.

PHÆNOMENON V.

Then the primary planets, by radii drawn to the earth, describe areas no wise proportional to the times; but that the areas which they describe by radii drawn to the sun are proportional to the times of description.

For to the earth they appear sometimes direct, sometimes stationary, nay, and sometimes retrograde. But from the sun they are always seen direct, and to proceed with a motion nearly uniform, that is to say, a little swifter in the perihelion and a little slower in the aphelion distances, so as to maintain an equality in the description of the areas. This a noted proposition among astronomers, and particularly demonstrable in Jupiter, from the eclipses of his satellites; by the help of which eclipses, as we have said, the heliocentric longitudes of that planet, and its distances from the sun, are determined.

PHÆNOMENON VI.

That the moon, by a radius drawn to the earth's centre, describes an area proportional to the time of description.

This we gather from the apparent motion of the moon, compared with its apparent diameter. It is true that the motion of the moon is a little disturbed by the action of the sun: but in laying down these Phænomena, I neglect those small and inconsiderable errors.

PROPOSITIONS.

PROPOSITION I. THEOREM I.

That the forces by which the circumjovial planets are continually drawn off from rectilinear motions, and retained in their proper orbits, tend to Jupiter's centre; and are reciprocally as the squares of the distances of the places of those planets *from that centre.*

The former part of this Proposition appears from Phæn. I, and Prop. II or III, Book I; the latter from Phæn. I, and Cor. 6, Prop. IV, of the same Book.

The same thing we are to understand of the planets which encompass Saturn, by Phæn. II.

PROPOSITION II. THEOREM II.

That the forces by which the primary planets are continually drawn off from rectilinear motions, and retained in their proper orbits, tend to the sun; and are reciprocally as the squares of the distances of the places of those planets *from the sun's centre.*

The former part of the Proposition is manifest from Phæn. V, and Prop. II, Book I; the latter from Phæn. IV, and *Cor.* 6, Prop. IV, of the same Book. But this part of the Proposition is, with great accuracy, demonstrable from the quiescence of the aphelion points; for a very small aberration from the *reciprocal* duplicate proportion would (by Cor. 1, Prop. XLV, Book I) produce a motion of the apsides sensible enough in every single revolution, and in many of them enormously great.

PROPOSITION III. THEOREM III.

That the force by which the moon is retained in its orbit tends to the earth; and is reciprocally as the square of the distance of its place from the earth's centre.

The former part of the Proposition is evident from Phæn. VI, and Prop. II or III, Book I; the latter from the very slow motion of the moon's apogee; which in every single revolution amounting but to 3° 3′ *in consequentia*, may be neglected. For (by Cor. 1, Prop. XLV, Book I) it appears, that, if the distance of the moon from the earth's centre is to the semi-diameter of the earth as D to 1, the force, from which such a motion will result, is reciprocally as $D^{2\frac{4}{243}}$, i. e., reciprocally as the power of D, whose exponent is $2\frac{4}{243}$; that is to say, in the proportion of the distance something greater than reciprocally duplicate, but which comes $59\frac{3}{4}$ times nearer to the duplicate than to the triplicate proportion. But in regard that this motion is owing to the action of the sun (as we shall afterwards

shew), it is here to be neglected. The action of the sun, attracting the moon from the earth, is nearly as the moon's distance from the earth; and therefore (by what we have shewed in Cor. 2, Prop. XLV, Book I) is to the centripetal force of the moon as 2 to 357,45, or nearly so; that is, as 1 to $178\frac{29}{40}$. And if we neglect so inconsiderable a force of the sun, the remaining force, by which the moon is retained in its orb, will be reciprocally as D^2. This will yet more fully appear from comparing this force with the force of gravity, as is done in the next Proposition.

Cor. If we augment the mean centripetal force by which the moon is retained in its orb, first in the proportion of $177\frac{29}{40}$ to $178\frac{29}{40}$, and then in the duplicate proportion of the semi-diameter of the earth to the mean distance of the centres of the moon and earth, we shall have the centripetal force of the moon at the surface of the earth; supposing this force, in descending to the earth's surface, continually to increase in the reciprocal duplicate proportion of the height.

PROPOSITION IV. THEOREM IV.

That the moon gravitates towards the earth, and by the force of gravity is continually drawn off from a rectilinear motion, and retained in its orbit.

The mean distance of the moon from the earth in the syzygies in semi-diameters of the earth, is, according to *Ptolemy* and most astronomers, 59; according to *Vendelin* and *Huygens*, 60; to *Copernicus*, $60\frac{1}{3}$; to *Street*, $60\frac{2}{3}$; and to *Tycho*, $56\frac{1}{2}$. But *Tycho*, and all that follow his tables of refraction, making the refractions of the sun and moon (altogether against the nature of light) to exceed the refractions of the fixed stars, and that by four or five minutes *near the horizon*, did thereby increase the moon's *horizontal* parallax by a like number of minutes, that is, by a twelfth or fifteenth part of the whole parallax. Correct this error, and the distance will become about $60\frac{1}{2}$ semi-diameters of the earth, near to what others have assigned. Let us assume the mean distance of 60 diameters in the syzygies; and suppose one revolution of the moon, in respect of the fixed stars, to be completed in $27^d.\ 7^h.\ 43'$, as astronomers have determined; and the circumference of the earth to amount to 123249600 *Paris* feet, as the French have found by mensuration. And now if we imagine the moon, deprived of all motion, to be let go, so as to descend towards the earth with the impulse of all that force by which (by Cor. Prop. III) it is retained in its orb, it will in the space of one minute of time, describe in its fall $15\frac{1}{12}$ *Paris* feet. This we gather by a calculus, founded either upon Prop. XXXVI, Book I, or (which comes to the same thing) upon Cor. 9, Prop. IV, of the same Book. For the versed sine of that arc, which the moon, in the space of one minute of time, would by its mean

motion describe at the distance of 60 semi-diameters of the earth, is nearly $15\frac{1}{12}$ *Paris* feet. or more accurately 15 feet, 1 inch, and 1 line $\frac{4}{9}$. Wherefore, since that force, in approaching to the earth, increases in the reciprocal duplicate proportion of the distance, and, upon that account, at the surface of the earth, is 60×60 times greater than at the moon, a body in our regions, falling with that force, ought in the space of one minute of time, to describe $60 \times 60 \times 15\frac{1}{12}$ *Paris* feet; and, in the space of one second of time, to describe $15\frac{1}{12}$ of those feet; or more accurately 15 feet, 1 inch, and 1 line $\frac{4}{9}$. And with this very force we actually find that bodies here upon earth do really descend; for a pendulum oscillating seconds in the latitude of *Paris* will be 3 *Paris* feet, and 8 lines $\frac{1}{2}$ in length, as Mr. *Huygens* has observed. And the space which a heavy body describes by falling in one second of time is to half the length of this pendulum in the duplicate ratio of the circumference of a circle to its diameter (as Mr. *Huygens* has also shewn), and is therefore 15 *Paris* feet, 1 inch, 1 line $\frac{7}{9}$. And therefore the force by which the moon is retained in its orbit becomes, at the very surface of the earth, equal to the force of gravity which we observe in heavy bodies there. And therefore (by Rule I and II) the force by which the moon is retained in its orbit is that very same force which we commonly call gravity; for, were gravity another force different from that, then bodies descending to the earth with the joint impulse of both forces would fall with a double velocity, and in the space of one second of time would describe $30\frac{1}{6}$ *Paris* feet; altogether against experience.

This calculus is founded on the hypothesis of the earth's standing still; for if both earth and moon move about the sun, and at the same time about their common centre of gravity, the distance of the centres of the moon and earth from one another will be $60\frac{1}{2}$ semi-diameters of the earth; as may be found by a computation from Prop. LX, Book I.

SCHOLIUM.

The demonstration of this Proposition may be more diffusely explained after the following manner. Suppose several moons to revolve about the earth, as in the system of Jupiter or Saturn; the periodic times of these moons (by the argument of induction) would observe the same law which *Kepler* found to obtain among the planets; and therefore their centripetal forces would be reciprocally as the squares of the distances from the centre of the earth, by Prop. I, of this Book. Now if the lowest of these were very small, and were so near the earth as almost to touch the tops of the highest mountains, the centripetal force thereof, retaining it in its orb, would be very nearly equal to the weights of any *terrestrial* bodies that should be found upon the tops of those mountains, as may be known by the foregoing computation. Therefore if the same little moon should be deserted by its centrifugal force that carries it through its orb, and so be

lisabled from going onward therein, it would descend to the earth; and that with the same velocity as heavy bodies do actually fall with upon the tops of those very mountains; because of the equality of the forces that oblige them both to descend. And if the force by which that lowest moon would descend were different from gravity, and if that moon were to gravitate towards the earth, as we find terrestrial bodies do upon the tops of mountains, it would then descend with twice the velocity, as being impelled by both these forces conspiring together. Therefore since both these forces, that is, the gravity of heavy bodies, and the centripetal forces of the moons, respect the centre of the earth, and are similar and equal between themselves, they will (by Rule I and II) have one and the same cause. And therefore the force which retains the moon in its orbit is that very force which we commonly call gravity; because otherwise this little moon at the top of a mountain must either be without gravity, or fall twice as swiftly as heavy bodies are wont to do.

PROPOSITION V. THEOREM V.

That the circumjovial planets gravitate towards Jupiter; the circumsaturnal towards Saturn; the circumsolar towards the sun; and by the forces of their gravity are drawn off from rectilinear motions, and retained in curvilinear orbits.

For the revolutions of the circumjovial planets about Jupiter, of the circumsaturnal about Saturn, and of Mercury and Venus, and the other circumsolar planets, about the sun, are appearances of the same sort with the revolution of the moon about the earth; and therefore, by Rule II, must be owing to the same sort of causes; especially since it has been demonstrated, that the forces upon which those revolutions depend tend to the centres of Jupiter, of Saturn, and of the sun; and that those forces, in receding from Jupiter, from Saturn, and from the sun, decrease in the same proportion, and according to the same law, as the force of gravity does in receding from the earth.

Cor. 1. There is, therefore, a power of gravity tending to all the planets; for, doubtless, Venus, Mercury, and the rest, are bodies of the same sort with Jupiter and Saturn. And since all attraction (by Law III) is mutual, Jupiter will therefore gravitate towards all his own satellites, Saturn towards his, the earth towards the moon, and the sun towards all the primary planets.

Cor. 2. The force of gravity which tends to any one planet is reciprocally as the square of the distance of places from that planet's centre.

Cor. 3. All the planets do mutually gravitate towards one another, by Cor. 1 and 2. And hence it is that Jupiter and Saturn, when near their

conjunction, by their mutual attractions sensibly disturb each other's motions. So the sun disturbs the motions of the moon; and both sun and moon disturb our sea, as we shall hereafter explain.

SCHOLIUM.

The force which retains the celestial bodies in their orbits has been hitherto called centripetal force; but it being now made plain that it can be no other than a gravitating force, we shall hereafter call it gravity. For the cause of that centripetal force which retains the moon in its orbit will extend itself to all the planets, by Rule I, II, and IV.

PROPOSITION VI. THEOREM VI.

That all bodies gravitate towards every planet; and that the weights of bodies towards any the same *planet, at equal distances from the centre of the planet, are proportional to the quantities of matter which they severally contain.*

It has been, now of a long time, observed by others, that all sorts of heavy bodies (allowance being made for the inequality of retardation which they suffer from a small power of resistance in the air) descend to the earth *from equal heights* in equal times; and that equality of times we may distinguish to a great accuracy, by the help of pendulums. I tried the thing in gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provided two wooden boxes, round and equal: I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the centre of oscillation of the other. The boxes hanging by equal threads of 11 feet made a couple of pendulums perfectly equal in weight and figure, and equally receiving the resistance of the air. And, placing the one by the other, I observed them to play together forward and backward, for a long time, with equal vibrations. And therefore the quantity of matter in the gold (by Cor. 1 and 6, Prop. XXIV, Book II) was to the quantity of matter in the wood as the action of the motive force (or *vis motrix*) upon all the gold to the action of the same upon all the wood; that is, as the weight of the one to the weight of the other: and the like happened in the other bodies. By these experiments, in bodies of the same weight, I could manifestly have discovered a difference of matter less than the thousandth part of the whole, had any such been. But, without all doubt, the nature of gravity towards the planets is the same as towards the earth. For, should we imagine our terrestrial bodies removed to the orb of the moon, and there, together with the moon, deprived of all motion, to be let go, so as to fall together towards the earth, it is certain, from what we have demonstrated before, that, in equal times, they would describe equal spaces with the moon, and of consequence are to the moon, in quantity of matter, as their weights to its weight. Moreover, since the satellites of Jupiter perform

their revolutions in times which observe the sesquiplicate proportion of their distances from Jupiter's centre, their accelerative gravities towards Jupiter will be reciprocally as the squares of their distances from Jupiter's centre; that is, equal, at equal distances. And, therefore, these satellites, if supposed to fall *towards Jupiter* from equal heights, would describe equal spaces in equal times, in like manner as heavy bodies do on our earth. And, by the same argument, if the circumsolar planets were supposed to be let fall at equal distances from the sun, they would, in their descent towards the sun, describe equal spaces in equal times. But forces which equally accelerate unequal bodies must be as those bodies: that is to say, the weights of the planets *towards the sun* must be as their quantities of matter. Further, that the weights of Jupiter and of his satellites towards the sun are proportional to the several quantities of their matter, appears from the exceedingly regular motions of the satellites (by Cor. 3, Prop. LXV, Book I). For if some of those bodies were more strongly attracted to the sun in proportion to their quantity of matter than others, the motions of the satellites would be disturbed by that inequality of attraction (by Cor. 2, Prop. LXV, Book I). If, at equal distances from the sun, any satellite, in proportion to the quantity of its matter, did gravitate towards the sun with a force greater than Jupiter in proportion to his, according to any given proportion, suppose of d to e; then the distance between the centres of the sun and of the satellite's orbit would be always greater than the distance between the centres of the sun and of Jupiter nearly in the subduplicate of that proportion: as by some computations I have found. And if the satellite did gravitate towards the sun with a force, lesser in the proportion of e to d, the distance of the centre of the satellite's orb from the sun would be less than the distance of the centre of Jupiter from the sun in the subduplicate of the same proportion. Therefore if, at equal distances from the sun, the accelerative gravity of any satellite towards the sun were greater or less than the accelerative gravity of Jupiter towards the sun but by one $\frac{1}{1000}$ part of the whole gravity, the distance of the centre of the satellite's orbit from the sun would be greater or less than the distance of Jupiter from the sun by one $\frac{1}{2000}$ part of the whole distance; that is, by a fifth part of the distance of the utmost satellite from the centre of Jupiter; an eccentricity of the orbit which would be very sensible. But the orbits of the satellites are concentric to Jupiter, and therefore the accelerative gravities of Jupiter, and of all its satellites towards the sun, are equal among themselves. And by the same argument, the weights of Saturn and of his satellites towards the sun, at equal distances from the sun, are as their several quantities of matter; and the weights of the moon and of the earth towards the sun are either none, or accurately proportional to the masses of matter which they contain. But some they are, by Cor. 1 and 3, Prop. V.

But further; the weights of all the parts of every planet towards any other

planet are one to another as the matter in the several parts; for if some parts did gravitate more, others less, than for the quantity of their matter, then the whole planet, according to the sort of parts with which it most abounds, would gravitate more or less than in proportion to the quantity of matter in the whole. Nor is it of any moment whether these parts are external or internal; for if, for example, we should imagine the terrestrial bodies with us to be raised up to the orb of the moon, to be there compared with its body: if the weights of such bodies were to the weights of the external parts of the moon as the quantities of matter in the one and in the other respectively; but to the weights of the internal parts in a greater or less proportion, then likewise the weights of those bodies would be to the weight of the whole moon in a greater or less proportion; against what we have shewed above.

Cor. 1. Hence the weights of bodies do not depend upon their forms and textures; for if the weights could be altered with the forms, they would be greater or less, according to the variety of forms, in equal matter; altogether against experience.

Cor. 2. Universally, all bodies about the earth gravitate towards the earth; and the weights of all, at equal distances from the earth's centre, are as the quantities of matter which they severally contain. This is the quality of all bodies within the reach of our experiments; and therefore (by Rule III) to be affirmed of all bodies whatsoever. If the *æther*, or any other body, were either altogether void of gravity, or were to gravitate less in proportion to its quantity of matter, then, because (according to *Aristotle, Des Cartes*, and others) there is no difference betwixt that and other bodies but in *mere* form of matter, by a successive change from form to form, it might be changed at last into a body of the same condition with those which gravitate most in proportion to their quantity of matter; and, on the other hand, the heaviest bodies, acquiring the first form of that body, might by degrees quite lose their gravity. And therefore the weights would depend upon the forms of bodies, and with those forms might be changed: contrary to what was proved in the preceding Corollary.

Cor. 3. All spaces are not equally full; for if all spaces were equally full, then the specific gravity of the fluid which fills the region of the air, on account of the extreme density of the matter, would fall nothing short of the specific gravity of quicksilver, or gold, or any other the most dense body; and, therefore, neither gold, nor any other body, could descend in air; for bodies do not descend in fluids, unless they are specifically heavier than the fluids. And if the quantity of matter in a given space can, by any rarefaction, be diminished, what should hinder a diminution to infinity?

Cor. 4. If all the solid particles of all bodies are of the same density, nor can be rarefied without pores, a void, space, or vacuum must be granted

By bodies of the same density, I mean those whose *vires inertiæ* are in the proportion of their bulks.

Cor. 5. The power of gravity is of a different nature from the power of magnetism; for the magnetic attraction is not as the matter attracted. Some bodies are attracted more by the magnet; others less; most bodies not at all. The power of magnetism in one and the same body may be increased and diminished; and is sometimes far stronger, for the quantity of matter, than the power of gravity; and in receding from the magnet decreases not in the duplicate but almost in the triplicate proportion of the distance, as nearly as I could judge from some rude observations.

PROPOSITION VII. THEOREM VII.

That there is a power of gravity tending to all bodies, proportional to the several quantities of matter which they contain.

That all the planets mutually gravitate one towards another, we have proved before; as well as that the force of gravity towards every one of them, considered apart, is reciprocally as the square of the distance of places from the centre of the planet. And thence (by Prop. LXIX, Book I, and its Corollaries) it follows, that the gravity tending towards all the planets is proportional to the matter which they contain.

Moreover, since all the parts of any planet A gravitate towards any other planet B; and the gravity of every part is to the gravity of the whole as the matter of the part to the matter of the whole; and (by Law III) to every action corresponds an equal re-action; therefore the planet B will, on the other hand, gravitate towards all the parts of the planet A; and its gravity towards any one part will be to the gravity towards the whole as the matter of the part to the matter of the whole. Q.E.D.

Cor. 1. Therefore the force of gravity towards any whole planet arises from, and is compounded of, the forces of gravity towards all its parts. Magnetic and electric attractions afford us examples of this; for all attraction towards the whole arises from the attractions towards the several parts. The thing may be easily understood in gravity, if we consider a greater planet, as formed of a number of lesser planets, meeting together in one globe; for *hence it would appear that* the force of the whole must arise from the forces of the component parts. If it is objected, that, according to this law, all bodies with us must mutually gravitate one towards another, whereas no such gravitation any where appears, I answer, that since the gravitation towards these bodies is to the gravitation towards the whole earth as these bodies are to the whole earth, the gravitation towards them must be far less than to fall under the observation of our senses.

Cor. 2. The force of gravity towards the several equal particles of any body is reciprocally as the square of the distance of places from the particles; as appears from Cor. 3, Prop. LXXIV, Book I.

PROPOSITION VIII. THEOREM VIII.

In two spheres mutually gravitating each towards the other, if the matter in places on all sides round about and equi-distant from the centres is similar, the weight of either sphere towards the other will be reciprocally as the square of the distance between their centres.

After I had found that the force of gravity towards a whole planet did arise from and was compounded of the forces of gravity towards all its parts, and towards every one part was in the reciprocal proportion of the squares of the distances from the part, I was yet in doubt whether that reciprocal duplicate proportion did accurately hold, or but nearly so, in the total force compounded of so many partial ones; for it might be that the proportion which accurately enough took place in greater distances should be wide of the truth near the surface of the planet, where the distances of the particles are unequal, and their situation dissimilar. But by the help of Prop. LXXV and LXXVI, Book I, and their Corollaries, I was at last satisfied of the truth of the Proposition, as it now lies before us.

Cor. 1. Hence we may find and compare together the weights of bodies towards different planets; for the weights of bodies revolving in circles about planets are (by Cor. 2, Prop. IV, Book I) as the diameters of the circles directly, and the squares of their periodic times reciprocally; and their weights at the surfaces of the planets, or at any other distances from their centres, are (by this Prop.) greater or less in the reciprocal duplicate proportion of the distances. Thus from the periodic times of Venus, revolving about the sun, in $224^{d}.\ 16\frac{3}{4}^{h}$, of the utmost circumjovial satellite revolving about Jupiter, in $16^{d}.\ 16\frac{8}{15}^{h}.$; of the Huygenian satellite about Saturn in $15^{d}.\ 22\frac{2}{3}^{h}.$; and of the moon about the earth in $27^{d}.\ 7^{h}.\ 43'$; compared with the mean distance of Venus from the sun, and with the greatest heliocentric elongations of the outmost circumjovial satellite from Jupiter's centre, $8'\ 16''$; of the Huygenian satellite from the centre of Saturn, $3'\ 4''$; and of the moon from the earth, $10'\ 33''$: by computation I found that the weight of equal bodies, at equal distances from the centres of the sun, of Jupiter, of Saturn, and of the earth, towards the sun, Jupiter, Saturn, and the earth, were one to another, as 1, $\frac{1}{1067}$, $\frac{1}{3021}$, and $\frac{1}{1694\frac{1}{2}}$ respectively. Then because as the distances are increased or diminished, the weights are diminished or increased in a duplicate ratio, the weights of equal bodies towards the sun, Jupiter, Saturn, and the earth, at the distances 10000, 997, 791, and 109 from their centres, that is, at their very superficies, will be as 10000, 943, 529, and 435 respectively. How much the weights of bodies are at the superficies of the moon, will be shewn hereafter.

Cor. 2. Hence likewise we discover the quantity of matter in the several

planets; for their quantities of matter are as the forces of gravity at equal distances from their centres; that is, in the sun, Jupiter, Saturn, and the earth, as 1, $\frac{1}{1067}$, $\frac{1}{3021}$, and $\frac{1}{169282}$ respectively. If the parallax of the sun be taken greater or less than 10″ 30‴, the quantity of matter in the earth must be augmented or diminished in the triplicate of that proportion.

Cor. 3. Hence also we find the densities of the planets; for (by Prop. LXXII, Book I) the weights of equal and similar bodies towards similar spheres are, at the surfaces of those spheres, as the diameters of the spheres; and therefore the densities of dissimilar spheres are as those weights applied to the diameters of the spheres. But the true diameters of the Sun, Jupiter, Saturn, and the earth, were one to another as 10000, 997, 791, and 109; and the weights towards the same as 10000, 943, 529, and 435 respectively; and therefore their densities are as 100, $94\frac{1}{2}$, 67, and 400. The density of the earth, which comes out by this computation, does not depend upon the parallax of the sun, but is determined by the parallax of the moon, and therefore is here truly defined. The sun, therefore, is a little denser than Jupiter, and Jupiter than Saturn, and the earth four times denser than the sun; for the sun, by its great heat, is kept in a sort of a rarefied state. The moon is denser than the earth, as shall appear afterward.

Cor. 4. The smaller the planets are, they are, *cæteris paribus*, of so much the greater density; for so the powers of gravity on their several surfaces come nearer to equality. They are likewise, *cæteris paribus*, of the greater density, as they are nearer to the sun. So Jupiter is more dense than Saturn, and the earth than Jupiter; for the planets were to be placed at different distances from the sun, that, according to their degrees of density, they might enjoy a greater or less proportion to the sun's heat. Our water, if it were removed as far as the orb of Saturn, would be converted into ice, and in the orb of Mercury would quickly fly away in vapour; for the light of the sun, to which its heat is proportional, is seven times denser in the orb of Mercury than with us: and by the thermometer I have found that a sevenfold heat of our summer sun will make water boil. Nor are we to doubt that the matter of Mercury is adapted to its heat, and is therefore more dense than the matter of our earth; since, in a denser matter, the operations of Nature require a stronger heat.

PROPOSITION IX. THEOREM IX.

That the force of gravity, considered downward from the surface of the planets, decreases nearly in the proportion of the distances from their centres.

If the matter of the planet were of an uniform density, this Proposition would be accurately true (by Prop. LXXIII. Book I). The error,

therefore, can be no greater than what may arise from the inequality of the density.

PROPOSITION X. THEOREM X.

That the motions of the planets in the heavens may subsist an exceedingly long time.

In the Scholium of Prop. XL, Book II, I have shewed that a globe of water frozen into ice, and moving freely in our air, in the time that it would describe the length of its semi-diameter, would lose by the resistance of the air $\frac{1}{4\,5\,8\,8}$ part of its motion; and the same proportion holds nearly in all globes, how great soever, and moved with whatever velocity. But that our globe of earth is of greater density than it would be if the whole consisted of water only, I thus make out. If the whole consisted of water only, whatever was of less density than water, because of its less specific gravity, would emerge and float above. And upon this account, if a globe of terrestrial matter, covered on all sides with water, was less dense than water, it would emerge somewhere; and, the subsiding water falling back, would be gathered to the opposite side. And such is the condition of our earth, which in a great measure is covered with seas. The earth, if it was not for its greater density, would emerge from the seas, and, according to its degree of levity, would be raised more or less above their surface, the water of the seas flowing backward to the opposite side. By the same argument, the spots of the sun, which float upon the lucid matter thereof, are lighter than that matter; and, however the planets have been formed while they were yet in fluid masses, all the heavier matter subsided to the centre. Since, therefore, the common matter of our earth on the surface thereof is about twice as heavy as water, and a little lower, in mines, is found about three, or four, or even five times more heavy, it is probable that the quantity of the whole matter of the earth may be five or six times greater than if it consisted all of water; especially since I have before shewed that the earth is about four times more dense than Jupiter. If, therefore, Jupiter is a little more dense than water, in the space of thirty days, in which that planet describes the length of 459 of its semi-diameters, it would, in a medium of the same density with our air, lose almost a tenth part of its motion. But since the resistance of mediums decreases in proportion to their weight or density, so that water, which is $13\frac{3}{5}$ times lighter than quicksilver, resists less in that proportion; and air, which is 860 times lighter than water, resists less in the same proportion; therefore in the heavens, where the weight of the medium in which the planets move is immensely diminished, the resistance will almost vanish.

It is shewn in the Scholium of Prop. XXII, Book II, that at the height of 200 miles above the earth the air is more rare than it is at the superficies of the earth in the ratio of 30 to 0,00000000000003998, or as

75000000000000 to 1 nearly. And hence the planet Jupiter, revolving in a medium of the same density with that superior air, would not lose by the resistance of the medium the 1000000th part of its motion in 1000000 years. In the spaces near the earth the resistance is produced only by the air, exhalations, and vapours. When these are carefully exhausted by the air-pump from under the receiver, heavy bodies fall within the receiver with perfect freedom, and without the least sensible resistance: gold itself, and the lightest down, let fall together, will descend with equal velocity; and though they fall through a space of four, six, and eight feet, they will come to the bottom at the same time; as appears from experiments. And therefore the celestial regions being perfectly void of air and exhalations, the planets and comets meeting no sensible resistance in those spaces will continue their motions through them for an immense tract of time.

HYPOTHESIS I.

That the centre of the system of the world is immovable.

This is acknowledged by all, while some contend that the earth, others that the sun, is fixed in that centre. Let us see what may from hence follow.

PROPOSITION XI. THEOREM XI.

That the common centre of gravity of the earth, the sun, and all the planets, is immovable.

For (by Cor. 4 of the Laws) that centre either is at rest, or moves uniformly forward in a right line; but if that centre moved, the centre of the world would move also, against the Hypothesis.

PROPOSITION XII. THEOREM XII.

That the sun is agitated by a perpetual motion, but never recedes far from the common centre of gravity of all the planets.

For since (by Cor. 2, Prop. VIII) the quantity of matter in the sun is to the quantity of matter in Jupiter as 1067 to 1; and the distance of Jupiter from the sun is to the semi-diameter of the sun in a proportion but a small matter greater, the common centre of gravity of Jupiter and the sun will fall upon a point a little without the surface of the sun. By the same argument, since the quantity of matter in the sun is to the quantity of matter in Saturn as 3021 to 1, and the distance of Saturn from the sun is to the semi-diameter of the sun in a proportion but a small matter less, the common centre of gravity of Saturn and the sun will fall upon a point a little within the surface of the sun. And, pursuing the principles of this computation, we should find that though the earth and all the planets were placed on one side of the sun, the distance of the common centre of gravity of all from the centre of the sun would scarcely amount to one diameter of

the sun. In other cases, the distances of those centres are always less; and therefore, since that centre of gravity is in perpetual rest, the sun, according to the various positions of the planets, must perpetually be moved every way, but will never recede far from that centre.

Cor. Hence the common centre of gravity of the earth, the sun, and all the planets, is to be esteemed the centre of the world; for since the earth, the sun, and all the planets, mutually gravitate one towards another, and are therefore, according to their powers of gravity, in perpetual agitation, as the Laws of Motion require, it is plain that their moveable centres cannot be taken for the immovable centre of the world. If that body were to be placed in the centre, towards which other bodies gravitate most (according to common opinion), that privilege ought to be allowed to the sun; but since the sun itself is moved, a fixed point is to be chosen from which the centre of the sun recedes least, and from which it would recede yet less if the body of the sun were denser and greater, and therefore less apt to be moved.

PROPOSITION XIII. THEOREM XIII.

The planets move in ellipses which have their common focus in the centre of the sun; and, by radii drawn to that centre, they describe areas proportional to the times of description.

We have discoursed above of these motions from the Phænomena. Now that we know the principles on which they depend, from those principles we deduce the motions of the heavens *à priori*. Because the weights of the planets towards the sun are reciprocally as the squares of their distances from the sun's centre, if the sun was at rest, and the other planets did not mutually act one upon another, their orbits would be ellipses, having the sun in their common focus; and they would describe areas proportional to the times *of description*, by Prop. I and XI, and Cor. 1, Prop. XIII, Book I. But the mutual actions of the planets one upon another are so very small, that they may be neglected; and by Prop. LXVI, Book I, they less disturb the motions of the planets around the sun in motion than if those motions were performed about the sun at rest.

It is true, that the action of Jupiter upon Saturn is not to be neglected: for the force of gravity towards Jupiter is to the force of gravity towards the sun (at equal distances, Cor. 2, Prop. VIII) as 1 to 1067; and therefore in the conjunction of Jupiter and Saturn, because the distance of Saturn from Jupiter is to the distance of Saturn from the sun almost as 4 to 9, the gravity of Saturn towards Jupiter will be to the gravity of Saturn towards the sun as 81 to 16 × 1067; or, as 1 to about 211. And hence arises a perturbation of the orb of Saturn in every conjunction of this planet with Jupiter, so sensible, that astronomers are puzzled with it. As the planet

is differently situated in these conjunctions, its eccentricity is sometimes augmented, sometimes diminished; its aphelion is sometimes carried forward, sometimes backward, and its mean motion is by turns accelerated and retarded; yet the whole error in its motion about the sun, though arising from so great a force, may be almost avoided (except in the mean motion) by placing the lower focus of its orbit in the common centre of gravity of Jupiter and the sun (according to Prop. LXVII, Book I), and therefore that error, when it is greatest, scarcely exceeds two minutes; and the greatest error in the mean motion scarcely exceeds two minutes yearly. But in the conjunction of Jupiter and Saturn, the accelerative forces of gravity of the sun towards Saturn, of Jupiter towards Saturn, and of Jupiter towards the sun, are almost as 16, 81, and $\frac{16 \times 81 \times 3021}{25}$, or 156609; and therefore the difference of the forces of gravity of the sun towards Saturn, and of Jupiter towards Saturn, is to the force of gravity of Jupiter towards the sun as 65 to 156609, or as 1 to 2409. But the greatest power of Saturn to disturb the motion of Jupiter is proportional to this difference; and therefore the perturbation of the orbit of Jupiter is much less than that of Saturn's. The perturbations of the other orbits are yet far less, except that the orbit of the earth is sensibly disturbed by the moon. The common centre of gravity of the earth and moon moves in an ellipsis about the sun in the focus thereof, and, by a radius drawn to the sun, describes areas proportional to the times of description. But the earth in the mean time by a menstrual motion is revolved about this common centre.

PROPOSITION XIV. THEOREM XIV.

The aphelions and nodes of the orbits of the planets are fixed.

The aphelions are immovable by Prop. XI, Book I; and so are the planes of the orbits, by Prop. I of the same Book. And if the planes are fixed, the nodes must be so too. It is true, that some inequalities may arise from the mutual actions of the planets and comets in their revolutions; but these will be so small, that they may be here passed by.

Cor. 1. The fixed stars are immovable, seeing they keep the same position to the aphelions and nodes of the planets.

Cor. 2. And since these stars are liable to no sensible parallax from the annual motion of the earth, they can have no force, because of their immense distance, to produce any sensible effect in our system. Not to mention that the fixed stars, every where promiscuously dispersed in the heavens, by their contrary attractions destroy their mutual actions, by Prop. LXX, Book I.

SCHOLIUM.

Since the planets near the sun (viz. Mercury, Venus, the Earth, and

Mars) are so small that they can act with but little force upon each other, therefore their aphelions and nodes must be fixed, excepting in so far as they are disturbed by the actions of Jupiter and Saturn, and other higher bodies. And hence we may find, by the theory of gravity, that their aphelions move a little *in consequentia*, in respect of the fixed stars, and that in the sesquiplicate proportion of their several distances from the sun. So that if the aphelion of Mars, in the space of a hundred years, is carried 33′ 20″ *in consequentia*, in respect of the fixed stars, the aphelions of the Earth, of Venus, and of Mercury, will in a hundred years be carried forwards 17′ 40″; 10′ 53′, and 4′ 16″, respectively. But these motions are so inconsiderable, that we have neglected them in this Proposition.

PROPOSITION XV. PROBLEM I.

To find the principal diameters of the orbits of the planets.

They are to be taken in the sub-sesquiplicate proportion of the periodic times, by Prop. XV, Book I, and then to be severally augmented in the proportion of the sum of the masses of matter in the sun and each planet to the first of two mean proportionals betwixt that sum and the quantity of matter in the sun, by Prop. LX, Book I.

PROPOSITION XVI. PROBLEM II.

To find the eccentricities and aphelions of the planets.

This Problem is resolved by Prop. XVIII, Book I.

PROPOSITION XVII. THEOREM XV.

That the diurnal motions of the planets are uniform, and that the libration of the moon arises from its diurnal motion.

The Proposition is proved from the first Law of Motion, and Cor. 22, Prop. LXVI, Book I. Jupiter, with respect to the fixed stars, revolves in $9^h.\ 56'$; Mars in $24^h.\ 39'$; Venus in about $23^h.$; the Earth in $23^h.\ 56'$; the Sun in $25\frac{1}{2}$ days, and the moon in 27 days, 7 hours, 43′. These things appear by the Phænomena. The spots in the sun's body return to the same situation on the sun's disk, with respect to the earth, in $27\frac{1}{2}$ days; and therefore with respect to the fixed stars the sun revolves in about $25\frac{1}{2}$ days. But because the lunar day, arising from its uniform revolution about its axis, is menstrual, *that is, equal to the time of its periodic revolution in its orb*, therefore the same face of the moon will be always nearly turned to the upper focus of its orb; but, as the situation of that focus requires, will deviate a little to one side and to the other from the earth in the lower focus; and this is the libration in longitude; for the libration in latitude arises from the moon's latitude, and the inclination of its axis to the plane of the ecliptic. This theory of the libration of the moon, Mr. *N. Mercator*

in his Astronomy, published at the beginning of the year 1676, explained more fully out of the letters I sent him. The utmost satellite of Saturn seems to revolve about its axis with a motion like this of the moon, respecting Saturn continually with the same face; for in its revolution round Saturn, as often as it comes to the eastern part of its orbit, it is scarcely visible, and generally quite disappears; which is like to be occasioned by some spots in that part of its body, which is then turned towards the earth, as M. *Cassini* has observed. So also the utmost satellite of Jupiter seems to revolve about its axis with a like motion, because in that part of its body which is turned from Jupiter it has a spot, which always appears as if it were in Jupiter's own body, whenever the satellite passes between Jupiter and our eye.

PROPOSITION XVIII. THEOREM XVI.

That the axes of the planets are less than the diameters drawn perpendicular to the axes.

The equal gravitation of the parts on all sides would give a spherical figure to the planets, if it was not for their diurnal revolution in a circle. By that circular motion it comes to pass that the parts receding from the axis endeavour to ascend about the equator; and therefore if the matter is in a fluid state, by its ascent towards the equator it will enlarge the diameters there, and by its descent towards the poles it will shorten the axis. So the diameter of Jupiter (by the concurring observations of astronomers) is found shorter betwixt pole and pole than from east to west. And, by the same argument, if our earth was not higher about the equator than at the poles, the seas would subside about the poles, and, rising towards the equator, would lay all things there under water.

PROPOSITION XIX. PROBLEM III

To find the proportion of the axis of a planet to the diameters perpendicular thereto.

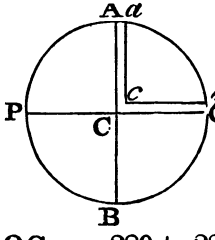
Our countryman, Mr. *Norwood*, measuring a distance of 905751 feet of *London* measure between *London* and *York*, in 1635, and observing the difference of latitudes to be 2° 28′, determined the measure of one degree to be 367196 feet of *London* measure, that is 57300 *Paris* toises. M. *Picart*, measuring an arc of one degree, and 22′ 55″ of the meridian between *Amiens* and *Malvoisine*, found an arc of one degree to be 57060 *Paris* toises. M. *Cassini*, the father, measured the distance upon the meridian from the town of *Collioure* in *Roussillon* to the Observatory of *Paris*; and his son added the distance from the Observatory to the Citadel of *Dunkirk*. The whole distance was 486156½ toises and the difference of the latitudes of *Collioure* and *Dunkirk* was 8 degrees, and 31

$11\frac{5}{6}''$. Hence an arc of one degree appears to be 57061 *Paris* toises. And from these measures we conclude that the circumference of the earth is 123249600, and its semi-diameter 19615800 *Paris* feet, upon the supposition that the earth is of a spherical figure.

In the latitude of *Paris* a heavy body falling in a second of time describes 15 *Paris* feet, 1 inch, $1\frac{7}{9}$ line, as above, that is, 2173 lines $\frac{7}{9}$. The weight of the body is diminished by the weight of the ambient air. Let us suppose the weight lost thereby to be $\frac{1}{1100}$ part of the whole weight; then that heavy body falling *in vacuo* will describe a height of 2174 lines in one second of time.

A body in every sidereal day of 23^h. 56′ 4″ uniformly revolving in a circle at the distance of 19615800 feet from the centre, in one second of time describes an arc of 1433,46 feet; the versed sine of which is 0,05236561 feet, or 7,54064 lines. And therefore the force with which bodies descend in the latitude of *Paris* is to the centrifugal force of bodies in the equator arising from the diurnal motion of the earth as 2174 to 7,54064.

The centrifugal force of bodies in the equator is to the centrifugal force with which bodies recede directly from the earth in the latitude of *Paris* 48° 50′ 10″ in the duplicate proportion of the radius to the cosine of the latitude, that is, as 7,54064 to 3,267. Add this force to the force with which bodies descend by their weight in the latitude of *Paris*, and a body, in the latitude of *Paris*, falling by its whole undiminished force of gravity, in the time of one second, will describe 2177,267 lines, or 15 *Paris* feet, 1 inch, and 5,267 lines. And the total force of gravity in that latitude will be to the centrifugal force of bodies in the equator of the earth as 2177,267 to 7,54064, or as 289 to 1.



Wherefore if APBQ represent the figure of the earth, now no longer spherical, but generated by the rotation of an ellipsis about its lesser axis PQ; and ACQ*qca* a canal full of water, reaching from the pole Q*q* to the centre C*c*, and thence rising to the equator A*a*; the weight of the water in the leg of the canal AC*ca* will be to the weight of water in the other leg QC*cq* as 289 to 288, because the centrifugal force arising from the circular motion sustains and takes off one of the 289 parts of the weight (in the one leg), and the weight of 288 in the other sustains the rest. But by computation (from Cor. 2, Prop. XCI, Book I) I find, that, if the matter of the earth was all uniform, and without any motion, and its axis PQ were to the diameter AB as 100 to 101, the force of gravity in the place Q towards the earth would be to the force of gravity in the same place Q towards a sphere described about the centre C with the radius PC, or QC, as 126 to 125. And, by the same argument, the force of gravity in the place A towards the spheroid generated by the rotation of

the ellipsis APBQ about the axis AB is to the force of gravity in the same place A, towards the sphere described about the centre C with the radius AC, as 125 to 126. But the force of gravity in the place A towards the earth is a mean proportional betwixt the forces of gravity towards the spheroid and this sphere; because the sphere, by having its diameter PQ diminished in the proportion of 101 to 100, is transformed into the figure of the earth; and this figure, by having a third diameter perpendicular to the two diameters AB and PQ diminished in the same proportion, is converted into the said spheroid; and the force of gravity in A, in either case, is diminished nearly in the same proportion. Therefore the force of gravity in A towards the sphere described about the centre C with the radius AC, is to the force of gravity in A towards the earth as 126 to $125\frac{1}{2}$. And the force of gravity in the place Q towards the sphere described about the centre C with the radius QC, is to the force of gravity in the place A towards the sphere described about the centre C, with the radius AC, in the proportion of the diameters (by Prop. LXXII, Book I), that is, as 100 to 101. If, therefore, we compound those three proportions 126 to 125, 126 to $125\frac{1}{2}$, and 100 to 101, into one, the force of gravity in the place Q towards the earth will be to the force of gravity in the place A towards the earth as $126 \times 126 \times 100$ to $125 \times 125\frac{1}{2} \times 101$; or as 501 to 500.

Now since (by Cor. 3, Prop. XCI, Book I) the force of gravity in either leg of the canal AC*ca*, or QC*cq*, is as the distance of the places from the centre of the earth, if those legs are conceived to be divided by transverse, parallel, and equidistant surfaces, into parts proportional to the wholes, the weights of any number of parts in the one leg AC*ca* will be to the weights of the same number of parts in the other leg as their magnitudes and the accelerative forces of their gravity conjunctly, that is, as 101 to 100, and 500 to 501, or as 505 to 501. And therefore if the centrifugal force of every part in the leg AC*ca*, arising from the diurnal motion, was to the weight of the same part as 4 to 505, so that from the weight of every part, conceived to be divided into 505 parts, the centrifugal force might take off four of those parts, the weights would remain equal in each leg, and therefore the fluid would rest in an equilibrium. But the centrifugal force of every part is to the weight of the same part as 1 to 289; that is, the centrifugal force, which should be $\frac{4}{505}$ parts of the weight, is only $\frac{1}{289}$ part thereof. And, therefore, I say, by the rule of proportion, that if the centrifugal force $\frac{4}{505}$ make the height of the water in the leg AC*ca* to exceed the height of the water in the leg QC*cq* by one $\frac{1}{100}$ part of its whole height, the centrifugal force $\frac{1}{289}$ will make the excess of the height in the leg AC*ca* only $\frac{1}{289}$ part of the height of the water in the other leg QC*cq*; and therefore the diameter of the earth at the equator, is to its diameter from pole to pole as 230 to 229. And since the mean semi-

diameter of the earth, according to *Picart's* mensuration, is 19615800 *Paris* feet, or 3923,16 miles (reckoning 5000 feet to a mile), the earth will be higher at the equator than at the poles by 85472 feet, or $17\frac{1}{10}$ miles. And its height at the equator will be about 19658600 feet, and at the poles 19573000 feet.

If, the density and periodic time of the diurnal revolution remaining the same, the planet was greater or less than the earth, the proportion of the centrifugal force to that of gravity, and therefore also of the diameter betwixt the poles to the diameter at the equator, would likewise remain the same. But if the diurnal motion was accelerated or retarded in any proportion, the centrifugal force would be augmented or diminished nearly in the same duplicate proportion; and therefore the difference of the diameters will be increased or diminished in the same duplicate ratio very nearly. And if the density of the planet was augmented or diminished in any proportion, the force of gravity tending towards it would also be augmented or diminished in the same proportion: and the difference of the diameters contrariwise would be diminished in proportion as the force of gravity is augmented, and augmented in proportion as the force of gravity is diminished. Wherefore, since the earth, in respect of the fixed stars, revolves in $23^h.\ 56'$, but Jupiter in $9^h.\ 56'$, and the squares of their periodic times are as 29 to 5, and their densities as 400 to $94\frac{1}{2}$, the difference of the diameters of Jupiter will be to its lesser diameter as $\frac{29}{5} \times \frac{400}{94\frac{1}{2}} \times \frac{1}{229}$ to 1, or as 1 to $9\frac{1}{3}$, nearly. Therefore the diameter of Jupiter from east to west is to its diameter from pole to pole nearly as $10\frac{1}{3}$ to $9\frac{1}{3}$. Therefore since its greatest diameter is $37''$, its lesser diameter lying between the poles will be $33''\ 25'''$. Add thereto about $3''$ for the irregular refraction of light, and the apparent diameters of this planet will become $40''$ and $36''\ 25'''$; which are to each other as $11\frac{1}{6}$ to $10\frac{1}{6}$, very nearly. These things are so upon the supposition that the body of Jupiter is uniformly dense. But now if its body be denser towards the plane of the equator than towards the poles, its diameters may be to each other as 12 to 11, or 13 to 12, or perhaps as 14 to 13.

And *Cassini* observed in the year 1691, that the diameter of Jupiter reaching from east to west is greater by about a fifteenth part than the other diameter. Mr. *Pound* with his 123 feet telescope, and an excellent micrometer, measured the diameters of Jupiter in the year 1719, and found them as follow.

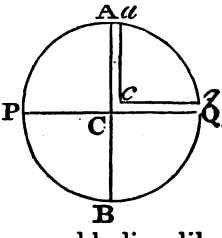
The Times.			Greatest diam.	Lesser diam.	The diam. to each other.		
	Day.	Hours.	Parts	Parts			
January	28	6	13,40	12,28	As 12	to	11
March	6	7	13,12	12,20	$13\frac{3}{4}$	to	$12\frac{3}{4}$
March	9	7	13,12	12,08	$12\frac{3}{4}$	to	$11\frac{3}{4}$
April	9	9	12,32	11,48	$14\frac{1}{2}$	to	$13\frac{1}{2}$

So that the theory agrees with the phænomena; for the planets are more heated by the sun's rays towards their equators, and therefore are a little more condensed by that heat than towards their poles.

Moreover, that there is a diminution of gravity occasioned by the diurnal rotation of the earth, and therefore the earth rises higher there than it does at the poles (supposing that its matter is uniformly dense), will appear by the experiments of pendulums related under the following Proposition.

PROPOSITION XX. PROBLEM IV.

To find and compare together the weights of bodies in the different regions of our earth.



Because the weights of the unequal legs of the canal of water ACQ*qca* are equal; and the weights of the parts proportional to the whole legs, and alike situated in them, are one to another as the weights of the wholes, and therefore equal betwixt themselves; the weights of equal parts, and alike situated in the legs, will be reciprocally as the legs, that is, reciprocally as 230 to 229. And the case is the same in all homogeneous equal bodies alike situated in the legs of the canal. Their weights are reciprocally as the legs, that is, reciprocally as the distances of the bodies from the centre of the earth. Therefore if the bodies are situated in the uppermost parts of the canals, or on the surface of the earth, their weights will be one to another reciprocally as their distances from the centre. And, by the same argument, the weights in all other places round the whole surface of the earth are reciprocally as the distances of the places from the centre; and, therefore, in the hypothesis of the earth's being a spheroid are given in proportion.

Whence arises this Theorem, that the increase of weight in passing from the equator to the poles is nearly as the versed sine of double the latitude; or, which comes to the same thing, as the square of the right sine of the latitude; and the arcs of the degrees of latitude in the meridian increase nearly in the same proportion. And, therefore, since the latitude of *Paris* is 48° 50′, that of places under the equator 00° 00′, and that of places under the poles 90°; and the versed sines of double those arcs are 11334,00000 and 20000, the radius being 10000; and the force of gravity at the pole is to the force of gravity at the equator as 230 to 229; and the excess of the force of gravity at the pole to the force of gravity at the equator as 1 to 229; the excess of the force of gravity in the latitude of *Paris* will be to the force of gravity at the equator as $1 \times \frac{11}{10}\frac{3}{10}\frac{3}{10}\frac{4}{10}$ to 229, or as 5667 to 2290000. And therefore the whole forces of gravity in those places will be one to the other as 2295667 to 2290000. Wherefore since the lengths of pendulums vibrating in equal times are as the forces of

gravity, and in the latitude of *Paris*, the length of a pendulum vibrating seconds is 3 *Paris* feet, and $8\frac{1}{2}$ lines, or rather because of the weight of the air, $8\frac{5}{9}$ lines, the length of a pendulum vibrating in the same time under the equator will be shorter by 1,087 lines. And by a like calculus the following table is made.

Latitude of the place.	Length of the pendulum		Measure of one degree in the meridian.
Deg.	Feet.	Lines.	Toises.
0	3 .	7,468	56637
5	3 .	7,482	56642
10	3 .	7,526	56659
15	3 .	7,596	56687
20	3 .	7,692	56724
25	3 .	7,812	56769
30	3 .	7,948	56823
35	3 .	8,099	56882
40	3 .	8,261	56945
1	3 .	8,294	56958
2	3 .	8,327	56971
3	3 .	8,361	56984
4	3 .	8,394	56997
45	3 .	8,428	57010
6	3 .	8,461	57022
7	3 .	8,494	57035
8	3 .	8,528	57048
9	3 .	8,561	57061
50	3 .	8,594	57074
55	3 .	8,756	57137
60	3 .	8,907	57196
65	3 .	9,044	57250
70	3 .	9,162	57295
75	3 .	9,258	57332
80	3 .	9,329	57360
85	3 .	9,372	57377
90	3 .	9,387	57382

By this table, therefore, it appears that the inequality of degrees is so small, that the figure of the earth, in geographical matters, may be considered as spherical; especially if the earth be a little denser towards the plane of the equator than towards the poles.

Now several astronomers, sent into remote countries to make astronomical observations, have found that pendulum clocks do accordingly move slower near the equator than in our climates. And, first of all, in the year 1672, M. *Richer* took notice of it in the island of *Cayenne*; for when, in the month of *August*, he was observing the transits of the fixed stars over the meridian, he found his clock to go slower than it ought in respect of the mean motion of the sun at the rate of 2′ 28″ a day. Therefore, fitting up a simple pendulum to vibrate in seconds, which were measured by an excellent clock, he observed the length of that simple pendulum; and this he did over and over every week for ten months together. And upon his return to *France*, comparing the length of that pendulum with the length

of the pendulum at *Paris* (which was 3 *Paris* feet and $8\frac{3}{5}$ lines), he found it shorter by $1\frac{1}{4}$ line.

Afterwards, our friend Dr. *Halley*, about the year 1677, arriving at the island of St. *Helena*, found his pendulum clock to go slower there than at *London* without marking the difference. But he shortened the rod of his clock by more than the $\frac{1}{8}$ of an inch, or $1\frac{1}{2}$ line; and to effect this, because the length of the screw at the lower end of the rod was not sufficient, he interposed a wooden ring betwixt the nut and the ball.

Then, in the year 1682, M. *Varin* and M. *des Hayes* found the length of a simple pendulum vibrating in seconds at the Royal Observatory of *Paris* to be 3 feet and $8\frac{5}{9}$ lines. And by the same method in the island of *Goree*, they found the length of an isochronal pendulum to be 3 feet and $6\frac{5}{9}$ lines, differing from the former by two lines. And in the same year, going to the islands of *Guadaloupe* and *Martinico*, they found that the length of an isochronal pendulum in those islands was 3 feet and $6\frac{1}{2}$ lines.

After this, M. *Couplet*, the son, in the month of *July* 1697, at the Royal Observatory of *Paris*, so fitted his pendulum clock to the mean motion of the sun, that for a considerable time together the clock agreed with the motion of the sun. In *November* following, upon his arrival at *Lisbon*, he found his clock to go slower than before at the rate of 2′ 13″ in 24 hours. And next *March* coming to *Paraiba*, he found his clock to go slower than at *Paris*, and at the rate 4′ 12″ in 24 hours; and he affirms, that the pendulum vibrating in seconds was shorter at *Lisbon* by $2\frac{1}{2}$ lines, and at *Paraiba* by $3\frac{2}{3}$ lines, than at *Paris*. He had done better to have reckoned those differences $1\frac{1}{3}$ and $2\frac{5}{9}$: for these differences correspond to the differences of the times 2′ 13″ and 4′ 12″. But this gentleman's observations are so gross, that we cannot confide in them.

In the following years, 1699, and 1700, M. *des Hayes*, making another voyage to *America*, determined that in the island of *Cayenne* and *Granada* the length of the pendulum vibrating in seconds was a small matter less than 3 feet and $6\frac{1}{2}$ lines; that in the island of St. *Christophers* it was 3 feet and $6\frac{3}{4}$ lines; and in the island of St. *Domingo* 3 feet and 7 lines.

And in the year 1704, P. *Feuillé*, at *Puerto Bello* in *America*, found that the length of the pendulum vibrating in seconds was 3 *Paris feet*, and only $5\frac{7}{12}$ lines, that is, almost 3 lines shorter than at *Paris*; but the observation was faulty. For afterward, going to the island of *Martinico*, he found the length of the isochronal pendulum there 3 *Paris* feet and $5\frac{9}{12}$ lines.

Now the latitude of *Paraiba* is 6° 38′ south; that of *Puerto Bello* 9° 33′ north; and the latitudes of the islands *Cayenne*, *Goree*, *Gaudaloupe*, *Martinico*, *Granada*, St. *Christophers*, and St. *Domingo*, are respectively 4° 55′, 14° 40″, 15° 00′, 14° 44′, 12° 06′, 17° 19′, and 19° 48′, north. And

the excesses of the length of the pendulum at *Paris* above the lengths of the isochronal pendulums observed in those latitudes are a little greater than by the table of the lengths of the pendulum before computed. And therefore the earth is a little higher under the equator than by the preceding calculus, and a little denser at the centre than in mines near the surface, unless, perhaps, the heats of the torrid zone have a little extended the length of the pendulums.

For M. *Picart* has observed, that a rod of iron, which in frosty weather in the winter season was one foot long, when heated by fire, was lengthened into one foot and $\frac{1}{4}$ line. Afterward M. *de la Hire* found that a rod of iron, which in the like winter season was 6 feet long, when exposed to the heat of the summer sun, was extended into 6 feet and $\frac{2}{3}$ line. In the former case the heat was greater than in the latter; but in the latter it was greater than the heat of the external parts of a human body; for metals exposed to the summer sun acquire a very considerable degree of heat. But the rod of a pendulum clock is never exposed to the heat of the summer sun, nor ever acquires a heat equal to that of the external parts of a human body; and, therefore, though the 3 feet rod of a pendulum clock will indeed be a little longer in the summer than in the winter season, yet the difference will scarcely amount to $\frac{1}{4}$ line. Therefore the total difference of the lengths of isochronal pendulums in different climates cannot be ascribed to the difference of heat; nor indeed to the mistakes of the *French* astronomers. For although there is not a perfect agreement betwixt their observations, yet the errors are so small that they may be neglected; and in this they all agree, that isochronal pendulums are shorter under the equator than at the Royal Observatory of *Paris*, by a difference not less than $1\frac{1}{4}$ line, nor greater than $2\frac{2}{3}$ lines. By the observations of M. *Richer*, in the island of *Cayenne*, the difference was $1\frac{1}{4}$ line. That difference being corrected by those of M. *des Hayes*, becomes $1\frac{1}{2}$ line or $1\frac{3}{4}$ line. By the less accurate observations of others, the same was made about two lines. And this disagreement might arise partly from the errors of the observations, partly from the dissimilitude of the internal parts of the earth, and the height of mountains; partly from the different heats of the air.

I take an iron rod of 3 feet long to be shorter by a sixth part of one line in winter time with us here in *England* than in the summer. Because of the great heats under the equator, subduct this quantity from the difference of one line and a quarter observed by M. *Richer*, and there will remain one line $\frac{1}{12}$, which agrees very well with $1\frac{87}{100}$ line collected, by the theory a little before. M. *Richer* repeated his observations, made in the island of *Cayenne*, every week for ten months together, and compared the lengths of the pendulum which he had there noted in the iron rods with the lengths thereof which he observed in *France*. This diligence and care seems to have been wanting to the other observers. If this gentleman's observations

are to be depended on, the earth is higher under the equator than at the poles, and that by an excess of about 17 miles; as appeared above by the theory.

PROPOSITION XXI. THEOREM XVII.

That the equinoctial points go backward, and that the axis of the earth, by a nutation in every annual revolution, twice vibrates towards the ecliptic, and as often returns to its former position.

The proposition appears from Cor. 20, Prop. LXVI, Book I; but that motion of nutation must be very small, and, indeed, scarcely perceptible.

PROPOSITION XXII. THEOREM XVIII.

That all the motions of the moon, and all the inequalities of those motions, follow from the principles which we have laid down.

That the greater planets, while they are carried about the sun, may in the mean time carry other lesser planets, revolving about them; and that those lesser planets must move in ellipses which have their foci in the centres of the greater, appears from Prop. LXV, Book I. But then their motions will be several ways disturbed by the action of the sun, and they will suffer such inequalities as are observed in our moon. Thus our moon (by Cor. 2, 3, 4, and 5, Prop. LXVI, Book I) moves faster, and, by a radius drawn to the earth, describes an area greater for the time, and has its orbit less curved, and therefore approaches nearer to the earth in the syzygies than in the quadratures, excepting in so far as these effects are hindered by the motion of eccentricity; for (by Cor. 9, Prop. LXVI, Book I) the eccentricity is greatest when the apogeon of the moon is in the syzygies, and least when the same is in the quadratures; and upon this account the perigeon moon is swifter, and nearer to us, but the apogeon moon slower, and farther from us, in the syzygies than in the quadratures. Moreover, the apogee goes forward, and the nodes backward; and this is done not with a regular but an unequal motion. For (by Cor. 7 and 8, Prop. LXVI, Book I) the apogee goes more swiftly forward in its syzygies, more slowly backward in its quadratures; and, by the excess of its progress above its regress, advances yearly *in consequentia*. But, contrariwise, the nodes (by Cor. 11, Prop. LXVI, Book I) are quiescent in their syzygies, and go fastest back in their quadratures. Farther, the greatest latitude of the moon (by Cor. 10, Prop. LXVI, Book I) is greater in the quadratures of the moon than in its syzygies. And (by Cor. 6, Prop. LXVI, Book I) the mean motion of the moon is slower in the perihelion of the earth than in its aphelion. And these are the principal inequalities (of the moon) taken notice of by astronomers.

But there are yet other inequalities not observed by former astronomers, by which the motions of the moon are so disturbed, that to this day we have not been able to bring them under any certain rule. For the velocities or horary motions of the apogee and nodes of the moon, and their equations, as well as the difference betwixt the greatest eccentricity in the syzygies, and the least eccentricity in the quadratures, and that inequality which we call the variation, are (by Cor. 14, Prop. LXVI, Book I) in the course of the year augmented and diminished in the triplicate proportion of the sun's apparent diameter. And besides (by Cor. 1 and 2, Lem. 10, and Cor. 16, Prop. LXVI, Book I) the variation is augmented and diminished nearly in the duplicate proportion of the time between the quadratures. But in astronomical calculations, this inequality is commonly thrown into and confounded with the equation of the moon's centre.

PROPOSITION XXIII. PROBLEM V.

To derive the unequal motions of the satellites of Jupiter and Saturn from the motions of our moon.

From the motions of our moon we deduce the corresponding motions of the moons or satellites of Jupiter in this manner, by Cor. 16, Prop. LXVI, Book I. The mean motion of the nodes of the outmost satellite of Jupiter is to the mean motion of the nodes of our moon in a proportion compounded of the duplicate proportion of the periodic times of the earth about the sun to the periodic times of Jupiter about the sun, and the simple proportion of the periodic time of the satellite about Jupiter to the periodic time of our moon about the earth; and, therefore, those nodes, in the space of a hundred years, are carried 8° 24′ backward, or *in antecedentia*. The mean motions of the nodes of the inner satellites are to the mean motion of the nodes of the outmost as their periodic times to the periodic time of the former, by the same Corollary, and are thence given. And the motion of the apsis of every satellite *in consequentia* is to the motion of its nodes *in antecedentia* as the motion of the apogee of our moon to the motion of its nodes (by the same Corollary), and is thence given. But the motions of the apsides thus found, must be diminished in the proportion of 5 to 9, or of about 1 to 2, on account of a cause which I cannot here descend to explain. The greatest equations of the nodes, and of the apsis of every satellite, are to the greatest equations of the nodes, and apogee of our moon respectively, as the motions of the nodes and apsides of the satellites, in the time of one revolution of the former equations, to the motions of the nodes and apogee of our moon, in the time of one revolution of the latter equations. The variation of a satellite seen from Jupiter is to the variation of our moon in the same proportion as the whole motions of their nodes

respectively during the times in which the satellite and our moon (after parting from) are revolved (again) to the sun, by the same Corollary; and therefore in the outmost satellite the variation does not exceed 5″ 12‴.

PROPOSITION XXIV. THEOREM XIX.

That the flux and reflux of the sea arise from the actions of the sun and moon.

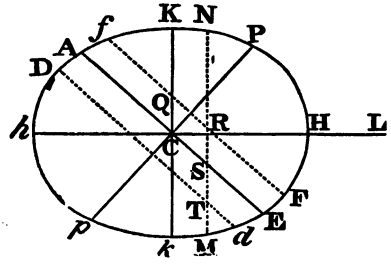
By Cor. 19 and 20, Prop. LXVI, Book I, it appears that the waters of the sea ought twice to rise and twice to fall every day, as well lunar as solar; and that the greatest height of the waters in the open and deep seas ought to follow the appulse of the luminaries to the meridian of the place by a less interval than 6 hours; as happens in all that eastern tract of the *Atlantic* and *Æthiopic* seas between *France* and the *Cape of Good Hope;* and on the coasts of *Chili* and *Peru* in the *South Sea;* in all which shores the flood falls out about the second, third, or fourth hour, unless where the motion propagated from the deep ocean is by the shallowness of the channels, through which it passes to some particular places, retarded to the fifth, sixth, or seventh hour, and even later. The hours I reckon from the appulse of each luminary to the meridian of the place, as well under as above the horizon; and by the hours of the lunar day I understand the 24th parts of that time which the moon, by its apparent diurnal motion, employs to come about again to the meridian of the place which it left the day before. The force of the sun or moon in raising the sea is greatest in the appulse of the luminary to the meridian of the place; but the force impressed upon the sea at that time continues a little while after the impression, and is afterwards increased by a new though less force still acting upon it. This makes the sea rise higher and higher, till this new force becoming too weak to raise it any more, the sea rises to its greatest height. And this will come to pass, perhaps, in one or two hours, but more frequently near the shores in about three hours, or even more, where the sea is shallow.

The two luminaries excite two motions, which will not appear distinctly, but between them will arise one mixed motion compounded out of both. In the conjunction or opposition of the luminaries their forces will be conjoined, and bring on the greatest flood and ebb. In the quadratures the sun will raise the waters which the moon depresses, and depress the waters which the moon raises, and from the difference of their forces the smallest of all tides will follow. And because (as experience tells us) the force of the moon is greater than that of the sun, the greatest height of the waters will happen about the third lunar hour. Out of the syzygies and quadratures, the greatest tide, which by the single force of the moon ought to fall out at the third lunar hour, and by the single force of the sun at the third solar hour, by the compounded forces of both must fall out in an interme-

diate time that aproaches nearer to the third hour of the moon than to that of the sun. And, therefore, while the moon is passing from the syzygies to the quadratures, during which time the 3d hour of the sun precedes the 3d hour of the moon, the greatest height of the waters will also precede the 3d hour of the moon, and that, by the greatest interval, a little after the octants of the moon; and, by like intervals, the greatest tide will follow the 3d lunar hour, while the moon is passing from the quadratures to the syzygies. Thus it happens in the open sea; for in the mouths of rivers the greater tides come later to their height.

But the effects of the luminaries depend upon their distances from the earth; for when they are less distant, their effects are greater, and when more distant, their effects are less, and that in the triplicate proportion of their apparent diameter. Therefore it is that the sun, in the winter time, being then in its perigee, has a greater effect, and makes the tides in the syzygies something greater, and those in the quadratures something less than in the summer season; and every month the moon, while in the perigee, raises greater tides than at the distance of 15 days before or after, when it is in its apogee. Whence it comes to pass that two highest tides do not follow one the other in two immediately succeeding syzygies.

The effect of either luminary doth likewise depend upon its declination or distance from the equator; for if the luminary was placed at the pole, it would constantly attract all the parts of the waters without any intension or remission of its action, and could cause no reciprocation of motion. And, therefore, as the luminaries decline from the equator towards either pole, they will, by degrees, lose their force, and on this account will excite lesser tides in the solstitial than in the equinoctial syzygies. But in the solstitial quadratures they will raise greater tides than in the quadratures about the equinoxes; because the force of the moon, then situated in the equator, most exceeds the force of the sun. Therefore the greatest tides fall out in those syzygies, and the least in those quadratures, which happen about the time of both equinoxes: and the greatest tide in the syzygies is always succeeded by the least tide in the quadratures, as we find by experience. But, because the sun is less distant from the earth in winter than in summer, it comes to pass that the greatest and least tides more frequently appear before than after the vernal equinox, and more frequently after than before the autumnal.



Moreover, the effects of the luminaries depend upon the latitudes of places. Let A*p*EP represent the earth covered with deep waters; C its centre; P, *p* its poles; AE the equator; F any place without the equator; F*f* the parallel of the place; D*d* the correspondent parallel on the

other side of the equator; L the place of the moon three hours before; H the place of the earth directly under it; *h* the opposite place; K, *k* the places at 90 degrees distance; CH, C*h*, the greatest heights of the sea from the centre of the earth; and CK, C*k*, its least heights: and if with the axes H*h*, K*k*, an ellipsis is described, and by the revolution of that ellipsis about its longer axis H*h* a spheroid HPK*hpk* is formed, this spheroid will nearly represent the figure of the sea; and CF, C*f*, CD, C*d*, will represent the heights of the sea in the places F*f*, D*d*. But farther; in the said revolution of the ellipsis any point N describes the circle NM cutting the parallels F*f*, D*d*, in any places RT, and the equator AE in S; CN will represent the height of the sea in all those places R, S, T, situated in this circle. Wherefore, in the diurnal revolution of any place F, the greatest flood will be in F, at the third hour after the appulse of the moon to the meridian above the horizon; and afterwards the greatest ebb in Q, at the third hour after the setting of the moon; and then the greatest flood in *f*, at the third hour after the appulse of the moon to the meridian under the horizon; and, lastly, the greatest ebb in Q, at the third hour after the rising of the moon; and the latter flood in *f* will be less than the preceding flood in F. For the whole sea is divided into two hemispherical floods, one in the hemisphere KH*k* on the north side, the other in the opposite hemisphere K*hk*, which we may therefore call the northern and the southern floods. These floods, being always opposite the one to the other, come by turns to the meridians of all places, after an interval of 12 lunar hours. And seeing the northern countries partake more of the northern flood, and the southern countries more of the southern flood, thence arise tides, alternately greater and less in all places without the equator, in which the luminaries rise and set. But the greatest tide will happen when the moon declines towards the vertex of the place, about the third hour after the appulse of the moon to the meridian above the horizon; and when the moon changes its declination *to the other side of the equator*, that which was the greater tide will be changed into a lesser. And the greatest difference of the floods will fall out about the times of the solstices; especially if the ascending node of the moon is about the first of Aries. So it is found by experience that the morning tides in winter exceed those of the evening, and the evening tides in summer exceed those of the morning; at *Plymouth* by the height of one foot, but at *Bristol* by the height of 15 inches, according to the observations of *Colepress* and *Sturmy*.

But the motions which we have been describing suffer some alteration from that force of reciprocation, which the waters, being once moved, retain a little while *by their vis insita*. Whence it comes to pass that the tides may continue for some time, though the actions of the luminaries should

cease. This power of retaining the impressed motion lessens the difference of the alternate tides, and makes those tides which immediately succeed after the syzygies greater, and those which follow next after the quadratures less. And hence it is that the alternate tides at *Plymouth* and *Bristol* do not differ much more one from the other than by the height of a foot or 15 inches, and that the greatest tides of all at those ports are not the first but the third after the syzygies. And, besides, all the motions are retarded in their passage through shallow channels, so that the greatest tides of all, in some straits and mouths of rivers, are the fourth or even the fifth after the syzygies.

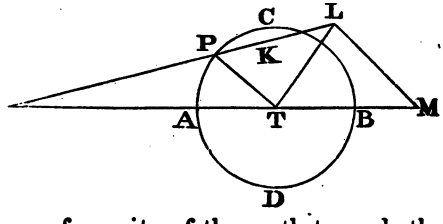
Farther, it may happen that the tide may be propagated from the ocean through different channels towards the same port, and may pass quicker through some channels than through others; in which case the same tide, divided into two or more succeeding one another, may compound new motions of different kinds. Let us suppose two equal tides flowing towards the same port from different places, the one preceding the other by 6 hours; and suppose the first tide to happen at the third hour of the appulse of the moon to the meridian of the port. If the moon at the time of the appulse to the meridian was in the equator, every 6 hours alternately there would arise equal floods, which, meeting with as many equal ebbs, would so balance one the other, that for that day, the water would stagnate and remain quiet. If the moon then declined from the equator, the tides in the ocean would be alternately greater and less, as was said; and from thence two greater and two lesser tides would be alternately propagated towards that port. But the two greater floods would make the greatest height of the waters to fall out in the middle time betwixt both; and the greater and lesser floods would make the waters to rise to a mean height in the middle time between them, and in the middle time between the two lesser floods the waters would rise to their least height. Thus in the space of 24 hours the waters would come, not twice, as commonly, but once only to their greatest, and once only to their least height; and their greatest height, if the moon declined towards the elevated pole, would happen at the 6th or 30th hour after the appulse of the moon to the meridian; and when the moon changed its declination, this flood would be changed into an ebb. An example of all which Dr. *Halley* has given us, from the observations of seamen in the port of *Batsham*, in the kingdom of *Tunquin*, in the latitude of 20° 50′ north. In that port, on the day which follows after the passage of the moon over the equator, the waters stagnate: when the moon declines to the north, they begin to flow and ebb, not twice, as in other ports, but once only every day: and the flood happens at the setting, and the greatest ebb at the rising of the moon. This tide increases with the declination of the moon till the 7th or 8th day; then for the 7 or 8 days following it

decreases at the same rate as it had increased before, and ceases when the moon changes its declination, crossing over the equator to the south. After which the flood is immediately changed into an ebb; and thenceforth the ebb happens at the setting and the flood at the rising of the moon; till the moon, again passing the equator, changes its declination. There are two inlets to this port and the neighboring channels, one from the seas of *China*, between the continent and the island of *Leuconia*; the other from the *Indian* sea, between the continent and the island of *Borneo*. But whether there be really two tides propagated through the said channels, one from the *Indian* sea in the space of 12 hours, and one from the sea of *China* in the space of 6 hours, which therefore happening at the 3d and 9th lunar hours, by being compounded together, produce those motions; or whether there be any other circumstances in the state of those seas, I leave to be determined by observations on the neighbouring shores.

Thus I have explained the causes of the motions of the moon and of the sea. Now it is fit to subjoin something concerning the quantity of those motions.

PROPOSITION XXV. PROBLEM VI.

To find the forces with which the sun disturbs the motions of the moon.

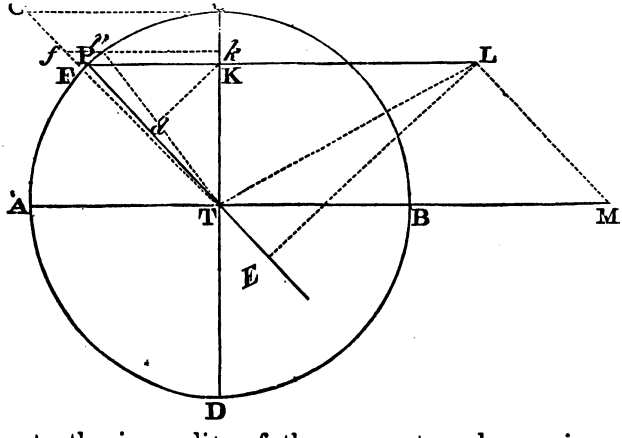


Let S represent the sun, T the earth, P the moon, CADB the moon's orbit. In SP take SK equal to ST; and let SL be to SK in the duplicate proportion of SK to SP: draw LM parallel to PT; and if ST or SK is supposed to represent the accelerated force of gravity of the earth towards the sun, SL will represent the accelerative force of gravity of the moon towards the sun. But that force is compounded of the parts SM and LM, of which the force LM, and that part of SM which is represented by TM, disturb the motion of the moon, as we have shewn in Prop. LXVI, Book I, and its Corollaries. Forasmuch as the earth and moon are revolved about their common centre of gravity, the motion of the earth about that centre will be also disturbed by the like forces; but we may consider the sums both of the forces and of the motions as in the moon, and represent the sum of the forces by the lines TM and ML, which are analogous to them both. The force ML (in its mean quantity) is to the centripetal force by which the moon may be retained in its orbit revolving about the earth at rest, at the distance PT, in the duplicate proportion of the periodic time of the moon about the earth to the periodic time of the earth about the sun (by Cor. 17, Prop. LXVI, Book I); that is, in the duplicate proportion of $27^d.\ 7^h.\ 43'$ to $365^d.\ 6^h.\ 9'$; or as 1000 to 178725; or as 1 to $178\frac{2}{4}\frac{9}{0}$. But in the

4th Prop. of this Book we found, that, if both earth and moon were revolved about their common centre of gravity, the mean distance of the one from the other would be nearly $60\frac{1}{2}$ mean semi-diameters of the earth; and the force by which the moon may be kept revolving in its orbit about the earth in rest at the distance PT of $60\frac{1}{2}$ semi-diameters of the earth, is to the force by which it may be revolved in the same time, at the distance of 60 semi-diameters, as $60\frac{1}{2}$ to 60: and this force is to the force of gravity with us very nearly as 1 to 60×60. Therefore the mean force ML is to the force of gravity on the surface of our earth as $1 \times 60\frac{1}{2}$ to $60 \times 60 \times 60 \times 178\frac{2}{4}\frac{9}{0}$, or as 1 to 638092,6; whence by the proportion of the lines TM, ML, the force TM is also given; and these are the forces with which the sun disturbs the motions of the moon. Q.E.I.

PROPOSITION XXVI. PROBLEM VII.

To find the horary increment of the area which the moon, by a radius drawn to the earth, describes in a circular orbit.



We have above shown that the area which the moon describes by a radius drawn to the earth is proportional to the time of description, excepting in so far as the moon's motion is disturbed by the action of the sun; and here we propose to investigate the inequality of the moment, or horary increment *of that area or motion so disturbed.* To render the calculus more easy, we shall suppose the orbit of the moon to be circular, and neglect all inequalities but that only which is now under consideration; and, because of the immense distance of the sun, we shall farther suppose that the lines SP and ST are parallel. By this means, the force LM will be always reduced to its mean quantity TP, as well as the force TM to its mean quantity 3PK. These forces (by Cor. 2 of the Laws of Motion) compose the force TL; and this force, by letting fall the perpendicular LE upon the radius TP, is resolved into the forces TE, EL; of which the force TE, acting constantly in the direction of the radius TP, neither accelerates nor retards the description of the area TPC made by that radius TP; but EL, *acting on the radius* TP in a perpendicular direction, accelerates or retards *the description of the area* in proportion as it accelerates or retards the moon.

That acceleration of the moon, in its passage from the quadrature C to the conjunction A, is in every moment of time as the *generating* accelerative force EL, that is, as $\frac{3PK \times TK}{TP}$. Let the time be represented by the mean motion of the moon, or (which comes to the same thing) by the angle CTP, or even by the arc CP. At right angles upon CT erect CG equal to CT; and, supposing the quadrantal arc AC to be divided into an infinite number of equal parts P*p*, &c., these *parts* may represent the like *infinite* number of the equal parts of time. Let fall *pk* perpendicular on CT, and draw TG meeting with KP, *kp* produced in F and *f*; then will FK be equal to TK, and K*k* be to PK as P*p* to T*p*, that is, in a given proportion; and therefore FK × K*k*, or the area FK*kf*, will be as $\frac{3PK \times TK}{TP}$, that is, as EL; and compounding, the whole area GCKF will be as the sum of all the forces EL impressed upon the moon in the whole time CP; and therefore also as the velocity generated by that sum, that is, as the acceleration of the description of the area CTP, or as the increment of the moment *thereof*. The force by which the moon may in its periodic time CADB of 27^{d}. 7^{h}. 43′ be retained revolving about the earth in rest at the distance TP, would cause a body falling in the time CT to describe the length $\frac{1}{2}$CT, and at the same time to acquire a velocity equal to that with which the moon is moved in its orbit. This appears from Cor. 9, Prop. IV., Book I. But since K*d*, drawn perpendicular on TP, is *but* a third part of EL, and *equal* to the half of TP, or ML, in the octants, the force EL in the octants, where it is greatest, will exceed the force ML in the proportion of 3 to 2; and therefore will be to that force by which the moon in its periodic time may be retained revolving about the earth at rest as 100 to $\frac{2}{3}$ × 17872$\frac{1}{2}$, or 11915; and in the time CT will generate a velocity equal to $\frac{1}{11915}$ parts of the velocity of the moon; but in the time CPA will generate a greater velocity in the proportion of CA to CT or TP. Let the greatest force EL in the octants be represented by the area FK × K*k*, or by the rectangle $\frac{1}{2}$TP × P*p*, which is equal thereto; and the velocity which that greatest force can generate in any time CP will be to the velocity which any other lesser force EL can generate in the same time as the rectangle $\frac{1}{2}$TP × CP to the area KCGF; but the velocities generated in the whole time CPA will be one to the other as the rectangle $\frac{1}{2}$TP × CA to the triangle TCG, or as the quadrantal arc CA to the radius TP; and therefore the latter velocity generated in the whole time will be $\frac{1}{11915}$ parts of the velocity of the moon. To this velocity of the moon, which is proportional to the mean moment of the area (supposing this mean moment to be represented by the number 11915), we add and subtract the half of the other velocity; the sum 11915 + 50, or 11965, will represent the greatest moment of the area in the syzygy A; and the

difference 11915 — 50, or 11865, the least moment thereof in the quadratures. Therefore the areas which in equal times are described in the syzygies and quadratures are one to the other as 11965 to 11865. And if to the least moment 11865 we add a moment which shall be to 100, the difference of the two former moments, as the trapezium FKCG to the triangle TCG, or, which comes to the same thing, as the square of the sine PK to the square of the radius TP (that is, as P*d* to TP), the sum will represent the moment of the area when the moon is in any intermediate place P.

But these things take place only in the hypothesis that the sun and the earth are at rest, and that the synodical revolution of the moon is finished in $27^d. 7^h. 43$. But since the moon's synodical period is really $29^d. 12^h. 44'$, the increments of the moments must be enlarged in the same proportion as the time is, that is, in the proportion of 1080853 to 1000000. Upon which account, the whole increment, which was $\frac{100}{11915}$ parts of the mean moment, will now become $\frac{100}{11023}$ parts thereof; and therefore the moment of the area in the quadrature of the moon will be to the moment thereof in the syzygy as 11023 — 50 to 11023 + 50; or as 10973 to 11073; and to the moment thereof, when the moon is in any intermediate place P, as 10973 to 10973 + P*d*; that is, supposing TP = 100.

The area, therefore, which the moon, by a radius drawn to the earth, describes in the several little equal parts of time, is nearly as the sum of the number 219,46, and the versed sine of the double distance of the moon from the nearest quadrature, considered in a circle which hath unity for its radius. Thus it is when the variation in the octants is in its mean quantity. But if the variation there is greater or less, that versed sine must be augmented or diminished in the same proportion.

PROPOSITION XXVII. PROBLEM VIII.

From the horary motion of the moon to find its distance from the earth.

The area which the moon, by a radius drawn to the earth, describes in every moment of time, is as the horary motion of the moon and the square of the distance of the moon from the earth conjunctly. And therefore the distance of the moon from the earth is in a proportion compounded of the subduplicate proportion of the area directly, and the subduplicate proportion of the horary motion inversely. Q.E.I.

COR. 1. Hence the apparent diameter of the moon is given; for it is reciprocally as the distance of the moon from the earth. Let astronomers try how accurately this rule agrees with the phænomena.

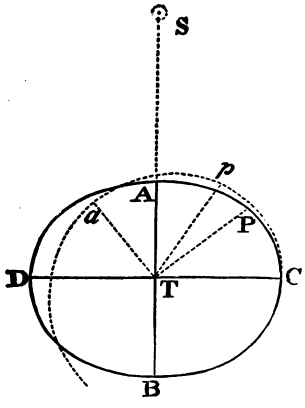
COR. 2. Hence also the orbit of the moon may be more exactly defined from the phænomena than hitherto could be done.

PROPOSITION XXVIII. PROBLEM IX.

To find the diameters of the orbit, in which, without eccentricity, the moon would move.

The curvature of the orbit which a body describes, if attracted in lines perpendicular to the orbit, is as the force of attraction directly, and the square of the velocity inversely. I estimate the curvatures of lines compared one with another according to the evanescent proportion of the sines or tangents of their angles of contact to equal radii, supposing those radii to be infinitely diminished. But the attraction of the moon towards the earth in the syzygies is the excess of its gravity towards the earth above the force of the sun 2PK (see Fig. Prop. XXV), by which force the accelerative gravity of the moon towards the sun exceeds the accelerative gravity of the earth towards the sun, or is exceeded by it. But in the quadratures that attraction is the sum of the gravity of the moon towards the earth, and the sun's force KT, by which the moon is attracted towards the earth. And these attractions, putting N for $\frac{AT + CT}{2}$, are nearly as $\frac{178725}{AT^2} - \frac{2000}{CT \times N}$ and $\frac{178725}{CT^2} + \frac{1000}{AT \times N}$, or as $178725N \times CT^2 - 2000AT^2 \times CT$, and $178725N \times AT^2 + 1000CT^2 \times AT$. For if the accelerative gravity of the moon towards the earth be represented by the number 178725, the mean force ML, which in the quadratures is PT or TK, and draws the moon towards the earth, will be 1000, and the mean force TM in the syzygies will be 3000; from which, if we subtract the mean force ML, there will remain 2000, the force by which the moon in the syzygies is drawn from the earth, and which we above called 2PK. But the velocity of the moon in the syzygies A and B is to its velocity in the quadratures C and D as CT to AT, and the moment of the area, which the moon by a radius drawn to the earth describes in the syzygies, to the moment of that area *described* in the quadratures conjunctly; that is, as 11073CT to 10973AT. Take this ratio twice inversely, and the former ratio once directly, and the curvature of the orb of the moon in the syzygies will be to the curvature thereof in the quadratures as $120406729 \times 178725AT^2 \times CT^2 \times N - 120406729 \times 2000AT^4 \times CT$ to $122611329 \times 178725AT^2 \times CT^2 \times N + 122611329 \times 1000CT^4 \times AT$, that is, as $2151969AT \times CT \times N - 24081AT^3$ to $2191371AT \times CT \times N + 12261CT^3$.

Because the figure of the moon's orbit is unknown, let us, in its stead, assume the ellipsis DBCA, in the centre of which we suppose the earth to be situated, and the greater axis DC to lie between the quadratures as the lesser AB between the syzygies. But since the plane of this ellipsis is revolved about the earth by an angular motion, and the orbit, whose curvature we now examine, should be described in a plane void of such motion



we are to consider the figure which the moon, while it is revolved in that ellipsis, describes in this plane, that is to say, the figure C*pa*, the several points *p* of which are found by assuming any point P in the ellipsis, which may represent the place of the moon, and drawing T*p* equal to TP in such manner that the angle P'T*p* may be equal to the apparent motion of the sun from the time of the last quadrature in C; or (which comes to the same thing) that the angle CT*p* may be to the angle CTP as the time of the synodic revolution of the moon to the time of the periodic revolution thereof, or as $29^d.\ 12^h.\ 44'$ to $27^d.\ 7^h.\ 43'$. If, therefore, in this proportion we take the angle CT*a* to the right angle CTA, and make T*a* of equal length with TA, we shall have *a* the lower and C the upper apsis of this orbit C*pa*. But, by computation, I find that the difference betwixt the curvature of this orbit C*pa* at the vertex *a*, and the curvature of a circle described about the centre T with the interval TA, is to the difference between the curvature of the ellipsis at the vertex A, and the curvature of the same circle, in the duplicate proportion of the angle CTP to the angle CT*p*; and that the curvature of the ellipsis in A is to the curvature of that circle in the duplicate proportion of TA to TC; and the curvature of that circle to the curvature of a circle described about the centre T with the interval TC as TC to TA; but that the curvature of this *last arch* is to the curvature of the ellipsis in C in the duplicate proportion of TA to TC; and that the difference betwixt the curvature of the ellipsis in the vertex C, and the curvature of this last circle, is to the difference betwixt the curvature of the figure C*pa*, at the vertex C, and the curvature of this same *last* circle, in the duplicate proportion of the angle CT*p* to the angle CTP; all which proportions are easily drawn from the sines of the angles of contact, and of the differences of those angles. But, by comparing those proportions together, we find the curvature of the figure C*pa* at *a* to be to its curvature at C as $AT^3.-\frac{16824}{100000}CT^2A'T$ to $CT^3 \div \frac{16824}{100000}AT^2 \times CT$; where the number $\frac{16824}{100000}$ represents the difference of the squares of the angles CTP and CT*p*, applied to the square of the lesser angle CTP; or (which is all one) the difference of the squares of the times $27^d.\ 7^h.\ 43'$, and $29^d.\ 12^h.\ 44'$, applied to the square of the time $27^d.\ 7^h.\ 43'$.

Since, therefore, *a* represents the syzygy of the moon, and C its quadrature, the proportion now found must be the same with that proportion of the curvature of the moon's orb in the syzygies to the curvature thereof in the quadratures, which we found above. Therefore, in order to find the

proportion of CT to AT, let us multiply the extremes and the means, and the terms which come out, applied to AT × CT, become $2062{,}79CT^4 - 2151969N \times CT^3 + 368676N \times AT \times CT^2 + 36342\ AT^2 \times CT^2 - 362047N \times AT^2 \times CT + 2191371N \times AT^3 + 4051{,}4AT^4 = 0$. Now if for the half sum N of the terms AT and CT we put 1, and x for their half difference, then CT will be $= 1 + x$, and AT $= 1 - x$. And substituting those values in the equation, after resolving thereof, we shall find $x = 0{,}00719$; and from thence the semi-diameter CT $= 1{,}00719$, and the semi-diameter AT $= 0{,}99281$, which numbers are nearly as $70\frac{1}{2}\frac{1}{4}$, and $69\frac{1}{2}\frac{1}{4}$. Therefore the moon's distance from the earth in the syzygies is to its distance in the quadratures (setting aside the consideration of eccentricity) as $69\frac{1}{2}\frac{1}{4}$ to $70\frac{1}{2}\frac{1}{4}$; or, in round numbers, as 69 to 70.

PROPOSITION XXIX. PROBLEM X.

To find the variation of the moon.

This inequality is owing partly to the elliptic figure of the moon's orbit, partly to the inequality of the moments of the area which the moon by a radius drawn to the earth describes. If the moon P revolved in the ellipsis DBCA about the earth quiescent in the centre of the ellipsis, and by the radius TP, drawn to the earth, described the area CTP, proportional to the time *of description;* and the greatest semi-diameter CT of the ellipsis was to the least TA as 70 to 69; the tangent of the angle CTP would be to the tangent of the angle of the mean motion, computed from the quadrature C, as the semi-diameter TA of the ellipsis to its semi-diameter TC, or as 69 to 70. But the description of the area CTP, as the moon advances from the quadrature to the syzygy, ought to be in such manner accelerated, that the moment of the area in the moon's syzygy may be to the moment thereof in its quadrature as 11073 to 10973; and that the excess of the moment in any intermediate place P above the moment in the quadrature may be as the square of the sine of the angle CTP; which we may effect with accuracy enough, if we diminish the tangent of the angle CTP in the subduplicate proportion of the number 10973 to the number 11073, that is, in proportion of the number 68,6877 to the number 69. Upon which account the tangent of the angle CTP will now be to the tangent of the mean motion as 68,6877 to 70; and the angle CTP in the octants, where the mean motion is 45°, will be found 44° 27′ 28″, which subtracted from 45°, the angle of the mean motion, leaves the greatest variation 32′ 32″. Thus it would be, if the moon, in passing from the quadrature to the syzygy, described an angle CTA of 90 degrees only. But because of the motion of the earth, by which the sun is apparently transferred *in consequentia*, the moon, before it overtakes the sun, describes an angle CT*a*, greater than a right angle, in the proportion of the time of the synodic revolution of the moon to the time of its periodic revolution, that

is, in the proportion of 29^{d}. 12^{h}. 44′ to 27^{d}. 7^{h}. 43′. Whence it comes to pass that all the angles about the centre T are dilated in the same proportion; and the greatest variation, which otherwise would be *but* 32′ 32″, now augmented in the said proportion, becomes 35′ 10″.

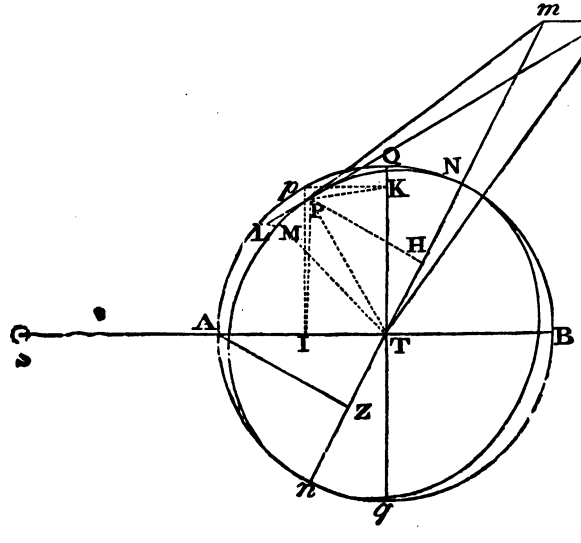
And this is its magnitude in the mean distance of the sun from the earth, neglecting the differences which may arise from the curvature of the *orbis magnus*, and the stronger action of the sun upon the moon when horned and new, than when gibbous and full. In other distances of the sun from the earth, the greatest variation is in a proportion compounded of the duplicate proportion of the time of the synodic revolution of the moon (the time of the year being given) directly, and the triplicate proportion of the distance of the sun from the earth inversely. And, therefore, in the apogee of the sun, the greatest variation is 33′ 14″, and in its perigee 37′ 11″, if the eccentricity of the sun is to the transverse semi-diameter of the *orbis magnus* as 16$\frac{1}{2}\frac{5}{8}$ to 1000.

Hitherto we have investigated the variation in an orb not eccentric, in which, to wit, the moon in its octants is always in its mean distance from the earth. If the moon, on account of its eccentricity, is more or less removed from the earth than if placed in this orb, the variation may be something greater, or something less, than according to this rule. But I leave the excess or defect to the determination of astronomers from the phænomena.

PROPOSITION XXX. PROBLEM XI.

To find the horary motion of the nodes of the moon in a circular orbit.

Let S represent the sun, T the earth, P the moon, NPn the orbit of the moon, Npn the orthographic projection of the orbit upon the plane of the ecliptic; N. n the nodes, nTNm the line of the nodes produced indefi-



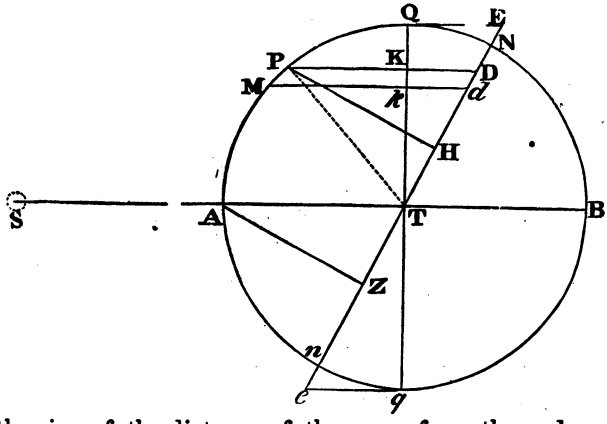
nitely; PI, PK perpendiculars upon the lines ST, Q*q*; P*p* a perpendicular upon the plane of the ecliptic; A, B the moon's syzygies in the plane of the ecliptic; AZ a perpendicular let fall upon N*n*, the line of the nodes; Q, *q* the quadratures of the moon in the plane of the ecliptic, and *p*K a perpendicular on the line Q*q* lying between the quadratures. The force of the sun to disturb the motion of the moon (by Prop. XXV) is twofold, one proportional to the line LM, the other to the line MT, in the scheme of that Proposition; and the moon by the former force is drawn towards the earth, by the latter towards the sun, in a direction parallel to the right line ST joining the earth and the sun. The former force LM acts in the direction of the plane of the moon's orbit, and therefore makes no change upon the situation thereof, and is upon that account to be neglected; the latter force MT, by which the plane of the moon's orbit is disturbed, is the same with the force 3PK or 3IT. And this force (by Prop. XXV) is to the force by which the moon may, in its periodic time, be uniformly revolved in a circle about the earth at rest, as 3IT to the radius of the circle multiplied by the number 178,725, or as IT to the radius thereof multiplied by 59,575. But in this calculus, and all that follows, I consider all the lines drawn from the moon to the sun as parallel to the line which joins the earth and the sun; because what inclination there is almost as much diminishes all effects in some cases as it augments them in others; and we are now inquiring after the mean motions of the nodes, neglecting such niceties as are of no moment, and would only serve to render the calculus more perplexed.

Now suppose PM to represent an arc which the moon describes in the least moment of time, and ML a little line, the half of which the moon, by the impulse of the said force 3IT, would describe in the same time; and joining PL, MP, let them be produced to *m* and *l*, where they cut the plane of the ecliptic, and upon T*m* let fall the perpendicular PH. Now, since the right line ML is parallel to the plane of the ecliptic, and therefore can never meet with the right line *ml* which lies in that plane, and yet both those right lines lie in one common plane LMP*ml*, they will be parallel, and upon that account the triangles LMP, *lm*P will be similar. And seeing MP*m* lies in the plane of the orbit, in which the moon did move while in the place P, the point *m* will fall upon the line N*n*, which passes through the nodes N, *n*, of that orbit. And because the force by which the half of the little line LM is generated, if the whole had been together, and at once impressed in the point P, would have generated that whole line, and caused the moon to move in the arc whose chord is LP; that is to say, would have transferred the moon from the plane MP*m*T into the plane LP*l*T; therefore the angular motion of the nodes generated by that force will be equal to the angle *m*T*l*. But *ml* is to *m*P as ML to MP; and since MP, because of the time given, is also given, *ml* will be as the rectan-

gle ML × *m*P, that is, as the rectangle IT × *m*P. And if T*ml* is a right angle, the angle *m*T*l* will be as $\frac{ml}{Tm}$, and therefore as $\frac{IT \times Pm}{Tm}$, that is (because T*m* and *m*P, TP and PH are proportional), as $\frac{IT \times PH}{TP}$; and, therefore, because TP is given, as IT × PH. But if the angle T*ml* or STN is oblique, the angle *m*T*l* will be yet less, in proportion of the sine of the angle STN to the radius, or AZ to AT. And therefore the velocity of the nodes is as IT × PH × AZ, or as the solid content of the sines of the three angles TPI, PTN, and STN.

If these are right angles, as happens when the nodes are in the quadratures, and the moon in the syzygy, the little line *ml* will be removed to an infinite distance, and the angle *m*T*l* will become equal to the angle *m*P*l*. But in this case the angle *m*P*l* is to the angle PTM, which the moon in the same time by its apparent motion describes about the earth, as 1 to 59,575. For the angle *m*P*l* is equal to the angle LPM, that is, to the angle of the moon's deflexion from a rectilinear path; which angle, if the gravity of the moon should have then ceased, the said force of the sun 3IT would by itself have generated in that given time; and the angle PTM is equal to the angle of the moon's deflexion from a rectilinear path; which angle, if the force of the sun 3IT should have then ceased, the force alone by which the moon is retained in its orbit would have generated in the same time. And these forces (as we have above shewn) are the one to the other as 1 to 59,575. Since, therefore, the mean horary motion of the moon (in respect of the fixed stars) is 32′ 56″ 27‴ 12$\frac{1}{2}$^{iv}. the horary motion of the node in this case will be 33″ 10‴ 33^{iv}. 12^{v}. But in other cases the horary motion will be to 33″ 10‴ 33^{iv}. 12^{v}. as the solid content of the sines of the three angles TPI, PTN, and STN (or of the distances of the moon from the quadrature, of the moon from the node, and of the node from the sun) to the cube of the radius. And as often as the sine of any angle is changed from positive to negative, and from negative to positive, so often must the regressive be changed into a progressive, and the progressive into a regressive motion. Whence it comes to pass that the nodes are progressive as often as the moon happens to be placed between either quadrature, and the node nearest to that quadrature. In other cases they are regressive, and by the excess of the regress above the progress, they are monthly transferred *in antecedentia.*

Cor. 1. Hence if from P and M, the extreme points of a least arc PM, on the line *Qq* joining the quadratures we let fall the perpendiculars PK M*k*, and produce the same till they cut the line of the nodes N*n* in D and *d*, the horary motion of the nodes will be as the area MPD*d*, and the square of the line AZ conjunctly. For let PK, PH, and AZ, be the three said sines, viz., PK the sine of the distance of the moon from the quadra-



ture, PH the sine of the distance of the moon from the node, and AZ the sine of the distance of the node from the sun; and the velocity of the node will be as the solid content of PK × PH × AZ. But PT is to PK as PM to K*k*; and, therefore, because PT and PM are given, K*k* will be as PK. Likewise AT is to PD as AZ to PH, and therefore PH is as the rectangle PD × AZ; and, by compounding those proportions, PK × PH is as the solid content K*k* × PD × AZ, and PK × PH × AZ as K*k* × PD × AZ^2; that is, as the area PD*d*M and AZ^1 conjunctly. Q.E.D.

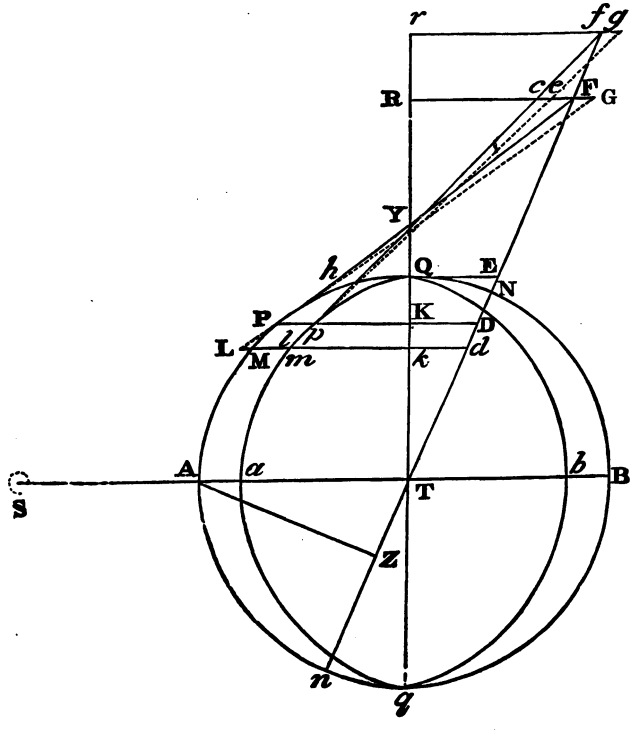
Cor. 2. In any given position of the nodes their mean horary motion is half their horary motion in the moon's syzygies; and therefore is to 16″ 35‴ 16ⁱᵛ. 36ᵛ. as the square of the sine of the distance of the nodes from the syzygies to the square of the radius, or as AZ^2 to AT^2. For if the moon, by an uniform motion, describes the semi-circle Q*Aq*, the sum of all the areas PD*d*M, during the time of the moon's passage from Q to M, will make up the area QM*d*E, terminating at the tangent QE of the circle; and by the time that the moon has arrived at the point *n*, that sum will make up the whole area EQA*n* described by the line PD: but when the moon proceeds from *n* to *q*, the line PD will fall without the circle, and describe the area *nqe*, terminating at the tangent *qe* of the circle, which area, because the nodes were before regressive, but are now progressive, must be subducted from the former area, and, being itself equal to the area QEN, will leave the semi-circle NQA*n*. While, therefore, the moon describes a semi-circle, the sum of all the areas PD*d*M will be the area of that semi-circle; and while the moon describes a complete circle, the sum of those areas will be the area of the whole circle. But the area PD*d*M, when the moon is in the syzygies, is the rectangle of the arc PM into the radius PT; and the sum of all the areas, *every one* equal to this area, in the time that the moon describes a complete circle, is the rectangle of the whole circumference into the radius of the circle; and this rectangle, being double the area of the circle, will be double the quantity of the former sum

If, therefore, the nodes went on with that velocity uniformly continued which they acquire in the moon's syzygies, they would describe a space double of that which they describe in fact; and, therefore, the mean motion, by which, if uniformly continued, they would describe the same space with that which they do in fact describe by an unequal motion, is *but* one-half of that motion which they are possessed of in the moon's syzygies. Wherefore since their greatest horary motion, if the nodes are in the quadratures, is 33″ 10‴ 33ⁱᵛ. 12ᵛ. their mean horary motion in this case will be 16″ 35‴ 16ⁱᵛ. 36ᵛ. And seeing the horary motion of the nodes is every where as AZ^2 and the area PD*d*M conjunctly, and, therefore, in the moon's syzygies, the horary motion of the nodes is as AZ^2 and the area PD*d*M conjunctly, that is (because the area PD*d*M described in the syzygies is given), as AZ^2, therefore the mean motion also will be as AZ^2; and, therefore, when the nodes are without the quadratures, this motion will be to 16″ 35‴ 16ⁱᵛ. 36ᵛ. as AZ^2 to AT'^2. Q.E.D.

PROPOSITION XXXI. PROBLEM XII.

To find the horary motion of the nodes of the moon in an elliptic orbit

Let Q*pmaq* represent an ellipsis described with the greater axis Q*q*, and the lesser axis *ab*; QA*q*B a circle circumscribed; T the earth in the common centre of both; S the sun; *p* the moon moving in this ellipsis; and



pm an arc which it describes in the least moment of time; N and *n* the nodes joined by the line N*n*; *p*K and *mk* perpendiculars upon the axis Q*q*, produced both ways till they meet the circle in P and M, and the line of the nodes in D and *d*. And if the moon, by a radius drawn to the earth, describes an area proportional to the time *of description*, the horary motion of the node in the ellipsis will be as the area *p*D*dm* and AZ^2 conjunctly.

For let PF touch the circle in P, and produced meet TN in F; and *pf* touch the ellipsis in *p*, and produced meet the same TN in *f*, and both tangents concur in the axis TQ at Y. And let ML represent the space which the moon, by the impulse of the above-mentioned force 3IT or 3PK, would describe with a transverse motion, in the meantime while revolving in the circle it describes the arc PM; and *ml* denote the space which the moon revolving in the ellipsis would describe in the same time by the impulse of the same force 3IT or 3PK; and let LP and *lp* be produced till they meet the plane of the ecliptic in G and *g*, and FG and *fg* be joined, of which FG produced may cut *pf*, *pg*, and TQ, in *c*, *e*, and R respectively; and *fg* produced may cut TQ in *r*. Because the force 3IT or 3PK in the circle is to the force 3IT or 3*p*K in the ellipsis as PK to *p*K, or as AT to *a*T, the space ML generated by the former force will be to the space *ml* generated by the latter as PK to *p*K; that is, because of the similar figures PYK*p* and FYR*c*, as FR to *c*R. But (because of the similar triangles PLM, PGF) ML is to FG as PL to PG, that is (on account of the parallels *lk*, PK, GR), as *pl* to *pe*, that is (because of the similar triangles *plm*, *cpe*), as *lm* to *ce*; and inversely as LM is to *lm*, or as FR is to *c*R, so is FG to *ce*. And therefore if *fg* was to *ce* as *fy* to *c*Y, that is, as *fr* to *c*R (that is, as *fr* to FR and FR to *c*R conjunctly, that is, as *f*T to FT, and FG to *ce* conjunctly), because the ratio of FG to *ce*, expunged on both sides, leaves the ratios *fg* to FG and *f*T to FT, *fg* would be to FG as *f*T to FT; and, therefore, the angles which FG and *fg* would subtend at the earth T would be equal to each other. But these angles (by what we have shewn in the preceding Proposition) are the motions of the nodes, while the moon describes in the circle the arc PM, in the ellipsis the arc *pm*; and therefore the motions of the nodes in the circle and in the ellipsis would be equal to each other. Thus, I say, it would be, if *fg* was to *ce* as *f*Y to *c*Y, that is, if *fg* was equal to $\frac{ce \times fY}{cY}$. But because of the similar triangles *fgp*, *cep*, *fg* is to *ce* as *fp* to *cp*; and therefore *fg* is equal to $\frac{ce}{cp} \times \frac{fp}{p}$; and therefore the angle which *fg* subtends in fact is to the former angle which FG subtends, that is to say, the motion of the nodes in the ellipsis is to the motion of the same in the circle as this *fg* or $\frac{ce \times fp}{cp}$ to the former *fg* or $\frac{ce \times fY}{cY}$, that is, as *fp* ×

cY to $fY \times cp$, or as fp to fY, and cY to cp; that is, if ph parallel to TN meet FP in h, as Fh to FY and FY to FP; that is, as Fh to FP or Dp to DP, and therefore as the area $Dpmd$ to the area DPMd. And, therefore, seeing (by Corol. 1, Prop. XXX) the latter area and AZ^2 conjunctly are proportional to the horary motion of the nodes in the circle, the former area and AZ^2 conjunctly will be proportional to the horary motion of the nodes in the ellipsis. Q.E.D.

COR. Since, therefore, in any given position of the nodes, the sum of all the areas $pDdm$, in the time while the moon is carried from the quadrature to any place m, is the area $mpQEd$ terminated at the tangent of the ellipsis QE; and the sum of all those areas, in one entire revolution, is the area of the whole ellipsis; the mean motion of the nodes in the ellipsis will be to the mean motion of the nodes in the circle as the ellipsis to the circle; that is, as Ta to TA, or 69 to 70. And, therefore, since (by Corol 2, Prop. XXX) the mean horary motion of the nodes in the circle is to $16'' 35''' 16^{iv}. 36^{v}.$ as AZ^2 to AT^2, if we take the angle $16'' 21''' 3^{iv}. 30^{v}.$ to the angle $16'' 35''' 16^{iv}. 36^{v}.$ as 69 to 70, the mean horary motion of the nodes in the ellipsis will be to $16'' 21''' 3^{iv}. 30^{v}.$ as AZ^2 to AT^2; that is, as the square of the sine of the distance of the node from the sun to the square of the radius.

But the moon, by a radius drawn to the earth, describes the area in the syzygies with a greater velocity than it does that in the quadratures, and upon that account the time is contracted in the syzygies, and prolonged in the quadratures; and together with the time the motion of the nodes is likewise augmented or diminished. But the moment of the area in the quadrature of the moon was to the moment thereof in the syzygies as 10973 to 11073; and therefore the mean moment in the octants is to the excess in the syzygies, and to the defect in the quadratures, as 11023, the half sum of those numbers, to their half difference 50. Wherefore since the time of the moon in the several little equal parts of its orbit is reciprocally as its velocity, the mean time in the octants will be to the excess of the time in the quadratures, and to the defect *of the time* in the syzygies arising from this cause, nearly as 11023 to 50. But, reckoning from the quadratures to the syzygies, I find that the excess of the moments of the area, in the several places above the least moment in the quadratures, is nearly as the square of the sine of the moon's distance from the quadratures; and therefore the difference betwixt the moment in any place, and the mean moment in the octants, is as the difference betwixt the square of the sine of the moon's distance from the quadratures, and the square of the sine of 45 degrees, or half the square of the radius; and the increment of the time in the several places between the octants and quadratures, and the decrement thereof between the octants and syzygies, is in the same proportion. But the motion of the nodes, while the moon describes the several little equal parts of its orbit, is accelerated or retarded

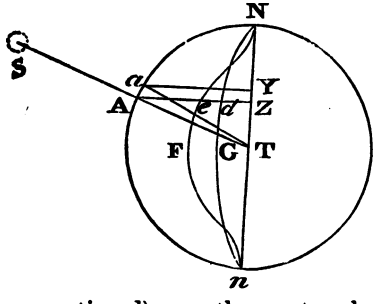
in the duplicate proportion of the time; for that motion, while the moon describes PM, is (*cæteris paribus*) as ML, and ML is in the duplicate proportion of the time. Wherefore the motion of the nodes in the syzygies, in the time while the moon describes given little parts of its orbit, is diminished in the duplicate proportion of the number 11073 to the number 11023; and the decrement is to the remaining motion as 100 to 10973; but to the whole motion as 100 to 11073 nearly. But the decrement in the places between the octants and syzygies, and the increment in the places between the octants and quadratures, is to this decrement nearly as the whole motion in these places to the whole motion in the syzygies, and the difference betwixt the square of the sine of the moon's distance from the quadrature, and the half square of the radius, to the half square of the radius conjunctly. Wherefore, if the nodes are in the quadratures, and we take two places, one on one side, one on the other, equally distant from the octant and other two distant by the same interval, one from the syzygy, the other from the quadrature, and from the decrements of the motions in the two places between the syzygy and octant we subtract the increments of the motions in the two other places between the octant and the quadrature, the remaining decrement will be equal to the decrement in the syzygy, as will easily appear by computation; and therefore the mean decrement, which ought to be subducted from the mean motion of the nodes, is the fourth part of the decrement in the syzygy. The whole horary motion of the nodes in the syzygies (when the moon by a radius drawn to the earth was supposed to describe an area proportional to the time) was $32'' 42''' 7^{iv}$. And we have shewn that the decrement of the motion of the nodes, in the time while the moon, now moving with greater velocity, describes the same space, was to this motion as 100 to 11073; and therefore this decrement is $17''' 43^{iv}. 11^{v}$. The fourth part of which $4''' 25^{iv}. 48^{v}$. subtracted from the mean horary motion above found, $16'' 21''' 3^{iv}. 30^{v}$. leaves $16'' 16''' 37^{iv}. 42^{v}$. their correct mean horary motion.

If the nodes are without the quadratures, and two places are considered, one on one side, one on the other, equally distant from the syzygies, the sum of the motions of the nodes, when the moon is in those places, will be to the sum of their motions, when the moon is in the same places and the nodes in the quadratures, as AZ^2 to AT^2. And the decrements of the motions arising from the causes but now explained will be mutually as the motions themselves, and therefore the remaining motions will be mutually betwixt themselves as AZ^2 to AT^2; and the mean motions will be as the remaining motions. And, therefore, in any given position of the nodes, their correct mean horary motion is to $16'' 16''' 37^{iv}. 42^{v}$. as AZ^2 to AT^2; that is, as the square of the sine of the distance of the nodes from the syzygies to the square of the radius.

PROPOSITION XXXII. PROBLEM XIII.

To find the mean motion of the nodes of the moon.

The yearly mean motion is the sum of all the mean horary motions throughout the course of the year. Suppose that the node is in N, and that, after every hour is elapsed, it is drawn back again to its former place; so that, notwithstanding its proper motion, it may constantly remain in the same situation with respect to the fixed stars; while in the mean time the sun S, by the motion of the earth, is seen to leave the node,



and to proceed till it completes its apparent annual course by an uniform motion. Let A*a* represent a given least arc, which the right line TS always drawn to the sun, by its intersection with the circle NA*n*, describes in the least given moment of time; and the mean horary motion (from what we have above shewn) will be as AZ^2, that is (because AZ and ZY are proportional), as the rectangle of AZ into ZY, that is, as the area AZY*a*; and the sum of all the mean horary motions from the beginning will be as the sum of all the areas *a*YZA, that is, as the area NAZ. But the greatest AZY*a* is equal to the rectangle of the arc A*a* into the radius of the circle; and therefore the sum of all these rectangles in the whole circle will be to the like sum of all the greatest rectangles as the area of the whole circle to the rectangle of the whole circumference into the radius, that is, as 1 to 2. But the horary motion corresponding to that greatest rectangle was 16″ 16‴ 37^{iv}. 42^{v}. and this motion in the complete course of the sidereal year, 365^{d}. 6^{h}. 9′, amounts to 39° 38′ 7″ 50‴, and therefore the half thereof, 19° 49′ 3″ 55‴, is the mean motion of the nodes corresponding to the whole circle. And the motion of the nodes, in the time while the sun is carried from N to A, is to 19° 49′ 3″ 55‴ as the area NAZ to the whole circle.

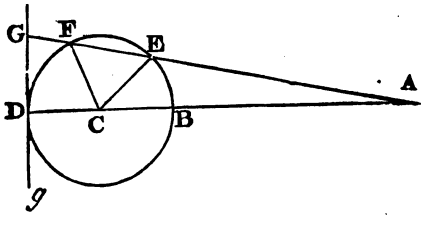
Thus it would be if the node was after every hour drawn back again to its former place, that so, after a complete revolution, the sun at the year's end would be found again in the same node which it had left when the year begun. But, because of the motion of the node in the mean time, the sun must needs meet the node sooner; and now it remains that we compute the abbreviation of the time Since, then, the sun, in the course of the year, travels 360 degrees, and the node in the same time by its greatest motion would be carried 39° 38′ 7″ 50‴, or 39,6355 degrees; and the mean motion of the node in any place N is to its mean motion in its quadratures as AZ^2 to AT^2 · the motion of the sun will be to the motion of the node

in N as $360AT^2$ to $39,6355AZ^2$; that is, as $9,0827646AT^2$ to AZ^2. Wherefore if we suppose the circumference NAn of the whole circle to be divided into little equal parts, such as Aa, the time in which the sun would describe the little arc Aa, if the circle was quiescent, will be to the time of which it would describe the same arc, supposing the circle together with the nodes to be revolved about the centre T, reciprocally as $9,0827646AT^2$ to $9,0827646AT^2 + AZ^2$; for the time is reciprocally as the velocity with which the little arc is described, and this velocity is the sum of the velocities of both sun and node. If, therefore, the sector NTA represent the time in which the sun by itself, without the motion of the node, would describe the arc NA, and the indefinitely small part ATa of the sector represent the little moment of the time in which it would describe the least arc Aa; and (letting fall aY perpendicular upon Nn) if in AZ we take dZ of such length that the rectangle of dZ into ZY may be to the least part ATa of the sector as AZ^2 to $9,0827646AT^2 + AZ^2$, that is to say, that dZ may be to $\frac{1}{2}$AZ as AT^2 to $9,0827646AT^2 + AZ^2$; the rectangle of dZ into ZY will represent the decrement of the time arising from the motion of the node, while the arc Aa is described; and if the curve NdGn is the locus where the point d is always found, the curvilinear area NdZ will be as the whole decrement *of time* while the whole arc NA is described; and, therefore, the excess of the sector NAT above the area NdZ will be as the whole time. But because the motion of the node in a less time is less in proportion of the time, the area AaYZ must also be diminished in the same proportion; which may be done by taking in AZ the line eZ of such length, that it may be to the length of AZ as AZ^2 to $9,0827646AT^2 + AZ^2$; for so the rectangle of eZ into ZY will be to the area AZYa as the decrement of the time in which the arc Aa is described to the whole time in which it would have been described, if the node had been quiescent; and, therefore, that rectangle will be as the decrement of the motion of the node. And if the curve NeFn is the locus of the point e, the whole area NeZ, which is the sum of all the decrements *of that motion*, will be as the whole decrement *thereof* during the time in which the arc AN is described; and the remaining area NAe will be as the remaining motion, which is the true motion of the node, during the time in which the whole arc NA is described by the joint motions of both sun and node. Now the area of the semi-circle is to the area of the figure NeFn found by the method of infinite series nearly as 795 to 6[illegible]. But the motion corresponding *or proportional* to the whole circle was 19° 49′ 3″ 55‴; and therefore the motion corresponding to double the figure NeFn is 1° 29′ 58″ 2‴, which taken from the former motion leaves 18° 19′ 5″ 53‴, the whole motion of the node with respect to the fixed stars in the interval between two of its conjunctions with the sun; and this motion subducted from the annual motion of the sun 360°, leaves 341° 40′ 54″ 7‴,

the motion of the sun in the interval between the same conjunctions. But as this motion is to the annual motion 360°, so is the motion of the node but just now found 18° 19′ 5″ 53‴ to its annual motion, which will therefore be 19° 18′ 1″ 23‴; and this is the mean motion of the nodes in the sidereal year. By astronomical tables, it is 19° 21′ 21″ 50‴. The difference is less than $\frac{1}{300}$ part of the whole motion, and seems to arise from the eccentricity of the moon's orbit, and its inclination to the plane of the ecliptic. By the eccentricity of this orbit the motion of the nodes is too much accelerated; and, on the other hand, by the inclination of the orbit, the motion of the nodes is something retarded, and reduced to its just velocity.

PROPOSITION XXXIII. PROBLEM XIV.

To find the true motion of the nodes of the moon.



In the time which is as the area NTA—N*dZ* (in the preceding Fig.) that motion is as the area NA*e*, and is thence given; but because the calculus is too difficult, it will be better to use the following construction of the Problem. About the centre C, with any interval CD, describe the circle BEFD; produce DC to A so as AB may be to AC as the mean motion to half the mean true motion when the nodes are in their quadratures (that is, as 19° 18′ 1″ 23‴ to 19° 49′ 3″ 55‴; and therefore BC to AC as the difference of those motions 0° 31′ 2″ 32‴ to the latter motion 19° 49′ 3″ 55‴, that is, as 1 to $38\frac{3}{10}$). Then through the point D draw the indefinite line G*g*, touching the circle in D; and if we take the angle BCE, or BCF, equal to the double distance of the sun from the place of the node, as found by the mean motion, and drawing AE or AF cutting the perpendicular DG in G, we take another angle which shall be to the whole motion of the node in the interval between its syzygies (that is, to 9° 11′ 3″) as the tangent DG to the whole circumference of the circle BED, and add this *last* angle (for which the angle DAG may be used) to the mean motion of the nodes, while they are passing from the quadratures to the syzygies, and subtract it from their mean motion while they are passing from the syzygies to the quadratures, we shall have their true motion; for the true motion so found will nearly agree with the true motion which comes out from assuming the times as the area NTA—N*dZ*, and the motion of the node as the area NA*e*; as whoever will please to examine and make the computations will find: and this is the semi-menstrual equation of the motion of the nodes. But there is also a menstrual equation, but which is by no means necessary for find-

ing of the moon's latitude; for since the variation of the inclination of the moon's orbit to the plane of the ecliptic is liable to a twofold inequality, the one semi-menstrual, the other menstrual, the menstrual inequality *of this variation*, and the menstrual equation of the nodes, so moderate and correct each other, that in computing the latitude of the moon both may be neglected.

COR. From this and the preceding Prop. it appears that the nodes are quiescent in their syzygies, but regressive in their quadratures, by an hourly motion of 16″ 19‴ 26ᴵᵛ.; and that the equation of the motion of the nodes in the octants is 1° 30′; all which exactly agree with the phænomena of the heavens.

SCHOLIUM.

Mr. *Machin*, Astron., Prof. Gresh., and Dr. *Henry Pemberton*, separately found out the motion of the nodes by a different method. Mention has been made of this method in another place. Their several papers, both of which I have seen, contained two Propositions, and exactly agreed with each other in both of them. Mr. *Machin's* paper coming first to my hands, I shall here insert it.

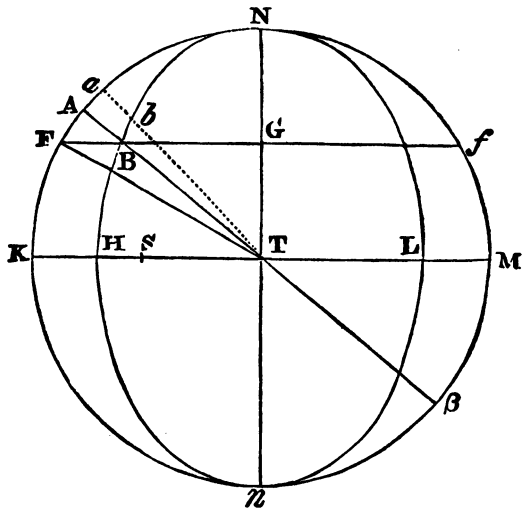
OF THE MOTION OF THE MOON'S NODES.

"PROPOSITION I.

The mean motion of the sun from the node is defined by a geometric mean proportional between the mean motion of the sun and that mean motion with which the sun recedes with the greatest swiftness from the node in the quadratures.

"Let T be the earth's place, N*n* the line of the moon's nodes at any given time, KTM a perpendicular thereto, TA a right line revolving about the centre with the same angular velocity with which the sun and the node recede from one another, in such sort that the angle between the quiescent right line N*n* and the revolving line TA may be always equal to the distance of the places of the sun and node. Now if any right line TK be divided into parts TS and SK, and those parts be taken as the mean horary motion of the sun to the mean horary motion of the node in the quadratures, and there be taken the right line TH, a mean proportional between the part TS and the whole TK, this right line will be proportional to the sun's mean motion from the node.

"For let there be described the circle NK*n*M from the centre T and with the radius TK, and about the same centre, with the semi-axis TH



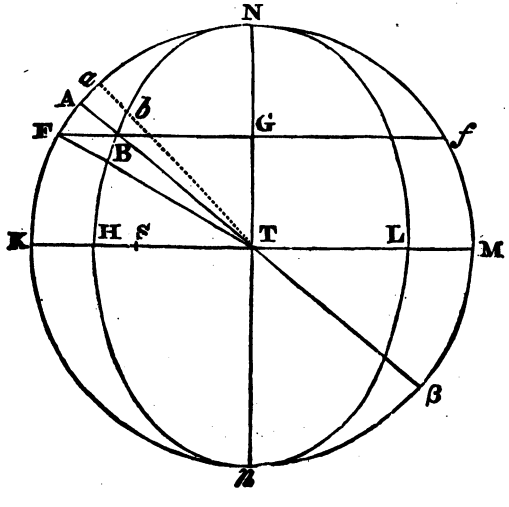
and TN, let there be described an ellipsis NH*n*L; and in the time in which the sun recedes from the node through the arc N*a*, if there be drawn the right line T*ba*, the area of the sector NT*a* will be the exponent of the sum of the motions of the sun and node in the same time. Let, therefore, the extremely small arc *a*A be that which the right line T*ba*, revolving according to the aforesaid law, will uniformly describe in a given particle of time, and the extremely small sector TA*a* will be as the sum of the velocities with which the sun and node are carried two different ways in that time. Now the sun's velocity is almost uniform, its inequality being so small as scarcely to produce the least inequality in the mean motion of the nodes. The other part of this sum, namely, the mean quantity of the velocity of the node, is increased in the recess from the syzygies in a duplicate ratio of the sine of its distance from the sun (by Cor. Prop. XXXI, of this Book), and, being greatest in its quadratures with the sun in K, is in the same ratio to the sun's velocity as SK to TS, that is, as (the difference of the squares of TK and TH, or) the rectangle KHM to TH^2. But the ellipsis NBH divides the sector AT*a*, the exponent of the sum of these two velocities, into two parts AB*ba* and BT*b*, proportional to the velocities. For produce BT to the circle in β, and from the point B let fall upon the greater axis the perpendicular BG, which being produced both ways may meet the circle in the points F and *f*; and because the space AB*ba* is to the sector TB*b* as the rectangle ABβ to BT^2 (that rectangle being equal to the difference of the squares of TA and TB, because the right line Aβ is equally cut in T, and unequally in B), therefore when the space AB*ba* is the greatest of all in K, this ratio will be the same as the ratio of the rectangle KHM to HT^2. But the greatest mean velocity of the node was shewn above to be in that very

ratio to the velocity of the sun; and therefore in the quadratures the sector AT*a* is divided into parts proportional to the velocities. And because the rectangle KHM is to HT^2 as FB*f* to BG^2, and the rectangle ABβ is equal to the rectangle FB*f*, therefore the little area AB*ba*, where it is greatest, is to the remaining sector TB*b* as the rectangle ABβ to BG^2 But the ratio of these little areas always was as the rectangle ABβ to BT^2; and therefore the little area AB*ba* in the place A is less than its correspondent little area in the quadratures in the duplicate ratio of BG to BT, that is, in the duplicate ratio of the sine of the sun's distance from the node. And therefore the sum of all the little areas AB*ba*, to wit, the space ABN, will be as the motion of the node in the time in which the sun hath been going over the arc NA since he left the node; and the remaining space, namely, the elliptic sector NTB, will be as the sun's mean motion in the same time. And because the mean annual motion of the node is that motion which it performs in the time that the sun completes one period of its course, the mean motion of the node from the sun will be to the mean motion of the sun itself as the area of the circle to the area of the ellipsis; that is, as the right line TK to the right line TH, which is a mean proportional between TK and TS; or, which comes to the same as the mean proportional TH to the right line TS.

PROPOSITION II.

"*The mean motion of the moon's nodes being given, to find their true motion.*

"Let the angle A be the distance of the sun from the mean place of the node, or the sun's mean motion from the node. Then if we take the angle B, whose tangent is to the tangent of the angle A as TH to TK, that is,



in the sub-duplicate ratio of the mean horary motion of the sun to the mean horary motion of the sun from the node, when the node is in the quadrature, that angle B will be the distance of the sun from the node's true place. For join FT, and, by the demonstration of the last Proportion, the angle FTN will be the distance of the sun from the mean place of the node, and the angle ATN the distance from the true place, and the tangents of these angles are between themselves as TK to TH.

"Cor. Hence the angle FTA is the equation of the moon's nodes; and the sine of this angle, where it is greatest in the octants, is to the radius as KH to TK + TH. But the sine of this equation in any other place A is to the greatest sine as the sine of the sums of the angles FTN + ATN to the radius; that is, nearly as the sine of double the distance of the sun from the mean place of the node (namely, 2FTN) to the radius.

"SCHOLIUM.

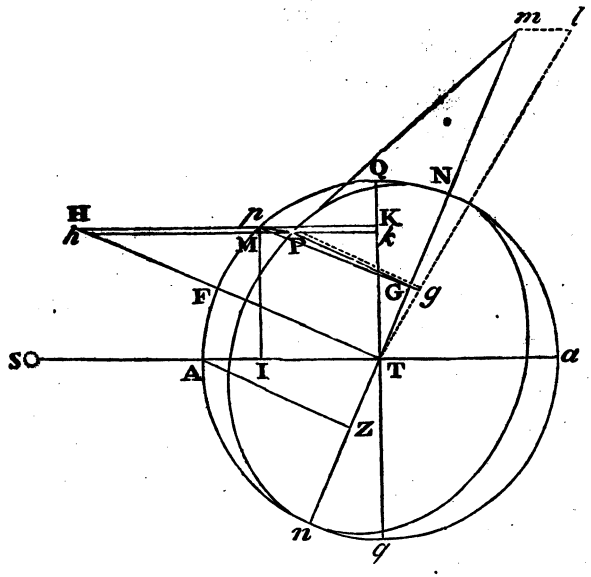
"If the mean horary motion of the nodes in the quadratures be 16″ 16‴ 37ⁱᵛ. 42ᵛ. that is, in a whole sidereal year, 39° 38′ 7″ 50‴, TH will be to TK in the subduplicate ratio of the number 9,0827646 to the number 10,0827646, that is, as 18,6524761 to 19,6524761. And, therefore, TH is to HK as 18,6524761 to 1; that is, as the motion of the sun in a sidereal year to the mean motion of the node 19° 18′ 1″ $23\frac{2}{3}$‴.

"But if the mean motion of the moon's nodes in 20 Julian years is 386° 50′ 15″, as is collected from the observations made use of in the theory of the moon, the mean motion of the nodes in one sidereal year will be 19° 20′ 31″ 58‴. and TH will be to HK as 360° to 19° 20′ 31″ 58‴; that is, as 18,61214 to 1: and from hence the mean horary motion of the nodes in the quadratures will come out 16″ 18‴ 48ⁱᵛ. And the greatest equation of the nodes in the octants will be 1° 29′ 57″."

PROPOSITION XXXIV. PROBLEM XV.

To find the horary variation of the inclination of the moon's orbit to the plane of the ecliptic.

Let A and *a* represent the syzygies; Q and *q* the quadratures; N and *n* the nodes; P the place of the moon in its orbit; *p* the orthographic projection of that place upon the plane of the ecliptic; and *m*T*l* the momentaneous motion of the nodes as above. If upon T*m* we let fall the perpendicular PG, and joining *p*G we produce it till it meet T*l* in *g*, and join also P*g*, the angle PG*p* will be the inclination of the moon's orbit to the plane of the ecliptic when the moon is in P; and the angle P*gp* will be the inclination of the same after a small moment of time is elapsed; and therefore the angle GP*g* will be the momentaneous variation of the inclination. But this angle GP*g* is to the angle GT*g* as TG to PG and P*p* to PG conjunctly. And, therefore, if for the moment of time we as-



sume an hour, since the angle GTg (by Prop. XXX) is to the angle 33′ 10‴ 33ᴵᵛ. as IT × PG × AZ to AT³, the angle GPg (or the horary variation of the inclination) will be to the angle 33″ 10‴ 33ᴵᵛ. as IT × AZ × TG × $\frac{Pp}{PG}$ to AT³. Q.E.I.

And thus it would be if the moon was uniformly revolved in a circular orbit. But if the orbit is elliptical, the mean motion of the nodes will be diminished in proportion of the lesser axis to the greater, as we have shewn above; and the variation of the inclination will be also diminished in the same proportion.

Cor. 1. Upon Nn erect the perpendicular TF, and let pM be the horary motion of the moon in the plane of the ecliptic; upon QT let fall the perpendiculars pK, Mk, and produce them till they meet TF in H and h; then IT will be to AT as Kk to Mp; and TG to Hp as TZ to AT; and, therefore, IT × TG will be equal to $\frac{K k \times H p \times TZ}{M p}$, that is, equal to the area HpMh multiplied into the ratio $\frac{TZ}{Mp}$: and therefore the horary variation of the inclination will be to 33″ 10‴ 33ᴵᵛ. as the area HpMh multiplied into AZ × $\frac{TZ}{Mp}$ × $\frac{Pp}{PG}$ to AT³.

Cor. 2. And, therefore, if the earth and nodes were after every hour drawn back from their new and instantly restored to their old places, so as their situation might continue given for a whole periodic month together, the whole variation of the inclination during that month would be to 33″

10‴ 33ᴵᵛ. as the aggregate of all the areas HpMh, generated in the time of one revolution of the point p (with due regard in summing to their proper signs + −), multiplied into $AZ \times TZ \times \frac{Pp}{PG}$ to $Mp \times AT^3$; that is, as the whole circle QAqa multiplied into $AZ \times TZ \times \frac{Pp}{PG}$ to $Mp \times AT^3$, that is, as the circumference QAqa multiplied into $AZ \times TZ \times \frac{Pp}{PG}$ to $2Mp \times AT^2$.

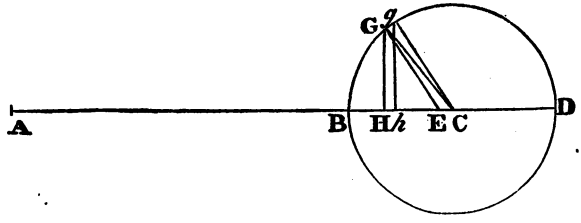
Cor. 3. And, therefore, in a given position of the nodes, the mean horary variation, from which, if uniformly continued through the whole month, that menstrual variation might be generated, is to 33″ 10‴ 33ᴵᵛ. as $AZ \times TZ \times \frac{Pp}{PG}$ to $2AT^2$, or as $Pp \times \frac{AZ \times TZ}{\frac{1}{2}AT}$ to $PG \times 4AT$; that is (because Pp is to PG as the sine of the aforesaid inclination to the radius, and $\frac{AZ \times TZ}{\frac{1}{2}AT}$ to 4AT as the sine of double the angle ATn to four times the radius), as the sine of the same inclination multiplied into the sine of double the distance of the nodes from the sun to four times the square of the radius.

Cor. 4. Seeing the horary variation of the inclination, when the nodes are in the quadratures, is (by this Prop.) to the angle 33″ 10‴ 33ᴵᵛ. as $IT \times AZ \times TG \times \frac{Pp}{PG}$ to AT^3, that is, as $\frac{IT \times TG}{\frac{1}{2}AT} \times \frac{Pp}{PG}$ to 2AT, that is, as the sine of double the distance of the moon from the quadratures multiplied into $\frac{Pp}{PG}$ to twice the radius, the sum of all the horary variations during the time that the moon, in this situation of the nodes, passes from the quadrature to the syzygy (that is, in the space of $177\frac{1}{6}$ hours) will be to the sum of as many angles 33″ 10‴ 33ᴵᵛ. or 5878″, as the sum of all the sines of double the distance of the moon from the quadratures multiplied into $\frac{Pp}{PG}$ to the sum of as many diameters; that is, as the diameter multiplied into $\frac{Pp}{PG}$ to the circumference; that is, if the inclination be 5° 1′, as $7 \times \frac{8}{10}\frac{1}{10}\frac{1}{6}$ to 22, or as 278 to 10000. And, therefore, the whole variation, composed out of the sum of all the horary variations in the aforesaid time, is 163″, or 2′ 43″.

PROPOSITION XXXV. PROBLEM XVI.

To a given time to find the inclination of the moon's orbit to the plane of the ecliptic.

Let AD be the sine of the greatest inclination, and AB the sine of the least. Bisect BD in C; and round the centre C, with the interval BC, describe the circle BGD. In AC take CE in the same proportion to EB



as EB to twice BA. And if to the time given we set off the angle AEG equal to double the distance of the nodes from the quadratures, and upon AD let fall the perpendicular GH, AH will be the sine of the inclination required.

For GE^2 is equal to $GH^2 + HE^2 = BHD + HE^2 = HBD + HE^2 - BH^2 = HBD + BE^2 - 2BH \times BE = BE^2 + 2EC \times BH = 2EC \times AB + 2EC \times BH = 2EC \times AH$; wherefore since 2EC is given, GE^2 will be as AH. Now let AEg represent double the distance of the nodes from the quadratures, in a given moment of time after, and the arc Gg, on account of the given angle GEg, will be as the distance GE. But Hh is to Gg as GH to GC, and, therefore, Hh is as the rectangle GH $\times$ Gg, or GH $\times$ GE, that is, as $\frac{GH}{GE} \times GE^2$, or $\frac{GH}{GE} \times AH$; that is, as AH and the sine of the angle AEG conjunctly. If, therefore, in any one case, AH be the sine of inclination, it will increase by the same increments as the sine of inclination doth, by Cor. 3 of the preceding Prop. and therefore will always continue equal to that sine. But when the point G falls upon either point B or D, AH is equal to this sine, and therefore remains always equal thereto. Q.E.D.

In this demonstration I have supposed that the angle BEG, representing double the distance of the nodes from the quadratures, increaseth uniformly; for I cannot descend to every minute circumstance of inequality. Now suppose that BEG is a right angle, and that Gg is in this case the horary increment of double the distance of the nodes from the sun; then, by Cor. 3 of the last Prop. the horary variation of the inclination in the same case will be to 33″ 10‴ 33ᴵᵛ. as the rectangle of AH, the sine of the inclination, into the sine of the right angle BEG, double the distance of the nodes from the sun, to four times the square of the radius; that is, as AH,

the sine of the mean inclination, to four times the radius; that is, seeing the mean inclination is about 5° $S\frac{1}{2}$, as its sine 896 to 40000, the quadruple of the radius, or as 224 to 10000. But the whole variation corresponding to BD, the difference of the sines, is to this horary variation as the diameter BD to the arc Gg, that is, conjunctly as the diameter BD to the semi-circumference BGD, and as the time of $2079\frac{7}{10}$ hours, in which the node proceeds from the quadratures to the syzygies, to one hour, that is as 7 to 11, and $2079\frac{7}{10}$ to 1. Wherefore, compounding all these proportions, we shall have the whole variation BD to 33″ 10‴ 33ᵛ. as 224 × 7 × $2079\frac{7}{10}$ to 110000, that is, as 29645 to 1000; and from thence that variation BD will come out 16′ $23\frac{1}{2}$″.

And this is the greatest variation of the inclination, abstracting from the situation of the moon in its orbit; for if the nodes are in the syzygies, the inclination suffers no change from the various positions of the moon. But if the nodes are in the quadratures, the inclination is less when the moon is in the syzygies than when it is in the quadratures by a difference of 2′ 43″, as we shewed in Cor. 4 of the preceding Prop.; and the whole mean variation BD, diminished by 1′ $21\frac{1}{2}$″, the half of this excess, becomes 15′ 2″, when the moon is in the quadratures; and increased by the same, becomes 17′ 45″ when the moon is in the syzygies. If, therefore, the moon be in the syzygies, the whole variation in the passage of the nodes from the quadratures to the syzygies will be 17′ 45″; and, therefore, if the inclination be 5° 17′ 20″, when the nodes are in the syzygies, it will be 4° 59′ 35″ when the nodes are in the quadratures and the moon in the syzygies. The truth of all which is confirmed by observations.

Now if the inclination of the orbit should be required when the moon is in the syzygies, and the nodes any where between them and the quadratures, let AB be to AD as the sine of 4° 59′ 35″ to the sine of 5° 17′ 20″, and take the angle AEG equal to double the distance of the nodes from the quadratures; and AH will be the sine of the inclination desired. To this inclination of the orbit the inclination of the same is equal, when the moon is 90° distant from the nodes. In other situations of the moon, this menstrual inequality, to which the variation of the inclination is obnoxious in the calculus of the moon's latitude, is balanced, and in a manner took off, by the menstrual inequality of the motion of the nodes (as we said before), and therefore may be neglected in the computation of the said latitude.

SCHOLIUM.

By these computations of the lunar motions I was willing to shew that by the theory of gravity the motions of the moon could be calculated from their physical causes. By the same theory I moreover found that the annual equation of the mean motion of the moon arises from the various

dilatation which the orbit of the moon suffers from the action of the sun according to Cor. 6, Prop. LXVI. Book 1. The force of this action is greater in the perigeon sun, and dilates the moon's orbit; in the apogeon sun it is less, and permits the orbit to be again contracted. The moon moves slower in the dilated and faster in the contracted orbit; and the annual equation, by which this inequality is regulated, vanishes in the apogee and perigee of the sun. In the mean distance of the sun from the earth it arises to about 11′ 50″; in other distances of the sun it is proportional to the equation of the sun's centre, and is added to the mean motion of the moon, while the earth is passing from its aphelion to its perihelion, and subducted while the earth is in the opposite semi-circle. Taking for the radius of the *orbis magnus* 1000, and $16\frac{7}{8}$ for the earth's eccentricity, this equation, when of the greatest magnitude, by the theory of gravity comes out 11′ 49″. But the eccentricity of the earth seems to be something greater, and with the eccentricity this equation will be augmented in the same proportion. Suppose the eccentricity $16\frac{1}{2}$, and the greatest equation will be 11′ 51″.

Farther; I found that the apogee and nodes of the moon move faster in the perihelion of the earth, where the force of the sun's action is greater, than in the aphelion thereof, and that in the reciprocal triplicate proportion of the earth's distance from the sun; and hence arise annual equations of those motions proportional to the equation of the sun's centre. Now the motion of the sun is in the reciprocal duplicate proportion of the earth's distance from the sun; and the greatest equation of the centre which this inequality generates is 1° 56′ 20″, corresponding to the above-mentioned eccentricity of the sun, $16\frac{1}{2}$. But if the motion of the sun had been in the reciprocal triplicate proportion of the distance, this inequality would have generated the greatest equation 2° 54′ 30″; and therefore the greatest equations which the inequalities of the motions of the moon's apogee and nodes do generate are to 2° 54′ 30″ as the mean diurnal motion of the moon's apogee and the mean diurnal motion of its nodes are to the mean diurnal motion of the sun. Whence the greatest equation of the mean motion of the apogee comes out 19′ 43″, and the greatest equation of the mean motion of the nodes 9′ 24″. The former equation is added, and the latter subducted, while the earth is passing from its perihelion to its aphelion, and contrariwise when the earth is in the opposite semi-circle.

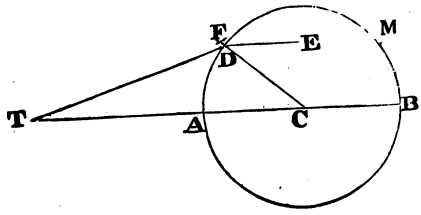
By the theory of gravity I likewise found that the action of the sun upon the moon is something greater when the transverse diameter of the moon's orbit passeth through the sun than when the same is perpendicular upon the line which joins the earth and the sun; and therefore the moon's orbit is something larger in the former than in the latter case. And hence arises another equation of the moon's mean motion, depending

upon the situation of the moon's apogee in respect of the sun, which is in its greatest quantity when the moon's apogee is in the octants of the sun, and vanishes when the apogee arrives at the quadratures or syzygies; and it is added to the mean motion while the moon's apogee is passing from the quadrature of the sun to the syzygy, and subducted while the apogee is passing from the syzygy to the quadrature. This equation, which I shall call the semi-annual, when greatest in the octants of the apogee, arises to about 3′ 45″, so far as I could collect from the phænomena: and this is its quantity in the mean distance of the sun from the earth. But it is increased and diminished in the reciprocal triplicate proportion of the sun's distance, and therefore is nearly 3′ 34″ when that distance is greatest, and 3′ 56″ when least. But when the moon's apogee is without the octants, it becomes less, and is to its greatest quantity as the sine of double the distance of the moon's apogee from the nearest syzygy or quadrature to the radius.

By the same theory of gravity, the action of the sun upon the moon is something greater when the line of the moon's nodes passes through the sun than when it is at right angles with the line which joins the sun and the earth; and hence arises another equation of the moon's mean motion, which I shall call the second semi-annual; and this is greatest when the nodes are in the octants of the sun, and vanishes when they are in the syzygies or quadratures; and in other positions of the nodes is proportional to the sine of double the distance of either node from the nearest syzygy or quadrature. And it is added to the mean motion of the moon, if the sun is *in antecedentia*, to the node which is nearest to him, and subducted if *in consequentia*; and in the octants, where it is of the greatest magnitude, it arises to 47″ in the mean distance of the sun from the earth, as I find from the theory of gravity. In other distances of the sun, this equation, greatest in the octants of the nodes, is reciprocally as the cube of the sun's distance from the earth; and therefore in the sun's perigee it comes to about 49″, and in its apogee to about 45″.

By the same theory of gravity, the moon's apogee goes forward at the greatest rate when it is either in conjunction with or in opposition to the sun, but in its quadratures with the sun it goes backward; and the eccentricity comes, in the former case, to its greatest quantity; in the latter to its least, by Cor. 7, 8, and 9, Prop. LXVI, Book 1. And those inequalities, by the Corollaries we have named, are very great, and generate the principal which I call the semi-annual equation of the apogee; and this semi-annual equation in its greatest quantity comes to about 12° 18′, as nearly as I could collect from the phænomena. Our countryman, *Horrox*, was the first who advanced the theory of the moon's moving in an ellipsis about the earth placed in its lower focus. Dr. *Halley* improved the notion, by putting the centre of the ellipsis in an epicycle whose cen-

tre is uniformly revolved about the earth; and from the motion in this epicycle the mentioned inequalities in the progress and regress of the apogee, and in the quantity of eccentricity, do arise. Suppose the mean distance of the moon from the earth to be divided into 100000 parts, and let T represent the earth, and TC the moon's mean eccentricity of 5505 such parts. Produce TC to B, so as CB may be the sine of the greatest semi-annual equation 12° 18′ to the radius TC; and the circle BDA described about the centre C, with the interval CB, will be the epicycle spoken of, in which the centre of the moon's orbit is placed, and revolved according to the order of the letters BDA. Set off the angle BCD equal to twice the annual argument, or twice the distance of the sun's true place from the place of the moon's apogee once equated, and CTD will be the semi-annual equation of the moon's apogee, and TD the eccentricity of its orbit, tending to the place of the apogee now twice equated. But, having the moon's mean motion, the place of its apogee, and its eccentricity, as well as the longer axis of its orbit 200000, from these *data* the true place of the moon in its orbit, together with its distance from the earth, may be determined by the methods commonly known.



In the perihelion of the earth, where the force of the sun is greatest, the centre of the moon's orbit moves faster about the centre C than in the aphelion, and that in the reciprocal triplicate proportion of the sun's distance from the earth. But, because the equation of the sun's centre is included in the annual argument, the centre of the moon's orbit moves faster in its epicycle BDA, in the reciprocal duplicate proportion of the sun's distance from the earth. Therefore, that it may move yet faster in the reciprocal simple proportion of the distance, suppose that from D, the centre of the orbit, a right line DE is drawn, tending towards the moon's apogee once equated, that is, parallel to TC; and set off the angle EDF equal to the excess of the aforesaid annual argument above the distance of the moon's apogee from the sun's perigee *in consequentia*; or, which comes to the same thing, take the angle CDF equal to the complement of the sun's true anomaly to 360°; and let DF be to DC as twice the eccentricity of the *orbis magnus* to the sun's mean distance from the earth, and the sun's mean diurnal motion from the moon's apogee to the sun's mean diurnal motion from its own apogee conjunctly, that is, as $33\frac{1}{3}$ to 1000, and 52′ 27″ 16‴ to 59′ 8″ 10‴ conjunctly, or as 3 to 100; and imagine the centre of the moon's orbit placed in the point F to be revolved in an epicycle whose centre is D, and radius DF, while the point D moves in the circumference of the circle DABD: for by this means the centre of

the moon's orbit comes to describe a certain curve line about the centre C, with a velocity which will be almost reciprocally as the cube of the sun's distance from the earth, as it ought to be.

The calculus of this motion is difficult, but may be rendered more easy by the following approximation. Assuming, as above, the moon's mean distance from the earth of 100000 parts, and the eccentricity TC of 5505 such parts, the line CB or CD will be found $1172\frac{3}{4}$, and DF $35\frac{1}{5}$ of those parts; and this line DF at the distance TC subtends the angle at the earth, which the removal of the centre of the orbit from the place D to the place F generates in the motion of this centre; and double this line DF in a parallel position, at the distance of the upper focus of the moon's orbit from the earth, subtends at the earth the same angle as DF did before, which that removal generates in the motion of this upper focus; but at the distance of the moon from the earth this double line 2DF at the upper focus, in a parallel position to the first line DF, subtends an angle at the moon, which the said removal generates in the motion of the moon, which angle may be therefore called the second equation of the moon's centre; and this equation, in the mean distance of the moon from the earth, is nearly as the sine of the angle which that line DF contains with the line drawn from the point F to the moon, and when in its greatest quantity amounts to 2′ 25″. But the angle which the line DF contains with the line drawn from the point F to the moon is found either by subtracting the angle EDF from the mean anomaly of the moon, or by adding the distance of the moon from the sun to the distance of the moon's apogee from the apogee of the sun; and as the radius to the sine of the angle thus found, so is 2′ 25″ to the second equation of the centre: to be added, if the forementioned sum be less than a semi-circle; to be subducted, if greater. And from the moon's place in its orbit thus corrected, its longitude may be found in the syzygies of the luminaries.

The atmosphere of the earth to the height of 35 or 40 miles refracts the sun's light. This refraction scatters and spreads the light over the earth's shadow; and the dissipated light near the limits of the shadow dilates the shadow. Upon which account, to the diameter of the shadow, as it comes out by the parallax, I add 1 or $1\frac{1}{3}$ minute in lunar eclipses.

But the theory of the moon ought to be examined and proved from the phænomena, first in the syzygies, then in the quadratures, and last of all in the octants; and whoever pleases to undertake the work will find it not amiss to assume the following mean motions of the sun and moon at the Royal Observatory of *Greenwich*, to the last day of *December* at noon, *anno* 1700, O.S. viz. The mean motion of the sun ♑ 20° 43′ 40″, and of its apogee ♋ 7° 44′ 30″; the mean motion of the moon ♌ 15° 21′ 00″; of its apogee, ♓ 8° 20′ 00″; and of its ascending node ☊ 27° 24′ 20″; and the difference of meridians betwixt the Observatory at *Greenwich* and

the Royal Observatory at *Paris*, $0^h.\ 9'\ 20''$: but the mean motion of the moon and of its apogee are not yet obtained with sufficient accuracy.

PROPOSITION XXXVI. PROBLEM XVII.

To find the force of the sun to move the sea.

The sun's force ML or PT' to disturb the motions of the moon, was (by Prop. XXV.) in the moon's quadratures, to the force of gravity with us, as 1 to 638092,6; and the force TM — LM or 2PK in the moon's syzygies is double that quantity. But, descending to the surface of the earth, these forces are diminished in proportion of the distances from the centre of the earth, that is, in the proportion of $60\frac{1}{2}$ to 1; and therefore the former force on the earth's surface is to the force of gravity as 1 to 38604600; and by this force the sea is depressed in such places as are 90 degrees distant from the sun. But by the other force, which is twice as great, the sea is raised not only in the places directly under the sun, but in those also which are directly opposed to it; and the sum of these forces is to the force of gravity as 1 to 12868200. And because the same force excites the same motion, whether it depresses the waters in those places which are 90 degrees distant from the sun, or raises them in the places which are directly under and directly opposed to the sun, the aforesaid sum will be the total force of the sun to disturb the sea, and will have the same effect as if the whole was employed in raising the sea in the places directly under and directly opposed to the sun, and did not act at all in the places which are 90 degrees removed from the sun.

And this is the force of the sun to disturb the sea in any given place, where the sun is at the same time both vertical, and in its mean distance from the earth. In other positions of the sun, its force to raise the sea is as the versed sine of double its altitude above the horizon of the place directly, and the cube of the distance from the earth reciprocally.

COR. Since the centrifugal force of the parts of the earth, arising from the earth's diurnal motion, which is to the force of gravity as 1 to 289, raises the waters under the equator to a height exceeding that under the poles by 85472 *Paris* feet, as above, in Prop. XIX., the force of the sun, which we have now shewed to be to the force of gravity as 1 to 12868200, and therefore is to that centrifugal force as 289 to 12868200, or as 1 to 44527, will be able to raise the waters in the places directly under and directly opposed to the sun to a height exceeding that in the places which are 90 degrees removed from the sun only by one *Paris* foot and $113\frac{1}{10}$ inches; for this measure is to the measure of 85472 feet as 1 to 44527.

PROPOSITION XXXVII. PROBLEM XVIII.

To find the force of the moon to move the sea.

The force of the moon to move the sea is to be deduced from its propor-

tion to the force of the sun, and this proportion is to be collected from the proportion of the motions of the sea, which are the effects of those forces. Before the mouth of the river *Avon*, three miles below *Bristol*, the height of the ascent of the water in the vernal and autumnal syzygies of the luminaries (by the observations of *Samuel Sturmy*) amounts to about 45 feet, but in the quadratures to 25 only. The former of those heights arises from the sum of the aforesaid forces, the latter from their difference. If, therefore, S and L are supposed to represent respectively the forces of the sun and moon while they are in the equator, as well as in their mean distances from the earth, we shall have L + S to L — S as 45 to 25, or as 9 to 5.

At *Plymouth* (by the observations of *Samuel Colepress*) the tide in its mean height rises to about 16 feet, and in the spring and autumn the height thereof in the syzygies may exceed that in the quadratures by more than 7 or 8 feet. Suppose the greatest difference of those heights to be 9 feet, and L + S will be to L — S as $20\frac{1}{2}$ to $11\frac{1}{2}$, or as 41 to 23; a proportion that agrees well enough with the former. But because of the great tide at *Bristol*, we are rather to depend upon the observations of *Sturmy*; and, therefore, till we procure something that is more certain, we shall use the proportion of 9 to 5.

But because of the reciprocal motions of the waters, the greatest tides do not happen at the times of the syzygies of the luminaries, but, as we have said before, are the third in order after the syzygies; or (reckoning from the syzygies) follow next after the third appulse of the moon to the meridian of the place after the syzygies; or rather (as *Sturmy* observes) are the third after the day of the new or full moon, or rather nearly after the twelfth hour from the new or full moon, and therefore fall nearly upon the forty-third hour after the new or full of the moon. But in this port they fall out about the seventh hour after the appulse of the moon to the meridian of the place; and therefore follow next after the appulse of the moon to the meridian, when the moon is distant from the sun, or from opposition with the sun by about 18 or 19 degrees *in consequentia*. So the summer and winter seasons come not to their height in the solstices themselves, but when the sun is advanced beyond the solstices by about a tenth part of its whole course, that is, by about 36 or 37 degrees. In like manner, the greatest tide is raised after the appulse of the moon to the meridian of the place, when the moon has passed by the sun, *or the opposition thereof*, by about a tenth part of the whole motion from *one greatest* tide to *the next following greatest* tide. Suppose that distance about $18\frac{1}{2}$ degrees; and the sun's force in this distance of the moon from the syzygies and quadratures will be of less moment to augment and diminish that part of the motion of the sea which proceeds from the motion of the moon than in the syzygies and quadratures themselves in the proportion of the radius to

the co-sine of double this distance, or of an angle of 37 degrees; that is, in proportion of 10000000 to 7985355; and, therefore, in the preceding analogy, in place of S we must put 0,7986355S.

But farther; the force of the moon in the quadratures must be diminished, on account of its declination from the equator; for the moon in those quadratures, or rather in 18½ degrees past the quadratures, declines from the equator by about 23° 13′; and the force of either luminary to move the sea is diminished as it declines from the equator nearly in the duplicate proportion of the co-sine of the declination; and therefore the force of the moon in those quadratures is only 0,8570327L; whence we have L+0,7986355S to 0,8570327L — 0,7986355S as 9 to 5.

Farther yet; the diameters of the orbit in which the moon should move, setting aside the consideration of eccentricity, are one to the other as 69 to 70; and therefore the moon's distance from the earth in the syzygies is to its distance in the quadratures, *cæteris paribus*, as 69 to 70; and its distances, when 18½ degrees advanced beyond the syzygies, where the greatest tide was excited, and when 18½ degrees passed by the quadratures, where the least tide was produced, are to its mean distance as 69,098747 and 69,897345 to 69½. But the force of the moon to move the sea is in the reciprocal triplicate proportion of its distance; and therefore its forces, in the greatest and least of those distances, are to its force in its mean distance as 0.9830427 and 1,017522 to 1. From whence we have 1,017522L × 0,7986355S to 0,9830427 × 0,8570327L — 0,7986355S as 9 to 5; and S to L as 1 to 4,4815. Wherefore since the force of the sun is to the force of gravity as 1 to 12868200, the moon's force will be to the force of gravity as 1 to 2871400.

Cor. 1. Since the waters excited by the sun's force rise to the height of a foot and $11\frac{1}{3}\frac{1}{6}$ inches, the moon's force will raise the same to the height of 8 feet and $7\frac{5}{3}\frac{5}{3}$ inches; and the joint forces of both will raise the same to the height of 10½ feet; and when the moon is in its perigee to the height of 12½ feet, and more, especially when the wind sets the same way as the tide. And a force of that quantity is abundantly sufficient to excite all the motions of the sea, and agrees well with the proportion of those motions; for in such seas as lie free and open from east to west, as in the *Pacific* sea, and in those tracts of the *Atlantic* and *Ethiopic* seas which lie without the tropics, the waters commonly rise to 6, 9, 12, or 15 feet; but in the *Pacific* sea, which is of a greater depth, as well as of a larger extent, the tides are said to be greater than in the *Atlantic* and *Ethiopic* seas; for to have a full tide raised, an extent of sea from east to west is required of no less than 90 degrees. In the *Ethiopic* sea, the waters rise to a less height within the tropics than in the temperate zones, because of the narrowness of the sea between *Africa* and the southern parts of *America*. In the middle of the open sea the waters cannot rise with-

out falling together, and at the same time, upon both the eastern and western shores, when, notwithstanding, in our narrow seas, they ought to fall on those shores by alternate turns; upon which account there is commonly but a small flood and ebb in such islands as lie far distant from the continent. On the contrary, in some ports, where to fill and empty the bays alternately the waters are with great violence forced in and out through shallow channels, the flood and ebb must be greater than ordinary; as at *Plymouth* and *Chepstow Bridge* in *England*, at the mountains of St. *Michael*, and the town of *Auranches*, in *Normandy*, and at *Cambaia* and *Pegu* in the *East Indies*. In these places the sea is hurried in and out with such violence, as sometimes to lay the shores under water, sometimes to leave them dry for many miles. Nor is this force of the influx and efflux to be broke till it has raised and depressed the waters to 30, 40, or 50 feet and above. And a like account is to be given of long and shallow channels or straits, such as the *Magellanic* straits, and those channels which environ *England*. The tide in such ports and straits, by the violence of the influx and efflux, is augmented above measure. But on such shores as lie towards the deep and open sea with a steep descent, where the waters may freely rise and fall without that precipitation of influx and efflux, the proportion of the tides agrees with the forces of the sun and moon.

Cor. 2. Since the moon's force to move the sea is to the force of gravity as 1 to 2871400, it is evident that this force is far less than to appear sensibly in statical or hydrostatical experiments, or even in those of pendulums. It is in the tides only that this force shews itself by any sensible effect.

Cor. 3. Because the force of the moon to move the sea is to the like force of the sun as 4,4815 to 1, and those forces (by Cor. 14, Prop. LXVI, Book 1) are as the densities of the bodies of the sun and moon and the cubes of their apparent diameters conjunctly, the density of the moon will be to the density of the sun as 4,4815 to 1 directly, and the cube of the moon's diameter to the cube of the sun's diameter inversely; that is (seeing the mean apparent diameters of the moon and sun are $31' 16\frac{1}{2}''$, and $32' 12''$), as 4891 to 1000. But the density of the sun was to the density of the earth as 1000 to 4000; and therefore the density of the moon is to the density of the earth as 4891 to 4000, or as 11 to 9. Therefore the body of the moon is more dense and more earthly than the earth itself.

Cor. 4. And since the true diameter of the moon (from the observations of astronomers) is to the true diameter of the earth as 100 to 365, the mass of matter in the moon will be to the mass of matter in the earth as 1 to 39,788.

Cor. 5. And the accelerative gravity on the surface of the moon will be

about three times less than the accelerative gravity on the surface of the earth.

Cor. 6. And the distance of the moon's centre from the centre of the earth will be to the distance of the moon's centre from the common centre of gravity of the earth and moon as 40,788 to 39,788.

Cor. 7. And the mean distance of the centre of the moon from the centre of the earth will be (in the moon's octants) nearly $60\frac{2}{5}$ of the greatest semi-diameters of the earth; for the greatest semi-diameter of the earth was 19658600 *Paris* feet, and the mean distance of the centres of the earth and moon, consisting of $60\frac{2}{5}$ such semi-diameters, is equal to 1187379440 feet. And this distance (by the preceding Cor.) is to the distance of the moon's centre from the common centre of gravity of the earth and moon as 40,788 to 39,788; which latter distance, therefore, is 1158268534 feet. And since the moon, in respect of the fixed stars, performs its revolution in $27^{d}.\ 7^{h}.\ 43\frac{1}{3}'$, the versed sine of that angle which the moon in a minute of time describes is 12752341 to the radius 1000,000000,000000; and as the radius is to this versed sine, so are 1158268534 feet to 14,7706353 feet. The moon, therefore, falling towards the earth by that force which retains it in its orbit, would in one minute of time describe 14,7706353 feet; and if we augment this force in the proportion of $178\frac{2}{4}\frac{9}{0}$ to $177\frac{2}{4}\frac{9}{0}$, we shall have the total force of gravity at the orbit of the moon, by Cor. Prop. III; and the moon falling by this force, in one minute of time would describe 14,8538067 feet. And at the 60th part of the distance of the moon from the earth's centre, that is, at the distance of 19789657 3 feet from the centre of the earth, a body falling by its weight, would, in one second of time, likewise describe 14,8538067 feet. And, therefore, at the distance of 19615800, which compose one mean semi-diameter of the earth, a heavy body would describe in falling 15,11175, or 15 feet, 1 inch, and $4\frac{1}{11}$ lines, in the same time. This will be the descent of bodies in the latitude of 45 degrees. And by the foregoing table, to be found under Prop. XX, the descent in the latitude of *Paris* will be a little greater by an excess of about $\frac{2}{3}$ parts of a line. Therefore, by this computation, heavy bodies in the latitude of *Paris* falling *in vacuo* will describe 15 *Paris* feet, 1 inch, $4\frac{2}{3}\frac{5}{3}$ lines, very nearly, in one second of time. And if the gravity be diminished by taking away a quantity equal to the centrifugal force arising in that latitude from the earth's diurnal motion, heavy bodies falling there will describe in one second of time 15 feet, 1 inch, and $1\frac{1}{2}$ line. And with this velocity heavy bodies do really fall in the latitude of *Paris*, as we have shewn above in Prop. IV and XIX.

Cor. 8. The mean distance of the centres of the earth and moon in the syzygies of the moon is equal to 60 of the greatest semi-diameters of the earth, subducting only about one 30th part of a semi-diameter: and in the

moon's quadratures the mean distance of the same centres is $60\frac{5}{6}$ such semi-diameters of the earth; for these two distances are to the mean distance of the moon in the octants as 69 and 70 to $69\frac{1}{2}$, by Prop. XXVIII.

Cor. 9. The mean distance of the centres of the earth and moon in the syzygies of the moon is 60 mean semi-diameters of the earth, and a 10th part of one semi-diameter; and in the moon's quadratures the mean distance of the same centres is 61 mean semi-diameters of the earth, subducting one 30th part of one semi-diameter.

Cor. 10. In the moon's syzygies its mean horizontal parallax in the latitudes of 0, 30, 38, 45, 52, 60, 90 degrees is 57′ 20″, 57′ 16″, 57′ 14″, 57′ 12″, 57′ 10″, 57′ 8″, 57′ 4″, respectively.

In these computations I do not consider the magnetic attraction of the earth, whose quantity is very small and unknown: if this quantity should ever be found out, and the measures of degrees upon the meridian, the lengths of isochronous pendulums in different parallels, the laws of the motions of the sea, and the moon's parallax, with the apparent diameters of the sun and moon, should be more exactly determined from phænomena: we should then be enabled to bring this calculation to a greater accuracy.

PROPOSITION XXXVIII. PROBLEM XIX.

To find the figure of the moon's body.

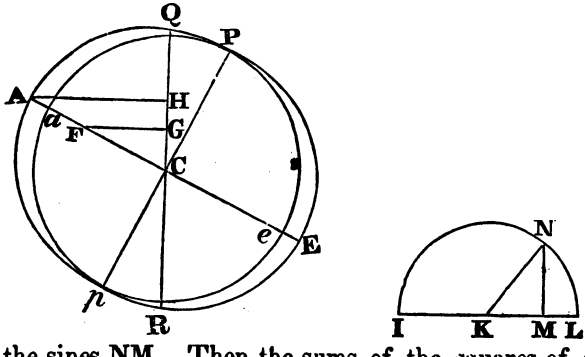
If the moon's body were fluid like our sea, the force of the earth to raise that fluid in the nearest and remotest parts would be to the force of the moon by which our sea is raised in the places under and opposite to the moon as the accelerative gravity of the moon towards the earth to the accelerative gravity of the earth towards the moon, and the diameter of the moon to the diameter of the earth conjunctly; that is, as 39,788 to 1, and 100 to 365 conjunctly, or as 1081 to 100. Wherefore, since our sea, by the force of the moon, is raised to $8\frac{2}{5}$ feet, the lunar fluid would be raised by the force of the earth to 93 feet; and upon this account the figure of the moon would be a spheroid, whose greatest diameter produced would pass through the centre of the earth, and exceed the diameters perpendicular thereto by 186 feet. Such a figure, therefore, the moon affects, and must have put on from the beginning. Q.E.I.

Cor. Hence it is that the same face of the moon always respects the earth; nor can the body of the moon possibly rest in any other position, but would return always by a libratory motion to this situation; but those librations, however, must be exceedingly slow, because of the weakness of the forces which excite them; so that the face of the moon, which should be always obverted to the earth, may, for the reason assigned in Prop. XVII. be turned towards the other focus of the moon's orbit, without being immediately drawn back, and converted again towards the earth.

LEMMA I.

If APEp *represent the earth uniformly dense, marked with the centre* C, *the poles* P, p, *and the equator* AE; *and if about the centre* C, *with the radius* CP, *we suppose the sphere* Pape *to be described, and* QR *to denote the plane on which a right line, drawn from the centre of the sun to the centre of the earth, insists at right angles; and further suppose that the several particles of the whole exterior earth* PapAPepE, *without the height of the said sphere, endeavour to recede towards this side and that side from the plane* QR, *every particle by a force proportional to its distance from that plane; I say, in the first place, that the whole force and efficacy of all the particles that are situate in* AE, *the circle of the equator, and disposed uniformly without the globe, encompassing the same after the manner of a ring, to wheel the earth about its centre, is to the whole force and efficacy of as many particles in that point* A *of the equator which is at the greatest distance from the plane* QR, *to wheel the earth about its centre with a like circular motion, as* 1 *to* 2. *And that circular motion will be performed about an axis lying in the common section of the equator and the plane* QR.

For let there be described from the centre K, with the diameter IL, the semi-circle INL. Suppose the semi-circumference INL to be divided into innumerable equal parts, and from the several parts N to the diameter



IL let fall the sines NM. Then the sums of the squares of all the sines NM will be equal to the sums of the squares of the sines KM, and both sums together will be equal to the sums of the squares of as many semi-diameters KN; and therefore the sum of the squares of all the sines NM will be but half so great as the sum of the squares of as many semi-diameters KN.

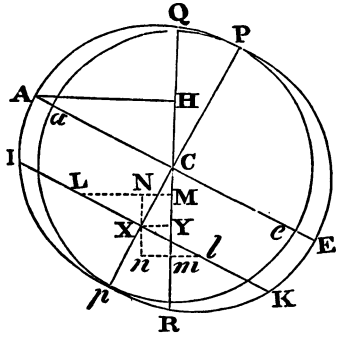
Suppose now the circumference of the circle AE to be divided into the like number of little equal parts, and from every such part F a perpendicular FG to be let fall upon the plane QR, as well as the perpendicular AH from the point A. Then the force by which the particle F recedes

from the plane QR will (by supposition) be as that perpendicular FG; and this force multiplied by the distance CG will represent the power of the particle F to turn the earth round its centre. And, therefore, the power of a particle in the place F will be to the power of a particle in the place A as FG × GC to AH × HC; that is, as FC^2 to AC^2: and therefore the whole power of all the particles F, in their proper places F, will be to the power of the like number of particles in the place A as the sum of all the FC^2 to the sum of all the AC^2, that is (by what we have demonstrated before), as 1 to 2. Q.E.D.

And because the action of those particles is exerted in the direction of lines perpendicularly receding from the plane QR, and that equally from each side of this plane, they will wheel about the circumference of the circle of the equator, together with the adherent body of the earth, round an axis which lies as well in the plane QR as in that of the equator.

LEMMA II.

The same things still supposed, I say, in the second place, that the total force or power of all the particles situated every where about the sphere to turn the earth about the said axis is to the whole force of the like number of particles, uniformly disposed round the whole circumference of the equator AE *in the fashion of a ring, to turn the whole earth about with the like circular motion, as* 2 *to* 5.



For let IK be any lesser circle parallel to the equator AE, and let L*l* be any two equal particles in this circle, situated without the sphere P*ape*; and if upon the plane QR, which is at right angles with a radius drawn to the sun, we let fall the perpendiculars LM, *lm*, the total forces by which these particles recede from the plane QR will be proportional to the perpendiculars LM, *lm*. Let the right line L*l* be drawn parallel to the plane P*ape*, and bisect the same in X; and through the point X draw N*n* parallel to the plane QR, and meeting the perpendiculars LM, *lm*, in N and *n*; and upon the plane QR let fall the perpendicular XY. And the contrary forces of the particles L and *l* to wheel about the earth contrariwise are as LM × MC, and *lm* × *m*C; that is, as LN × MC + NM × MC, and *ln* × *m*C — *nm* × *m*C; or LN × MC + NM × MC, and LN × *m*C — NM × *m*C, and LN × M*m* — NM × MC + *m*C, the difference of the two, is the force of both taken together to turn the earth round. The affirmative part of this difference LN × M*m*, or 2LN × NX, is to 2AH × HC, the force of two particles of the same size situated in A, as LX^2 to AC^2; and the negative part NM

$\times$ MC $+$ mC, or 2XY $\times$ CY, is to 2AH $\times$ HC, the force of the same two particles situated in A, as CX^2 to AC^2. And therefore the difference of the parts, that is, the force of the two particles L and l, taken together, to wheel the earth about, is to the force of two particles, equal to the former and situated in the place A, to turn in like manner the earth round, as $LX^2 - CX^2$ to AC^2. But if the circumference IK of the circle IK is supposed to be divided into an infinite number of little equal parts L, all the LX^2 will be to the like number of IX^2 as 1 to 2 (by Lem. 1); and to the same number of AC^2 as IX^2 to $2AC^2$; and the same number of CX^2 to as many AC^2 as $2CX^2$ to $2AC^2$. Wherefore the united forces of all the particles in the circumference of the circle IK are to the joint forces of as many particles in the place A as $IX^2 - 2CX^2$ to $2AC^2$; and therefore (by Lem. 1) to the united forces of as many particles in the circumference of the circle AE as $IX^2 - 2CX^2$ to AC^2.

Now if Pp, the diameter of the sphere, is conceived to be divided into an infinite number of equal parts, upon which a like number of circles IK are supposed to insist, the matter in the circumference of every circle IK will be as IX^2; and therefore the force of that matter to turn the earth about will be as IX^2 into $IX^2 - 2CX^2$; and the force of the same matter, if it was situated in the circumference of the circle AE, would be as IX^2 into AC^2. And therefore the force of all the particles of the whole matter situated without the sphere in the circumferences of all the circles is to the force of the like number of particles situated in the circumference of the greatest circle AE as all the IX^2 into $IX^2 - 2CX^2$ to as many IX^2 into AC^2; that is, as all the $AC^2 - CX^2$ into $AC^2 - 3CX^2$ to as many $AC^2 - CX^2$ into AC^2; that is, as all the $AC^4 - 4AC^2 \times CX^2 + 3CX^4$ to as many $AC^4 - AC^2 \times CX^2$; that is, as the whole fluent quantity, whose fluxion is $AC^4 - 4AC^2 \times CX^2 + 3CX^4$, to the whole fluent quantity, whose fluxion is $AC^4 - AC^2 \times CX^2$; and, therefore, by the method of fluxions, as $AC^4 \times CX - \frac{4}{3}AC^2 \times CX^3 + \frac{3}{5}CX^5$ to $AC^4 \times CX - \frac{1}{3}AC^2 \times CX^3$; that is, if for CX we write the whole Cp, or AC, as $\frac{4}{15}AC^5$ to $\frac{2}{3}AC^5$; that is, as 2 to 5. Q.E.D.

LEMMA III.

The same things still supposed, I say, in the third place, that the motion of the whole earth about the axis above-named arising from the motions of all the particles, will be to the motion of the aforesaid ring about the same axis in a proportion compounded of the proportion of the matter in the earth to the matter in the ring; and the proportion of three squares of the quadrantal arc of any circle to two squares of its diameter, that is, in the proportion of the matter to the matter, and of the number 925275 *to the number* 1000000.

For the motion of a cylinder revolved about its quiescent axis is to the

motion of the inscribed sphere revolved together with it as any four equal squares to three circles inscribed in three of those squares; and the motion of this cylinder is to the motion of an exceedingly thin ring surrounding both sphere and cylinder in their common contact as double the matter in the cylinder to triple the matter in the ring; and this motion of the ring, uniformly continued about the axis of the cylinder, is to the uniform motion of the same about its own diameter performed in the same periodic time as the circumference of a circle to double its diameter.

HYPOTHESIS II.

If the other parts of the earth were taken away, and the remaining ring was carried alone about the sun in the orbit of the earth by the annual motion, while by the diurnal motion it was in the mean time revolved about its own axis inclined to the plane of the ecliptic by an angle of $23\frac{1}{2}$ degrees, the motion of the equinoctial points would be the same, whether the ring were fluid, or whether it consisted of a hard and rigid matter.

PROPOSITION XXXIX. PROBLEM XX.

To find the precession of the equinoxes.

The middle horary motion of the moon's nodes in a circular orbit, when the nodes are in the quadratures, was $16'' 35''' 16^{iv}. 36^{v}.$; the half of which, $8'' 17''' 38^{v}. 18^{v}.$ (for the reasons above explained) is the mean horary motion of the nodes in such an orbit, which motion in a whole sidereal year becomes $20° 11' 46''$. Because, therefore, the nodes of the moon in such an orbit would be yearly transferred $20° 11' 46''$ *in antecedentia;* and, if there were more moons, the motion of the nodes of every one (by Cor. 16, Prop. LXVI, Book 1) would be as its periodic time; if upon the surface of the earth a moon was revolved in the time of a sidereal day, the annual motion of the nodes of this moon would be to $20° 11' 46''$ as $23^{h}. 56'$, the sidereal day, to $27^{d}. 7^{h}. 43'$, the periodic time of our moon, that is, as 1436 to 39343. And the same thing would happen to the nodes of a ring of moons encompassing the earth, whether these moons did not mutually touch each the other, or whether they were molten, and formed into a continued ring, or whether that ring should become rigid and inflexible.

Let us, then, suppose that this ring is in quantity of matter equal to the whole exterior earth $PapAPepE$, which lies without the sphere $Pape$ (see fig. Lem. II); and because this sphere is to that exterior earth as aC^2 to $AC^2 - aC^2$, that is (seeing PC or aC the least semi-diameter of the earth is to AC the greatest semi-diameter of the same as 229 to 230), as 52441 to 459; if this ring encompassed the earth round the equator, and both together were revolved about the diameter of the ring, the motion of

the ring (by Lem. III) would be to the motion of the inner sphere as 459 to 52441 and 1000000 to 925275 conjunctly, that is, as 4590 to 485223; and therefore the motion of the ring would be to the sum of the motions of both ring and sphere as 4590 to 489813. Wherefore if the ring adheres to the sphere, and communicates its motion to the sphere, by which its nodes or equinoctial points recede, the motion remaining in the ring will be to its former motion as 4590 to 489813; upon which account the motion of the equinoctial points will be diminished in the same proportion. Wherefore the annual motion of the equinoctial points of the body, composed of both ring and sphere, will be to the motion 20° 11′ 46″ as 1436 to 39343 and 4590 to 489813 conjunctly, that is, as 100 to 292369. But the forces by which the nodes of a number of moons (as we explained above), and therefore by which the equinoctial points of the ring recede (that is, the forces 3IT, in fig. Prop. XXX), are in the several particles as the distances of those particles from the plane QR; and by these forces the particles recede from that plane: and therefore (by Lem. II) if the matter of the ring was spread all over the surface of the sphere, after the fashion of the figure P*ap*AP*ep*E, in order to make up that exterior part of the earth, the total force or power of all the particles to wheel about the earth round any diameter of the equator, and therefore to move the equinoctial points, would become less than before in the proportion of 2 to 5. Wherefore the annual regress of the equinoxes now would be to 20° 11′ 46″ as 10 to 73092; that is, would be 9″ 56‴ 50^{iv}.

But because the plane of the equator is inclined to that of the ecliptic, this motion is to be diminished in the proportion of the sine 91706 (which is the co-sine of 23½ deg.) to the radius 100000; and the remaining motion will now be 9″ 7‴ 20^{iv}. which is the annual precession of the equinoxes arising from the force of the sun.

But the force of the moon to move the sea was to the force of the sun nearly as 4,4815 to 1; and the force of the moon to move the equinoxes is to that of the sun in the same proportion. Whence the annual precession of the equinoxes proceeding from the force of the moon comes out 40″ 52‴ 52^{iv}. and the total annual precession arising from the united forces of both will be 50″ 00‴ 12^{iv}. the quantity of which motion agrees with the phænomena; for the precession of the equinoxes, by astronomical observations, is about 50″ yearly.

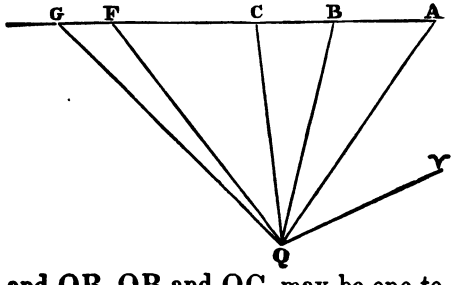
If the height of the earth at the equator exceeds its height at the poles by more than 17⅙ miles, the matter thereof will be more rare near the surface than at the centre; and the precession of the equinoxes will be augmented by the excess of height, and diminished by the greater rarity.

And now we have described the system of the sun, the earth, moon, and planets, it remains that we add something about the comets.

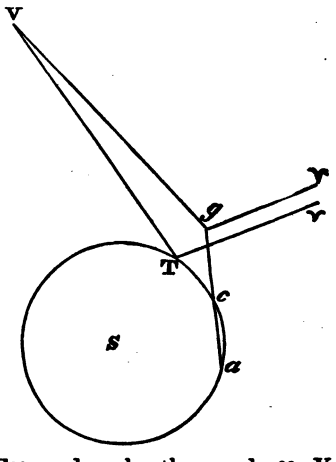
LEMMA IV

That the comets are higher than the moon, and in the regions of the planets.

As the comets were placed by astronomers above the moon, because they were found to have no diurnal parallax, so their annual parallax is a convincing proof of their descending into the regions of the planets; for all the comets which move in a direct course according to the order of the signs, about the end of their appearance become more than ordinarily slow or retrograde, if the earth is between them and the sun; and more than ordinarily swift, if the earth is approaching to a heliocentric opposition with them; whereas, on the other hand, those which move against the order of the signs, towards the end of their appearance appear swifter than they ought to be, if the earth is between them and the sun; and slower, and perhaps retrograde, if the earth is in the other side of its orbit. And these appearances proceed chiefly from the diverse situations which the earth acquires in the course of its motion, after the same manner as it happens to the planets, which appear sometimes retrograde, sometimes more slowly, and sometimes more swiftly, progressive, according as the motion of the earth falls in with that of the planet, or is directed the contrary way. If the earth move the same way with the comet, but, by an angular motion about the sun, so much swifter that right lines drawn from the earth to the comet converge towards the parts beyond the comet, the comet seen from the earth, because of its slower motion, will appear retrograde; and even if the earth is slower than the comet, the motion of the earth being subducted, the motion of the comet will at least appear retarded; but if the earth tends the contrary way to that of the comet, the motion of the comet will from thence appear accelerated; and from this apparent acceleration, or retardation, or regressive motion, the distance of the comet may be inferred in this manner. Let ϒQA, ϒQB, ϒQC, be three observed longitudes of the comet about the time of its first appearing, and ϒQF its last observed longitude before its disappearing. Draw the right line ABC, whose parts AB, BC, intercepted between the right lines QA and QB, QB and QC, may be one to the other as the two times between the three first observations. Produce AC to G, so as AG may be to AB as the time between the first and last observation to the time between the first and second; and join QG. Now if the comet did move uniformly in a right line, and the earth either stood still, or was likewise carried forwards in a right line by an uniform motion, the angle ϒQG would be the



longitude of the comet at the time of the last observation. The angle, therefore, FQG, which is the difference of the longitude, proceeds from the inequality of the motions of the comet and the earth; and this angle, if the earth and comet move contrary ways, is added to the angle ϒQG, and accelerates the apparent motion of the comet; but if the comet move the same way with the earth, it is subtracted, and either retards the motion of the comet, or perhaps renders it retrograde, as we have but now explained. This angle, therefore, proceeding chiefly from the motion of the earth, is justly to be esteemed the parallax of the comet; neglecting, to wit, some little increment or decrement that may arise from the unequal motion of the comet in its orbit; and from this parallax we thus deduce the distance of the comet. Let S represent the sun, *ac*ϒ the *orbis magnus*, *a* the earth's place in the first observation, *c* the place of the earth in the third observation, T the place of the earth in the last observation, and Tϒ a right line drawn to the beginning of Aries. Set off the angle ϒTV equal to the angle ϒQF, that is, equal to the longitude of the comet at the time when the earth is in T; join *ac*, and produce it to *g*, so as *ag* may be to *ac* as AG to AC; and *g* will be the place at which the earth would have arrived in the time of the last observation, if it had continued to move uniformly in the right line *ac*. Wherefore, if we draw *g*ϒ parallel to Tϒ, and make the angle ϒ*g*V equal to the angle ϒQG, this angle ϒ*g*V will be equal to the longitude of the comet seen from the place *g*, and the angle TV*g* will be the parallax which arises from the earth's being transferred from the place *g* into the place T; and therefore V will be the place of the comet in the plane of the ecliptic. And this place V is commonly lower than the orb of Jupiter.



The same thing may be deduced from the incurvation of the way of the comets; for these bodies move almost in great circles, while their velocity is great; but about the end of their course, when that part of their apparent motion which arises from the parallax bears a greater proportion to their whole apparent motion, they commonly deviate from those circles, and when the earth goes to one side, they deviate to the other; and this deflexion, because of its corresponding with the motion of the earth, must arise chiefly from the parallax; and the quantity thereof is so considerable, as, by my computation, to place the disappearing comets a good deal lower than Jupiter. Whence it follows that when they approach nearer to us in their perigees and perihelions they often descend below the orbs of Mars and the inferior planets.

The near approach of the comets is farther confirmed from the light of their heads; for the light of a celestial body, illuminated by the sun, and receding to remote parts, is diminished in the quadruplicate proportion of the distance; to wit, in one duplicate proportion, on account of the increase of the distance from the sun, and in another duplicate proportion, on account of the decrease of the apparent diameter. Wherefore if both the quantity of light and the apparent diameter of a comet are given, its distance will be also given, by taking the distance of the comet to the distance of a planet in the direct proportion of their diameters and the reciprocal subduplicate proportion of their lights. Thus, in the comet of the year 1682, Mr. *Flamsted* observed with a telescope of 16 feet, and measured with a micrometer, the least diameter of its head, 2′ 00; but the nucleus or star in the middle of the head scarcely amounted to the tenth part of this measure; and therefore its diameter was only 11″ or 12″; but in the light and splendor of its head it surpassed that of the comet in the year 1680, and might be compared with the stars of the first or second magnitude. Let us suppose that Saturn with its ring was about four times more lucid; and because the light of the ring was almost equal to the light of the globe within, and the apparent diameter of the globe is about 21″, and therefore the united light of both globe and ring would be equal to the light of a globe whose diameter is 30″, it follows that the distance of th comet was to the distance of Saturn as 1 to $\sqrt{4}$ inversely, and 12″ to 30 directly; that is, as 24 to 30, or 4 to 5. Again; the comet in the month of *April* 1665, as *Hevelius* informs us, excelled almost all the fixed stars in splendor, and even Saturn itself, as being of a much more vivid colour; for this comet was more lucid than that other which had appeared about the end of the preceding year, and had been compared to the stars of the first magnitude. The diameter of its head was about 6′; but the nucleus, compared with the planets by means of a telescope, was plainly less than Jupiter; and sometimes judged less, sometimes judged equal, to the globe of Saturn within the ring. Since, then, the diameters of the heads of the comets seldom exceed 8′ or 12′, and the diameter of the nucleus or central star is but about a tenth or perhaps fifteenth part of the diameter of the head, it appears that these stars are generally of about the same apparent magnitude with the planets. But in regard that their light may be often compared with the light of Saturn, yea, and sometimes exceeds it, it is evident that all comets in their perihelions must either be placed below or not far above Saturn; and they are much mistaken who remove them almost as far as the fixed stars; for if it was so, the comets could receive no more light from our sun than our planets do from the fixed stars.

So far we have gone, without considering the obscuration which comets suffer from that plenty of thick smoke which encompasseth their heads, and through which the heads always shew dull, as through a cloud; for by

how much the more a body is obscured by this smoke, by so much the more near it must be allowed to come to the sun, that it may vie with the planets in the quantity of light which it reflects. Whence it is probable that the comets descend far below the orb of Saturn, as we proved before from their parallax. But, above all, the thing is evinced from their tails, which must be owing either to the sun's light reflected by a smoke arising from them, and dispersing itself through the æther, or to the light of their own heads. In the former case, we must shorten the distance of the comets, lest we be obliged to allow that the smoke arising from their heads is propagated through such a vast extent of space, and with such a velocity and expansion as will seem altogether incredible; in the latter case, the whole light of both head and tail is to be ascribed to the central nucleus. But, then, if we suppose all this light to be united and condensed within the disk of the nucleus, certainly the nucleus will by far exceed Jupiter itself in splendor, especially when it emits a very large and lucid tail If, therefore, under a less apparent diameter, it reflects more light, it must be much more illuminated by the sun, and therefore much nearer to it; and the same argument will bring down the heads of comets sometimes within the orb of Venus, viz., when, being hid under the sun's rays, they emit such huge and splendid tails, like beams of fire, as sometimes they do; for if all that light was supposed to be gathered together into one star, it would sometimes exceed not one Venus only, but a great many such united into one.

Lastly; the same thing is inferred from the light of the heads, which increases in the recess of the comets from the earth towards the sun, and decreases in their return from the sun towards the earth; for so the comet of the year 1665 (by the observations of *Hevelius*), from the time that it was first seen, was always losing of its apparent motion, and therefore had already passed its perigee; but yet the splendor of its head was daily increasing, till, being hid under the sun's rays, the comet ceased to appear. The comet of the year 1683 (by the observations of the same *Hevelius*), about the end of *July*, when it first appeared, moved at a very slow rate, advancing only about 40 or 45 minutes in its orb in a day's time; but from that time its diurnal motion was continually upon the increase, till *September* 4, when it arose to about 5 degrees; and therefore, in all this interval of time, the comet was approaching to the earth. Which is likewise proved from the diameter of its head, measured with a micrometer; for, *August* 6, *Hevelius* found it only 6′ 05″, including the *coma*, which, *September* 2 he observed to be 9′ 07″, and therefore its head appeared far less about the beginning than towards the end of the motion; though about the beginning, because nearer to the sun, it appeared far more lucid than towards the end, as the same *Hevelius* declares. Wherefore in all this interval of time. on account of its recess from the sun, it decreased

in splendor, notwithstanding its access towards the earth. The comet of the year 1618, about the middle of *December*, and that of the year 1680, about the end of the same month, did both move with their greatest velocity, and were therefore then in their perigees; but the greatest splendor of their heads was seen two weeks before, when they had just got clear of the sun's rays; and the greatest splendor of their tails a little more early, when yet nearer to the sun. The head of the former comet (according to the observations of *Cysatus*), *December* 1, appeared greater than the stars of the first magnitude; and, *December* 16 (then in the perigee), it was but little diminished in magnitude, but in the splendor and brightness of its light a great deal. *January* 7, *Kepler*, being uncertain about the head, left off observing. *December* 12, the head of the latter comet was seen and observed by Mr. *Flamsted*, when but 9 degrees distant from the sun; which is scarcely to be done in a star of the third magnitude. *December* 15 and 17, it appeared as a star of the third magnitude, its lustre being diminished by the brightness of the clouds near the setting sun. *December* 26, when it moved with the greatest velocity, being almost in its perigee, it was less than the mouth of *Pegasus*, a star of the third magnitude. *January* 3, it appeared as a star of the fourth. *January* 9, as one of the fifth. *January* 13, it was hid by the splendor of the moon, then in her increase. *January* 25, it was scarcely equal to the stars of the seventh magnitude. If we compare equal intervals of time on one side and on the other from the perigee, we shall find that the head of the comet, which at both intervals of time was far, but yet equally, removed from the earth, and should have therefore shone with equal splendor, appeared brightest on the side of the perigee towards the sun, and disappeared on the other. Therefore, from the great difference of light in the one situation and in the other, we conclude the great vicinity of the sun and comet in the former; for the light of comets uses to be regular, and to appear greatest when the heads move fastest, and are therefore in their perigees; excepting in so far as it is increased by their nearness to the sun.

COR. 1. Therefore the comets shine by the sun's light, which they reflect.

COR. 2. From what has been said, we may likewise understand why comets are so frequently seen in that hemisphere in which the sun is, and so seldom in the other. If they were visible in the regions far above Saturn, they would appear more frequently in the parts opposite to the sun; for such as were in those parts would be nearer to the earth, whereas the presence of the sun must obscure and hide those that appear in the hemisphere in which he is. Yet, looking over the history of comets, I find that four or five times more have been seen in the hemisphere towards the sun than in the opposite hemisphere; besides, without doubt, not a few, which have been hid by the light of the sun: for comets descending

into our parts neither emit tails, nor are so well illuminated by the sun, as to discover themselves to our naked eyes, until they are come nearer to us than Jupiter. But the far greater part of that spherical space, which is described about the sun with so small an interval, lies on that side of the earth which regards the sun; and the comets in that greater part are commonly more strongly illuminated, as being for the most part nearer to the sun.

Cor. 3. Hence also it is evident that the celestial spaces are void of resistance; for though the comets are carried in oblique paths, and sometimes contrary to the course of the planets, yet they move every way with the greatest freedom, and preserve their motions for an exceeding long time, even where contrary to the course of the planets. I am out in my judgment if they are not a sort of planets revolving in orbits returning into themselves with a perpetual motion; for, as to what some writers contend, that they are no other than meteors, led into this opinion by the perpetual changes that happen to their heads, it seems to have no foundation; for the heads of comets are encompassed with huge atmospheres, and the lowermost parts of these atmospheres must be the densest; and therefore it is in the clouds only, not in the bodies of the comets themselves, that these changes are seen. Thus the earth, if it was viewed from the planets, would, without all doubt, shine by the light of its clouds, and the solid body would scarcely appear through the surrounding clouds. Thus also the belts of Jupiter are formed in the clouds of that planet, for they change their position one to another, and the solid body of Jupiter is hardly to be seen through them; and much more must the bodies of comets be hid under their atmospheres, which are both deeper and thicker.

PROPOSITION XL. THEOREM XX.

That the comets move in some of the conic sections, having their foci in the centre of the sun; and by radii drawn to the sun describe areas proportional to the times.

This proposition appears from Cor. 1, Prop. XIII, Book 1, compared with Prop. VIII, XII, and XIII, Book III.

Cor. 1. Hence if comets are revolved in orbits returning into themselves, those orbits will be ellipses; and their periodic times be to the periodic times of the planets in the sesquiplicate proportion of their principal axes. And therefore the comets, which for the most part of their course are higher than the planets, and upon that account describe orbits with greater axes, will require a longer time to finish their revolutions. Thus if the axis of a comet's orbit was four times greater than the axis of the orbit of Saturn, the time of the revolution of the comet would be to the time of the revolution of Saturn, that is, to 30 years, as $4\sqrt{4}$ (or 8) to 1, and would therefore be 240 years.

Cor. 2. But their orbits will be so near to parabolas, that parabolas may be used for them without sensible error.

Cor. 3. And, therefore, by Cor. 7, Prop. XVI, Book 1, the velocity of every comet will always be to the velocity of any planet, supposed to be revolved at the same distance in a circle about the sun, nearly in the subduplicate proportion of double the distance of the planet from the centre of the sun to the distance of the comet from the sun's centre, very nearly. Let us suppose the radius of the *orbis magnus*, or the greatest semidiameter of the ellipsis which the earth describes, to consist of 100000000 parts; and then the earth by its mean diurnal motion will describe 1720212 of those parts, and 71675½ by its horary motion. And therefore the comet, at the same mean distance of the earth from the sun, with a velocity which is to the velocity of the earth as $\sqrt{2}$ to 1, would by its diurnal motion describe 2432747 parts, and 101364½ parts by its horary motion. But at greater or less distances both the diurnal and horary motion will be to this diurnal and horary motion in the reciprocal subduplicate proportion of the distances, and is therefore given.

Cor. 4. Wherefore if the *latus rectum* of the parabola is quadruple of the radius of the *orbis magnus*, and the square of that radius is supposed to consist of 100000000 parts, the area which the comet will daily describe by a radius drawn to the sun will be 1216373½ parts, and the horary area will be 50682¼ parts. But, if the *latus rectum* is greater or less in any proportion, the diurnal and horary area will be less or greater in the subduplicate of the same proportion reciprocally.

LEMMA V.

To find a curve line of the parabolic kind which shall pass through any given number of points.

Let those points be A, B, C, D, E, F, &c., and from the same to any right line HN, given in position, let fall as many perpendiculars AH, BI, CK, DL, EM, FN, &c.

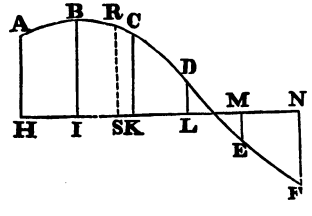
b 2*b* 3*b* 4*b* 5*b*

c 2*c* 3*c* 4*c*

d 2*d* 3*d*

e 2*e*

f



Case 1. If HI, IK, KL, &c., the intervals of the points H, I, K, L, M N, &c., are equal, take *b*, 2*b*, 3*b*, 4*b*, 5*b*, &c., the first differences of the perpendiculars AH, BI, CK, &c.; their second differences *c*, 2*c*, 3*c*, 4*c*, &c.; their third, *d*, 2*d*, 3*d*, &c., that is to say, so as AH — BI may be = *b*, BI

— CK = $2b$, CK — DL = $3b$, DL + EM = $4b$, — EM + FN = $5b$, &c.; then $b - 2b = c$, &c., and so on to the last difference, which is here f. Then, erecting any perpendicular RS, which may be considered as an ordinate of the curve required, in order to find the length of this ordinate, suppose the intervals HI, IK, KL, LM, &c., to be units, and let AH = a, — HS = p, $\frac{1}{2}p$ into — IS = q, $\frac{1}{3}q$ into + SK = r, $\frac{1}{4}r$ into + SL = s, $\frac{1}{5}s$ into + SM = t; proceeding, to wit, to ME, the last perpendicular but one, and prefixing negative signs before the terms HS, IS, &c., which lie from S towards A; and affirmative signs before the terms SK, SL, &c., which lie on the other side of the point S; and, observing well the signs, RS will be = $a + bp + cq + dr + es + ft$, + &c.

Case 2. But if HI, IK, &c., the intervals of the points H, I, K, L, &c., are unequal, take b, $2b$, $3b$, $4b$, $5b$, &c., the first differences of the perpendiculars AH, BI, CK, &c., divided by the intervals between those perpendiculars; c, $2c$, $3c$, $4c$, &c., their second differences, divided by the intervals between every two; d, $2d$, $3d$, &c., their third differences, divided by the intervals between every three; e, $2e$, &c., their fourth differences, divided by the intervals between every four; and so forth; that is, in such manner, that b may be = $\frac{AH - BI}{HI}$, $2b = \frac{BI - CK}{IK}$, $3b = \frac{CK - DL}{KL}$, &c., then $c = \frac{b - 2b}{HK}$, $2c = \frac{2b - 3b}{IL}$, $3c = \frac{3b - 4b}{KM}$, &c., then $d = \frac{c - 2c}{HL}$, $2d = \frac{2c - 3c}{IM}$, &c. And those differences being found, let AH be = a, — HS = p, p into — IS = q, q into + SK = r, r into + SL = s, s into + SM = t; proceeding, to wit, to ME, the last perpendicular but one: and the ordinate RS will be = $a + bp + cq + dr + es + ft$, + &c.

Cor. Hence the areas of all curves may be nearly found; for if some number of points of the curve to be squared are found, and a parabola be supposed to be drawn through those points, the area of this parabola will be nearly the same with the area of the curvilinear figure proposed to be squared: but the parabola can be always squared geometrically by methods vulgarly known.

LEMMA VI.

Certain observed places of a comet being given, to find the place of the same to any intermediate given time.

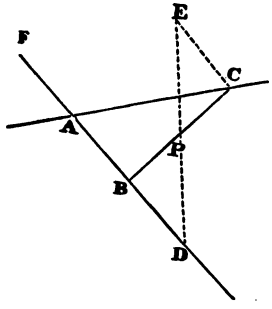
Let HI, IK, KL, LM (in the preceding Fig.), represent the times between the observations; HA, IB, KC, LD, ME, five observed longitudes of the comet; and HS the given time between the first observation and the longitude required. Then if a regular curve ABCDE is supposed to be drawn through the points A, B, C, D, E, and the ordinate RS is found out by the preceding lemma, RS will be the longitude required.

After the same method, from five observed latitudes, we may find the latitude to a given time.

If the differences of the observed longitudes are small, suppose of 4 or 5 degrees, three or four observations will be sufficient to find a new longitude and latitude; but if the differences are greater, as of 10 or 20 degrees, five observations ought to be used.

LEMMA VII.

Through a given point P *to draw a right line* BC, *whose parts* PB, PC, *cut off by two right lines* AB, AC, *given in position, may be one to the other in a given proportion.*

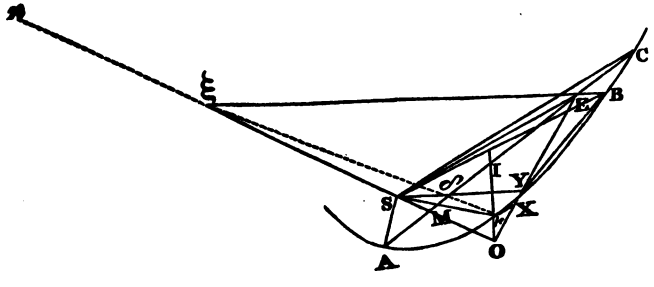


From the given point P suppose any right line PD to be drawn to either of the right lines given, as AB; and produce the same towards AC, the other given right line, as far as E, so as PE may be to PD in the given proportion. Let EC be parallel to AD. Draw CPB, and PC will be to PB as PE to PD. Q.E.F.

LEMMA VIII.

Let ABC *be a parabola, having its focus in* S. *By the chord* AC *bisected in* I *cut off the segment* ABCI, *whose diameter is* Iμ *and vertex* μ. *In* Iμ *produced take* μO *equal to one half of* Iμ. *Join* OS, *and produce it to* ξ, *so as* Sξ *may be equal to* 2SO. *Now, supposing a comet to revolve in the arc* CBA, *draw* ξB, *cutting* AC *in* E; *I say, the point* E *will cut off from the chord* AC *the segment* AE, *nearly proportional to the time.*

For if we join EO, cutting the parabolic arc ABC in Y, and draw μX touching the same arc in the vertex μ, and meeting EO in X, the curvilinear area AEXμA will be to the curvilinear area ACYμA as AE to AC; and, therefore, since the triangle ASE is to the triangle ASC in the same proportion, the whole area ASEXμA will be to the whole area ASCYμA as



AE to AC. But, because ξO is to SO as 3 to 1, and EO to XC in the same proportion, SX will be parallel to EB; and, therefore, joining BX, the triangle SEB will be equal to the triangle XEB. Wherefore if to the area ASEXμA we add the triangle EXB, and from the sum subduct the triangle SEB, there will remain the area ASBXμA, equal to the area ASEXμA, and therefore in proportion to the area ASCYμA as AE to AC. But the area ASBYμA is nearly equal to the area ASBXμA; and this area ASBYμA is to the area ASCYμA as the time of description of the arc AB to the time of description of the whole arc AC; and, therefore, AE is to AC nearly in the proportion of the times. Q.E.D.

Cor. When the point B falls upon the vertex μ of the parabola, AE is to AC accurately in the proportion of the times.

SCHOLIUM.

If we join $\mu\xi$ cutting AC in δ, and in it take ξn in proportion to μB as 27MI to 16Mμ, and draw Bn, this Bn will cut the chord AC, in the proportion of the times, more accurately than before; but the point n is to be taken beyond or on this side the point ξ, according as the point B is more or less distant from the principal vertex of the parabola than the point μ.

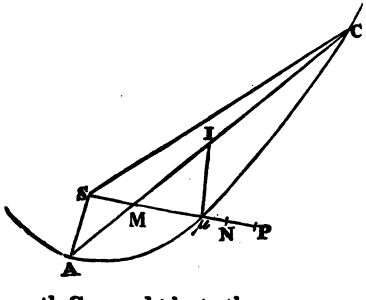
LEMMA IX.

The right lines Iμ *and* μM, *and the length* $\frac{AI^2}{4S\mu}$, *are equal among themselves.*

For 4Sμ is the latus rectum of the parabola belonging to the vertex μ.

LEMMA X.

Produce Sμ *to* N *and* P, *so as* μN *may be one third of* μI, *and* SP *may be to* SN *as* SN *to* Sμ; *and in the time that a comet would describe the arc* AμC, *if it was supposed to move always forwards with the velocity which it hath in a height equal to* SP, *it would describe a length equal to the chord* AC.



For if the comet with the velocity which it hath in μ was in the said time supposed to move uniformly forward in the right line which touches the parabola in μ, the area which it would describe by a radius drawn to the point S would be equal to the parabolic area ASCμA; and therefore the space contained under the length described in the tangent and the length Sμ would be to the space contained under the lengths AC and SM as the

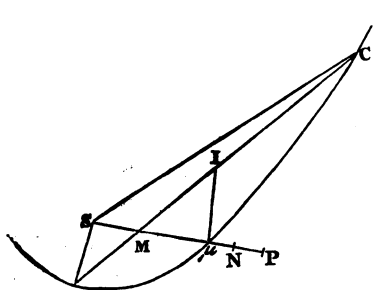
area ASCμA to the triangle ASC, that is, as SN to SM. Wherefore AC is to the length described in the tangent as Sμ to SN. But since the velocity of the comet in the height SP (by Cor. 6, Prop. XVI., Book I) is to the velocity of the same in the height Sμ in the reciprocal subduplicate proportion of SP to Sμ, that is, in the proportion of Sμ to SN, the length described with this velocity will be to the length in the same time described in the tangent as Sμ to SN. Wherefore since AC, and the length described with this new velocity, are in the same proportion to the length described in the tangent, they must be equal betwixt themselves. Q.E.D.

Cor. Therefore a comet, with that velocity which it hath in the height Sμ + $\frac{2}{3}$Iμ, would in the same time describe the chord AC nearly.

LEMMA XI.

If a comet void of all motion was let fall from the height SN, *or* Sμ + $\frac{1}{2}$Iμ, *towards the sun, and was still impelled to the sun by the same force uniformly continued by which it was impelled at first, the same, in one half of that time in which it might describe the arc* AC *in its own orbit, would in descending describe a space equal to the length* Iμ.

For in the same time that the comet would require to describe the parabolic arc AC, it would (by the last Lemma), with that velocity which it hath in the height SP, describe the chord AC; and, therefore (by Cor. 7, Prop. XVI, Book 1), if it was in the same time supposed to revolve by the force of its own gravity in a circle whose semi-diameter was SP, it would describe an arc of that circle, the length of which would be to the chord of the parabolic arc AC in the subduplicate proportion of 1 to 2. Wherefore if with that weight, which in the height SP it hath towards the sun, it should fall from that height towards the sun, it would (by Cor. 9, Prop. XVI, Book 1) in half the said time describe a space equal to the square of half the said chord applied to quadruple the height SP, that is, it would describe the space $\frac{AI^2}{4SP}$. But since the weight of the comet towards the sun in the height SN is to the weight of the same towards the sun in the height SP as SP to Sμ, the comet, by the weight which it hath in the height SN, in falling from that height towards the sun, would in the same time describe the space $\frac{AI^2}{4S\mu}$; that is, a space equal to the length Iμ or μM. Q.E.D

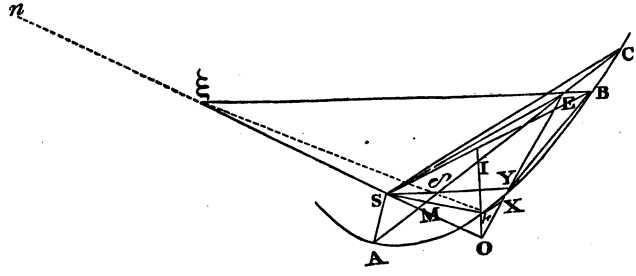


PROPOSITION XLI. PROBLEM XXI.

From three observations given to determine the orbit of a comet moving in a parabola.

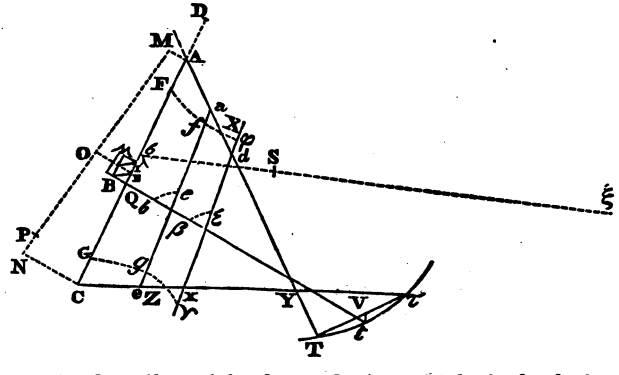
This being a Problem of very great difficulty, I tried many methods of resolving it; and several of these Problems, the composition whereof I have given in the first Book, tended to this purpose. But afterwards I contrived the following solution, which is something more simple.

Select three observations distant one from another by intervals of time nearly equal; but let that interval of time in which the comet moves more slowly be somewhat greater than the other; so, to wit, that the difference of the times may be to the sum of the times as the sum of the



times to about 600 days; or that the point E may fall upon M nearly, and may err therefrom rather towards I than towards A. If such direct observations are not at hand, a new place of the comet must be found, by Lem. VI.

Let S represent the sun; T, *t*, τ, three places of the earth in the *orbis magnus;* TA, *t*B, τC, three observed longitudes of the comet; V the time between the first observation and the second; W the time between the second and the third; X the length which in the whole time V + W



the comet might describe with that velocity which it hath in the mean distance of the earth from the sun, which length is to be found by Cor. 3,

Prop. XL, Book III; and tV a perpendicular upon the chord Tτ. In the mean observed longitude tB take at pleasure the point B, for the place of the comet in the plane of the ecliptic; and from thence, towards the sun S, draw the line BE, which may be to the perpendicular tV as the content under SB and St^2 to the cube of the hypothenuse of the right angled triangle, whose sides are SB, and the tangent of the latitude of the comet in the second observation to the radius tB. And through the point E (by Lemma VII) draw the right line AEC, whose parts AE and EC, terminating in the right lines TA and τC, may be one to the other as the times V and W: then A and C will be nearly the places of the comet in the plane of the ecliptic in the first and third observations, if B was its place rightly assumed in the second.

Upon AC, bisected in I, erect the perpendicular Ii. Through B draw the obscure line Bi parallel to AC. Join the obscure line Si, cutting AC in λ, and complete the parallelogram iI $\lambda\mu$. Take Iσ equal to 3Iλ; and through the sun S draw the obscure line $\sigma\xi$ equal to 3Sσ + 3 $i\lambda$. Then, cancelling the letters A, E, C, I, from the point B towards the point ξ, draw the new obscure line BE, which may be to the former BE in the duplicate proportion of the distance BS to the quantity Sμ + $\frac{1}{3}$ $i\lambda$. And through the point E draw again the right line AEC by the same rule as before; that is, so as its parts AE and EC may be one to the other as the times V and W between the observations. Thus A and C will be the places of the comet more accurately.

Upon AC, bisected in I, erect the perpendiculars AM, CN, IO, of which AM and CN may be the tangents of the latitudes in the first and third observations, to the radii TA and τC. Join MN, cutting IO in O. Draw the rectangular parallelogram $i\lambda\mu$, as before. In IA produced take ID equal to Sμ + $\frac{2}{3}$ $i\lambda$. Then in MN, towards N, take MP, which may be to the above found length X in the subduplicate proportion of the mean distance of the earth from the sun (or of the semi-diameter of the *orbis magnus*) to the distance OD. If the point P fall upon the point N; A, B, and C, will be three places of the comet, through which its orbit is to be described in the plane of the ecliptic. But if the point P falls not upon the point N, in the right line AC take CG equal to NP, so as the points G and P may lie on the same side of the line NC.

By the same method as the points E, A, C, G, were found from the assumed point B, from other points b and β assumed at pleasure, find out the new points e, a, c, g; and ε, a, κ, γ. Then through G, g, and γ, draw the circumference of a circle G$g\gamma$, cutting the right line τC in Z: and Z will be one place of the comet in the plane of the ecliptic. And in AC, ac, $a\kappa$, taking AF, af, $a\phi$, equal respectively to CG, cg, $\kappa\gamma$; through the points F, f, and ϕ, draw the circumference of a circle F$f\phi$, cutting the right line AT in X; and the point X will be another place of the comet in the plane of

the ecliptic. And at the points X and Z, erecting the tangents of the latitudes of the comet to the radii TX and τZ, two places of the comet in its own orbit will be determined. Lastly, if (by Prop. XIX., Book 1) to the focus S a parabola is described passing through those two places, this parabola will be the orbit of the comet. Q.E.I.

The demonstration of this construction follows from the preceding Lemmas, because the right line AC is cut in E in the proportion of the times, by Lem. VII., as it ought to be, by Lem. VIII.; and BE, by Lem. XI., is a portion of the right line BS or Bς in the plane of the ecliptic, intercepted between the arc ABC and the chord AEC; and MP (by Cor. Lem. X.) is the length of the chord of that arc, which the comet should describe in its proper orbit between the first and third observation, and therefore is equal to MN, providing B is a true place of the comet in the plane of the ecliptic.

But it will be convenient to assume the points B, *b*, *β*, not at random, but nearly true. If the angle AQ*t*, at which the projection of the orbit in the plane of the ecliptic cuts the right line *t*B, is rudely known, at that angle with B*t* draw the obscure line AC, which may be to $\frac{4}{3}$Tτ in the subduplicate proportion of SQ to S*t*; and, drawing the right line SEB so as its part EB may be equal to the length V*t*, the point B will be determined, which we are to use for the first time. Then, cancelling the right line AC, and drawing anew AC according to the preceding construction, and, moreover, finding the length MP, in *t*B take the point *b*, by this rule, that, if TA and τC intersect each other in Y, the distance Y*b* may be to the distance YB in a proportion compounded of the proportion of MP to MN, and the subduplicate proportion of SB to S*b*. And by the same method you may find the third point *β*, if you please to repeat the operation the third time; but if this method is followed, two operations generally will be sufficient; for if the distance B*b* happens to be very small, after the points F, *f*, and G, *g*, are found, draw the right lines F*f* and G*g*, and they will cut TA and τC in the points required, X and Z.

EXAMPLE.

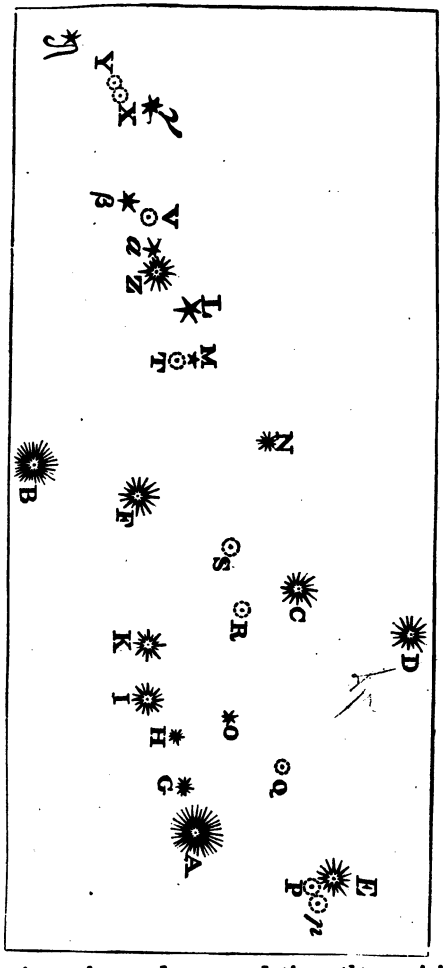
Let the comet of the year 1680 be proposed. The following table shews the motion thereof, as observed by *Flamsted*, and calculated afterwards by him from his observations, and corrected by Dr. *Halley* from the same observations.

	Time.		Sun's	Comet's	
	Appar.	True.	Longitude.	Longitude.	Lat. N.
	h. ′	h. ′ ″	° ′ ″	° ′ ″	° ′ ″
1680, *Dec.* 12	4.46	4.46. 0	♑ 1.51.23	♑ 6.32.30	8.28. 0
21	6.32½	6.36.59	11.06 44	♒ 5.08.12	21.42.13
24	6.12	6.17.52	14.09.26	18.49.23	25.23. 5
26	5.14	5 20.44	16.09.22	28.24.13	27.00.52
29	7.55	8.03.02	19.19.43	♓ 13.10.41	28.09.58
30	8.02	8.10.26	20.21.09	17.38.20	28.11.53
1681, *Jan.* 5	5.51	6.01.38	26.22.18	♈ 8.48.53	26.15. 7
9	6.49	7.00.53	♒ 0.29.02	18.44.04	24.11.56
10	5.54	6.06.10	1.27.43	20.40.50	23.43.52
13	6.56	7.08.55	4.33.20	25.59.48	22.17.28
25	7.44	7.58.42	16.45.36	♉ 9.35. 0	17.56.30
30	8.07	8.21.53	21.49.58	13.19.51	16.42.18
Feb. 2	6.20	6.34.51	24.46.59	15.13 53	16.04. 1
5	6.50	7.04.41	27.49.51	16.59 06	15.27. 3

To these you may add some observations of mine.

	Ap. Time	Comet's	
		Longitude.	Lat. N.
	h. ′	° ′ ″	° ′ ″
1681, *Feb.* 25	8.30	♉ 26.18.35	12.46.46
27	8.15	27.04.30	12.36.12
Mar. 1	11. 0	27.52.42	12.23.40
2	8. 0	28.12.48	12.19.38
5	11.30	29.18. 0	12.03.16
7	9.30	♊ 0. 4. 0	11.57. 0
9	8.30	0.43. 4	11.45.52

These observations were made by a telescope of 7 feet, with a micrometer and threads placed in the focus of the telescope; by which instruments we determined the positions both of the fixed stars among themselves, and of the comet in respect of the fixed stars. Let A represent the star of the fourth magnitude in the left heel of *Perseus* (*Bayer's o*), B the following star of the third magnitude in the left foot (*Bayer's* ζ), C a star of the sixth magnitude (*Bayer's n*) in the heel of the same foot, and D, E, F, G, H, I, K, L, M, N, O, Z, α, β, γ, δ, other smaller stars in the same foot; and let *p*, P, Q, R, S, T, V, X, represent the places of the comet in the observations above set down; and, reckoning the distance AB of $80\frac{7}{12}$ parts, AC was $52\frac{1}{4}$ of those parts; BC, $58\frac{5}{6}$; AD, $57\frac{5}{12}$; BD, $82\frac{6}{11}$; CD, $23\frac{3}{8}$; AE, $29\frac{4}{7}$; CE, $57\frac{1}{2}$; DE, $49\frac{11}{12}$; AI, $27\frac{7}{12}$; BI, $52\frac{1}{6}$; CI, $36\frac{7}{12}$; DI, $53\frac{5}{11}$; AK, $38\frac{2}{3}$; BK, 43; CK, $31\frac{5}{9}$; FK, 29; FB, 23; FC, $36\frac{1}{4}$; AH, $18\frac{4}{7}$; DH, $50\frac{1}{8}$; BN, $46\frac{5}{12}$; CN, $31\frac{1}{3}$; BL, $45\frac{5}{12}$; NL, $31\frac{5}{7}$. HO was to HI as 7 to 6, and, produced, did pass between the stars D and E, so as the distance of the star D from this right line was $\frac{1}{6}$CD. LM was to LN as 2 to 9, and, produced, did pass through the star H. Thus were the positions of the fixed stars determined in respect of one another.



Mr. *Pound* has since observed a second time the positions of these fixed stars amongst themselves, and collected their longitudes and latitudes according to the following table.

The fixed stars.	Their Longitudes	Latitude North.	The fixed stars.	Their Longitudes	Latitude North.
	° ′ ″	° ′ ″		° ′ ″	° ′ ″
A	♉ 26.41.50	12. 8.36	L	♉ 29.33.34	12. 7.48
B	28.40.23	11.17.54	M	29.18.54	12. 7.20
C	27.58.30	12.40.25	N	28.48.29	12.31. 9
E	26.27.17	12.52. 7	Z	29.44.48	11.57.13
F	28.28.37	11.52.22	α	29.52. 3	11.55.48
G	26.56. 8	12. 4.58	β	♊ 0. 8.23	11.48.56
H	27.11.45	12. 2. 1	γ	0.40.10	11.55.18
I	27.25. 2	11.53.11	δ	1. 3.20	11.30.42
K	27.42. 7	11.53.26			

The positions of the comet to these fixed stars were observed to be as follow:

Friday, *February* 25, O.S. at $8\frac{1}{2}^h$. P. M. the distance of the comet in p from the star E was less than $\frac{3}{13}$AE, and greater than $\frac{1}{5}$AE, and therefore nearly equal to $\frac{3}{14}$AE; and the angle ApE was a little obtuse, but almost right. For from A, letting fall a perpendicular on pE, the distance of the comet from that perpendicular was $\frac{1}{5}p$E.

The same night, at $9\frac{1}{2}^h$., the distance of the comet in P from the star E was greater than $\frac{1}{4\frac{1}{2}}$ AE, and less than $\frac{1}{5\frac{1}{4}}$ AE, and therefore nearly equal to $\frac{1}{4\frac{7}{8}}$ of AE, or $\frac{8}{39}$AE. But the distance of the comet from the perpendicular let fall from the star A upon the right line PE was $\frac{4}{5}$PE.

Sunday, *February* 27, $8\frac{1}{4}^h$. P. M. the distance of the comet in Q from the star O was equal to the distance of the stars O and H; and the right line QO produced passed between the stars K and B. I could not, by reason of intervening clouds, determine the position of the star to greater accuracy.

Tuesday, *March* 1, 11^h. P. M. the comet in R lay exactly in a line between the stars K and C, so as the part CR of the right line CRK was a little greater than $\frac{1}{3}$CK, and a little less than $\frac{1}{3}$CK + $\frac{1}{8}$CR, and therefore = $\frac{1}{3}$CK + $\frac{1}{16}$ CR, or $\frac{1}{4}\frac{6}{5}$CK.

Wednesday, *March* 2, 8^h. P. M. the distance of the comet in S from the star C was nearly $\frac{4}{6}$FC; the distance of the star F from the right line CS produced was $\frac{1}{2\frac{1}{4}}$FC; and the distance of the star B from the same right line was five times greater than the distance of the star F; and the right line NS produced passed between the stars H and I five or six times nearer to the star H than to the star I.

Saturday, *March* 5, $11\frac{1}{2}^h$. P. M. when the comet was in T, the right line MT was equal to $\frac{1}{2}$ML, and the right line LT produced passed between B and F four or five times nearer to F than to B, cutting off from BF a fifth or sixth part thereof towards F: and MT produced passed on the outside of the space BF towards the star B four times nearer to the star B than to the star F. M was a very small star, scarcely to be seen by the telescope; but the star L was greater, and of about the eighth magnitude.

Monday, *March* 7, $9\frac{1}{2}^h$. P. M. the comet being in V, the right line Vα produced did pass between B and F, cutting off, from BF towards F, $\frac{1}{16}$ of BF, and was to the right line Vβ as 5 to 4. And the distance of the comet from the right line $\alpha\beta$ was $\frac{1}{2}$Vβ.

Wednesday, *March* 9, $8\frac{1}{2}^h$. P. M. the comet being in X, the right line γX was equal to $\frac{1}{4}\gamma\delta$; and the perpendicular let fall from the star δ upon the right γX was $\frac{2}{5}$ of $\gamma\delta$.

The same night, at 12^h. the comet being in Y, the right line γY was

equal to $\frac{1}{3}$ of $\gamma\delta$, or a little less, as perhaps $\frac{5}{7}$ of $\gamma\delta$; and a perpendicular let fall from the star δ on the right line γY was equal to about $\frac{1}{6}$ or $\frac{1}{7}$ $\gamma\delta$. But the comet being then extremely near the horizon, was scarcely discernible, and therefore its place could not be determined with that certainty as in the foregoing observations.

From these observations, by constructions of figures and calculations, I deduced the longitudes and latitudes of the comet; and Mr. *Pound*, by correcting the places of the fixed stars, hath determined more correctly the places of the comet, which correct places are set down above. Though my micrometer was none of the best, yet the errors in longitude and latitude (as derived from my observations) scarcely exceed one minute. The comet (according to my observations), about the end of its motion, began to decline sensibly towards the north, from the parallel which it described about the end of *February*.

Now, in order to determine the orbit of the comet out of the observations above described, I selected those three which *Flamsted* made, *Dec.* 21, *Jan.* 5, and *Jan.* 25; from which I found S*t* of 9842,1 parts, and V*t* of 455, such as the semi-diameter of the *orbis magnus* contains 10000. Then for the first observation, assuming *t*B of 5657 of those parts, I found SB 9747, BE for the first time 412, Sμ 9503, *i*λ 413, BE for the second time 421, OD 10186, X 8528,4, PM 8450, MN 8475, NP 25; from whence, by the second operation, I collected the distance *tb* 5640; and by this operation I at last deduced the distances TX 4775 and τZ 11322. From which, limiting the orbit, I found its descending node in ♋, and ascending node in ♈ 1° 53′; the inclination of its plane to the plane of the ecliptic 61° $20\frac{1}{3}$, the vertex thereof (or the perihelion of the comet) distant from the node 8° 38′, and in ♌ 27° 43′, with latitude 7° 34′ south; its *latus rectum* 236,8; and the diurnal area described by a radius drawn to the sun 93585, supposing the square of the semi-diameter of the *orbis magnus* 100000000; that the comet in this orbit moved directly according to the order of the signs, and on *Dec.* 8^{d}. 00^{h}. 04′ P. M was in the vertex or perihelion of its orbit. All which I determined by scale and compass, and the chords of angles, taken from the table of natural sines, in a pretty large figure, in which, to wit, the radius of the *orbis magnus* (consisting of 10000 parts) was equal to $16\frac{1}{3}$ inches of an *English* foot.

Lastly, in order to discover whether the comet did truly move in the orbit so determined, I investigated its places in this orbit partly by arithmetical operations, and partly by scale and compass, to the times of some of the observations, as may be seen in the following table:—

The Comet's							
	Dist. from sun.	Longitude computed.	Latitud. computed.	Longitude observed.	Latitude observed	Dif Lo.	Dif. Lat.
Dec. 12	2792	♍ 6°.32′	8°.18½	♍ 6° 31½	8°.26	+1	− 7⅓
29	8403	♓ 13 .13⅔	28. 00	♓ 13 .11¾	28 .10 1/12	+2	−10 1/12
Feb. 5	16669	♉ 17 .00	15. 29⅔	♉ 16 .59⅞	15 .27⅗	+0	+ 2¼
Mar. 5	21737	29 .19¾	12. 4	29 .20⅙	12 . 3½	−1	+ ½

But afterwards Dr. *Halley* did determine the orbit to a greater accuracy by an arithmetical calculus than could be done by linear descriptions; and, retaining the place of the nodes in ♋ and ♍ 1° 53′, and the inclination of the plane of the orbit to the ecliptic 61° 20⅓′, as well as the time of the comet's being in perihelio, *Dec.* 8ᵈ. 00ʰ. 04′, he found the distance of the perihelion from the ascending node measured in the comet's orbit 9° 20′, and the *latus rectum* of the parabola 2430 parts, supposing the mean distance of the sun from the earth to be 100000 parts; and from these *data*, by an accurate arithmetical calculus, he computed the places of the comet to the times of the observations as follows:—

	True time.	Dist from the sun.	The Comet's Longitude computed.	Latitude computed.	Errors in Long.	Errors in Lat.
	d h. ′ ″		° ′ ″	° ′ ″	′ ″	′ ″
Dec.	12. 4.46.	28028	♍ 6.29.25	8.26. 0 bor.	— 3. 5	— 2. 0
	21. 6.37.	61076	♒ 5. 6.30	21.43.20	— 1.42	+ 1. 7
	24. 6.18.	70008	18.48.20	25.22.40	— 1. 3	— 0.25
	26. 5.20.	75576	28.22.45	27. 1.36	— 1.28	+ 0.44
	29. 8. 3.	84021	♓ 13.12.40	28.10.10	+ 1.59	+ 0.12
	30. 8.10.	86661	17.40. 5	28.11.20	+ 1.45	— 0.33
Jan.	5. 6. 1½	101440	♈ 8.49.49	26.15.15	+ 0.56	+ 0. 8
	9. 7. 0.	110959	18.44.36	24.12.54	+ 0.32	+ 0.58
	10. 6. 6.	113162	20.41. 0	23.44.10	+ 0.10	+ 0.18
	13. 7. 9.	120000	26. 0.21	22.17.30	+ 0.33	+ 0. 2
	25. 7.59.	145370	♉ 9.33.40	17.57.55	— 1.20	+ 1.25
	30. 8.22.	155303	13.17.41	16.42. 7	— 2.10	— 0.11
Feb.	2. 6.35.	160951	15.11.11	16. 4.15	— 2.42	+ 0.14
	5. 7. 4½	166686	16.58.55	15.29.13	— 0.41	+ 2. 0
	25. 8.41.	202570	26.15.46	12.48. 0	— 2.49	+ 1.10
Mar.	5.11.39.	216205	29.18.35	12. 5.40	+ 0.35	+ 2.14

This comet also appeared in the *November* before, and at *Coburg*, in *Saxony*, was observed by Mr. *Gottfried Kirch*, on the 4th of that month, on the 6th and 11th O. S.; from its positions to the nearest fixed stars observed with sufficient accuracy, sometimes with a two feet, and sometimes with a ten feet telescope; from the difference of longitudes of *Coburg* and *London*, 11°; and from the places of the fixed stars observed by Mr. *Pound*, Dr. *Halley* has determined the places of the comet as follows:—

Nov. 3, 17^h. 2′, apparent time at *London*, the comet was in ☊ 29 deg. 51′, with 1 deg. 17′ 45″ latitude north.

November 5, 15^h. 58′ the comet was in ♍ 3° 23′, with 1° 6′ nortl. lat.

November 10, 16^h. 31′, the comet was equally distant from two stars in ☊, which are σ and τ in *Bayer*; but it had not quite touched the right line that joins them, but was very little distant from it. In *Flamsted's* catalogue this star σ was then in ♍ 14° 15′, with 1 deg. 41′ lat. north nearly, and τ in ♍ 17° 3½′ with 0 deg. 34′ lat. south; and the middle point between those stars was ♍ 15° 39¼′, with 0° 33½′ lat. north. Let the distance of the comet from that right line be about 10′ or 12′; and the difference of the longitude of the comet and that middle point will be 7′; and the difference of the latitude nearly 7½′; and thence it follows that the comet was in ♍ 15° 32′, with about 26′ lat. north.

The first observation from the position of the comet with respect to certain small fixed stars had all the exactness that could be desired; the second also was accurate enough. In the third observation, which was the least accurate, there might be an error of 6 or 7 minutes, but hardly greater. The longitude of the comet, as found in the first and most accurate observation, being computed in the aforesaid parabolic orbit, comes out ☊ 29° 30′ 22″, its latitude north 1° 25′ 7″, and its distance from the sun 115546.

Moreover, Dr. *Halley*, observing that a remarkable comet had appeared four times at equal intervals of 575 years (that is, in the month of *September* after *Julius Cæsar* was killed; *An. Chr.* 531, in the consulate of *Lampadius* and *Orestes*; *An. Chr.* 1106, in the month of *February*; and at the end of the year 1680; and that with a long and remarkable tail, except when it was seen after *Cæsar's* death, at which time, by reason of the inconvenient situation of the earth, the tail was not so conspicuous), set himself to find out an elliptic orbit whose greater axis should be 1382957 parts, the mean distance of the earth from the sun containing 10000 such; in which orbit a comet might revolve in 575 years; and, placing the ascending node in ♋ 2° 2′, the inclination of the plane of the orbit to the plane of the ecliptic in an angle of 61° 6′ 48″, the perihelion of the comet in this plane in ♑ 22° 44′ 25″, the equal time of the perihelion *December* 7^d. 23^h. 9′, the distance of the perihelion from the ascending node in the plane of the ecliptic 9° 17′ 35″, and its conjugate axis 18481,2, he computed the motions of the comet in this elliptic orbit. The places of the comet, as deduced from the observations, and as arising from computation made in this orbit, may be seen in the following table.

True time.	Longitude observed.	Latit'de North obs.	Longitude comp.	Latitude computed.	Errors in Long.	Errors in Lat.
d. h. ′	☌ ′ ″	° ′ ″	☌ ′ ″	° ′ ″	′ ″	′ ″
Nov. 3.16.47	♌ 29.51. 0	1.17.45	♌ 29.51.22	1.17.32 N	+ 0.22	— 0.13
5.15.37	♍ 3.23. 0	1. 6. 0	♍ 3.24.32	1. 6. 9	+ 1.32	+ 0. 9
10.16.18	15.32. 0	0.27. 0	15.33. 2	0 25. 7	+ 1. 2	— 1.53
16.17.00			♎ 8.16.45	0.53. 7 S		
18.21.34			18.52.15	1.26.54		
20.17. 0			28.10.36	1.53.35		
23.17. 5			♏ 13.22.42	2.29. 0		
Dec. 12. 4.46	♑ 6.32.30	8.28. 0	♑ 6.31.20	8.29. 6 N	— 1.10	+ 1. 6
21. 6.37	♒ 5. 8.12	21.42.13	♒ 5. 6.14	21.44.42	— 1.58	+ 2.29
24. 6.18	18.49.23	25.23. 5	18.47.30	25.23 35	— 1.53	+ 0.30
26. 5.21	28.24.13	27. 0.52	28.21.42	27. 2. 1	— 2.31	+ 1. 9
29. 8. 3	♓ 13.10.41	28. 9.58	♓ 13.11.14	28.10.38	+ 0.33	+ 0.40
30. 8.10	17.38. 0	28.11.53	17.38 27	28.11.37	+ 0. 7	— 0.16
Jan. 5. 6. 1½	♈ 8.48.53	26.15. 7	♈ 8.48.51	26.14.57	— 0. 2	— 0.10
9. 7. 1	18.44. 4	24.11.56	18.43.51	24.12.17	— 0.13	+ 0.21
10. 6. 6	20.40.50	23.43.32	20.40.23	23.43.25	— 0.27	— 0. 7
13. 7. 9	25.59.48	22 17.28	26. 0. 8	22.16 32	+ 0.20	— 0.56
25. 7.59	♉ 9.35. 0	17.56.30	♉ 9.34.11	17.56. 6	— 0.49	— 0.24
30. 8.22	13.19.51	16.42.18	13.18.28	16.40. 5	— 1.23	— 2.13
Feb. 2. 6.35	15.13.53	16. 4. 1	15.11.59	16. 2.17	— 1.54	— 1.54
5. 7. 4½	16.59. 6	15.27. 3	16.59.17	15.27. 0	+ 0.11	— 0. 3
25. 8.41	26.18.35	12.46.46	26.16.59	12.45.22	— 1.36	— 1.24
Mar. 1.11.10	27.52.42	12.23.40	27.51.47	12.22.28	— 0.55	— 1.12
5.11.39	29.18. 0	12. 3.16	29 20.11	12. 2 50	+ 2.11	— 0.26
9. 8.38	♊ 0.43. 4	11.45.52	♊ 0.42.43	11.45.35	— 0.21	— 0.17

The observations of this comet from the beginning to the end agree as perfectly with the motion of the comet in the orbit just now described as the motions of the planets do with the theories from whence they are calculated; and by this agreement plainly evince that it was one and the same comet that appeared all that time, and also that the orbit of that comet is here rightly defined.

In the foregoing table we have omitted the observations of *Nov.* 16, 18, 20, and 23, as not sufficiently accurate, for at those times several persons had observed the comet. *Nov.* 17, O. S. *Ponthæus* and his companions, at 6^h. in the morning at *Rome* (that is, 5^h. 10′ at *London*), by threads directed to the fixed stars, observed the comet in ♎ 8° 30′, with latitude 0° 40′ south. Their observations may be seen in a treatise which *Ponthæus* published concerning this comet. *Cellius*, who was present, and communicated his observations in a letter to *Cassini*, saw the comet at the same hour in ♎ 8° 30′, with latitude 0° 30′ south. It was likewise seen by *Galletius* at the same hour at *Avignon* (that is, at 5^h. 42′ morning at *London*) in ♎ 8° without latitude. But by the theory the comet was at that time in ♎ 8° 16′ 45″, and its latitude was 0° 53′ 7″ south.

Nov. 18, at 6^h. 30′ in the morning at *Rome* (that is, at 5^h. 40′ at *London*), *Ponthæus* observed the comet in ♎ 13° 30′, with latitude 1° 20

south; and *Cellius* in ♎ 13° 30′, with latitude 1° 00′ south. But at 5ʰ. 30′ in the morning at *Avignon*, *Galletius* saw it in ♎ 13° 00′, with latitude 1° 00′ south. In the University of *La Fleche*, in *France*, at 5ʰ. in the morning (that is, at 5ʰ. 9′ at *London*), it was seen by *P. Ango*, in the middle between two small stars, one of which is the middle of the three which lie in a right line in the southern hand of Virgo, *Bayers* ψ; and the other is the outmost of the wing, *Bayer's* θ. Whence the comet was then in ♎ 12° 46′ with latitude 50′ south. And I was informed by Dr. *Halley*, that on the same day at *Boston* in *New England*, in the latitude of 42½ deg. at 5ʰ. in the morning (that is, at 9ʰ. 44′ in the morning at *London*), the comet was seen near ♎ 14°, with latitude 1° 30′ south.

Nov. 19, at 4½ʰ. at *Cambridge*, the comet (by the observation of a young man) was distant from *Spica* ♍ about 2° towards the north west. Now the spike was at that time in ♎ 19° 23′ 47″, with latitude 2° 1′ 59″ south. The same day, at 5ʰ. in the morning, at *Boston* in *New England*, the comet was distant from *Spica* ♍ 1°, with the difference of 40′ in latitude. The same day, in the island of *Jamaica*, it was about 1° distant from *Spica* ♍. The same day, Mr. *Arthur Storer*, at the river *Patuxent*, near *Hunting Creek*, in *Maryland*, in the confines of *Virginia*, in lat. 38½°, at 5 in the morning (that is, at 10ʰ. at *London*), saw the comet above *Spica* ♍, and very nearly joined with it, the distance between them being about ¾ of one deg. And from these observations compared, I conclude, that at 9ʰ. 44′ at *London* the comet was in ♎ 18° 50′, with about 1° 25′ latitude south. Now by the theory the comet was at that time in ♎ 18° 52′ 15″, with 1° 26′ 54″ lat. south.

Nov. 20, *Montenari*, professor of astronomy at *Padua*, at 6ʰ. in the morning at *Venice* (that is, 5ʰ. 10′ at *London*), saw the comet in ♎ 23°, with latitude 1° 30′ south. The same day, at *Boston*, it was distant from *Spica* ♍ by about 4° of longitude east, and therefore was in ♎ 23° 24′ nearly.

Nov. 21, *Ponthœus* and his companions, at 7¼ʰ. in the morning, observed the comet in ♎ 27° 50′, with latitude 1° 16′ south; *Cellius*, in ♎ 28°; *P. Ango* at 5ʰ. in the morning, in ♎ 27° 45′; *Montenari* in ♎ 27° 51′. The same day, in the island of *Jamaica*, it was seen near the beginning of ♏, and of about the same latitude with *Spica* ♍, that is, 2° 2′. The same day, at 5ʰ. morning, at *Ballasore*, in the *East Indies* (that is, at 11ʰ. 20′ of the night preceding at *London*), the distance of the comet from *Spica* ♍ was taken 7° 35′ to the east. It was in a right line between the spike and the balance, and therefore was then in ♎ 26° 58′, with about 1° 11′ lat. south; and after 5ʰ. 40′ (that is, at 5ʰ. morning at *London*), it was in ♎ 28° 12′, with 1° 16′ lat. south. Now by the theory the comet was then in ♎ 28° 10′ 36″, with 1° 53′ 35″ lat. south.

Nov. 22, the comet was seen by *Montenari* in ♏ 2° 33′; but at *Boston*

in *New England*, it was found in about ♏ 3°, and with almost the same latitude as before, that is, 1° 30′. The same day, at 5ʰ. morning at *Ballasore*, the comet was observed in ♏ 1° 50′; and therefore at 5ʰ. morning at *London*, the comet was in ♏ 3° 5′ nearly. The same day, at 6½ʰ. in the morning at *London*, Dr. *Hook* observed it in about ♏ 3° 30′, and that in the right line which passeth through *Spica* ♍ and *Cor Leonis*; not, indeed, exactly, but deviating a little from that line towards the north. *Montenari* likewise observed, that this day, and some days after, a right line drawn from the comet through *Spica* passed by the south side of *Cor Leonis* at a very small distance therefrom. The right line through *Cor Leonis* and *Spica* ♍ did cut the ecliptic in ♍ 3° 46′ at an angle of 2° 51′; and if the comet had been in this line and in ♏ 3°, its latitude would have been 2° 26′; but since *Hook* and *Montenari* agree that the comet was at some small distance from this line towards the north, its latitude must have been something less. On the 20th, by the observation of *Montenari*, its latitude was almost the same with that of *Spica* ♍, that is, about 1° 30′. But by the agreement of *Hook*, *Montenari*, and *Ango*, the latitude was continually increasing, and therefore must now, on the 22d, be sensibly greater than 1° 30′; and, taking a mean between the extreme limits but now stated, 2° 26′ and 1° 30′, the latitude will be about 1° 58′. *Hook* and *Montenari* agree that the tail of the comet was directed towards *Spica* ♍, declining a little from that star towards the south according to *Hook*, but towards the north according to *Montenari*; and, therefore, that declination was scarcely sensible; and the tail, lying nearly parallel to the equator, deviated a little from the opposition of the sun towards the north.

Nov. 23, O. S. at 5ʰ. morning, at *Nuremberg* (that is, at 4½ʰ. at *London*), Mr. *Zimmerman* saw the comet in ♏ 8° 8′, with 2° 31′ south lat. its place being collected by taking its distances from fixed stars.

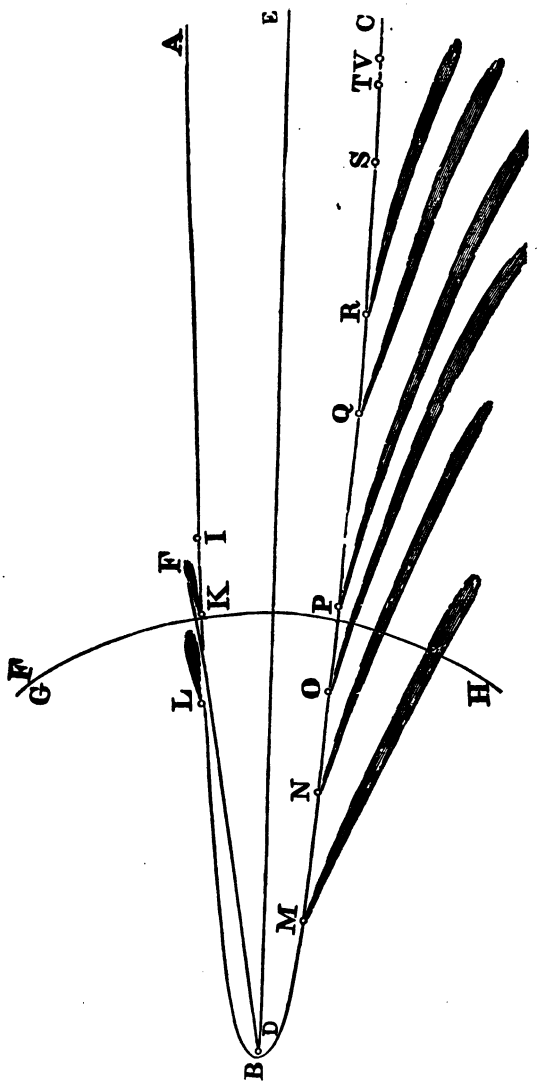
Nov. 24, before sun-rising, the comet was seen by *Montenari* in ♏ 12° 52′ on the north side of the right line through *Cor Leonis* and *Spica* ♍, and therefore its latitude was something less than 2° 38′; and since the latitude, as we said, by the concurring observations of *Montenari*, *Ango*, and *Hook*, was continually increasing, therefore, it was now, on the 24th, something greater than 1° 58′; and, taking the mean quantity, may be reckoned 2° 18′, without any considerable error. *Ponthæus* and *Galletius* will have it that the latitude was now decreasing; and *Cellius*, and the observer in *New England*, that it continued the same, viz., of about 1°, or 1½°. The observations of *Ponthæus* and *Cellius* are more rude, especially those which were made by taking the azimuths and altitudes; as are also the observations of *Galletius*. Those are better which were made by taking the position of the comet to the fixed stars by *Montenari*, *Hook*, *Ango*, and the observer in *New England*, and sometimes by

Ponthæus and *Cellius*. The same day, at 5h. morning, at *Ballasore*, the comet was observed in ♏ 11° 45′; and, therefore, at 5h. morning at *London*, was in ♏ 13° nearly. And, by the theory, the comet was at that time in ♏ 13° 22′ 42″.

Nov. 25, before sunrise, *Montenari* observed the comet in ♏ 17¾ nearly; and *Cellius* observed at the same time that the comet was in a right line between the bright star in the right thigh of Virgo and the southern scale of Libra; and this right line cuts the comet's way in ♏ 18° 36′. And, by the theory, the comet was in ♏ 18⅓° nearly.

From all this it is plain that these observations agree with the theory, so far as they agree with one another; and by this agreement it is made clear that it was one and the same comet that appeared all the time from *Nov.* 4 to *Mar.* 9. The path of this comet did twice cut the plane of the ecliptic, and therefore was not a right line. It did cut the ecliptic not in opposite parts of the heavens, but in the end of Virgo and beginning of Capricorn, including an arc of about 98°; and therefore the way of the comet did very much deviate from the path of a great circle; for in the month of *Nov.* it declined at least 3° from the ecliptic towards the south; and in the month of *Dec.* following it declined 29° from the ecliptic towards the north; the two parts of the orbit in which the comet descended towards the sun, and ascended again from the sun, declining one from the other by an apparent angle of above 30°, as observed by *Montenari*. This comet travelled over 9 signs, to wit, from the last deg. of ♌ to the beginning of ♎, beside the sign of ♌, through which it passed before it began to be seen; and there is no other theory by which a comet can go over so great a part of the heavens with a regular motion. The motion of this comet was very unequable; for about the 20th of *Nov.* it described about 5° a day. Then its motion being retarded between *Nov.* 26 and *Dec.* 12, to wit, in the space of 15½ days, it described only 40°. But the motion thereof being afterwards accelerated, it described near 5° a day, till its motion began to be again retarded. And the theory which justly corresponds with a motion so unequable, and through so great a part of the heavens, which observes the same laws with the theory of the planets, and which accurately agrees with accurate astronomical observations, cannot be otherwise than true.

And, thinking it would not be improper, I have given a true representation of the orbit which this comet described, and of the tail which it emitted in several places, in the annexed figure; protracted in the plane of the trajectory. In this scheme ABC represents the trajectory of the comet, D the sun DE the axis of the trajectory, DF the line of the nodes, GH the intersection of the sphere of the *orbis magnus* with the plane of the trajectory, I the place of the comet *Nov.* 4, *Ann.* 1680; K the place of the same *Nov.* 11; L the place of the same *Nov.* 19; M its place *Dec.* 12; N



its place *Dec.* 21; O its place *Dec.* 29; P its place *Jan.* 5 following; Q its place *Jan.* 25; R its place *Feb.* 5; S its place *Feb.* 25; T its place *March* 5; and V its place *March* 9. In determining the length of the tail, I made the following observations.

Nov. 4 and 6, the tail did not appear; *Nov.* 11, the tail just begun to shew itself, but did not appear above $\frac{1}{2}$ deg. long through a 10 feet telescope; *Nov.* 17, the tail was seen by *Ponthæus* more than 15° long; *Nov.* 18, in *New-England*, the tail appeared 30° long, and directly opposite to the sun, extending itself to the planet Mars, which was then in ♍, 9° 54′; *Nov.* 19. in *Maryland*, the tail was found 15° or 20° long; *Dec.* 10 (by

the observation of Mr. *Flamsted*), the tail passed through the middle of the distance intercepted between the tail of the Serpent of *Ophiuchus* and the star δ in the south wing of *Aquila*, and did terminate near the stars A, ω, b, in *Bayer's* tables. Therefore the end of the tail was in ♑ 19½°, with latitude about 34½° north; *Dec* 11, it ascended to the head of *Sagitta* (*Bayer's* α, β), terminating in ♑ 26° 43′, with latitude 38° 34′ north; *Dec.* 12, it passed through the middle of *Sagitta*, nor did it reach much farther; terminating in ♒ 4°, with latitude 42½° north nearly. But these things are to be understood of the length of the brighter part of the tail; for with a more faint light, observed, too, perhaps, in a serener sky, at *Rome*, *Dec.* 12, 5ʰ. 40′, by the observation of *Ponthæus*, the tail arose to 10° above the rump of the Swan, and the side thereof towards the west and towards the north was 45′ distant from this star. But about that time the tail was 3° broad towards the upper end; and therefore the middle thereof was 2° 15′ distant from that star towards the south, and the upper end was ♓ in 22°, with latitude 61° north; and thence the tail was about 70° long; *Dec.* 21, it extended almost to *Cassiopeia's* chair, equally distant from β and from *Schedir*, so as its distance from either of the two was equal to the distance of the one from the other, and therefore did terminate in ♈ 24°, with latitude 47½°; *Dec.* 29, it reached to a contact with *Scheat* on its left, and exactly filled up the space between the two stars in the northern foot of *Andromeda*, being 54° in length; and therefore terminated in ♉ 19°, with 35° of latitude; *Jan.* 5, it touched the star π in the breast of *Andromeda* on its right side, and the star μ of the girdle on its left; and, according to our observations, was 40° long; but it was curved, and the convex side thereof lay to the south; and near the head of the comet it made an angle of 4° with the circle which passed through the sun and the comet's head; but towards the other end it was inclined to that circle in an angle of about 10° or 11°; and the chord of the tail contained with that circle an angle of 8°. *Jan.* 13, the tail terminated between *Alamech* and *Algol*, with a light that was sensible enough; but with a faint light it ended over against the star κ in *Perseus's* side. The distance of the end of the tail from the circle passing through the sun and the comet was 3° 50′; and the inclination of the chord of the tail to that circle was 8½°. *Jan.* 25 and 26, it shone with a faint light to the length of 6° or 7°; and for a night or two after, when there was a very clear sky, it extended to the length of 12°, or something more, with a light that was very faint and very hardly to be seen; but the axis thereof was exactly directed to the bright star in the eastern shoulder of *Auriga*, and therefore deviated from the opposition of the sun towards the north by an angle of 10°. Lastly, *Feb.* 10, with a telescope I observed the tail 2° long; for that fainter light which I spoke of did not appear through the glasses. But *Ponthæus* writes, that, on *Feb.* 7, he saw the tail 12° long. *Feb.* 25, the comet was without a tail, and so continued till it disappeared.

Now if one reflects upon the orbit described, and duly considers the other appearances of this comet, he will be easily satisfied that the bodies of comets are solid, compact, fixed, and durable, like the bodies of the planets; for if they were nothing else but the vapours or exhalations of the earth, of the sun, and other planets, this comet, in its passage by the neighbourhood of the sun, would have been immediately dissipated; for the heat of the sun is as the density of its rays, that is, reciprocally as the square of the distance of the places from the sun. Therefore, since on *Dec.* 8, when the comet was in its perihelion, the distance thereof from the centre of the sun was to the distance of the earth from the same as about 6 to 1000, the sun's heat on the comet was at that time to the heat of the summer-sun with us as 1000000 to 36, or as 28000 to 1. But the heat of boiling water is about 3 times greater than the heat which dry earth acquires from the summer-sun, as I have tried; and the heat of red-hot iron (if my conjecture is right) is about three or four times greater than the heat of boiling water. And therefore the heat which dry earth on the comet, while in its perihelion, might have conceived from the rays of the sun, was about 2000 times greater than the heat of red-hot iron. But by so fierce a heat, vapours and exhalations, and every volatile matter, must have been immediately consumed and dissipated.

This comet, therefore, must have conceived an immense heat from the sun, and retained that heat for an exceeding long time; for a globe of iron of an inch in diameter, exposed red-hot to the open air, will scarcely lose all its heat in an hour's time; but a greater globe would retain its heat longer in the proportion of its diameter, because the surface (in proportion to which it is cooled by the contact of the ambient air) is in that proportion less in respect of the quantity of the included hot matter; and therefore a globe of red hot iron equal to our earth, that is, about 40000000 feet in diameter, would scarcely cool in an equal number of days, or in above 50000 years. But I suspect that the duration of heat may, on account of some latent causes, increase in a yet less proportion than that of the diameter; and I should be glad that the true proportion was investigated by experiments.

It is farther to be observed, that the comet in the month of *December*, just after it had been heated by the sun, did emit a much longer tail, and much more splendid, than in the month of *November* before, when it had not yet arrived at its perihelion; and, universally, the greatest and most fulgent tails always arise from comets immediately after their passing by the neighbourhood of the sun. Therefore the heat received by the comet conduces to the greatness of the tail: from whence, I think I may infer, that the tail is nothing else but a very fine vapour, which the head or nucleus of the comet emits by its heat.

But we have had three several opinions about the tails of comets; for

some will have it that they are nothing else but the beams of the sun's light transmitted through the comets' heads, which they suppose to be transparent; others, that they proceed from the refraction which light suffers in passing from the comet's head to the earth: and, lastly, others, that they are a sort of clouds or vapour constantly rising from the comets' heads, and tending towards the parts opposite to the sun. The first is the opinion of such as are yet unacquainted with optics; for the beams of the sun are seen in a darkened room only in consequence of the light that is reflected from them by the little particles of dust and smoke which are always flying about in the air; and, for that reason, in air impregnated with thick smoke, those beams appear with great brightness, and move the sense vigorously; in a yet finer air they appear more faint, and are less easily discerned; but in the heavens, where there is no matter to reflect the light, they can never be seen at all. Light is not seen as it is in the beam, but as it is thence reflected to our eyes; for vision can be no otherwise produced than by rays falling upon the eyes; and, therefore, there must be some reflecting matter in those parts where the tails of the comets are seen: for otherwise, since all the celestial spaces are equally illuminated by the sun's light, no part of the heavens could appear with more splendor than another. The second opinion is liable to many difficulties. The tails of comets are never seen variegated with those colours which commonly are inseparable from refraction; and the distinct transmission of the light of the fixed stars and planets to us is a demonstration that the æther or celestial medium is not endowed with any refractive power: for as to what is alleged, that the fixed stars have been sometimes seen by the *Egyptians* environed with a *Coma* or *Capillitium*, because that has but rarely happened, it is rather to be ascribed to a casual refraction of clouds; and so the radiation and scintillation of the fixed stars to the refractions both of the eyes and air; for upon laying a telescope to the eye, those radiations and scintillations immediately disappear. By the tremulous agitation of the air and ascending vapours, it happens that the rays of light are alternately turned aside from the narrow space of the pupil of the eye; but no such thing can have place in the much wider aperture of the object-glass of a telescope; and hence it is that a scintillation is occasioned in the former case, which ceases in the latter; and this cessation in the latter case is a demonstration of the regular transmission of light through the heavens, without any sensible refraction. But, to obviate an objection that may be made from the appearing of no tail in such comets as shine but with a faint light, as if the secondary rays were then too weak to affect the eyes, and for that reason it is that the tails of the fixed stars do not appear, we are to consider, that by the means of telescopes the light of the fixed stars may be augmented above an hundred fold, and yet no tails are seen; that the light of the planets is yet more copious without any

tail; but that comets are seen sometimes with huge tails, when the light of their heads is but faint and dull. For so it happened in the comet of the year 1680, when in the month of *December* it was scarcely equal in light to the stars of the second magnitude, and yet emitted a notable tail, extending to the length of 40°, 50°, 60°, or 70°, and upwards; and afterwards, on the 27th and 28th of *January*, when the head appeared but as a star of the 7th magnitude, yet the tail (as we said above), with a light that was sensible enough, though faint, was stretched out to 6 or 7 degrees in length, and with a languishing light that was more difficultly seen, even to 12°, and upwards. But on the 9th and 10th of *February*, when to the naked eye the head appeared no more, through a telescope I viewed the tail of 2° in length. But farther; if the tail was owing to the refraction of the celestial matter, and did deviate from the opposition of the sun, according to the figure of the heavens, that deviation in the same places of the heavens should be always directed towards the same parts. But the comet of the year 1680, *December* 28d. 8½h. P. M. at *London*, was seen in ♓ 8° 41′, with latitude north 28° 6′; while the sun was in ♑ 18° 26′. And the comet of the year 1577, *December* 29d. was in ♓ 8° 41′, with latitude north 28 40′, and the sun, as before, in about ♑ 18° 26′. In both cases the situation of the earth was the same, and the comet appeared in the same place of the heavens; yet in the former case the tail of the comet (as well by my observations as by the observations of others) deviated from the opposition of the sun towards the north by an angle of 4½ degrees; whereas in the latter there was (according to the observations of *Tycho*) a deviation of 21 degrees towards the south. The refraction, therefore, of the heavens being thus disproved, it remains that the *phænomena* of the tails of comets must be derived from some reflecting matter.

And that the tails of comets do arise from their heads, and tend towards the parts opposite to the sun, is farther confirmed from the laws which the tails observe. As that, lying in the planes of the comets' orbits which pass through the sun, they constantly deviate from the opposition of the sun towards the parts which the comets' heads in their progress along these orbits have left. That to a spectator, placed in those planes, they appear in the parts directly opposite to the sun; but, as the spectator recedes from those planes, their deviation begins to appear, and daily becomes greater. That the deviation, *cæteris paribus*, appears less when the tail is more oblique to the orbit of the comet, as well as when the head of the comet approaches nearer to the sun, especially if the angle of deviation is estimated near the head of the comet. That the tails which have no deviation appear straight, but the tails which deviate are likewise bended into a certain curvature. That this curvature is greater when the deviation is greater; and is more sensible when the tail, *cæteris paribus* is longer; for in the shorter tails the curvature is hardly to be per-

ceived. That the angle of deviation is less near the comet's head, but greater towards the other end of the tail; and that because the convex side of the tail regards the parts from which the deviation is made, and which lie in a right line drawn out infinitely from the sun through the comet's head. And that the tails that are long and broad, and shine with a stronger light, appear more resplendent and more exactly defined on the convex than on the concave side. Upon which accounts it is plain that the *phænomena* of the tails of comets depend upon the motions of their heads, and by no means upon the places of the heavens in which their heads are seen; and that, therefore, the tails of comets do not proceed from the refraction of the heavens, but from their own heads, which furnish the matter that forms the tail. For, as in our air, the smoke of a heated body ascends either perpendicularly if the body is at rest, or obliquely if the body is moved obliquely, so in the heavens, where all bodies gravitate towards the sun, smoke and vapour must (as we have already said) ascend from the sun, and either rise perpendicularly if the smoking body is at rest, or obliquely if the body, in all the progress of its motion, is always leaving those places from which the upper or higher parts of the vapour had risen before; and that obliquity will be least where the vapour ascends with most velocity, to wit, near the smoking body, when that is near the sun. But, because the obliquity varies, the column of vapour will be incurvated; and because the vapour in the preceding sides is something more recent, that is, has ascended something more late from the body, it will therefore be something more dense on that side, and must on that account reflect more light, as well as be better defined. I add nothing concerning the sudden uncertain agitation of the tails of comets, and their irregular figures, which authors sometimes describe, because they may arise from the mutations of our air, and the motions of our clouds, in part obscuring those tails; or, perhaps, from parts of the *Via Lactea*, which might have been confounded with and mistaken for parts of the tails of the comets as they passed by.

But that the atmospheres of comets may furnish a supply of vapour great enough to fill so immense spaces, we may easily understand from the rarity of our own air; for the air near the surface of our earth possesses a space 850 times greater than water of the same weight; and therefore a cylinder of air 850 feet high is of equal weight with a cylinder of water of the same breadth, and but one foot high. But a cylinder of air reaching to the top of the atmosphere is of equal weight with a cylinder of water about 33 feet high: and, therefore, if from the whole cylinder of air the lower part of 850 feet high is taken away, the remaining upper part will be of equal weight with a cylinder of water 32 feet high: and from thence (and by the hypothesis, confirmed by many experiments, that the compression of air is as the weight of the incumbent atmosphere, and

that the force of gravity is reciprocally as the square of the distance from the centre of the earth) raising a calculus, by Cor. Prop. XXII, Book II, I found, that, at the height of one semi-diameter of the earth, reckoned from the earth's surface, the air is more rare than with us in a far greater proportion than of the whole space within the orb of Saturn to a spherical space of one inch in diameter; and therefore if a sphere of our air of but one inch in thickness was equally rarefied with the air at the height of one semi-diameter of the earth from the earth's surface, it would fill all the regions of the planets to the orb of Saturn, and far beyond it. Wherefore since the air at greater distances is immensely rarefied, and the *coma* or atmosphere of comets is ordinarily about ten times higher, reckoning from their centres, than the surface of the nucleus, and the tails rise yet higher, they must therefore be exceedingly rare; and though, on account of the much thicker atmospheres of comets, and the great gravitation of their bodies towards the sun, as well as of the particles of their air and vapours mutually one towards another, it may happen that the air in the celestial spaces and in the tails of comets is not so vastly rarefied, yet from this computation it is plain that a very small quantity of air and vapour is abundantly sufficient to produce all the appearances of the tails of comets; for that they are, indeed, of a very notable rarity appears from the shining of the stars through them. The atmosphere of the earth, illuminated by the sun's light, though but of a few miles in thickness, quite obscures and extinguishes the light not only of all the stars, but even of the moon itself; whereas the smallest stars are seen to shine through the immense thickness of the tails of comets, likewise illuminated by the sun, without the least diminution of their splendor. Nor is the brightness of the tails of most comets ordinarily greater than that of our air, an inch or two in thickness, reflecting in a darkened room the light of the sun-beams let in by a hole of the window-shutter.

And we may pretty nearly determine the time spent during the ascent of the vapour from the comet's head to the extremity of the tail, by drawing a right line from the extremity of the tail to the sun, and marking the place where that right line intersects the comet's orbit: for the vapour that is now in the extremity of the tail, if it has ascended in a right line from the sun, must have begun to rise from the head at the time when the head was in the point of intersection. It is true, the vapour does not rise in a right line from the sun, but, retaining the motion which it had from the comet before its ascent, and compounding that motion with its motion of ascent, arises obliquely; and, therefore, the solution of the Problem will be more exact, if we draw the line which intersects the orbit parallel to the length of the tail; or rather (because of the curvilinear motion of the comet) diverging a little from the line or length of the tail. And by means of this principle I found that the vapour which, *January* 25, was

in the extremity of the tail, had begun to rise from the head before *December* 11, and therefore had spent in its whole ascent 45 days; but that the whole tail which appeared on *December* 10 had finished its ascent in the space of the two days then elapsed from the time of the comet's being in its perihelion. The vapour, therefore, about the beginning and in the neighbourhood of the sun rose with the greatest velocity, and afterwards continued to ascend with a motion constantly retarded by its own gravity; and the higher it ascended, the more it added to the length of the tail; and while the tail continued to be seen, it was made up of almost all that vapour which had risen since the time of the comet's being in its perihelion; nor did that part of the vapour which had risen first, and which formed the extremity of the tail, cease to appear, till its too great distance, as well from the sun, from which it received its light, as from our eyes, rendered it invisible. Whence also it is that the tails of other comets which are short do not rise from their heads with a swift and continued motion, and soon after disappear, but are permanent and lasting columns of vapours and exhalations, which, ascending from the heads with a slow motion of many days, and partaking of the motion of the heads which they had from the beginning, continue to go along together with them through the heavens. From whence again we have another argument proving the celestial spaces to be free, and without resistance, since in them not only the solid bodies of the planets and comets, but also the extremely rare vapours of comets' tails, maintain their rapid motions with great freedom, and for an exceeding long time.

Kepler ascribes the ascent of the tails of the comets to the atmospheres of their heads; and their direction towards the parts opposite to the sun to the action of the rays of light carrying along with them the matter of the comets' tails; and without any great incongruity we may suppose, that, in so free spaces, so fine a matter as that of the æther may yield to the action of the rays of the sun's light, though those rays are not able sensibly to move the gross substances in our parts, which are clogged with so palpable a resistance. Another author thinks that there may be a sort of particles of matter endowed with a principle of levity, as well as others are with a power of gravity; that the matter of the tails of comets may be of the former sort, and that its ascent from the sun may be owing to its levity; but, considering that the gravity of terrestrial bodies is as the matter of the bodies, and therefore can be neither more nor less in the same quantity of matter, I am inclined to believe that this ascent may rather proceed from the rarefaction of the matter of the comets' tails. The ascent of smoke in a chimney is owing to the impulse of the air with which it is entangled. The air rarefied by heat ascends, because its specific gravity is diminished, and in its ascent carries along with it the smoke with which it is engaged; and why may not the tail of a comet rise from the sun after the same man-

ner? For the sun's rays do not act upon the mediums which they pervade otherwise than by reflection and refraction; and those reflecting particles heated by this action, heat the matter of the æther which is involved with them. That matter is rarefied by the heat which it acquires, and because, by this rarefaction, the specific gravity with which it tended towards the sun before is diminished, it will ascend therefrom, and carry along with it the reflecting particles of which the tail of the comet is composed. But the ascent of the vapours is further promoted by their circumgyration about the sun, in consequence whereof they endeavour to recede from the sun, while the sun's atmosphere and the other matter of the heavens are either altogether quiescent, or are only moved with a slower circumgyration derived from the rotation of the sun. And these are the causes of the ascent of the tails of the comets in the neighbourhood of the sun, where their orbits are bent into a greater curvature, and the comets themselves are plunged into the denser and therefore heavier parts of the sun's atmosphere: upon which account they do then emit tails of an huge length; for the tails which then arise, retaining their own proper motion, and in the mean time gravitating towards the sun, must be revolved in ellipses about the sun in like manner as the heads are, and by that motion must always accompany the heads, and freely adhere to them. For the gravitation of the vapours towards the sun can no more force the tails to abandon the heads, and descend to the sun, than the gravitation of the heads can oblige them to fall from the tails. They must by their common gravity either fall together towards the sun, or be retarded together in their common ascent therefrom; and, therefore (whether from the causes already described, or from any others), the tails and heads of comets may easily acquire and freely retain any position one to the other, without disturbance or impediment from that common gravitation.

The tails, therefore, that rise in the perihelion positions of the comets will go along with their heads into far remote parts, and together with the heads will either return again from thence to us, after a long course of years, or rather will be there rarefied, and by degrees quite vanish away; for afterwards, in the descent of the heads towards the sun, new short tails will be emitted from the heads with a slow motion; and those tails by degrees will be augmented immensely, especially in such comets as in their perihelion distances descend as low as the sun's atmosphere; for all vapour in those free spaces is in a perpetual state of rarefaction and dilatation; and from hence it is that the tails of all comets are broader at their upper extremity than near their heads. And it is not unlikely but that the vapour, thus perpetually rarefied and dilated, may be at last dissipated and scattered through the whole heavens, and by little and little be attracted towards the planets by its gravity, and mixed with their atmosphere; for as the seas are absolutely necessary to the constitution of our earth, that

from them, the sun, by its heat, may exhale a sufficient quantity of vapours, which, being gathered together into clouds, may drop down in rain, for watering of the earth, and for the production and nourishment of vegetables; or, being condensed with cold on the tops of mountains (as some philosophers with reason judge), may run down in springs and rivers; so for the conservation of the seas, and fluids of the planets, comets seem to be required, that, from their exhalations and vapours condensed, the wastes of the planetary fluids spent upon vegetation and putrefaction, and converted into dry earth, may be continually supplied and made up; for all vegetables entirely derive their growths from fluids, and afterwards, in great measure, are turned into dry earth by putrefaction; and a sort of slime is always found to settle at the bottom of putrefied fluids; and hence it is that the bulk of the solid earth is continually increased; and the fluids, if they are not supplied from without, must be in a continual decrease, and quite fail at last. I suspect, moreover, that it is chiefly from the comets that spirit comes, which is indeed the smallest but the most subtle and useful part of our air, and so much required to sustain the life of all things with us.

The atmospheres of comets, in their descent towards the sun, by running out into the tails, are spent and diminished, and become narrower, at least on that side which regards the sun; and in receding from the sun, when they less run out into the tails, they are again enlarged, if *Hevelius* has justly marked their appearances. But they are seen least of all just after they have been most heated by the sun, and on that account then emit the longest and most resplendent tails; and, perhaps, at the same time, the nuclei are environed with a denser and blacker smoke in the lowermost parts of their atmosphere; for smoke that is raised by a great and intense heat is commonly the denser and blacker. Thus the head of that comet which we have been describing, at equal distances both from the sun and from the earth, appeared darker after it had passed by its perihelion than it did before; for in the month of *December* it was commonly compared with the stars of the third magnitude, but in *November* with those of the first or second; and such as saw both appearances have described the first as of another and greater comet than the second. For, *November* 19, this comet appeared to a young man at *Cambridge*, though with a pale and dull light, yet equal to *Spica Virginis;* and at that time it shone with greater brightness than it did afterwards. And *Montenari*, *November* 20, st. vet. observed it larger than the stars of the first magnitude, its tail being then 2 degrees long. And Mr. *Storer* (by letters which have come into my hands) writes, that in the month of *December*, when the tail appeared of the greatest bulk and splendor, the head was but small, and far less than that which was seen in the month of *November* before sun-rising; and, conjecturing at the cause of the appearance, he judged it to proceed

from there being a greater quantity of matter in the head at first, which was afterwards gradually spent.

And, which farther makes for the same purpose, I find, that the heads of other comets, which did put forth tails of the greatest bulk and splendor, have appeared but obscure and small. For in *Brazil, March* 5, 1668, 7^{h}. P. M., *St. N. P. Valentinus Estancius* saw a comet near the horizon, and towards the south west, with a head so small as scarcely to be discerned, but with a tail above measure splendid, so that the reflection thereof from the sea was easily seen by those who stood upon the shore; and it looked like a fiery beam extended 23° in length from the west to south, almost parallel to the horizon. But this excessive splendor continued only three days, decreasing apace afterwards; and while the splendor was decreasing, the bulk of the tail increased: whence in *Portugal* it is said to have taken up one quarter of the heavens, that is, 45 degrees, extending from west to east with a very notable splendor, though the whole tail was not seen in those parts, because the head was always hid under the horizon: and from the increase of the bulk and decrease of the splendor of the tail, it appears that the head was then in its recess from the sun, and had been very near to it in its perihelion, as the comet of 1680 was. And we read, in the *Saxon* Chronicle, of a like comet appearing in the year 1106, *the star whereof was small and obscure* (as that of 1680), *but the splendour of its tail was very bright, and like a huge fiery beam stretched out in a direction between the east and north*, as *Hevelius* has it also from *Simeon*, the monk of *Durham*. This comet appeared in the beginning of *February*, about the evening, and towards the south west part of heaven; from whence, and from the position of the tail, we infer that the head was near the sun. *Matthew Paris* says, *It was distant from the sun by about a cubit, from three of the clock* (rather six) *till nine, putting forth a long tail.* Such also was that most resplendent comet described by *Aristotle*, lib. 1, *Meteor.* 6. *The head whereof could not be seen, because it had set before the sun, or at least was hid under the sun's rays; but next day it was seen as well as might be; for, having left the sun but a very little way, it set immediately after it. And the scattered light of the head, obscured by the too great splendour* (of the tail) *did not yet appear. But afterwards* (as *Aristotle says*) *when the splendour* (of the tail) *was now diminished* (the head of), *the comet recovered its native brightness; and the splendour* (of its tail) *reached now to a third part of the heavens* (that is, to 60°). *This appearance was in the winter season* (an. 4, Olymp. 101), *and, rising to* Orion's *girdle, it there vanished away.* It is true that the comet of 1618, which came out directly from under the sun's rays with a very large tail, seemed to equal, if not to exceed, the stars of the first magnitude; but, then, abundance of other comets have appeared yet greater than this, that put forth shorter tails; some of which are said

to have appeared as big as Jupiter, others as big as Venus, or even as the moon.

We have said, that comets are a sort of planets revolved in very eccentric orbits about the sun; and as, in the planets which are without tails, those are commonly less which are revolved in lesser orbits, and nearer to the sun, so in comets it is probable that those which in their perihelion approach nearer to the sun are generally of less magnitude, that they may not agitate the sun too much by their attractions. But as to the transverse diameters of their orbits, and the periodic times of their revolutions, I leave them to be determined by comparing comets together which after long intervals of time return again in the same orbit. In the mean time, the following Proposition may give some light in that inquiry.

PROPOSITION XLII. PROBLEM XXII.

To correct a comet's trajectory found as above.

Operation 1. Assume that position of the plane of the trajectory which was determined according to the preceding proposition; and select three places of the comet, deduced from very accurate observations, and at great distances one from the other. Then suppose A to represent the time between the first observation and the second, and B the time between the second and the third; but it will be convenient that in one of those times the comet be in its perigeon, or at least not far from it. From those apparent places find, by trigonometric operations, the three true places of the comet in that assumed plane of the trajectory; then through the places found, and about the centre of the sun as the focus, describe a conic section by arithmetical operations, according to Prop. XXI., Book 1. Let the areas of this figure which are terminated by radii drawn from the sun to the places found be D and E; to wit, D the area between the first observation and the second, and E the area between the second and third; and let T represent the whole time in which the whole area D + E should be described with the velocity of the comet found by Prop. XVI., Book 1.

Oper. 2. Retaining the inclination of the plane of the trajectory to the plane of the ecliptic, let the longitude of the nodes of the plane of the trajectory be increased by the addition of 20 or 30 minutes, which call P. Then from the aforesaid three observed places of the comet let the three true places be found (as before) in this new plane; as also the orbit passing through those places, and the two areas of the same described between the two observations, which call d and e; and let t be the whole time in which the whole area $d + e$ should be described.

Oper. 3. Retaining the longitude of the nodes in the first operation, let the inclination of the plane of the trajectory to the plane of the ecliptic be increased by adding thereto 20′ or 30′, which call Q. Then from the

aforesaid three observed apparent places of the comet let the three true places be found in this new plane, as well as the orbit passing through them, and the two areas of the same described between the observation, which call δ and ε; and let τ be the whole time in which the whole area $\delta + \varepsilon$ should be described.

Then taking C to 1 as A to B; and G to 1 as D to E; and g to 1 as d to e; and γ to 1 as δ to ε; let S be the true time between the first observation and the third; and, observing well the signs + and —, let such numbers m and n be found out as will make $2G - 2C, = mG - mg + nG - n\gamma$; and $2T - 2S = mT - ml + nT - n\tau$. And if, in the first operation, I represents the inclination of the plane of the trajectory to the plane of the ecliptic, and K the longitude of either node, then $I + nQ$ will be the true inclination of the plane of the trajectory to the plane of the ecliptic, and $K + mP$ the true longitude of the node. And, lastly, if in the first, second, and third operations, the quantities R, r, and ρ, represent the parameters of the trajectory, and the quantities $\frac{1}{L}, \frac{1}{l}, \frac{1}{\lambda}$, the transverse diameters of the same, then $R + mr - mR + n\rho - nR$ will be the true parameter, and $\frac{1}{L + ml - mL + n\lambda - nL}$ will be the true transverse diameter of the trajectory which the comet describes; and from the transverse diameter given the periodic time of the comet is also given. Q.E.I. But the periodic times of the revolutions of comets, and the transverse diameters of their orbits, cannot be accurately enough determined but by comparing comets together which appear at different times. If, after equal intervals of time, several comets are found to have described the same orbit, we may thence conclude that they are all but one and the same comet revolved in the same orbit; and then from the times of their revolutions the transverse diameters of their orbits will be given, and from those diameters the elliptic orbits themselves will be determined.

To this purpose the trajectories of many comets ought to be computed, supposing those trajectories to be parabolic; for such trajectories will always nearly agree with the *phænomena*, as appears not only from the parabolic trajectory of the comet of the year 1680, which I compared above with the observations, but likewise from that of the notable comet which appeared in the year 1664 and 1665, and was observed by *Hevelius*, who, from his own observations, calculated the longitudes and latitudes thereof, though with little accuracy. But from the same observations Dr. *Halley* did again compute its places; and from those new places determined its trajectory, finding its ascending node in ☊ 21° 13′ 55″; the inclination of the orbit to the plane of the ecliptic 21° 18′ 40″; the distance of its perihelion from the node, estimated in the comet's orbit, 49° 27′ 30″ its perihelion in ☊ 8° 40′ 30″, with heliocentric latitude south

16° 01′ 45″; the comet to have been in its perihelion *November* 24d. 11h. 52′ P.M. equal time at *London*, or 13h. 8′ at *Dantzick*, O. S.; and that the *latus rectum* of the parabola was 410286 such parts as the sun's mean distance from the earth is supposed to contain 100000. And how nearly the places of the comet computed in this orbit agree with the observations, will appear from the annexed table, calculated by Dr. *Halley*.

Appar. Time at Dantzick.	The observed Distances of the Comet from		The observed Places.	The Places computed in the Orb.
December d. h. ′		° ′ ″	° ′ ″	° ′ ″
3.18.29½	The Lion's heart	46.24.20	Long. ♎ 7.01.00	♎ 7. 1.29
	The Virgin's spike	22.52.10	Lat. S. 21.39. 0	21.38.50
4.18. 1½	The Lion's heart	46. 2.45	Long. ♎ 6.15. 0	♎ 6.16. 5
	The Virgin's spike	23.52.40	Lat. S. 22.24. 0	22.24. 0
7.17.48	The Lion's heart	44.48. 0	Long. ♎ 3. 6. 0	♎ 3. 7.33
	The Virgin's spike	27.56.40	Lat. S. 25.22. 0	25.21.40
17.14.43	The Lion's heart	53.15.15	Long. ♌ 2.56. 0	♌ 2.56. 0
	Orion's right shoulder	45.43.30	Lat. S. 49.25. 0	49.25. 0
19. 9.25	Procyon	35.13.50	Long. ♊ 28.40.30	♊ 28.43. 0
	Bright star of Whale's jaw	52 56. 0	Lat. S. 45.48. 0	45.46. 0
20. 9.53½	Procyon	40.49. 0	Long. ♊ 13.03. 0	♊ 13. 5. 0
	Bright star of Whale's jaw	40.04. 0	Lat. S. 39.54. 0	39.53. 0
21. 9. 9½	Orion's right shoulder	26.21.25	Long. ♊ 2.16. 0	♊ 2 18.30
	Bright star of Whale's jaw	29.28. 0	Lat. S. 33.41. 0	33.39.40
22. 9. 0	Orion's right shoulder	29.47. 0	Long. ♉ 24.24. 0	♉ 24.27. 0
	Bright star of Whale's jaw	20.29.30	Lat. S. 27.45. 0	27.46. 0
26. 7.58	The bright star of Aries	23.20. 0	Long. ♉ 9. 0. 0	♉ 9. 2.28
	Aldebaran	26.44. 0	Lat. S. 12.36. 0	12.34.13
27. 6.45	The bright star of Aries	20.45. 0	Long. ♉ 7. 5.40	♉ 7. 8.45
	Aldebaran	28.10. 0	Lat. S. 10.23. 0	10.23.15
28. 7.39	The bright star of Aries	18.29. 0	Long. ♉ 5.24.45	♉ 5.27.52
	Palilicium	29.37. 0	Lat. S. 8.22.50	8.23.37
31. 6.45	Andromeda's girdle	30.48.10	Long. ♉ 2. 7.40	♉ 2. 8.20
	Palilicium	32.53.30	Lat. S. 4.13. 0	4.16.25
Jan. 1665	Andromeda's girdle	25.11. 0	Long. ♈ 28.24.47	♈ 28.24. 0
7. 7.37½	Palilicium	37.12.25	Lat. N. 0.54. 0	0.53. 0
13. 7. 0	Andromeda's head	28. 7.10	Long. ♈ 27. 6.54	♈ 27. 6.39
	Palilicium	38.55.20	Lat. N. 3. 6.50	3. 7.40
24. 7.29	Andromeda's girdle	20.32.15	Long. ♈ 26.29.15	♈ 26.28.50
	Palilicium	40. 5. 0	Lat. N. 5.25.50	5.26. 0
Feb. 7. 8.37			Long. ♈ 27. 4.46	♈ 27.24.55
			Lat. N. 7. 3.29	7. 3.15
22. 8.46			Long. ♈ 28.29.46	♈ 28.29.58
			Lat. N. 8.12.36	8.10.25
March. 1. 8.16			Long. ♈ 29.18 15	♈ 29.18.20
			Lat. N. 8.36.26	8.36.12
7. 8.37			Long. ♉ 0. 2.48	♉ 0. 2.42
			Lat. N. 8.56.30	8.56.56

In *February*, the beginning of the year 1665, the first star of Aries, which I shall hereafter call γ, was in ♈ 28° 30′ 15″, with 7° 8′ 58″ north

lat.; the second star of Aries was in ♈ 29° 17′ 18″, with 8° 28′ 16″ north lat.; and another star of the seventh magnitude, which I call A, was in ♈ 28° 24′ 45″, with 8° 28′ 33″ north lat. The comet *Feb.* 7ᵈ. 7ʰ. 30′ at *Paris* (that is, *Feb.* 7ᵈ. 8ʰ. 37′ at *Dantzick*) O. S. made a triangle with those stars γ and A, which was right-angled in γ; and the distance of the comet from the star γ was equal to the distance of the stars γ and A, that is, 1° 19′ 46″ of a great circle; and therefore in the parallel of the latitude of the star γ it was 1° 20′ 26″. Therefore if from the longitude of the star γ there be subducted the longitude 1° 20′ 26″, there will remain the longitude of the comet ♈ 27° 9′ 49″. M. *Auzout*, from this observation of his, placed the comet in ♈ 27° 0′, nearly; and, by the scheme in which Dr. *Hooke* delineated its motion, it was then in ♈ 26° 59′ 24′. I place it in ♈ 27° 4′ 46″, taking the middle between the two extremes.

From the same observations, M. *Auzout* made the latitude of the comet at that time 7° and 4′ or 5′ to the north; but he had done better to have made it 7° 3′ 29″, the difference of the latitudes of the comet and the star γ being equal to the difference of the longitude of the stars γ and A.

February 22ᵈ. 7ʰ. 30′ at *London*, that is, *February* 22ᵈ. 8ʰ. 46′ at *Dantzick*, the distance of the comet from the star A, according to Dr. *Hooke's* observation, as was delineated by himself in a scheme, and also by the observations of M. *Auzout*, delineated in like manner by M. *Petit*, was a fifth part of the distance between the star A and the first star of Aries, or 15′ 57″; and the distance of the comet from a right line joining the star A and the first of Aries was a fourth part of the same fifth part, that is, 4′; and therefore the comet was in ♈ 28° 29′ 46″, with 8° 12′ 36″ north lat.

March 1, 7ʰ. 0′ at *London*, that is, *March* 1, 8ʰ. 16′ at *Dantzick*, the comet was observed near the second star in Aries, the distance between them being to the distance between the first and second stars in Aries, that is, to 1° 33′, as 4 to 45 according to Dr. *Hooke*, or as 2 to 23 according to M. *Gottignies*. And, therefore, the distance of the comet from the second star in Aries was 8′ 16″ according to Dr. *Hooke*, or 8′ 5″ according to M. *Gottignies*; or, taking a mean between both, 8′ 10″. But, according to M. *Gottignies*, the comet had gone beyond the second star of Aries about a fourth or a fifth part of the space that it commonly went over in a day, to wit, about 1′ 35″ (in which he agrees very well with M. *Auzout*); or, according to Dr. *Hooke*, not quite so much, as perhaps only 1′. Wherefore if to the longitude of the first star in Aries we add 1′, and 8′ 10″ to its latitude, we shall have the longitude of the comet ♈ 29° 18′, with 8° 36′ 26″ north lat.

March 7, 7ʰ. 30′ at *Paris* (that is, *March* 7, 8ʰ. 37′ at *Dantzick*), from the observations of M. *Auzout*, the distance of the comet from the second star in Aries was equal to the distance of that star from the star

A, that is, 52,'29"; and the difference of the longitude of the comet and the second star in Aries was 45' or 46', or, taking a mean quantity, 45' 30"; and therefore the comet was in ♉ 0° 2' 48". From the scheme of the observations of M. *Auzout*, constructed by M. *Petit*, *Hevelius* collected the latitude of the comet 8° 54'. But the engraver did not rightly trace the curvature of the comet's way towards the end of the motion; and *Hevelius*, in the scheme of M. *Auzout's* observations which he constructed himself, corrected this irregular curvature, and so made the latitude of the comet 8° 55' 30". And, by farther correcting this irregularity, the latitude may become 8° 56', or 8° 57'.

This comet was also seen *March* 9, and at that time its place must have been in ♉ 0° 18', with 9° 3½' north lat. nearly.

This comet appeared three months together, in which space of time it travelled over almost six signs, and in one of the days thereof described almost 20 deg. Its course did very much deviate from a great circle, bending towards the north, and its motion towards the end from retrograde became direct; and, notwithstanding its course was so uncommon, yet by the table it appears that the theory, from beginning to end, agrees with the observations no less accurately than the theories of the planets usually do with the observations of them; but we are to subduct about 2' when the comet was swiftest, which we may effect by taking off 12" from the angle between the ascending node and the perihelion, or by making that angle 49° 27' 18". The annual parallax of both these comets (this and the preceding) was very conspicuous, and by its quantity demonstrates the annual motion of the earth in the *orbis magnus*.

This theory is likewise confirmed by the motion of that comet, which in the year 1683 appeared retrograde, in an orbit whose plane contained almost a right angle with the plane of the ecliptic, and whose ascending node (by the computation of Dr. *Halley*) was in ♏ 23° 23'; the inclination of its orbit to the ecliptic 83° 11'; its perihelion in ♊ 25° 29' 30" its perihelion distance from the sun 56020 of such parts as the radius of the *orbis magnus* contains 100000; and the time of its perihelion *July* 2d. 3h. 50'. And the places thereof, computed by Dr. *Halley* in this orbit, are compared with the places of the same observed by Mr. *Flamsted*. in the following table:—

1683 Eq. time.	Sun's place	Comet's Long. com.	Lat.Nor. comput.	Comet's Long. obs'd	Lat.Nor. observ'd	Diff. Long.	Diff. Lat.
d. h. ′	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ″	′ ″	′ ″
July 13.12.55	♌ 1.02.30	♋ 13.05.42	29.28.13	♋ 13. 6.42	29.28.20	+ 1.00	+ 0.07
15.11.15	2.53.12	11.37.48	29.34. 0	11.39.43	29.34.50	+ 1.55	+ 0.50
17.10.20	4.45.45	10. 7. 6	29.33.30	10. 8.40	29.34. 0	+ 1.34	+ 0.30
23.13 40	10.38.21	5.10.27	28.51.42	5.11.30	28.50.28	+ 1.03	— 1.14
25.14. 5	12.35.28	3.27.53	24.24.47	3.27. 0	28.23.40	— 0.53	— 1. 7
31. 9.42	18.09.22	♊ 27.55. 3	26.22.52	♊ 27.54.24	26.22.25	— 0.39	— 0.27
31.14.55	18.21.53	27.41. 7	26.16.57	27.41. 8	26.14.50	+ 0. 1	— 2. 7
Aug. 2.14 56	20.17.16	25.29.32	25.16.19	25.28.46	25.17.28	— 0.46	+ 1. 9
4.10.49	22.02.50	23.18.20	24.10.49	23.16.55	24.12.19	— 1.25	+ 1.30
6.10. 9	23.56.45	20.42.23	22.47. 5	20.40.32	22.49. 5	— 1.51	+ 2. 0
9.10.26	26.50.52	16 7.57	20. 6.37	16. 5.55	20. 6.10	— 2. 2	— 0.27
15.14. 1	♍ 2.47.13	3.30.48	11.37.33	3.26.18	11.32. 1	— 4.30	— 5.32
16.15.10	3.48. 2	0.43. 7	9.34.16	0.41.55	9.34.13	— 1.12	— 0. 3
18.15.44	5.45.33	♉ 24.52.53	5.11.15 South.	♉ 24.49. 5	5. 9.11 South	— 3.48	— 2. 4
22.14.44	9.35.49	11. 7.14	5.16.58	11.07.12	5.16.58	— 0. 2	— 0. 3
23 15.52	10.36.48	7. 2.18	8.17. 9	7. 1.17	8.16.41	— 1. 1	— 0.28
26.16. 2	13.31.10	♈ 24.45.31	16.38. 0	♈ 24.44.00	16.38.20	— 1.31	+ 0.20

This theory is yet farther confirmed by the motion of that retrograde comet which appeared in the year 1682. The ascending node of this (by Dr. *Halley's* computation) was in ♉ 21° 16′ 30″; the inclination of its orbit to the plane of the ecliptic 17° 56′ 00″; its perihelion in ♒ 2° 52′ 50″; its perihelion distance from the sun 58328 parts, of which the radius of the *orbis magnus* contains 100000; the equal time of the comet's being in its perihelion *Sept.* 4ᵈ. 7ʰ. 39′. And its places, collected from Mr. *Flamsted's* observations, are compared with its places computed from our theory in the following table:—

1682 App. Time.	Sun's place	Comet's Lon. comp.	Lat.Nor comp.	Com. Long. observed.	Lat Nor observ.	Diff. Long.	Diff. Lat.
d. h. ′	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	′ ″
Aug 19.16.38	♍ 7. 0. 7	♌ 18.14 28	25.50. 7	♌ 18.14.40	25.49.55	— 0.12	+ 0.12
20.15.38	7.55.52	24.46.23	26.14.42	24.46.22	26.12.52	+ 0. 1	+ 1.50
21. 8.21	8.36.14	29.37.15	26.20. 3	29.38.02	26.17.37	— 0.47	+ 2.26
22. 8. 8	9.33.55	♍ 6.29.53	26. 8.42	♍ 6.30. 3	26. 7.12	— 0.10	+ 1.30
29.08.20	16 22.40	♎ 12.37.54	18.37.47	♎ 12.37.49	18.34. 5	+ 0. 5	+ 3.42
30. 7.45	17.19.41	15.36. 1	17.26.43	15.35.18	17.27.17	+ 0.43	— 0.34
Sept. 1. 7.33	19.16. 9	20.30.53	15.13. 0	20.27. 4	15. 9.49	+ 3.49	+ 3.11
4. 7.22	22.11.28	25.42. 0	12.23.48	25.40 58	12.22. 0	+ 1. 2	+ 1.48
5. 7.32	23 10.29	27. 0.46	11.33.08	26.59.24	11.33.51	+ 1.22	— 0.43
8. 7.16	26. 5.58	29.58.44	9.26.46	29.58.45	9.26.43	— 0. 1	+ 0. 3
9. 7.26	27. 5. 9	♏ 0.44 10	8.49.10	♏ 0 44. 4	8.48.25	+ 0. 6	+ 0.45

This theory is also confirmed by the retrograde motion of the comet that appeared in the year 1723. The ascending node of this comet (according to the computation of Mr. *Bradley*, Savilian Professor of Astronomy at *Oxford*) was in ♈ 14° 16′. The inclination of the orbit to the plane of the ecliptic 49° 59′. Its perihelion was in ♉ 12° 15′ 20″. Its perihelion distance from the sun 998651 parts, of which the radius of the *orbis magnus* contains 1000000, and the equal time of its perihelion *September* 16ᵈ

16h. 10′. The places of this comet computed in this orbit by Mr. *Bradley*, and compared with the places observed by himself, his uncle Mr. *Pound*, and Dr. *Halley*, may be seen in the following table.

1723 Eq. time.	Comet's Long. obs.	Lat.Nor obs.	Comet's Lon. com.	Lat.Nor comp.	Diff. Lon.	Diff. Lat.
d. h. ′	♌ ° ′ ″	° ′ ″	♌ ° ′ ″	° ′ ″	″	″
Oct. 9.8. 5	♌ 7.22.15	5. 2. 0	♌ 7.21.26	5. 2.47	+ 49	— 47
10.6.21	6.41.12	7.44.13	6.41.42	7.43.18	— 50	+ 55
12.7.22	5 39.58	11.55. 0	5.40.19	11 54 55	— 21	+ 5
14.8.57	4.59.49	14.43.50	5. 0.37	14.44. 1	— 48	— 11
15.6.35	4.47.41	15.40.51	4.47.45	15.40.55	— 4	— 4
21.6 22	4. 2.32	19.41.49	4. 2.21	19.42. 3	+ 11	— 14
22.6.24	3.59. 2	20. 8.12	3.59.10	20. 8.17	— 8	— 5
24.8. 2	3.55.29	20.55.18	3 55.11	20.55. 9	+ 18	+ 9
29.8.56	3.56.17	22.20.27	3.56.42	22.20.10	— 25	+ 17
30.6.20	3.58. 9	22.32.28	3.58.17	22.32.12	— 8	+ 16
Nov. 5.5.53	4.16.30	23.38.33	4.16.23	23.38. 7	+ 7	+ 26
8.7. 6	4.29.36	24. 4.30	4.29.54	24. 4.40	— 18	— 10
14.6.20	5. 2.16	24.48.46	5. 2.51	24.48 16	— 35	+ 30
20.7.45	5.42.20	25.24.45	5.43.13	25.25.17	— 53	— 32
Dec. 7.6.45	8. 4.13	26.54.18	8. 3.55	26.53.42	+ 18	+ 36

From these examples it is abundantly evident that the motions of comets are no less accurately represented by our theory than the motions of the planets commonly are by the theories of them; and, therefore, by means of this theory, we may enumerate the orbits of comets, and so discover the periodic time of a comet's revolution in any orbit; whence, at last, we shall have the transverse diameters of their elliptic orbits and their aphelion distances.

That retrograde comet which appeared in the year 1607 described an orbit whose ascending node (according to Dr. *Halley's* computation) was in ♉ 20° 21′; and the inclination of the plane of the orbit to the plane of the ecliptic 17° 2′; whose perihelion was in ♌ 2° 16′; and its perihelion distance from the sun 58680 of such parts as the radius of the *orbis magnus* contains 100000; and the comet was in its perihelion *October* 16d. 3h. 50′; which orbit agrees very nearly with the orbit of the comet which was seen in 1682. If these were not two different comets, but one and the same, that comet will finish one revolution in the space of 75 years; and the greater axis of its orbit will be to the greater axis of the *orbis magnus* as $\sqrt[3]{:75 \times 75}$ to 1, or as 1778 to 100, nearly. And the aphelion distance of this comet from the sun will be to the mean distance of the earth from the sun as about 35 to 1; from which data it will be no hard matter to determine the elliptic orbit of this comet. But these things are to be supposed on condition, that, after the space of 75 years, the same comet shall return again in the same orbit. The other comets seem to ascend to greater heights, and to require a longer time to perform their revolutions.

But, because of the great number of comets, of the great distance of their

aphelions from the sun, and of the slowness of their motions in the aphelions, they will, by their mutual gravitations, disturb each other; so that their eccentricities and the times of their revolutions will be sometimes a little increased, and sometimes diminished. Therefore we are not to expect that the same comet will return exactly in the same orbit, and in the same periodic times: it will be sufficient if we find the changes no greater than may arise from the causes just spoken of.

And hence a reason may be assigned why comets are not comprehended within the limits of a zodiac, as the planets are; but, being confined to no bounds, are with various motions dispersed all over the heavens; namely, to this purpose, that in their aphelions, where their motions are exceedingly slow, receding to greater distances one from another, they may suffer less disturbance from their mutual gravitations: and hence it is that the comets which descend the lowest, and therefore move the slowest in their aphelions, ought also to ascend the highest.

The comet which appeared in the year 1680 was in its perihelion less distant from the sun than by a sixth part of the sun's diameter; and because of its extreme velocity in that proximity to the sun, and some density of the sun's atmosphere, it must have suffered some resistance and retardation; and therefore, being attracted something nearer to the sun in every revolution, will at last fall down upon the body of the sun. Nay, in its aphelion, where it moves the slowest, it may sometimes happen to be yet farther retarded by the attractions of other comets, and in consequence of this retardation descend to the sun. So fixed stars, that have been gradually wasted by the light and vapours emitted from them for a long time, may be recruited by comets that fall upon them; and from this fresh supply of new fuel those old stars, acquiring new splendor, may pass for new stars. Of this kind are such fixed stars as appear on a sudden, and shine with a wonderful brightness at first, and afterwards vanish by little and little. Such was that star which appeared in *Cassiopeia's* chair; which *Cornelius Gemma* did not see upon the 8th of *November*, 1572, though he was observing that part of the heavens upon that very night, and the sky was perfectly serene; but the next night (*November* 9) he saw it shining much brighter than any of the fixed stars, and scarcely inferior to *Venus* in splendor. *Tycho Brahe* saw it upon the 11th of the same month, when it shone with the greatest lustre; and from that time he observed it to decay by little and little; and in 16 months' time it entirely disappeared. In the month of *November*, when it first appeared, its light was equal to that of *Venus*. In the month of *December* its light was a little diminished, and was now become equal to that of *Jupiter*. In *January* 1573 it was less than *Jupiter*, and greater than *Sirius*; and about the end of *February* and the beginning of *March* became equal to that star. In the months of *April* and *May* it was equal to a star of the second mag-

nitude; in *June*, *July*, and *August*, to a star of the third magnitude; in *September*, *October*, and *November*, to those of the fourth magnitude; in *December* and *January* 1574 to those of the fifth; in *February* to those of the sixth magnitude; and in *March* it entirely vanished. Its colour at the beginning was clear, bright, and inclining to white; afterwards it turned a little yellow; and in *March* 1573 it became ruddy, like *Mars* or *Aldebaran*: in *May* it turned to a kind of dusky whiteness, like that we observe in *Saturn*; and that colour it retained ever after, but growing always more and more obscure. Such also was the star in the right foot of *Serpentarius*, which *Kepler's* scholars first observed *September* 30, O.S. 1604, with a light exceeding that of *Jupiter*, though the night before it was not to be seen; and from that time it decreased by little and little, and in 15 or 16 months entirely disappeared. Such a new star appearing with an unusual splendor is said to have moved *Hipparchus* to observe, and make a catalogue of, the fixed stars. As to those fixed stars that appear and disappear by turns, and increase slowly and by degrees, and scarcely ever exceed the stars of the third magnitude, they seem to be of another kind, which revolve about their axes, and, having a light and a dark side, shew those two different sides by turns. The vapours which arise from the sun, the fixed stars, and the tails of the comets, may meet at last with, and fall into, the atmospheres of the planets by their gravity, and there be condensed and turned into water and humid spirits; and from thence, by a slow heat, pass gradually into the form of salts, and sulphurs, and tinctures, and mud, and clay, and sand, and stones, and coral, and other terrestrial substances.

GENERAL SCHOLIUM.

The hypothesis of vortices is pressed with many difficulties. That every planet by a radius drawn to the sun may describe areas proportional to the times of description, the periodic times of the several parts of the vortices should observe the duplicate proportion of their distances from the sun; but that the periodic times of the planets may obtain the sesquiplicate proportion of their distances from the sun, the periodic times of the parts of the vortex ought to be in the sesquiplicate proportion of their distances. That the smaller vortices may maintain their lesser revolutions about *Saturn*, *Jupiter*, and other planets, and swim quietly and undisturbed in the greater vortex of the sun, the periodic times of the parts of the sun's vortex should be equal; but the rotation of the sun and planets about their axes, which ought to correspond with the motions of their vortices, recede far from all these proportions. The motions of the comets are exceedingly regular, are governed by the same laws with the motions of the planets, and can by no means be accounted for by the hypothesis of vortices; for comets are carried with very eccentric motions through all parts of the

heavens indifferently, with a freedom that is incompatible with the notion of a vortex.

Bodies projected in our air suffer no resistance but from the air. Withdraw the air, as is done in Mr. *Boyle's* vacuum, and the resistance ceases; for in this void a bit of fine down and a piece of solid gold descend with equal velocity. And the parity of reason must take place in the celestial spaces above the earth's atmosphere; in which spaces, where there is no air to resist their motions, all bodies will move with the greatest freedom; and the planets and comets will constantly pursue their revolutions in orbits given in kind and position, according to the laws above explained; but though these bodies may, indeed, persevere in their orbits by the mere laws of gravity, yet they could by no means have at first derived the regular position of the orbits themselves from those laws.

The six primary planets are revolved about the sun in circles concentric with the sun, and with motions directed towards the same parts, and almost in the same plane. Ten moons are revolved about the earth, Jupiter and Saturn, in circles concentric with them, with the same direction of motion, and nearly in the planes of the orbits of those planets; but it is not to be conceived that mere mechanical causes could give birth to so many regular motions, since the comets range over all parts of the heavens in very eccentric orbits; for by that kind of motion they pass easily through the orbs of the planets, and with great rapidity; and in their aphelions, where they move the slowest, and are detained the longest, they recede to the greatest distances from each other, and thence suffer the least disturbance from their mutual attractions. This most beautiful system of the sun, planets, and comets, could only proceed from the counsel and dominion of an intelligent and powerful Being. And if the fixed stars are the centres of other like systems, these, being formed by the like wise counsel, must be all subject to the dominion of One; especially since the light of the fixed stars is of the same nature with the light of the sun, and from every system light passes into all the other systems: and lest the systems of the fixed stars should, by their gravity, fall on each other mutually, he hath placed those systems at immense distances one from another.

This Being governs all things, not as the soul of the world, but as Lord over all; and on account of his dominion he is wont to be called *Lord God παντοκράτωρ*, or *Universal Ruler;* for *God* is a relative word, and has a respect to servants; and *Deity* is the dominion of God not over his own body, as those imagine who fancy God to be the soul of the world, but over servants. The Supreme God is a Being eternal, infinite, absolutely perfect; but a being, however perfect, without dominion, cannot be said to be Lord God; for we say, my God, your God, the God of *Israel*, the God of Gods, and Lord of Lords; but we do not say, my Eternal, your Eternal, the Eternal of *Israel*, the Eternal of Gods; we do not say, my Infinite, or

my Perfect: these are titles which have no respect to servants. The word *God** usually signifies *Lord;* but every lord is not a God. It is the dominion of a spiritual being which constitutes a God: a true, supreme, or imaginary dominion makes a true, supreme, or imaginary God And from his true dominion it follows that the true God is a living, intelligent, and powerful Being; and, from his other perfections, that he is supreme, or most perfect. He is eternal and infinite, omnipotent and omniscient; that is, his duration reaches from eternity to eternity; his presence from infinity to infinity; he governs all things, and knows all things that are or can be done. He is not eternity or infinity, but eternal and infinite; he is not duration or space, but he endures and is present. He endures for ever, and is every where present; and by existing always and every where, he constitutes duration and space. Since every particle of space is *always*, and every indivisible moment of duration is *every where*, certainly the Maker and Lord of all things cannot be *never* and *no where*. Every soul that has perception is, though in different times and in different organs of sense and motion, still the same indivisible person. There are given successive parts in duration, co-existent parts in space, but neither the one nor the other in the person of a man, or his thinking principle; and much less can they be found in the thinking substance of God. Every man, so far as he is a thing that has perception, is one and the same man during his whole life, in all and each of his organs of sense. God is the same God, always and every where. He is omnipresent not *virtually* only, but also *substantially;* for virtue cannot subsist without substance. In him† are all things contained and moved; yet neither affects the other: God suffers nothing from the motion of bodies; bodies find no resistance from the omnipresence of God. It is allowed by all that the Supreme God exists necessarily; and by the same necessity he exists *always* and *every where*. Whence also he is all similar, all eye, all ear, all brain, all arm, all power to perceive, to understand, and to act; but in a manner not at all human, in a manner not at all corporeal, in a manner utterly unknown to us. As a blind man has no idea of colours, so have we no idea of the manner by

* Dr. *Pocock* derives the Latin word *Deus* from the *Arabic du* (in the oblique case *di*), which signifies *Lord*. And in this sense princes are called *gods*, *Psal.* lxxxii. ver. 6; and *John* x. ver. 35. And *Moses* is called a *god* to his brother *Aaron*, and a *god* to *Pharaoh* (*Exod.* iv. ver. 16; and vii. ver. 1). And in the same sense the souls of dead princes were formerly, by the Heathens, called *gods*, but falsely, because of their want of dominion.

† This was the opinion of the Ancients. So *Pythagoras*, in *Cicer. de Nat. Deor.* lib. i *Thales*, *Anaxagoras*, *Virgil*, Georg. lib. iv. ver. 220; and Æneid, lib. vi. ver. 721. *Philo Allegor*, at the beginning of lib. i. *Aratus*, in his Phænom. at the beginning. So also the sacred writers; as St. *Paul*, *Acts*, xvii. ver. 27, 28. St. *John's* Gosp. chap. xiv. ver. 2. *Moses*, in *Deut.* iv. ver. 39; and x. ver. 14. *David*, *Psal.* cxxxix. ver. 7, 8, 9. *Solomon*, 1 *Kings*, viii. ver. 27. *Job*, xxii. ver. 12, 13, 14. *Jeremiah*, xxiii. ver. 23, 24. The Idolaters supposed the sun, moon, and stars, the souls of men, and other parts of the world, to be parts of the Supreme God, and therefore to be worshipped; but erroneously.

which the all-wise God perceives and understands all things. He is utterly void of all body and bodily figure, and can therefore neither be seen, nor heard, nor touched; nor ought he to be worshipped under the representation of any corporeal thing. We have ideas of his attributes, but what the real substance of any thing is we know not. In bodies, we see only their figures and colours, we hear only the sounds, we touch only their outward surfaces, we smell only the smells, and taste the savours; but their inward substances are not to be known either by our senses, or by any reflex act of our minds: much less, then, have we any idea of the substance of God. We know him only by his most wise and excellent contrivances of things, and final causes; we admire him for his perfections; but we reverence and adore him on account of his dominion: for we adore him as his servants; and a god without dominion, providence, and final causes, is nothing else but Fate and Nature. Blind metaphysical necessity, which is certainly the same always and every where, could produce no variety of things. All that diversity of natural things which we find suited to different times and places could arise from nothing but the ideas and will of a Being necessarily existing. But, by way of allegory, God is said to see, to speak, to laugh, to love, to hate, to desire, to give, to receive, to rejoice, to be angry, to fight, to frame, to work, to build; for all our notions of God are taken from the ways of mankind by a certain similitude, which, though not perfect, has some likeness, however. And thus much concerning God; to discourse of whom from the appearances of things, does certainly belong to Natural Philosophy.

Hitherto we have explained the phænomena of the heavens and of our sea by the power of gravity, but have not yet assigned the cause of this power. This is certain, that it must proceed from a cause that penetrates to the very centres of the sun and planets, without suffering the least diminution of its force; that operates not according to the quantity of the surfaces of the particles upon which it acts (as mechanical causes use to do), but according to the quantity of the solid matter which they contain, and propagates its virtue on all sides to immense distances, decreasing always in the duplicate proportion of the distances. Gravitation towards the sun is made up out of the gravitations towards the several particles of which the body of the sun is composed; and in receding from the sun decreases accurately in the duplicate proportion of the distances as far as the orb of Saturn, as evidently appears from the quiescence of the aphelions of the planets; nay, and even to the remotest aphelions of the comets, if those aphelions are also quiescent. But hitherto I have not been able to discover the cause of those properties of gravity from phænomena, and I frame no hypotheses; for whatever is not deduced from the phænomena is to be called an hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in ex

perimental philosophy. In this philosophy particular propositions are inferred from the phænomena, and afterwards rendered general by induction. Thus it was that the impenetrability, the mobility, and the impulsive force of bodies, and the laws of motion and of gravitation, were discovered. And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea.

And now we might add something concerning a certain most subtle Spirit which pervades and lies hid in all gross bodies; by the force and action of which Spirit the particles of bodies mutually attract one another at near distances, and cohere, if contiguous; and electric bodies operate to greater distances, as well repelling as attracting the neighbouring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the members of animal bodies move at the command of the will, namely, by the vibrations of this Spirit, mutually propagated along the solid filaments of the nerves, from the outward organs of sense to the brain, and from the brain into the muscles. But these are things that cannot be explained in few words, nor are we furnished with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which this electric and elastic Spirit operates.

END OF THE MATHEMATICAL PRINCIPLES.

THE SYSTEM OF THE WORLD.

THE

SYSTEM OF THE WORLD.

It was the ancient opinion of not a few, in the earliest ages of philosophy, that the fixed stars stood immoveable in the highest parts of the world; that under the fixed stars the planets were carried about the sun; that the earth, as one of the planets, described an annual course about the sun, while by a diurnal motion it was in the mean time revolved about its own axis; and that the sun, as the common fire which served to warm the whole, was fixed in the centre of the universe.

This was the philosophy taught of old by *Philolaus*, *Aristarchus* of *Samos*, *Plato* in his riper years, and the whole sect of the *Pythagoreans;* and this was the judgment of *Anaximander*, more ancient than any of them; and of that wise king of the *Romans*, *Numa Pompilius*, who, as a symbol of the figure of the world with the sun in the centre, erected a temple in honour of *Vesta*, of a round form, and ordained perpetual fire to be kept in the middle of it.

The *Egyptians* were early observers of the heavens; and from them, probably, this philosophy was spread abroad among other nations; for from them it was, and the nations about them, that the *Greeks*, a people of themselves more addicted to the study of philology than of nature, derived their first, as well as soundest, notions of philosophy; and in the vestal ceremonies we may yet trace the ancient spirit of the *Egyptians;* for it was their way to deliver their mysteries, that is, their philosophy of things above the vulgar way of thinking, under the veil of religious rites and hieroglyphic symbols.

It is not to be denied but that *Anaxagoras*, *Democritus*, and others, did now and then start up, who would have it that the earth possessed the centre of the world, and that the stars of all sorts were revolved towards the west about the earth quiescent in the centre, some at a swifter, others at a slower rate.

However, it was agreed on both sides that the motions of the celestial bodies were performed in spaces altogether free and void of resistance. The whim of solid orbs was of a later date, introduced by *Eudoxus*, *Calippus*, and *Aristotle;* when the ancient philosophy began to decline, and to give place to the new prevailing fictions of the Greeks.

But, above all things, the phænomena of comets can by no means consist

with the notion of solid orbs. The *Chaldeans*, the most learned astronomers of their time, looked upon the comets (which of ancient times before had been numbered among the celestial bodies) as a particular sort of planets, which, describing very eccentric orbits, presented themselves to our view only by turns, viz., once in a revolution, when they descended into the lower parts of their orbits.

And as it was the unavoidable consequence of the hypothesis of solid orbs, while it prevailed, that the comets should be thrust down below the moon, so no sooner had the late observations of astronomers restored the comets to their ancient places in the higher heavens, but these celestial spaces were at once cleared of the incumbrance of solid orbs, which by these observations were broke into pieces, and discarded for ever.

Whence it was that the planets came to be retained within any certain bounds in these free spaces, and to be drawn off from the rectilinear courses, which, left to themselves, they should have pursued, into regular revolutions in curvilinear orbits, are questions which we do not know how the ancients explained; and probably it was to give some sort of satisfaction to this difficulty that solid orbs were introduced.

The later philosophers pretend to account for it either by the action of certain vortices, as *Kepler* and *Des Cartes;* or by some other principle of impulse or attraction, as *Borelli*, *Hooke*, and others of our nation; for, from the laws of motion, it is most certain that these effects must proceed from the action of some force or other.

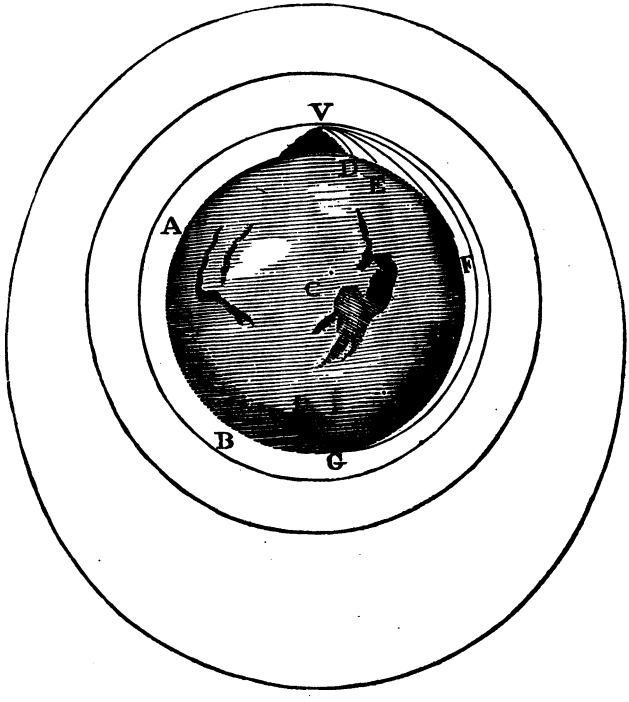
But our purpose is only to trace out the quantity and properties of this force from the phænomena (p. 218), and to apply what we discover in some simple cases as principles, by which, in a mathematical way, we may estimate the effects thereof in more involved cases; for it would be endless and impossible to bring every particular to direct and immediate observation.

We said, *in a mathematical way*, to avoid all questions about the nature or quality of this force, which we would not be understood to determine by any hypothesis; and therefore call it by the general name of a centripetal force, as it is a force which is directed towards some centre; and as it regards more particularly a body in that centre, we call it circum-solar, circum-terrestrial, circum-jovial; and in like manner in respect of other central bodies.

That by means of centripetal forces the planets may be retained in certain orbits, we may easily understand, if we consider the motions of projectiles (p. 75, 76, 77); for a stone projected is by the pressure of its own weight forced out of the rectilinear path, which by the projection alone it should have pursued, and made to describe a curve line in the air; and through that crooked way is at last brought down to the ground; and the greater the velocity is with which it is projected, the farther it goes before it falls to the earth. We may therefore suppose the velocity to be so in

creased, that it would describe an arc of 1, 2, 5, 10, 100, 1000 miles before it arrived at the earth, till at last, exceeding the limits of the earth, it should pass quite by without touching it.

Let AFB represent the surface of the earth, C its centre, VD, VE, VF, the curve lines which a body would describe, if projected in an horizontal direction from the top of an high mountain successively with more and



more velocity (p. 400); and, because the celestial motions are scarcely retarded by the little or no resistance of the spaces in which they are performed, to keep up the parity of cases, let us suppose either that there is no air about the earth, or at least that it is endowed with little or no power of resisting; and for the same reason that the body projected with a less velocity describes the lesser arc VD, and with a greater velocity the greater arc VE, and, augmenting the velocity, it goes farther and farther to F and G, if the velocity was still more and more augmented, it would reach at last quite beyond the circumference of the earth, and return to the mountain from which it was projected.

And since the areas which by this motion it describes by a radius drawn to the centre of the earth are (by Prop. 1, Book 1, *Princip. Math.*) proportional to the times in which they are described, its velocity, when it returns to the mountain, will be no less than it was at first; and, retaining the same velocity, it will describe the same curve over and over, by the same law

But if we now imagine bodies to be projected in the directions of lines parallel to the horizon from greater heights, as of 5, 10, 100, 1000, or more miles, or rather as many semi-diameters of the earth, those bodies, according to their different velocity, and the different force of gravity in different heights, will describe arcs either concentric with the earth, or variously eccentric, and go on revolving through the heavens in those trajectories, just as the planets do in their orbs.

As when a stone is projected obliquely, that is, any way but in the perpendicular direction, the perpetual deflection thereof towards the earth from the right line in which it was projected is a proof of its gravitation to the earth, no less certain than its direct descent when only suffered to fall freely from rest; so the deviation of bodies moving in free spaces from rectilinear paths, and perpetual deflection therefrom towards any place, is a sure indication of the existence of some force which from all quarters impels those bodies towards that place.

And as, from the supposed existence of gravity, it necessarily follows that all bodies about the earth must press downwards, and therefore must either descend directly to the earth, if they are let fall from rest, or at least perpetually deviate from right lines towards the earth, if they are projected obliquely; so from the supposed existence of a force directed to any centre, it will follow, by the like necessity, that all bodies upon which this force acts must either descend directly to that centre, or at least deviate perpetually towards it from right lines, if otherwise they should have moved obliquely in these right lines.

And how from the motions given we may infer the forces, or from the forces given we may determine the motions, is shewn in the two first Books of our *Principles of Philosophy.*

If the earth is supposed to stand still, and the fixed stars to be revolved in free spaces in the space of 24 hours, it is certain the forces by which the fixed stars are retained in their orbs are not directed to the earth, but to the centres of the several orbs, that is, of the several parallel circles, which the fixed stars, declining to one side and the other from the equator, describe daily; also that by radii drawn to the centres of those orbs the fixed stars describe areas exactly proportional to the times of description. Then, because the periodic times are equal (by Cor. III, Prop. IV, Book 1), it follows that the centripetal forces are as the radii of the several orbs, and that they will perpetually revolve in the same orbs. And the like consequences may be drawn from the supposed diurnal motion of the planets.

That forces should be directed to no body on which they physically depend, but to innumerable imaginary points in the axis of the earth, is an hypothesis too incongruous. It is more incongruous still that those forces should increase exactly in proportion of the distances from this axis; for

this is an indi ation of an increase to immensity, or rather to infinity; whereas the forces of natural things commonly decrease in receding from the fountain from which they flow. But, what is yet more absurd, neither are the areas described by the same star proportional to the times, nor are its revolutions performed in the same orb; for as the star recedes from the neighbouring pole, both areas and orb increase; and from the increase of the area it is demonstrated that the forces are not directed to the axis of the earth. And this difficulty (Cor. 1, Prop. II) arises from the twofold motion that is observed in the fixed stars, one diurnal round the axis of the earth, the other exceedingly slow round the axis of the ecliptic. And the explication thereof requires a composition of forces so perplexed and so variable, that it is hardly to be reconciled with any physical theory.

That there are centripetal forces actually directed to the bodies of the sun, of the earth, and other planets, I thus infer.

The moon revolves about our earth, and by radii drawn to its centre (p. 390) describes areas nearly proportional to the times in which they are described, as is evident from its velocity compared with its apparent diameter; for its motion is slower when its diameter is less (and therefore its distance greater), and its motion is swifter when its diameter is greater.

The revolutions of the satellites of Jupiter about that planet are more regular (p. 386); for they describe circles concentric with Jupiter by equable motions, as exactly as our senses can distinguish.

And so the satellites of Saturn are revolved about this planet with motions nearly (p. 387) circular and equable, scarcely disturbed by any eccentricity hitherto observed.

That Venus and Mercury are revolved about the sun, is demonstrable from their moon-like appearances (p. 388): when they shine with a full face, they are in those parts of their orbs which in respect of the earth lie beyond the sun; when they appear half full, they are in those parts which lie over against the sun; when horned, in those parts which lie between the earth and the sun; and sometimes they pass over the sun's disk, when directly interposed between the earth and the sun.

And Venus, with a motion almost uniform, describes an orb nearly circular and concentric with the sun.

But Mercury, with a more eccentric motion, makes remarkable approaches to the sun, and goes off again by turns; but it is always swifter as it is near to the sun, and therefore by a radius drawn to the sun still describes areas proportional to the times.

Lastly, that the earth describes about the sun, or the sun about the earth, by a radius from the one to the other, areas exactly proportional to the times, is demonstrable from the apparent diameter of the sun compared with its apparent motion.

These are astronomical experiments; from which it follows, by Prop. I,

II, III, in the first Book of our *Principles*, and their Corollaries (p. 212, 213, 214). that there are centripetal forces actually directed (either accurately or without considerable error) to the centres of the earth, of Jupiter, of Saturn, and of the sun. In Mercury, Venus, Mars, and the lesser planets, where experiments are wanting, the arguments from analogy must be allowed in their place.

That those forces (p. 212, 213, 214) decrease in the duplicate proportion of the distances from the centre of every planet, appears by Cor. VI, Prop. IV, Book 1; for the periodic times of the satellites of Jupiter are one to another (p. 386, 387) in the sesquiplicate proportion of their distances from the centre of this planet.

This proportion has been long ago observed in those satellites; and Mr. *Flamsted*, who had often measured their distances from Jupiter by the micrometer, and by the eclipses of the satellites, wrote to me, that it holds to all the accuracy that possibly can be discerned by our senses. And he sent me the dimensions of their orbits taken by the micrometer, and reduced to the mean distance of Jupiter from the earth, or from the sun, together with the times of their revolutions, as follows:—

The greatest elongation of the satellites from the centre of Jupiter as seen from the sun.	The periodic times of their revolutions.
′ ″ ″	d h ′ ″
1st 1 48 or 108	1 18 28 36
2d 3 01 or 181	3 13 17 54
3d 4 46 or 286	7 03 59 36
4th 8 13½ or 493½	16 18 5 13

Whence the sesquiplicate proportion may be easily seen. For example; the $16^d\ 18^h.\ 05'\ 13''$ is to the time $1^d.\ 18^h.\ 28'\ 36''$ as $493\frac{1}{2}'' \times \sqrt{493\frac{1}{2}''}$ to $108'' \times \sqrt{108''}$, neglecting those small fractions which, in observing, cannot be certainly determined.

Before the invention of the micrometer, the same distances were determined by semi-diameters of Jupiter thus:—

Distance of the	1st	2d	3d	4th
By Galileo, . . .	6	10	16	28
" Simon Marius .	6	10	16	26
" Cassini . . .	5	8	13	23
" Borelli, more exactly . . .	5⅔	8⅔	14	24⅔

After the invention of the micrometer:—

By Townley . . .	5,51	8,78	13,47	24,72
" Flamsted . . .	5,31	8,85	13,98	24,23
More accurately by the eclipses . .	5,578	8,876	14.159	24,903

And the periodic times of those satellities, by the observations of Mr. *Flamsted*, are $1^d.\ 18^h.\ 28'\ 36''$ | $3^d.\ 13^h.\ 17'\ 54''$ | $7^d.\ 3^h.\ 59'\ 36''$ | $16^d.\ 18^h.\ 5'\ 13''$, as above.

And the distances thence computed are 5,578 | 8,878 | 14,168 | 24,968, accurately agreeing with the distances by observation.

Cassini assures us (p. 388, 389) that the same proportion is observed in the circum-saturnal planets. But a longer course of observations is required before we can have a certain and accurate theory of those planets.

In the circum-solar planets, Mercury and Venus, the same proportion holds with great accuracy, according to the dimensions of their orbs, as determined by the observations of the best astronomers.

That Mars is revolved about the sun is demonstrated from the phases which it shews, and the proportion of its apparent diameters (p. 388, 389, and 390); for from its appearing full near conjunction with the sun, and gibbous in its quadratures, it is certain that it surrounds the sun.

And since its diameter appears about five times greater when in opposition to the sun than when in conjunction therewith, and its distance from the earth is reciprocally as its apparent diameter, that distance will be about five times less when in opposition to than when in conjunction with the sun; but in both cases its distance from the sun will be nearly about the same with the distance which is inferred from its gibbous appearance in the quadratures. And as it encompasses the sun at almost equal dist nces, but in respect of the earth is very unequally distant, so by radii drawn to the sun it describes areas nearly uniform; but by radii drawn to the earth, it is sometimes swift, sometimes stationary, and sometimes retrograde.

That Jupiter, in a higher orb than Mars, is likewise revolved about the sun, with a motion nearly equable, as well in distance as in the areas described, I infer thus.

Mr. *Flamsted* assured me, by letters, that all the eclipses of the innermost satellite which hitherto have been well observed do agree with his theory so nearly, as never to differ therefrom by two minutes of time; that in the outmost the error is little greater; in the outmost but one, scarcely three times greater; that in the innermost but one the difference is indeed much greater, yet so as to agree as nearly with his computations as the moon does with the common tables; and that he computes those eclipses only from the mean motions corrected by the equation of light discovered and introduced by Mr. *Romer*. Supposing, then, that the theory differs by a less error than that of 2′ from the motion of the outmost satellite as hitherto described, and taking as the periodic time $16^d.\ 18^h.\ 5'\ 13''$ to 2′ in time, so is the whole circle or 360° to the arc 1′ 48″, the error of Mr. *Flamsted's* computation, reduced to the satellite's orbit, will be less than 1′ 48″; that is, the longitude of the satellite, as seen from the centre of Jupiter, will be determined with a less error than 1′ 48″. But when

the satellite is in the middle of the shadow, that longitude is the same with the heliocentric longitude of Jupiter; and, therefore, the hypothesis which Mr. *Flamsted* follows, viz., the *Copernican*, as improved by *Kepler*, and (as to the motion of Jupiter) lately corrected by himself, rightly represents that longitude within a less error than 1′ 48″; but by this longitude, together with the geocentric longitude, which is always easily found, the distance of Jupiter from the sun is determined; which must, therefore, be the very same with that which the hypothesis exhibits. For that greatest error of 1′ 48″ that can happen in the heliocentric longitude is almost insensible, and quite to be neglected, and perhaps may arise from some yet undiscovered eccentricity of the satellite; but since both longitude and distance are rightly determined, it follows of necessity that Jupiter, by radii drawn to the sun, describes areas so conditioned as the hypothesis requires, that is, proportional to the times.

And the same thing may be concluded of Saturn from his satellite, by the observations of Mr. *Huygens* and Dr. *Halley;* though a longer series of observations is yet wanting to confirm the thing, and to bring it under a sufficiently exact computation.

For if Jupiter was viewed from the sun, it would never appear retrograde nor stationary, as it is seen sometimes from the earth, but always to go forward with a motion nearly uniform (p. 389). And from the very great inequality of its apparent geocentric motion, we infer (by Prop. III Cor. IV) that the force by which Jupiter is turned out of a rectilinear course, and made to revolve in an orb, is not directed to the centre of the earth. And the same argument holds good in Mars and in Saturn. Another centre of these forces is therefore to be looked for (by Prop. II and III, and the Corollaries of the latter), about which the areas described by radii intervening may be equable; and that this is the sun, we have proved already in Mars and Saturn nearly, but accurately enough in Jupiter. It may be alledged that the sun and planets are impelled by some other force equally and in the direction of parallel lines; but by such a force (by Cor. VI of the Laws of Motion) no change would happen in the situation of the planets one to another, nor any sensible effect follow: but our business is with the causes of sensible effects. Let us, therefore, neglect every such force as imaginary and precarious, and of no use in the phænomena of the heavens; and the whole remaining force by which Jupiter is impelled will be directed (by Prop. III, Cor. I) to the centre of the sun.

The distances of the planets from the sun come out the same, whether, with *Tycho*, we place the earth in the centre of the system, or the sun with *Copernicus:* and we have already proved that these distances are true in Jupiter.

Kepler and *Bullialdus* have, with great care (p. 388), determined the distances of the planets from the sun; and hence it is that their tables

agree best with the heavens. And in all the planets, in Jupiter and Mars, in Saturn and the earth, as well as in Venus and Mercury, the cubes of their distances are as the squares of their periodic times; and therefore (by Cor. VI, Prop. IV) the centripetal circum-solar force throughout all the planetary regions decreases in the duplicate proportion of the distances from the sun. In examining this proportion, we are to use the mean distances, or the transverse semi-axes of the orbits (by Prop. XV), and to neglect those little fractions, which, in defining the orbits, may have arisen from the insensible errors of observation, or may be ascribed to other causes which we shall afterwards explain. And thus we shall always find the said proportion to hold exactly; for the distances of Saturn, Jupiter, Mars, the Earth, Venus, and Mercury, from the sun, drawn from the observations of astronomers, are, according to the computation of *Kepler*, as the numbers 951000, 519650, 152350, 100000, 72400, 38806; by the computation of *Bullialdus*, as the numbers 954198, 522520, 152350, 100000, 72398, 38585; and from the periodic times they come out 953806, 520116, 152399, 100000, 72333, 38710. Their distances, according to *Kepler* and *Bullialdus*, scarcely differ by any sensible quantity, and where they differ most the distances drawn from the periodic times, fall in between them.

That the circum-terrestrial force likewise decreases in the duplicate proportion of the distances, I infer thus.

The mean distance of the moon from the centre of the earth, is, in semi-diameters of the earth, according to *Ptolemy*, *Kepler* in his *Ephemerides*, *Bullialdus*, *Hevelius*, and *Ricciolus*, 59; according to *Flamsted*, $59\frac{1}{3}$; according to *Tycho*, $56\frac{1}{2}$; to *Vendelin*, 60; to *Copernicus*, $60\frac{1}{3}$; to *Kircher*, $62\frac{1}{2}$ (p. 391, 392, 393).

But *Tycho*, and all that follow his tables of refraction, making the refractions of the sun and moon (altogether against the nature of light) to exceed those of the fixed stars, and that by about four or five minutes in the horizon, did thereby augment the horizontal parallax of the moon by about the like number of minutes; that is, by about the 12th or 15th part of the whole parallax. Correct this error, and the distance will become 60 or 61 semi-diameters of the earth, nearly agreeing with what others have determined.

Let us, then, assume the mean distance of the moon 60 semi-diameters of the earth, and its periodic time in respect of the fixed stars $27^{d}.\ 7^{h}.\ 43'$, as astronomers have determined it. And (by Cor. VI, Prop. IV) a body revolved in our air, near the surface of the earth supposed at rest, by means of a centripetal force which should be to the same force at the distance of the moon in the reciprocal duplicate proportion of the distances from the centre of the earth, that is, as 3600 to 1, would (secluding the resistance of the air) complete a revolution in $1^{h}.\ 24'\ 27''$.

Suppose the circumference of the earth to be 123249600 *Paris* feet, as

has been determined by the late mensuration of the *French* (vide p. 406); then the same body, deprived of its circular motion, and falling by the impulse of the same centripetal force as before, would, in one second of time, describe $15\frac{1}{12}$ *Paris* feet.

This we infer by a calculus formed upon Prop. XXXVI, and it agrees with what we observe in all bodies about the earth. For by the experiments of pendulums, and a computation raised thereon, Mr. *Huygens* has demonstrated that bodies falling by all that centripetal force with which (of whatever nature it is) they are impelled near the surface of the earth, do, in one second of time, describe $15\frac{1}{12}$ *Paris* feet.

But if the earth is supposed to move, the earth and moon together (by Cor. IV of the Laws of Motion, and Prop. LVII) will be revolved about their common centre of gravity. And the moon (by Prop. LX) will in the same periodic time, $27^{d}.\ 7^{h}.\ 43'$, with the same circum-terrestrial force diminished in the duplicate proportion of the distance, describe an orbit whose semi-diameter is to the semi-diameter of the former orbit, that is, to 60 semi-diameters of the earth, as the sum of both the bodies of the earth and moon to the first of two mean proportionals between this sum and the body of the earth; that is, if we suppose the moon (on account of its mean apparent diameter $31\frac{1}{2}'$) to be about $\frac{1}{4\frac{1}{2}}$ of the earth, as 43 to $\sqrt[3]{42 + \overline{43}|^2}$, or as about 128 to 127. And therefore the semi-diameter of the orbit, that is, the distance between the centres of the moon and earth, will in this case be $60\frac{1}{2}$ semi-diameters of the earth, almost the same with that assigned by *Copernicus*, which the *Tychonic* observations by no means disprove; and, therefore, the duplicate proportion of the decrement of the force holds good in this distance. I have neglected the increment of the orbit which arises from the action of the sun as inconsiderable; but if that is subducted, the true distance will remain about $60\frac{1}{3}$ semi-diameters of the earth.

But farther (p. 390); this proportion of the decrement of the forces is confirmed from the eccentricity of the planets, and the very slow motion of their apses; for (by the Corollaries of Prop. XLV) in no other proportion could the circum-solar planets once in every revolution descend to their least and once ascend to their greatest distance from the sun, and the places of those distances remain immoveable. A small error from the duplicate proportion would produce a motion of the apses considerable in every revolution, but in many enormous.

But now, after innumerable revolutions, hardly any such motion has been perceived in the orbs of the circum-solar planets. Some astronomers affirm that there is no such motion; others reckon it no greater than what may easily arise from the causes hereafter to be assigned, and is of no moment in the present question.

We may even neglect the motion of the moon's apsis (p. 390, 391), which is far greater than in the circum-solar planets, amounting in every revolution to three degrees; and from this motion it is demonstrable that the circum-terrestrial force decreases in no less than the duplicate, but far less than the triplicate proportion of the distance; for if the duplicate proportion was gradually changed into the triplicate, the motion of the apsis would thereby increase to infinity; and, therefore, by a very small mutation, would exceed the motion of the moon's apsis. This slow motion arises from the action of the circum-solar force, as we shall afterwards explain. But, secluding this cause, the apsis or apogeon of the moon will be fixed, and the duplicate proportion of the decrease of the circum-terrestrial force in different distances from the earth will accurately take place.

Now that this proportion has been established, we may compare the forces of the several planets among themselves (p. 391).

In the mean distance of Jupiter from the earth, the greatest elongation of the outmost satellite from Jupiter's centre (by the observations of Mr. *Flamsted*) is 8′ 13″; and therefore the distance of the satellite from the centre of Jupiter is to the mean distance of Jupiter from the centre of the sun as 124 to 52012, but to the mean distance of Venus from the centre of the sun as 124 to 7234; and their periodic times are $16\frac{3}{4}$d. and $224\frac{2}{3}$d.; and from hence (according to Cor. II, Prop. IV), dividing the distances by the squares of the times, we infer that the force by which the satellite is impelled towards Jupiter is to the force by which Venus is impelled towards the sun as 442 to 143; and if we diminish the force by which the satellite is impelled in the duplicate proportion of the distance 124 to 7234, we shall have the circum-jovial force in the distance of Venus from the sun to the circum-solar force by which Venus is impelled as $\frac{143}{1100}$ to 143, or as 1 to 1100; wherefore at equal distances the circum-solar force is 1100 times greater than the circum-jovial.

And, by the like computation, from the periodic time of the satellite of Saturn 15d. 22h. and its greatest elongation from Saturn, while that planet is in its mean distance from us, 3′ 20″, it follows that the distance of this satellite from Saturn's centre is to the distance of Venus from the sun as $92\frac{2}{5}$ to 7234; and from thence that the absolute circum-solar force is 2360 times greater than the absolute circum-saturnal.

From the regularity of the heliocentric and irregularity of the geocentric motions of Venus, of Jupiter, and the other planets, it is evident (by Cor. IV, Prop. III) that the circum-terrestrial force, compared with the circum-solar, is very small.

Ricciolus and *Vendelin* have severally tried to determine the sun's parallax from the moon's dichotomies observed by the telescope, and they agree that it does not exceed half a minute.

Kepler, from *Tycho's* observations and his own, found the parallax of

Mars insensible, even in opposition to the sun, when that parallax is something greater than the sun's.

Flamsted attempted the same parallax with the micrometer in the perigeon position of Mars, but never found it above 25″; and thence concluded the sun's parallax at most 10″.

Whence it follows that the distance of the moon from the earth bears no greater proportion to the distance of the earth from the sun than 29 to 10000; nor to the distance of Venus from the sun than 29 to 7233.

From which distances, together with the periodic times, by the method above explained, it is easy to infer that the absolute circum-solar force is greater than the absolute circum-terrestrial force at least 229400 times.

And though we were only certain, from the observations of *Ricciolus* and *Vendelin*, that the sun's parallax was less than half a minute, yet from this it will follow that the absolute circum-solar force exceeds the absolute circum-terrestrial force 8500 times.

By the like computations I happened to discover an analogy, that is observed between the forces and the bodies of the planets; but, before I explain this analogy, the apparent diameters of the planets in their mean distances from the earth must be first determined.

Mr. *Flamsted* (p. 387), by the micrometer, measured the diameter of Jupiter 40″ or 41″; the diameter of Saturn's ring 50″; and the diameter of the sun about 32′ 13″ (p. 387).

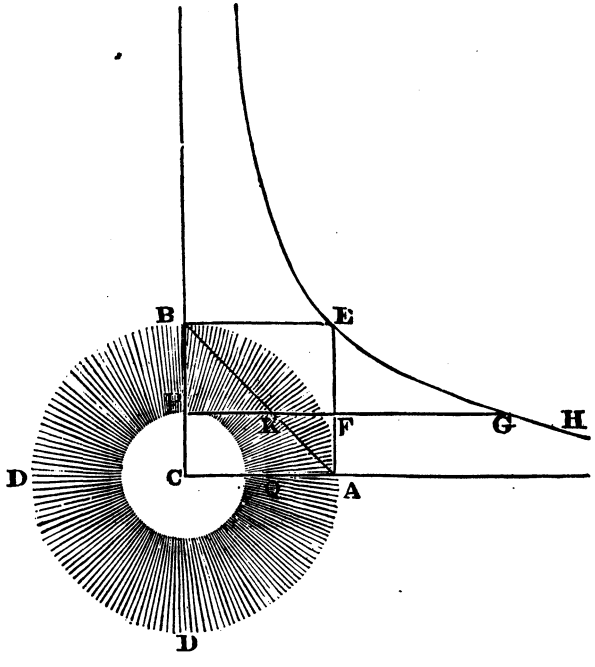
But the diameter of Saturn is to the diameter of the ring, according to Mr. *Huygens* and Dr. *Halley*, as 4 to 9; according to *Galletus*, as 4 to 10; and according to *Hooke* (by a telescope of 60 feet), as 5 to 12. And from the mean proportion, 5 to 12, the diameter of Saturn's body is inferred about 21″.

Such as we have said are the apparent magnitudes; but, because of the unequal refrangibility of light, all lucid points are dilated by the telescope, and in the focus of the object-glass possess a circular space whose breadth is about the 50th part of the aperture of the glass.

It is true, that towards the circumference the light is so rare as hardly to move the sense; but towards the middle, where it is of greater density, and is sensible enough, it makes a small lucid circle, whose breadth varies according to the splendor of the lucid point, but is generally about the 3d, or 4th, or 5th part of the breadth of the whole.

Let ABD represent the circle of the whole light; PQ the small circle of the denser and clearer light; C the centre of both; CA, CB, semi-diameters of the greater circle containing a right angle at C; ACBE the square comprehended under these semi-diameters; AB the diagonal of that square; EGH an hyperbola with the centre C and asymptotes CA, CB PG a perpendicular erected from any point P of the line BC, and meeting the hyperbola in G, and the right lines AB, AE, in K and F: and the

density of the light in any place P, will, by my computation, be as the line FG, and therefore at the centre infinite, but near the circumference very small. And the whole light within the small circle PQ is to the whole without as the area of the quadrilateral figure CAKP to the trian-



gle PKB. And we are to understand the small circle PQ to be there terminated, where FG, the density of the light, begins to be less than what is required to move the sense.

Hence it was, that, at the distance of 191382 feet, a fire of 3 feet in diameter, through a telescope of 3 feet, appeared to Mr. *Picart* of 8″ in breadth, when it should have appeared only of 3″ 14‴; and hence it is that the brighter fixed stars appear through the telescope as of 5″ or 6″ in diameter, and that with a good full light; but with a fainter light they appear to run out to a greater breadth. Hence, likewise, it was that *Hevelius*, by diminishing the aperture of the telescope, did cut off a great part of the light towards the circumference, and brought the disk of the star to be more distinctly defined, which, though hereby diminished, did yet appear as of 5″ or 6″ in diameter. But Mr. *Huygens*, only by clouding the eye-glass with a little smoke, did so effectually extinguish this scattered light, that the fixed stars appeared as mere points, void of all sensible breadth. Hence also it was that Mr. *Huygens*, from the breadth of bodies interposed to intercept the whole light of the planets, reckoned their diameters greater than others have measured them by the micrometer; for the

scattered light, which could not be seen before for the stronger light of the planet, when the planet is hid, appears every way farther spread. Lastly, from hence it is that the planets appear so small in the disk of the sun, being lessened by the dilated light. For to *Hevelius*, *Galletius*, and Dr. *Halley*, Mercury did not seem to exceed 12″ or 15″; and Venus appeared to Mr. *Crabtrie* only 1′ 3″; to *Horrox* but 1′ 12″; though by the mensurations of *Hevelius* and *Hugenius* without the sun's disk, it ought to have been seen at least 1′ 24″. Thus the apparent diameter of the moon, which in 1684, a few days both before and after the sun's eclipse, was measured at the observatory of *Paris* 31′ 30″, in the eclipse itself did not seem to exceed 30′ or 30′ 05″; and therefore the diameters of the planets are to be diminished when without the sun, and to be augmented when within it, by some seconds. But the errors seem to be less than usual in the mensurations that are made by the micrometer. So from the diameter of the shadow, determined by the eclipses of the satellites, Mr. *Flamsted* found that the semi-diameter of Jupiter was to the greatest elongation of the outmost satellite as 1 to 24,903. Wherefore since that elongation is 8′ 13″, the diameter of Jupiter will be 39½″; and, rejecting the scattered light, the diameter found by the micrometer 40″ or 41″ will be reduced to 39½″; and the diameter of Saturn 21″ is to be diminished by the like correction, and to be reckoned 20″, or something less. But (if I am not mistaken) the diameter of the sun, because of its stronger light, is to be diminished something more, and to be reckoned about 32′, or 32′ 6″.

That bodies so different in magnitude should come so near to an analogy with their forces, is not without some mystery (p. 400).

It may be that the remoter planets, for want of heat, have not those metallic substances and ponderous minerals with which our earth abounds; and that the bodies of Venus and Mercury, as they are more exposed to the sun's heat, are also harder baked, and more compact.

For, from the experiment of the burning-glass, we see that the heat increases with the density of light; and this density increases in the reciprocal duplicate proportion of the distance from the sun; from whence the sun's heat in Mercury is proved to be sevenfold its heat in our summer seasons. But with this heat our water boils; and those heavy fluids, quicksilver and the spirit of vitriol, gently evaporate, as I have tried by the thermometer; and therefore there can be no fluids in Mercury but what are heavy, and able to bear a great heat, and from which substances of great density may be nourished.

And why not, if God has placed different bodies at different distances from the sun, so as the denser bodies always possess the nearer places, and each body enjoys a degree of heat suitable to its condition, and proper for its nourishment? From this consideration it will best appear that the weights of all the planets are one to another as their forces.

But I should be glad the diameters of the planets were more accurately measured; and that may be done, if a lamp, set at a great distance, is made to shine through a circular hole, and both the hole and the light of the lamp are so diminished that the spectrum may appear through the telescope just like the planet, and may be defined by the same measure: then the diameter of the hole will be to its distance from the objective glass as the true diameter of the planet to its distance from us. The light of the lamp may be diminished by the interposition either of pieces of cloth, or of smoked glass.

Of kin to the analogy we have been describing, there is another observed between the forces and the bodies attracted (p. 395, 396, 397). Since the action of the centripetal force upon the planets decreases in the duplicate proportion of the distance, and the periodic time increases in the sesquiplicate thereof, it is evident that the actions of the centripetal force, and therefore the periodic times, would be equal in equal planets at equal distances from the sun; and in equal distances of unequal planets the total actions of the centripetal force would be as the bodies of the planets; for if the actions were not proportional to the bodies to be moved, they could not equally retract these bodies from the tangents of their orbs in equal times: nor could the motions of the satellites of Jupiter be so regular, if it was not that the circum-solar force was equally exerted upon Jupiter and all its satellites in proportion of their several weights. And the same thing is to be said of Saturn in respect of its satellites, and of our earth in respect of the moon, as appears from Cor. II and III, Prop. LXV. And, therefore, at equal distances, the actions of the centripetal force are equal upon all the planets in proportion of their bodies, or of the quantities of matter in their several bodies; and for the same reason must be the same upon all the particles of the same size of which the planet is composed; for if the action was greater upon some sort of particles than upon others than in proportion to their quantity of matter, it would be also greater or less upon the whole planets not in proportion to the quantity only, but likewise of the sort of the matter more copiously found in one and more sparingly in another.

In such bodies as are found on our earth of very different sorts, I examined this analogy with great accuracy (p. 343, 344).

If the action of the circum-terrestrial force is proportional to the bodies to be moved, it will (by the Second Law of Motion) move them with equal velocity in equal times, and will make all bodies let fall to descend through equal spaces in equal times, and all bodies hung by equal threads to vibrate in equal times. If the action of the force was greater, the times would be less; if that was less, these would be greater.

But it has been long ago observed by others, that (allowance being made for the small resistance of the air) all bodies descend through equal spaces

in equal times; and, by the help of pendulums, that equality of times may be distinguished to great exactness.

I tried the thing in gold, silver, lead, glass, sand, common salt wood, water, and wheat. I provided two equal wooden boxes. I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the centre of oscillation of the other. The boxes, hung by equal threads of 11 feet, made a couple of pendulums perfectly equal in weight and figure, and equally exposed to the resistance of the air: and, placing the one by the other, I observed them to play together forwards and backwards for a long while, with equal vibrations. And therefore (by Cor. 1 and VI, Prop. XXIV, Book II) the quantity of matter in the gold was to the quantity of matter in the wood as the action of the motive force upon all the gold to the action of the same upon all the wood; that is, as the weight of the one to the weight of the other.

And by these experiments, in bodies of the same weight, could have discovered a difference of matter less than the thousandth part of the whole.

Since the action of the centripetal force upon the bodies attracted is, at equal distances, proportional to the quantities of matter in those bodies, reason requires that it should be also proportional to the quantity of matter in the body attracting.

For all action is mutual, and (p. 83, 93, by the Third Law of Motion) makes the bodies mutually to approach one to the other, and therefore must be the same in both bodies. It is true that we may consider one body as attracting, another as attracted; but this distinction is more mathematical than natural. The attraction is really common of either to other, and therefore of the same kind in both.

And hence it is that the attractive force is found in both. The sun attracts Jupiter and the other planets; Jupiter attracts its satellites; and, for the same reason, the satellites act as well one upon another as upon Jupiter, and all the planets mutually one upon another.

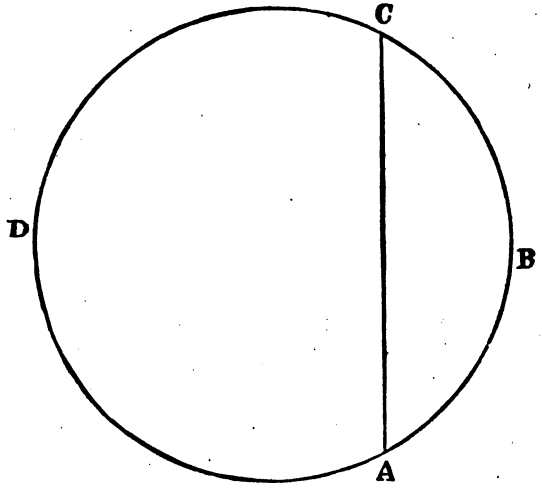
And though the mutual actions of two planets may be distinguished and considered as two, by which each attracts the other, yet, as those actions are intermediate, they do not make two but one operation between two terms. Two bodies may be mutually attracted each to the other by the contraction of a cord interposed. There is a double cause of action, to wit, the disposition of both bodies, as well as a double action in so far as the action is considered as upon two bodies; but as betwixt two bodies it is but one single one. It is not one action by which the sun attracts Jupiter, and another by which Jupiter attracts the sun; but it is one action by which the sun and Jupiter mutually endeavour to approach each the other. By the action with which the sun attracts Jupiter, Jupiter and the sun endeavours to come nearer together (by the Third Law of Motion); and by the action with which Jupiter attracts the sun, likewise Ju-

piter and the sun endeavor to come nearer together. But the sun is not attracted towards Jupiter by a twofold action, nor Jupiter by a twofold action towards the sun; but it is one single intermediate action, by which both approach nearer together.

Thus iron draws the load-stone (p. 93), as well as the load-stone draws the iron; for all iron in the neighbourhood of the load-stone draws other iron. But the action betwixt the load-stone and iron is single, and is considered as single by the philosophers. The action of iron upon the load-stone, is, indeed, the action of the load-stone betwixt itself and the iron, by which both endeavour to come nearer together: and so it manifestly appears; for if you remove the load-stone, the whole force of the iron almost ceases.

In this sense it is that we are to conceive one single action to be exerted betwixt two planets, arising from the conspiring natures of both: and this action standing in the same relation to both, if it is proportional to the quantity of matter in the one, it will be also proportional to the quantity of matter in the other.

Perhaps it may be objected, that, according to this philosophy (p. 398), all bodies should mutually attract one another, contrary to the evidence of experiments in terrestrial bodies; but I answer, that the experiments in terrestrial bodies come to no account; for the attraction of homogeneous spheres near their surfaces are (by Prop. LXXII) as their diameters. Whence a sphere of one foot in diameter, and of a like nature to the earth, would attract a small body placed near its surface with a force 20000000 times less than the earth would do if placed near its surface; but so small a force could produce no sensible effect. If two such spheres were distant but by $\frac{1}{4}$ of an inch, they would not, even in spaces void of



resistance, come together by the force of their mutual attraction in less than a month's time; and less spheres will come together at a rate yet slower, viz., in the proportion of their diameters. Nay, whole mountains will not be sufficient to produce any sensible effect. A mountain of an hemispherical figure, three miles high, and six broad, will not, by its attraction, draw the pendulum two minutes out of the true perpendicular; and it is only in the great bodies of the planets that these forces are to be perceived, unless we may reason about smaller bodies in manner following.

Let ABCD (p. 93) represent the globe of the earth cut by any plane AC into two parts ACB, and ACD. The part ACB bearing upon the part ACD presses it with its whole weight; nor can the part ACD sustain this pressure and continue unmoved, if it is not opposed by an equal contrary pressure. And therefore the parts equally press each other by their weights, that is, equally attract each other, according to the third Law of Motion; and, if separated and let go, would fall towards each other with velocities reciprocally as the bodies. All which we may try and see in the load-stone, whose attracted part does not propel the part attracting, but is only stopped and sustained thereby.

Suppose now that ACB represents some small body on the earth's surface; then, because the mutual attractions of this particle, and of the remaining part ACD of the earth towards each other, are equal, but the attraction of the particle towards the earth (or its weight) is as the matter of the particle (as we have proved by the experiment of the pendulums), the attraction of the earth towards the particle will likewise be as the matter of the particle; and therefore the attractive forces of all terrestrial bodies will be as their several quantities of matter.

The forces (p. 396), which are as the matter in terrestrial bodies of all forms, and therefore are not mutable with the forms, must be found in all sorts of bodies whatsoever, celestial as well as terrestrial, and be in all proportional to their quantities of matter, because among all there is no difference of substance, but of modes and forms only. But in the celestial bodies the same thing is likewise proved thus. We have shewn that the action of the circum-solar force upon all the planets (reduced to equal distances) is as the matter of the planets; that the action of the circum-jovial force upon the satellites of Jupiter observes the same law; and the same thing is to be said of the attraction of all the planets towards every planet: but thence it follows (by Prop. LXIX) that their attractive forces are as their several quantities of matter.

As the parts of the earth mutually attract one another, so do those of all the planets. If Jupiter and its satellites were brought together, and formed into one globe, without doubt they would continue mutually to attract one another as before. And, on the other hand, if the body of Jupiter was broke into more globes, to be sure, these would no less attract

one another than they do the satellites now. From these attractions it is that the bodies of the earth and all the planets effect a spherical figure, and their parts cohere, and are not dispersed through the æther. But we have before proved that these forces arise from the universal nature of matter (p. 398), and that, therefore, the force of any whole globe is made up of the several forces of all its parts. And from thence it follows (by Cor. III, Prop. LXXIV) that the force of every particle decreases in the duplicate proportion of the distance from that particle; and (by Prop. LXXIII and LXXV) that the force of an entire globe, reckoning from the surface outwards, decreases in the duplicate, but, reckoning inwards, in the simple proportion of the distances from the centres, if the matter of the globe be uniform. And though the matter of the globe, reckoning from the centre towards the surface, is not uniform (p. 398, 399), yet the decrease in the duplicate proportion of the distance outwards would (by Prop. LXXVI) take place, provided that difformity is similar in places round about at equal distances from the centre. And two such globes will (by the same Proposition) attract one the other with a force decreasing in the duplicate proportion of the distance between their centres.

Wherefore the absolute force of every globe is as the quantity of matter which the globe contains; but the motive force by which every globe is attracted towards another, and which, in terrestrial bodies, we commonly call their weight, is as the content under the quantities of matter in both globes applied to the square of the distance between their centres (by Cor. IV, Prop. LXXVI), to which force the quantity of motion, by which each globe in a given time will be carried towards the other, is proportional. And the accelerative force, by which every globe according to its quantity of matter is attracted towards another, is as the quantity of matter in that other globe applied to the square of the distance between the centres of the two (by Cor. II, Prop. LXXVI); to which force, the velocity by which the attracted globe will, in a given time, be carried towards the other is proportional. And from these principles well understood, it will be now easy to determine the motions of the celestial bodies among themselves.

From comparing the forces of the planets one with another, we have above seen that the circum-solar does more than a thousand times exceed all the rest; but by the action of a force so great it is unavoidable but that all bodies within, nay, and far beyond, the bounds of the planetary system must descend directly to the sun, unless by other motions they are impelled towards other parts: nor is our earth to be excluded from the number of such bodies; for certainly the moon is a body of the same nature with the planets, and subject to the same attractions with the other planets, seeing it is by the circum-terrestrial force that it is retained in its orbit. But that the earth and moon are equally attracted towards the sun, we have above proved; we have likewise before proved that all bodies are subject to

the said common laws of attraction. Nay, supposing any of those bodies to be deprived of its circular motion about the sun, by having its distance from the sun, we may find (by Prop. XXXVI) in what space of time it would in its descent arrive at the sun; to wit, in half that periodic time in which the body might be revolved at one half of its former distance; or in a space of time that is to the periodic time of the planet as 1 to $4\sqrt{2}$; as that Venus in its descent would arrive at the sun in the space of 40 days, Jupiter in the space of two years and one month, and the earth and moon together in the space of 66 days and 19 hours. But, since no such thing happens, it must needs be, that those bodies are moved towards other parts (p. 75), nor is every motion sufficient for this purpose. To hinder such a descent, a due proportion of velocity is required. And hence depends the force of the argument drawn from the retardation of the motions of the planets. Unless the circum-solar force decreased in the duplicate ratio of their increasing slowness, the excess thereof would force those bodies to descend to the sun; for instance, if the motion (*cæteris paribus*) was retarded by one half, the planet would be retained in its orb by one fourth of the former circum-solar force, and by the excess of the other three fourths would descend to the sun. And therefore the planets (Saturn, Jupiter, Mars, Venus, and Mercury) are not really retarded in their perigees, nor become really stationary, or regressive with slow motions. All these are but apparent, and the absolute motions, by which the planets continue to revolve in their orbits, are always direct, and nearly equable. But that such motions are performed about the sun, we have already proved; and therefore the sun, as the centre of the absolute motions, is quiescent. For we can by no means allow quiescence to the earth, lest the planets in their perigees should indeed be truly retarded, and become truly stationary and regressive, and so for want of motion should descend to the sun. But farther; since the planets (Venus, Mars, Jupiter, and the rest) by radii drawn to the sun describe regular orbits, and areas (as we have shewn) nearly and to sense proportional to the times, it follows (by Prop. III, and Cor. III, Prop. LXV) that the sun is moved with no notable force, unless perhaps with such as all the planets are equally moved with, according to their several quantities of matter, in parallel lines, and so the whole system is transferred in right lines. Reject that translation of the whole system, and the sun will be almost quiescent in the centre thereof. If the sun was revolved about the earth, and carried the other planets round about itself, the earth ought to attract the sun with a great force, but the circum-solar planets with no force producing any sensible effect, which is contrary to Cor. III, Prop. LXV. Add to this, that if hitherto the earth, because of the gravitation of its parts, has been placed by most authors in the lowermost region of the universe; now, for better reason, the sun possessed of a centripetal force exceeding our terrestrial gravitation a thousand

times and more, ought to be depressed into the lowermost place, and to be held for the centre of the system. And thus the true disposition of the whole system will be more fully and more exactly understood.

Because the fixed stars are quiescent one in respect of another (p. 401, 402), we may consider the sun, earth, and planets, as one system of bodies carried hither and thither by various motions among themselves; and the common centre of gravity of all (by Cor. IV of the Laws of Motion) will either be quiescent, or move uniformly forward in a right line: in which case the whole system will likewise move uniformly forward in right lines. But this is an hypothesis hardly to be admitted; and, therefore, setting it aside, that common centre will be quiescent: and from it the sun is never far removed. The common centre of gravity of the sun and Jupiter falls on the surface of the sun; and though all the planets were placed towards the same parts from the sun with Jupiter the common centre of the sun and all of them would scarcely recede twice as far from the sun's centre; and, therefore, though the sun, according to the various situation of the planets, is variously agitated, and always wandering to and fro with a slow motion of libration, yet it never recedes one entire diameter of its own body from the quiescent centre of the whole system. But from the weights of the sun and planets above determined, and the situation of all among themselves, their common centre of gravity may be found; and, this being given, the sun's place to any supposed time may be obtained.

About the sun thus librated the other planets are revolved in elliptic orbits (p 403), and, by radii drawn to the sun, describe areas nearly proportional to the times, as is explained in Prop. LXV. If the sun was quiescent, and the other planets did not act mutually one upon another, their orbits would be elliptic, and the areas exactly proportional to the times (by Prop. XI, and Cor. 1, Prop. XIII). But the actions of the planets among themselves, compared with the actions of the sun on the planets, are of no moment, and produce no sensible errors. And those errors are less in revolutions about the sun agitated in the manner but now described than if those revolutions were made about the sun quiescent (by Prop. LXVI, and Cor. Prop. LXVIII), especially if the focus of every orbit is placed in the common centre of gravity of all the lower included planets; viz., the focus of the orbit of Mercury in the centre of the sun; the focus of the orbit of Venus in the common centre of gravity of Mercury and the sun; the focus of the orbit of the earth in the common centre of gravity of Venus, Mercury, and the sun; and so of the rest. And by this means the foci of the orbits of all the planets, except Saturn, will not be sensibly removed from the centre of the sun, nor will the focus of the orbit of Saturn recede sensibly from the common centre of gravity of Jupiter and the sun. And therefore astronomers are not far from the truth, when they reckon the sun's centre the common focus of all the planetary orbits. In Saturn itself

the error thence arising does not exceed 1′ 45″. And if its orbit, by placing the focus thereof in the common centre of gravity of Jupiter and the sun, shall happen to agree better with the phænomena, from thence all that we have said will be farther confirmed.

If the sun was quiescent, and the planets did not act one on another, the aphelions and nodes of their orbits would likewise (by Prop. 1, XI, and Cor. Prop. XIII) be quiescent. And the longer axes of their elliptic orbits would (by Prop. XV) be as the cubic roots of the squares of their periodic times: and therefore from the given periodic times would be also given. But those times are to be measured not from the equinoctial points, which are moveable, but from the first star of Aries. Put the semi-axis of the earth's orbit 100000, and the semi-axes of the orbits of Saturn, Jupiter, Mars, Venus, and Mercury, from their periodic times, will come out 953806, 520116, 152399, 72333, 38710 respectively. But from the sun's motion every semi-axis is increased (by Prop. LX) by about one third of the distance of the sun's centre from the common centre of gravity of the sun and planet (p. 405, 406.) And from the actions of the exterior planets on the interior, the periodic times of the interior are something protracted, though scarcely by any sensible quantity; and their aphelions are transferred (by Cor. VI and VII, Prop. LXVI) by very slow motions in *consequentia*. And on the like account the periodic times of all, especially of the exterior planets, will be prolonged by the actions of the comets, if any such there are, without the orb of Saturn, and the aphelions of all will be thereby carried forwards in *consequentia*. But from the progress of the aphelions the regress of the nodes follows (by Cor. XI, XIII, Prop. LXVI). And if the plane of the ecliptic is quiescent, the regress of the nodes (by Cor. XVI, Prop. LXVI) will be to the progress of the aphelion in every orbit as the regress of the nodes of the moon's orbit to the progress of its apogeon nearly, that is, as about 10 to 21. But astronomical observations seem to confirm a very slow progress of the aphelions, and a regress of the nodes in respect of the fixed stars. And hence it is probable that there are comets in the regions beyond the planets, which, revolving in very eccentric orbs, quickly fly through their perihelion parts, and, by an exceedingly slow motion in their aphelions, spend almost their whole time in the regions beyond the planets; as we shall afterwards explain more at large.

The planets thus revolved about the sun (p. 413, 414, 415) may at the same time carry others revolving about themselves as satellites or moons, as appears by Prop. LXVI. But from the action of the sun our moon must move with greater velocity, and, by a radius drawn to the earth, describe an area greater for the time; it must have its orbit less curve, and therefore approach nearer to the earth in the syzygies than in the quadratures, except in so far as the motion of eccentricity hinders those effects.

For the eccentricity is greatest when the moon's apogeon is in the syzygies, and least when the same is in the quadratures; and hence it is that the perigeon moon is swifter and nearer to us, but the apogeon moon slower and farther from us, in the syzygies than in the quadratures. But farther; the apogeon has a progressive and the nodes a regressive motion, both unequable. For the apogeon is more swiftly progressive in its syzygies, more slowly regressive in its quadratures, and by the excess of its progress above its regress is yearly transferred in *consequentia;* but the nodes are quiescent in their syzygies, and most swiftly regressive in their quadratures. But farther, still, the greatest latitude of the moon is greater in its quadratures than in its syzygies; and the mean motion swifter in the aphelion of the earth than in its perihelion. More inequalities in the moon's motion have not hitherto been taken notice of by astronomers: but all these follow from our principles in Cor. II, III, IV, V, VI, VII, VIII, IX, X, XI, XII, XIII, Prop. LXVI, and are known really to exist in the heavens. And this may seen in that most ingenious, and if I mistake not, of all, the most acccurate, hypothesis of Mr. *Horrox*, which Mr. *Flamsted* has fitted to the heavens; but the astronomical hypotheses are to be corrected in the motion of the nodes; for the nodes admit the greatest equation or prosthaphæresis in their octants, and this inequality is most conspicuous when the moon is in the nodes, and therefore also in the octants; and hence it was that *Tycho*, and others after him, referred this inequality to the octants of the moon, and made it menstrual; but the reasons by us adduced prove that it ought to be referred to the octants of the nodes, and to be made annual.

Beside those inequalities taken notice of by astronomers (p. 414, 445, 447,) there are yet some others, by which the moon's motions are so disturbed, that hitherto by no law could they be reduced to any certain regulation. For the velocities or horary motions of the apogee and nodes of the moon, and their equations, as well as the difference betwixt the greatest eccentricity in the syzygies and the least in the quadratures, and that inequality which we call the variation, in the progress of the year are augmented and diminished (by Cor. XIV, Prop. LXVI) in the triplicate ratio of the sun's apparent diameter. Beside that, the variation is mutable nearly in the duplicate ratio of the time between the quadratures (by Cor. I and II, Lem. X, and Cor. XVI, Prop. LXVI); and all those inequalities are something greater in that part of the orbit which respects the sun than in the opposite part, but by a difference that is scarcely or not at all perceptible.

By a computation (p. 422), which for brevity's sake I do not describe, I also find that the area which the moon by a radius drawn to the earth describes in the several equal moments of time is nearly as the sum of the number $237\frac{3}{10}$, and versed sine of the double distance of the moon from

the nearest quadrature in a circle whose radius is unity; and therefore that the square of the moon's distance from the earth is as that sum divided by the horary motion of the moon. Thus it is when the variation in the octants is in its mean quantity; but if the variation is greater or less, that versed sine must be augmented or diminished in the same ratio. Let astronomers try how exactly the distances thus found will agree with the moon's apparent diameters.

From the motions of our moon we may derive the motions of the moons or satellites of Jupiter and Saturn (p. 413); for the mean motion of the nodes of the outmost satellite of Jupiter is to the mean motion of the nodes of our moon in a proportion compounded of the duplicate proportion of the periodic time of the earth about the sun to the periodic time of Jupiter about the sun, and the simple proportion of the periodic time of the satellite about Jupiter to the periodic time of our moon about the earth (by Cor. XVI, Prop. LXVI): and therefore those nodes, in the space of a hundred years, are carried S° 24′ backwards, or in *antecedentia*. The mean motions of the nodes of the inner satellites are to the (mean) motion of (the nodes of) the outmost as their periodic times to the periodic time of this, by the same corollary, and are thence given. And the motion of the apsis of every satellite in *consequentia* is to the motion of its nodes in *antecedentia*, as the motion of the apogee of our moon to the motion of its nodes (by the same Corollary), and is thence given. The greatest equations of the nodes and line of the apses of each satellite are to the greatest equations of the nodes and the line of the apses of the moon respectively as the motion of the nodes and line of the apses of the satellites in the time of one revolution of the first equations to the motion of the nodes and apogeon of the moon in the time of one revolution of the last equations. The variation of a satellite seen from Jupiter is to the variation of our moon in the same proportion as the whole motions of their nodes respectively, during the times in which the satellite and our moon (after parting from) are revolved (again) to the sun, by the same Corollary; and therefore in the outmost satellite the variation does not exceed 5″ 12‴. From the small quantity of those inequalities, and the slowness of the motions, it happens that the motions of the satellites are found to be so regular, that the more modern astronomers either deny all motion to the nodes, or affirm them to be very slowly regressive.

(P. 404). While the planets are thus revolved in orbits about remote centres, in the mean time they make their several rotations about their proper axes; the sun in 26 days; Jupiter in 9^h. 56′; Mars in $24\frac{2}{3}^h$.; Venus in 23^h.; and that in planes not much inclined to the plane of the ecliptic, and according to the order of the signs, as astronomers determine from the spots or maculæ that by turns present themselves to our sight in their bodies; and there is a like revolution of our earth performed in 24^h.;

and those motions are neither accelerated nor retarded by the actions of the centripetal forces, as appears by Cor. XXII, Prop. LXVI; and therefore of all others they are the most equable and most fit for the mensuration of time; but those revolutions are to be reckoned equable not from their return to the sun, but to some fixed star: for as the position of the planets to the sun is unequably varied, the revolutions of those planets from sun to sun are rendered unequable.

In like manner is the moon revolved about its axis by a motion most equable in respect of the fixed stars, viz., in $27^{d}.\ 7^{h}.\ 43'$, that is, in the space of a sidereal month; so that this diurnal motion is equal to the mean motion of the moon in its orbit; upon which account the same face of the moon always respects the centre about which this mean motion is performed, that is, the exterior focus of the moon's orbit nearly; and hence arises a deflection of the moon's face from the earth, sometimes towards the east, and other times towards the west, according to the position of the focus which it respects; and this deflection is equal to the equation of the moon's orbit, or to the difference betwixt its mean and true motions; and this is the moon's libration in longitude: but it is likewise affected with a libration in latitude arising from the inclination of the moon's axis to the plane of the orbit in which the moon is revolved about the earth; for that axis retains the same position to the fixed stars nearly, and hence the poles present themselves to our view by turns, as we may understand from the example of the motion of the earth, whose poles, by reason of the inclination of its axis to the plane of the ecliptic, are by turns illuminated by the sun. To determine exactly the position of the moon's axis to the fixed stars, and the variation of this position, is a problem worthy of an astronomer.

By reason of the diurnal revolutions of the planets, the matter which they contain endeavours to recede from the axis of this motion; and hence the fluid parts rising higher towards the equator than about the poles (p. 405), would lay the solid parts about the equator under water, if those parts did not rise also (p. 405, 409): upon which account the planets are something thicker about the equator than about the poles; and their equinoctial points (p. 413) thence become regressive; and their axes, by a motion of nutation, twice in every revolution, librate towards their ecliptics, and twice return again to their former inclination, as is explained in Cor. XVIII, Prop. LXVI; and hence it is that Jupiter, viewed through very long telescopes, does not appear altogether round (p. 409), but having its diameter that lies parallel to the ecliptic something longer than that which is drawn from north to south.

And from the diurnal motion and the attractions (p. 415, 418) of the sun and moon our sea ought twice to rise and twice to fall every day, as well lunar as solar (by Cor. XIX, XX, Prop. LXVI), and the greatest

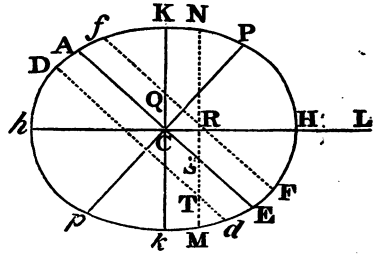
height of the water to happen before the sixth hour of either day and after the twelfth hour preceding. By the slowness of the diurnal motion the flood is retracted to the twelfth hour; and by the force of the motion of reciprocation it is protracted and deferred till a time nearer to the sixth hour. But till that time is more certainly determined by the phænomena, choosing the middle between those extremes, why may we not conjecture the greatest height of the water to happen at the third hour? for thus the water will rise all that time in which the force of the luminaries to raise it is greater, and will fall all that time in which their force is less; viz., from the ninth to the third hour when that force is greater, and from the third to the ninth when it is less. The hours I reckon from the appulse of each luminary to the meridian of the place, as well under as above the horizon; and by the hours of the lunar day I understand the twenty-fourth parts of that time which the moon spends before it comes about again by its apparent diurnal motion to the meridian of the place which it left the day before.

But the two motions which the two luminaries raise will not appear distinguished, but will make a certain mixed motion. In the conjunction or opposition of the luminaries their forces will be conjoined, and bring on the greatest flood and ebb. In the quadratures the sun will raise the waters which the moon depresseth, and depress the waters which the moon raiseth; and from the difference of their forces the smallest of all tides will follow. And because (as experience tells us) the force of the moon is greater than that of the sun, the greatest height of the water will happen about the third lunar hour. Out of the syzygies and quadratures the greatest tide which by the single force of the moon ought to fall out at the third lunar hour, and by the single force of the sun at the third solar hour, by the compounded forces of both must fall out in an intermediate time that approaches nearer to the third hour of the moon than to that of the sun; and, therefore, while the moon is passing from the syzygies to the quadratures, during which time the third hour of the sun precedes the third of the moon, the greatest tide will precede the third lunar hour, and that by the greatest interval a little after the octants of the moon; and by like intervals the greatest tide will follow the third lunar hour, while the moon is passing from the quadratures to the syzygies.

But the effects of the luminaries depend upon their distances from the earth; for when they are less distant their effects are greater, and when more distant their effects are less, and that in the triplicate proportion of their apparent diameters. Therefore it is that the sun in the winter time, being then in its perigee, has a greater effect, and makes the tides in the syzygies something greater, and those in the quadratures something less, *cæteris paribus*, than in the summer season; and every month the moon, while in the perigee, raiseth greater tides than at the distance of 15 days

before or after, when it is in its apogee. Whence it comes to pass that two highest tides do not follow one the other in two immediately succeeding syzygies.

The effect of either luminary doth likewise depend upon its declination or distance from the equator; for if the luminary was placed at the pole, it would constantly attract all the parts of the waters, without any intension or remission of its action, and could cause no reciprocation of motion; and, therefore, as the luminaries decline from the equator towards either pole, they will by degrees lose their force, and on this account will excite lesser tides in the solstitial than in the equinoctial syzygies. But in the solstitial quadratures they will raise greater tides than in the quadratures about the equinoxes; because the effect of the moon, then situated in the equator, most exceeds the effect of the sun; therefore the greatest tides fall out in those syzygies, and the least in those quadratures, which happen about the time of both equinoxes; and the greatest tide in the syzygies is always succeeded by the least tide in the quadratures, as we find by experience. But because the sun is less distant from the earth in winter than in summer, it comes to pass that the greatest and least tides more frequently appear before than after the vernal equinox, and more frequently after than before the autumnal.



Moreover, the effects of the luminaries depend upon the latitudes of places. Let $ApEP$ represent the earth on all sides covered with deep waters; C its centre; P, *p*, its poles; AE the equator; F any place without the equator; F*f* the parallel of the place; D*d* the correspondent parallel on the other side of the equator; L the place which the moon possessed three hours before H the place of the earth directly under it; *h* the opposite place; K, *k*, the places at 90 degrees distance; CH, C*h*, the greatest heights of the sea from the centre of the earth; and CK, C*k*, the least heights; and if with the axes H*h*, K*k*, an ellipsis is described, and by the revolution of that ellipsis about its longer axis H*h* a spheroid HPK*hpk* is formed, this spheroid will nearly represent the figure of the sea; and CF, C*f*, CD, C*d*, will represent the sea in the places F, *f*, D, *d*. But farther; if in the said revolution of the ellipsis any point N describes the circle NM, cutting the parallels F*f*, D*d*, in any places R, T, and the equator AE in S, CN will represent the height of the sea in all those places R, S, T, situated in this circle. Wherefore in the diurnal revolution of any place F the greatest flood will be in F, at the third hour after the appulse of the moon to the meridian above the horizon; and afterwards the greatest ebb in Q, at the third hour after the setting of the moon; and then the greatest flood in *f*,

at the third hour after the appulse of the moon to the meridian under the horizon, and, lastly, the greatest ebb in Q, at the third hour after the rising of the moon; and the latter flood in f will be less than the preceding flood in F For the whole sea is divided into two huge and hemispherical floods, one in the hemisphere KHkC on the north side, the other in the opposite hemisphere KHkC, which we may therefore call the northern and the southern floods: these floods being always opposite the one to the other, come by turns to the meridians of all places after the interval of twelve lunar hours; and, seeing the northern countries partake more of the northern flood, and the southern countries more of the southern flood, thence arise tides alternately greater and less in all places without the equator in which the luminaries rise and set. But the greater tide will happen when the moon declines towards the vertex of the place, about the third hour after the appulse of the moon to the meridian above the horizon; and when the moon changes its declination, that which was the greater tide will be changed into a lesser; and the greatest difference of the floods will fall out about the times of the solstices, especially if the ascending node of the moon is about the first of Aries. So the morning tides in winter exceed those of the evening, and the evening tides exceed those of the morning in summer; at *Plymouth* by the height of one foot, but at *Bristol* by the height of 15 inches, according to the observations of *Colepress* and *Sturmy*.

But the motions which we have been describing suffer some alteration from that force of reciprocation which the waters [having once received] retain a little while by their *vis insita;* whence it comes to pass that the tides may continue for some time, though the actions of the luminaries should cease. This power of retaining the impressed motion lessens the difference of the alternate tides, and makes those tides which immediately succeed after the syzygies greater, and those which follow next after the quadratures less. And hence it is that the alternate tides at *Plymouth* and *Bristol* do not differ much more one from the other than by the height of a foot, or of 15 inches; and that the greatest tides of all at those ports are not the first but the third after the syzygies.

And, besides, all the motions are retarded in their passage through shallow channels, so that the greatest tides of all, in some straits and mouths of rivers, are the fourth, or even the fifth, after the syzygies.

It may also happen that the greatest tide may be the fourth or fifth after the syzygies, or fall out yet later, because the motions of the sea are retarded in passing through shallow places towards the shores; for so the tide arrives at the western coast of *Ireland* at the third lunar hour, and an hour or two after at the ports in the southern coast of the same island; as also at the islands *Cassiterides*, commonly *Sorlings;* then successively at *Falmouth*, *Plymouth*, *Portland*, the isle of *Wight*, *Winchester*, *Dover*,

the mouth of the *Thames*, and *London Bridge*, spending twelve hours in this passage. But farther; the propagation of the tides may be obstructed even by the channels of the ocean itself, when they are not of depth enough, for the flood happens at the third lunar hour in the *Canary* islands; and at all those western coasts that lie towards the Atlantic ocean, as of *Ireland*, *France*, *Spain*, and all *Africa*, to the *Cape* of *Good Hope*, except in some shallow places, where it is impeded, and falls out later; and in the straits of *Gibraltar*, where, by reason of a motion propagated from the Mediterranean sea, it flows sooner. But, passing from those coasts over the breadth of the ocean to the coasts of *America*, the flood arrives first at the most eastern shores of *Brazil*, about the fourth or fifth lunar hour; then at the mouth of the river of the *Amazons* at the sixth hour, but at the neighbouring islands at the fourth hour; afterwards at the islands of *Bermudas* at the seventh hour, and at port St. *Augustin* in *Florida* at seven and a half. And therefore the tide is propagated through the ocean with a slower motion than it should be according to the course of the moon; and this retardation is very necessary, that the sea at the same time may fall between *Brazil* and *New France*, and rise at the *Canary* islands, and on the coasts of *Europe* and *Africa*, and *vice versa*: for the sea cannot rise in one place but by falling in another. And it is probable that the *Pacific* sea is agitated by the same laws; for in the coasts of *Chili* and *Peru* the highest flood is said to happen at the third lunar hour. But with what velocity it is thence propagated to the eastern coasts of *Japan*, the *Philippine* and other islands adjacent to *China*, I have not yet learned.

Farther; it may happen (p. 418) that the tide may be propagated from the ocean through different channels towards the same port, and may pass quicker through some channels than through others, in which case the same tide, divided into two or more succeeding one another, may compound new motions of different kinds. Let us suppose one tide to be divided into two equal tides, the former whereof precedes the other by the space of six hours, and happens at the third or twenty-seventh hour from the appulse of the moon to the meridian of the port. If the moon at the time of this appulse to the meridian was in the equator, every six hours alternately there would arise equal floods, which, meeting with as many equal ebbs, would so balance one the other, that, for that day, the water would stagnate, and remain quiet. If the moon then declined from the equator, the tides in the ocean would be alternately greater and less, as was said; and from hence two greater and two lesser tides would be alternately propagated towards that port. But the two greater floods would make the greatest height of the waters to fall out in the middle time betwixt both, and the greater and lesser floods would make the waters to rise to a mean height in the middle time between them; and in the middle time between

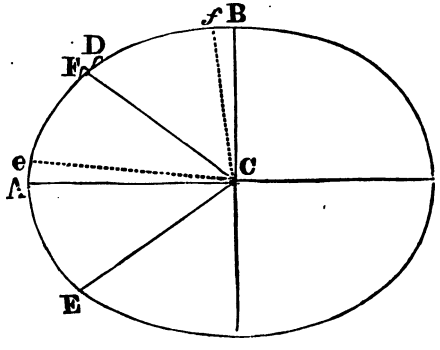
the two lesser floods the waters would rise to their least height. Thus in the space of twenty-four hours the waters would come, not twice, but once only to their greatest, and once only to their least height; and their greatest height, if the moon declined towards the elevated pole, would happen at the sixth or thirtieth hour after the appulse of the moon to the meridian; and when the moon changed its declination, this flood would be changed into an ebb.

Of all which we have an example in the port of *Batsham*, in the kingdom of *Tunquin*, in the latitude of 20° 50′ north. In that port, on the day which follows after the passage of the moon over the equator, the waters stagnate; when the moon declines to the north, they begin to flow and ebb, not twice, as in other ports, but once only every day; and the flood happens at the setting, and the greatest ebb at the rising of the moon. This tide increaseth with the declination of the moon till the seventh or eighth day; then for the seventh or eighth day following it decreaseth at the same rate as it had increased before, and ceaseth when the moon changeth its declination. After which the flood is immediately changed into an ebb; and thenceforth the ebb happens at the setting and the flood at the rising of the moon, till the moon again changes its declination. There are two inlets from the ocean to this port; one more direct and short between the island *Hainan* and the coast of *Quantung*, a province of *China;* the other round about between the same island and the coast of *Cochim;* and through the shorter passage the tide is sooner propagated to *Batsham.*

In the channels of rivers the influx and reflux depends upon the current of the rivers, which obstructs the ingress of the waters from the sea, and promotes their egress to the sea, making the ingress later and slower, and the egress sooner and faster; and hence it is that the reflux is of longer duration that the influx, especially far up the rivers, where the force of the sea is less. So *Sturmy* tells us, that in the river *Avon*, three miles below *Bristol*, the water flows only five hours, but ebbs seven; and without doubt the difference is yet greater above *Bristol*, as at *Caresham* or the *Bath.* This difference does likewise depend upon the quantity of the flux and reflux; for the more vehement motion of the sea near the syzygies of the luminaries more easily overcoming the resistance of the rivers, will make the ingress of the water to happen sooner and to continue longer, and will therefore diminish this difference. But while the moon is approaching to the syzygies, the rivers will be more plentifully filled, their currents being obstructed by the greatness of the tides, and therefore will something more retard the reflux of the sea a little after than a little before the syzygies. Upon which account the slowest tides of all will not happen in the syzygies, but precede them a little; and I observed above that the tides before the syzygies were also retarded by the force of the sun; and from both

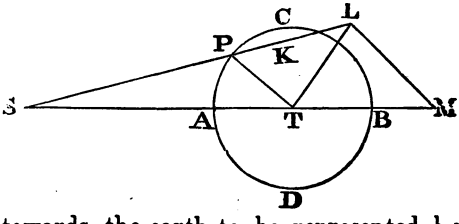
causes conjoined the retardation of the tides will be both greater and sooner before the syzygies. All which I find to be so, by the tide-tables which *Flamsted* has composed from a great many observations.

By the laws we have been describing, the times of the tides are governed; but the greatness of the tides depends upon the greatness of the seas. Let C represent the centre of the earth, EADB the oval figure of the seas, CA the longer semi-axis of this oval, CB the shorter insisting at right angles upon the former, D the middle point between A and B, and ECF or *eCf* the angle at the centre of the earth, subtended by the breadth of the sea that terminates in the shores E, F, or *e, f*. Now, supposing that the point



A is in the middle between the points E, F, and the point D in the middle between the points *e, f*, if the difference of the heights CA, CB, represent the quantity of the tide in a very deep sea surrounding the whole earth, the excess of the height CA above the height CE or CF will represent the quantity of the tide in the middle of the sea EF, terminated by the shores E, F; and the excess of the height *Ce* above the height *Cf* will nearly represent the quantity of the tide on the shores *f* of the same sea. Whence it appears that the tides are far less in the middle of the sea than at the shores; and that the tides at the shores are nearly as EF (p. 451, 452), the breadth of the sea not exceeding a quadrantal arc. And hence it is that near the equator, where the sea between *Africa* and *America* is narrow, the tides are far less than towards either side in the temperate zones, where the seas are extended wider; or on almost all the shores of the Pacific sea, as well towards *America* as towards *China*, and within as well as without the tropics; and that in islands in the middle of the sea they scarcely rise higher than two or three feet, but on the shores of great continents are three or four times greater, and above, especially if the motions propagated from the ocean are by degrees contracted into a narrow space, and the water, to fill and empty the bays alternately, is forced to flow and ebb with great violence through shallow places; as *Plymouth* and *Chepstow Bridge* in *England*, at the mount of *St. Michael* and town of *Avranches* in *Normandy*, and at *Cambaia* and *Pegu* in the *East Indies*. In which places

the sea, hurried in and out with great violence, sometimes lays the shores under water, sometimes leaves them dry, for many miles. Nor is the force of the influx and efflux to be broke till it has raised or depressed the water to forty or fifty feet and more. Thus also long and shallow straits that open to the sea with mouths wider and deeper than the rest of their channel (such as those about *Britain* and the *Magellanic Straits* at the eastern entry) will have a greater flood and ebb, or will more intend and remit their course, and therefore will rise higher and be depressed lower. On the coast of *South America* it is said that the *Pacific* sea in its reflux sometimes retreats two miles, and gets out of sight of those that stand on shore. Whence in these places the floods will be also higher; but in deeper waters the velocity of influx and efflux is always less, and therefore the ascent and descent is so too. Nor in such places is the ocean known to ascend to more than six, eight, or ten feet. The quantity of the ascent I compute in the following manner

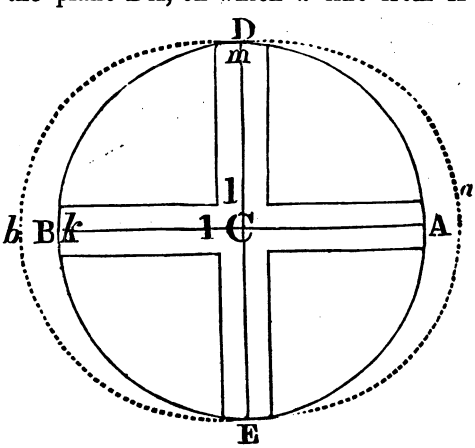


Let S represent the sun, T the earth (419, 420), P the moon, PAGB the moon's orbit. In SP take SK equal to ST and SL to SK in the duplicate ratio of SK to SP. Parallel to PT draw LM; and, supposing the mean quantity of the circum-solar force directed towards the earth to be represented by the distance ST or SK, SL will represent the quantity thereof directed towards the moon. But that force is compounded of the parts SM, LM; of which the force LM and that part of SM which is represented by TM, do disturb the motion of the moon (as appears from Prop. LXVI, and its Corollaries) In so far as the earth and moon are revolved about their common centre of gravity, the earth will be liable to the action of the like forces. But we may refer the sums as well of the forces as of the motions to the moon, and represent the sums of the forces by the lines TM and ML, which are proportional to them. The force LM, in its mean quantity, is to the force by which the moon may be revolved in an orbit, about the earth quiescent, at the distance PT in the duplicate ratio of the moon's periodic time about the earth to the earth's periodic time about the sun (by Cor. XVII, Prop. LXVI); that is, in the duplicate ratio of $27^d.\ 7^h.\ 43'$ to $365^d.\ 6^h.\ 9'$; or as 1000 to 178725, or 1 to $178\frac{2}{4}\frac{9}{0}$. The force by which the moon may be revolved in its orb about the earth in rest, at the distance PT of $60\frac{1}{2}$ semi-diameters of the earth, is to the force by which it may revolve in the same time at the distance of 60 semi-diameters as $60\frac{1}{2}$ to 60; and this force is to the force of gravity with us as 1 to 60×60 nearly; and therefore the mean force ML is to the force of gravity at the surface of the earth as $1 \times 60\frac{1}{2}$ to $60 \times 60 \times 178\frac{2}{4}\frac{9}{0}$, or 1 to

638092,6. Whence the force TM will be also given from the proportion of the lines TM, ML. And these are the forces of the sun, by which the moon's motions are disturbed.

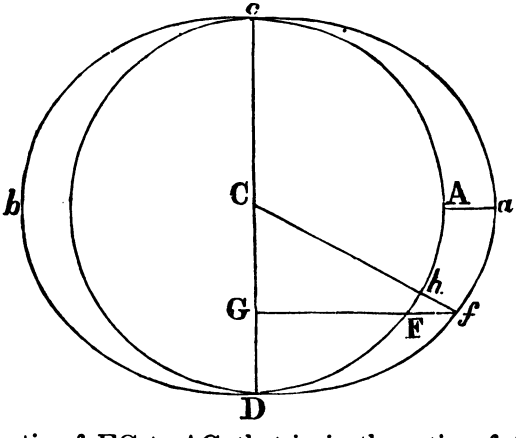
If from the moon's orbit (p. 449) we descend to the earth's surface, those forces will be diminished in the ratio of the distances $60\frac{1}{2}$ and 1; and therefore the force LM will then become 38604600 times less than the force of gravity. But this force acting equally every where upon the earth, will scarcely effect any change on the motion of the sea, and therefore may be neglected in the explication of that motion. The other force TM, in places where the sun is vertical, or in their nadir, is triple the quantity of the force ML, and therefore but 12868200 times less than the force of gravity.

Suppose now ADBE to represent the spherical surface of the earth, *a*D*b*E the surface of the water overspreading it, C the centre of both, A the place to which the sun is vertical, B the place opposite; D, E, places at 90 degrees distance from the former; ACE*mlk* a right angled cylindric canal passing through the earth's centre. The force TM in any place is as the distance of the place from the plane DE, on which a line from A to C insists at right angles, and therefore in the part of the canal which is represented by EC *lm* is of no quantity, but in the other part AC*lk* is as the gravity at the several heights; for in descending towards the centre of the earth, gravity is (by Prop. LXXIII) every where as the height; and therefore the force TM drawing the water upwards will diminish its gravity in the leg AC*lk* of the canal in a given ratio: upon which account the water will ascend in this leg, till its defect of gravity is supplied by its greater height; nor will it rest in an equilibrium till its total gravity becomes equal to the total gravity in EC*lm*, the other leg of the canal. Because the gravity of every particle is as its distance from the earth's centre, the weight of the whole water in either leg will increase in the duplicate ratio of the height; and therefore the height of the water in the leg AC*lk* will be to the height thereof in the leg C*lm*E in the subduplicate ratio of the number 12868201 to 12868200, or in the ratio of the number 25623053 to the number 25623052, and the height of the water in the leg EC*lm* to the difference of the heights, as 25623052 to 1. But the height in the leg EC*lm* is of 19615800 *Paris* feet, as has been lately



found by the mensuration of the *French*; and, therefore, by the preceding analogy, the difference of the heights comes out $9\frac{1}{5}$ inches of the *Paris* foot; and the sun's force will make the height of the sea at A to exceed the height of the same at E by 9 inches. And though the water of the canal ACE*mlk* be supposed to be frozen into a hard and solid consistence, yet the heights thereof at A and E, and all other intermediate places, would still remain the same.

Let A*a* (in the following figure) represent that excess of height of 9 inches at A, and *hf* the excess of height at any other place *h*; and upon DC let fall the perpendicular *f*G, meeting the globe of the earth in F; and because the distance of the sun is so great that all the right lines drawn thereto may be considered as parallel, the force TM in any place *f* will be to the same force in the place A as the sine FG to the radius AC. And, therefore, since those forces tend to the sun in the direction of parallel lines, they will generate the parallel heights F*f*, A*a*, in the same ratio; and therefore the figure of the water D*faeb* will be a spheroid made by the revolution of an ellipsis about its longer axis *ab*. And the perpendicular height *fh* will be to the oblique height F*f* as *f*G to *f*C, or as FG to AC: and therefore the height *fh* is to the height A*a* in the duplicate ratio of FG to AC, that is, in the ratio of the versed sine of double the angle DC*f* to double the radius, and is thence given. And hence to the several moments of the apparent revolution of the sun about the earth we may infer the proportion of the ascent and descent of the waters at any given place under the equator, as well as of the diminution of that ascent and descent, whether arising from the latitude of places or from the sun's declination; viz., that on account of the latitude of places, the ascent and descent of the sea is in all places diminished in the duplicate ratio of the co-sines of latitude; and on account of the sun's declination, the ascent and descent under the equator is diminished in the duplicate ratio of the co-sine of declination. And in places without the equator the half sum of the morning and evening ascents (that is, the mean ascent) is diminished nearly in the same ratio.



Let S and L respectively represent the forces of the sun and moon placed in the equator, and at their mean distances from the earth; R the radius; T and V the versed sines of double the complements of the sun

and moon's declinations to any given time; D and E the mean apparent diameters of the sun and moon: and, supposing F and G to be their apparent diameters to that given time, their forces to raise the tides under the equator will be, in the syzygies $\frac{VG^3}{2RE^3}L + \frac{TF^3}{2RD^3}S$; in the quadratures, $\frac{VG^3}{2RE^3}L - \frac{TF^3}{2RD^3}S$. And if the same ratio is likewise observed under the parallels, from observations accurately made in our northern climates we may determine the proportion of the forces L and S; and then by means of this rule predict the quantities of the tides to every syzygy and quadrature.

At the mouth of the river *Avon*, three miles below *Bristol* (p. 450 to 453), in spring and autumn, the whole ascent of the water in the conjunction or opposition of the luminaries (by the observation of *Sturmy*) is about 45 feet, but in the quadratures only 25. Because the apparent diameters of the luminaries are not here determined, let us assume them in their mean quantities, as well as the moon's declination in the equinoctial quadratures in its mean quantity, that is, $23\frac{1}{2}°$; and the versed sine of double its complement will be 1682, supposing the radius to be 1000. But the declinations of the sun in the equinoxes and of the moon in the syzygies are of no quantity, and the versed sines of double the complements are each 2000. Whence those forces become L + S in the syzygies, and $\frac{1682}{2000}L - S$ in the quadratures; respectively proportional to the heights of the tides of 45 and 25 feet, or of 9 and 5 paces. And, therefore, multiplying the extremes and the means, we have $5L + 5S = \frac{15138}{2000}L -$ 9S, or $L = \frac{28000}{5138}S = 5\frac{5}{11}S$.

But farther; I remember to have been told that in summer the ascent of the sea in the syzygies is to the ascent thereof in the quadratures as about 5 to 4. In the solstices themselves it is probable that the proportion may be something less, as about 6 to 5; whence it would follow that L is $= 5\frac{1}{3}S$ [for then the proportion is $\frac{1682}{2000}L + \frac{1682}{2000}S : L - \frac{1682}{2000}S :: 6 : 5$]. Till we can more certainly determine the proportion from observation, let us assume $L = 5\frac{1}{3}S$; and since the heights of the tides are as the forces which excite them, and the force of the sun is able to raise the tides to the height of nine inches, the moon's force will be sufficient to raise the same to the height of four feet. And if we allow that this height may be doubled, or perhaps tripled, by that force of reciprocation which we observe in the motion of the waters, and by which their motion once begun is kept

up for some time, there will be force enough to generate all that quantity of tides which we really find in the ocean.

Thus we have seen that these forces are sufficient to move the sea. But, so far as I can observe, they will not be able to produce any other effect sensible on our earth; for since the weight of one grain in 4000 is not sensible in the nicest balance; and the sun's force to move the tides is 12868200 less than the force of gravity; and the sum of the forces of both moon and sun, exceeding the sun's force only in the ratio of $6\frac{1}{3}$ to 1, is still 2032890 times less than the force of gravity; it is evident that both forces together are 500 times less than what is required sensibly to increase or diminish the weight of any body in a balance. And, therefore, they will not sensibly move any suspended body; nor will they produce any sensible effect on pendulums, barometers, bodies swimming in stagnant water, or in the like statical experiments. In the atmosphere, indeed, they will excite such a flux and reflux as they do in the sea, but with so small a motion that no sensible wind will be thence produced.

If the effects of both moon and sun in raising the tides (p. 454), as well as their apparent diameters, were equal among themselves, their absolute forces would (by Cor. XIV, Prop. LXVI) be as their magnitudes. But the effect of the moon is to the effect of the sun as about $5\frac{1}{3}$ to 1; and the moon's diameter less than the sun's in the ratio of $31\frac{1}{2}$ to $32\frac{1}{3}$, or of 45 to 46. Now the force of the moon is to be increased in the ratio of the effect directly, and in the triplicate ratio of the diameter inversely. Whence the force of the moon compared with its magnitude will be to the force of the sun compared with its magnitude in the ratio compounded of $5\frac{1}{3}$ to 1, and the triplicate of 45 to 46 inversely, that is, in the ratio of about $5\frac{7}{16}$ to 1. And therefore the moon, in respect of the magnitude of its body, has an absolute centripetal force greater than the sun in respect of the magnitude of its body in the ratio to $5\frac{7}{16}$ to 1, and is therefore more dense in the same ratio.

In the time of $27^{d}.\ 7^{h}.\ 43'$, in which the moon makes its revolution about the earth, a planet may be revolved about the sun at the distance of 18,951 diameters of the sun from the sun's centre, supposing the mean apparent diameter of the sun to be $32\frac{1}{5}'$; and in the same time the moon may be revolved about the earth at rest, at the distance of 30 of the earth's diameters. If in both cases the number of diameters was the same, the absolute circum-terrestrial force would (by Cor. II, Prop. LXXII) be to the absolute circum-solar force as the magnitude of the earth to the magnitude of the sun. Because the number of the earth's diameters is greater in the ratio of 30 to 18,954, the body of the earth will be less in the triplicate of that ratio, that is, in the ratio of $3\frac{2}{3}\frac{8}{9}$ to 1. Wherefore the earth's force, for the magnitude of its body, is to the sun's force, for the magnitude of its body, as $3\frac{2}{3}\frac{8}{9}$ to 1; and consequently the earth's density to the sun's will be in

the same ratio. Since, then, the moon's density is to the sun's density as $5\frac{7}{10}$ to 1, the moon's density will be to the earth's density as $5\frac{7}{10}$ to $3\frac{2}{3}\frac{3}{3}$, or as 23 to 16. Wherefore since the moon's magnitude is to the earth's magnitude as about 1 to $41\frac{1}{2}$, the moon's absolute centripetal force will be to the earth's absolute centripetal force as about 1 to 29, and the quantity of matter in the moon to the quantity of matter in the earth in the same ratio. And hence the common centre of gravity of the earth and moon is more exactly determined than hitherto has been done; from the knowledge of which we may now infer the moon's distance from the earth with greater accuracy. But I would rather wait till the proportion of the bodies of the moon and earth one to the other is more exactly defined from the phænomena of the tides, hoping that in the mean time the circumference of the earth may be measured from more distant stations than any body has yet employed for this purpose.

Thus I have given an account of the system of the planets. As to the fixed stars, the smallness of their annual parallax proves them to be removed to immense distances from the system of the planets: that this parallax is less than one minute is most certain; and from thence it follows that the distance of the fixed stars is above 360 times greater than the distance of Saturn from the sun. Such as reckon the earth one of the planets, and the sun one of the fixed stars, may remove the fixed stars to yet greater distances by the following arguments: from the annual motion of the earth there would happen an apparent transposition of the fixed stars, one in respect of another, almost equal to their double parallax; but the greater and nearer stars, in respect of the more remote, which are only seen by the telescope, have not hitherto been observed to have the least motion. If we should suppose that motion to be but less than 20″, the distance of the nearer fixed stars would exceed the mean distance of Saturn by above 2000 times. Again; the disk of Saturn, which is only 17″ or 18″ in diameter, receives but about $\frac{1}{2100000000}$ of the sun's light; for so much less is that disk than the whole spherical surface of the orb of Saturn. Now if we suppose Saturn to reflect about $\frac{1}{4}$ of this light, the whole light reflected from its illuminated hemisphere will be about $\frac{1}{4200000000}$ of the whole light emitted from the sun's hemisphere; and, therefore, since light is rarefied in the duplicate ratio of the distance from the luminous body, if the sun was $10000\sqrt{42}$ times more distant than Saturn, it would yet appear as lucid as Saturn now does without its ring, that is, something more lucid than a fixed star of the first magnitude. Let us, therefore, suppose that the distance from which the sun would shine as a fixed star exceeds that of Saturn by about 100,000 times, and its apparent diameter will be 7ᵛ. 16ᵛⁱ. and its parallax arising from the annual motion of the earth 13‴: and so great will be the distance, the apparent diameter, and the parallax of the fixed stars of the first magnitude, in bulk and light equal to our sun

Some may, perhaps, imagine that a great part of the light of the fixed stars is intercepted and lost in its passage through so vast spaces, and upon that account pretend to place the fixed stars at nearer distances; but at this rate the remoter stars could be scarcely seen. Suppose, for example, that $\frac{3}{4}$ of the light perish in its passage from the nearest fixed stars to us; then $\frac{3}{4}$ will twice perish in its passage through a double space, thrice through a triple, and so forth. And, therefore, the fixed stars that are at a double distance will be 16 times more obscure, viz., 4 times more obscure on account of the diminished apparent diameter; and, again, 4 times more on account of the lost light. And, by the same argument, the fixed stars at a triple distance will be $9 \times 4 \times 4$, or 144 times more obscure; and those at a quadruple distance will be $16 \times 4 \times 4 \times 4$, or 1024 times more obscure; but so great a diminution of light is no ways consistent with the phænomena and with that hypothesis which places the fixed stars at different distances.

The fixed stars being, therefore, at such vast distances from one another (p. 460, 461), can neither attract each other sensibly, nor be attracted by our sun. But the comets must unavoidably be acted on by the circumsolar force; for as the comets were placed by astronomers above the moon, because they were found to have no diurnal parallax, so their annual parallax is a convincing proof of their descending into the regions of the planets. For all the comets which move in a direct course, according to the order of the signs, about the end of their appearance become more than ordinarily slow, or retrograde, if the earth is between them and the sun; and more than ordinarily swift if the earth is approaching to a heliocentric opposition with them. Whereas, on the other hand, those which move against the order of the signs, towards the end of their appearance, appear swifter than they ought to be if the earth is between them and the sun; and slower, and perhaps retrograde, if the earth is in the other side of its orbit. This is occasioned by the motion of the earth in different situations. If the earth go the same way with the comet, with a swifter motion, the comet becomes retrograde; if with a slower motion, the comet becomes slower, however; and if the earth move the contrary way, it becomes swifter; and by collecting the differences between the slower and swifter motions, and the sums of the more swift and retrograde motions, and comparing them with the situation and motion of the earth from whence they arise, I found, by means of this parallax, that the distances of the comets at the time they cease to be visible to the naked eye are always less than the distance of Saturn, and generally even less than the distance of Jupiter.

The same thing may be collected from the curvature of the way of the comets (p. 462). These bodies go on nearly in great circles while their motion continues swift; but about the end of their course, when that part

of their apparent motion which arises from the parallax bears a greater proportion to their whole apparent motion, they commonly deviate from those circles; and when the earth goes to one side, they deviate to the other; and this deflection, because of its corresponding with the motion of the earth, must arise chiefly from the parallax; and the quantity thereof is so considerable, as, by my computation, to place the disappearing comets a good deal lower than Jupiter. Whence it follows, that, when they approach nearer to us in their perigees and perihelions, they often descend below the orbits of Mars and the inferior planets.

Moreover, this nearness of the comets is confirmed by the annual parallax of the orbit, in so far as the same is pretty nearly collected by the supposition that the comets move uniformly in right lines. The method of collecting the distance of a comet according to this hypothesis from four observations (first attempted by *Kepler*, and perfected by Dr. *Wallis* and Sir *Christopher Wren*) is well known; and the comets reduced to this regularity generally pass through the middle of the planetary region. So the comets of the year 1607 and 1618, as their motions are defined by *Kepler*, passed between the sun and the earth; that of the year 1664 below the orbit of Mars; and that in 1680 below the orbit of Mercury, as its motion was defined by Sir *Christopher Wren* and others. By a like rectilinear hypothesis, *Hevelius* placed all the comets about which we have any observations below the orbit of Jupiter. It is a false notion, therefore, and contrary to astronomical calculation, which some have entertained, who, from the regular motion of the comets, either remove them into the regions of the fixed stars, or deny the motion of the earth; whereas their motions cannot be reduced to perfect regularity, unless we suppose them to pass through the regions near the earth in motion; and these are the arguments drawn from the parallax, so far as it can be determined without an exact knowledge of the orbits and motions of the comets.

The near approach of the comets is farther confirmed from the light of their heads (p. 463, 465); for the light of a celestial body, illuminated by the sun, and receding to remote parts, is diminished in the quadruplicate proportion of the distance; to wit, in one duplicate proportion on account of the increase of the distance from the sun; and in another duplicate proportion on account of the decrease of the apparent diameter. Hence it may be inferred, that Saturn being at a double distance, and having its apparent diameter nearly half of that of Jupiter, must appear about 16 times more obscure; and that, if its distance were 4 times greater, its light would be 256 times less; and therefore would be hardly perceivable to the naked eye. But now the comets often equal Saturn's light, without exceeding him in their apparent diameters. So the comet of the year 1668, according to Dr. *Hooke's* observations, equalled in brightness the light of a fixed star of the first magnitude; and its head, or the star in

the middle of the coma, appeared, through a telescope of 15 feet, as lucid as Saturn near the horizon; but the diameter of the head was only 25″ that is, almost the same with the diameter of a circle equal to Saturn and his ring. The coma or hair surrounding the head was about ten times as broad; namely, $4\frac{1}{6}$ min. Again; the least diameter of the hair of the comet of the year 1682, observed by Mr. *Flamsted* with a tube of 16 feet and measured with the micrometer, was 2′ 0″; but the nucleus, or star in the middle, scarcely possessed the tenth part of this breadth, and was therefore only 11″ or 12″ broad; but the light and clearness of its head exceeded that of the year 1680, and was equal to that of the stars of the first or second magnitude. Moreover, the comet of the year 1665, in *April*, as *Hevelius* informs us, exceeded almost all the fixed stars in splendor, and even Saturn itself, as being of a much more vivid colour; for this comet was more lucid than that which appeared at the end of the foregoing year and was compared to the stars of the first magnitude. The diameter of the coma was about 6′; but the nucleus, compared with the planets by means of a telescope, was plainly less than Jupiter, and was sometimes thought less, sometimes equal to the body of Saturn within the ring. To this breadth add that of the ring, and the whole face of Saturn will be twice as great as that of the comet, with a light not at all more intense; and therefore the comet was nearer to the sun than Saturn. From the proportion of the nucleus to the whole head found by these observations, and from its breadth, which seldom exceeds 8′ or 12′, it appears that the stars of the comets are most commonly of the same apparent magnitude as the planets; but that their light may be compared oftentimes with that of Saturn, and sometimes exceeds it. And hence it is certain that in their perihelia their distances can scarcely be greater than that of Saturn. At twice that distance, the light would be four times less, which besides by its dim paleness would be as much inferior to the light of Saturn as the light of Saturn is to the splendor of Jupiter: but this difference would be easily observed. At a distance ten times greater, their bodies must be greater than that of the sun; but their light would be 100 times fainter than that of Saturn. And at distances still greater, their bodies would far exceed the sun; but, being in such dark regions, they must be no longer visible. So impossible is it to place the comets in the middle regions between the sun and fixed stars, accounting the sun as one of the fixed stars; for certainly they would receive no more light there from the sun than we do from the greatest of the fixed stars.

So far we have gone without considering that obscuration which comets suffer from that plenty of thick smoke which encompasseth their heads, and through which the heads always shew dull as through a cloud; for by how much the more a body is obscured by this smoke, by so much the more near it must be allowed to come to the sun, that it may vie with the

planets in the quantity of light which it reflects; whence it is probable that the comets descend far below the orbit of Saturn, as we proved before from their parallax. But, above all, the thing is evinced from their tails, which must be owing either to the sun's light reflected from a smoke arising from them, and dispersing itself through the æther, or to the light of their own heads.

In the former case we must shorten the distance of the comets, lest we be obliged to allow that the smoke arising from their heads is propagated through such a vast extent of space, and with such a velocity of expansion, as will seem altogether incredible; in the latter case the whole light of both head and tail must be ascribed to the central nucleus. But, then, if we suppose all this light to be united and condensed within the disk of the nucleus, certainly the nucleus will by far exceed Jupiter itself in splendor, especially when it emits a very large and lucid tail. If, therefore, under a less apparent diameter, it reflects more light, it must be much more illuminated by the sun, and therefore much nearer to it. So the comet that appeared *Dec.* 12 and 15, O.S. *Anno* 1679, at the time it emitted a very shining tail, whose splendor was equal to that of many stars like Jupiter, if their light were dilated and spread through so great a space, was, as to the magnitude of its nucleus, less than Jupiter (as Mr. *Flamsted* observed), and therefore was much nearer to the sun: nay, it was even less than Mercury. For on the 17th of that month, when it was nearer to the earth, it appeared to *Cassini* through a telescope of 35 feet a little less than the globe of Saturn. On the 8th of this month, in the morning, Dr. *Halley* saw the tail, appearing broad and very short, and as if it rose from the body of the sun itself, at that time very near its rising. Its form was like that of an extraordinary bright cloud; nor did it disappear till the sun itself began to be seen above the horizon. Its splendor, therefore, exceeded the light of the clouds till the sun rose, and far surpassed that of all the stars together, as yielding only to the immediate brightness of the sun itself. Neither Mercury, nor Venus, nor the moon itself, are seen so near the rising sun. Imagine all this dilated light collected together, and to be crowded into the orbit of the comet's nucleus which was less than Mercury; by its splendor, thus increased, becoming so much more conspicuous, it will vastly exceed Mercury, and therefore must be nearer to the sun. On the 12th and 15th of the same month, this tail, extending itself over a much greater space, appeared more rare; but its light was still so vigorous as to become visible when the fixed stars were hardly to be seen, and soon after to appear like a fiery beam shining in a wonderful manner. From its length, which was 40 or 50 degrees, and its breadth of 2 degrees, we may compute what the light of the whole must be

This near approach of the comets to the sun is confirmed from the situation they are seen in when their tails appear most resplendent; for when

the head passes by the sun, and lies hid under the solar rays, very bright and shining tails, like fiery beams, are said to issue from the horizon; but afterwards, when the head begins to appear, and is got farther from the sun, that splendor always decreases, and turns by degrees into a paleness like to that of the milky way, but much more sensible at first; after that vanishing gradually. Such was that most resplendent comet described by *Aristotle*, Lib. 1, Meteor. 6. "The head thereof could not be seen, because it set before the sun, or at least was hid under the sun's rays; but the next day it was seen as well as might be; for, having left the sun but a very little way, it set immediately after it; and the scattered light of the head obscured by the too great splendour (of the tail) did not yet appear. But afterwards (says *Aristotle*), when the splendour of the tail was now diminished (the head of), the comet recovered its native brightness. And the splendour of its tail reached now to a third part of the heavens (that is, to 60°). It appeared in the winter season, and, rising to *Orion's* girdle, there vanished away." Two comets of the same kind are described by *Justin*, Lib. 37, which, according to his account, "shined so bright, that the whole heaven seemed to be on fire; and by their greatness filled up a fourth part of the heavens, and by their splendour exceeded that of the sun." By which last words a near position of these bright comets and the rising or setting sun is intimated (p. 494, 495). We may add to these the comet of the year 1101 or 1106, "the star of which was small and obscure (like that of 1680); but the splendour arising from it extremely bright, reaching like a fiery beam to the east and north," as *Hevelius* has it from *Simeon*, the monk of *Durham*. It appeared at the beginning of *February* about the evening in the south-west. From this and from the situation of the tail we may infer that the head was near the sun. *Matthew Paris* says, "it was about one cubit from the sun; from the third [or rather the sixth] to the ninth hour sending out a long stream of light." The comet of 1264, in *July*, or about the solstice, preceded the rising sun, sending out its beams with a great light towards the west as far as the middle of the heavens; and at the beginning it ascended a little above the horizon: but as the sun went forwards it retired every day farther from the horizon, till it passed by the very middle of the heavens. It is said to have been at the beginning large and bright, having a large coma, which decayed from day to day. It is described in *Append. Matth. Paris, Hist. Ang.* after this manner: "*An. Christi* 1265, there appeared a comet so wonderful, that none then living had ever seen the like; for, rising from the east with a great brightness, it extended itself with a great light as far as the middle of the hemisphere towards the west." The Latin original being somewhat barbarous and obscure, it is here subjoined. *Ab oriente enim cum magno fulgore surgens, usque ad medium hemisphærii versus occidentem, omnia perlucide pertrahebat.*

"In the year 1401 or 1402, the sun being got below the horizon, there appeared in the west a bright and shining comet, sending out a tail upwards, in splendor like a flame of fire, and in form like a spear, darting its rays from west to east. When the sun was sunk below the horizon, by the lustre of its own rays it enlightened all the borders of the earth, not permitting the other stars to shew their light, or the shades of night to darken the air, because its light exceeded that of the others, and extended itself to the upper part of the heavens, flaming," &c., *Hist. Byzant. Duc. Mich. Nepot.* From the situation of the tail of this comet, and the time of its first appearance, we may infer that the head was then near the sun, and went farther from him every day; for that comet continued three months. In the year 1527, *Aug.* 11, about four in the morning, there was seen almost throughout *Europe* a terrible comet in *Leo*, which continued flaming an hour and a quarter every day. It rose from the east, and ascended to the south and west to a prodigious length. It was most conspicuous to the north, and its cloud (that is, its tail) was very terrible; having, according to the fancies of the vulgar, the form of an arm a little bent holding a sword of a vast magnitude. In the year 1618, in the end of *November*, there began a rumour, that there appeared about sun-rising a bright beam, which was the tail of a comet whose head was yet concealed within the brightness of the solar rays. On *Nov.* 24, and from that time, the comet itself appeared with a bright light, its head and tail being extremely resplendent. The length of the tail, which was at first 20 or 30 deg., increased till *December* 9, when it arose to 75 deg., but with a light much more faint and dilute than at the beginning. In the year 1668, *March* 5, N. S., about 7 in the evening, *P. Valent. Estancius*, being in *Brazil*, saw a comet near the horizon in the south-west. Its head was small, and scarcely discernible, but its tail extremely bright and refulgent, so that the reflection of it from the sea was easily seen by those who stood upon the shore. This great splendor lasted but three days, decreasing very remarkably from that time. The tail at the beginning extended itself from west to south, and in a situation almost parallel to the horizon, appearing like a shining beam 23 deg. in length. Afterwards, the light decreasing, its magnitude increased till the comet ceased to be visible; so that *Cassini*, at *Bologna*, saw it (*Mar.* 10, 11, 12) rising from the horizon 32 deg. in length. In *Portugal* it is said to have taken up a fourth part of the heavens (that is, 45 deg.), extending itself from west to east with a notable brightness; though the whole of it was not seen, because the head in this part of the world always lay hid below the horizon. From the increase of the tail it is plain that the head receded from the sun, and was nearest to it at the beginning, when the tail appeared brightest.

To all these we may add the comet of 1680, whose wonderful splendor at the conjunction of the head with the sun was above described. But so

great a splendor argues the comets of this kind to have really passed near the fountain of light, especially since the tails never shine so much in their opposition to the sun; nor do we read that fiery beams have ever appeared there.

Lastly, the same thing is inferred (p. 466, 467) from the light of the heads increasing in the recess of the comets from the earth towards the sun, and decreasing in their return from the sun towards the earth; for so the last comet of the year 1665 (by the observation of *Hevelius*), from the time that it was first seen, was always losing of its apparent motion, and therefore had already passed its perigee; yet the splendor of its head was daily increasing, till, being hid by the sun's rays, the comet ceased to appear. The comet of the year 1683 (by the observation of the same *Hevelius*), about the end of *July*, when it first appeared, moved at a very slow rate, advancing only about 40 or 45 minutes in its orbit in a day's time. But from that time its diurnal motion was continually upon the increase till *September* 4, when it arose to about 5 degrees; and therefore in all this interval of time the comet was approaching to the earth. Which is likewise proved from the diameter of its head measured with a micrometer; for, *August* the 6th, *Hevelius* found it only 6′ 5″, including the coma; which, *September* 2, he observed 9′ 7″. And therefore its head appeared far less about the beginning than towards the end of its motion, though about the beginning, because nearer to the sun, it appeared far more lucid than towards the end, as the same *Hevelius* declares. Wherefore in all this interval of time, on account of its recess from the sun, it decreased in splendor, notwithstanding its access towards the earth. The comet of the year 1618, about the middle of *December*, and that of the year 1680, about the end of the same month, did both move with their greatest velocity, and were therefore then in their perigees; but the greatest splendor of their heads was seen two weeks before, when they had just got clear of the sun's rays; and the greatest splendor of their tails a little more early, when yet nearer to the sun. The head of the former comet, according to the observations of *Cysatus*, *Dec.* 1, appeared greater than the stars of the first magnitude; and, *Dec.* 16 (being then in its perigee), of a small magnitude, and the splendor or clearness was much diminished. *Jan.* 7, *Kepler*, being uncertain about the head, left off observing. *Dec.* 12, the head of the last comet was seen and observed by *Flamsted* at the distance of 9 degrees from the sun, which a star of the third magnitude could hardly have been. *December* 15 and 17, the same appeared like a star of the third magnitude, its splendor being diminished by the bright clouds near the setting sun. *Dec.* 26, when it moved with the greatest swiftness, and was almost in its perigee, it was inferior to *Os Pegasi*, a star of the third magnitude. *Jan.* 3, it appeared like a star of the fourth; *Jan.* 9, like a star of the fifth. *Jan.* 13, it disappeared, by reason of the

brightness of the moon, which was then in its increase. *Jan.* 25, it was scarcely equal to the stars of the seventh magnitude. If we take equal times on each hand of the perigee, the heads placed at remote distances would have shined equally before and after, because of their equal distances from the earth. That in one case they shined very bright, and in the other vanished, is to be ascribed to the nearness of the sun in the first case, and his distance in the other; and from the great difference of the light in these two cases we infer its great nearness in the first of them; for the light of the comets uses to be regular, and to appear greatest when their heads move the swiftest, and are therefore in their perigees, excepting in so far as it is increased by their nearness to the sun.

From these things I at last discovered why the comets frequent so much the region of the sun. If they were to be seen in the regions a great way beyond Saturn, they must appear oftener in those parts of the heavens that are opposite to the sun; for those which are in that situation would be nearer to the earth, and the interposition of the sun would obscure the others: but, looking over the history of comets, I find that four or five times more have been seen in the hemisphere towards the sun than in the opposite hemisphere; besides, without doubt, not a few which have been hid by the light of the sun; for comets descending into our parts neither emit tails, nor are so well illuminated by the sun, as to discover themselves to our naked eyes, till they are come nearer to us than Jupiter. But the far greater part of that spherical space, which is described about the sun with so small an interval, lies on that side of the earth which regards the sun, and the comets in that greater part are more strongly illuminated, as being for the most part nearer to the sun: besides, from the remarkable eccentricity of their orbits, it comes to pass that their lower apsides are much nearer to the sun than if their revolutions were performed in circles concentric to the sun.

Hence also we understand why the tails of the comets, while their heads are descending towards the sun, always appear short and rare, and are seldom said to have exceeded 15 or 20 deg. in length; but in the recess of the heads from the sun often shine like fiery beams, and soon after reach to 40, 50, 60, 70 deg. in length, or more. This great splendor and length of the tails arises from the heat which the sun communicates to the comet as it passes near it. And thence, I think, it may be concluded, that all the comets that have had such tails have passed very near the sun.

Hence also we may collect that the tails arise from the atmospheres of the heads (p. 487 to 488): but we have had three several opinions about the tails of comets; for some will have it that they are nothing else but the beams of the sun's light transmitted through the comets' heads, which they suppose to be transparent; others, that they proceed from the refraction which light suffers in passing from the comet's head to the earth;

and, lastly, others, that they are a sort of clouds or vapour constantly rising from the comets' heads, and tending towards the parts opposite to the sun. The first is the opinion of such as are yet unacquainted with optics; for the beams of the sun are not seen in a darkened room, but in consequence of the light that is reflected from them by the little particles of dust and smoke which are always flying about in the air; and hence it is that in air impregnated with thick smoke they appear with greater brightness, and are more faintly and more difficultly seen in a finer air; but in the heavens, where there is no matter to reflect the light, they are not to be seen at all. Light is not seen as it is in the beams, but as it is thence reflected to our eyes; for vision is not made but by rays falling upon the eyes, and therefore there must be some reflecting matter in those parts where the tails of comets are seen; and so the argument turns upon the third opinion; for that reflecting matter can be no where found but in the place of the tail, because otherwise, since all the celestial spaces are equally illuminated by the sun's light, no part of the heavens could appear with more splendor than another. The second opinion is liable to many difficulties. The tails of comets are never seen variegated with those colours which ever use to be inseparable from refraction; and the distinct transmission of the light of the fixed stars and planets to us is a demonstration that the æther or celestial medium is not endowed with any refractive power. For as to what is alledged that the fixed stars have been sometimes seen by the *Egyptians* environed with a coma or capillitium because that has but rarely happened, it is rather to be ascribed to a casual refraction of clouds, as well as the radiation and scintillation of the fixed stars to the refractions both of the eyes and air; for upon applying a telescope to the eye, those radiations and scintillations immediately disappear. By the tremulous agitation of the air and ascending vapours, it happens that the rays of light are alternately turned aside from the narrow space of the pupil of the eye; but no such thing can have place in the much wider aperture of the object-glass of a telescope; and hence it is that a scintillation is occasioned in the former case which ceases in the latter; and this cessation in the latter case is a demonstration of the regular transmission of light through the heavens without any sensible refraction. But, to obviate an objection that may be made from the appearing of no tail in such comets as shine but with a faint light, as if the secondary rays were then too weak to affect the eyes, and for this reason it is that the tails of the fixed stars do not appear, we are to consider that by the means of telescopes the light of the fixed stars may be augmented above an hundred fold and yet no tails are seen; that the light of the planets is yet more copious without any tail, but that comets are seen sometimes with huge tails when the light of their heads is but faint and dull; for so it happened in the comet of the year 1680, when in the month of *De-*

cember it was scarcely equal in light to the stars of the second magnitude, and yet emitted a notable tail, extending to the length of 40°, 50°, 60°, or 70°, and upwards; and afterwards, on the 27th and 28th of *January*, the head appeared but as a star of the seventh magnitude; but the tail (as was said above), with a light that was sensible enough, though faint, was stretched out to 6 or 7 degrees in length, and with a languishing light that was more difficultly seen, even to 12° and upwards. But on the 9th and 10th of *February*, when to the naked eye the head appeared no more, I saw through a telescope the tail of 2° in length. But farther; if the tail was owing to the refraction of the celestial matter, and did deviate from the opposition of the sun, according as the figure of the heavens requires, that deviation, in the same places of the heavens, should be always directed towards the same parts: but the comet of the year 1680, *December* 28^{d}. 8$\frac{1}{2}^{h}$. P. M. at *London*, was seen in *Pisces*, 8° 41′, with latitude north 28° 6′, while the sun was in *Capricorn* 18° 26′. And the comet of the year 1577, *December* 29, was in *Pisces* 8° 41′, with latitude north 28° 40′; and the sun, as before, in about *Capricorn* 18° 26′. In both cases the situation of the earth was the same, and the comet appeared in the same place of the heavens; yet in the former case the tail of the comet (as well by my observations as by the observations of others) deviated from the opposition of the sun towards the north by an angle of 4$\frac{1}{2}$ degrees, whereas in the latter there was (according to the observation of *Tycho*) a deviation of 21 degrees towards the south. The refraction, therefore, of the heavens being thus disproved, it remains that the phænomena of the tails of comets must be derived from some reflecting matter. That vapours sufficient to fill such immense spaces may arise from the comet's atmospheres, may be easily understood from what follows.

It is well known that the air near the surface of our earth possesses a space about 1200 times greater than water of the same weight; and therefore a cylindric column of air 1200 feet high is of equal weight with a cylinder of water of the same breadth, and but one foot high. But a cylinder of air reaching to the top of the atmosphere is of equal weight with a cylinder of water about 33 feet high; and therefore if from the whole cylinder of air the lower part of 1200 feet high is taken away, the remaining upper part will be of equal weight with a cylinder of water 32 feet high. Wherefore at the height of 1200 feet, or two furlongs, the weight of the incumbent air is less, and consequently the rarity of the compressed air greater, than near the surface of the earth in the ratio of 33 to 32. And, having this ratio, we may compute the rarity of the air in all places whatsoever (by the help of Cor. Prop. XXII, Book II), supposing the expansion thereof to be reciprocally proportional to its compression; and this proportion has been proved by the experiments of *Hooke* and others. The result of the computation I have set down in the follow-

ing table, in the first column of which you have the height o the air in miles, whereof 4000 make a semi-diameter of the earth; in the second the compression of the air, or the incumbent weight; in the third its rarity or expansion, supposing gravity to decrease in the duplicate ratio of the distances from the earth's centre. And the *Latin* numeral characters are here used for certain numbers of ciphers, as 0,xvii 1224 for 0,00000000000000001224, and 26956 xv for 2695600000000000000.

AIR's

Height.	Compression.	Expansion.
0	33	1
5	17,8515	1,8486
10	9,6717	3,4151
20	2,852	11,571
40	0,2525	136,83
400	0,xvii 1224	26956 xv
4000	0,cv. 4465	73907 cii
40000	0,cxcii 1628	20263 clxxxix
400000	0,ccx 7895	41798 ccvii
4000000	0,ccxii 9878	33414 ccix
Infinite.	0,ccxii 6041	54622 ccix

But from this table it appears that the air, in proceeding upwards, is rarefied in such manner, that a sphere of that air which is nearest to the earth, of but one inch in diameter, if dilated with that rarefaction which it would have at the height of one semi-diameter of the earth, would fill all the planetary regions as far as the sphere of Saturn, and a great way beyond; and at the height of ten semi-diameters of the earth would fill up more space than is contained in the whole heavens on this side the fixed stars, according to the preceding computation of their distance. And though, by reason of the far greater thickness of the atmospheres of comets, and the great quantity of the circum-solar centripetal force, it may happen that the air in the celestial spaces, and in the tails of comets, is not so vastly rarefied, yet from this computation it is plain that a very small quantity of air and vapour is abundantly sufficient to produce all the appearances of the tails of comets; for that they are indeed of a very notable rarity appears from the shining of the stars through them. The atmosphere of the earth, illuminated by the sun's light, though but of a few miles in thickness, obscures and extinguishes the light not only of all the stars, but even of the moon itself; whereas the smallest stars are seen to shine through the immense thickness of the tails of comets, likewise illuminated by the sun, without the least diminution of their splendor.

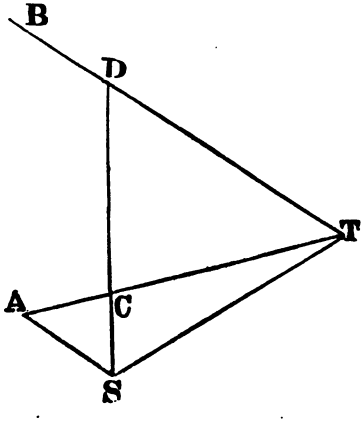
Kepler ascribes the ascent of the tails of comets to the atmospheres of their heads, and their direction towards the parts opposite to the sun to the action of the rays of light carrying along with them the matter of the comets' tails; and without any great incongruity we may suppose that, in so free spaces, so fine a matter as that of the æther may yield to the action

of the rays of the sun's light, though those rays are not able sensibly to move the gross substances in our parts, which are clogged with so palpable a resistance. Another author thinks that there may be a sort of particles of matter endowed with a principle of levity as well as others are with a power of gravity; that the matter of the tails of comets may be of the former sort, and that its ascent from the sun may be owing to its levity; but, considering the gravity of terrestrial bodies is as the matter of the bodies, and therefore can be neither more nor less in the same quantity of matter, I am inclined to believe that this ascent may rather proceed from the rarefaction of the matter of the comets' tails. The ascent of smoke in a chimney is owing to the impulse of the air with which it is entangled. The air rarefied by heat ascends, because its specific gravity is diminished, and in its ascent carries along with it the smoke with which it is engaged. And why may not the tail of a comet rise from the sun after the same manner? for the sun's rays do not act any way upon the mediums which they pervade but by reflection and refraction; and those reflecting particles heated by this action, heat the matter of the æther which is involved with them. That matter is rarefied by the heat which it acquires, and because by this rarefaction the specific gravity, with which it tended towards the sun before, is diminished, it will ascend therefrom like a stream, and carry along with it the reflecting particles of which the tail of the comet is composed; the impulse of the sun's light, as we have said, promoting the ascent.

But that the tails of comets do arise from their heads (p. 488), and tend towards the parts opposite to the sun, is farther confirmed from the laws which the tails observe; for, lying in the planes of the comets' orbits which pass through the sun, they constantly deviate from the opposition of the sun towards the parts which the comets' heads in their progress along those orbits have left; and to a spectator placed in those planes they appear in the parts directly opposite to the sun; but as the spectator recedes from those planes, their deviation begins to appear, and daily becomes greater. And the deviation, *cæteris paribus*, appears less when the tail is more oblique to the orbit of the comet, as well as when the head of the comet approaches nearer to the sun; especially if the angle of deviation is estimated near the head of the comet. Farther; the tails which have no deviation appear straight, but the tails which deviate are likewise bended into a certain curvature; and this curvature is greater when the deviation is greater, and is more sensible when the tail, *cæteris paribus*, is longer; for in the shorter tails the curvature is hardly to be perceived. And the angle of deviation is less near the comet's head, but greater towards the other end of the tail, and that because the lower side of the tail regards the parts from which the deviation is made, and which lie in a right line drawn out infinitely from the sun through the comet's head. And the tails that are

longer and broader, and shine with a stronger light, appear more resplendent and more exactly defined on the convex than on the concave side. Upon which accounts it is plain that the phænomena of the tails of comets depend upon the motions of their heads, and by no means upon the places of the heavens in which their heads are seen; and that, therefore, the tails of the comets do not proceed from the refraction of the heavens, but from their own heads, which furnish the matter that forms the tail; for as in our air the smoke of a heated body ascends either perpendicularly, if the body is at rest, or obliquely if the body is moved obliquely, so in the heavens, where all the bodies gravitate towards the sun, smoke and vapour must (as we have already said) ascend from the sun, and either rise perpendicularly, if the smoking body is at rest, or obliquely, if the body, in the progress of its motion, is always leaving those places from which the upper or higher parts of the vapours had risen before. And that obliquity will be less where the vapour ascends with more velocity, to wit, near the smoking body, when that is near the sun; for there the force of the sun by which the vapour ascends is stronger. But because the obliquity is varied, the column of vapour will be incurvated; and because the vapour in the preceding side is something more recent, that is, has ascended something more lately from the body, it will therefore be something more dense on that side, and must on that account reflect more light, as well as be better defined; the vapour on the other side languishing by degrees, and vanishing out of sight.

But it is none of our present business to explain the causes of the appearances of nature. Let those things which we have last said be true or false, we have at least made out, in the preceding discourse, that the rays of light are directly propagated from the tails of comets in right lines through the heavens, in which those tails appear to the spectators wherever placed; and consequently the tails must ascend from the heads of the comets towards the parts opposite to the sun. And from this principle we may determine anew the limits of their distances in manner following. Let S represent the sun, T the earth, STA the elongation of a comet from the sun, and ATB the apparent length of its tail; and because the light is propagated from the extremity of the tail in the direction of the right line TB, that extremity must lie somewhere in the line TB. Suppose it in D, and join DS cutting TA in C. Then, because the tail is always stretched out towards the parts nearly opposite to the sun, and therefore



the sun, the head of the comet, and the extremity of the tail, lie in a right line, the comet's head will be found in C. Parallel to TB draw SA, meeting the line TA in A, and the comet's head C must necessarily be found between T and A, because the extremity of the tail lies somewhere in the infinite line TB; and all the lines SD which can possibly be drawn from the point S to the line TB must cut the line TA somewhere between T and A. Wherefore the distance of the comet from the earth cannot exceed the interval TA, nor its distance from the sun the interval SA beyond, or ST on this side the sun. For instance: the elongation of the comet of 1680 from the sun, *Dec.* 12, was 9°, and the length of its tail 35° at least. If, therefore, a triangle TSA is made, whose angle T is equal to the elongation 9°, and angle A equal to ATB, or to the length of the tail, viz., 35°, then SA will be to ST, that is, the limit of the greatest possible distance of the comet from the sun to the semi-diameter of the *orbis magnus*, as the sine of the angle T to the sine of the angle A, that is, as about 3 to 11. And therefore the comet at that time was less distant from the sun than by $\frac{3}{11}$ of the earth's distance from the sun, and consequently either was within the orb of Mercury, or between that orb and the earth. Again, *Dec.* 21, the elongation of the comet from the sun was $32\frac{2}{3}°$, and the length of its tail 70°. Wherefore as the sine of $32\frac{2}{3}°$ to the sine of 70°, that is, as 4 to 7, so was the limit of the comet's distance from the sun to the distance of the earth from the sun, and consequently the comet had not then got without the orb of Venus. *Dec.* 28, the elongation of the comet from the sun was 55°, and the length of its tail 56°; and therefore the limit of the comet's distance from the sun was not yet equal to the distance of the earth from the same, and consequently the comet had not then got without the earth's orbit. But from its parallax we find that its egress from the orbit happened about *Jan.* 5, as well as that it had descended far within the orbit of Mercury. Let us suppose it to have been in its perihelion *Dec.* the 8th, when it was in conjunction with the sun; and it will follow that in the journey from its perihelion to its exit out of the earth's orbit it had spent 28 days; and consequently that in the 26 or 27 days following, in which it ceased to be farther seen by the naked eye, it had scarcely doubled its distance from the sun; and by limiting the distances of other comets by the like arguments, we come at last to this conclusion,—that all comets, during the time in which they are visible by us, are within the compass of a spherical space described about the sun as a centre, with a radius double, or at most triple, of the distance of the earth from the sun.

And hence it follows that the comets, during the whole time of their appearance unto us, being within the sphere of activity of the circumsolar force, and therefore agitated by the impulse of that force, will (by Cor. 1, Prop. XII, Book I, for the same reason as the planets) be made to

move in conic sections that have one focus in the centre of the sun, and by radii drawn to the sun, to describe areas proportional to the times; for that force is propagated to an immense distance, and will govern the motions of bodies far beyond the orbit of Saturn.

There are three hypotheses about comets (p. 466); for some will have it that they are generated and perish as often as they appear and vanish; others, that they come from the regions of the fixed stars, and are seen by us in their passage through the system of our planets; and, lastly, others, that they are bodies perpetually revolving about the sun in very eccentric orbits. In the first case, the comets, according to their different velocities, will move in conic sections of all sorts: in the second, they will describe hyperbolas, and in either of the two will frequent indifferently all quarters of the heavens, as well those about the poles as those towards the ecliptic; in the third, their motions will be performed in ellipses very eccentric, and very nearly approaching to parabolas. But (if the law of the planets is observed) their orbits will not much decline from the plane of the ecliptic; and, so far as I could hitherto observe, the third case obtains; for the comets do, indeed, chiefly frequent the zodiac, and scarcely ever attain to a heliocentric latitude of 40°. And that they move in orbits very nearly parabolical, I infer from their velocity; for the velocity with which a parabola is described is every where to the velocity with which a comet or planet may be revolved about the sun in a circle at the same distance in the subduplicate ratio of 2 to 1 (by Cor. VII, Prop. XVI); and, by my computation, the velocity of comets is found to be much about the same. I examined the thing by inferring nearly the velocities from the distances, and the distances both from the parallaxes and the phænomena of the tails, and never found the errors of excess or defect in the velocities greater than what might have arose from the errors in the distances collected after that manner. But I likewise made use of the reasoning that follows.

Supposing the radius of the *orbis magnus* to be divided into 1000 parts: let the numbers in the first column of the following table represent the distance of the vertex of the parabola from the sun's centre, expressed by those parts: and a comet in the times expressed in col. 2, will pass from its perihelion to the surface of the sphere which is described about the sun as a centre with the radius of the *orbis magnus;* and in the times expressed in col. 3, 4, and 5, it will double, triple, and quadruple, that its distance from the sun.

TABLE I.

The distance of a comet's perihelion from the Sun's centre.	The time of a comet's passage from its perihelion to a distance from the sun equal to											
	The radius of the *orbis magnus*.			To its double.			To its triple			To its Quadruple.		
	d.	h.	′	d.	h.	′	d.	h.	′	d.	h.	′
0	27	11	12	77	16	28	142	17	14	219	17	30
5	27	16	07	77	23	14						
10	27	21	00	78	06	24						
20	28	06	40	78	20	13	144	03	19	221	08	54
40	29	01	32	79	23	34						
80	30	13	25	82	04	56						
160	33	05	29	86	10	26	153	16	08	232	12	20
320	37	13	46	93	23	38						
640	37	09	49	105	01	28						
1280				106	06	35	200	06	43	297	03	46
2560							147	22	31	300	06	03

[This table, here corrected, is made on the supposition that the earth's diurnal motion is just 59′, and the measure of one minute loosely 0,2909, in respect of the radius 1000. If those measures are taken true, the true numbers of the table will all come out less. But the difference, even when greatest, and to the quadruple of the earth's distance from the sun, amounts only to 16^h. 55′.]

The time of a comet's ingress into the sphere of the *orbis magnus*, or of its egress from the same, may be inferred nearly from its parallax, but with more expedition by the following

TABLE II.

The apparent elongation of a comet from the sun.	Its apparent diurnal motion in its own orbit.		Its distance from the earth in parts whereof the radius of the *orbis magnus* contains 1000.
	Direct.	Retrog.	
60°	2° 18′	00° 20′	1000
65	2 33	00 35	845
70	2 55	00 57	684
72	3 07	01 09	618
74	3 23	01 25	551
76	3 43	01 45	484
78	4 10	02 12	416
80	4 57	02 49	347
82	5 45	03 47	278
84	7 18	05 20	209
86	10 27	08 19	140
88	18 37	16 39	70
90	Infi'ite	Infi'ite	00

The ingress of a comet into the sphere of the *orbis magnus*, or its egress from the same, happens at the time of its elongation from the sun, expressed in col. 1, against its diurnal motion. So in the comet of 1681, *Jan.* 4, O.S. the apparent diurnal motion in its orbit was about 3° 5′, and the corresponding elongation 71⅔°; and the comet had acquired this elongation from the sun *Jan.* 4, about six in the evening. Again, in the year 1680, *Nov.* 11, the diurnal motion of the comet that then appeared was about 4⅔°; and the corresponding elongation 79⅔ happened *Nov.* 10, a little before midnight. Now at the times named these comets had arrived at an equal distance from the sun with the earth, and the earth was then almost in its perihelion. But the first table is fitted to the earth's mean distance from the sun assumed of 1000 parts; and this distance is greater by such an excess of space as the earth might describe by its annual motion in one day's time, or the comet by its motion in 16 hours. To reduce the comet to this mean distance of 1000 parts, we add those 16 hours to the former time, and subduct them from the latter; and thus the former becomes *Jan.* 4ᵈ. 10ʰ. afternoon; the latter *Nov.* 10, about six in the morning. But from the tenor and progress of the diurnal motions it appears that both comets were in conjunction with the sun between *Dec.* 7 and *Dec.* 8; and from thence to *Jan.* 4ᵈ. 10ʰ. afternoon on one side, and to *Nov.* 10ᵈ. 6ʰ. of the morning on the other, there are about 28 days. And so many days (by Table 1) the motions in parabolic trajectories do require.

But though we have hitherto considered those comets as two, yet, from the coincidence of their perihelions and agreement of their velocities, it is probable that in effect they were but one and the same; and if so, the orbit of this comet must have either been a parabola, or at least a conic section very little differing from a parabola, and at its vertex almost in contact with the surface of the sun. For (by Tab. 2) the distance of the comet from the earth, *Nov.* 10, was about 360 parts, and *Jan.* 4, about 630. From which distances, together with its longitudes and latitudes, we infer the distance of the places in which the comet was at those times to have been about 280: the half of which, viz., 140, is an ordinate to the comet's orbit, cutting off a portion of its axis nearly equal to the radius of the *orbis magnus*, that is, to 1000 parts. And, therefore, dividing the square of the ordinate 140 by 1000, the segment of the axis, we find the *latus rectum* 19, 16, or in a round number 20; the fourth part whereof, 5, is the distance of the vertex of the orbit from the sun's centre. But the time corresponding to the distance of 5 parts in Tab. 1 is 27ᵈ. 16ʰ. 7′. In which time, if the comet moved in a parabolic orbit, it would have been carried from its perihelion to the surface of the sphere of the *orbis magnus* described with the radius 1000, and would have spent the double of that time, viz., 55ᵈ. 8¼ʰ. in the whole course of its motion within that sphere: and so in fact it did; for from *Nov.* 10ᵈ. 6ʰ. of the morning, the

time of the comet's ingress into the sphere of the *orbis magnus*, to *Jan.* 4^{d}. 10^{h}. afternoon, the time of its egress from the same, there are 55^{d}. 16^{h}. The small difference of $7\frac{3}{4}$^{h}. in this rude way of computing is to be neglected, and perhaps may arise from the comet's motion being some small matter slower, as it must have been if the true orbit in which it was carried was an ellipsis. The middle time between its ingress and egress was *December* 8^{d}. 2^{h}. of the morning; and therefore at this time the comet ought to have been in its perihelion. And accordingly that very day, just before sunrising, Dr. *Halley* (as we said) saw the tail short and broad, but very bright, rising perpendicularly from the horizon. From the position of the tail it is certain that the comet had then crossed over the ecliptic, and got into north latitude, and therefore had passed by its perihelion, which lay on the other side of the ecliptic, though it had not yet come into conjunction with the sun; and the comet [see more of this famous comet, p. 475 to 486] being at this time between its perihelion and its conjunction with the sun, must have been in its perihelion a few hours before; for in so near a distance from the sun it must have been carried with great velocity, and have apparently described almost half a degree every hour.

By like computations I find that the comet of 1618 entered the sphere of the *orbis magnus December* 7, towards sun-setting; but its conjunction with the sun was *Nov.* 9, or 10, about 28 days intervening, as in the preceding comet; for from the size of the tail of this, in which it was equal to the preceding, it is probable that this comet likewise did come almost into a contact with the sun. Four comets were seen that year of which this was the last. The second, which made its first appearance *October* 31, in the neighbourhood of the rising sun, and was soon after hid under the sun's rays, I suspect to have been the same with the fourth, which emerged out of the sun's rays about *Nov.* 9. To these we may add the comet of 1607, which entered the sphere of the *orbis magnus Sept.* 14, O.S. and arrived at its perihelion distance from the sun about *October* 19, 35 days intervening. Its perihelion distance subtended an apparent angle at the earth of about 23 degrees, and was therefore of 390 parts. And to this number of parts about 34 days correspond in Tab. 1. Farther; the comet of 1665 entered the sphere of the *orbis magnus* about *March* 17, and came to its perihelion about *April* 16, 30 days intervening. Its perihelion distance subtended an angle at the earth of about seven degrees, and therefore was of 122 parts: and corresponding to this number of parts, in Tab. 1, we find 30 days. Again; the comet of 1682 entered the sphere of the *orbis magnus* about *Aug.* 11, and arrived at its perihelion about *Sep.* 16, being then distant from the sun by about 350 parts, to which, in Tab. 1, belong $33\frac{1}{2}$ days. Lastly; that memorable comet of *Regiomontanus*, which in 1472 was carried through the circum-polar parts of our northern hemisphere with such rapidity as to describe 40

degrees in one day, entered the sphere of the *orbis magnus Jan.* 21, about the time that it was passing by the pole, and, hastening from thence towards the sun, was hid under the sun's rays about the end of *Feb.*, whence it is probable that 30 days, or a few more, were spent between its ingress into the sphere of the *orbis magnus* and its perihelion. Nor did this comet truly move with more velocity than other comets, but owed the greatness of its apparent velocity to its passing by the earth at a near distance.

It appears, then, that the velocity of comets (p. 471), so far as it can be determined by these rude ways of computing, is that very velocity with which parabolas, or ellipses near to parabolas, ought to be described; and therefore the distance between a comet and the sun being given, the velocity of the comet is nearly given. And hence arises this problem.

PROBLEM.

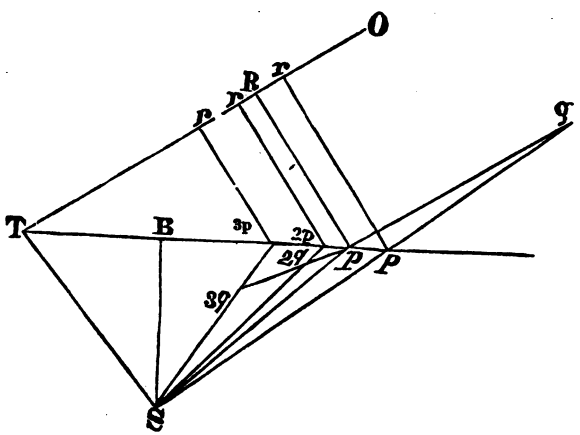
The relation betwixt the velocity of a comet and its distance from the sun's centre being given, the comet's trajectory is required.

If this problem was resolved, we should thence have a method of determining the trajectories of comets to the greatest accuracy; for if that relation be twice assumed, and from thence the trajectory be twice computed, and the error of each trajectory be found from observations, the assumption may be corrected by the Rule of False, and a third trajectory may thence be found that will exactly agree with the observations. And by determining the trajectories of comets after this method, we may come, at last, to a more exact knowledge of the parts through which those bodies travel, of the velocities with which they are carried, what sort of trajectories they describe, and what are the true magnitudes and forms of their tails according to the various distances of their heads from the sun; whether, after certain intervals of time, the same comets do return again, and in what periods they complete their several revolutions. But the problem may be resolved by determining, first, the hourly motion of a comet to a given time from three or more observations, and then deriving the trajectory from this motion. And thus the invention of the trajectory, depending on one observation, and its hourly motion at the time of this observation, will either confirm or disprove itself; for the conclusion that is drawn from the motion only of an hour or two and a false hypothesis, will never agree with the motions of the comets from beginning to end. The method of the whole computation is this.

LEMMA I.

To cut two right lines OR, TP, *given in position, by a third right line* RP, *so as* TRP *may be a right angle; and, if another right line* SP *is drawn to any given point* S, *the solid contained under this line* SP, *and the square of the right line* OR *terminated at a given point* O, *may be of a given magnitude.*

It is done by linear description thus. Let the given magnitude of the solid be $M^2 \times N$; from any point r of the right line OR erect the per-



pendicular rp meeting TP in p. Then through the point Sp draw the line Sq equal to $\frac{M^2 \times N}{Or^2}$. In like manner draw three or more right lines S2q, S3q, &c.; and a regular line $q2q3q$, drawn through all the points $q2q3q$, &c., will cut the right line TP in the point P, from which the perpendicular PR is to be let fall. Q.E.F.

By trigonometry thus. Assuming the right line TP as found by the preceding method, the perpendiculars TR, SB, in the triangles TPR, TPS, will be thence given; and the side SP in the triangle SBP, as well as the error $\frac{M^2 \times N}{OR^2} - SP$. Let this error, suppose D, be to a new error, suppose E, as the error $2p2q \pm 3p3q$ to the error $2p3p$; or as the error $2p2q \pm D$ to the error $2pP$; and this new error added to or subducted from the length TP, will give the correct length TP $\pm$ E. The inspection of the figure will shew whether we are to add to or subtract; and if at any time there should be use for a farther correction, the operation may be repeated

By arithmetic thus. Let us suppose the thing done, and let TP + e be the correct length of the right line TP as found out by delineation; and thence the correct lengths of the lines OR, BP, and SP, will be OR $-\frac{TR}{TP}e$, BP + e, and $\sqrt{SP^2 + 2BPe + ee} = \frac{M^2N}{OR^2 + \frac{2OR \times TR}{TP}e + \frac{TR^2}{TP^2}ee}$.

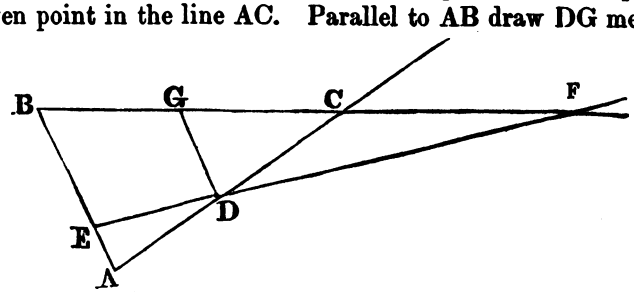
Whence, by the method of converging series, we have SP $+ \frac{BP}{SP}e + \frac{SB^2}{2SP^3}$ ee, &c., $= \frac{M^2N}{OR^2} + \frac{2TR}{TP} \times \frac{M^2N}{OR^3}e + \frac{3TR^2}{TP^2} \times \frac{M^2N}{OR^4}ee$, &c. For the given co-efficients $\frac{M^2N}{OR^2} - SP$, $\frac{2TR}{TP} \times \frac{M^2N}{OR^3} - \frac{BP}{SP} \frac{3TR^2}{TP^2} \times \frac{M^2N}{OR^4} - \frac{SB^2}{2SP^3}$, putting F, $\frac{F}{G}$ $\frac{F}{GH}$, and carefully observing the signs, we find F $+ \frac{F}{G}e +$ $\frac{F}{GH}ee = 0$, and $e + \frac{ee}{H} = -$ G. Whence, neglecting the very small term $\frac{e^2}{H}$, e comes out equal to — G. If the error $\frac{e^2}{H}$ is not despicable, take $-G - \frac{G^2}{H} = e$.

And it is to be observed that here a general method is hinted at for solving the more intricate sort of problems, as well by trigonometry as by arithmetic, without those perplexed computations and resolutions of affected equations which hitherto have been in use.

LEMMA II.

To cut three right lines given in position by a fourth right line that shall pass through a point assigned in any of the three, and so as its intercepted parts shall be in a given ratio one to the other.

Let AB, AC, BC, be the right lines given in position, and suppose D to be the given point in the line AC. Parallel to AB draw DG meeting BC



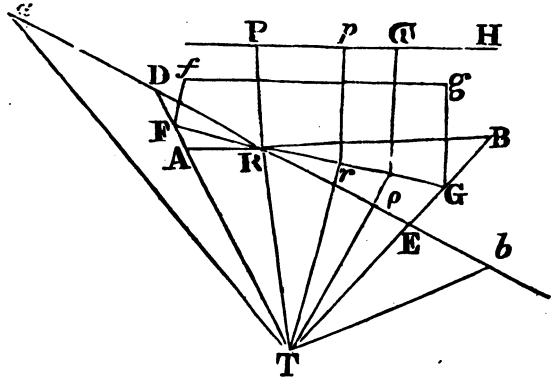
in G; and, taking GF to BG in the given ratio, draw FDE; and FD will be to DE as FG to BG. Q.E.F.

By trigonometry thus. In the triangle CGD all the angles and the side CD are given, and from thence its remaining sides are found; and from the given ratios the lines GF and BE are also given.

LEMMA III.

To find and represent by a linear description the hourly motion of a comet to any given time.

From observations of the best credit, let three longitudes of the comet be given, and, supposing ATR, RTB, to be their differences, let the hourly motion be required to the time of the middle observation TR. By *Lem* II, draw the right line ARB, so as its intercepted parts AR, RB, may be



as the times between the observations; and if we suppose a body in the whole time to describe the whole line AB with an equal motion, and to be in the mean time viewed from the place T, the apparent motion of that body about the point R will be nearly the same with that of the comet at the time of the observation TR.

The same more accurately.

Let T*a*, T*b*, be two longitudes given at a greater distance on one side and on the other; and by *Lem.* II draw the right line *a*R*b* so as its intercepted parts *a*R, R*b* may be as the times between the observations *a*TR, RT*b*. Suppose this to cut the lines TA, TB, in D and E; and because the error of the inclination TR*a* increases nearly in the duplicate ratio of the time between the observations, draw FRG, so as either the angle DRF may be to the angle ARF, or the line DF to the line AF, in the duplicate ratio of the whole time between the observations *a*TB to the whole time between the observations ATB, and use the line thus found FG in place of the line AB found above.

It will be convenient that the angles ATR, RTB, *a*TA, BT*b*, be no less than of ten or fifteen degrees, the times corresponding no greater than

of eight or twelve days, and the longitudes taken when the comet moves with the greatest velocity; for thus the errors of the observations will bear a less proportion to the differences of the longitudes.

LEMMA IV.

To find the longitudes of a comet to any given times.

It is done by taking in the line FG the distances R*r*, Rρ, proportional to the times, and drawing the lines T*r*, Tρ. The way of working by trigonometry is manifest.

LEMMA V.

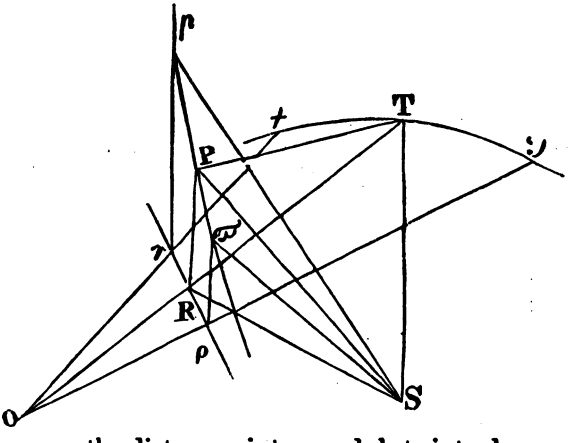
To find the latitudes.

On TF, TR, TG, as radiuses, at right angles erect F*f*, RP, G*g*, tangents of the observed latitudes; and parallel to *fg* draw PH. The perpendiculars *rp*, $\rho\tilde{\omega}$, meeting PH, will be the tangents of the sought latitudes to T*r* and Tρ as radiuses.

PROBLEM I.

From the assumed ratio of the velocity to determine the trajectory of a comet.

Let S represent the sun; *t*, T, τ, three places of the earth in its orbit at equal distances; *p*, P, $\tilde{\omega}$, as many corresponding places of the comet in



its trajectory, so as the distances interposed betwixt place and place may answer to the motion of one hour; *pr*, PR, $\tilde{\omega}\rho$, perpendiculars let fall on the plane of the ecliptic, and *r*Rρ the vestige of the trajectory in this plane. Join S*p*, SP, S$\tilde{\omega}$, SR, ST, *tr*, TR, $\tau\rho$, TP, and let *tr*, $\tau\rho$, meet in O, TR will nearly converge to the same point O, or the error will be inconsiderable. By the premised lemmas the angles *r*OR, ROρ, are given, as well as the ratios *pr* to *tr*, PR to TR, and $\tilde{\omega}\rho$ to $\tau\rho$. The figure *t*TτO

is likewise given both in magnitude and position, together with the distance ST, and the angles STR, PTR, STP. Let us assume the velocity of the comet in the place P to be to the velocity of a planet revolved about the sun in a circle, at the same distance SP, as V to 1; and we shall have a line $p\mathrm{P}\tilde{\omega}$ to be determined, of this condition, that the space $p\tilde{\omega}$, described by the comet in two hours, may be to the space $\mathrm{V} \times t\tau$ (that is, to the space which the earth describes in the same time multiplied by the number V) in the subduplicate ratio of ST, the distance of the earth from the sun, to SP, the distance of the comet from the sun; and that the space $p\mathrm{P}$, described by the comet in the first hour, may be to the space $\mathrm{P}\tilde{\omega}$, described by the comet in the second hour, as the velocity in p to the velocity in P; that is, in the subduplicate ratio of the distance SP to the distance $\mathrm{S}p$, or in the ratio of $2\mathrm{S}p$ to $\mathrm{SP} + \mathrm{S}p$; for in this whole work I neglect small fractions that can produce no sensible error.

In the first place, then, as mathematicians, in the resolution of affected equations, are wont, for the first essay, to assume the root by conjecture, so, in this analytical operation, I judge of the sought distance TR as I best can by conjecture. Then, by *Lem.* II. I draw $r\rho$, first supposing $r\mathrm{R}$ equal to $\mathrm{R}\rho$, and again (after the ratio of SP to $\mathrm{S}p$ is discovered) so as $r\mathrm{R}$ may be to $\mathrm{R}\rho$ as $2\mathrm{SP}$ to $\mathrm{SP} + \mathrm{S}p$, and I find the ratios of the lines $p\tilde{\omega}$, $r\rho$, and OR, one to the other. Let M be to $\mathrm{V} \times t\tau$ as OR to $p\tilde{\omega}$; and because the square of $p\tilde{\omega}$ is to the square of $\mathrm{V} \times t\tau$ as ST to SP, we shall have, *ex æquo*, OR^2 to M^2 as ST to SP, and therefore the solid $\mathrm{OR}^2 \times \mathrm{SP}$ equal to the given solid $\mathrm{M}^2 \times \mathrm{ST}$; whence (supposing the triangles STP, PTR, to be now placed in the same plane) TR, TP, SP, PR, will be given, by *Lem.* I. All this I do, first by delineation in a rude and hasty way; then by a new delineation with greater care; and, lastly, by an arithmetical computation. Then I proceed to determine the position of the lines $r\rho$, $p\tilde{\omega}$, with the greatest accuracy, together with the nodes and inclination of the plane $\mathrm{S}p\tilde{\omega}$ to the plane of the ecliptic; and in that plane $\mathrm{S}p\tilde{\omega}$ I describe the trajectory in which a body let go from the place P in the direction of the given right line $p\tilde{\omega}$ would be carried with a velocity that is to the velocity of the earth as $p\tilde{\omega}$ to $\mathrm{V} \times t\tau$. Q.E.F.

PROBLEM II.

To correct the assumed ratio of the velocity and the trajectory thence found.

Take an observation of the comet about the end of its appearance, or any other observation at a very great distance from the observations used before, and find the intersection of a right line drawn to the comet, in that observation with the plane $\mathrm{S}p\tilde{\omega}$, as well as the comet's place in its trajectory to the time of the observation. If that intersection happens in this place, it is a proof that the trajectory was rightly determined; if other-

wise, a new number V is to be assumed, and a new trajectory to be found; and then the place of the comet in this trajectory to the time of that probatory observation, and the intersection of a right line drawn to the comet with the plane of the trajectory, are to be determined as before; and by comparing the variation of the error with the variation of the other quantities, we may conclude, by the Rule of Three, how far those other quantities ought to be varied or corrected, so as the error may become as small as possible. And by means of these corrections we may have the trajectory exactly, providing the observations upon which the computation was founded were exact, and that we did not err much in the assumption of the quantity V; for if we did, the operation is to be repeated till the trajectory is exactly enough determined. Q.E.F.

END OF THE SYSTEM OF THE WORLD.

CONTENTS

OF

THE SYSTEM OF THE WORLD.

That the matter of the heavens is fluid, ... 511
The principle of circular motion in free spaces, ... 512
The effects of centripetal forces, ... 512
The certainty of the argument, ... 514
What follows from the supposed diurnal motion of the stars, ... 514
The incongruous consequences of this supposition, ... 514
That there is a centripetal force really directed to the centre of every planet, ... 515
Centripetal forces decrease in duplicate proportion of distances from the centre of every planet, 516
That the superior planets are revolved about the sun, and by radii drawn to the sun describe areas proportional to the times, ... 517
That the force which governs the superior planets is directed not to the earth, but to the sun, . 518
That the circum-solar force throughout all the regions of the planets decreaseth in the duplicate proportion of the distances from the sun, ... 519
That the circum-terrestrial force decreases in the duplicate proportion of the distances from the earth proved in the hypothesis of the earth's being at rest, ... 519
The same proved in the hypothesis of the earth's motion, ... 520
The decrement of the forces in the duplicate proportion of the distances from the earth and planets, proved from the eccentricity of the planets, and the very slow motion of their apses, . 520
The quantity of the forces tending towards the several planets: the circum-solar very great, . 521
The circum-terrestrial force very small, ... 521
The apparent diameters of the planets, ... 521
The correction of the apparent diameters, ... 522
Why the density is greater in some of the planets and less in others; but the forces in all are as their quantities of matter, ... 524
Another analogy between the forces and bodies, proved in the celestial bodies, ... 525
Proved in terrestrial bodies, ... 525
The affinity of those analogies, ... 526
And coincidence, ... 526
That the forces of small bodies are insensible, ... 527
Which, notwithstanding, there are forces tending towards all terrestrial bodies proportional to their quantities of matter, ... 528
Proved that the same forces tend towards the celestial bodies, ... 528
That from the surfaces of the planets, reckoning outward, their forces decrease in the duplicate; but, reckoning inward, in the simple proportion of the distances from their centres, . 528
The quantities of the forces and of the motions arising in the several cases, ... 529
That all the planets revolve about the sun, ... 529
That the common centre of gravity of all the planets is quiescent. That the sun is agitated with a very slow motion. This motion defined, ... 531
That the planets, nevertheless, are revolved in ellipses having their foci in the sun; and by radii drawn to the sun describe areas proportional to the times, ... 531
Of the dimensions of the orbits, and of the motions of their aphelions and nodes, ... 532
All the motions of the moon that have hitherto been observed by astronomers derived from the foregoing principles, ... 532
As also some other unequable motions that hitherto have not been observed, ... 533
And the distance of the moon from the earth to any given time, ... 533
The motions of the satellites of Jupiter and Saturn derived from the motions of our moon, . 534
That the planets, in respect of the fixed stars, are revolved by equable motions about their proper axes. And that (perhaps) those motions are the most fit for the equation of time, 534
The moon likewise is revolved by a diurnal motion about its axis, and its libration thence arises, 535
That the sea ought twice to flow, and twice to ebb, every day; that the highest water must fall out in the third hour after the appulse of the luminaries to the meridian of the place, . 536

The precession of the equinoxes, and the libratory motion of the axes of the earth and planets, 535
That the greatest tides happen in the syzygies of the luminaries, the least in their quadratures; and that at the third hour after the appulse of the moon to the meridian of the place. But that out of the syzygies and quadratures those greatest and least tides deviate a little from that third hour towards the third hour after the appulse of the sun to the meridian, . 536
That the tides are greatest when the luminaries are in their perigees, ... 536
That the tides are greatest about the equinoxes, ... 536
That out of the equator the tides are greater and less alternately, ... 537
That, by the conservation of the impressed motion, the difference of the tides is diminished; and that hence it may happen that the greatest menstrual tide will be the third after the syzygy, 538
That the motions of the sea may be retarded by impediments in its channels, ... 538
That from the impediments of channels and shores various phænomena do arise, as that the sea may flow but once every day, ... 539
That the times of the tides within the channels of rivers are more unequal than in the ocean, . 540
That the tides are greater in greater and deeper seas; greater on the shores of continents than of islands in the middle of the sea; and yet greater in shallow bays that open with wide inlets to the sea, ... 540
The force of the sun to disturb the motions of the moon, computed from the foregoing principles, 542
The force of the sun to move the sea computed, ... 543
The height of the tide under the equator arising from the force of the sun computed, . . . 543
The height of the tides under the parallels arising from the sun's force computed, ... 544
The proportion of the tides under the equator, in the syzygies and quadratures, arising from the joint forces of both sun and moon, ... 545
The force of the moon to excite tides, and the height of the water thence arising, computed, . 545
That those forces of the sun and moon are scarcely sensible by any other effect beside the tides which they raise in the sea, ... 546
That the body of the moon is about six times more dense than the body of the sun, ... 547
That the moon is more dense than the earth in a ratio of about three to two, ... 547
Of the distance of the fixed stars, ... 547
That the comets, as often as they become visible to us, are nearer than Jupiter, proved from their parallax in longitude, ... 548
The same proved from their parallax in latitude, ... 549
The same proved otherwise by the parallax, ... 550
From the light of the comets' heads it is proved that they descend to the orbit of Saturn, . 550
And also below the orb of Jupiter, and sometimes below the orb of the earth, ... 551
The same proved from the extraordinary splendor of their tails when they are near the sun, . 551
The same proved from the light of their heads, as being greater, *cæteris paribus*, when they come near to the sun, ... 553
The same confirmed by the great number of comets seen in the region of the sun, ... 555
This also confirmed by the greater magnitude and splendor of the tails after the conjunction of the heads with the sun than before, ... 555
That the tails arise from the atmospheres of the comets, ... 556
That the air and vapour in the celestial spaces is of an immense rarity; and that a small quantity of vapour may be sufficient to explain all the phænomena of the tails of comets, . . 556
After what manner the tails of comets may arise from the atmospheres of their heads, . . 559
That the tails do indeed arise from those atmospheres, proved from several of their phænomena, 559
That comets do sometimes descend below the orbit of Mercury, proved from their tails, . 560
That the comets move in conic sections, having one focus in the centre of the sun, and by radii drawn to that centre do describe areas proportional to the times, ... 561
That those conic sections are near to parabolas, proved from the velocity of the comets, . 561
In what space of time comets describing parabolic trajectories pass through the sphere of the *orbis magnus*, ... 562
At what time comets enter into and pass out of the sphere of the *orbis magnus*, ... 563
With what velocity the comets of 1680 passed through the sphere of the *orbis magnus*, . . 564
That these were not two, but one and the same comet. In what orbit and with what velocity this comet was carried through the heavens described more exactly, ... 564
With what velocity comets are carried, shewed by more examples, ... 565
The investigation of the trajectory of comets proposed, ... 566
Lemmas premised to the solution of the problem, ... 567
The problem resolved, ... 570

INDEX TO THE PRINCIPIA.

EQUINOXES, their præcession—the cause of that motion shewn, ... 413
" the quantity of that motion computed from the causes, ... 453
AIR, its density at any height, collected by Prop. XXII, Book II, and its density at the height of one semi-diameter of the earth, shewn, ... 489
" its elastic force, what cause it may be attributed to, ... 302
" its gravity compared with that of water, ... 489
" its resistance, collected by experiments of pendulums, ... 315
" the same more accurately by experiments of falling bodies, and a theory, ... 355
ANGLES of contact not all of the same kind, but some infinitely less than others, ... 101
APSIDES, their motion shewn, ... 172, 173
AREAS which revolving bodies, by radii drawn to the centre of force describe, compared with the times of description, ... 103, 105, 106, 195, 209
As, the mathematical signification of this word defined, ... 100
ATTRACTION of all bodies demonstrated, ... 397
" the certainty of this demonstration shewn, ... 384
" the cause or manner thereof no where defined by the author, ... 507
" the common centre of gravity of the earth, sun, and all the planets, is at rest, confirmed by Cor. 2, Prop. XIV, Book III, ... 401
" the common centre of gravity of the earth and moon goes round the *orbis magnus*, 402
" its distance from the earth and from the moon, ... 452
CENTRE, the common centre of gravity of many bodies does not alter its state of motion or rest by the actions of the bodies among themselves, ... 87
" of the forces by which revolving bodies are retained in their orbits, how indicated by the description of areas, ... 107
" how found by the given velocities of the revolving bodies, ... 110
CIRCLE, by what law of centripetal force tending to any given point its circumference may be described, ... 108, 111, 114
COMETS, a sort of planets, not meteors, ... 465, 486
" higher than the moon, and in the planetary regions, ... 460
" their distance how collected very nearly by observations, ... 461
" more of them observed in the hemisphere towards the sun than in the opposite hemisphere; and how this comes to pass, ... 464
" shine by the sun's light reflected from them, ... 464
" surrounded with vast atmospheres, ... 463, 465
" those which come nearest to the sun probably the least, ... 495
" why they are not comprehended within a zodiac, like the planets, but move differently into all parts of the heavens, ... 502
" may sometimes fall into the sun, and afford a new supply of fire, ... 502
" the use of them hinted, ... 492
" move in conic sections, having their foci in the sun's centre, and by radii drawn to the sun describe areas proportional to the times. Move in ellipses if they come round again in their orbits, but these ellipses will be near to parabolas, ... 466
COMET'S parabolic trajectory found from three observations given, ... 472
" corrected when found, ... 495
" place in a parabola found to a given time, ... 466
" velocity compared with the velocity of the planets, ... 466
COMETS' TAILS directed from the sun, ... 489
" " brightest and largest immediately after their passage through the neighbourhood of the sun, ... 487
" " their wonderful rarity, ... 490
" " their origin and nature, ... 465
" " in what space of time they ascend from their heads, ... 490

COMET of the years 1664 and 1665—the observations of its motion compared with the theory, . 496
" of the years 1680 and 1681—observations of its motion, ... 474
" its motion computed in a parabolic orbit, ... 478
" in an elliptic orbit, ... 479
" its trajectory, and its tail in the several parts of its orbit, delineated, ... 484
" of the year 1682—its motion compared with the theory, ... 500
" seems to have appeared in the year 1607, and likely to return again after a period of 75 years, ... 501, 502
" of the year 1683—its motion compared with the theory, ... 499
" of the year 1723—its motion compared with the theory, ... 501
CONIC SECTIONS, by what law of centripetal force tending to any given point they may be described by revolving bodies, ... 125
" the geometrical description of them when the foci are given, ... 125
" when the foci are not given, ... 131
" when the centres or asymptotes are given, ... 147
CURVATURE of figures how estimated, ... 271, 423
CURVES distinguished into geometrically rational and geometrically irrational, ... 157
CYCLOID, or EPICYCLOID, its rectification, ... 184
" " its evoluta, ... 185
CYLINDER. the attraction of a cylinder composed of attracting particles, whose forces are reciprocally as the square of the distances, ... 239
DESCENT of heavy bodies in vacuo, how much it is, ... 405
" and ascent of bodies in resisting mediums, ... 252, 265, 281, 283, 345
DESCENT or ASCENT rectilinear, the spaces described, the times of decription, and the velocities acquired in such ascent or descent, compared, on the supposition of any kind of centripetal force, ... 160
EARTH, its dimension by *Norwood*, by *Picart*, and by *Cassini*, ... 405
" its figure discovered, with the proportion of its diameters, and the measure of the degrees upon the meridian, ... 405, 409
" the excess of its height at the equator above its height at the poles, ... 407, 412
" its greatest and least semi-diameter, ... 407
" its mean semi-diameter, ... 407
" the globe of the earth more dense than if it was entirely water, ... 400
" the nutation of its axis, ... 413
" the annual motion thereof in the *orbis magnus* demonstrated, ... 498
" the eccentricity thereof how much, ... 452
" the motion of its aphelion how much, ... 404
ELLIPSES, by what law of centripetal force tending to the centre of the figure it is described by a revolving body, ... 114
" by what law of centripetal force tending to the focus of the figure it is described by a revolving body, ... 116
FLUID, the definition thereof, ... 108
FLUIDS, the laws of their density and compression shewn, ... 293
" their motion in running out at a hole in a vessel determined, ... 331
FORCES, their composition and resolution, ... 84
" attractive forces of spherical bodies, composed of particles attracting according to any law, determined, ... 218
" attractive forces of bodies not spherical, composed of particles attracting according to any law, determined, ... 233
" the invention of the centripetal forces, when a body is revolved in a non-resisting space about an immoveable centre in any orbit, ... 103, 116
" the centripetal forces tending to any point by which any figure may be described by a revolving body being given, the centripetal forces tending to any other point by which the same figure may be described in the same periodic time are also given, . . . 112
" the centripetal forces by which any figure is described by a revolving body being given, there are given the forces by which a new figure may be described, if the ordinates are augmented or diminished in any given ratio, or the angle of their inclination be any how changed, the periodic time remaining the same, ... 116
" centripetal forces decreasing in the duplicate proportion of the distances, what figures may be described by them, ... 120 19

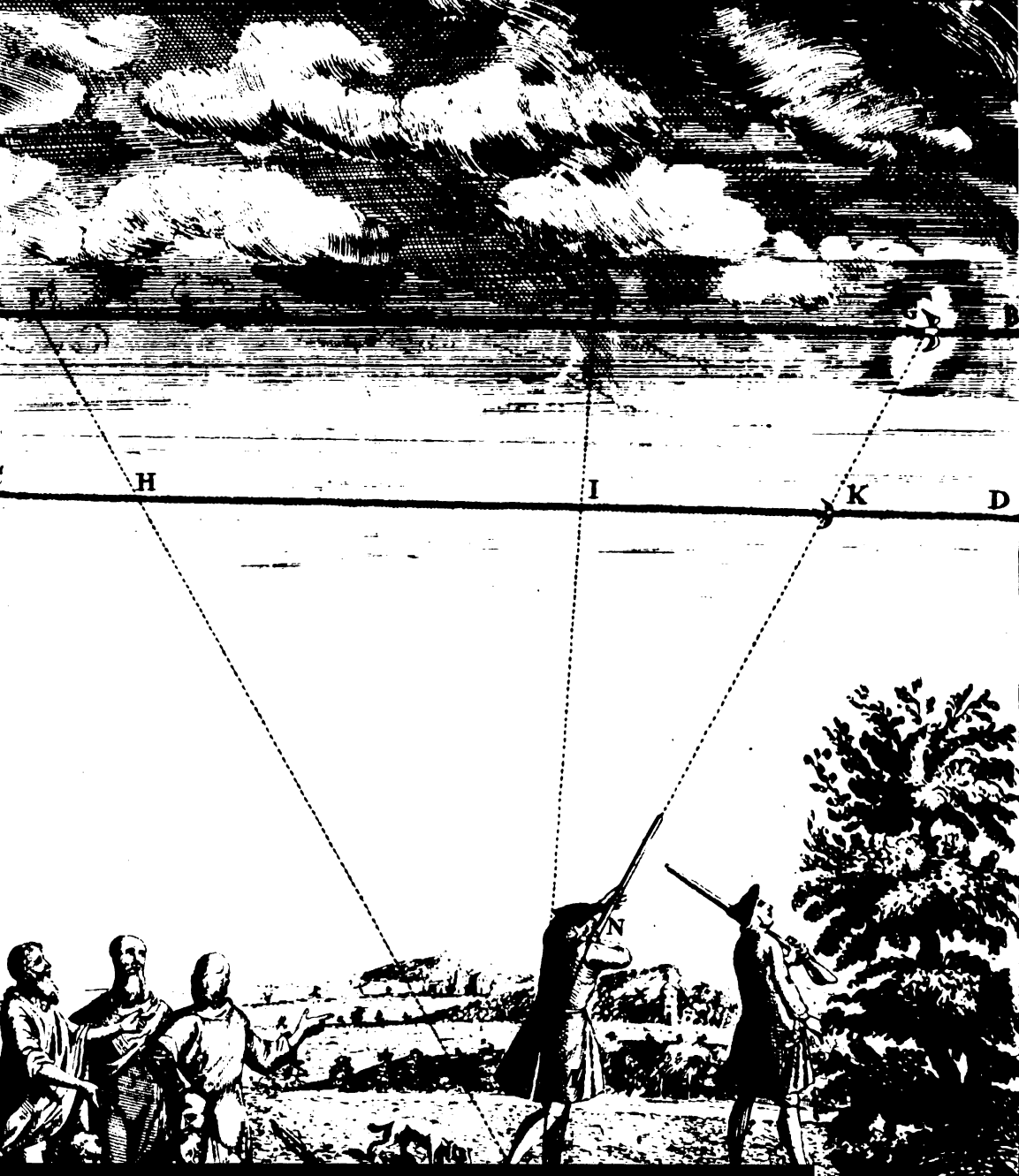
Force, centripetal force defined, ... 74
" the absolute quantity of centripetal force defined, ... 75
" the accelerative quantity of the same defined, ... 76
" the motive quantity of the same defined, ... 76
" the proportion thereof to any known force how collected, ... 109
" a centripetal force that is reciprocally as the cube of the ordinate tending to a vastly remote centre of force will cause a body to move in any given conic section, . . 114
" a centripetal force that is as the cube of the ordinate tending to a vastly remote centre of force will cause a body to move in an hyperbola, ... 243
" centrifugal force of bodies on the earth's equator, how great, ... 405
God, his nature, ... 506
Gravity mutual between the earth and its parts, ... 94
" of a different nature from magnetical force, ... 397
" the cause of it not assigned, ... 507
" tends towards all the planets, ... 393
" from the surfaces of the planets upwards decreases in the duplicate ratio of the distances from the centre, ... 400
" from the same downwards decreases nearly in the simple ratio of the same, . . 400
" tends towards all bodies, and is proportional to the quantity of matter in each, . 397
" is the force by which the moon is retained in its orbit, ... 391
" the same proved by an accurate calculus, ... 153
" is the force by which the primary planets and the satellites of Jupiter and Saturn are retained in their orbits, ... 393
Heat, an iron rod increases in length by heat, ... 112
" of the sun, how great at different distances from the sun, ... 186
" how great in Mercury, ... 400
" how great in the comet of 1680, when in its perihelion, ... 186
Heavens are void of any sensible resistance, 401, 445, 492; and, therefore, of almost any corporeal fluid whatever, ... 355, 356
" suffer light to pass through them without any refraction, ... 485
Hydrostatics, the principles thereof delivered, ... 293
Hyperbola, by what law of centrifugal force tending from the centre of the figure it is described by a revolving body, ... 116
" by what law of centrifugal force tending from the focus of the figure it is described by a revolving body, ... 117
" by what law of centripetal force tending to the focus of the figure it is described by a revolving body, ... 118
Hypotheses of what kind soever rejected from this philosophy, ... 508
Jupiter, its periodic time, ... 388
" its distance from the sun, ... 388
" its apparent diameter, ... 386
" its true diameter, ... 399
" its attractive force, how great, ... 398
" the weights of bodies on its surface, ... 399
" its density, ... 399
" its quantity of matter, ... 399
" its perturbation by Saturn, how much, ... 403
" the proportion of its diameters exhibited by computation, ... 409
" and compared with observations, ... 409
" its rotation about its axis, in what time performed, ... 409
" the cause of its belts hinted at, ... 445
Light, its propagation not instantaneous, ... 246
" its velocity different in different mediums, ... 245
" a certain reflection it sometimes suffers explained, ... 245
" its refraction explained, ... 243
" refraction is not made in the single point of incidence, ... 247
" an incurvation of light about the extremities of bodies observed by experiments, . . 246
" not caused by the agitation of any ethereal medium, ... 368
Magnetic force, ... 94, 304, 397, 454

MARS, its periodic time, ... 388
" its distance from the sun, ... 389
" the motion of its aphelion, ... 405
MATTER, its quantity of matter defined, ... 73
" its *vis insita* defined, ... 74
" its impressed force defined, ... 74
" its extension, hardness, impenetrability, mobility, *vis inertiæ*, gravity, how discovered, 385
" subtle matter of *Descartes* inquired into, ... 320
MECHANICAL POWERS explained and demonstrated, ... 94
MERCURY, its periodic time, ... 388
" its distance from the sun, ... 389
" the motion of its aphelion, ... 405
METHOD of first and last ratios, ... 95
" of transforming figures into others of the same analytical order, ... 141
" of fluxions, ... 261
" differential, ... 447
" of finding the quadratures of all curves very nearly true, ... 448
" of converging series applied to the solution of difficult problems, . . . 271, 436
MOON, the inclination of its orbit to the ecliptic greatest in the syzygies of the node with the sun, and least in the quadratures, ... 208
" the figure of its body collected by calculation, ... 454
" its librations explained, ... 405
" its mean apparent diameter, ... 453
" its true diameter, ... 453
" weight of bodies on its surface, ... 453
" its density, ... 453
" its quantity of matter, ... 453
" its mean distance from the earth, how many greatest semi-diameters of the earth contained therein, ... 453
" how many mean semi-diameters, ... 454
" its force to move the sea how great, ... 449
" not perceptible in experiments of pendulums, or any statical or hydrostatical observations, 452
" its periodic time, ... 454
" the time of its synodical revolution, ... 422
" its motions, and the inequalities of the same derived from their causes, . . 413, 444
" revolves more slowly, in a dilated orbit, when the earth is in its perihelion; and more swiftly in the aphelion the same, its orbit being contracted, ... 413, 444, 445
" revolves more slowly, in a dilated orbit, when the apogæon is in the syzygies with the sun; and more swiftly, in a contracted orbit, when the apogæon is in the quadratures, . 445
" revolves more slowly, in a dilated orbit, when the node is in the syzygies with the sun; and more swiftly, in a contracted orbit, when the node is in the quadratures, . . 446
" moves slower in its quadratures with the sun, swifter in the syzygies; and by a radius drawn to the earth describes an area, in the first case less in proportion to the time, in the last case greater, ... 413
" the inequality of those areas computed, ... 420
" its orbit is more curve, and goes farther from the earth in the first case; in the last case its orbit is less curve, and comes nearer to the earth, ... 415
" the figure of this orbit, and the proportion of its diameters collected by computation, . 423
" a method of finding the moon's distance from the earth by its horary motion, . . 423
" its apogæon moves more slowly when the earth is in its aphelion, more swiftly in the perihelion, ... 414, 445
" its apogæon goes forward most swiftly when in the syzygies with the sun; and goes backward in the quadratures, ... 414, 446
" its eccentricity greatest when the apogæon is in the syzygies with the sun; least when the same is in the quadratures, ... 414, 446
" its nodes move more slowly when the earth is in its aphelion, and more swiftly in the perihelion, ... 414, 445
" its nodes are at rest in their syzygies with the sun, and go back most swiftly in the quadratures, ... 414

Moon, the motions of the nodes and the inequalities of its motions computed from the theory of gravity, ... 427, 430, 434, 436
" the same from a different principle, ... 437
" the variations of the inclination computed from the theory of gravity, ... 441, 443
" the equations of the moon's motions for astronomical uses, ... 445
" the annual equation of the moon's mean motion, ... 445
" the first semi-annual equation of the same, ... 446
" the second semi-annual equation of the same, ... 447
" the first equation of the moon's centre, ... 447
" the second equation of the moon's centre, ... 448
Moon's first variation, ... 425
" the annual equation of the mean motion of its apogee, ... 445
" the semi-annual equation of the same, ... 447
" the semi-annual equation of its eccentricity, ... 447
" the annual equation of the mean motion of its nodes, ... 445
" the semi-annual equation of the same, ... 437
" the semi-annual equation of the inclination of the orbit to the ecliptic, ... 444
" the method of fixing the theory of the lunar motions from observations, ... 464
Motion, its quantity defined, ... 73
" absolute and relative, ... 78
" absolute and relative, the separation of one from the other possible, demonstrated by an example 82
" laws thereof, ... 83
" of concurring bodies after their reflection, by what experiments collected, ... 91
" of bodies in eccentric sections, ... 116
" in moveable orbits, ... 172
" in given superficies, and of the reciprocal motion of pendulums, ... 183
" of bodies tending to each other with centripetal forces, ... 194
" of very small bodies agitated by centripetal forces tending to each part of some very great body, ... 233
" of bodies resisted in the ratio of the velocities, ... 251
" in the duplicate ratio of the velocity, ... 258
" partly in the simple and partly in the duplicate ratio of the same, ... 280
" of bodies proceeding by their *vis insita* alone in resisting mediums, 251, 258, 259, 280, 281, 330
" of bodies ascending or descending in right lines in resisting mediums, and acted on by an uniform force of gravity, ... 252, 265, 281, 283
" of bodies projected in resisting mediums, and acted on by an uniform force of gravity, 255, 268
" of bodies revolving in resisting mediums, ... 287
" of funependulous bodies in resisting mediums, ... 304
" and resistance of fluids, ... 323
" propagated through fluids, ... 356
" of fluids after the manner of a vortex, or circular, ... 370
Motions, composition and resolution of them, ... 84
Ovals for optic uses, the method of finding them which Cartesius concealed, ... 246
" a general solution of Cartesius's problem, ... 247, 248
Orbits, the invention of those which are described by bodies going off from a given place with a given velocity according to a given right line, when the centripetal force is reciprocally as the square of the distance, and the absolute quantity of that force is known, . 123
" of those which are described by bodies when the centripetal force is reciprocally as the cube of the distance, ... 114, 171, 176
" of those which are described by bodies agitated by any centripetal forces whatever, 169
Parabola, by what law of centripetal force tending to the focus of the figure the same may be described, ... 120
Pendulums, their properties explained, ... 186, 190, 304
" the diverse lengths of isochronous pendulums in different latitudes compared among themselves, both by observations and by the theory of gravity, . . . 409 to 413
Place defined, and distinguished into absolute and relative, ... 78
Places of bodies moving in conic sections found to any assigned time, ... 153
Planets not carried about by corporeal vortices, ... 378

PLANETS, their periodic times, ... 386
" their distances from the sun, ... 389
" the aphelia and nodes of their orbits do almost rest, ... 405
" their orbits determined, ... 406
" the way of finding their places in their orbits, ... 347 to 350
" their density suited to the heat they receive from the sun, ... 400
" their diurnal revolutions equable, ... 406
" their axes less than the diameters that stand upon them at right angles, . . . 406
PLANETS, PRIMARY, surround the sun, ... 387
" " move in ellipses whose focus is in the sun's centre, ... 403
" " by radii drawn to the sun describe areas proportional to the times, . 388, 403
" " revolve in periodic times that are in the sesquiplicate proportion of the distances from the sun, ... 387
" " are retained in their orbits by a force of gravity which respects the sun, and is reciprocally as the square of the distance from the sun's centre, 389, 393
PLANETS, SECONDARY, move in ellipses having their focus in the centre of the primary, . 413
" " by radii drawn to their primary describe areas proportional to the times, ... 386, 387, 390
" " revolve in periodic times that are in the sesquiplicate proportion of their distances from the primary, ... 386, 387
PROBLEM KEPLERIAN, solved by the trochoid and by approximations, ... 157 to 160
" " of the ancients, of four lines, related by Pappus, and attempted by Cartesius, by an algebraic calculus solved by a geometrical composition, . 135
PROJECTILES move in parabolas when the resistance of the medium is taken away, 91, 115, 243, 273
" their motions in resisting mediums, ... 255, 268
PULSES of the air, by which sounds are propagated, their intervals or breadths determined, 368, 370
" these intervals in sounds made by open pipes probably equal to twice the length of the pipes, ... 370
QUADRATURES general of oval figures not to be obtained by finite terms, ... 153
QUALITIES of bodies how discovered, and when to be supposed universal, ... 384
RESISTANCE, the quantity thereof in mediums not continued, ... 329
" in continued mediums, ... 409
" in mediums of any kind whatever, ... 33.
" of mediums is as their density, *cæteris paribus*, . . 320, 321, 324, 329, 344, 355
" is in the duplicate proportion of the velocity of the bodies resisted, *cæteris paribus*, ... 258, 314, 324, 329, 344, 351
" is in the duplicate proportion of the diameters of spherical bodies resisted, *cæteris paribus*, ... 317, 318, 329, 344
" of fluids threefold, arises either from the inactivity of the fluid matter, or the tenacity of its parts, or friction, ... 286
" the resistance found in fluids, almost all of the first kind, ... 321, 354
" cannot be diminished by the subtilty of the parts of the fluid, if the density remain, 355
" of a globe, what proportion it bears to that of a cylinder, in mediums not continued, 327
" in compressed mediums, ... 343
" of a globe in mediums not continued, ... 329
" in compressed mediums, ... 344
" how found by experiments, ... 345 to 355
" to a frustum of a cone, how made the least possible, ... 328
" what kind of solid it is that meets with the least, ... 329
RESISTANCES, the theory thereof confirmed by experiments of pendulums, . . . 313 to 321
" by experiments of falling bodies, ... 345 to 356
REST, true and relative, ... 78
RULES of philosophy, ... 384
SATELLITES, the greatest heliocentric elongation of Jupiter's satellites, ... 387
" the greatest heliocentric elongation of the *Huygenian* satellite from Saturn's centre, 398
" the periodic times of Jupiter's satellites, and their distances from his centre, . 386, 387
" the periodic times of Saturn's satellites, and their distances from his centre, 387, 388
" the inequalities of the motions of the satellites of Jupiter and Saturn derived from the motions of the moon, ... 413
SESQUIPLICATE proportion defined, ... 101

SATURN, its periodic time, ... 388
" its distance from the sun, ... 388
" its apparent diameter, ... 388
" its true diameter, ... 399
" its attractive force, how great, ... 398
" the weight of bodies on its surface, ... 399
" its density, ... 399
" its quantity of matter, ... 399
" its perturbation by the approach of Jupiter how great, ... 403
" the apparent diameter of its ring, ... 388
SHADOW of the earth to be augmented in lunar eclipses, because of the refraction of the atmosphere, ... 447
SOUNDS, their nature explained, ... 360, 363, 365, 366, 367, 368, 369
" not propagated *in directum*, ... 359
" caused by the agitation of the air, ... 368
" their velocity computed, ... 368, 369
" somewhat swifter by the theory in summer than in winter, ... 370
" cease immediately, when the motion of the sonorous body ceases, ... 365
" how augmented in speaking trumpets, ... 370
SPACE, absolute and relative, ... 78, 79
" not equally full, ... 396
SPHEROID, the attraction of the same when the forces of its particles are reciprocally as the squares of the distances, ... 239
SPIRAL cutting all its radii in a given angle, by what law of centripetal force tending to the centre thereof it may be described by a revolving body, ... 107, 287, 291
SPIRIT pervading all bodies, and concealed within them, hinted at, as required to solve a great many phænomena of Nature, ... 508
STARS, the fixed stars demonstrated to be at rest, ... 404
" their twinkling what to be ascribed to, ... 487
" new stars, whence they may arise, ... 502
SUBSTANCES of all things unknown, ... 507
SUN, moves round the common centre of gravity of all the planets, ... 401
" the periodic time of its revolution about its axis, ... 405
" its mean apparent diameter, ... 453
" its true diameter, ... 398
" its horizontal parallax, ... 398
" has a menstrual parallax, ... 403
" its attractive force how great, ... 398
" the weight of bodies on its surface, ... 399
" its density, ... 399
" its quantity of matter, ... 399
" its force to disturb the motions of the moon, ... 391, 419
" its force to move the sea, ... 448
TIDES of the sea derived from their cause, ... 415, 448, 449
TIME, absolute and relative, ... 78, 79
" the astronomical equation thereof proved by pendulum clocks, and the eclipses of Jupiter's satellites, ... 79
A VACUUM proved, or that all spaces (if said to be full) are not equally full, ... 396
VELOCITIES of bodies moving in conic sections, where the centripetal force tends to the focus, . 121
VELOCITY, the greatest that a globe falling in a resisting medium can acquire, ... 344
VENUS, its periodic time, ... 388
" its distance from the sun, ... 388
" the motion of its aphelion, ... 405
VORTICES, their nature and constitution examined, ... 504
WAVES, the velocity with which they are propagated on the superficies of stagnant water, . 361
WEIGHTS of bodies towards the sun, the earth, or any planet, are, at equal distances from the centre, as the quantities of matter in the bodies, ... 394
" they do not depend upon the forms and textures of bodies, ... 395
" of bodies in different regions of the earth found out, and compared together, . . 409



The method of fluxions and infinite series

Isaac Newton

THE

METHOD of FLUXIONS

AND

INFINITE SERIES;

WITH ITS

Application to the Geometry of CURVE-LINES.

By the INVENTOR

Sir ISAAC NEWTON, *Kt.*

Late Preſident of the Royal Society.

Tranſlated from the AUTHOR's LATIN ORIGINAL
not yet made publick.

To which is ſubjoin'd,

A PERPETUAL COMMENT upon the whole Work,

Conſiſting of

ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,

In order to make this Treatiſe

A compleat Inſtitution for the uſe of LEARNERS.

By *JOHN COLSON*, M. A. and F. R. S.

Maſter of Sir *Joſeph Williamſon*'s free Mathematical-School at *Rocheſter*.

LONDON:

Printed by HENRY WOODFALL;

And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.

M.DCC.XXXVI.

TO

William Jones Eſq; F.R.S.

SIR,

IT was a laudable cuſtom among the ancient Geometers, and very worthy to be imitated by their Succeſſors, to addreſs their Mathematical labours, not ſo much to Men of eminent rank and ſtation in the world, as to Perſons of diſtinguiſh'd merit and proficience in the ſame Studies. For they knew very well, that ſuch only could be competent Judges of their Works, and would receive them with the eſteem they might deſerve. So far at leaſt I can copy after thoſe great Originals, as to chuſe a Patron for theſe Speculations, whoſe known skill and abilities in ſuch matters will enable him to judge, and whoſe known candor will incline him to judge favourably, of the ſhare I have had in the preſent performance. For as to the fundamental part of the Work, of which I am only the Interpreter, I know it cannot but pleaſe you; it will need no protection, nor can it receive a greater recommendation, than to bear the name of its illuſtrious Author. However, it very naturally applies itſelf to you, who had the honour (for I am ſure you think it ſo) of the Author's friendſhip and familiarity in his life-time; who had his own conſent to publiſh an elegant edition of ſome of his pieces, of a nature not very different from this; and who have ſo juſt an eſteem for, as well as knowledge of, his other moſt ſublime, moſt admirable, and juſtly celebrated Works.

But beſides theſe motives of a publick nature, I had others that more nearly concern myſelf. The many perſonal obligations I have received from you, and your generous manner of conferring them, require all the teſtimonies of gratitude in my power. Among the reſt, give me leave to mention one, (tho' it be a privilege I have enjoy'd in common with many others, who have the happineſs of your acquaintance,) which is, the free acceſs you have always allow'd me, to your copious Collection of whatever is choice and excellent in the Mathematicks. Your judgment and induſtry, in collecting thoſe valuable κειμήλια, are not more conſpicuous, than the freedom and readineſs with which you communicate them, to all ſuch who you know will apply them to their proper uſe, that is, to the general improvement of Science.

Before I take my leave, permit me, good Sir, to join my wiſhes to thoſe of the publick, that your own uſeful Lucubrations may ſee the light, with all convenient ſpeed; which, if I rightly conceive of them, will be an excellent methodical Introduction, not only to the mathematical Sciences in general, but alſo to theſe, as well as to the other curious and abſtruſe Speculations of our great Author. You are very well apprized, as all other good Judges muſt be, that to illuſtrate him is to cultivate real Science, and to make his Diſcoveries eaſy and familiar, will be no ſmall improvement in Mathematicks and Philoſophy.

That you will receive this addreſs with your uſual candor, and with that favour and friendſhip I have ſo long and often experienced, is the earneſt requeſt of,

SIR,

Your moſt obedient humble Servant,

J. COLSON.

THE PREFACE.

I Cannot but very much congratulate with my Mathematical Readers, and think it one of the moſt fortunate circumſtances of my Life, that I have it in my power to preſent the publick with a moſt valuable Anecdote, of the greateſt Maſter in Mathematical and Philoſophical Knowledge, that ever appear'd in the World. And ſo much the more, becauſe this Anecdote is of an elementary nature, preparatory and introductory to his other moſt arduous and ſublime Speculations, and intended by himſelf for the inſtruction of Novices and Learners. I therefore gladly embraced the opportunity that was put into my hands, of publiſhing this poſthumous Work, becauſe I found it had been compoſed with that view and deſign. And that my own Country-men might firſt enjoy the benefit of this publication, I reſolved upon giving it in an *Engliſh* Tranſlation, with ſome additional Remarks of my own. I thought it highly injurious to the memory and reputation of the great Author, as well as invidious to the glory of our own Nation, that ſo curious and uſeful a piece ſhould be any longer ſuppreſs'd, and confined to a few private hands, which ought to be communicated to all the learned World for general Inſtruction. And more eſpecially at a time when the Principles of the Method here taught have been ſcrupulouſly ſifted and examin'd, have been vigorouſly oppoſed and (we may ſay) ignominiouſly rejected as inſufficient, by ſome Mathematical Gentlemen, who ſeem not to have derived their knowledge of them from their only true Source, that is, from our Author's own Treatiſe wrote expreſsly to explain them. And on the other hand, the Principles of this Method have been zealouſly and commendably defended by other Mathematical Gentlemen, who yet

ſeem to have been as little acquainted with this Work, (or at leaſt to have over-look'd it,) the only genuine and original Fountain of this kind of knowledge. For what has been elſewhere deliver'd by our Author, concerning this Method, was only accidental and occaſional, and far from that copiouſneſs with which he treats of it here, and illuſtrates it with a great variety of choice Examples.

The learned and ingenious Dr. *Pemberton*, as he acquaints us in his View of Sir *Iſaac Newton*'s Philoſophy, had once a deſign of publiſhing this Work, with the conſent and under the inſpection of the Author himſelf; which if he had then accompliſh'd, he would certainly have deſerved and received the thanks of all lovers of Science. The Work would have then appear'd with a double advantage, as receiving the laſt Emendations of its great Author, and likewiſe in paſſing through the hands of ſo able an Editor. And among the other good effects of this publication, poſſibly it might have prevented all or a great part of thoſe Diſputes, which have ſince been raiſed, and which have been ſo ſtrenuouſly and warmly purſued on both ſides, concerning the validity of the Principles of this Method. They would doubtleſs have been placed in ſo good a light, as would have cleared them from any imputation of being in any wiſe defective, or not ſufficiently demonſtrated. But ſince the Author's Death, as the Doctor informs us, prevented the execution of that deſign, and ſince he has not thought fit to reſume it hitherto, it became needful that this publication ſhould be undertook by another, tho' a much inferior hand.

For it was now become highly neceſſary, that at laſt the great Sir *Iſaac* himſelf ſhould interpoſe, ſhould produce his genuine Method of Fluxions, and bring it to the teſt of all impartial and conſiderate Mathematicians; to ſhew its evidence and ſimplicity, to maintain and defend it in his own way, to convince his Opponents, and to teach his Diſciples and Followers upon what grounds they ſhould proceed in vindication of the Truth and Himſelf. And that this might be done the more eaſily and readily, I reſolved to accompany it with an ample Commentary, according to the beſt of my ſkill, and (I believe) according to the mind and intention of the Author, wherever I thought it needful; and particularly with an Eye to the fore-mention'd Controverſy. In which I have endeavour'd to obviate the difficulties that have been raiſed, and to explain every thing in ſo full a manner, as to remove all the objections of any force, that have been any where made, at leaſt ſuch as have occur'd to my obſervation. If what is here advanced, as there is good reaſon

ſon to hope, ſhall prove to the ſatisfaction of thoſe Gentlemen, who firſt ſtarted theſe objections, and who (I am willing to ſuppoſe) had only the cauſe of Truth at heart; I ſhall be very glad to have contributed any thing, towards the removing of their Scruples. But if it ſhall happen otherwiſe, and what is here offer'd ſhould not appear to be ſufficient evidence, conviction, and demonſtration to them; yet I am perſuaded it will be ſuch to moſt other thinking Readers, who ſhall apply themſelves to it with unprejudiced and impartial minds; and then I ſhall not think my labour ill beſtow'd. It ſhould however be well conſider'd by thoſe Gentlemen, that the great number of Examples they will find here, to which the Method of Fluxions is ſucceſsfully apply'd, are ſo many vouchers for the truth of the Principles, on which that Method is founded. For the Deductions are always conformable to what has been derived from other uncontroverted Principles, and therefore muſt be acknowledg'd as true. This argument ſhould have its due weight, even with ſuch as cannot, as well as with ſuch as will not, enter into the proof of the Principles themſelves. And the *hypotheſis* that has been advanced to evade this concluſion, of one error in reaſoning being ſtill corrected by another equal and contrary to it, and that ſo regularly, conſtantly, and frequently, as it muſt be ſuppos'd to do here; this *hypotheſis*, I ſay, ought not to be ſeriouſly refuted, becauſe I can hardly think it is ſeriouſly propoſed.

The chief Principle, upon which the Method of Fluxions is here built, is this very ſimple one, taken from the Rational Mechanicks; which is, That Mathematical Quantity, particularly Extenſion, may be conceived as generated by continued local Motion; and that all Quantities whatever, at leaſt by analogy and accommodation, may be conceived as generated after a like manner. Conſequently there muſt be comparative Velocities of increaſe and decreaſe, during ſuch generations, whoſe Relations are fixt and determinable, and may therefore (problematically) be propoſed to be found. This Problem our Author here ſolves by the help of another Principle, not leſs evident; which ſuppoſes that Quantity is infinitely diviſible, or that it may (mentally at leaſt) ſo far continually diminiſh, as at laſt, before it is totally extinguiſh'd, to arrive at Quantities that may be call'd vaniſhing Quantities, or which are infinitely little, and leſs than any aſſignable Quantity. Or it ſuppoſes that we may form a Notion, not indeed of abſolute, but of relative and comparative infinity. 'Tis a very juſt exception to the Method of Indiviſibles, as alſo to the foreign infiniteſimal Method, that they have recourſe at once to

infinitely little Quantities, and infinite orders and gradations of theſe, not relatively but abſolutely ſuch. They aſſume theſe Quantities *ſimul & ſemel*, without any ceremony, as Quantities that actually and obviouſly exiſt, and make Computations with them accordingly; the reſult of which muſt needs be as precarious, as the abſolute exiſtence of the Quantities they aſſume. And ſome late Geometricians have carry'd theſe Speculations, about real and abſolute Infinity, ſtill much farther, and have raiſed imaginary Syſtems of infinitely great and infinitely little Quantities, and their ſeveral orders and properties; which, to all ſober Inquirers into mathematical Truths, muſt certainly appear very notional and viſionary.

Theſe will be the inconveniencies that will ariſe, if we do not rightly diſtinguiſh between abſolute and relative Infinity. Abſolute Infinity, as ſuch, can hardly be the object either of our Conceptions or Calculations, but relative Infinity may, under a proper regulation. Our Author obſerves this diſtinction very ſtrictly, and introduces none but infinitely little Quantities that are relatively ſo; which he arrives at by beginning with finite Quantities, and proceeding by a gradual and neceſſary progreſs of diminution. His Computations always commence by finite and intelligible Quantities; and then at laſt he inquires what will be the reſult in certain circumſtances, when ſuch or ſuch Quantities are diminiſh'd *in infinitum*. This is a conſtant practice even in common Algebra and Geometry, and is no more than deſcending from a general Propoſition, to a particular Caſe which is certainly included in it. And from theſe eaſy Principles, managed with a vaſt deal of ſkill and ſagacity, he deduces his Method of Fluxions; which if we conſider only ſo far as he himſelf has carry'd it, together with the application he has made of it, either here or elſewhere, directly or indirectly, expreſly or tacitely, to the moſt curious Diſcoveries in Art and Nature, and to the ſublimeſt Theories: We may deſervedly eſteem it as the greateſt Work of Genius, and as the nobleſt Effort that ever was made by the Human Mind. Indeed it muſt be own'd, that many uſeful Improvements, and new Applications, have been ſince made by others, and probably will be ſtill made every day. For it is no mean excellence of this Method, that it is doubtleſs ſtill capable of a greater degree of perfection; and will always afford an inexhauſtible fund of curious matter, to reward the pains of the ingenious and induſtrious Analyſt.

As I am deſirous to make this as ſatisfactory as poſſible, eſpecially to the very learned and ingenious Author of the Diſcourſe call'd *The Analyſt*, whoſe eminent Talents I acknowledge myſelf to have a

great veneration for; I ſhall here endeavour to obviate ſome of his principal Objections to the Method of Fluxions, particularly ſuch as I have not touch'd upon in my Comment, which is ſoon to follow.

He thinks our Author has not proceeded in a demonſtrative and ſcientifical matter, in his *Princip. lib.* 2. *lem.* 2. where he deduces the Moment of a Rectangle, whoſe Sides are ſuppoſed to be variable Lines. I ſhall repreſent the matter Analytically thus, agreeably (I think) to the mind of the Author.

Let X and Y be two variable Lines, or Quantities, which at different periods of time acquire different values, by flowing or increaſing continually, either equably or alike inequably. For inſtance, let there be three periods of time, at which X becomes $A - \frac{1}{2}a$, A, and $A + \frac{1}{2}a$; and Y becomes $B - \frac{1}{2}b$, B, and $B + \frac{1}{2}b$ ſucceſſively and reſpectively; where A, a, B, b, are any quantities that may be aſſumed at pleaſure. Then at the ſame periods of time the variable Product or Rectangle XY will become $\overline{A - \frac{1}{2}a} \times \overline{B - \frac{1}{2}b}$, AB, and $\overline{A + \frac{1}{2}a} \times \overline{B + \frac{1}{2}b}$, that is, $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$, AB, and $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. Now in the interval from the firſt period of time to the ſecond, in which X from being $A - \frac{1}{2}a$ is become A, and in which Y from being $B - \frac{1}{2}b$ is become B, the Product XY from being $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$ becomes AB; that is, by Subtraction, its whole Increment during that interval is $\frac{1}{2}aB + \frac{1}{2}bA - \frac{1}{4}ab$. And in the interval from the ſecond period of time to the third, in which X from being A becomes $A + \frac{1}{2}a$, and in which Y from being B becomes $B + \frac{1}{2}b$, the Product XY from being AB becomes $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$; that is, by Subtraction, its whole Increment during that interval is $\frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. Add theſe two Increments together, and we ſhall have $aB + bA$ for the compleat Increment of the Product XY, during the whole interval of time, while X flow'd from the value $A - \frac{1}{2}a$ to $A + \frac{1}{2}a$, or Y flow'd from the value $B - \frac{1}{2}b$ to $B + \frac{1}{2}b$. Or it might have been found by one Operation, thus: While X flows from $A - \frac{1}{2}a$ to A, and thence to $A + \frac{1}{2}a$, or Y flows from $B - \frac{1}{2}b$ to B, and thence to $B + \frac{1}{2}b$, the Product XY will flow from $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$ to AB, and thence to $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$; therefore by Subtraction the whole Increment during that interval of time will be $aB + bA$. Q. E. D.

This may eaſily be illuſtrated by Numbers thus: Make A, a, B, b, equal to 9, 4, 15, 6, reſpectively; (or any other Numbers to be aſſumed at pleaſure.) Then the three ſucceſſive values of X will be 7, 9, 11, and the three ſucceſſive values of Y will be 12, 15, 18, reſpectively.

reſpectively. Alſo the three ſucceſſive values of the Product XY will be 84, 135, 198. But $aB + bA = 4 \times 15 + 6 \times 9 = 114 = 198 - 84$. Q. E. O.

Thus the Lemma will be true of any conceivable finite Increments whatever; and therefore by way of Corollary, it will be true of infinitely little Increments, which are call'd Moments, and which was the thing the Author principally intended here to demonſtrate. But in the caſe of Moments it is to be conſider'd, that X, or definitely $A - \frac{1}{2}a$, A, and $A + \frac{1}{2}a$, are to be taken indifferently for the ſame Quantity; as alſo Y, and definitely $B - \frac{1}{2}b$, B, $B + \frac{1}{2}b$. And the want of this Conſideration has occaſion'd not a few perplexities.

Now from hence the reſt of our Author's Concluſions, in the ſame Lemma, may be thus derived ſomething more explicitely. The Moment of the Rectangle AB being found to be $Ab + aB$, when the contemporary Moments of A and B are repreſented by a and b reſpectively; make $B = A$, and therefore $b = a$, and then the Moment of $A \times A$, or A^2, will be $Aa + aA$, or $2aA$. Again, make $B = A^2$, and therefore $b = 2aA$, and then the Moment of $A \times A^2$, or A^3, will be $2aA^2 + aA^2$, or $3aA^2$. Again, make $B = A^3$, and therefore $b = 3aA^2$, and then the Moment of $A \times A^3$, or A^4, will be $3aA^3 + aA^3$, or $4aA^3$. Again, make $B = A^4$, and therefore $b = 4aA^3$, and then the Moment of $A \times A^4$, or A^5, will be $4aA^4 + aA^4$, or $5aA^4$. And ſo on *in infinitum*. Therefore in general, aſſuming m to repreſent any integer affirmative Number, the Moment of A^m will be maA^{m-1}.

Now becauſe $A^m \times A^{-m} = 1$, (where m is any integer affirmative Number,) and becauſe the Moment of Unity, or any other conſtant quantity, is $= 0$; we ſhall have $A^m \times$ Mom. $A^{-m} + A^{-m} \times$ Mom. $A^m = 0$, or Mom. $A^{-m} = -A^{-2m} \times$ Mom. A^m. But Mom. $A^m = maA^{m-1}$, as found before; therefore Mom. $A^{-m} = -A^{-2m} \times maA^{m-1} = -maA^{-m-1}$. Therefore the Moment of A^m will be maA^{m-1}, when m is any integer Number, whether affirmative or negative.

And univerſally, if we put $A^{\frac{m}{n}} = B$, or $A^m = B^n$, where m and n may be any integer Numbers, affirmative or negative; then we ſhall have $maA^{m-1} = nbB^{n-1}$, or $b = \frac{maA^{m-1}}{nA^{\frac{mn-n}{n}}} = \frac{m}{n}aA^{\frac{m}{n}-1}$, which is the Moment of B, or of $A^{\frac{m}{n}}$. So that the Moment of A^m will be

be ſtill maA^{m-1}, whether m be affirmative or negative, integer or fraction.

The Moment of AB being $bA + aB$, and the Moment of CD being $dC + cD$; ſuppoſe $D = AB$, and therefore $d = bA + aB$, and then by Subſtitution the Moment of ABC will be $\overline{bA + aB} \times C + cAB = bAC + aBC + cAB$. And likewiſe the Moment of $A^n B^n$ will be $nbB^{n-1}A^n + maA^{m-1}B^n$. And ſo of any others.

Now there is ſo near a connexion between the Method of Moments and the Method of Fluxions, that it will be very eaſy to paſs from the one to the other. For the Fluxions or Velocities of increaſe, are always proportional to the contemporary Moments. Thus if for A, B, C, &c. we write x, y, z, &c. for a, b, c, &c. we may write $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. Then the Fluxion of xy will be $\dot{x}y + x\dot{y}$, the Fluxion of x^m will be $m\dot{x}x^{m-1}$, whether m be integer or fraction, affirmative or negative; the Fluxion of xyz will be $\dot{x}yz + x\dot{y}z + xy\dot{z}$, and the Fluxion of $x^m y^n$ will be $m\dot{x}x^{m-1}y^n + nx^m\dot{y}y^{n-1}$. And ſo of the reſt.

Or the former Inquiry may be placed in another view, thus: Let A and $A + a$ be two ſucceſſive values of the variable Quantity X, as alſo B and $B + b$ be two ſucceſſive and contemporary values of Y; then will AB and $AB + aB + bA + ab$ be two ſucceſſive and contemporary values of the variable Product XY. And while X, by increaſing perpetually, flows from its value A to $A + a$, or Y flows from B to $B + b$; XY at the ſame time will flow from AB to $AB + aB + bA + ab$, during which time its whole Increment, as appears by Subtraction, will become $aB + bA + ab$. Or in Numbers thus: Let A, a, B, b, be equal to 7, 4, 12, 6, reſpectively; then will the two ſucceſſive values of X be 7, 11, and the two ſucceſſive values of Y will be 12, 18. Alſo the two ſucceſſive values of the Product XY will be 84, 198. But the Increment $aB + bA + ab = 48 + 42 + 24 = 114 = 198 - 84$, as before.

And thus it will be as to all finite Increments: But when the Increments become Moments, that is, when a and b are ſo far diminiſh'd, as to become infinitely leſs than A and B; at the ſame time ab will become infinitely leſs than either aB or bA, (for aB. ab :: B. b, and bA. ab :: A. a,) and therefore it will vaniſh in reſpect of them. In which caſe the Moment of the Product or Rectangle will be $aB + bA$, as before. This perhaps is the more obvious and direct way of proceeding, in the preſent Inquiry; but, as there was room for choice, our Author thought fit to chuſe the former way,

as

as the more elegant, and in which he was under no neceſſity of having recourſe to that Principle, that quantities ariſing in an Equation, which are infinitely leſs than the others, may be neglected or expunged in compariſon of thoſe others. Now to avoid the uſe of this Principle, tho' otherwiſe a true one, was all the Artifice uſed on this occaſion, which certainly was a very fair and juſtifiable one.

I ſhall conclude my Obſervations with conſidering and obviating the Objections that have been made, to the uſual Method of finding the Increment, Moment, or Fluxion of any indefinite power x^n of the variable quantity x, by giving that Inveſtigation in ſuch a manner, as to leave (I think) no room for any juſt exceptions to it. And the rather becauſe this is a leading point, and has been ſtrangely perverted and miſrepreſented.

In order to find the Increment of the variable quantity or power x^n, (or rather its relation to the Increment of x, conſider'd as given; becauſe Increments and Moments can be known only by compariſon with other Increments and Moments, as alſo Fluxions by compariſon with other Fluxions;) let us make $x^n = y$, and let X and Y be any ſynchronous Augments of x and y. Then by the hypotheſis we ſhall have the Equation $\overline{x + X}|^n = y + Y$; for in any Equation the variable Quantities may always be increaſed by their ſynchronous Augments, and yet the Equation will ſtill hold good. Then by our Author's famous Binomial Theorem we ſhall have $y + Y = x^n + nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}X^3$, &c. or removing the equal Quantities y and x^n, it will be $Y = nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}X^3$, &c. So that when X denotes the given Increment of the variable quantity x, Y will here denote the ſynchronous Increment of the indefinite power y or x^n; whoſe value therefore, in all caſes, may be had from this Series. Now that we may be ſure we proceed regularly, we will verify this thus far, by a particular and familiar inſtance or two. Suppoſe $n = 2$, then $Y = 2xX + X^2$. That is, while x flows or increaſes to $x + X$, x^2 in the ſame time, by its Increment $Y = 2xX + X^2$, will increaſe to $x^2 + 2xX + X^2$, which we otherwiſe know to be true. Again, ſuppoſe $n = 3$, then $Y = 3x^2X + 3xX^2 + X^3$. Or while x increaſes to $x + X$, x^3 by its Increment $Y = 3x^2X + 3xX^2 + X^3$ will increaſe to $x^3 + 3x^2X + 3xX^2 + X^3$. And ſo in all other particular caſes, whereby we may plainly perceive, that this general Concluſion muſt be certain and indubitable.

This

This Series therefore will be always true, let the Augments X and Y be ever ſo great, or ever ſo little; for the truth does not at all depend on the circumſtance of their magnitude. Nay, when they are infinitely little, or when they become Moments, it muſt be true alſo, by virtue of the general Concluſion. But when X and Y are diminiſh'd *in infinitum*, ſo as to become at laſt infinitely little, the greater powers of X muſt needs vaniſh firſt, as being relatively of an infinitely leſs value than the ſmaller powers. So that when they are all expunged, we ſhall neceſſarily obtain the Equation $Y = nx^{n-1}X$; where the remaining Terms are likewiſe infinitely little, and conſequently would vaniſh, if there were other Terms in the Equation, which were (relatively) infinitely greater than themſelves. But as there are not, we may ſecurely retain this Equation, as having an undoubted right ſo to do; and eſpecially as it gives us an uſeful piece of information, that X and Y, tho' themſelves infinitely little, or vaniſhing quantities, yet they vaniſh in proportion to each other as 1 to nx^{n-1}. We have therefore learn'd at laſt, that the Moment by which x increaſes, or X, is to the contemporary Moment by which x^n increaſes, or Y, as 1 is to nx^{n-1}. And their Fluxions, or Velocities of increaſe, being in the ſame proportion as their ſynchronous Moments, we ſhall have $nx^{n-1}\dot{x}$ for the Fluxion of x^n, when the Fluxion of x is denoted by $\dot{x}$.

I cannot conceive there can be any pretence to inſinuate here, that any unfair artifices, any leger-de-main tricks, or any ſhifting of the hypotheſis, that have been ſo ſeverely complain'd of, are at all made uſe of in this Inveſtigation. We have legitimately derived this general Concluſion in finite Quantities, that in all caſes the relation of the Increments will be $Y = nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2$, &c. of which one particular caſe is, when X and Y are ſuppoſed continually to decreaſe, till they finally terminate in nothing. But by thus continually decreaſing, they approach nearer and nearer to the Ratio of 1 to nx^{n-1}, which they attain to at the very inſtant of their vaniſhing, and not before. This therefore is their ultimate Ratio, the Ratio of their Moments, Fluxions, or Velocities, by which x and x^n continually increaſe or decreaſe. Now to argue from a general Theorem to a particular caſe contain'd under it, is certainly one of the moſt legitimate and logical, as well as one of the moſt uſual and uſeful ways of arguing, in the whole compaſs of the Mathematicks. To object here, that after we have made X and Y to ſtand for ſome quantity, we are not at liberty to make them nothing, or no quantity, or vaniſhing quantities, is not an Objection againſt the

Method of Fluxions, but againſt the common Analyticks. This Method only adopts this way of arguing, as a conſtant practice in the vulgar Algebra, and refers us thither for the proof of it. If we have an Equation any how compos'd of the general Numbers *a*, *b*, *c*, &c. it has always been taught, that we may interpret theſe by any particular Numbers at pleaſure, or even by 0, provided that the Equation, or the Conditions of the Queſtion, do not expreſsly require the contrary. For general Numbers, as ſuch, may ſtand for any definite Numbers in the whole Numerical Scale; which Scale (I think) may be thus commodiouſly repreſented, &c. — 3, — 2, — 1, 0, 1, 2, 3, 4, &c. where all poſſible fractional Numbers, intermediate to theſe here expreſs'd, are to be conceived as interpolated. But in this Scale the Term 0 is as much a Term or Number as any other, and has its analogous properties in common with the reſt. We are likewiſe told, that we may not give ſuch values to general Symbols afterwards, as they could not receive at firſt; which if admitted is, I think, nothing to the preſent purpoſe. It is always moſt eaſy and natural, as well as moſt regular, inſtructive, and elegant, to make our Inquiries as much in general Terms as may be, and to deſcend to particular caſes by degrees, when the Problem is nearly brought to a concluſion. But this is a point of convenience only, and not a point of neceſſity. Thus in the preſent caſe, inſtead of deſcending from finite Increments to infinitely little Moments, or vaniſhing Quantities, we might begin our Computation with thoſe Moments themſelves, and yet we ſhould arrive at the ſame Concluſions. As a proof of which we may conſult our Author's own Demonſtration of his Method, in pag. 24. of this Treatiſe. In ſhort, to require this is juſt the ſame thing as to inſiſt, that a Problem, which naturally belongs to Algebra, ſhould be ſolved by common Arithmetick; which tho' poſſible to be done, by purſuing backwards all the ſteps of the general proceſs, yet would be very troubleſome and operoſe, and not ſo inſtructive, or according to the true Rules of Art.

But I am apt to ſuſpect, that all our doubts and ſcruples about Mathematical Inferences and Argumentations, eſpecially when we are ſatisfied that they have been juſtly and legitimately conducted, may be ultimately reſolved into a ſpecies of infidelity and diſtruſt. Not in reſpect of any implicite faith we ought to repoſe on meer human authority, tho' ever ſo great, (for that, in Mathematicks, we ſhould utterly diſclaim,) but in reſpect of the Science itſelf. We are hardly brought to believe, that the Science is ſo perfectly regular and uniform,

form, ſo infinitely conſiſtent, conſtant, and accurate, as we ſhall really find it to be, when after long experience and reflexion we ſhall have overcome this prejudice, and ſhall learn to purſue it rightly. We do not readily admit, or eaſily comprehend, that Quantities have an infinite number of curious and ſubtile properties, ſome near and obvious, others remote and abſtruſe, which are all link'd together by a neceſſary connexion, or by a perpetual chain, and are then only diſcoverable when regularly and cloſely purſued; and require our truſt and confidence in the Science, as well as our induſtry, application, and obſtinate perſeverance, our ſagacity and penetration, in order to their being brought into full light. That Nature is ever conſiſtent with herſelf, and never proceeds in theſe Speculations *per ſaltum*, or at random, but is infinitely ſcrupulous and ſolicitous, as we may ſay, in adhering to Rule and Analogy. That whenever we make any regular Poſitions, and purſue them through ever ſo great a variety of Operations, according to the ſtrict Rules of Art; we ſhall always proceed through a ſeries of regular and well-connected tranſmutations, (if we would but attend to 'em,) till at laſt we arrive at regular and juſt Concluſions. That no properties of Quantity are intirely deſtructible, or are totally loſt and aboliſh'd, even tho' proſecuted to infinity itſelf; for if we ſuppoſe ſome Quantities to become infinitely great, or infinitely little, or nothing, or leſs than nothing, yet other Quantities that have a certain relation to them will only undergo proportional, and often finite alterations, will ſympathize with them, and conform to 'em in all their changes; and will always preſerve their analogical nature, form, or magnitude, which will be faithfully exhibited and diſcover'd by the reſult. This we may collect from a great variety of Mathematical Speculations, and more particularly when we adapt Geometry to Analyticks, and Curve-lines to Algebraical Equations. That when we purſue general Inquiries, Nature is infinitely prolifick in particulars that will reſult from them, whether in a direct ſubordination, or whether they branch out collaterally; or even in particular Problems, we may often perceive that theſe are only certain caſes of ſomething more general, and may afford good hints and aſſiſtances to a ſagacious Analyſt, for aſcending gradually to higher and higher Diſquiſitions, which may be proſecuted more univerſally than was at firſt expected or intended. Theſe are ſome of thoſe Mathematical Principles, of a higher order, which we find a difficulty to admit, and which we ſhall never be fully convinced of, or know the whole uſe of, but from much practice and attentive conſideration; but more eſpecially by a diligent

perusal, and close examination, of this and the other Works of our illustrious Author. He abounded in these sublime views and inquiries, had acquired an accurate and habitual knowledge of all these, and of many more general Laws, or Mathematical Principles of a superior kind, which may not improperly be call'd *The Philosophy of Quantity*; and which, assisted by his great Genius and Sagacity, together with his great natural application, enabled him to become so compleat a Master in the higher Geometry, and particularly in the Art of Invention. This Art, which he possest in the greatest perfection imaginable, is indeed the sublimest, as well as the most difficult of all Arts, if it properly may be call'd such; as not being reducible to any certain Rules, nor can be deliver'd by any Precepts, but is wholly owing to a happy sagacity, or rather to a kind of divine Enthusiasm. To improve Inventions already made, to carry them on, when begun, to farther perfection, is certainly a very useful and excellent Talent; but however is far inferior to the Art of Discovery, as having a πῶ ϛῶ, or certain *data* to proceed upon, and where just method, close reasoning, strict attention, and the Rules of Analogy, may do very much. But to strike out new lights, to adventure where no footsteps had ever been set before, *nullius antè trita solo*; this is the noblest Endowment that a human Mind is capable of, is reserved for the chosen few *quos Jupiter æquus amavit*, and was the peculiar and distinguishing Character of our great Mathematical Philosopher. He had acquired a compleat knowledge of the Philosophy of Quantity, or of its most essential and most general Laws; had consider'd it in all views, had pursued it through all its disguises, and had traced it through all its Labyrinths and Recesses; in a word, it may be said of him not improperly, that he tortured and tormented Quantities all possible ways, to make them confess their Secrets, and discover their Properties.

The Method of Fluxions, as it is here deliver'd in this Treatise, is a very pregnant and remarkable instance of all these particulars. To take a cursory view of which, we may conveniently enough divide it into these three parts. The first will be the Introduction, or the Method of infinite Series. The second is the Method of Fluxions, properly so call'd. The third is the application of both these Methods to some very general and curious Speculations, chiefly in the Geometry of Curve-lines.

As to the first, which is the Method of infinite Series, in this the Author opens a new kind of Arithmetick, (new at least at the time of his writing this,) or rather he vastly improves the old. For he

he extends the received Notation, making it compleatly univerſal, and ſhews, that as our common Arithmetick of Integers received a great Improvement by the introduction of decimal Fractions; ſo the common Algebra or Analyticks, as an univerſal Arithmetick, will receive a like Improvement by the admiſſion of his Doctrine of infinite Series, by which the ſame analogy will be ſtill carry'd on, and farther advanced towards perfection. Then he ſhews how all complicate Algebraical Expreſſions may be reduced to ſuch Series, as will continually converge to the true values of thoſe complex quantities, or their Roots, and may therefore be uſed in their ſtead: whether thoſe quantities are Fractions having multinomial Denominators, which are therefore to be reſolved into ſimple Terms by a perpetual Diviſion; or whether they are Roots of pure Powers, or of affected Equations, which are therefore to be reſolved by a perpetual Extraction. And by the way, he teaches us a very general and commodious Method for extracting the Roots of affected Equations in Numbers. And this is chiefly the ſubſtance of his Method of infinite Series.

The Method of Fluxions comes next to be deliver'd, which indeed is principally intended, and to which the other is only preparatory and ſubſervient. Here the Author diſplays his whole ſkill, and ſhews the great extent of his Genius. The chief difficulties of this he reduces to the Solution of two Problems, belonging to the abſtract or Rational Mechanicks. For the direct Method of Fluxions, as it is now call'd, amounts to this Mechanical Problem, *The length of the Space deſcribed being continually given, to find the Velocity of the Motion at any time propoſed.* Alſo the inverſe Method of Fluxions has, for a foundation, the Reverſe of this Problem, which is, *The Velocity of the Motion being continually given, to find the Space deſcribed at any time propoſed.* So that upon the compleat Analytical or Geometrical Solution of theſe two Problems, in all their varieties, he builds his whole Method.

His firſt Problem, which is, *The relation of the flowing Quantities being given, to determine the relation of their Fluxions,* he diſpatches very generally. He does not propoſe this, as is uſually done, *A flowing Quantity being given, to find its Fluxion*; for this gives us too lax and vague an Idea of the thing, and does not ſufficiently ſhew that Compariſon, which is here always to be underſtood. Fluents and Fluxions are things of a relative nature, and ſuppoſe two at leaſt, whoſe relation or relations ſhould always be expreſs'd by Equations. He requires therefore that all ſhould be reduced to Equations, by which the relation of the flowing Quantities will be exhibited, and their comparative

comparative magnitudes will be more eaſily eſtimated; as alſo the comparative magnitudes of their Fluxions. And beſides, by this means he has an opportunity of reſolving the Problem much more generally than is commonly done. For in the uſual way of taking Fluxions, we are confined to the Indices of the Powers, which are to be made Coefficients; whereas the Problem in its full extent will allow us to take any Arithmetical Progreſſions whatever. By this means we may have an infinite variety of Solutions, which tho' different in form, will yet all agree in the main; and we may always chuſe the ſimpleſt, or that which will beſt ſerve the preſent purpoſe. He ſhews alſo how the given Equation may comprehend ſeveral variable Quantities, and by that means the Fluxional Equation may be found, notwithſtanding any ſurd quantities that may occur, or even any other quantities that are irreducible, or Geometrically irrational. And all this is derived and demonſtrated from the properties of Moments. He does not here proceed to ſecond, or higher Orders of Fluxions, for a reaſon which will be aſſign'd in another place.

His next Problem is, *An Equation being propoſed exhibiting the relation of the Fluxions of Quantities, to find the relation of thoſe Quantities, or Fluents, to one another*; which is the direct Converſe of the foregoing Problem. This indeed is an operoſe and difficult Problem, taking it in its full extent, and requires all our Author's ſkill and addreſs; which yet he ſolves very generally, chiefly by the aſſiſtance of his Method of infinite Series. He firſt teaches how we may return from the Fluxional Equation given, to its correſponding finite Fluential or Algebraical Equation, when that can be done. But when it cannot be done, or when there is no ſuch finite Algebraical Equation, as is moſt commonly the caſe, yet however he finds the Root of that Equation by an infinite converging Series, which anſwers the ſame purpoſe. And often he ſhews how to find the Root, or Fluent required, by an infinite number of ſuch Series. His proceſſes for extracting theſe Roots are peculiar to himſelf, and always contrived with much ſubtilty and ingenuity.

The reſt of his Problems are an application or an exemplification of the foregoing. As when he determines the *Maxima* and *Minima* of quantities in all caſes. When he ſhews the Method of drawing Tangents to Curves, whether Geometrical or Mechanical; or however the nature of the Curve may be defined, or refer'd to right Lines or other Curves. Then he ſhews how to find the Center or Radius of Curvature, of any Curve whatever, and that in a ſimple but general manner; which he illuſtrates by many curious Examples,

and

and purſues many other ingenious Problems, that offer themſelves by the way. After which he diſcuſſes another very ſubtile and intirely new Problem about Curves, which is, to determine the quality of the Curvity of any Curve, or how its Curvature varies in its progreſs through the different parts, in reſpect of equability or inequability.

He then applies himſelf to conſider the Areas of Curves, and ſhews us how we may find as many Quadrable Curves as we pleaſe, or ſuch whoſe Areas may be compared with thoſe of right-lined Figures. Then he teaches us to find as many Curves as we pleaſe, whoſe Areas may be compared with that of the Circle, or of the Hyperbola, or of any other Curve that ſhall be aſſign'd; which he extends to Mechanical as well as Geometrical Curves. He then determines the Area in general of any Curve that may be propoſed, chiefly by the help of infinite Series; and gives many uſeful Rules for aſcertaining the Limits of ſuch Areas. And by the way he ſquares the Circle and Hyperbola, and applies the Quadrature of this to the conſtructing of a Canon of Logarithms. But chiefly he collects very general and uſeful Tables of Quadratures, for readily finding the Areas of Curves, or for comparing them with the Areas of the Conic Sections; which Tables are the ſame as thoſe he has publiſh'd himſelf, in his Treatiſe of Quadratures. The uſe and application of theſe he ſhews in an ample manner, and derives from them many curious Geometrical Conſtructions, with their Demonſtrations.

Laſtly, he applies himſelf to the Rectification of Curves, and ſhews us how we may find as many Curves as we pleaſe, whoſe Curve-lines are capable of Rectification; or whoſe Curve-lines, as to length, may be compared with the Curve-lines of any Curves, that ſhall be aſſign'd. And concludes in general, with rectifying any Curve-lines that may be propoſed, either by the aſſiſtance of his Tables of Quadratures, when that can be done, or however by infinite Series. And this is chiefly the ſubſtance of the preſent Work. As to the account that perhaps may be expected, of what I have added in my Annotations; I ſhall refer the inquiſitive Reader to the Preface, which will go before that part of the Work.

THE

CONTENTS.

THE Introduction, or the Method of reſolving complex Quantities into infinite Series of ſimple Terms. —— pag. 1

Prob. 1. *From the given Fluents to find the Fluxions.* —— p. 21

Prob. 2. *From the given Fluxions to find the Fluents.* —— p. 25

Prob. 3. *To determine the* Maxima *and* Minima *of Quantities.* p. 44

Prob. 4. *To draw Tangents to Curves.* —— p. 46

Prob. 5. *To find the Quantity of Curvature in any Curve.* p. 59

Prob. 6. *To find the Quality of Curvature in any Curve.* p. 75

Prob. 7. *To find any number of Quadrable Curves.* p. 80

Prob. 8. *To find Curves whoſe Areas may be compared to thoſe of the Conic Sections.* —— p. 81

Prob. 9. *To find the Quadrature of any Curve aſſign'd.* p. 86

Prob. 10. *To find any number of rectifiable Curves.* p. 124

Prob. 11. *To find Curves whoſe Lines may be compared with any Curve-lines aſſign'd.* —— p. 129

Prob. 12. *To rectify any Curve-lines aſſign'd.* —— p. 134

THE METHOD of FLUXIONS, AND INFINITE SERIES.

INTRODUCTION: *Or, the Reſolution of Equations by Infinite Series.*

1. HAVING obſerved that moſt of our modern Geometricians, neglecting the Synthetical Method of the Ancients; have apply'd themſelves chiefly to the cultivating of the Analytical Art; by the aſſiſtance of which they have been able to overcome ſo many and ſo great difficulties, that they ſeem to have exhauſted all the Speculations of Geometry, excepting the Quadrature of Curves, and ſome other matters of a like nature, not yet intirely diſcuſs'd: I thought it not amiſs, for the ſake of young Students in this Science, to compoſe the following Treatiſe, in which I have endeavour'd to enlarge the Boundaries of Analyticks, and to improve the Doctrine of Curve-lines.

2. Since there is a great conformity between the Operations in Species, and the ſame Operations in common Numbers; nor do they ſeem to differ, except in the Characters by which they are re-

preſented, the firſt being general and indefinite, and the other definite and particular: I cannot but wonder that no body has thought of accommodating the lately-diſcover'd Doctrine of Decimal Fractions in like manner to Species, (unleſs you will except the Quadrature of the Hyberbola by Mr. *Nicolas Mercator*;) eſpecially ſince it might have open'd a way to more abſtruſe Diſcoveries. But ſince this Doctrine of Species, has the ſame relation to Algebra, as the Doctrine of Decimal Numbers has to common Arithmetick; the Operations of Addition, Subtraction, Multiplication, Diviſion, and Extraction of Roots, may eaſily be learned from thence, if the Learner be but ſkill'd in Decimal Arithmetick, and the Vulgar Algebra, and obſerves the correſpondence that obtains between Decimal Fractions and Algebraick Terms infinitely continued. For as in Numbers, the Places towards the right-hand continually decreaſe in a Decimal or Subdecuple Proportion; ſo it is in Species reſpectively, when the Terms are diſpoſed, (as is often enjoin'd in what follows,) in an uniform Progreſſion infinitely continued, according to the Order of the Dimenſions of any Numerator or Denominator. And as the convenience of Decimals is this, that all vulgar Fractions and Radicals, being reduced to them, in ſome meaſure acquire the nature of Integers, and may be managed as ſuch; ſo it is a convenience attending infinite Series in Species, that all kinds of complicate Terms, (ſuch as Fractions whoſe Denominators are compound Quantities, the Roots of compound Quantities, or of affected Equations, and the like,) may be reduced to the Claſs of ſimple Quantities; that is, to an infinite Series of Fractions, whoſe Numerators and Denominators are ſimple Terms; which will no longer labour under thoſe difficulties, that in the other form ſeem'd almoſt inſuperable. Firſt therefore I ſhall ſhew how theſe Reductions are to be perform'd, or how any compound Quantities may be reduced to ſuch ſimple Terms, eſpecially when the Methods of computing are not obvious. Then I ſhall apply this Analyſis to the Solution of Problems.

3. Reduction by Diviſion and Extraction of Roots will be plain from the following Examples, when you compare like Methods of Operation in Decimal and in Specious Arithmetick.

Examples

Examples of Reduction by Diviſion.

4. The Fraction $\frac{aa}{b+x}$ being propoſed, divide aa by $b+x$ in the following manner:

$$b+x)\ aa+o\ \left(\frac{aa}{b}-\frac{aax}{b^2}+\frac{aax^2}{b^3}-\frac{aax^3}{b^4}+\frac{aax^4}{b^5},\ \&c.\right.$$

$$\begin{array}{l}
aa+\frac{aax}{b} \\
\hline
o-\frac{aax}{b}+o \\
\quad -\frac{aax}{b}-\frac{aax^2}{b^2} \\
\hline
\quad o+\frac{a^2x^2}{b^2}+o \\
\quad\quad +\frac{a^2x^2}{b^2}+\frac{a^2x^3}{b^3} \\
\hline
\quad\quad o-\frac{a^2x^3}{b^3}+o \\
\quad\quad\quad -\frac{a^2x^3}{b^3}-\frac{a^2x^4}{b^4} \\
\hline
\quad\quad\quad o+\frac{a^2x^4}{b^4}\ \&c.
\end{array}$$

The Quotient therefore is $\frac{aa}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{a^2x^3}{b^4}+\frac{a^2x^4}{b^5}$, &c. which Series, being infinitely continued, will be equivalent to $\frac{aa}{b+x}$. Or making x the firſt Term of the Diviſor, in this manner, $x+b)\ aa+o$ (the Quotient will be $\frac{aa}{x}-\frac{aab}{x^2}+\frac{aab^2}{x^3}-\frac{a^2b^3}{x^4}$, &c. found as by the foregoing Proceſs.

5. In like manner the Fraction $\frac{1}{1+xx}$ will be reduced to $1-x^2+x^4-x^6+x^8$, &c. or to $x^{-2}-x^{-4}+x^{-6}-x^{-8}$, &c.

6. And the Fraction $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}$ will be reduced to $2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^2+34x^{\frac{5}{2}}$, &c.

7. Here it will be proper to obſerve, that I make uſe of x^{-1}, x^{-2}, x^{-3}, x^{-4}, &c. for $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$, $\frac{1}{x^4}$ &c. of $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{1}{4}}$, $x^{\frac{2}{3}}$, $x^{\frac{3}{4}}$, &c. for $\sqrt{x}$, $\sqrt{x^3}$, $\sqrt{x^5}$, $\sqrt[3]{x}$, $\sqrt[3]{x^2}$, &c. and of $x^{-\frac{1}{2}}$, $x^{-\frac{2}{3}}$, $x^{-\frac{1}{4}}$, &c. for $\frac{1}{\sqrt{x}}$, $\frac{1}{\sqrt[3]{x^2}}$, $\frac{1}{\sqrt[4]{x}}$ &c. And this by the Rule of Analogy, as may be apprehended from ſuch Geometrical Progreſſions as theſe; x^1, $x^{\frac{1}{2}}$, x^0, $x^{\frac{1}{2}}$, x, $x^{\frac{3}{2}}$, x^0 (or 1,) $x^{-\frac{1}{2}}$, x^{-1}, $x^{-\frac{3}{2}}$, x^{-2}, &c.

8. In the ſame manner for $\frac{aa}{x} - \frac{a^2b}{x^2} + \frac{a^2b^2}{x^3}$, &c. may be wrote $a^2x^{-1} - a^2bx^{-2} + a^2b^2x^{-3}$, &c.

9. And thus inſtead of $\sqrt{aa - xx}$ may be wrote $\overline{aa - xx}|^{\frac{1}{2}}$; and $\overline{aa - xx}|^2$ inſtead of the Square of $aa - xx$; and $\overline{\frac{abb - y^3}{by + yy}}|^{\frac{2}{3}}$ inſtead of $\sqrt[3]{\frac{ab^2 - y^3}{by + yy}}$: And the like of others.

10. So that we may not improperly diſtinguiſh Powers into Affirmative and Negative, Integral and Fractional.

Examples of Reduction by Extraction of Roots.

11. The Quantity $aa + xx$ being propoſed, you may thus extract its Square-Root.

$$aa + xx \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9} - \frac{21x^{12}}{1024a^{11}}, \text{ &c.}\right.$$

$$\begin{array}{l}
aa \\
\hline
0 + xx \\
\quad + xx + \frac{x^4}{4a^2} \\
\hline
\qquad - \frac{x^4}{4a^2} \\
\qquad - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6} \\
\hline
\qquad\quad + \frac{x^6}{8a^4} - \frac{x^8}{64a^6} \\
\qquad\quad + \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8} + \frac{x^{12}}{256a^{10}} \\
\hline
\qquad\qquad - \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}}, \text{ &c.} \\
\qquad\qquad - \frac{5x^8}{64a^6} - \frac{5x^{10}}{128a^8} + \frac{5x^{12}}{512a^{10}} \\
\hline
\qquad\qquad\quad + \frac{7x^{10}}{128a^8} - \frac{7x^{12}}{512a^{10}}, \text{ &c.} \\
\qquad\qquad\quad + \frac{7x^{10}}{128a^8} + \frac{7x^{12}}{256a^{10}} \\
\hline
\qquad\qquad\qquad - \frac{21x^{12}}{512a^{10}}, \text{ &c.}
\end{array}$$

So that the Root is found to be $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$, &c. Where it may be obſerved, that towards the end of the Operation I neglect all thoſe Terms, whoſe Dimenſions would exceed the Dimenſions of the laſt Term, to which I intend only to continue the Root, ſuppoſe to $\frac{x^{12}}{a^{11}}$.

12.

12. Alſo the Order of the Terms may be inverted in this manner $xx + aa$, in which caſe the Root will be found to be $x + \frac{aa}{2x} - \frac{a^4}{8x^3} + \frac{a^6}{16x^5} - \frac{5a^8}{128x^7}$ &c.

13. Thus the Root of $aa - xx$ is $a - \frac{xx}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c.

14. The Root of $x - xx$ is $x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c.

15. Of $aa + bx - xx$ is $a + \frac{bx}{2a} - \frac{xx}{2a} - \frac{b^2x^2}{8a^3}$, &c.

16. And $\sqrt{\frac{1+axx}{1-bxx}}$ is $\frac{1+\frac{1}{2}ax^2-\frac{1}{8}a^2x^4+\frac{1}{16}a^3x^6, \text{ &c.}}{1-\frac{1}{2}bx^2-\frac{1}{8}b^2x^4-\frac{1}{16}b^3x^6, \text{ &c.}}$ and moreover by actually dividing, it becomes

$$\begin{array}{llll} 1 + \frac{1}{2}bx^2 & + \frac{3}{8}b^2x^4 & + \frac{5}{16}b^3x^6, & \text{&c.} \\ + \frac{1}{2}a & + \frac{1}{4}ab & + \frac{3}{16}ab^2 & \\ & - \frac{1}{8}a^2 & - \frac{1}{16}a^2b & \\ & & + \frac{1}{16}a^3 & \end{array}$$

17. But theſe Operations, by due preparation, may very often be abbreviated; as in the foregoing Example to find $\sqrt{\frac{1+axx}{1-bxx}}$, if the Form of the Numerator and Denominator had not been the ſame, I might have multiply'd each by $\sqrt{1-bxx}$, which would have produced $\frac{\sqrt{1+ax^2-abx^4}}{1-bxx}$, and the reſt of the work might have been performed by extracting the Root of the Numerator only, and then dividing by the Denominator.

18. From hence I imagine it will ſufficiently appear, by what means any other Roots may be extracted, and how any compound Quantities, however entangled with Radicals or Denominators, (ſuch as $x^3 + \frac{\sqrt{x-\sqrt{1-xx}}}{\sqrt[3]{axx+x^3}} - \frac{\sqrt[5]{x^3+2x^5-x^{\frac{3}{2}}}}{\sqrt[3]{x+xx}-\sqrt{2x-x^2}}$) may be reduced to infinite Series conſiſting of ſimple Terms.

Of the Reduction of affected Equations.

19. As to affected Equations, we muſt be ſomething more particular in explaining how their Roots are to be reduced to ſuch Series as theſe; becauſe their Doctrine in Numbers, as hitherto deliver'd by Mathematicians, is very perplexed, and incumber'd with ſuperfluous Operations, ſo as not to afford proper Specimens for performing the Work in Species. I ſhall therefore firſt ſhew how the Reſolu-

Reſolution of affected Equations may be compendiouſly perform'd in Numbers, and then I ſhall apply the ſame to Species.

20. Let this Equation $y^3 - 2y - 5 = 0$ be propoſed to be reſolved, and let 2 be a Number (any how found) which differs from the true Root leſs than by a tenth part of itſelf. Then I make $2 + p = y$, and ſubſtitute $2 + p$ for y in the given Equation, by which is produced a new Equation $p^3 + 6p^2 + 10p - 1 = 0$, whoſe Root is to be ſought for, that it may be added to the Quote. Thus rejecting $p^3 + 6p^2$ becauſe of its ſmallneſs, the remaining Equation $10p - 1 = 0$, or $p = 0,1$, will approach very near to the truth. Therefore I write this in the Quote, and ſuppoſe $0,1 + q = p$, and ſubſtitute this fictitious Value of p as before, which produces $q^3 + 6,3q^2 + 11,23q + 0,0601 = 0$. And ſince $11,23q + 0,0601 = 0$ is near the truth, or $q = -0,0054$ nearly, (that is, dividing 0,061 by 11,23, till ſo many Figures ariſe as there are places between the firſt Figures of this, and of the principal Quote excluſively, as here there are two places between 2 and 0,005) I write $-0,0054$ in the lower part of the Quote, as being negative; and ſuppoſing $-0,0054 + r = q$, I ſubſtitute this as before. And thus I continue the Operation as far as I pleaſe, in the manner of the following Diagram:

$y^3 - 2y - 5 = 0$	$+2,10000000$ $-0,00544852$ $+2,09455148$, &c. $= y$
$2 + p = y$. $+y^3$ $-2y$ -5	$+8 + 12p + 6p^2 + p^3$ $-4 - 2p$ -5
The Sum	$-1 + 10p + 6p^2 + p^3$
$0,1 + q = p$. $+p^3$ $+6p^2$ $+10p$ -1	$+0,001 + 0,03q + 0,3q^2 + q^3$ $+0,06 + 1,2 + 6,$ $+1, + 10,$ $-1,$
The Sum	$0,061 + 11,23q + 6,3q^2 + q^3$
$-0,0054 + r = q$. q^3 $+6,3q^2$ $+11,23q$ $+0,061$	$-0,000000157464 + 0,00008748r - 0,01621r^2 + r^3$ $+0,00018378 - 0,06804 + 6,3$ $-0,060642 + 11,23$ $+0,061$
The Sum	$+0,00054416 + 11,162r$
$-0,00004852 + s = r$.	

21. But the Work may be much abbreviated towards the end by this Method, eſpecially in Equations of many Dimenſions. Having firſt determin'd how far you intend to extract the Root, count ſo many places after the firſt Figure of the Coefficient of the laſt Term but one, of the Equations that reſult on the right ſide of the Diagram, as there remain places to be fill'd up in the Quote, and reject the Decimals that follow. But in the laſt Term the Decimals may be neglected, after ſo many more places as are the decimal places that are fill'd up in the Quote. And in the antepenultimate Term reject all that are after ſo many fewer places. And ſo on, by proceeding Arithmetically, according to that Interval of places: Or, which is the ſame thing, you may cut off every where ſo many Figures as in the penultimate Term, ſo that their loweſt places may be in Arithmetical Progreſſion, according to the Series of the Terms, or are to be ſuppos'd to be ſupply'd with Cyphers, when it happens otherwiſe. Thus in the preſent Example, if I deſired to continue the Quote no farther than to the eighth place of Decimals, when I ſubſtituted $0{,}00054 + r$ for q, where four decimal places are compleated in the Quote, and as many remain to be compleated, I might have omitted the Figures in the five inferior places, which therefore I have mark'd or cancell'd by little Lines drawn through them; and indeed I might alſo have omitted the firſt Term r^3, although its Coefficient be 0,99999. Thoſe Figures therefore being expunged, for the following Operation there ariſes the Sum $0{,}0005416 + 11{,}162r$, which by Diviſion, continued as far as the Term preſcribed, gives $-0{,}00004852$ for r, which compleats the Quote to the Period required. Then ſubtracting the negative part of the Quote from the affirmative part, there ariſes 2,09455148 for the Root of the propoſed Equation.

22. It may likewiſe be obſerved, that at the beginning of the Work, if I had doubted whether $0{,}1 + p$ was a ſufficient Approximation to the Root, inſtead of $10p - 1 = 0$, I might have ſuppos'd that $6p^2 + 10p - 1 = 0$, and ſo have wrote the firſt Figure of its Root in the Quote, as being nearer to nothing. And in this manner it may be convenient to find the ſecond, or even the third Figure of the Quote, when in the ſecondary Equation, about which you are converſant, the Square of the Coefficient of the penultimate Term is not ten times greater than the Product of the laſt Term multiply'd into the Coefficient of the antepenultimate Term. And indeed you will often ſave ſome pains, eſpecially in Equations of many Dimenſions, if you ſeek for all the Figures

to

to be added to the Quote in this manner; that is, if you extract the leſſer Root out of the three laſt Terms of its ſecondary Equation: For thus you will obtain, at every time, as many Figures again in the Quote.

23. And now from the Reſolution of numeral Equations, I ſhall proceed to explain the like Operations in Species; concerning which, it is neceſſary to obſerve what follows.

24. Firſt, that ſome one of the ſpecious or literal Coefficients, if there are more than one, ſhould be diſtinguiſh'd from the reſt, which either is, or may be ſuppos'd to be, much the leaſt or greateſt of all, or neareſt to a given Quantity. The reaſon of which is, that becauſe of its Dimenſions continually increaſing in the Numerators, or the Denominators of the Terms of the Quote, thoſe Terms may grow leſs and leſs, and therefore the Quote may conſtantly approach to the Root required; as may appear from what is ſaid before of the Species x, in the Examples of Reduction by Diviſion and Extraction of Roots. And for this Species, in what follows, I ſhall generally make uſe of x or z; as alſo I ſhall uſe y, p, q, r, s, &c. for the Radical Species to be extracted.

25. Secondly, when any complex Fractions, or ſurd Quantities, happen to occur in the propoſed Equation, or to ariſe afterwards in the Proceſs, they ought to be removed by ſuch Methods as are ſufficiently known to Analyſts. As if we ſhould have $y^3 + \frac{bb}{b-x}y^2 - x^3 = 0$, multiply by $b-x$, and from the Product $by^3 - xy^3 + b^2y^2 - bx^3 + x^4 = 0$ extract the Root y. Or we might ſuppoſe $y \times b-x = v$, and then writing $\frac{v}{b-x}$ for y, we ſhould have $v^3 + b^2v^2 - b^3x^3 + 3b^2x^4 - 3bx^5 + x^6 = 0$, whence extracting the Root v, we might divide the Quote by $b-x$, in order to obtain y. Alſo if the Equation $y^5 - xy^{\frac{5}{2}} + x^{\frac{5}{2}} = 0$ were propoſed, we might put $y^{\frac{1}{2}} = v$, and $x^{\frac{1}{2}} = z$, and ſo writing vv for y, and z^3 for x, there will ariſe $v^6 - z^3v + z^{6} = 0$; which Equation being reſolved, y and x may be reſtored. For the Root will be found $v = z + z^3 + 6z^5$, &c. and reſtoring y and x, we have $y^{\frac{1}{2}} = x^{\frac{1}{2}} + x + 6x^{\frac{5}{2}}$, &c. then ſquaring, $y = x^{\frac{2}{3}} + 2x^{\frac{4}{3}} + 13x^2$, &c.

26. After the ſame manner if there ſhould be found negative Dimenſions of x and y, they may be removed by multiplying by the ſame x and y. As if we had the Equation $x^3 + 3x^2y^{-1} - 2x^{-1} - 16y^{-3} = 0$, multiply by x and y^3, and there would ariſe $x^4y^3 + 3x^3y^2 - 2y^3 - 16x = 0$. And if the Equation were $x = \frac{aa}{y} - \frac{2a^3}{y^2} + \frac{3a^4}{y^3}$,

by

by multiplying into y^3 there would arise $xy^3 = a^2y^2 - 2a^3y + 3a^4$. And so of others.

27. Thirdly, when the Equation is thus prepared, the work begins by finding the first Term of the Quote; concerning which, as also for finding the following Terms, we have this general Rule, when the indefinite Species (x or z) is supposed to be small; to which Case the other two Cases are reducible.

28. Of all the Terms, in which the Radical Species (y, p, q, or r, &c.) is not found, chuse the lowest in respect of the Dimensions of the indefinite Species (x or z, &c.) then chuse another Term in which that Radical Species is found, such as that the Progression of the Dimensions of each of the fore-mentioned Species, being continued from the Term first assumed to this Term, may descend as much as may be, or ascend as little as may be. And if there are any other Terms, whose Dimensions may fall in with this Progression continued at pleasure, they must be taken in likewise. Lastly, from these Terms thus selected, and made equal to nothing, find the Value of the said Radical Species, and write it in the Quote.

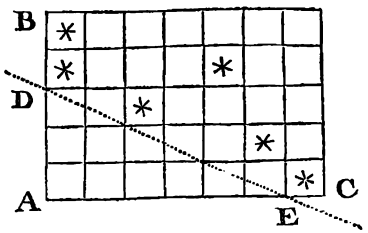
29. But that this Rule may be more clearly apprehended, I shall explain it farther by help of the following Diagram. Making a right Angle BAC, divide its sides AB, AC, into equal parts, and raising Perpendiculars, distribute the Angular Space into equal Squares or Parallelograms, which you may conceive to be denominated from the Dimensions of the Species x and y, as they are here inscribed. Then, when any Equation is proposed, mark such of the Parallelograms as correspond to all its Terms, and let a Ruler be apply'd to two, or perhaps more, of the Parallelograms so mark'd, of which let one be the lowest in the left-hand Column at AB, the other touching the Ruler towards the right-hand; and let all the rest, not touching the Ruler, lie above it. Then select those Terms of the Equation which are represented by the Parallelograms that touch the Ruler, and from them find the Quantity to be put in the Quote.

B

x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy	xy^2	xy^3	xy^4
1	y	y^2	y^3	y^4

A C

30. Thus to extract the Root y out of the Equation $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$, I mark the Parallelograms belong-



ing to the Terms of this Equation with the Mark *, as you ſee here done. Then I apply the Ruler DE to the lower of the Parallelograms mark'd in the left-hand Column, and I make it turn round towards the right-hand from the lower to the upper, till it begins in like manner to touch another, or perhaps more, of the Parallelograms that are mark'd; and I ſee that the places ſo touch'd belong to x^3, x^2y^2, and y^6. Therefore from the Terms $y^6 - 7a^2x^2y^2 + 6a^3x^3$, as if equal to nothing, (and moreover, if you pleaſe, reduced to $v^6 - 7v^2 + 6 = 0$, by making $y = v\sqrt{ax}$,) I ſeek the Value of y, and find it to be four-fold, $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$, and $-\sqrt{2ax}$, of which I may take any one for the initial Term of the Quote, according as I deſign to extract this or that Root of the given Equation.

31. Thus having the Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$, I chuſe the Terms $-by^2 + 9bx^2$, and thence I obtain $+3x$ for the initial Term of the Quote.

32. And having $y^3 + axy + aay - x^3 - 2a^3 = 0$, I make choice of $y^3 + a^2y - 2a^3$, and its Root $+a$ I write in the Quote.

33. Alſo having $x^2y^5 - 3c^4xy^2 - c^5x^2 + c^7 = 0$, I ſelect $x^2y^5 + c^7$, which gives $-\sqrt[5]{\frac{c^7}{x^2}}$ for the firſt Term of the Quote. And the like of others.

34. But when this Term is found, if its Power ſhould happen to be negative, I depreſs the Equation by the ſame Power of the indefinite Species, that there may be no need of depreſſing it in the Reſolution; and beſides, that the Rule hereafter deliver'd, for the ſuppreſſion of ſuperfluous Terms, may be conveniently apply'd. Thus the Equation $8z^6y^3 + az^6y^2 - 27a^9 = 0$ being propoſed, whoſe Root is to begin by the Term $\frac{3c^3}{2z^2}$, I depreſs by z^2, that it may become $8z^4y^3 + az^4y^2 - 27a^9z^{-2} = 0$, before I attempt the Reſolution.

35. The ſubſequent Terms of the Quotes are derived by the ſame Method, in the Progreſs of the Work, from their ſeveral ſecondary Equations, but commonly with leſs trouble. For the whole affair is perform'd by dividing the loweſt of the Terms affected with the indefinitely ſmall Species, (x, x^2, x^3, &c.) without the Radical Species, (p, q, r, &c.) by the Quantity with which that radical Species

of one Dimenſion only is affected, without the other indefinite Species, and by writing the Reſult in the Quote. So in the following Example, the Terms $\frac{x}{4}$, $\frac{xx}{64a}$, $\frac{131x^3}{512a^2}$, &c. are produced by dividing a^2x, $\frac{1}{16}ax^2$, $\frac{1}{128}\frac{3}{2}\frac{1}{8}x^3$, &c. by $4aa$.

36. Theſe things being premiſed, it remains now to exhibit the Praxis of Reſolution. Therefore let the Equation $y^3+a^2y+axy-2a^3-x^3=0$ be propoſed to be reſolved. And from its Terms $y^3+a^2y-2a^3=0$, being a fictitious Equation, by the third of the foregoing Premiſes, I obtain $y-a=0$, and therefore I write $+a$ in the Quote. Then becauſe $+a$ is not the compleat Value of y, I put $a+p=y$, and inſtead of y, in the Terms of the Equation written in the Margin, I ſubſtitute $a+p$, and the Terms reſulting (p^3+3ap^2+axp, &c.) I again write in the Margin; from which again, according to the third of the Premiſes, I ſelect the Terms $+4a^2p+a^2x=0$ for a fictitious Equation, which giving $p=-\frac{1}{4}x$, I write $-\frac{1}{4}x$ in the Quote. Then becauſe $-\frac{1}{4}x$ is not the accurate Value of p, I put $-\frac{1}{4}x+q=p$, and in the marginal Terms for p I ſubſtitute $-\frac{1}{4}x+q$, and the reſulting Terms ($q^3-\frac{3}{4}xq^2+3aq^2$, &c.) I again write in the Margin, out of which, according to the foregoing Rule, I again ſelect the Terms $4a^2q-\frac{1}{16}ax^2=0$ for a fictitious Equation, which giving $q=\frac{xx}{64a}$, I write $\frac{xx}{64a}$ in the Quote. Again, ſince $\frac{xx}{64a}$ is not the accurate Value of q, I make $\frac{xx}{64a}+r=q$, and inſtead of q I ſubſtitute $\frac{xx}{64a}+r$ in the marginal Terms. And thus I continue the Proceſs at pleaſure, as the following Diagram exhibits to view.

$y^3+a^2y-2a^3+axy-x^3=0.$		$y=a-\frac{x}{4}+\frac{x^2}{64a}+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$ &c.
$+a+p=y.$	$+y^3$	$+a^3+3a^2p+3ap^2+p^3.$
	$+axy$	$+a^2x+axp$
	$+a^2y$	$+a^3+a^2p$
	$-x^3$	$-x^3$
	$-2a^3$	$-2a^3$
$-\frac{1}{4}x+q=p.$	$+p^3$	$-\frac{1}{64}x^3+\frac{3}{160}x^2q-\frac{3}{4}xq^2+q^3.$
	$+3ap^2$	$+\frac{3}{160}ax^2-\frac{3}{8}axq+3aq^2.$
	$+axp$	$-\frac{1}{4}ax^2+axq$
	$+4a^2p$	$-a^2x+4a^2q$
	$+a^2x$	$+a^2x$
	$-x^3$	$-x^3$
$+\frac{x^2}{64a}+r=q.$	$+q^3$	*
	$-\frac{3}{4}xq^2$	*
	$+3aq^2$	$+\frac{3x^4}{4096a}$ * $+\frac{3}{32}x^2r+3ar^2.$
	$+\frac{3}{160}x^2q$	$+\frac{3x^4}{1024a}$ * $+\frac{3}{160}x^2r$
	$-\frac{1}{8}axq$	$-\frac{1}{1280}x^3$ $-\frac{1}{8}axr$
	$+4a^2q$	$+\frac{1}{160}ax^2$ $+4a^2r$
	$-\frac{655}{64}x^3$	$-\frac{655}{64}x^3$
	$-\frac{1}{160}ax^2$	$-\frac{1}{160}ax^2$
$+4a^2-\frac{1}{2}ax+\frac{9}{32}x^2)+\frac{13}{128}\frac{1}{0}x^3-\frac{15x^4}{4096a}$		$(+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$

37. If it were required to continue the Quote only to a certain Period, that x, for inſtance, in the laſt Term ſhould not aſcend beyond a given Dimenſion; as I ſubſtitute the Terms, I omit ſuch as I foreſee will be of no uſe. For which this is the Rule, that after the firſt Term reſulting in the collateral Margin from every Quantity, ſo many Terms are to be added to the right-hand, as the Index of the higheſt Power required in the Quote exceeds the Index of that firſt reſulting Term.

38. As in the preſent Example, if I deſired that the Quote, (or the Species x in the Quote,) ſhould aſcend no higher than to four Dimenſions, I omit all the Terms after x^4, and put only one after x^3.

Therefore

Therefore the Terms after the Mark * are to be conceived to be expunged. And thus the Work being continued till at laſt we come to the Terms $\frac{15x^4}{4096a} - \frac{131x^3}{128} + 4a^2r - \frac{1}{2}axr$, in which $p, q, r,$ or s, &c. repreſenting the Supplement of the Root to be extracted, are only of one Dimenſion; we may find ſo many Terms by Diviſion, $\left(+\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}\right)$ as we ſhall ſee wanting to compleat the Quote. So that at laſt we ſhall have $y = a - \frac{1}{4}x + \frac{xx}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ &c.

39. For the ſake of farther Illuſtration, I ſhall propoſe another Example to be reſolved. From the Equation $\frac{1}{3}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y - z = 0$, let the Quote be found only to the fifth Dimenſion, and the ſuperfluous Terms be rejected after the Mark, *&c.*

$\frac{1}{3}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y - z = 0$. $y = z + \frac{1}{4}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c.		
$z + p = y$.	$+\frac{1}{3}y^5$	$+\frac{1}{3}z^5$, &c.
	$-\frac{1}{4}y^4$	$-\frac{1}{4}z^4 - z^3p$, &c.
	$+\frac{1}{3}y^3$	$+\frac{1}{3}z^3 + z^2p + zp^2$, &c.
	$-\frac{1}{2}y^2$	$-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
	$+y$	$+z + p$
	$-z$	$-z$
$\frac{1}{2}z^2 + q = p$.	$+zp^2$	$+\frac{1}{4}z^5$, &c.
	$-\frac{1}{2}p^2$	$-\frac{1}{8}z^4 - \frac{1}{2}z^2q$, &c.
	$-z^3p$	$-\frac{1}{4}z^5$, &c.
	$+z^2p$	$+\frac{1}{2}z^4 + z^2q$
	$-zp$	$-\frac{1}{2}z^3 - zq$
	$+p$	$+\frac{1}{2}z^2 + q$
	$+\frac{1}{5}z^5$	$+\frac{1}{5}z^5$
	$-\frac{1}{4}z^4$	$-\frac{1}{4}z^4$
	$+\frac{1}{3}z^3$	$+\frac{1}{3}z^3$
	$-\frac{1}{2}z^2$	$-\frac{1}{2}z^2$
$1 - z + \frac{1}{2}z^2)\ \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{120}z^5\ \left(\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5\right.$		

40. And thus if we propoſe the Equation $\frac{63}{2816}y^{11} + \frac{35}{1152}y^9 + \frac{5}{112}y^7 + \frac{3}{40}y^5 + \frac{1}{6}y^3 + y - z = 0$, to be reſolved only to the ninth Dimenſion of the Quote; before the Work begins we may reject the Term $\frac{63}{2816}y^{11}$; then as we operate we may reject all the Terms beyond z^9, beyond z^7 we may admit but one, and two only after z^5;

z^5; becauſe we may obſerve, that the Quote ought always to aſcend by the Interval of two Units, in this manner, z, z^3, z^5, &c. Then at laſt we ſhall have $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c.

41. And hence an Artifice is diſcover'd, by which Equations, tho' affected *in infinitum*, and conſiſting of an infinite number of Terms, may however be reſolved. And that is, before the Work begins all the Terms are to be rejected, in which the Dimenſion of the indefinitely ſmall Species, not affected by the radical Species, exceeds the greateſt Dimenſion required in the Quote; or from which, by ſubſtituting inſtead of the radical Species, the firſt Term of the Quote found by the Parallelogram as before, none but ſuch exceeding Terms can ariſe. Thus in the laſt Example I ſhould have omitted all the Terms beyond y^3, though they went on *ad infinitum*. And ſo in this Equation

$$0 = \begin{cases} -8 + z^2 - 4z^4 + 9z^6 - 16z^8, \text{ &c.} \\ +y \text{ in } z^2 - 2z^4 + 3z^6 - 4z^8, \text{ &c.} \\ -y^2 \text{ in } z^2 - z^4 + z^6 - z^8, \text{ &c.} \\ +y^3 \text{ in } z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6 - \frac{1}{4}z^8, \text{ &c.} \end{cases}$$

that the Cubick Root may be extracted only to four Dimenſions of z, I omit all the Terms *in infinitum* beyond $+y^3$ in $z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6$, and all beyond $-y^2$ in $z^2 - z^4 + z^6$, and all beyond $+y$ in $z^2 - 2z^4$, and beyond $-8 + z^2 - 4z^4$. And therefore I aſſume this Equation only to be reſolved, $\frac{1}{3}z^6y^3 - \frac{1}{2}z^4y^3 + z^2y^3 - z^6y^2 + z^4y^2 - z^2y^2 - 2z^4y + z^2y - 4z^4 + z^2 - 8 = 0$. Becauſe $2z^{\frac{2}{3}}$, (the firſt Term of the Quote,) being ſubſtituted inſtead of y in the reſt of the Equation depreſs'd by $z^{\frac{2}{3}}$, gives every where more than four Dimenſions.

42. What I have ſaid of higher Equations may alſo be apply'd to Quadraticks. As if I deſired the Root of this Equation

$$0 = \begin{cases} y^2 \\ -y \text{ in } a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} \text{ &c.} \\ +\frac{x^4}{4a^2} \end{cases}$$

as far as the Period x^6, I omit all the Terms *in infinitum*, beyond $-y$ in $a + x + \frac{x^2}{a}$, and aſſume only this Equation, $y^2 - ay - xy - \frac{x^2}{a}y + \frac{x^4}{4a^2} = 0$. This I reſolve either in the uſual manner, by making

y

$y=\frac{1}{2}a+\frac{1}{2}x+\frac{x^2}{2a}-\sqrt{\frac{1}{4}a^2+\frac{1}{2}ax+\frac{3}{4}x^2+\frac{x^3}{2a}}$; or more expeditiously by the Method of affected Equations deliver'd before, by which we shall have $y=\frac{x^4}{4a^3}-\frac{x^5}{4a^4}$ *, where the last Term required vanishes, or becomes equal to nothing.

43. Now after that Roots are extracted to a convenient Period, they may sometimes be continued at pleasure, only by observing the Analogy of the Series. So you may for ever continue this $z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\frac{1}{120}z^5$, &c. (which is the Root of the infinite Equation $z=y+\frac{1}{2}y^2+\frac{1}{2}y^3+\frac{1}{4}y^4$, &c.) by dividing the last Term by these Numbers in order 2, 3, 4, 5, 6, *&c.* And this, $z-\frac{1}{6}z^3+\frac{1}{120}z^5-\frac{1}{3040}z^7+\frac{1}{36288}z^9$, &c. may be continued by dividing by these Numbers 2×3, 4×5, 6×7, 8×9, *&c.* Again, the Series $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$, &c. may be continued at pleasure, by multiplying the Terms respectively by these Fractions, $\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{3}{6}$, $-\frac{5}{8}$, $-\frac{7}{10}$, *&c.* And so of others.

44. But in discovering the first Term of the Quote, and sometimes of the second or third, there may still remain a difficulty to be overcome. For its Value, sought for as before, may happen to be surd, or the inextricable Root of an high affected Equation. Which when it happens, provided it be not also impossible, you may represent it by some Letter, and then proceed as if it were known. As in the Example $y^3+axy+a^2y-x^3-2a^3=0$: If the Root of this Equation $y^3+a^2y-2a^3=0$, had been surd, or unknown, I should have put any Letter b for it, and then have perform'd the Resolution as follows, suppose the Quote found only to the third Dimension.

$y^3+aay+axy-2a^3-x^3=0$. Make $a^2+3b^2=c^2$, then
$y=b-\frac{abx}{c^2}+\frac{a^4b^2x^2}{c^6}+\frac{x^3}{c^2}+\frac{a^3b^3x^3}{c^8}-\frac{a^5bx^3}{c^8}+\frac{a^5b^3x^3}{c^{10}}$ &c.

$b+p=y.$	$+y^3$	$+b^3+3b^2p+3bp^2+p^3.$
	$+axy$	$+abx+axp$
	$+a^2y$	$+a^2b+a^2p$
	$-x^3$	$-x^3$
	$-2a^3$	$-2a^3$
$-\frac{abx}{c^2}+q=p.$	$+p^3$	$-\frac{a^3b^3x^3}{c^6}$ &c.
	$+3bp^2$	$+\frac{3a^2b^3x^2}{c^4}-\frac{6ab^2x}{c^2}q$ &c.
	$+axp$	$-\frac{a^2bx^2}{c^2}+axq$
	$+c^2p$	$-abx+c^2q$
	$-x^3$	$-x^3$
	$+abx$	$+abx$

$c^2+ax-\frac{6ab^2x}{c^2}\Big)\frac{a^4b^2x^2}{c^4}+x^3+\frac{a^3b^3x^3}{c^6}\Big(\frac{a^4b^2x^2}{c^6}+\frac{x^3}{c^2}+\frac{a^3b^3x^3}{c^8}$ &c.

45. Here writing b in the Quote, I ſuppoſe $b+p=y$, and then for y I ſubſtitute as you ſee. Whence proceeds p^3+3bp^2, &c. rejecting the Terms $b^3+a^2b-2a^3$, as being equal to nothing: For b is ſuppos'd to be a Root of this Equation $y^3+a^2y-2a^3=0$. Then the Terms $3b^2p+a^2p+abx$ give $\frac{-abx}{3b^2+a^2}$ to be ſet in the Quote, and $\frac{-abx}{a^2+3b^2}+q$ to be ſubſtituted for p.

46. But for brevity's ſake I write cc for $aa+3bb$, yet with this caution, that $aa+3bb$ may be reſtored, whenever I perceive that the Terms may be abbreviated by it. When the Work is finiſh'd, I aſſume ſome Number for a, and reſolve this Equation $y^3+a^2y-2a^3=0$, as is ſhewn above concerning Numeral Equations; and I ſubſtitute for b any one of its Roots, if it has three Roots. Or rather, I deliver ſuch Equations from Species, as far as I can, eſpecially from the indefinite Species, and that after the manner before inſinuated. And for the reſt only, if any remain that cannot be expunged, I put Numbers. Thus $y^3+a^2y-2a^3=0$ will be freed from a, by dividing the Root by a, and it will become $y^3+y-2=0$, whoſe Root being found, and multiply'd by a, muſt be ſubſtituted inſtead of b.

47.

47. Hitherto I have ſuppos'd the indefinite Species to be little. But if it be ſuppos'd to approach nearly to a given Quantity, for that indefinitely ſmall difference I put ſome Species, and that being ſubſtituted, I ſolve the Equation as before. Thus in the Equation $\frac{1}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y + a - x = 0$, it being known or ſuppos'd that x is nearly of the ſame Quantity as a, I ſuppoſe z to be their difference; and then writing $a+z$ or $a-z$ for x, there will ariſe $\frac{1}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y \pm z = 0$, which is to be ſolved as before.

48. But if that Species be ſuppos'd to be indefinitely great, for its Reciprocal, which will therefore be indefinitely little, I put ſome Species, which being ſubſtituted, I proceed in the Reſolution as before. Thus having $y^3 + y^2 + y - x^2 = 0$, where x is known or ſuppos'd to be very great, for the reciprocally little Quantity $\frac{1}{x}$ I put z, and ſubſtituting $\frac{1}{z}$ for x, there will ariſe $y^3 + y^2 + y - \frac{1}{z^2} = 0$, whoſe Root is $y = \frac{1}{z} - \frac{1}{3} - \frac{2}{9}z + \frac{7}{81}z^2 + \frac{5}{81}z^3$, &c. where x being reſtored, if you pleaſe, it will be $y = x - \frac{1}{3} + \frac{2}{9x} + \frac{7}{81x^2} + \frac{5}{81x^3}$, &c.

49. If it ſhould happen that none of theſe Expedients ſhould ſucceed to your deſire, you may have recourſe to another. Thus in the Equation $y^4 - x^2y^2 + xy^2 + 2y^2 - 2y + 1 = 0$, whereas the firſt Term ought to be obtain'd from the Suppoſition that $y^4 + 2y^2 - 2y + 1 = 0$, which yet admits of no poſſible Root; you may try what can be done another way. As you may ſuppoſe that x is but little different from $+2$, or that $2+z = x$. Then ſubſtituting $2+z$ inſtead of x, there will ariſe $y^4 - z^2y^2 - 3zy^2 - 2y + 1 = 0$, and the Quote will begin from $+1$. Or if you ſuppoſe x to be indefinitely great, or $\frac{1}{x} = z$, you will have $y^4 - \frac{y^2}{z^2} + \frac{y^2}{z} + 2y^2 - 2y + 1 = 0$, and $+z$ for the initial Term of the Quote.

50. And thus by proceeding according to ſeveral Suppoſitions, you may extract and expreſs Roots after various ways.

51. If you ſhould deſire to find after how many ways this may be done, you muſt try what Quantities, when ſubſtituted for the indefinite Species in the propoſed Equation, will make it diviſible by y, $+$ or $-$ ſome Quantity, or by y alone. Which, for Example ſake, will happen in the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$,

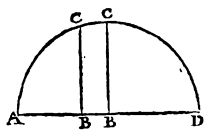
by ſubſtituting $+a$, or $-a$, or $-2a$, or $\overline{-2a^3}|^{\frac{1}{3}}$, &c. inſtead of x. And thus you may conveniently ſuppoſe the Quantity x to differ little from $+a$, or $-a$, or $-2a$, or $\overline{-2a^3}|^{\frac{1}{3}}$, and thence you may extract the Root of the Equation propoſed after ſo many ways. And perhaps alſo after ſo many other ways, by ſuppoſing thoſe differences to be indefinitely great. Beſides, if you take for the indefinite Quantity this or that of the Species which expreſs the Root, you may perhaps obtain your deſire after other ways. And farther ſtill, by ſubſtituting any fictitious Values for the indefinite Species, ſuch as $az+bz^2$, $\frac{a}{b+z}$, $\frac{a+cz}{b+z}$, &c. and then proceeding as before in the Equations that will reſult.

52. But now that the truth of theſe Concluſions may be manifeſt; that is, that the Quotes thus extracted, and produced *ad libitum*, approach ſo near to the Root of the Equation, as at laſt to differ from it by leſs than any aſſignable Quantity, and therefore when infinitely continued, do not at all differ from it: You are to conſider, that the Quantities in the left-hand Column of the right-hand ſide of the Diagrams, are the laſt Terms of the Equations whoſe Roots are p, q, r, s, &c. and that as they vaniſh, the Roots p, q, r, s, &c. that is, the differences between the Quote and the Root ſought, vaniſh at the ſame time. So that the Quote will not then differ from the true Root. Wherefore at the beginning of the Work, if you ſee that the Terms in the ſaid Column will all deſtroy one another, you may conclude, that the Quote ſo far extracted is the perfect Root of the Equation. But if it be otherwiſe, you will ſee however, that the Terms in which the indefinitely ſmall Species is of few Dimenſions, that is, the greateſt Terms, are continually taken out of that Column, and that at laſt none will remain there, unleſs ſuch as are leſs than any given Quantity, and therefore not greater than nothing when the Work is continued *ad infinitum*. So that the Quote, when infinitely extracted, will at laſt be the true Root.

53. Laſtly, altho' the Species, which for the ſake of perſpicuity I have hitherto ſuppos'd to be indefinitely little, ſhould however be ſuppos'd to be as great as you pleaſe, yet the Quotes will ſtill be true, though they may not converge ſo faſt to the true Root. This is manifeſt from the Analogy of the thing. But here the Limits of the Roots, or the greateſt and leaſt Quantities, come to be conſider'd. For theſe Properties are in common both to finite and infinite Equations. The Root in theſe is then greateſt or leaſt, when

when there is the greateſt or leaſt difference between the Sums of the affirmative Terms, and of the negative Terms; and is limited when the indefinite Quantity, (which therefore not improperly I ſuppos'd to be ſmall,) cannot be taken greater, but that the Magnitude of the Root will immediately become infinite, that is, will become impoſſible.

54. To illuſtrate this, let ACD be a Semicircle deſcribed on the Diameter AD, and BC be an Ordinate.



Make AB $= x$, BC $= y$, AD $= a$. Then $y = \sqrt{ax - xx} = \sqrt{ax} - \frac{x}{2a}\sqrt{ax} - \frac{x^2}{8a^2}\sqrt{ax} - \frac{x^3}{16a^3}\sqrt{ax}$, &c. as before. Therefore BC, or y, then becomes greateſt when $\sqrt{ax}$ moſt exceeds all the Terms $\frac{x}{2a}\sqrt{ax} + \frac{x^2}{8a^2}\sqrt{ax} + \frac{x^3}{16a^3}\sqrt{ax}$, &c. that is, when $x = \frac{1}{2}a$; but it will be terminated when $x = a$. For if we take x greater than a, the Sum of all the Terms $-\frac{x}{2a}\sqrt{ax} - \frac{x^2}{8a^2}\sqrt{ax} - \frac{x^3}{16a^3}\sqrt{ax}$, &c. will be infinite. There is another Limit alſo, when $x = 0$, by reaſon of the impoſſibility of the Radical $\sqrt{-ax}$; to which Terms or Limits, the Limits of the Semicircle A, B, and D, are correſpondent.

Tranſition to the METHOD OF FLUXIONS.

55. And thus much for the Methods of Computation, of which I ſhall make frequent uſe in what follows. Now it remains, that for an Illuſtration of the Analytick Art, I ſhould give ſome Specimens of Problems, eſpecially ſuch as the nature of Curves will ſupply. But firſt it may be obſerved, that all the difficulties of theſe may be reduced to theſe two Problems only, which I ſhall propoſe concerning a Space deſcribed by local Motion, any how accelerated or retarded.

56. I. *The Length of the Space deſcribed being continually (that is, at all Times) given; to find the Velocity of the Motion at any Time propoſed.*

57. II. *The Velocity of the Motion being continually given; to find the Length of the Space deſcribed at any Time propoſed.*

58. Thus in the Equation $xx = y$, if y repreſents the Length of the Space at any time deſcribed, which (time) another Space x, by increaſing with an uniform Celerity $\dot{x}$, meaſures and exhibits as

deſcribed: Then $2\dot{x}x$ will repreſent the Celerity by which the Space y, at the ſame moment of Time, proceeds to be deſcribed; and contrary-wiſe. And hence it is, that in what follows, I conſider Quantities as if they were generated by continual Increaſe, after the manner of a Space, which a Body or Thing in Motion deſcribes.

59. But whereas we need not conſider the Time here, any farther than as it is expounded and meaſured by an equable local Motion; and beſides, whereas only Quantities of the ſame kind can be compared together, and alſo their Velocities of Increaſe and Decreaſe: Therefore in what follows I ſhall have no regard to Time formally conſider'd, but I ſhall ſuppoſe ſome one of the Quantities propoſed, being of the ſame kind, to be increaſed by an equable Fluxion, to which the reſt may be referr'd, as it were to Time; and therefore, by way of Analogy, it may not improperly receive the name of Time. Whenever therefore the word *Time* occurs in what follows, (which for the ſake of perſpicuity and diſtinction I have ſometimes uſed,) by that Word I would not have it underſtood as if I meant Time in its formal Acceptation, but only that other Quantity, by the equable Increaſe or Fluxion whereof, Time is expounded and meaſured.

60. Now thoſe Quantities which I conſider as gradually and indefinitely increaſing, I ſhall hereafter call *Fluents*, or *Flowing Quantities*, and ſhall repreſent them by the final Letters of the Alphabet v, x, y, and z; that I may diſtinguiſh them from other Quantities, which in Equations are to be conſider'd as known and determinate, and which therefore are repreſented by the initial Letters a, b, c, &c. And the Velocities by which every Fluent is increaſed by its generating Motion, (which I may call *Fluxions*, or ſimply Velocities or Celerities,) I ſhall repreſent by the ſame Letters pointed thus $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$. That is, for the Celerity of the Quantity v I ſhall put $\dot{v}$, and ſo for the Celerities of the other Quantities x, y, and z, I ſhall put $\dot{x}$, $\dot{y}$, and $\dot{z}$ reſpectively.

61. Theſe things being premiſed, I ſhall now forthwith proceed to the matter in hand; and firſt I ſhall give the Solution of the two Problems juſt now propoſed.

PROB.

PROB. I.

The Relation of the Flowing Quantities to one another being given, to determine the Relation of their Fluxions.

SOLUTION.

1. Diſpoſe the Equation, by which the given Relation is expreſs'd, according to the Dimenſions of ſome one of its flowing Quantities, ſuppoſe x, and multiply its Terms by any Arithmetical Progreſſion, and then by $\frac{\dot{x}}{x}$. And perform this Operation ſeparately for every one of the flowing Quantities. Then make the Sum of all the Products equal to nothing, and you will have the Equation required.

2. EXAMPLE 1. If the Relation of the flowing Quantities x and y be $x^3 - ax^2 + axy - y^3 = 0$; firſt diſpoſe the Terms according to x, and then according to y, and multiply them in the following manner.

Mult.	x^3	$-ax^2$	$+axy$	$-y^3$	$-y^3$	$+axy$	$\frac{-ax^2}{+x^3}$
by	$\frac{3\dot{x}}{x}$	$\frac{2\dot{x}}{x}$	$\frac{\dot{x}}{x}$	0	$\frac{3\dot{y}}{y}$	$\frac{\dot{y}}{y}$	0
makes	$3\dot{x}x^2$	$-2a\dot{x}x$	$+a\dot{x}y$	$*$	$-3\dot{y}y^2$	$+a\dot{y}x$	$*$

The Sum of the Products is $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, which Equation gives the Relation between the Fluxions $\dot{x}$ and $\dot{y}$. For if you take x at pleaſure, the Equation $x^3 - ax^2 + axy - y^3 = 0$ will give y. Which being determined, it will be $\dot{x} : \dot{y} :: 3y^2 - ax : 3x^2 - 2ax + ay$.

3. Ex. 2. If the Relation of the Quantities x, y, and z, be expreſs'd by the Equation $2y^3 + x^2y - 2cyz + 3yz^2 - z^3 = 0$;

Mult.	$2y^3 + xx \times y - z^3$ $- 2cz$ $+ 3z^2$	$yx^2 + 2y^3$ $- 2cyz$ $+ 3yz^2$ $- z^3$	$-z^3 + 3yz^2 - 2cyz + x^2y$ $+ 2y^3$
by	$\frac{2\dot{y}}{y} \cdot 0 \cdot \frac{\dot{y}}{y}$	$\frac{2\dot{x}}{x} \cdot 0 \cdot$	$\frac{3\dot{z}}{z} \cdot \frac{2\dot{z}}{z} \cdot \frac{\dot{z}}{z} \cdot 0$
makes	$4\dot{y}y^2 \quad * \quad + \frac{yz^3}{y}$	$2\dot{x}xy \quad *$	$-3\dot{z}z^2 + 6\dot{z}zy - 2c\dot{z}y \quad *$

Where-

Wherefore the Relation of the Celerities of Flowing, or of the Fluxions x, y, and z, is $4yy^2 + \frac{yz^3}{y} + 2xxy - 3zz^2 + 6zzy - 2czy = 0$.

4. But ſince there are here three flowing Quantities, x, y, and z, another Equation ought alſo to be given, by which the Relation among them, as alſo among their Fluxions, may be intirely determined. As if it were ſuppoſed that $x + y - z = 0$. From whence another Relation among the Fluxions $\dot{x} + \dot{y} - \dot{z} = 0$ would be found by this Rule. Now compare theſe with the foregoing Equations, by expunging any one of the three Quantities, and alſo any one of the Fluxions, and then you will obtain an Equation which will intirely determine the Relation of the reſt.

5. In the Equation propos'd, whenever there are complex Fractions, or ſurd Quantities, I put ſo many Letters for each, and ſuppoſing them to repreſent flowing Quantities, I work as before. Afterwards I ſuppreſs and exterminate the aſſumed Letters, as you ſee done here.

6. Ex. 3. If the Relation of the Quantities x and y be $yy - aa - x\sqrt{aa - xx} = 0$; for $x\sqrt{aa - xx}$ I write z, and thence I have the two Equations $yy - aa - z = 0$, and $a^2x^2 - x^4 - z^2 = 0$, of which the firſt will give $2y\dot{y} - \dot{z} = 0$, as before, for the Relation of the Celerities $\dot{y}$ and $\dot{z}$, and the latter will give $2a^2x\dot{x} - 4\dot{x}x^3 - 2z\dot{z} = 0$, or $\frac{a^2x\dot{x} - z\dot{x}x^3}{z} = \dot{z}$, for the Relation of the Celerities $\dot{x}$ and $\dot{z}$. Now $\dot{z}$ being expunged, it will be $2y\dot{y} \frac{-a^2\dot{x} + 2x\dot{x}^3}{z} = 0$, and then reſtoring $x\sqrt{aa - xx}$ for z, we ſhall have $2y\dot{y} \frac{-a^2\dot{x} + 2x\dot{x}^3}{\sqrt{aa - xx}} = 0$, for the Relation between $\dot{x}$ and $\dot{y}$, as was required.

7. Ex. 4. If $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay + xx} = 0$, expreſſes the Relation that is between x and y: I make $\frac{by^3}{a+y} = z$, and $xx\sqrt{ay + xx} = v$, from whence I ſhall have the three Equations $x^3 - ay^2 + z - v = 0$, $az + yz - by^3 = 0$, and $ax^4y + x^6 - vv = 0$. The firſt gives $3x\dot{x}^2 - 2ay\dot{y} + \dot{z} - \dot{v} = 0$, the ſecond gives $a\dot{z} + z\dot{y} + y\dot{z} - 3by^2\dot{y} = 0$, and the third gives $4ax\dot{x}^3y + 6x\dot{x}^5 + ayx^4 - 2v\dot{v} = 0$, for the Relations of the Velocities $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$. But the

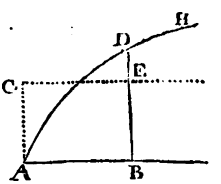
the Values of $\dot{z}$ and $\dot{v}$, found by the ſecond and third Equations, (that is, $\frac{3b\dot{y}\dot{y}^2 - y\dot{z}}{a+y}$ for $\dot{z}$, and $\frac{4a\dot{x}\dot{x}^3\dot{y} + 6\dot{x}\dot{x}^3 + a\dot{y}\dot{x}^4}{2v}$ for $\dot{v}$) I ſubſtitute in the firſt Equation, and there ariſes $3\dot{x}\dot{x}^2 - 2a\dot{y}\dot{y} + \frac{3b\dot{y}\dot{y}^2 - y\dot{z}}{a+y} - \frac{4a\dot{x}\dot{x}^3\dot{y} - 6\dot{x}\dot{x}^3 - a\dot{y}\dot{x}^4}{2v} = 0$. Then inſtead of z and v reſtoring their Values $\frac{by^3}{a+y}$ and $xx\sqrt{ay + xx}$, there will ariſe the Equation ſought $3\dot{x}\dot{x}^2 - 2a\dot{y}\dot{y} + \frac{3aby\dot{y}^2 + 2by\dot{y}^3}{aa + 2ay + yy} - \frac{4a\dot{x}\dot{x}y - 6\dot{x}\dot{x}^3 - a\dot{y}\dot{x}\dot{x}}{2\sqrt{ay + xx}} = 0$, by which the Relation of the Velocities $\dot{x}$ and $\dot{y}$ will be expreſs'd.

8. After what manner the Operation is to be perform'd in other Caſes, I believe is manifeſt from hence; as when in the Equation propos'd there are found ſurd Denominators, Cubick Radicals, Radicals within Radicals, as $\sqrt{ax + \sqrt{aa - xx}}$, or any other complicate Terms of the like kind.

9. Furthermore, altho' in the Equation propoſed there ſhould be Quantities involved, which cannot be determined or expreſs'd by any Geometrical Method, ſuch as Curvilinear Areas or the Lengths of Curve-lines; yet the Relations of their Fluxions may be found, as will appear from the following Example.

Preparation for EXAMPLE 5.

10. Suppoſe BD to be an Ordinate at right Angles to AB, and that ADH be any Curve, which is defined by the Relation between AB and BD exhibited by an Equation. Let AB be called x, and the Area of the Curve ADB, apply'd to Unity, be call'd z. Then erect the Perpendicular AC equal to Unity, and thro' C draw CE parallel to AB, and meeting BD in E. Then conceiving theſe two Superficies ADB and ACEB to be generated by the Motion of the right Line BED; it is manifeſt that their Fluxions, (that is, the Fluxions of the Quantities $1 \times z$, and $1 \times x$, or of the Quantities z and x,) are to each other as the generating Lines BD and BE. Therefore $\dot{z} : \dot{x} ::$ BD : BE or 1, and therefore $\dot{z} = \dot{x} \times$ BD.



11. And hence it is, that z may be involved in any Equation, expreſſing the Relation between x and any other flowing Quantity y; and yet the Relation of the Fluxions $\dot{x}$ and $\dot{y}$ may however be diſcover'd.

12.

12. Ex. 5. As if the Equation $zz + axz - y^4 = 0$ were propos'd to expreſs the Relation between x and y, as alſo $\sqrt{ax - xx} = BD$, for determining a Curve, which therefore will be a Circle. The Equation $zz + axz - y^4 = 0$, as before, will give $2\dot{z}z + a\dot{z}x + a\dot{x}z - 4\dot{y}y^3 = 0$, for the Relation of the Celerities $\dot{x}$, $\dot{y}$, and $\dot{z}$. And therefore ſince it is $\dot{z} = \dot{x} \times BD$ or $= \dot{x}\sqrt{ax - xx}$, ſubſtitute this Value inſtead of it, and there will ariſe the Equation $2\dot{x}z + a\dot{x}x\sqrt{ax - xx} + a\dot{x}z - 4\dot{y}y^3 = 0$, which determines the Relation of the Celerities $\dot{x}$ and $\dot{y}$.

DEMONSTRATION *of the Solution.*

13. The Moments of flowing Quantities, (that is, their indefinitely ſmall Parts, by the acceſſion of which, in indefinitely ſmall portions of Time, they are continually increaſed,) are as the Velocities of their Flowing or Increaſing.

14. Wherefore if the Moment of any one, as x, be repreſented by the Product of its Celerity $\dot{x}$ into an indefinitely ſmall Quantity o (that is, by $\dot{x}o$,) the Moments of the others v, y, z, will be repreſented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; becauſe $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.

15. Now ſince the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little acceſſions of the flowing Quantities x and y, by which thoſe Quantities are increaſed through the ſeveral indefinitely little intervals of Time; it follows, that thoſe Quantities x and y, after any indefinitely ſmall interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$. And therefore the Equation, which at all times indifferently expreſſes the Relation of the flowing Quantities, will as well expreſs the Relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y: So that $x + \dot{x}o$ and $y + \dot{y}o$ may be ſubſtituted in the ſame Equation for thoſe Quantities, inſtead of x and y.

16. Therefore let any Equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and ſubſtitute $x + \dot{x}o$ for x, and $y + \dot{y}o$ for y, and there will ariſe

$$\left.\begin{array}{llll} x^3 & + 3\dot{x}ox^2 & + 3\dot{x}^2oox & + \dot{x}^3o^3 \\ -ax^2 & - 2a\dot{x}ox & - a\dot{x}^2oo & \\ + axy & + a\dot{x}oy & + a\dot{y}ox & + a\dot{x}\dot{y}oo \\ -y^3 & - 3\dot{y}oy^2 & - 3\dot{y}^2ooy & - \dot{y}^3o^3 \end{array}\right\} = 0.$$

17.

17. Now by Suppoſition $x^3 - ax^2 + axy - y^3 = 0$, which therefore being expunged, and the remaining Terms being divided by o, there will remain $3\dot{x}\dot{x}^2 + 3x^2\dot{o}\dot{x} + x^3\dot{o}\dot{o} - 2a\dot{x}\dot{x} - ax^2\dot{o} + a\dot{x}\dot{y} + a\dot{y}\dot{x} + ax\dot{y}\dot{o} - 3\dot{y}\dot{y}^2 - 3y^2\dot{o}\dot{y} - y^3\dot{o}\dot{o} = 0$. But whereas o is ſuppoſed to be infinitely little, that it may repreſent the Moments of Quantities; the Terms that are multiply'd by it will be nothing in reſpect of the reſt. Therefore I reject them, and there remains $3\dot{x}\dot{x}^2 - 2a\dot{x}\dot{x} + a\dot{x}\dot{y} + a\dot{y}\dot{x} - 3\dot{y}\dot{y}^2 = 0$, as above in Examp. 1.

18. Here we may obſerve, that the Terms that are not multiply'd by o will always vaniſh, as alſo thoſe Terms that are multiply'd by o of more than one Dimenſion. And that the reſt of the Terms being divided by o, will always acquire the form that they ought to have by the foregoing Rule: Which was the thing to be proved.

19. And this being now ſhewn, the other things included in the Rule will eaſily follow. As that in the propos'd Equation ſeveral flowing Quantities may be involved; and that the Terms may be multiply'd, not only by the Number of the Dimenſions of the flowing Quantities, but alſo by any other Arithmetical Progreſſions; ſo that in the Operation there may be the ſame difference of the Terms according to any of the flowing Quantities, and the Progreſſion be diſpos'd according to the ſame order of the Dimenſions of each of them. And theſe things being allow'd, what is taught beſides in Examp. 3, 4, and 5, will be plain enough of itſelf.

PROB. II.

An Equation being propoſed, including the Fluxions of Quantities, to find the Relations of thoſe Quantities to one another.

A PARTICULAR SOLUTION.

1. As this Problem is the Converſe of the foregoing, it muſt be ſolved by proceeding in a contrary manner. That is, the Terms multiply'd by $\dot{x}$ being diſpoſed according to the Dimenſions of x; they muſt be divided by $\frac{\dot{x}}{x}$, and then by the number of their Dimenſions, or perhaps by ſome other Arithmetical Progreſſion. Then the ſame work muſt be repeated with the Terms multiply'd by $\dot{v}$, $\dot{y}$,

or $\dot{z}$, and the Sum resulting must be made equal to nothing, rejecting the Terms that are redundant.

2. EXAMPLE. Let the Equation proposed be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$. The Operation will be after this manner:

Divide	$3\dot{x}x^2$	$-2a\dot{x}x$	$+a\dot{x}y$	Divide	$-3\dot{y}y^2$	$*$	$+a\dot{y}x$
by $\frac{\dot{x}}{x}$. Quot.	$3x^3$	$-2ax^2$	$+ayx$	by $\frac{\dot{y}}{y}$. Quot.	$-3y^3$	$*$	$+axy$
Divide by	3 .	2 .	1.	Divide by	3 .	2 .	1.
Quote	x^3	$-ax^2$	$+ayx$	Quote	$-y^3$	$*$	$+axy$

Therefore the Sum $x^3 - ax^2 + axy - y^3 = 0$, will be the required Relation of the Quantities x and y. Where it is to be observed, that tho' the Term axy occurs twice, yet I do not put it twice in the Sum $x^3 - ax^2 + axy - y^3 = 0$, but I reject the redundant Term. And so whenever any Term recurs twice, (or oftener when there are several flowing Quantities concern'd,) it must be wrote only once in the Sum of the Terms.

3. There are other Circumstances to be observed, which I shall leave to the Sagacity of the Artist; for it would be needless to dwell too long upon this matter, because the Problem cannot always be solved by this Artifice. I shall add however, that after the Relation of the Fluents is obtain'd by this Method, if we can return, by Prob. 1. to the proposed Equation involving the Fluxions, then the work is right, otherwise not. Thus in the Example proposed, after I have found the Equation $x^3 - ax^2 + axy - y^3 = 0$, if from thence I seek the Relation of the Fluxions $\dot{x}$ and $\dot{y}$ by the first Problem, I shall arrive at the proposed Equation $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$. Whence it is plain, that the Equation $x^3 - ax^2 + axy - y^3 = 0$ is rightly found. But if the Equation $\dot{x}x - \dot{x}y + a\dot{y} = 0$ were proposed, by the prescribed Method I should obtain this $\frac{1}{2}x^2 - xy + ay = 0$, for the Relation between x and y; which Conclusion would be erroneous: Since by Prob. 1. the Equation $\dot{x}x - \dot{x}y - \dot{y}x + a\dot{y} = 0$ would be produced, which is different from the former Equation.

4. Having therefore premised this in a perfunctory manner, I shall now undertake the general Solution.

A

A PREPARATION FOR THE GENERAL SOLUTION.

5. Firſt it muſt be obſerved, that in the propoſed Equation the Symbols of the Fluxions, (ſince they are Quantities of a different kind from the Quantities of which they are the Fluxions,) ought to aſcend in every Term to the ſame number of Dimenſions: And when it happens otherwiſe, another Fluxion of ſome flowing Quantity muſt be underſtood to be Unity, by which the lower Terms are ſo often to be multiply'd, till the Symbols of the Fluxions ariſe to the ſame number of Dimenſions in all the Terms. As if the Equation $\dot{x} + \dot{x}\dot{y}x - axx = 0$ were propoſed, the Fluxion $\dot{z}$ of ſome third flowing Quantity z muſt be underſtood to be Unity, by which the firſt Term $\dot{x}$ muſt be multiply'd once, and the laſt axx twice, that the Fluxions in them may aſcend to as many Dimenſions as in the ſecond Term $\dot{x}\dot{y}x$: As if the propoſed Equation had been derived from this $\dot{x}\dot{z} + \dot{x}\dot{y}x - az\dot{z}x^2 = 0$, by putting $\dot{z} = 1$. And thus in the Equation $\dot{y}\dot{x} = y\dot{y}$, you ought to imagine $\dot{x}$ to be Unity, by which the Term $y\dot{y}$ is multiply'd.

6. Now Equations, in which there are only two flowing Quantities, which every where ariſe to the ſame number of Dimenſions, may always be reduced to ſuch a form, as that on one ſide may be had the Ratio of the Fluxions, $\left(\text{as } \frac{\dot{y}}{\dot{x}}, \text{ or } \frac{\dot{x}}{\dot{y}}, \text{ or } \frac{\dot{z}}{\dot{x}}, \&c.\right)$ and on the other ſide the Value of that Ratio, expreſs'd by ſimple Algebraic Terms; as you may ſee here, $\frac{\dot{y}}{\dot{x}} = 2 + 2x - y$. And when the foregoing particular Solution will not take place, it is required that you ſhould bring the Equations to this form.

7. Wherefore when in the Value of that Ratio any Term is denominated by a Compound quantity, or is Radical, or if that Ratio be the Root of an affected Equation; the Reduction muſt be perform'd either by Diviſion, or by Extraction of Roots, or by the Reſolution of an affected Equation, as has been before ſhewn.

8. As if the Equation $\dot{y}a - \dot{y}x - \dot{x}a + \dot{x}x - \dot{x}y = 0$ were propoſed; firſt by Reduction this becomes $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a-x}$, or $\frac{\dot{x}}{\dot{y}} = \frac{a-x}{a-x+y}$. And in the firſt Caſe, if I reduce the Term $\frac{y}{a-x}$, denominated by the compound Quantity $a - x$, to an infinite Series of

ſimple Terms $\frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$ &c. by dividing the Numerator y by the Denominator $a - x$, I ſhall have $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$ &c. by the help of which the Relation between x and y is to be determined.

9. So the Equation $\ddot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}\dot{x}x$ being given, or $\frac{\ddot{y}\dot{y}}{\dot{x}\dot{x}} = \frac{\dot{y}}{\dot{x}} + xx$, and by a farther Reduction $\frac{\dot{y}}{\dot{x}} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + xx}$: I extract the ſquare Root out of the Terms $\frac{1}{4} + xx$, and obtain the infinite Series $\frac{1}{2} + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. which if I ſubſtitute for $\sqrt{\frac{1}{4} + xx}$, I ſhall have $\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6 - 5x^8$, &c. or $\frac{\dot{y}}{\dot{x}} = -x^2 + x^4 - 2x^6 + 5x^8$, &c. according as $\sqrt{\frac{1}{4} + xx}$ is either added to $\frac{1}{2}$, or ſubtracted from it.

10. And thus if the Equation $\dot{y}^3 + ax\dot{x}^2\dot{y} + a^2\dot{x}^2\dot{y} - x^3\dot{x}^3 - 2\dot{x}^3a^3 = 0$ were propoſed, or $\frac{\dot{y}^3}{\dot{x}^3} + ax\frac{\dot{y}}{\dot{x}} + a^2\frac{\dot{y}}{\dot{x}} - x^3 - 2a^3 = 0$, I extract the Root of the affected Cubick Equation, and there ariſes $\frac{\dot{y}}{\dot{x}} = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ &c. as may be ſeen before.

11. But here it may be obſerved, that I look upon thoſe Terms only as compounded, which are compounded in reſpect of flowing Quantities. For I eſteem thoſe as ſimple Quantities which are compounded only in reſpect of given Quantities. For they may be reduced to ſimple Quantities by ſuppoſing them equal to other given Quantities. Thus I conſider the Quantities $\frac{ax+bx}{c}$, $\frac{x}{a+b}$, $\frac{bcc}{ax+bx}$, $\frac{l^4}{ax^2+bx^2}$, $\sqrt{ax+bx}$, &c. as ſimple Quantities, becauſe they may may all be reduced to the ſimple Quantities $\frac{ex}{c}$, $\frac{x}{e}$, $\frac{b^2}{ex}$, $\frac{b^4}{cx^2}$, $\sqrt{ex}$ (or $e^{\frac{1}{2}}x^{\frac{1}{2}}$) &c. by ſuppoſing $a + b = e$.

12. Moreover, that the flowing Quantities may the more eaſily be diſtinguiſh'd from one another, the Fluxion that is put in the Numerator of the Ratio, or the Antecedent of the Ratio, may not improperly be call'd the *Relate Quantity*, and the other in the Denominator, to which it is compared, the *Correlate*: Alſo the flowing

flowing Quantities may be diſtinguiſh'd by the ſame Names reſpectively. And for the better underſtanding of what follows, you may conceive, that the Correlate Quantity is Time, or rather any other Quantity that flows equably, by which Time is expounded and meaſured. And that the other, or the Relate Quantity, is Space, which the moving Thing, or Point, any how accelerated or retarded, deſcribes in that Time. And that it is the Intention of the Problem, that from the Velocity of the Motion, being given at every Inſtant of Time, the Space deſcribed in the whole Time may be determined.

13. But in reſpect of this Problem Equations may be diſtinguiſh'd into three Orders.

14. Firſt: In which two Fluxions of Quantities, and only one of their flowing Quantities are involved.

15. Second: In which the two flowing Quantities are involved, together with their Fluxions.

16. Third: In which the Fluxions of more than two Quantities are involved.

17. With theſe Premiſes I ſhall attempt the Solution of the Problem, according to theſe three Caſes.

SOLUTION OF CASE I.

18. Suppoſe the flowing Quantity, which alone is contain'd in the Equation, to be the Correlate, and the Equation being accordingly diſpos'd, (that is, by making on one ſide to be only the Ratio of the Fluxion of the other to the Fluxion of this, and on the other ſide to be the Value of this Ratio in ſimple Terms,) multiply the Value of the Ratio of the Fluxions by the Correlate Quantity, then divide each of its Terms by the number of Dimenſions with which that Quantity is there affected, and what ariſes will be equivalent to the other flowing Quantity.

19. So propoſing the Equation $\dot{y}\dot{y} = \dot{x}y + \dot{x}xxx$; I ſuppoſe x to be the Correlate Quantity, and the Equation being accordingly reduced, we ſhall have $\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6$, &c. Now I multiply the Value of $\frac{\dot{y}}{\dot{x}}$ into x, and there ariſes $x + x^3 - x^5 + 2x^7$, &c. which Terms I divide ſeverally by their number of Dimenſions, and the Reſult $x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7$, &c. I put $= y$. And by this

this Equation will be defined the Relation between x and y, as was required.

20. Let the Equation be $\frac{\dot{y}}{\dot{x}} = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512a^2}$ &c. there will arise $y = ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$ &c. for determining the Relation between x and y.

21. And thus the Equation $\frac{\dot{y}}{\dot{x}} = \frac{1}{x^3} - \frac{1}{x^2} + \frac{a}{x^{\frac{1}{2}}} - x^{\frac{1}{2}} + x^{\frac{3}{2}}$, gives $y = -\frac{1}{2x^2} + \frac{1}{x} + 2ax^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}$. For multiply the Value of $\frac{\dot{y}}{\dot{x}}$ into x, and it becomes $\frac{1}{xx} - \frac{1}{x} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}$, or $x^{-2} - x^{-1} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}$, which Terms being divided by the number of Dimensions, the Value of y will arise as before.

22. After the same manner the Equation $\frac{\dot{x}}{\dot{y}} = \frac{2b^2c}{\sqrt{ay^3}} + \frac{3y^2}{a+b} + \sqrt{by^3 + cy^3}$, gives $x = -\frac{4b^2c}{\sqrt{ay}} + \frac{y^3}{a+b} + \frac{2}{3}\sqrt{by^3 + cy^3}$. For the Value of $\frac{\dot{x}}{\dot{y}}$ being multiply'd by y, there arises $\frac{2b^2c}{\sqrt{ay}} + \frac{3y^3}{a+b} + \sqrt{by^3 + cy^3}$ or $2b^2ca^{-\frac{1}{2}}y^{-\frac{1}{2}} + \frac{3}{a+b}y^3 + \sqrt{b+c} \times y^{\frac{3}{2}}$. And thence the Value of x results, by dividing by the number of the Dimensions of each Term.

23. And so $\frac{\dot{y}}{\dot{z}} = z^{\frac{2}{3}}$, gives $y = \frac{3}{5}z^{\frac{5}{3}}$. And $\frac{\dot{y}}{\dot{x}} = \frac{ab}{cx^{\frac{1}{3}}}$, gives $y = \frac{3abx^{\frac{2}{3}}}{2c}$. But the Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, gives $y = \frac{a}{0}$. For $\frac{a}{x}$ multiply'd into x makes a, which being divided by the number of Dimensions, which is 0, there arises $\frac{a}{0}$, an infinite Quantity for the Value of y.

24. Wherefore, whenever a like Term shall occur in the Value of $\frac{\dot{y}}{\dot{x}}$, whose Denominator involves the Correlate Quantity of one Dimension only; instead of the Correlate Quantity, substitute the Sum or the Difference between the same and some other given Quantity to be assumed at pleasure. For there will be the same Relation of Flowing, of the Fluents in the Equation so produced, as of the Equation at first proposed; and the infinite Relate Quantity

tity by this means will be diminiſh'd by an infinite part of itſelf, and will become finite, but yet conſiſting of Terms infinite in number.

25. Therefore the Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$ being propoſed, if for x I write $b + x$, aſſuming the Quantity b at pleaſure, there will ariſe $\frac{\dot{y}}{\dot{x}} = \frac{a}{b+x}$; and by Diviſion $\frac{\dot{y}}{\dot{x}} = \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4}$ &c. And now the Rule aforegoing will give $y = \frac{ax}{b} - \frac{ax^2}{2b^2} + \frac{ax^3}{3b^3} - \frac{ax^4}{4b^4}$ &c. for the Relation between x and y.

26. So if you have the Equation $\frac{\dot{y}}{\dot{x}} = \frac{2}{x} + 3 - xx$; becauſe of the Term $\frac{2}{x}$, if you write $1 + x$ for x, there will ariſe $\frac{\dot{y}}{\dot{x}} = \frac{2}{1+x} + 2 - 2x - xx$. Then reducing the Term $\frac{2}{1+x}$ into an infinite Series $+ 2 - 2x + 2x^2 - 2x^3 + 2x^4$, &c. you will have $\frac{\dot{y}}{\dot{x}} = 4 - 4x + x^2 - 2x^3 + 2x^4$, &c. And then according to the Rule $y = 4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{2}{5}x^5$, &c. for the Relation of x and y.

27. And thus if the Equation $\frac{\dot{y}}{\dot{x}} = x^{-\frac{1}{2}} + x^{-1} - x^{\frac{2}{3}}$ were propoſed; becauſe I here obſerve the Term x^{-1} (or $\frac{1}{x}$) to be found, I tranſmute x, by ſubſtituting $1 - x$ for it, and there ariſes $\frac{\dot{y}}{\dot{x}} = \frac{1}{\sqrt{1-x}} + \frac{1}{1-x} - \sqrt{1-x}$. Now the Term $\frac{1}{1-x}$ produces $1 + x + x^2 + x^3$, &c. and the Term $\sqrt{1-x}$ is equivalent to $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$, and therefore $\frac{1}{\sqrt{1-x}}$ or $\frac{1}{1-\frac{1}{2}x-\frac{1}{8}x^2}$, &c. is the ſame as $1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3$, &c. So that when theſe Values are ſubſtituted, I ſhall have $\frac{\dot{y}}{\dot{x}} = 1 + 2x + \frac{3}{2}x^2 + \frac{17}{16}x^3$, &c. And then by the Rule $y = x + x^2 + \frac{1}{2}x^3 + \frac{17}{64}x^4$, &c. And ſo of others.

28. Alſo in other Caſes the Equation may ſometimes be conveniently reduced, by ſuch a Tranſmutation of the flowing Quantity. As if this Equation were propoſed $\frac{\dot{y}}{\dot{x}} = \frac{c^3 x}{c^3 - 3c^2x + 3cx^2 - x^3}$; inſtead of

of x I write $c - x$, and then I ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{c^3 - c^2x}{x^3}$ or $\frac{c^3}{x^3} - \frac{c^2}{x^2}$; and then by the Rule $y = -\frac{c^3}{2xx} + \frac{c^2}{x}$. But the uſe of ſuch Tranſmutations will appear more plainly in what follows.

SOLUTION OF CASE II.

29. PREPARATION. And ſo much for Equations that involve only one Fluent. But when each of them are found in the Equation, firſt it muſt be reduced to the Form preſcribed, by making, that on one ſide may be had the Ratio of the Fluxions, equal to an aggregate of ſimple Terms on the other ſide.

30. And beſides, if in the Equations ſo reduced there be any Fractions denominated by the flowing Quantity, they muſt be freed from thoſe Denominators, by the above-mentioned Tranſmutation of the flowing Quantity.

31. So the Equation $\dot{y}ax - \dot{x}xy - aa\dot{x} = 0$ being propoſed, or $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{x}$; becauſe of the Term $\frac{a}{x}$, I aſſume b at pleaſure, and for x I either write $b + x$, or $b - x$, or $x - b$. As if I ſhould write $b + x$, it will become $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b+x}$. And then the Term $\frac{a}{b+x}$ being converted by Diviſion into an infinite Series, we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4}$, &c.

32. And after the ſame manner the Equation $\frac{\dot{y}}{\dot{x}} = 3y - 2x + \frac{x}{y} - \frac{2y}{xx}$ being propoſed; if, by reaſon of the Terms $\frac{x}{y}$ and $\frac{2y}{xx}$, I write $1 - y$ for y, and $1 - x$ for x, there will ariſe $\frac{\dot{y}}{\dot{x}} = 1 - 3y + 2x + \frac{1-x}{1-y} + \frac{2y-2}{1-2x+x^2}$. But the Term $\frac{1-x}{1-y}$ by infinite Diviſion gives $1 - x + y - xy + y^2 - xy^2 + y^3 - xy^3$, &c. and the Term $\frac{2y-2}{1-2x+xx}$ by a like Diviſion gives $2y - 2 + 4xy - 4x + 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10x^4y - 10x^4$, &c. Therefore $\frac{\dot{y}}{\dot{x}} = -3x + 3xy + y^2 - xy^2 + y^3 - xy^3$, &c. $+ 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10x^4y - 10x^4$, &c.

33.

33. RULE. The Equation being thus prepared, when need requires, diſpoſe the Terms according to the Dimenſions of the flowing Quantities, by ſetting down firſt thoſe that are not affected by the Relate Quantity, then thoſe that are affected by its leaſt Dimenſion, and ſo on. In like manner alſo diſpoſe the Terms in each of theſe Claſſes according to the Dimenſions of the other Correlate Quantity, and thoſe in the firſt Claſs, (or ſuch as are not affected by the Relate Quantity,) write in a collateral order, proceeding towards the right hand, and the reſt in a deſcending Series in the left-hand Column, as the following Diagrams indicate. The work being thus prepared, multiply the firſt or the loweſt of the Terms in the firſt Claſs by the Correlate Quantity, and divide by the number of Dimenſions, and put this in the Quote for the initial Term of the Value of the Relate Quantity. Then ſubſtitute this into the Terms of the Equation that are diſpoſed in the left-hand Column, inſtead of the Relate Quantity, and from the next loweſt Terms you will obtain the ſecond Term of the Quote, after the ſame manner as you obtain'd the firſt. And by repeating the Operation you may continue the Quote as far as you pleaſe. But this will appear plainer by an Example or two.

34. EXAMP. 1. Let the Equation $\frac{\dot{y}}{\dot{x}} = 1 - 3x + y + x^2 + xy$ be propoſed, whoſe Terms $1 - 3x + x^2$, which are not affected by the Relate Quantity y, you ſee diſpos'd collaterally in the up-

	$+1 - 3x + xx$
$+y$	$* + x \quad - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$, &c.
$+xy$	$* \quad * \quad + xx - x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 + \frac{1}{30}x^6$, &c
The Sum	$1 - 2x + xx - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{4}{30}x^5$, &c.
$y =$	$x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, &c.

permoſt Row, and the reſt y and xy in the left-hand Column. And firſt I multiply the initial Term 1 into the Correlate Quantity x, and it makes x, which being divided by the number of Dimenſions 1, I place it in the Quote under-written. Then ſubſtituting this Term inſtead of y in the marginal Terms $+y$ and $+xy$, I have $+x$ and $+xx$, which I write over againſt them to the right hand. Then from the reſt I take the loweſt Terms $-3x$ and $+x$, whoſe aggregate $-2x$ multiply'd into x becomes $-2xx$, and

being divided by the number of Dimenſions 2, gives $-\frac{1}{2}xx$ for the ſecond Term of the Value of y in the Quote. Then this Term being likewiſe aſſumed to compleat the Value of the Marginals $+y$ and $+xy$, there will ariſe alſo $-\frac{1}{2}xx$ and $-\frac{1}{2}x^3$, to be added to the Terms $+x$ and $+\frac{1}{2}xx$ that were before inſerted. Which being done, I again aſſume the next loweſt Terms $+\frac{1}{2}xx$, $-\frac{1}{2}xx$, and $+\frac{1}{2}xx$, which I collect into one Sum $\frac{1}{2}xx$, and thence I derive (as before) the third Term $+\frac{1}{3}x^3$, to be put in the Value of y. Again, taking this Term $\frac{1}{3}x^3$ into the Values of the marginal Terms, from the next loweſt $+\frac{1}{3}x^3$ and $-\frac{1}{2}x^3$ added together, I obtain $-\frac{1}{6}x^4$ for the fourth Term of the Value of y. And ſo on *in infinitum*.

35. EXAMP. 2. In like manner if it were required to determine the Relation of x and y in this Equation, $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, &c. which Series is ſuppoſed to proceed *ad infinitum*; I put 1 in the beginning, and the other Terms in the left-hand Column, and then purſue the work according to the following Diagram.

	$+1$						
$+\frac{y}{a}$	$*$	$+\frac{x}{a}$	$+\frac{x^2}{2a^2}$	$+\frac{x^3}{2a^3}$	$+\frac{x^4}{2a^4}$	$+\frac{x^5}{2a^5}$,	&c.
$+\frac{xy}{a^2}$	$*$	$*$	$\frac{x^2}{a^2}$	$+\frac{x^3}{2a^3}$	$+\frac{x^4}{2a^4}$	$+\frac{x^5}{2a^5}$,	&c.
$+\frac{x^2y}{a^3}$	$*$	$*$	$*$	$+\frac{x^3}{a^3}$	$+\frac{x^4}{2a^4}$	$+\frac{x^5}{2a^5}$,	&c.
$+\frac{x^3y}{a^4}$	$*$	$*$	$*$	$*$	$+\frac{x^4}{a^4}$	$+\frac{x^5}{2a^5}$,	&c.
$+\frac{x^4y}{a^5}$	$*$	$*$	$*$	$*$	$*$	$+\frac{x^5}{a^5}$,	&c.
&c.							
Sum	$1+$	$\frac{x}{a}+$	$\frac{3x^2}{2a^2}+$	$\frac{2x^3}{a^3}+$	$\frac{5x^4}{2a^4}+$	$\frac{3x^5}{a^5}$,	&c.
$y =$	$x+$	$\frac{x^2}{2a}+$	$\frac{x^3}{2a^2}+$	$\frac{x^4}{2a^3}+$	$\frac{x^5}{2a^4}+$	$\frac{x^6}{2a^5}$,	&c.

36. As I here propoſed to extract the Value of y as far as ſix Dimenſions of x only; for that reaſon I omit all the Terms in the Operation which I foreſee will contribute nothing to my purpoſe, as is intimated by the Mark, *&c.* which I have ſubjoin'd to the Series that are cut off.

37. EXAMP. 3. In like manner if this Equation were propoſed $\frac{y}{x} = -3x + 3xy + y^2 - xy^2 + y^3 - xy^3 + y^4 - xy^4$, &c. $+6x^2y - 6x^2 + 8x^3y - 8x^3 + 10xy^4 - 10x^4$, &c. and it is intended to extract the Value of y as far as ſeven Dimenſions of x. I place the Terms in order, according to the following Diagram, and I work as before, only with this exception, that ſince in the left-hand Column y is not only of one, but alſo of two and three Dimenſions; (or of more than three, if I intended to produce the Value of y beyond the degree of x^7,) I ſubjoin the ſecond and third Powers of the Value of y, ſo far gradually produced, that when they are ſubſtituted by degrees to the right-hand, in the Values of the Marginals

	$-3x$	$-6x^2$	$-8x^3$	$-10x^4$	$-12x^5$	$-14x^6$, &c.
$+3xy$	*	*	$-\frac{9}{2}x^3$	$-6x^4$	$-\frac{75}{5}x^5$	$-\frac{273}{20}x^6$, &c.
$+6x^2y$	*	*	*	$-9x^4$	$-12x^5$	$-\frac{75}{4}x^6$, &c.
$+8x^3y$	*	*	*	*	$-12x^5$	$-16x^6$, &c.
$+10x^4y$ &c.	*	*	*	*	*	$-15x^6$, &c.
$+y^2$	*	*	*	$+\frac{9}{4}x^4$	$+6x^5$	$+\frac{107}{8}x^6$, &c.
$-xy^2$ &c.	*	*	*	*	$-\frac{9}{4}x^5$	$-6x^6$, &c.
$+y^3$	*	*	*	*	*	$-\frac{27}{8}x^6$, &c.
Sum	$-3x$	$-6x^2$	$-\frac{25}{2}x^3$	$-\frac{91}{4}x^4$	$-\frac{333}{8}x^5$	$-\frac{367}{5}x^6$, &c.
$y =$	$-\frac{3}{2}x^2$	$-2x^3$	$-\frac{25}{8}x^4$	$-\frac{91}{20}x^5$	$-\frac{111}{16}x^6$	$-\frac{367}{35}x^7$, &c.
$y^2 =$	$+\frac{9}{4}x^4$	$+6x^5$	$+\frac{107}{8}x^6$, &c.			
$y^3 =$	$-\frac{27}{8}x^6$, &c.					

to the left, Terms may ariſe of ſo many Dimenſions as I obſerve to be required for the following Operation. And by this Method there ariſes at length $y = -\frac{3}{2}x^2 - 6x^3 - \frac{25}{8}x^4$, &c. which is the

Equation required. But whereas this Value is negative, it appears that one of the Quantities x or y decreaſes, while the other increaſes. And the ſame thing is alſo to be concluded, when one of the Fluxions is affirmative, and the other negative.

38. EXAMP. 4. You may proceed in like manner to reſolve the Equation, when the Relate Quantity is affected with fractional Dimenſions. As if it were propoſed to extract the Value of x from this Equation, $\frac{\dot{x}}{\dot{y}} = \frac{1}{2}y - 4y^2 + 2yx^{\frac{1}{2}} - \frac{4}{5}x^2 + 7y^{\frac{5}{2}} + 2y^5$, in

	$+\frac{1}{2}y$	*	$-4y^2+7y^{\frac{5}{2}}+2y^3$			
$2yx^{\frac{1}{2}}$	*	*	$+y^2$	*	$-2y^3+4y^{\frac{7}{2}}-2y^4$, &c.	
$-\frac{4}{5}x^2$	*	*	*	*	*	$-\frac{1}{10}y^4$, &c.
Sum	$+\frac{1}{2}y$	*	$-3y^2+7y^{\frac{5}{2}}$	*	$+4y^{\frac{7}{2}}-\frac{41}{10}y^4$, &c.	
$x =$	$+\frac{1}{4}y^2-y^3+2y^{\frac{7}{2}}$	*	$+\frac{8}{9}y^{\frac{9}{2}}-\frac{41}{100}y^5$, &c.			
$x^{\frac{1}{2}} =$	$+\frac{1}{2}y-y^2+2y^{\frac{5}{2}}-y^3$, &c.					
$x^2 =$	$\frac{1}{16}y^4$, &c.					

which x in the Term $2yx^{\frac{1}{2}}$ (or $2y\sqrt{x}$) is affected with the Fractional Dimenſion $\frac{1}{2}$. From the Value of x I derive by degrees the Value of $x^{\frac{1}{2}}$, (that is, by extracting its ſquare-Root,) as may be obſerved in the lower part of this Diagram; that it may be inſerted and transfer'd gradually into the Value of the marginal Term $2yx^{\frac{1}{2}}$. And ſo at laſt I ſhall have the Equation $x = \frac{1}{4}y^2 - y^3 + 2y^{\frac{7}{2}} + \frac{8}{9}y^{\frac{9}{2}} - \frac{41}{100}y^5$, &c. by which x is expreſs'd indefinitely in reſpect of y. And thus you may operate in any other caſe whatſoever.

39. I ſaid before, that theſe Solutions may be perform'd by an infinite variety of ways. This may be done if you aſſume at pleaſure not only the initial quantity of the upper Series, but any other given quantity for the firſt Term of the Quote, and then you may proceed as before. Thus in the firſt of the preceding Examples, if you aſſume 1 for the firſt Term of the Value of y, and ſubſtitute it for y in the marginal Terms $+y$ and $+xy$, and purſue the reſt of the Operation as before, (of which I have here given a

+ 1

	$+1-3x+xx$
$+y$	$+1+2x \quad * \quad +x^3+\frac{1}{4}x^4$, &c.
$+xy$	$\quad *+x+2x^2 \quad * \quad +x^4$, &c.
Sum	$+2 \quad * \quad +3x^2+x^3+\frac{5}{4}x^4$, &c.
$y=$	$1+2x \quad * \quad +x^3+\frac{1}{4}x^4+\frac{1}{4}x^5$, &c.

Specimen,) another Value of y will ariſe, $1+2x+x^3+\frac{1}{4}x^4$, &c. And thus another and another Value may be produced, by aſſuming 2, or 3, or any other number for its firſt Term. Or if you make uſe of any Symbol, as a, to repreſent the firſt Term indefinitely, by the ſame method of Operation, (which I ſhall here ſet down,) you will find $y=a+x+ax-xx+axx+\frac{1}{3}x^3+\frac{2}{3}ax^3$, &c. which being found, for a you may ſubſtitute 1, 2, 0, $\frac{1}{2}$, or any other Number, and thereby obtain the Relation between x and y an infinite variety of ways.

	$+1-3x+xx$
$+y$	$+a+x \quad -xx \quad +\frac{1}{3}x^3$, &c.
	$\quad +ax+ax^2+\frac{2}{3}ax^3$, &c.
$+xy$	$\quad *+ax+x^2 \quad -x^3$, &c.
	$\quad +ax^2+ax^3$, &c.
Sum	$+1-2x+x^2 \quad -\frac{1}{3}x^3$, &c.
	$+a+2ax+2ax^2+\frac{5}{3}ax^3$, &c.
$y=$	$a+x \quad -x^2 \quad +\frac{1}{3}x^3 \quad -\frac{1}{6}x^4$, &c.
	$\quad +ax+ax^2+\frac{2}{3}ax^3+\frac{5}{12}ax^4$, &c.

40. And it is to be obſerved, that when the Quantity to be extracted is affected with a Fractional Dimenſion, (as you ſee in the fourth of the preceding Examples,) then it is convenient to take Unity, or ſome other proper Number, for its firſt Term. And indeed this is neceſſary, when to obtain the Value of that fractional Dimenſion, the Root cannot otherwiſe be extracted, becauſe of the negative Sign; as alſo when there are no Terms to be diſpoſed in the firſt or capital Claſs, from which that initial Term may be deduced.

41.

41. And thus at laſt I have compleated this moſt troubleſome and of all others moſt difficult Problem, when only two flowing Quantities, together with their Fluxions, are comprehended in an Equation. But beſides this general Method, in which I have taken in all the Difficulties, there are others which are generally ſhorter, by which the Work may often be eaſed; to give ſome Specimens of which, *ex abundanti*, perhaps will not be diſagreeable to the Reader.

42. I. If it happen that the Quantity to be reſolved has in ſome places negative Dimenſions, it is not of abſolute neceſſity that therefore the Equation ſhould be reduced to another form. For thus the Equation $y = \frac{1}{y} - xx$ being propoſed, where y is of one negative Dimenſion, I might indeed reduce it to another Form, as by writing $1 + y$ for y; but the Reſolution will be more expedite as you have it in the following Diagram.

1	* * $- xx$
$\frac{1}{y}$	$1 - x + \frac{3}{2}xx$, &c.
Sum	$1 - x + \frac{1}{2}xx$
$y =$	$-1 + x - \frac{1}{2}xx + \frac{1}{6}x^3$, &c.
$\frac{1}{y} =$	$1 - x + \frac{3}{2}xx$, &c.

43. Here aſſuming 1 for the initial Term of the Value of y, I extract the reſt of the Terms as before, and in the mean time I deduce from thence, by degrees, the Value of $\frac{1}{y}$ by Diviſion, and inſert it in the Value of the marginal Term.

44. II. Neither is it neceſſary that the Dimenſions of the other flowing Quantity ſhould be always affirmative. For from the Equation $y = 3 + 2y - \frac{yy}{x}$, without the preſcribed Reduction of the Term $\frac{yy}{x}$, there will ariſe $y = 3x - \frac{3}{2}xx + 2x^3$, &c.

45. And from the Equation $y = -y + \frac{1}{x} - \frac{1}{xx}$, the Value of y will be found $y = \frac{1}{x}$, if the Operation be perform'd after the Manner of the following Specimen.

$-\frac{1}{xx}$

	$-\frac{1}{xx}+\frac{1}{x^3}-$
$-y$	$\cdot\ -\frac{1}{x}-$
Sum	$-\frac{1}{xx}$ o
$y=$	$\frac{1}{x}$

46. Here we may obſerve by the way, that among the infinite manners by which any Equation may be reſolved, it often happens that there are ſome, that terminate at a finite Value of the Quantity to be extracted, as in the foregoing Example. And theſe are not difficult to find, if ſome Symbol be aſſumed for the firſt Term. For when the Reſolution is perform'd, then ſome proper Value may be given to that Symbol, which may render the whole finite.

47. III. Again, if the Value of y is to be extracted from this Equation $y = \frac{y}{2x} + 1 - 2x + \frac{1}{4}xx$, it may be done conveniently enough, without any Reduction of the Term $\frac{y}{2x}$, by ſuppoſing (after the manner of Analyſts,) that to be given which is required. Thus for the firſt Term of the Value of y I put $2ex$, taking $2e$ for the numeral Coefficient which is yet unknown. And ſubſtituting $2ex$ inſtead of y, in the marginal Term, there ariſes e, which I write on the right-hand; and the Sum $1+e$ will give $x+ex$ for the ſame firſt Term of the Value of y, which I had firſt repreſented by the Term $2ex$. Therefore I make $2ex = x+ex$, and thence I deduce $e = 1$. So that the firſt Term $2ex$ of the Value of y is $2x$. After the ſame manner I make uſe of the fictitious Term $2fx^2$ to repreſent the ſecond Term of the Value of y, and thence at laſt I derive $-\frac{1}{3}$ for the Value of f, and therefore that ſecond Term is $-\frac{1}{3}xx$. And ſo the fictitious Coefficient g in the third Term will give $\frac{1}{10}$, and b in the fourth Term will be o. Wherefore ſince there are no other Terms remaining, I conclude the work is finiſh'd, and that the Value of y is exactly $2x - \frac{1}{5}x^2 + \frac{1}{2}x^3$. See the Operation in the following Diagram.

	$1 - 2x + \frac{1}{2}xx$
$\frac{y}{2x}$	$e + fx + gxx + hx^3$
Sum	$+1 - 2x + \frac{1}{2}xx$ $+e + fx + gx^2 + hx^3$
Hypothetically $y =$	$2ex + 2fx^2 + 2gx^3 + 2hx^4$
	$\parallel \quad \parallel \quad \parallel \quad \parallel$
Conſequentially $y =$	$+x - x^2 + \frac{1}{6}x^3 + \frac{1}{4}hx^4$ $+cx + \frac{1}{2}fx^2 + \frac{1}{3}gx^3$
Real Value $y =$	$2x - \frac{4}{3}x^2 + \frac{1}{3}x^3$

48. Much after the ſame manner, if it were $\dot{y} = \frac{3y}{4x}$; ſuppoſe $y = ex^s$, where e denotes the unknown Coefficient, and s the number of Dimenſions, which is alſo unknown. And ex^s being ſubſtituted for y, there will ariſe $\dot{y} = \frac{3ex^{s-1}}{4}$, and thence again $y = \frac{3ex^s}{4s}$. Compare theſe two Values of y, and you will find $\frac{3e}{4s} = e$, and therefore $s = \frac{3}{4}$, and e will be indefinite. Therefore aſſuming e at pleaſure, you will have $y = ex^{\frac{3}{4}}$.

49. IV. Sometimes alſo the Operation may be begun from the higheſt Dimenſion of the equable Quantity, and continually proceed to the lower Powers. As if this Equation were given, $\dot{y} = \frac{y}{xx} + \frac{1}{xx} + 3 + 2x - \frac{4}{x}$, and we would begin from the higheſt Term $2x$, by diſpoſing the capital Series in an order contrary to the foregoing; there will ariſe at laſt $y = xx + 4x - \frac{1}{x}$, &c. as may be ſeen in the form of working here ſet down.

	$+2x + 3 - \frac{4}{x} + \frac{1}{xx}$
$+\frac{y}{xx}$	$* + 1 + \frac{4}{x} \quad * - \frac{1}{x^3} + \frac{1}{2x^4}$, &c.
Sum	$+2x + 4 \quad * + \frac{1}{xx} - \frac{1}{x^3} + \frac{1}{2x^4}$, &c.
$y =$	$x^2 + 4x \quad * - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3}$, &c.

50.

50. And here it may be obſerved by the way, that as the Operation proceeded, I might have inſerted any given Quantity between the Terms $4x$ and $-\frac{1}{x}$, for the intermediate Term that is deficient, and ſo the Value of y might have been exhibited an infinite variety of ways.

51. V. If there are beſides any fractional Indices of the Dimenſions of the Relate Quantity, they may be reduced to Integers by ſuppoſing that Quantity, which is affected by its fractional Dimenſion, to be equal to any third Fluent; and then by ſubſtituting that Quantity, as alſo its Fluxion, ariſing from that fictitious Equation, inſtead of the Relate Quantity and its Fluxion.

52. As if the Equation $\dot{y} = 3x\dot{y}^{\frac{2}{3}} + y$ were propoſed, where the Relate Quantity is affected with the fractional Index $\frac{2}{3}$ of its Dimenſion; a Fluent z being aſſumed at pleaſure, ſuppoſe $y^{\frac{1}{3}} = z$, or $y = z^3$; the Relation of the Fluxions, by Prob. 1. will be $\dot{y} = 3z\dot{z}z^2$. Therefore ſubſtituting $3z\dot{z}z^2$ for $\dot{y}$, as alſo z^3 for y, and z^2 for $y^{\frac{2}{3}}$, there will ariſe $3z\dot{z}z^2 = 3xz^2 + z^3$, or $\dot{z} = x + \frac{1}{3}z$, where z performs the office of the Relate Quantity. But after the Value of z is extracted, as $z = \frac{1}{2}x^2 + \frac{x^3}{18} + \frac{x^4}{216} + \frac{x^5}{3240}$, &c. inſtead of z reſtore $y^{\frac{1}{3}}$, and you will have the deſired Relation between x and y; that is, $y^{\frac{1}{3}} = \frac{1}{2}x^2 + \frac{1}{18}x^3 + \frac{1}{216}x^4$, &c. and by Cubing each ſide, $y = \frac{1}{8}x^6 + \frac{1}{24}x^7 + \frac{1}{288}x^8$, &c.

53. In like manner if the Equation $\dot{y} = \sqrt{4y} + \sqrt{xy}$ were given, or $\dot{y} = 2y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}}$; I make $z = y^{\frac{1}{2}}$, or $z\dot{z} = \dot{y}$, and thence by Prob. 1. $2z\dot{z} = \dot{y}$, and by conſequence $2z\dot{z} = 2z + x^{\frac{1}{2}}z$, or $\dot{z} = 1 + \frac{1}{2}x^{\frac{1}{2}}$. Therefore by the firſt Caſe of this 'tis $z = x + \frac{1}{3}x^{\frac{3}{2}}$, or $y^{\frac{1}{2}} = x + \frac{1}{3}x^{\frac{3}{2}}$, then by ſquaring each ſide, $y = xx + \frac{1}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$. But if you ſhould deſire to have the Value of y exhibited an infinite number of ways, make $z = c + x + \frac{1}{3}x^{\frac{3}{2}}$, aſſuming any initial Term c, and it will be $z\dot{z}$, that is $\dot{y}$, $= c^2 + 2cx + \frac{2}{3}cx^{\frac{3}{2}} + x^2 + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$. But perhaps I may ſeem too minute, in treating of ſuch things as will but ſeldom come into practice.

SOLUTION OF CASE III.

54. The Reſolution of the Problem will ſoon be diſpatch'd, when the Equation involves three or more Fluxions of Quantities. For

between any two of thoſe Quantities any Relation may be aſſumed, when it is not determined by the State of the Queſtion, and the Relation of their Fluxions may be found from thence; ſo that either of them, together with its Fluxion, may be exterminated. For which reaſon if there are found the Fluxions of three Quantities, only one Equation need to be aſſumed, two if there be four, and ſo on; that the Equation propos'd may finally be transform'd into another Equation, in which only two Fluxions may be found. And then this Equation being reſolved as before, the Relations of the other Quantities may be diſcover'd.

55. Let the Equation propoſed be $2\dot{x} - \dot{z} + \dot{y}x = 0$; that I may obtain the Relation of the Quantities x, y, and z, whoſe Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$ are contained in the Equation; I form a Relation at pleaſure between any two of them, as x and y, ſuppoſing that $x = y$, or $2y = a + z$, or $x = yy$, &c. But ſuppoſe at preſent $x = yy$, and thence $\dot{x} = 2y\dot{y}$. Therefore writing $2y\dot{y}$ for $\dot{x}$, and $y\dot{y}$ for $\dot{x}$, the Equation propoſed will be transform'd into this: $4y\dot{y} - \dot{z} + y\dot{y}^2 = 0$. And thence the Relation between y and z will ariſe, $2yy + \frac{1}{3}y^3 = z$. In which if x be written for yy, and $x^{\frac{3}{2}}$ for y^3, we ſhall have $2x + \frac{1}{3}x^{\frac{3}{2}} = z$. So that among the infinite ways in which x, y, and z may be related to each other, one of them is here found, which is repreſented by theſe Equations, $x = yy$, $2y^2 + \frac{1}{3}y^3 = z$, and $2x + \frac{1}{3}x^{\frac{3}{2}} = z$.

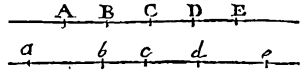
DEMONSTRATION.

56. And thus we have ſolved the Problem, but the Demonſtration is ſtill behind. And in ſo great a variety of matters, that we may not derive it ſynthetically, and with too great perplexity, from its genuine foundations, it may be ſufficient to point it out thus in ſhort, by way of Analyſis. That is, when any Equation is propos'd, after you have finiſh'd the work, you may try whether from the derived Equation you can return back to the Equation propos'd, by Prob. 1. And therefore, the Relation of the Quantities in the derived Equation requires the Relation of the Fluxions in the propoſed Equation, and contrary-wiſe: which was to be ſhewn.

57. So if the Equation propoſed were $\dot{y} = \dot{x}$, the derived Equation will be $y = \frac{1}{2}x^2$; and on the contrary, by Prob. 1. we have $\dot{y} = x\dot{x}$, that is, $\dot{y} = x$, becauſe $\dot{x}$ is ſuppoſed Unity. And thus from

from $\dot{y} = 1 - 3x + y + xx + xy$ is derived $y = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, &c. And thence by Prob. 1. $\dot{y} = 1 - 2x + x^2 - \frac{1}{3}x^3 + \frac{1}{6}x^4 - \frac{1}{15}x^5$, &c. Which two Values of $\dot{y}$ agree with each other, as appears by ſubſtituting $x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$, &c. inſtead of y in the firſt Value.

58. But in the Reduction of Equations I made uſe of an Operation, of which alſo it will be convenient to give ſome account. And that is, the Tranſmutation of a flowing Quantity by its connexion with a given Quantity. Let AE and *ae* be two Lines indefinitely extended each way, along which two moving Things or Points may paſs from afar, and at the ſame time may reach the places A and *a*, B and *b*, C and *c*, D and *d*, &c. and let B be the Point, by its diſtance from which, the Motion of the moving thing or point in AE is eſtimated; ſo that — BA, BC, BD, BE, ſucceſſively, may be the flowing Quantities, when the moving thing is in the places A, C, D, E. Likewiſe let *b* be a like point in the other Line. Then will — BA and — *ba* be contemporaneous Fluents, as alſo BC and *bc*, BD and *bd*, BE and *be*, &c. Now if inſtead of the points B and *b*, be ſubſtituted A and *c*, to which, as at reſt, the Motions are refer'd; then o and — *ca*, AB and — *cb*, AC and o, AD and *cd*, AE and *ce*, will be contemporaneous flowing Quantities. Therefore the flowing Quantities are changed by the Addition and Subtraction of the given Quantities AB and *ac*; but they are not changed as to the Celerity of their Motions, and the mutual reſpect of their Fluxion. For the contemporaneous parts AB and *ab*, BC and *bc*, CD and *cd*, DE and *de*, are of the ſame length in both caſes. And thus in Equations in which theſe Quantities are repreſented, the contemporaneous parts of Quantities are not therefore changed, notwithſtanding their abſolute magnitude may be increaſed or diminiſhed by ſome given Quantity. Hence the thing propoſed is manifeſt: For the only Scope of this Problem is, to determine the contemporaneous Parts, or the contemporary Differences of the abſolute Quantities *v*, *x*, *y*, or *z*, deſcribed with a given Rate of Flowing. And it is all one of what abſolute magnitude thoſe Quantities are, ſo that their contemporary or correſpondent Differences may agree with the propoſed Relation of the Fluxions.



59. The reaſon of this matter may alſo be thus explain'd Algebraically. Let the Equation $\dot{y} = \dot{x}xy$ be propoſed, and ſup-

poſe $x = 1 + z$. Then by Prob. 1. $\dot{x} = \dot{z}$. So that for $y = \dot{x}xy$, may be wrote $\dot{y} = \dot{x}y + \dot{x}zy$. Now ſince $\dot{x} = \dot{z}$, it is plain, that though the Quantities x and z be not of the ſame length, yet that they flow alike in reſpect of y, and that they have equal contemporaneous parts. Why therefore may I not repreſent by the ſame Symbols Quantities that agree in their Rate of Flowing; and to determine their contemporaneous Differences, why may not I uſe $\dot{y} = \dot{x}y + \dot{x}xy$ inſtead of $\dot{y} = \dot{x}xy$?

60. Laſtly it appears plainly in what manner the contemporary parts may be found, from an Equation involving flowing Quantities. Thus if $y = \frac{1}{x} + x$ be the Equation, when $x = 2$, then $y = 2\frac{1}{2}$. But when $x = 3$, then $y = 3\frac{1}{3}$. Therefore while x flows from 2 to 3, y will flow from $2\frac{1}{2}$ to $3\frac{1}{3}$. So that the parts deſcribed in this time are $3 - 2 = 1$, and $3\frac{1}{3} - 2\frac{1}{2} = \frac{5}{6}$.

61. This Foundation being thus laid for what follows, I ſhall now proceed to more particular Problems.

PROB. III.

To determine the Maxima *and* Minima *of Quantities.*

1. When a Quantity is the greateſt or the leaſt that it can be, at that moment it neither flows backwards or forwards. For if it flows forwards, or increaſes, that proves it was leſs, and will preſently be greater than it is. And the contrary if it flows backwards, or decreaſes. Wherefore find its Fluxion, by Prob. 1. and ſuppoſe it to be nothing.

2. EXAMP. 1. If in the Equation $x^3 - ax^2 + axy - y^3 = 0$ the greateſt Value of x be required; find the Relation of the Fluxions of x and y, and you will have $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$. Then making $\dot{x} = 0$, there will remain $-3\dot{y}y^2 + a\dot{y}x = 0$, or $3y^2 = ax$. By the help of this you may exterminate either x or y out of the primary Equation, and by the reſulting Equation you may determine the other, and then both of them by $-3y^2 + ax = 0$.

3. This Operation is the ſame, as if you had multiply'd the Terms of the propoſed Equation by the number of the Dimenſions of the other flowing Quantity y. From whence we may derive the famous

famous Rule of *Huddenius*, that, in order to obtain the greateſt or leaſt Relate Quantity, the Equation muſt be diſpoſed according to the Dimenſions of the Correlate Quantity, and then the Terms are to be multiply'd by any Arithmetical Progreſſion. But ſince neither this Rule, nor any other that I know yet publiſhed, extends to Equations affected with ſurd Quantities, without a previous Reduction; I ſhall give the following Example for that purpoſe.

4. EXAMP. 2. If the greateſt Quantity y in the Equation $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$ be to be determin'd, ſeek the Fluxions of x and y, and there will ariſe the Equation $3\dot{x}x^2 - 2a\dot{y}y + \frac{3ab\dot{y}y^2 + 2b\dot{y}y^3}{a^2+2ay+y^2} - \frac{4a\dot{x}xy + 6\dot{x}x^3 + a\dot{y}x^2}{2\sqrt{ay+xx}} = 0$. And ſince by ſuppoſition $\dot{y} = 0$, omit the Terms multiply'd by $\dot{y}$, (which, to ſhorten the labour, might have been done before, in the Operation,) and divide the reſt by $\dot{x}x$, and there will remain $3x - \frac{2ay+3xx}{\sqrt{ay+xx}} = 0$. When the Reduction is made, there will ariſe $4ay + 3xx = 0$, by help of which you may exterminate either of the quantities x or y out of the propos'd Equation, and then from the reſulting Equation, which will be Cubical, you may extract the Value of the other.

5. From this Problem may be had the Solution of theſe following.

I. *In a given Triangle, or in a Segment of any given Curve, to inſcribe the greateſt Rectangle.*

II. *To draw the greateſt or the leaſt right Line, which can lie between a given Point, and a Curve given in poſition. Or, to draw a Perpendicular to a Curve from a given Point.*

III. *To draw the greateſt or the leaſt right Lines, which paſſing through a given Point, can lie between two others, either right Lines or Curves.*

IV. *From a given Point within a Parabola, to draw a right Line, which ſhall cut the Parabola more obliquely than any other. And to do the ſame in other Curves.*

V. *To determine the Vertices of Curves, their greateſt or leaſt Breadths, the Points in which revolving parts cut each other,* &c.

VI. *To find the Points in Curves, where they have the greateſt or leaſt Curvature.*

VII. *To find the leaſt Angle in a given Ellipſis, in which the Ordinates can cut their Diameters.*

VIII.

VIII. *Of Ellipſes that paſs through four given Points, to determine the greateſt, or that which approaches neareſt to a Circle.*

IX. *To determine ſuch a part of a Spherical Superficies, which can be illuminated, in its farther part, by Light coming from a great diſtance, and which is refracted by the nearer Hemiſphere.*

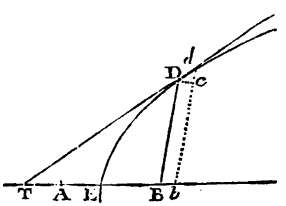
And many other Problems of a like nature may more eaſily be propoſed than reſolved, becauſe of the labour of Computation.

PROB. IV.

To draw Tangents to Curves.

Firſt Manner.

1. Tangents may be variouſly drawn, according to the various Relations of Curves to right Lines. And firſt let BD be a right Line, or Ordinate, in a given Angle to another right Line AB, as a Baſe or Abſciſs, and terminated at the Curve ED. Let this Ordinate move through an indefinitely ſmall Space to the place *bd*, ſo that it may be increaſed by the Moment *cd*, while AB is increaſed by the Moment B*b*, to which D*c* is equal and parallel. Let D*d* be produced till it meets with AB in T, and this Line will touch the Curve in D or *d*; and the Triangles *dc*D, DBT will be ſimilar. So that it is TB : BD :: D*c* (or B*b*) : *cd*.



2. Since therefore the Relation of BD to AB is exhibited by the Equation, by which the nature of the Curve is determined; ſeek for the Relation of the Fluxions, by Prob. 1. Then take TB to BD in the Ratio of the Fluxion of AB to the Fluxion of BD, and TD will touch the Curve in the Point D.

3. Ex. 1. Calling AB $= x$, and BD $= y$, let their Relation be $x^3 - ax^2 + axy - y^3 = 0$. And the Relation of the Fluxions will be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + ay\dot{x} = 0$. So that $\dot{y} : \dot{x} :: 3xx - 2ax + ay : 3y^2 - ax$ :: BD (y) : BT. Therefore BT $= \frac{3y^2 - axy}{3x^2 - 2ax + ay}$. Therefore the Point D being given, and thence DB and AB, or y and x, the length BT will be given, by which the Tangent TD is determined.

4.

4. But this Method of Operation may be thus concinnated. Make the Terms of the propoſed Equation equal to nothing: multiply by the proper number of the Dimenſions of the Ordinate, and put the Reſult in the Numerator: Then multiply the Terms of the ſame Equation by the proper number of the Dimenſions of the Abſciſs, and put the Product divided by the Abſciſs, in the Denominator of the Value of BT. Then take BT towards A, if its Value be affirmative, but the contrary way if that Value be negative.

5. Thus the Equation $\overset{0}{x^3} - \overset{0}{ax^2} + \overset{1}{axy} - \overset{3}{y^3} = 0$, being multiply'd by the upper Numbers, gives $\overset{3}{axy} - \overset{2}{3y^3}$ for the Numerator; and multiply'd by the lower Numbers, and then divided by x, gives $3x^2 - 2ax + ay$ for the Denominator of the Value of BT.

6. Thus the Equation $y^3 - by^2 - cdy + bcd + dxy = 0$, (which denotes a Parabola of the ſecond kind, by help of which *Des Cartes* conſtructed Equations of ſix Dimenſions; ſee his Geometry, *p.* 42. *Amſterd. Ed. An.* 1659.) by Inſpection gives $\frac{3y^3 - 2by^2 - cdy + dxy}{dy}$, or $\frac{3y}{d} - \frac{2b}{d} - c + x = \text{BT}$.

7. And thus $a^2 - \frac{r}{q}x^2 - y^2 = 0$, (which denotes an Ellipſis whoſe Center is A,) gives $\frac{-2y}{-\frac{2r}{q}x}$, or $\frac{qy}{rx} = \text{BT}$. And ſo in others.

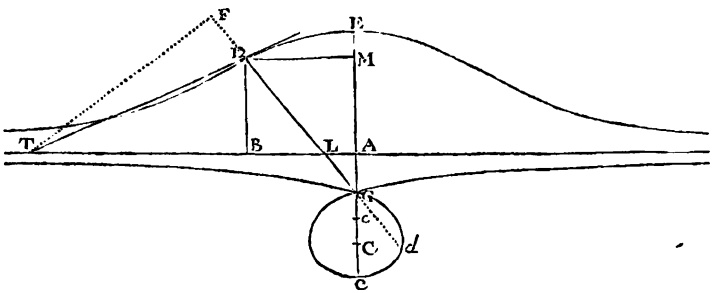
8. And you may take notice, that it matters not of what quantity the Angle of Ordination ABD may be.

9. But as this Rule does not extend to Equations affected by ſurd Quantities, or to mechanical Curves; in theſe Caſes we muſt have recourſe to the fundamental Method.

10. Ex. 2. Let $x^3 - ay^2 + \frac{b^3}{a+y} - xx\sqrt{ay + xx} = 0$ be the Equation expreſſing the Relation between AB and BD; and by Prob. 1. the Relation of the Fluxions will be $3\dot{x}x^2 - 2a\dot{y}y + \frac{3ab\dot{y}^2 + 2b\dot{y}^3}{aa + 2ay + yy} - \frac{4a\dot{x}xy - 6\dot{x}x^3 - a\dot{y}x^2}{2\sqrt{ay + xx}} = 0$. Therefore it will be $3xx \frac{-4axy - 6x^3}{2\sqrt{ay + xx}} : 2ay \frac{-3ab\dot{y} + 2b\dot{y}^3}{aa + 2ay + yy} + \frac{a\dot{x}x}{2\sqrt{ay + xx}} :: (\dot{y} : \dot{x} ::)$ BD : BT.

11.

11. EX. 3. Let ED be the Conchoid of *Nicomedes*, deſcribed with the Pole G, the Aſymptote AT, and the Diſtance LD; and let



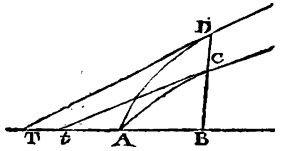
GA $= b$, LD $= c$, AB $= x$, and BD $= y$. And becauſe of ſimilar Triangles DBL and DMG, it will be LB : BD :: DM : MG; that is, $\sqrt{cc-yy} : y :: x : b+y$, and therefore $b+y\sqrt{cc-yy} = yx$. Having got this Equation, I ſuppoſe $\sqrt{cc-yy} = z$, and thus I ſhall have two Equations $bz + yz = yx$, and $zz = cc - yy$. By the help of theſe I find the Fluxions of the Quantities x, y, and z, by Prob. 1. From the firſt ariſes $\dot{b}z + \dot{y}z + y\dot{z} = \dot{y}x + x\dot{y}$, and from the ſecond $2z\dot{z} = -2y\dot{y}$, or $z\dot{z} + y\dot{y} = 0$. Out of theſe if we exterminate $\dot{z}$, there will ariſe $-\frac{b\dot{y}y}{z} - \frac{y\dot{y}^2}{z} + \dot{y}z = \dot{y}x + x\dot{y}$, which being reſolved it will be $\dot{y} : \dot{z} - \frac{b\dot{y}}{z} - \frac{y\dot{y}}{z} - \dot{x} ::$ ($\dot{y} : \dot{x} ::$) BD : BT. But as BD is y, therefore BT $= z - x - \frac{by-yy}{z}$. That is, $-$BT $=$ AL $+ \frac{\text{BD} \times \text{GM}}{\text{BL}}$; where the Sign $-$ prefixt to BT denotes, that the Point T muſt be taken contrary to the Point A.

12. SCHOLIUM. And hence it appears by the bye, how that point of the Conchoid may be found, which ſeparates the concave from the convex part. For when AT is the leaſt poſſible, D will be that point. Therefore make AT $= v$; and ſince BT $= -z + x + \frac{by+yy}{z}$, then $v = -z + 2x + \frac{by+yy}{z}$. Here to ſhorten the work, for x ſubſtitute $\frac{bz+yz}{y}$, which Value is derived from what is before, and it will be $\frac{2bz}{y} + z + \frac{by+yy}{z} = v$. Whence the Fluxions $\dot{v}$, $\dot{y}$, and $\dot{z}$ being found by Prob. 1. and ſuppoſing $\dot{v} = 0$, by

by Prob. 3. there will arise $\frac{2b\dot{z}}{y} - \frac{2by\dot{z}}{yy} + \dot{z} + \frac{b\dot{y} + 2z\dot{y}}{z} - \frac{bz\dot{y} + zz\dot{y}}{zz} = \dot{v} = 0$. Lastly, substituting in this $\frac{-yy}{z}$ for $\dot{z}$, and $cc - yy$ for zz, (which values of $\dot{z}$ and zz are had from what goes before,) and making a due Reduction, you will have $y^3 + 3by^2 - 2bc^2 = 0$. By the Construction of which Equation y or AM, will be given. Then thro' M drawing MD parallel to AB, it will fall upon the Point D of contrary Flexure.

13. Now if the Curve be Mechanical whose Tangent is to be drawn, the Fluxions of the Quantities are to be found, as in Examp. 5. of Prob. 1. and then the rest is to be perform'd as before.

14. Ex. 4. Let AC and AD be two Curves, which are cut in the Points C and D by the right Line BCD, apply'd to the Abscis AB in a given Angle. Let AB $= x$, BD $= y$, and $\frac{\text{Area ACB}}{1} = z$. Then (by Prob. 1. Preparat. to Examp. 5.) it will be $\dot{z} = \dot{x} \times$ BC.



15. Now let AC be a Circle, or any known Curve; and to determine the other Curve AD, let any Equation be proposed, in which z is involved, as $zz + axz = y^4$. Then by Prob. 1. $2z\dot{z} + a\dot{x}z + ax\dot{z} = 4\dot{y}y^3$. And writing $\dot{x} \times$ BC for $\dot{z}$, it will be $2\dot{x}z \times$ BC $+ a\dot{x}x \times$ BC $+ ax\dot{z} = 4\dot{y}y^3$. Therefore $2z \times$ BC $+ ax \times$ BC $+ az : 4y^3 :: (\dot{y} : \dot{x} ::)$ BD : BT. So that if the nature of the Curve AC be given, the Ordinate BC, and the Area ACB or z; the Point T will be given, through which the Tangent DT will pass.

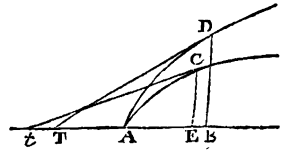
16. After the same manner, if $3z = 2y$ be the Equation to the Curve AD; 'twill be $(3\dot{z})\ 3\dot{x} \times \text{BC} = 2\dot{y}$. So that 3BC : z :: $(\dot{y} : \dot{x} ::)$ BD : BT. And so in others.

17. Ex. 5. Let AB $= x$, BD $= y$, as before, and let the length of any Curve AC be z. And drawing a Tangent to it, as Ct, 'twill be Bt : Ct :: $\dot{x} : \dot{z}$, or $\dot{z} = \frac{\dot{x} \times Ct}{Bt}$.

18. Now for determining the other Curve AD, whose Tangent is to be drawn, let there be given any Equation in which z is involved, suppose $z = y$. Then it will be $\dot{z} = \dot{y}$, so that Ct : Bt :: $(\dot{y} : \dot{x} ::)$: BD : BT. But the Point T being found, the Tangent DT may be drawn.

19. Thus ſuppoſing $xz = yy$, 'twill be $\dot{x}z + \dot{z}x = 2y\dot{y}$; and for $\dot{z}$ writing $\frac{\dot{x} \times Ct}{Bt}$, there will ariſe $\dot{x}z + \frac{\dot{x}x \times Ct}{Bt} = 2y\dot{y}$. Therefore $z + \frac{x \times Ct}{Bt} : 2y :: BD : DT$.

20. Ex. 6. Let AC be a Circle, or any other known Curve, whoſe Tangent is Ct, and let AD be any other Curve whoſe Tangent DT is to be drawn, and let it be defin'd by aſſuming AB = to the Arch AC; and (CE, BD being Ordinates to AB in a given Angle,) let the Relation of BD to CE or AE be expreſs'd by any Equation.



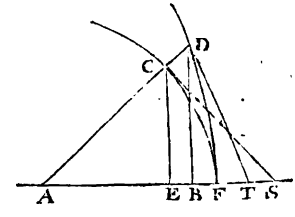
21. Therefore call AB or AC $= x$, BD $= y$, AE $= z$, and CE $= v$. And it is plain that $\dot{v}$, $\dot{x}$, and $\dot{z}$, the Fluxions of CE, AC, and AE, are to each other as CE, Ct, and Et. Therefore $\dot{x} \times \frac{CE}{Ct} = \dot{v}$, and $\dot{x} \times \frac{Et}{Ct} = \dot{z}$.

22. Now let any Equation be given to define the Curve AD, as $y = z$. Then $\dot{y} = \dot{z}$; and therefore $Et : Ct :: (\dot{y} : \dot{x} ::)$ BD : BT.

23. Or let the Equation be $y = z + v - x$, and it will be $\dot{y} = (\dot{v} + \dot{z} - \dot{x} =) \dot{x} \times \frac{CE + Et - Ct}{Ct}$. And therefore $CE + Et - Ct : Ct :: (\dot{y} : \dot{x} ::)$ BD : BT.

24. Or finally, let the Equation be $ayy = v^3$, and it will be $2a\dot{y}y = (3\dot{v}v^2 =) 3\dot{x}v^2 \times \frac{CE}{Ct}$. So that $3v^2 \times CE : 2ay \times Ct ::$ BD : BT.

25. Ex. 7. Let FC be a Circle, which is touched by CS in C; and let FD be a Curve, which is defined by aſſuming any Relation of the Ordinate DB to the Arch FC, which is intercepted by DA drawn to the Center. Then letting fall CE, the Ordinate in the Circle, call AC or AF $= 1$, AB $= x$, DB $= y$, AE $= z$, CE $= v$, CF $= t$; and it will be $t\dot{z} = (\dot{t} \times \frac{CE}{CS} =) \dot{v}$, and $-\dot{t}v = (\dot{t} \times \frac{-ES}{CS} =) \dot{z}$. Here I put $\dot{z}$ negatively, becauſe AE is diminiſh'd while EC is increaſed. And beſides AE : EC :: AB :

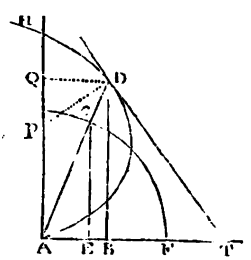


AB : BD, ſo that $\dot{z}y = v\dot{x}$, and thence by Prob. 1. $\dot{z}\dot{y} + \dot{y}\dot{z} = \dot{v}\dot{x} + \dot{x}\dot{v}$. Then exterminating $\dot{v}$, $\dot{z}$, and v, 'tis $\dot{y}\dot{x} - \dot{t}\dot{y}^2 - \dot{t}\dot{x}^2 = \dot{x}\dot{y}$.

26. Now let the Curve DF be defined by any Equation, from which the Value of t may be derived, to be ſubſtituted here. Suppoſe let $t = y$, (an Equation to the firſt Quadratrix,) and by Prob. 1. it will be $\dot{t} = \dot{y}$, ſo that $\dot{y}\dot{x} - \dot{y}\dot{y}^2 - \dot{y}\dot{x}^2 = \dot{x}\dot{y}$. Whence $y : xx + yy - x :: (\dot{y} : -\dot{x} ::)$ BD (y) : BT. Therefore BT $= x^2 + y^2 - x$; and AT $= xx + yy = \frac{AD^2}{AF}$.

27. After the ſame manner, if it is $tt = by$, there will ariſe $2\dot{t}\dot{t} = b\dot{y}$; and thence AT $= \frac{b}{2t} \times \frac{AD^2}{AF}$. And ſo of others.

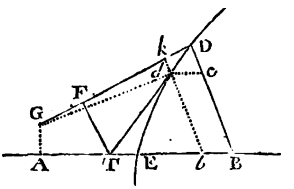
28. Ex. 8. Now if AD be taken equal to the Arch FC, the Curve ADH being then the Spiral of *Archimedes*; the ſame names of the Lines ſtill remaining as were put afore: Becauſe of the right Angle ABD 'tis $xx + yy = tt$, and therefore (by Prob. 1.) $\dot{x}x + \dot{y}y = \dot{t}t$. 'Tis alſo AD : AC :: DB : CE, ſo that $tv = y$, and thence (by Prob. 1.) $\dot{t}v + t\dot{v} = \dot{y}$. Laſtly, the Fluxion of the Arch FC is to the Fluxion of the right Line CE, as AC to AE, or as AD to AB, that is, $\dot{t} : \dot{v} :: t : x$, and thence $\dot{t}x = v\dot{t}$. Compare the Equations now found, and you will ſee 'tis $\dot{t}v + \dot{t}x = \dot{y}$, and thence $\dot{x}x + \dot{y}y = (\dot{t}t =) \frac{y\dot{t}}{v+x}$. And therefore compleating the Parallelogram ABDQ, if you make QD : QP :: (BD : BT :: $\dot{y} : -\dot{x} ::$) $x : y - \frac{t}{v+x}$; that is, if you take AP $= \frac{t}{v+x}$, PD will be perpendicular to the Spiral.



29. And from hence (I imagine) it will be ſufficiently manifeſt, by what methods the Tangents of all ſorts of Curves are to be drawn. However it may not be foreign from the purpoſe, if I alſo ſhew how the Problem may be perform'd, when the Curves are refer'd to right Lines, after any other manner whatever: So that having the choice of ſeveral Methods, the eaſieſt and moſt ſimple may always be uſed.

Second Manner.

30. Let D be a point in the Curve, from which the Subtenſe DG is drawn to a given Point G, and let DB be an Ordinate in any given Angle to the Abſciſs AB. Now let the Point D flow for an infinitely ſmall ſpace D*d* in the Curve, and in GD let G*k* be taken equal to G*d*, and let the Parallelogram *dc*B*b* be compleated. Then D*k* and D*c* will be the contemporary Moments of GD and BD, by which they are diminiſh'd while D is transfer'd to *d*. Now let the right Line D*d* be produced, till it meets with AB in T, and from the Point T to the Subtenſe GD let fall the perpendicular TF, and then the Trapezia D*cdk* and DBTF will be like; and therefore DB : DF :: D*c* : D*k*.



31. Since then the Relation of BD to GD is exhibited by the Equation for determining the Curve; find the Relation of the Fluxions, and take FD to DB in the Ratio of the Fluxion of GD to the Fluxion of BD. Then from F raiſe the perpendicular FT, which may meet with AB in T, and DT being drawn will touch the Curve in D. But DT muſt be taken towards G, if it be affirmative, and the contrary way if negative.

32. Ex. 1. Call GD $= x$, and BD $= y$, and let their Relation be $x^3 - ax^2 + axy - y^3 = 0$. Then the Relation of the Fluxions will be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + ay\dot{x} - 3\dot{y}y^2 = 0$. Therefore $3\dot{x}x - 2a\dot{x}x + a\dot{y}y : 3\dot{y}y - a\dot{x}x :: (\dot{y} : \dot{x} ::)$ DB $(\dot{y})$: DF. So that DF $= \frac{3y^2 - ay}{3x^2 - 2ax + ay}$. Then any Point D in the Curve being given, and thence BD and GD or y and x, the Point F will be given alſo. From whence if the Perpendicular FT be raiſed, from its concourſe T with the Abſciſs AB, the Tangent DT may be drawn.

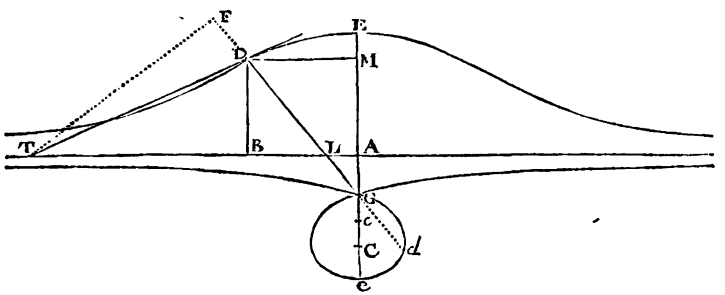
33. And hence it appears, that a Rule might be derived here, as well as in the former Caſe. For having diſpoſed all the Terms of the given Equation on one ſide, multiply by the Dimenſions of the Ordinate y, and place the reſult in the Numerator of a Fraction. Then multiply its Terms ſeverally by the Dimenſions of the Subtenſe x, and dividing the reſult by that Subtenſe x, place the Quotient in the Denominator of the Value of DF. And take the ſame Line DF towards G if it be affirmative, otherwiſe the contrary way. Where you

you may obſerve, that it is no matter how far diſtant the Point G is from the Abſciſs AB, or if it be at all diſtant, nor what is the Angle of Ordination ABD.

34. Let the Equation be as before $x^3 - ax^2 + axy - y^3 = 0$; it gives immediately $axy - 3y^3$ for the Numerator, and $3x^2 - 2ax + ay$ for the Denominator of the Value of DF.

35. Let alſo $a + \frac{b}{a}x - y = 0$, (which Equation is to a Conick Section,) it gives $-y$ for the Numerator, and $\frac{b}{a}$ for the Denominator of the Value of DF, which therefore will be $-\frac{ay}{b}$.

36. And thus in the Conchoid, (wherein theſe things will be perform'd more expeditiouſly than before,) putting GA $= b$,



LD $= c$, GD $= x$, and BD $= y$, it will be BD (y) : DL (c) :: GA (b) : GL ($x - c$). Therefore $xy - cy = cb$, or $xy - cy - cb = 0$. This Equation according to the Rule gives $\frac{xy - cy}{y}$, that is, $x - c =$ DF. Therefore prolong GD to F, ſo that DF $=$ LG, and at F raiſe the perpendicular FT meeting the Aſymptote AB in T, and DT being drawn will touch the Conchoid.

37. But when compound or ſurd Quantities are found in the Equation, you muſt have recourſe to the general Method, except you ſhould chuſe rather to reduce the Equation.

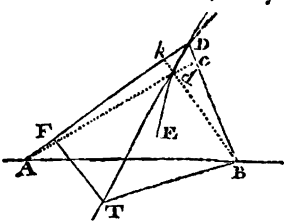
38. Ex. 2. If the Equation $b + y \times \sqrt{cc - yy} = yx$, were given for the Relation between GD and BD; (ſee the foregoing Figure, p. 52.) find the Relation of the Fluxions by Prob. 1. As ſuppoſing $\sqrt{cc - yy} = z$, you will have the Equations $bz + yz = yx$, and $cc - yy = zz$, and thence the Relation of the Fluxions $bz + y\dot{z} + \dot{y}z = y\dot{x} + \dot{y}x$, and $-2\dot{y}y = 2\dot{z}z$. And now $\dot{z}$ and z being

exterminated, there will ariſe $\dot{y}\sqrt{cc-yy} - \frac{b\dot{y}y+\dot{v}v^2}{\sqrt{cc-yy}} - \dot{y}x = \dot{x}y$.

Therefore $y : \sqrt{cc-yy} - \frac{by+v}{\sqrt{cc-yy}} - x :: (\dot{y} : \dot{x} ::)$ BD (y) : DF.

Third Manner.

39. Moreover, if the Curve be refer'd to two Subtenſes AD and BD, which being drawn from two given Points A and B, may meet at the Curve: Conceive that Point D to flow on through an infinitely little Space D*d* in the Curve; and in AD and BD take A*k* = A*d*, and B*c* = B*d*; and then *k*D and *c*D will be contemporaneous Moments of the Lines AD and BD. Take therefore DF to BD in the Ratio of the Moment D*k* to the Moment D*c*, (that is, in the Ratio of the Fluxion of the Line AD to the Fluxion of the Line BD,) and draw BT, FT perpendicular to BD, AD, meeting in T. Then the Trapezia DFTB and D*kdc* will be ſimilar, and therefore the Diagonal DT will touch the Curve.

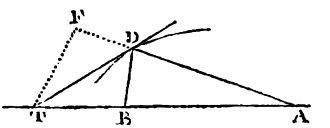


40. Therefore from the Equation, by which the Relation is defined between AD and BD, find the Relation of the Fluxions by Prob. 1. and take FD to BD in the ſame Ratio.

41. EXAMP. Suppoſing AD = x, and BD = y, let their Relation be $a + \frac{ex}{d} - y = 0$. This Equation is to the Ellipſes of the ſecond Order, whoſe Properties for Refracting of Light are ſhewn by *Des Cartes*, in the ſecond Book of his Geometry. Then the Relation of the Fluxions will be $\frac{e\dot{x}}{d} - \dot{y} = 0$. 'Tis therefore $e : d :: (\dot{y} : \dot{x} ::)$ BD : DF.

42. And for the ſame reaſon if $a - \frac{ex}{d} - y = 0$, 'twill be $e : -d ::$ BD : DF. In the firſt Caſe take DF towards A, and contrary-wiſe in the other caſe.

43. COROL. 1. Hence if $d = e$, (in which caſe the Curve becomes a Conick Section,) 'twill be DF = DB. And therefore the Triangles DFT and DBT being equal, the Angle FDB will be biſected by the Tangent.

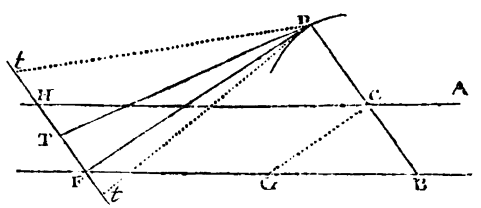


44. COROL. 2. And hence alſo thoſe things will be manifeſt of themſelves, which are demonſtrated, in a very prolix manner, by *Des Cartes* concerning the Refraction of theſe Curves. For as much as DF and DB, (which are in the given Ratio of d to e,) in reſpect of the Radius DT, are the Sines of the Angles DTF and DTB, that is, of the Ray of Incidence AD upon the Surface of the Curve, and of its Reflexion or Refraction DB. And there is a like reaſoning concerning the Refractions of the Conick Sections, ſuppoſing that either of the Points A or B be conceived to be at an infinite diſtance.

45. It would be eaſy to modify this Rule in the manner of the foregoing, and to give more Examples of it: As alſo when Curves are refer'd to Right lines after any other manner, and cannot commodiouſly be reduced to the foregoing, it will be very eaſy to find out other Methods in imitation of theſe, as occaſion ſhall require.

Fourth Manner.

46. As if the right Line BCD ſhould revolve about a given Point B, and one of its Points D ſhould deſcribe a Curve, and another Point C ſhould be the interſection of the right Line BCD, with another right Line AC given in poſition. Then the Relation of BC and BD being expreſs'd by any Equation; draw BF parallel to AC, ſo as to meet DF, perpendicular to BD, in F. Alſo erect FT perpendicular to DF; and take FT in the ſame Ratio to BC, that the Fluxion of BD has to the Fluxion of BC. Then DT being drawn will touch the Curve.



Fifth Manner.

47. But if the Point A being given, the Equation ſhould expreſs the Relation between AC and BD; draw CG parallel to DF, and take FT in the ſame Ratio to BG, that the Fluxion of BD has to the Fluxion of AC.

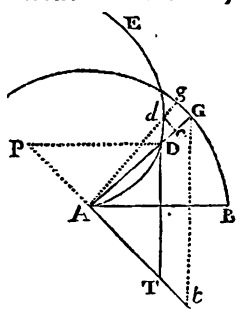
Sixth Manner.

48. Or again, if the Equation expreſſes the Relation between AC and CD; let AC and FT meet in H; and take HT in the ſame Ratio to BG, that the Fluxion of CD has to the Fluxion of AC. And the like in others.

Seventh

Seventh Manner: For Spirals.

49. The Problem is not otherwiſe perform'd, when the Curves are refer'd, not to right Lines, but to other Curve-lines, as is uſual in Mechanick Curves. Let BG be the Circumference of a Circle, in whoſe Semidiameter AG, while it revolves about the Center A, let the Point D be conceived to move any how, ſo as to deſcribe the Spiral ADE. And ſuppoſe D*d* to be an infinitely little part of the Curve thro' which D flows, and in AD take A*c* = A*d*, then *c*D and G*g* will be contemporaneous Moments of the right Line AD and of the Periphery BG. Therefore draw A*t* parallel to *cd*, that is, perpendicular to AD, and let the Tangent DT meet it in T; then it will be *c*D : *cd* :: AD : AT. Alſo let G*t* be parallel to the Tangent DT, and it will be *cd* : G*g* :: (A*d* or AD : AG ::) AT : A*t*.



50. Therefore any Equation being propoſed, by which the Relation is expreſs'd between BG and AD; find the Relation of their Fluxions by Prob. 1. and take A*t* in the ſame Ratio to AD: And then G*t* will be parallel to the Tangent.

51. Ex. 1. Calling BG = x, and AD = y, let their Relation be $x^3 - ax^2 + axy - y^3 = 0$, and by Prob. 1. $3x^2 - 2ax + ay : 3y^2 - ax$ :: ($\dot{y} : \dot{x}$ ::) AD : A*t*. The Point *t* being thus found, draw G*t*, and DT parallel to it, which will touch the Curve.

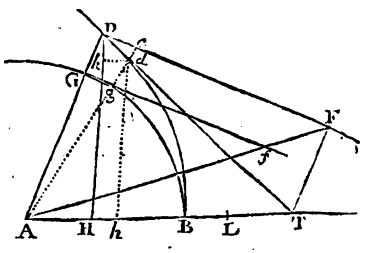
52. Ex. 2. If 'tis $\frac{ax}{b} = y$, (which is the Equation to the Spiral of *Archimedes*,) 'twill be $\frac{a\dot{x}}{b} = \dot{y}$, and therefore $a : b$ :: ($\dot{y} : \dot{x}$ ::) AD : A*t*. Wherefore by the way, if TA be produced to P, that it may be AP : AB :: $a : b$, PD will be perpendicular to the Curve.

53. Ex. 3. If $xx = by$, then $2\dot{x}x = b\dot{y}$, and $2x : b$ :: AD : A*t*. And thus Tangents may be eaſily drawn to any Spirals whatever.

Eighth

Eighth Manner : For Quadratrices.

54. Now if the Curve be ſuch, that any Line AGD, being drawn from the Center A, may meet the Circular Arch in G, and the Curve in D; and if the Relation between the Arch BG, and the right Line DH, which is an Ordinate to the Baſe or Abſciſs AH in a given Angle, be determin'd by any Equation whatever: Conceive the Point D to move in the Curve for an infinitely ſmall Interval to *d*, and the Parallelogram *db*H*k* being compleated, produce A*d* to *c*, ſo that A*c* = AD; then G*g* and D*k* will be contemporaneous Moments of the Arch BG and of the Ordinate DH. Now produce D*d* ſtrait on to T, where it may meet with AB, and from thence let fall the Perpendicular TF on D*c*F. Then the Trapezia D*kdc* and DHTF will be ſimilar; and therefore D*k* : D*c* :: DH : DF. And beſides if G*f* be raiſed perpendicular to AG, and meets AF in *f*; becauſe of the Parallels DF and G*f*, it will be D*c* : G*g* :: DF : G*f*. Therefore *ex æquo*, 'tis D*k* : G*g* :: DH : G*f*, that is, as the Moments or Fluxions of the Lines DH and BG.



55. Therefore by the Equation which expreſſes the Relation of BG to DH, find the Relation of the Fluxions (by Prob. 1.) and in that Ratio take G*f*, the Tangent of the Circle BG, to DH. Draw DF parallel to G*f*, which may meet A*f* produced in F. And at F erect the perpendicular FT, meeting AB in T; and the right Line DT being drawn, will touch the Quadratrix.

56. Ex. 1. Making BG = x, and DH = y, let it be $xx = by$; then (by Prob. 1.) $2x\dot{x} = b\dot{y}$. Therefore $2x : b :: (\dot{y} : \dot{x} ::)$ DH : G*f*; and the Point *f* being found, the reſt will be determin'd as above.

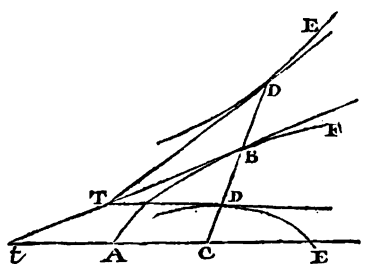
But perhaps this Rule may be thus made ſomething neater: Make $x : y ::$ AB : AL. Then AL : AD :: AD : AT, and then DT will touch the Curve. For becauſe of equal Triangles AFD and ATD, 'tis AD × DF = AT × DH, and therefore AT : AD :: (DF or $\frac{AD}{AG} \times Gf$: DH or $\frac{\dot{y}}{\dot{x}}$ G*f* ::) AD : ($\frac{\dot{y}}{\dot{x}}$ AG or) AL.

57. Ex. 2. Let $x = y$, (which is the Equation to the Quadratrix of the Ancients,) then $\dot{x} = \dot{y}$. Therefore AB : AD :: AD : AT.

58. Ex. 3. Let $axx = y^3$, then $2a\dot{x}x = 3y\dot{y}^2$. Therefore make $3y^2 : 2ax :: (\dot{x} : \dot{y} ::)$ AB : AL. Then AL : AD :: AD : AT. And thus you may determine expeditiouſly the Tangents of any other Quadratrices, howſoever compounded.

Ninth Manner.

59. Laſtly, if ABF be any given Curve, which is touch'd by the right Line B*t*; and a part BD of the right Line BC, (being an Ordinate in any given Angle to the Abſciſs AC,) intercepted between this and another Curve DE, has a Relation to the portion of the Curve AB, which is expreſs'd by any Equation: You may draw a Tangent DT to the other Curve, by taking (in the Tangent of this Curve,) BT in the ſame Ratio to BD, as the Fluxion of the Curve AB hath to the Fluxion of the right Line BD.



60. Ex. 1. Calling AB $= x$, and BD $= y$; let it be $ax = yy$, and therefore $a\dot{x} = 2y\dot{y}$. Then $a : 2y :: (\dot{y} : \dot{x} ::)$ BD : BT.

61. Ex. 2. Let $\frac{a}{b}x = y$, (the Equation to the Trochoid, if ABF be a Circle,) then $\frac{a}{b}\dot{x} = \dot{y}$, and $a : b ::$ BD : BT.

62. And with the ſame eaſe may Tangents be drawn, when the Relation of BD to AC, or to BC, is expreſs'd by any Equation; or when the Curves are refer'd to right Lines, or to any other Curves, after any other manner whatever.

63. There are alſo many other Problems, whoſe Solutions are to be derived from the ſame Principles; ſuch as theſe following.

I. *To find a Point of a Curve, where the Tangent is parallel to the Abſciſs, or to any other right Line given in poſition; or is perpendicular to it, or inclined to it in any given Angle.*

II. *To find the Point where the Tangent is moſt or leaſt inclined to the Abſciſs, or to any other right Line given in poſition. That is, to find the confine of contrary Flexure.* Of this I have already given a Specimen in the Conchoid.

III. *From any given Point without the Perimeter of a Curve, to draw a right Line, which with the Perimeter may make an Angle of Contact,*

Contact, or a right Angle, or any other given Angle. That is, from a given Point, to draw Tangents, or Perpendiculars, or right Lines that ſhall have any other Inclination to a Curve-line.

IV. *From any given Point within a Parabola, to draw a right Line, which may make with the Perimeter the greateſt or leaſt Angle poſſible. And ſo of all Curves whatever.*

V. *To draw a right Line which may touch two Curves given in poſition, or the ſame Curve in two Points, when that can be done.*

VI. *To draw any Curve with given Conditions, which may touch another Curve given in poſition, in a given Point.*

VII. *To determine the Refraction of any Ray of Light, that falls upon any Curve Superficies.*

The Reſolution of theſe, or of any other the like Problems, will not be ſo difficult, abating the tedioufneſs of Computation, as that there is any occaſion to dwell upon them here: And I imagine it may be more agreeable to Geometricians barely to have mention'd them.

PROB. V.

At any given Point of a given Curve, to find the Quantity of Curvature.

1. There are few Problems concerning Curves more elegant than this, or that give a greater Inſight into their nature. In order to its Reſolution, I muſt premiſe theſe following general Conſiderations.

2. I. The ſame Circle has every where the ſame Curvature, and in different Circles it is reciprocally proportional to their Diameters. If the Diameter of any Circle is as little again as the Diameter of another, the Curvature of its Periphery will be as great again. If the Diameter be one-third of the other, the Curvature will be thrice as much, *&c.*

3. II. If a Circle touches any Curve on its concave ſide, in any given Point, and if it be of ſuch magnitude, that no other tangent Circle can be interſcribed in the Angles of Contact near that Point; that Circle will be of the ſame Curvature as the Curve is of, in that Point of Contact. For the Circle that comes between the Curve and another Circle at the Point of Contact, varies leſs from the Curve, and makes a nearer approach to its Curvature, than that other Circle does. And therefore that Circle approaches neareſt to its

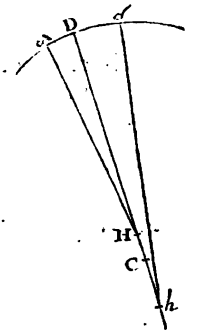
Curvature, between which and the Curve no other Circle can intervene.

4. III. Therefore the Center of Curvature to any Point of a Curve, is the Center of a Circle equally curved. And thus the Radius or Semidiameter of Curvature is part of the Perpendicular to the Curve, which is terminated at that Center.

5. IV. And the proportion of Curvature at different Points will be known from the proportion of Curvature of æqui-curve Circles, or from the reciprocal proportion of the Radii of Curvature.

6. Therefore the Problem is reduced to this, that the Radius, or Center of Curvature may be found.

7. Imagine therefore that at three Points of the Curve ∂, D, and *d*, Perpendiculars are drawn, of which thoſe that are at D and ∂ meet in H, and thoſe that are at D and *d* meet in *h*: And the Point D being in the middle, if there is a greater Curvity at the part D∂ than at D*d*, then DH will be leſs than *dh*. But by how much the Perpendiculars ∂H and *dh* are nearer the intermediate Perpendicular, ſo much the leſs will the diſtance be of the Points H and *h*: And at laſt when the Perpendiculars meet, thoſe Points will coincide. Let them coincide in the Point C, then will C be the Center of Curvature, at the Point D of the Curve, on which the Perpendiculars ſtand; which is manifeſt of itſelf.



8. But there are ſeveral Symptoms or Properties of this Point C, which may be of uſe to its determination.

9. I. That it is the Concourſe of Perpendiculars that are on each ſide at an infinitely little diſtance from DC.

10. II. That the Interſections of Perpendiculars, at any little finite diſtance on each ſide, are ſeparated and divided by it; ſo that thoſe which are on the more curved ſide D∂ ſooner meet at H, and thoſe which are on the other leſs curved ſide D*d* meet more remotely at *h*.

11. III. If DC be conceived to move, while it inſiſts perpendicularly on the Curve, that point of it C, (if you except the motion of approaching to or receding from the Point of Inſiſtence C,) will be leaſt moved, but will be as it were the Center of Motion.

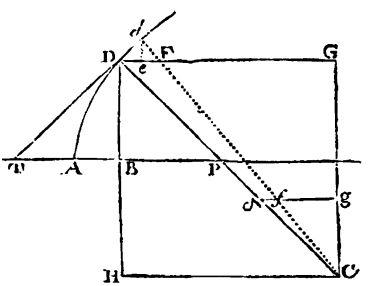
12. IV. If a Circle be deſcribed with the Center C, and the diſtance DC, no other Circle can be deſcribed, that can lie between at the Contact.

13.

13. V. Laſtly, if the Center H or *h* of any other touching Circle approaches by degrees to C the Center of this, till at laſt it coincides with it; then any of the points in which that Circle ſhall cut the Curve, will coincide with the point of Contact D.

14. And each of theſe Properties may ſupply the means of ſolving the Problem different ways: But we ſhall here make choice of the firſt, as being the moſt ſimple.

15. At any Point D of the Curve let DT be a Tangent, DC a Perpendicular, and C the Center of Curvature, as before. And let AB be the Abſciſs, to which let DB be apply'd at right Angles, and which DC meets in P. Draw DG parallel to AB, and CG perpendicular to it, in which take C*g* of any given Magnitude, and draw *g*δ perpendicular to it, which meets DC in δ. Then it will be C*g* : *g*δ :: (TB : BD ::) the Fluxion of the Abſciſs, to the Fluxion of the Ordinate. Likewiſe imagine the Point D to move in the Curve an infinitely little diſtance D*d*, and drawing *de* perpendicular to DG, and C*d* perpendicular to the Curve, let C*d* meet DG in F, and δg in *f*. Then will D*e* be the Momentum of the Abſciſs, *de* the Momentum of the Ordinate, and δf the contemporaneous Momentum of the right Line *g*δ. Therefore DF $= De + \frac{de \times de}{De}$. Having therefore the Ratio's of theſe Moments, or, which is the ſame thing, of their generating Fluxions, you will have the Ratio of CG to the given Line C*g*, (which is the ſame as that of DF to δf,) and thence the Point C will be determined.



16. Therefore let AB $= x$, BD $= y$, C*g* $= 1$, and *g*δ $= z$; then it will be $1 : z :: \dot{x} : \dot{y}$, or $z = \frac{\dot{y}}{\dot{x}}$. Now let the Momentum δf of z be $\dot{z} \times o$, (that is, the Product of the Velocity and of an infinitely ſmall Quantity o,) and therefore the Moments $De = \dot{x} \times o$, $de = \dot{y} \times o$, and thence $DF = \dot{x}o + \frac{\ddot{y}y}{\dot{x}}$. Therefore 'tis C*g* (1) : CG :: (δf : DF ::) $\dot{z}o : \dot{x}o + \frac{\ddot{y}yo}{\dot{x}}$. That is, $CG = \frac{\dot{x}\dot{x} + \ddot{y}y}{\dot{x}\dot{z}}$.

17.

17. And whereas we are at liberty to aſcribe whatever Velocity we pleaſe to the Fluxion of the Abſciſs $\dot{x}$, (to which, as to an equable Fluxion, the reſt may be referr'd;) make $\dot{x} = 1$, and then $\dot{y} = z$, and $CG = \frac{1+zz}{z}$. And thence $DG = \frac{z+z^3}{z}$, and $DC = \frac{\overline{1+zz}\sqrt{1+zz}}{z}$.

18. Therefore any Equation being propoſed, in which the Relation of BD to AB is expreſs'd for defining the Curve; firſt find the Relation betwixt $\dot{x}$ and $\dot{y}$, by Prob. 1. and at the ſame time ſubſtitute 1 for $\dot{x}$, and z for $\dot{y}$. Then from the Equation that ariſes, by the ſame Prob. 1. find the Relation between $\dot{x}$, $\dot{y}$, and $\dot{z}$, and at the ſame time ſubſtitute 1 for $\dot{x}$, and z for $\dot{y}$, as before. And thus by the former operation you will obtain the Value of z, and by the latter you will have the Value of $\dot{z}$; which being obtain'd, produce DB to H, towards the concave part of the Curve, that it may be $DH = \frac{1+zz}{\dot{z}}$, and draw HC parallel to AB, and meeting the Perpendicular DC in C; then will C be the Center of Curvature at the Point D of the Curve. Or ſince it is $1 + zz = \frac{PT}{BT}$, make $DH = \frac{PT}{\dot{z} \times BT}$, or $DC = \frac{\overline{DP}^3}{z \times \overline{DB}^2}$.

19. Ex. 1. Thus the Equation $ax + bx^2 - y^2 = 0$ being propoſed, (which is an Equation to the Hyperbola whoſe Latus rectum is a, and Tranſverſum $\frac{a}{b}$;) there will ariſe (by Prob. 1.) $a + 2bx - 2zy = 0$, (writing 1 for $\dot{x}$, and z for $\dot{y}$ in the reſulting Equation, which otherwiſe would have been $a\dot{x} + 2b\dot{x}x - 2\dot{y}y = 0$;) and hence again there ariſes $2b - 2zz - 2\dot{z}y = 0$, (1 and $\dot{z}$ being again wrote for $\dot{x}$ and $\dot{y}$.) By the firſt we have $z = \frac{a+2bx}{2y}$, and by the latter $\dot{z} = \frac{b-zz}{y}$. Therefore any Point D of the Curve being given, and conſequently x and y, from thence z and $\dot{z}$ will be given, which being known, make $\frac{1+zz}{\dot{z}} = GC$ or DH, and draw HC.

20. As if definitely you make $a = 3$, and $b = 1$, ſo that $3x + xx = yy$ may be the condition of the Hyperbola. And if you aſſume $x = 1$, then $y = 2$, $z = \frac{5}{4}$, $\dot{z} = -\frac{9}{32}$, and $DH = -9\frac{1}{8}$. H being found, raiſe the Perpendicular HC meeting the Perpendicular

cular DC before drawn; or, which is the ſame thing, make HD : HC :: (1 : z ::) 1 : $\frac{5}{4}$. Then draw DC the Radius of Curvature.

21. When you think the Computation will not be too perplex, you may ſubſtitute the indefinite Values of $\dot{z}$ and z into $\frac{1+zz}{\dot{z}}$, the Value of CG. Thus in the preſent Example, by a due Reduction you will have DH $= y + \frac{4y^3 + 4by^3}{aa}$. Yet the Value of DH by Calculation comes out negative, as may be ſeen in the numeral Example. But this only ſhews, that DH muſt be taken towards B; for if it had come out affirmative, it ought to have been drawn the contrary way.

22. COROL. Hence let the Sign prefixt to the Symbol $+b$ be changed, that it may be $ax - bxx - yy = 0$, (an Equation to the Ellipſis,) then DH $= y + \frac{4y^3 - 4by^3}{aa}$.

23. But ſuppoſing $b = 0$, that the Equation may become $ax - yy = 0$, (an Equation to the Parabola,) then DH $= y + \frac{4y^3}{aa}$; and thence DG $= \frac{1}{2}a + 2x$.

24. From theſe ſeveral Expreſſions it may eaſily be concluded, that the Radius of Curvature of any Conick Section is always $\frac{\overline{4DP}^3}{aa}$.

25. Ex. 2. If $x^3 = ay^2 - xy^2$ be propoſed, (which is the Equation to the Ciſſoid of *Diocles*,) by Prob. 1. it will be firſt $3x^2 = 2axy - 2xxy - y^2$; and then $6\dot{x} = 2a\dot{x}y + 2ax\dot{z} - 2\dot{z}y - 2x\dot{z}y - 2xx\dot{z} - 2\dot{z}y$: So that $\dot{z} = \frac{3xx + yy}{2ay - 2xy}$, and $\dot{z} = \frac{3x - azz + 2zv + xzz}{ay - xy}$. Therefore any Point of the Ciſſoid being given, and thence x and y, there will be given alſo z and $\dot{z}$; which being known, make $\frac{1+zz}{\dot{z}} = $ CG.

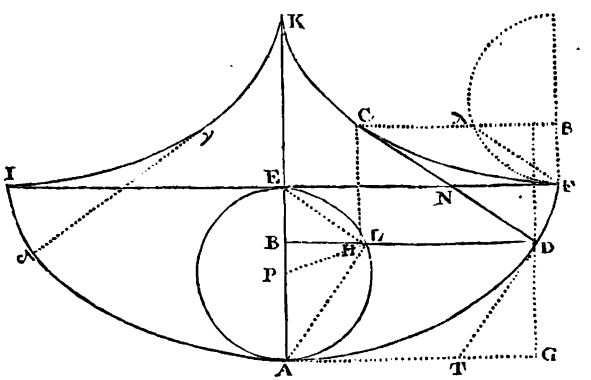
26. Ex. 3. If $\overline{b+y}\sqrt{cc-yy} = xy$ were given, (which is the Equation to the Conchoid, in pag. 48;) make $\sqrt{cc-yy} = v$, and there will ariſe $bv + yv = xy$. Now the firſt of theſe, ($cc - yy = vv$,) will give (by Prob. 1.) $-2y\dot{z} = 2\dot{v}v$, (writing z for y;) and the latter will give $b\dot{v} + y\dot{v} + \dot{z}v = \dot{y} + x\dot{z}$. And from theſe Equations rightly diſpoſed $\dot{v}$ and $\dot{z}$ will be determined. But that $\dot{z}$ may alſo be found; out of the laſt Equation exterminate the Fluxion $\dot{v}$, by ſubſtituting $-\frac{yz}{v}$, and there will ariſe $-\frac{b\dot{z}z}{v} - \frac{yz\dot{z}}{v} + \dot{z}v = \dot{y}$

$= y + xz$, an Equation that comprehends the flowing Quantities, without any of their Fluxions, as the Reſolution of the firſt Problem requires. Hence therefore by Prob. 1. we ſhall have $-\frac{bz^2}{v} - \frac{byz}{v} + \frac{byz\dot{v}}{v\dot{v}} - \frac{2yzz}{v} - \frac{yz\dot{z}}{v} + \frac{yz\dot{z}v}{v\dot{v}} + \dot{z}v + z\dot{v} = 2z + x\dot{z}$. This Equation being reduced, and diſpoſed in order, will give $\dot{z}$. But when z and $\dot{z}$ are known, make $\frac{1+zz}{\dot{z}} = CG$.

27. If we had divided the laſt Equation but one by z, then by Prob. 1. we ſhould have had $-\frac{bz}{v} + \frac{by\dot{v}}{v\dot{v}} - \frac{2yz}{v} + \frac{yy\dot{v}}{v\dot{v}} + \dot{v} = 2 - \frac{y\dot{z}}{z\dot{z}}$; which would have been a more ſimple Equation than the former, for determining $\dot{z}$.

28. I have given this Example, that it may appear, how the operation is to be perform'd in ſurd Equations: But the Curvature of the Conchoid may be thus found a ſhorter way. The parts of the Equation $b+y\sqrt{cc-yy} = xy$ being ſquared, and divided by yy, there ariſes $\frac{b^2c^2}{y^2} + \frac{2bc^2}{y} + c^2 - 2by - y^2 = x^2$, and thence by Prob. 1. $-\frac{2b^2c^2z}{y^3} - \frac{2bc^2z}{y^2} - 2bz - 2yz = 2x$, or $-\frac{b^2c^2}{y^3} - \frac{bc^2}{y^2} - b - y = \frac{x}{z}$. And hence again by Prob. 1. $\frac{3b^2c^2z}{y^4} + \frac{2bc^2z}{y^3} - z = \frac{1}{z} - \frac{xz}{z\dot{z}}$. By the firſt reſult z is determined, and $\dot{z}$ by the latter.

29. Ex. 4. Let ADF be a Trochoid [or Cycloid] belonging to the Circle ALE, whoſe Diameter is AE; and making the Ordinate BD to cut the Circle in L, call AE $= a$, AB $= x$, BD $= y$, BL $= v$, and the Arch AL $= t$, and the Fluxion of the ſame Arch $= \dot{t}$. And firſt (drawing the Semidiameter PL,) the Fluxion of the Baſe or Abſciſs AB will be to the Fluxion of the Arch AL, as BL



to

to PL; that is, $\dot{x}$ or $1 : \dot{t} :: v : \frac{1}{2}a$. And therefore $\frac{a}{2v} = \dot{t}$. Then from the nature of the Circle $ax - xx = vv$, and therefore by Prob. 1. $a - 2x = 2\dot{v}v$, or $\frac{a-2x}{2v} = \dot{v}$.

30. Moreover from the nature of the Trochoid, 'tis LD = Arch AL, and therefore $v + t = y$. And thence (by Prob. 1) $\dot{v} + \dot{t} = \dot{z}$. Laſtly, inſtead of the Fluxions $\dot{v}$ and $\dot{t}$ let their Values be ſubſtituted, and there will ariſe $\frac{a-x}{v} = \dot{z}$. Whence (by Prob. 1.) is derived $-\frac{a\dot{v}}{vv} + \frac{x\dot{v}}{vv} - \frac{\dot{x}}{v} = \dot{z}$. And theſe being found, make $\frac{1+zz}{z} = -$ DH, and raiſe the perpendicular HC.

31. COR. 1. Now it follows from hence, that DH = 2BL, and CH = 2BE, or that EF biſects the radius of Curvature CD in N. And this will appear by ſubſtituting the values of $\dot{z}$ and $\dot{x}$ now found, in the Equation $\frac{1+zz}{\dot{z}} =$ DH, and by a proper reduction of the reſult.

32. COR. 2. Hence the Curve FCK, deſcribed indefinitely by the Center of Curvature of ADF, is another Trochoid equal to this, whoſe Vertices at I and F adjoin to the Cuſpids of this. For let the Circle Fλ, equal and alike poſited to ALE, be deſcribed, and let Cβ be drawn parallel to EF, meeting the Circle in λ: Then will Arch Fλ = (Arch EL = NF =) Cλ.

33. COR. 3. The right Line CD, which is at right Angles to the Trochoid IAF, will touch the Trochoid IKF in the point C.

34. COR. 4. Hence (in the inverted Trochoids,) if at the Cuſpid K of the upper Trochoid, a Weight be hung by a Thread at the diſtance KA or 2EA, and while the Weight vibrates, the Thread be ſuppos'd to apply itſelf to the parts of the Trochoid KF and KI, which reſiſt it on each ſide, that it may not be extended into a right Line, but compel it (as it departs from the Perpendicular) to be by degrees inflected above, into the Figure of the Trochoid, while the lower part CD, from the loweſt Point of Contact, ſtill remains a right Line: The Weight will move in the Perimeter of the lower Trochoid, becauſe the Thread CD will always be perpendicular to it.

35. COR. 5. Therefore the whole Length of the Thread KA is equal to the Perimeter of the Trochoid KCF, and its part CD is equal to the part of the Perimeter CF.

36. COR. 6. Since the Thread by its oſcillating Motion revolves about the moveable Point C, as a Center; the Superficies through which the whole Line CD continually paſſes, will be to the Superficies through which the part CN above the right Line IF paſſes at the ſame time, as $\overline{CD}^2$ to $\overline{CN}^2$, that is, as 4 to 1. Therefore the Area CFN is a fourth part of the Area CFD; and the Area KCNE is a fourth part of the Area AKCD.

37. COR. 7. Alſo ſince the ſubtenſe EL is equal and parallel to CN, and is converted about the immoveable Center E, juſt as CN moves about the moveable Center C; the Superficies will be equal through which they paſs in the ſame time, that is, the Area CFN, and the Segment of the Circle EL. And thence the Area NFD will be the triple of that Segment, and the whole area EADF will be the triple of the Semicircle.

38. COR. 8. When the Weight D arrives at the point F, the whole Thread will be wound about the Perimeter of the Trochoid KCF, and the Radius of Curvature will there be nothing. Wherefore the Trochoid IAF is more curved, at its Cuſpid F, than any Circle; and makes an Angle of Contact, with the Tangent βF produced, infinitely greater than a Circle can make with a right Line.

39. But there are Angles of Contact that are infinitely greater than Trochoidal ones, and others infinitely greater than theſe, and ſo on *in infinitum*; and yet the greateſt of them all are infinitely leſs than right-lined Angles. Thus $xx = ay$, $x^3 = by^2$, $x^4 = cy^3$, $x^5 = dy^4$, &c. denote a Series of Curves, of which every ſucceeding one makes an Angle of Contact with its Abſciſs, which is infinitely greater than the preceding can make with the ſame Abſciſs. And the Angle of Contact which the firſt $xx = ay$ makes, is of the ſame kind with Circular ones; and that which the ſecond $x^3 = by^2$ makes, is of the ſame kind with Trochoidals. And tho' the Angles of the ſucceeding Curves do always infinitely exceed the Angles of the preceding, yet they can never arrive at the magnitude of a right-lined Angle.

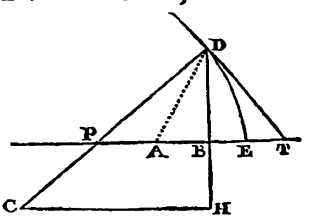
40. After the ſame manner $x = y$, $xx = ay$, $x^3 = b^2y$, $x^4 = c^3y$, &c. denote a Series of Lines, of which the Angles of the ſubſequents, made with their Abſciſs's at the Vertices, are always infinitely leſs than the Angles of the preceding. Moreover, between the Angles of Contact of any two of theſe kinds, other Angles of Contact may be found *ad infinitum*, that ſhall infinitely exceed each other.

41. Now it appears, that Angles of Contact of one kind are infinitely greater than thoſe of another kind; ſince a Curve of one kind, however great it may be, cannot, at the Point of Contact,

lie between the Tangent and a Curve of another kind, however ſmall that Curve may be. Or an Angle of Contact of one kind cannot neceſſarily contain an Angle of Contact of another kind, as the whole contains a part. Thus the Angle of Contact of the Curve $x^4 = cy^3$, or the Angle which it makes with its Abſciſs, neceſſarily includes the Angle of Contact of the Curve $x^3 = by^2$, and can never be contain'd by it. For Angles that can mutually exceed each other are of the ſame kind, as it happens with the aforeſaid Angles of the Trochoid, and of this Curve $x^3 = by^2$.

42. And hence it appears, that Curves, in ſome Points, may be infinitely more ſtraight, or infinitely more curved, than any Circle, and yet not, on that account, loſe the form of Curve-lines. But all this by the way only.

43. Ex. 5. Let ED be the Quadratrix to the Circle, deſcribed from Center A; and letting fall DB perpendicular to AE, make $AB = x$, $BD = y$, and $AE = 1$. Then 'twill be $\dot{y}x - y\dot{y}^2 - y\dot{x}^2 = \dot{x}y$, as before. Then writing 1 for $\dot{x}$, and z for $\dot{y}$, the Equation becomes $zx - zy^2 - zx^2 = y$; and thence, by Prob. 1. $\dot{z}x - \dot{z}y^2 - \dot{z}x^2 + z\dot{x} - 2z\dot{x}x - 2zy\dot{y} = \dot{y}$. Then reducing, and again writing 1 for $\dot{x}$, and z for $\dot{y}$, there ariſes $\dot{z} = \frac{2z^2y + z\dot{x}x}{x - xx - yy}$. But z and $\dot{z}$ being found, make $\frac{1 + zz}{\dot{z}} = DH$, and draw HC as above.

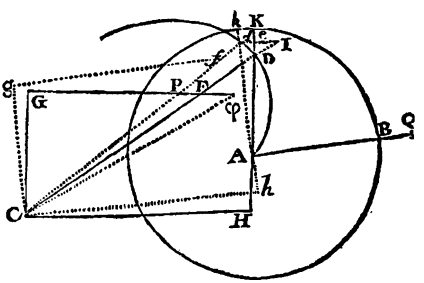


44. If you deſire a Conſtruction of the Problem, you will find it very ſhort. Thus draw DP perpendicular to DT, meeting AT in P, and make 2AP : AE :: PT : CH. For $z = \left(\frac{y}{x - xx - yy}\right) = \frac{BD}{-BT}$, and $zy = \frac{BD^q}{-BT} = -BP$; and $\dot{z}y + x = -AP$, and $\frac{\dot{z}z}{x - xx - yy}$ into $\dot{z}y + x = \frac{2BD}{AE \times BT^q}$ into $-AP = \dot{z}$. Moreover it is $1 + zz = \frac{PT}{BT}$, (becauſe $= 1 + \frac{BD^q}{BT^q} = \frac{DT^q}{BT^q}$,) and therefore $\frac{1 + zz}{\dot{z}} = \frac{PT \times AE \times BT}{-2BD \times AP} = DH$. Laſtly, it is BT : BD :: DH : CH $= \frac{PT \times AE}{-2AP}$. Here the negative Value only ſhews, that CH muſt be taken the ſame way as AB from DH.

45. In the ſame manner the Curvature of Spirals, or of any other Curves whatever, may be determined by a very ſhort Calculation.

46. Furthermore, to determine the Curvature without any previous reduction, when the Curves are refer'd to right Lines in any other manner, this Method might have been apply'd, as has been done already for drawing Tangents. But as all Geometrical Curves, as alſo Mechanical, (eſpecially when the defining conditions are reduced to infinite Equations, as I ſhall ſhew hereafter,) may be refer'd to rectangular Ordinates, I think I have done enough in this matter. He that deſires more, may eaſily ſupply it by his own induſtry; eſpecially if for a farther illuſtration I ſhall add the Method for Spirals.

47. Let BK be a Circle, A its Center, and B a given Point in its Circumference. Let AD*d* be a Spiral, DC its Perpendicular, and C the Center of Curvature at the Point D. Then drawing the right Line ADK, and CG parallel and equal to AK, as alſo the Perpendicular GF meeting CD in F: Make AB or AK $=$ 1 $=$ CG, BK $= x$, AD $= y$, and GF $= z$. Then conceive the Point D to move in the Spiral for an infinitely little Space D*d*, and then through *d* draw the Semidiameter A*k*, and C*g* parallel and equal to it, draw *gf* perpendicular to *g*C, ſo that C*d* cuts *gf* in *f*, and GF in P; produce GF to φ, ſo that Gφ $= gf$, and draw *de* perpendicular to AK, and produce it till it meets CD at I. Then the contemporaneous Moments of BK, AD, and Gφ, will be K*k*, D*e* and Fφ, which therefore may be call'd $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$.



48. Now it is AK : A*e* (AD) :: *k*K : *de* $= \dot{y}o$, where I aſſume $\dot{x} = 1$, as above. Alſo CG : GF :: *de* : *e*D $= \dot{o}y\dot{z}$, and therefore $y\dot{z} = \dot{y}$. Beſides CG : CF :: *de* : *d*D $= \dot{o}y \times$ CF :: *d*D : *d*I $= \dot{o}y \times$ CFq. Moreover, becauſe $\angle$PCφ ($= \angle$GCg) $= \angle$DA*d*, and $\angle$CPφ ($= \angle$C*d*I $= \angle$ *ed*D + Rect.) $= \angle$ AD*d*, the Triangles CPφ and AD*d* are ſimilar, and thence AD : D*d* :: CP (CF) : Pφ $= o \times$ CFq. From whence take Fφ, and there will remain PF $= o \times$ CF$q - o \times z$. Laſtly, letting fall CH perpendicular to AD, 'tis PF : *d*I :: CG : *e*H or DH $= \frac{y \times \text{CF}q}{\text{CF}q - z}$. Or ſubſtituting $1 + zz$ for CFq, 'twill be DH $= \frac{y + yzz}{1 + zz - z}$. Here it may be obſerved, that

that in this kind of Computations, I take thoſe Quantities (AD and A*e*) for equal, the Ratio of which differs but infinitely little from the Ratio of Equality.

49. Now from hence ariſes the following Rule. The Relation of x and y being exhibited by any Equation, find the Relation of the Fluxions $\dot{x}$ and $\dot{y}$, (by Prob. 1.) and ſubſtitute 1 for $\dot{x}$, and yz for $\dot{y}$. Then from the reſulting Equation find again, (by Prob. 1.) the Relation between $\dot{x}$, $\dot{y}$, and z, and again ſubſtitute 1 for $\dot{x}$. The firſt reſult by due reduction will give $\dot{y}$ and z, and the latter will give $\dot{z}$; which being known, make $\frac{y+yzz}{1+zz-z} = DH$, and raiſe the Perpendicular HC, meeting the Perpendicular to the Spiral DC before drawn in C, and C will be the Center of Curvature. Or which comes to the ſame thing, take CH : HD :: z : 1, and draw CD.

50. Ex. 1. If the Equation be $ax = y$, (which will belong to the Spiral of *Archimedes*,) then (by Prob. 1.) $a\dot{x} = \dot{y}$, or (writing 1 for $\dot{x}$, and yz for $\dot{y}$,) $a = yz$. And hence again (by Prob 1.) $0 = \dot{y}z + y\dot{z}$. Wherefore any Point D of the Spiral being given, and thence the length AD or y, there will be given $z = \frac{a}{y}$, and $\dot{z} = \left(-\frac{\dot{y}z}{y} \text{ or}\right) -\frac{a\dot{z}}{y}$. Which being known, make $1+zz-\dot{z}$: $1+zz$ :: DA (y) : DH. And 1 : z :: DH : CH.

And hence you will eaſily deduce the following Conſtruction. Produce AB to Q, ſo that AB : Arch BK :: Arch BK : BQ, and make AB + AQ : AQ :: DA : DH :: a : HC.

51. Ex. 2. If $ax^2 = y^3$ be the Equation that determines the Relation between BK and AD; (by Prob. 1.) you will have $2a\dot{x}x = 3\dot{y}y^2$, or $2a\dot{x} = 3z\dot{y}^3$. Thence again $2a\dot{x} = 3\dot{z}\dot{y}^3 + 9z\dot{y}y^2$. 'Tis therefore $z = \frac{2ax}{3y^3}$, and $\dot{z} = \frac{2a-9z\dot{y}^3}{3y^3}$. Theſe being known, make $1+zz-\dot{z}$: $1+zz$ :: DA : DH. Or, the work being reduced to a better form, make $9xx+10 : 9xx+4$ :: DA : DH.

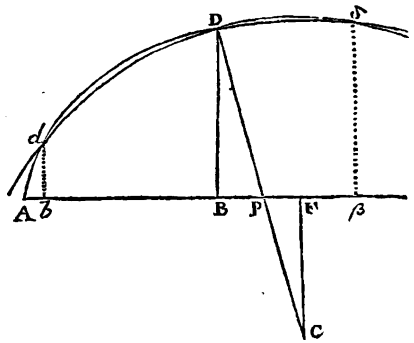
52. Ex. 3. After the ſame manner, if $ax^2 - bxy = y^3$ determines the Relation of BK to AD; there will ariſe $\frac{2ax-by}{bxy+3y^3} = z$, and $\frac{2a\dot{x}-2b\dot{x}y-bz^2\dot{x}y-9z\dot{x}y^3}{bxy+3y^3} = \dot{z}$. From which DH, and thence the Point C, is determined as before.

53.

53. And thus you will eaſily determine the Curvatures of any other Spirals; or invent Rules for any other kinds of Curves, in imitation of theſe already given.

54. And now I have finiſh'd the Problem; but having made uſe of a Method which is pretty different from the common ways of operation, and as the Problem itſelf is of the number of thoſe which are not very frequent among Geometricians: For the illuſtration and confirmation of the Solution here given, I ſhall not think much to give a hint of another, which is more obvious, and has a nearer relation to the uſual Methods of drawing Tangents. Thus if from any Center, and with any Radius, a Circle be conceived to be deſcribed, which may cut any Curve in ſeveral Points; if that Circle be ſuppos'd to be contracted, or enlarged, till two of the Points of interſection coincide, it will there touch the Curve. And beſides, if its Center be ſuppos'd to approach towards, or recede from, the Point of Contact, till the third Point of interſection ſhall meet with the former in the Point of Contact; then will that Circle be æquicurved with the Curve in that Point of Contact: In like manner as I inſinuated before, in the laſt of the five Properties of the Center of Curvature, by the help of each of which I affirm'd the Problem might be ſolved in a different manner.

55. Therefore with Center C, and Radius CD, let a Circle be deſcribed, that cuts the Curve in the Points *d*, D, and δ; and letting fall the Perpendiculars DB, *db*, $\delta\beta$, and CF, to the Abſciſs AB; call AB $= x$, BD $= y$, AF $= v$, FC $= t$, and DC $= s$. Then BF $= v - x$, and DB $+$ FC $= y + t$. The ſum of the Squares of theſe is equal to the Square of DC; that is, $v^2 - 2vx + x^2 + y^2 + 2yt + t^2 = ss$. If you would abbreviate this, make $v^2 + t^2 - s^2 = q^2$, (any Symbol at pleaſure,) and it becomes $x^2 - 2vx + y^2 + 2ty + q^2 = 0$. After you have found t, v, and q^2, you will have $s = \sqrt{v^2 + t^2 - q^2}$.



56. Now let any Equation be propoſed for defining the Curve, the quantity of whoſe Curvature is to be found. By the help of this Equation you may exterminate either of the Quantities x or y, and

and there will ariſe an Equation, the Roots of which, (*db*, DB, *δβ*, &c. if you exterminate *x*; or A*b*, AB, A*β*, &c. if you exterminate *y*,) are at the Points of interſection *d*, D, *δ*, &c. Wherefore ſince three of them become equal, the Circle both touches the Curve, and will alſo be of the ſame degree of Curvature as the Curve, in the point of Contact. But they will become equal by comparing the Equation with another fictitious Equation of the ſame number of Dimenſions, which has three equal Roots; as *Des Cartes* has ſhew'd. Or more expeditiouſly by multiplying its Terms twice by an Arithmetical Progreſſion.

57. EXAMPLE. Let the Equation be $ax = yy$, (which is an Equation to the Parabola,) and exterminating *x*, (that is, ſubſtituting its Value $\frac{yy}{a}$ in the foregoing Equation,) there will ariſe Three of whoſe Roots *y* are to be made equal. And for this purpoſe I multiply the Terms twice by an Arithmetical Progreſſion, as you ſee done here; and there ariſes

$$\frac{y^4}{aa} * - \frac{2v}{a}y^2 + 2ty + q^2 = 0.$$
$$+ y^2$$

4	*	2	1	0
3	*	1	0	−1

$$\frac{12y^4}{aa} - \frac{4v}{a}y^2 + 2y^2 = 0.$$

Or $v = \frac{3y^2}{a} + \frac{1}{2}a$. Whence it is eaſily infer'd, that $BF = 2x + \frac{1}{2}a$, as before.

58. Wherefore any Point D of the Parabola being given, draw the Perpendicular DP to the Curve, and in the Axis take $PF = 2AB$, and erect FC Perpendicular to FA, meeting DP in C; then will C be the Center of Curvity deſired.

59. The ſame may be perform'd in the Ellipſis and Hyperbola, but the Calculation will be troubleſome enough, and in other Curves generally very tedious.

Of Queſtions that have ſome Affinity to the preceding Problem.

60. From the Reſolution of the preceding Problem ſome others may be perform'd; ſuch are,

I. *To find the Point where the Curve has a given degree of Curvature.*

61. Thus in the Parabola, $ax = yy$, if the Point be required whoſe Radius of Curvature is of a given length *f*: From the Center of Curvature, found as before, you will determine the Radius

to

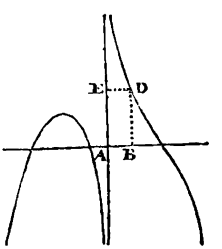
to be $\frac{a+4x}{2a}\sqrt{aa+4ax}$, which muſt be made equal to f. Then by reduction there ariſes $x = -\frac{1}{4}a + \sqrt[3]{\frac{1}{16}aff}$.

II. *To find the Point of Rectitude.*

62. I call that the *Point of Rectitude*, in which the Radius of Flexure becomes infinite, or its Center at an infinite diſtance: Such it is at the Vertex of the Parabola $a^3x = y^4$. And this ſame Point is commonly the Limit of contrary Flexure, whoſe Determination I have exhibited before. But another Determination, and that not inelegant, may be derived from this Problem. Which is, the longer the Radius of Flexure is, ſo much the leſs the Angle DC*d* (*Fig. pag.* 61.) becomes, and alſo the Moment δf; ſo that the Fluxion of the Quantity z is diminiſh'd along with it, and by the Infinitude of that Radius, altogether vaniſhes. Therefore find the Fluxion $\dot{z}$, and ſuppoſe it to become nothing.

63. As if we would determine the Limit of contrary Flexure in the Parabola of the ſecond kind, by the help of which *Carteſius* conſtructed Equations of ſix Dimenſions; the Equation to that Curve is $x^3 - bx^2 - cdx + bcd + dxy = 0$. And hence (by Prob. 1.) ariſes $3\dot{x}xx - 2b\dot{x}x - cd\dot{x} + d\dot{x}y + dx\dot{y} = 0$. Now writing 1 for $\dot{x}$, and z for $\dot{y}$, it becomes $3x^2 - 2bx - cd + dy + dxz = 0$; whence again (by Prob. 1.) $6\dot{x}x - 2b\dot{x} + d\dot{y} + d\dot{x}z + dx\dot{z} = 0$. Here again writing 1 for $\dot{x}$, z for $\dot{y}$, and 0 for $\dot{z}$, it becomes $6x - 2b + 2dz = 0$. And exterminating z, by putting $b - 3x$ for dz in the Equation $3xx - 2bx - cd + dy + dxz = 0$, there will ariſe $-bx - cd + dy = 0$, or $y = c + \frac{bx}{d}$; this being ſubſtituted in the room of y in the Equation of the Curve, we ſhall have $x^3 + bcd = 0$; which will determine the Confine of contrary Flexure.

64. By a like Method you may determine the Points of Rectitude, which do not come between parts of contrary Flexure. As if the Equation $x^4 - 4ax^3 + 6a^2x^2 - b^3y = 0$ expreſs'd the nature of a Curve; you have firſt, (by Prob. 1.) $4\dot{x}x^3 - 12a\dot{x}x^2 + 12a^2\dot{x}x - b^3\dot{z} = 0$, and hence again $12\dot{x}x^2 - 24a\dot{x}x + 12a^2\dot{x} - b^3\dot{z} = 0$. Here ſuppoſe $\dot{z} = 0$, and by Reduction there will ariſe $x = a$. Wherefore take AB $= a$, and erect the perpendicular BD; this will meet the Curve in the Point of Rectitude D, as was required.



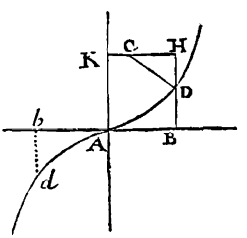
III.

III. *To find the Point of infinite Flexure.*

65. Find the Radius of Curvature, and ſuppoſe it to be nothing. Thus to the Parabola of the ſecond kind, whoſe Equation is $x^3 = ay^2$, that Radius will be CD $= \frac{4a+9x}{6a}\sqrt{4ax+9xx}$; which becomes nothing when $x = 0$.

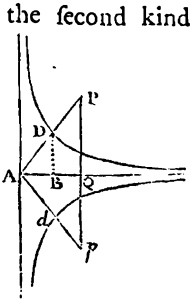
IV. *To determine the Point of the greateſt or leaſt Flexure.*

66. At theſe Points the Radius of Curvature becomes either the greateſt or leaſt. Wherefore the Center of Curvature, at that moment of Time, neither moves towards the point of Contact, nor the contrary way, but is intirely at reſt. Therefore let the Fluxion of the Radius CD be found; or more expeditiouſly, let the Fluxion of either of the Lines BH or AK be found, and let it be made equal to nothing.



67. As if the Queſtion were propoſed concerning the Parabola of the ſecond kind $x^3 = a^2y$; firſt to determine the Center of Curvature you will find DH $= \frac{aa+9xy}{6x}$, and therefore BH $= \frac{aa+15xy}{6x}$; make BH $= v$, then $\frac{aa}{6x} + \frac{5}{2}y = v$. Hence (by Prob. 1.) $-\frac{a^2\dot{x}}{6xx} + \frac{5}{2}\dot{y} = \dot{v}$. But now ſuppoſe $\dot{v}$, or the Fluxion of BH, to be nothing; and beſides, ſince by Hypotheſis $x^3 = a^2y$, and thence (by Prob. 1.) $3\dot{x}x^2 = a^2\dot{y}$, putting $\dot{x} = 1$, ſubſtitute $\frac{3xx}{aa}$ for $\dot{y}$, and there will ariſe $45x^4 = a^4$. Take therefore AB $= a\sqrt[4]{\frac{1}{45}} = a \times \overline{45}|^{-\frac{1}{4}}$, and raiſing the perpendicular BD, it will meet the Curve in the Point of the greateſt Curvature. Or, which is the ſame thing, make AB : BD :: $3\sqrt{5}$: 1.

68. After the ſame manner the Hyperbola of the ſecond kind repreſented by the Equation $xy^2 = a^3$, will be moſt inflected in the points D and d, which you may determine by taking in the Abſciſs AQ $= 1$, and erecting the Perpendicular QP $= \sqrt{5}$, and Qp equal to it on the other ſide. Then drawing AP and Ap, they will meet the Curve in the points D and d required.



V. *To determine the Locus of the Center of Curvature, or to deſcribe the Curve, in which that Center is always found.*

69. We have already ſhewn, that the Center of Curvature of the Trochoid is always found in another Trochoid. And thus the Center of Curvature of the Parabola is found in another Parabola of the ſecond kind, repreſented by the Equation $axx = y^3$, as will eaſily appear from Calculation.

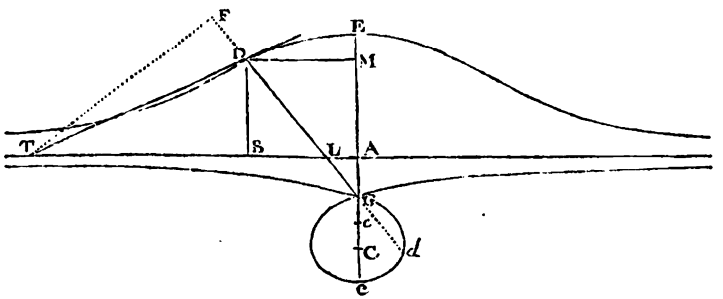
VI. *Light falling upon any Curve, to find its Focus, or the Concourſe of the Rays that are refracted at any of its Points.*

70. Find the Curvature at that Point of the Curve, and deſcribe a Circle from the Center, and with the Radius of Curvature. Then find the Concourſe of the Rays, when they are refracted by a Circle about that Point: For the ſame is the Concourſe of the refracted Rays in the propoſed Curve.

71. To theſe may be added a particular Invention of the Curvature at the Vertices of Curves, where they cut their Abſciſſes at right Angles. For the Point in which the Perpendicular to the Curve, meeting with the Abſciſs, cuts it ultimately, is the Center of its Curvature. So that having the relation between the Abſciſs x, and the rectangular Ordinate y, and thence (by Prob. 1.) the relation between the Fluxions $\dot{x}$ and $\dot{y}$; the Value $\dot{y}\dot{y}$, if you ſubſtitute 1 for $\dot{x}$ into it, and make $y = 0$, will be the Radius of Curvature.

72. Thus in the Ellipſis $ax - \frac{a}{b}xx = yy$, it is $\frac{a\dot{x}}{2} - \frac{a\dot{x}\dot{x}}{b} = \dot{y}\dot{y}$; which Value of $\dot{y}\dot{y}$, if we ſuppoſe $y = 0$, and conſequently $x = b$, writing 1 for $\dot{x}$, becomes $\frac{1}{2}a$ for the Radius of Curvature. And ſo at the Vertices of the Hyperbola and Parabola, the Radius of Curvature will be always half of the Latus rectum.

73. And in like manner for the Conchoid, defined by the Equation $\frac{b^2z^2}{xx} + \frac{2bcc}{x} + \frac{cc}{-bb} - 2bx - xx = yy$, the Value of yy, (found by

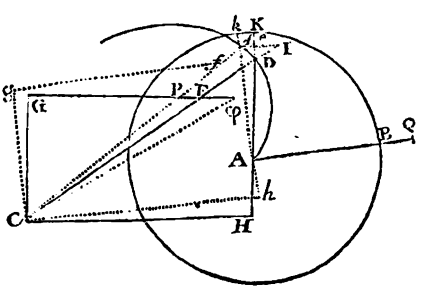


Prob. 1.) will be $-\frac{b^2z^2}{x^3} - \frac{bc^2}{x^2} - b - x$. Now ſuppoſing $y = 0$, and thence $x = c$ or $-c$, we ſhall have $-\frac{bb}{c} - 2b - c$, or $\frac{bb}{c} - 2b + c$, for the Radius of Curvature. Therefore make AE : EG :: EG : EC, and Ae : eG :: eG : ec, and you will have the Centers of Curvature C and c, at the Vertices E and e of the Conjugate Conchoids.

PROB. VI.

To determine the Quality of the Curvature, at a given Point of any Curve.

1. By the *Quality of Curvature* I mean its Form, as it is more or leſs inequable, or as it is varied more or leſs, in its progreſs thro' different parts of the Curve. So if it were demanded, what is the Quality of the Curvature of the Circle? it might be anſwer'd, that it is uniform, or invariable. And thus if it were demanded, what is the Quality of the Curvature of the Spiral, which is deſcribed by the motion of the point D, proceeding from A in AD with an accelerated velocity, while the right Line AK moves with an uniform rotation about the Center A; the acceleration of



which Velocity is ſuch, that the right Line AD has the ſame ratio to the Arch BK, deſcribed from a given point B, as a Number has to its Logarithm: I ſay, if it be aſk'd, What is the Quality of the Curvature of this Spiral? It may be anſwer'd, that it is uniformly varied, or that it is equably inequable. And thus other Curves, in their ſeveral Points, may be denominated inequably inequable, according to the variation of their Curvature.

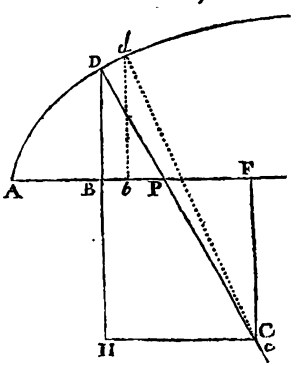
2. Therefore the Inequability or Variation of Curvature is required at any Point of a Curve. Concerning which it may be obſerved,

3. I. That at Points placed alike in like Curves, there is a like Inequability or Variation of Curvature.

4. II. And that the Moments of the Radii of Curvature, at thoſe Points, are proportional to the contemporaneous Moments of the Curves, and the Fluxions to the Fluxions.

5. III. And therefore, that where thoſe Fluxions are not proportional, the Inequability of the Curvature will be unlike. For there will be a greater Inequability, where the Ratio of the Fluxion of the Radius of Curvature to the Fluxion of the Curve is greater. And therefore that ratio of the Fluxions may not improperly be call'd the Index of the Inequability or of the Variation of Curvature.

6. At the points D and *d*, infinitely near to each other, in the Curve AD*d*, let there be drawn the Radii of Curvature DC and *dc*; and D*d* being the Moment of the Curve, C*c* will be the contemporaneous Moment of the Radius of Curvature, and $\frac{Cc}{Dd}$ will be the Index of the Inequability of Curvature. For the Inequability may be call'd ſuch and ſo great, as the quantity of that ratio $\frac{Cc}{Dd}$ ſhews it to be: Or the Curvature may be ſaid to be ſo much the more unlike to the uniform Curvature of a Circle.



7. Now letting fall the perpendicular Ordinates DB and *db*, to any line AB meeting DC in P; make AB $= x$, BD $= y$, DP $= t$, DC $= v$, and thence B*b* $= \dot{x}o$, it will be C*c* $= \dot{v}o$; and BD : DP :: B*b* : D*d* $= \frac{\dot{x}ot}{y}$, and $\frac{Cc}{Dd} = \frac{\dot{v}v}{\dot{x}t} = \frac{\dot{v}y}{t}$, making $\dot{x} = 1$.

Wherefore

Wherefore the relation between x and y being exhibited by any Equation, and thence, (according to Prob. 4. and 5.) the Perpendicular DP or t, being found, and the Radius of Curvature v, and the Fluxion $\dot{v}$ of that Radius, (by Prob. 1.) the Index $\frac{\dot{v}y}{\dot{t}}$ of the Inequability of Curvature will be given alſo.

8. Ex. 1. Let the Equation to the Parabola $2ax = yy$ be given; then (by Prob. 4.) BP $= a$, and therefore DP $= \sqrt{aa+yy} = t$. Alſo (by Prob. 5.) BF $= a + 2x$, and BP : DP :: BF : DC $= \frac{at+2tx}{a} = v$. Now the Equations $2ax = yy$, $aa+yy = tt$, and $\frac{at+2tx}{a} = v$, (by Prob. 1.) give $2a\dot{x} = 2\dot{y}y$, and $2\dot{y}y = 2\dot{t}t$, and $\frac{a\dot{t}+2t\dot{x}+2\dot{t}x}{a} = \dot{v}$. Which being reduced to order, and putting $\dot{x} = 1$, there will ariſe $\dot{y} = \frac{a}{y}$, $\dot{t} = \left(\frac{\dot{y}}{\dot{t}} =\right) \frac{a}{t}$, and $\dot{v} = \frac{a\dot{t}+2t\dot{x}+2\dot{t}}{a}$. And thus $\dot{y}$, $\dot{t}$, and $\dot{v}$ being found, there will be had $\frac{\dot{v}y}{\dot{t}}$ the Index of the Inequability of Curvature.

9. As if in Numbers it were determin'd, that $a = 1$, or $2x = yy$, and $\dot{x} = \frac{1}{2}$; then $\dot{y} (= \sqrt{2\dot{x}}) = 1$, $\dot{y} \left(= \frac{a}{y}\right) = 1$, $t \left(= \sqrt{aa+yy}\right.$ $\left.=\right) \sqrt{2}$, $\dot{t} \left(= \frac{a}{t}\right) = \sqrt{\frac{1}{2}}$, and $\dot{v} = \left(\frac{a\dot{t}+2t\dot{x}+2\dot{t}}{a} =\right) 3\sqrt{2}$. So that $\frac{\dot{v}y}{\dot{t}} = 3$, which therefore is the Index of Inequability.

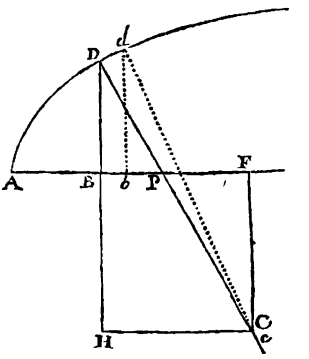
10. But if it were determin'd, that $x = 2$, then $y = 2$, $\dot{y} = \frac{1}{2}$, $t = \sqrt{5}$, $\dot{t} = \sqrt{\frac{1}{5}}$, and $\dot{v} = 3\sqrt{5}$. So that $\left(\frac{\dot{v}y}{\dot{t}} =\right) 6$ will be here the Index of Inequability.

11. Wherefore the Inequability of Curvature at the Point of the Curve, from whence an Ordinate, equal to the Latus rectum of the Parabola, being drawn perpendicular to the Axis, will be double to the Inequability at that Point, from whence the Ordinate ſo drawn is half the Latus rectum; that is, the Curvature at the firſt Point is as unlike again to the Curvature of the Circle, as the Curvature at the ſecond Point.

12. Ex. 2. Let the Equation be $2ax - bxx = yy$, and (by Prob. 4.) it will be $a - bx =$ BP, and thence $tt = (aa - 2abx + bbxx + yy$ $=) aa - byy + yy$. Alſo (by Prob. 5.) it is DH $= y + \frac{a^3 - b^2 s}{aa}$, where, if for $yy - byy$ you ſubſtitute $tt - aa$, there ariſes DH $= \frac{ttv}{aa}$. 'Tis alſo BD : DP :: DH : DC $= \frac{t^3}{a^2} = v$. Now (by Prob. 1.) the Equations $2ax - bxx = yy$, $aa - byy + yy = tt$, and $\frac{t^3}{a^4} = v$, give

give $a - bx = \dot{y}y$, and $\dot{y}y - b\dot{y}y = \dot{t}t$, and $\frac{3\dot{t}tt}{aa} = \dot{v}$. And thus $\dot{v}$ being found, the Index $\frac{\dot{v}y}{\dot{t}}$ of the Inequability of Curvature, will alſo be known.

13. Thus in the Ellipſis $2x - 3xx = yy$, where it is $a = 1$, and $b = 3$; if we make $x = \frac{1}{2}$, then $y = \frac{1}{2}$, $\dot{y} = -1$, $\dot{t} = \sqrt{\frac{1}{2}}$, $\dot{t} = \sqrt{2}$, $\dot{v} = 3\sqrt{\frac{1}{2}}$, and therefore $\frac{\dot{v}y}{\dot{t}} = \frac{3}{2}$, which is the Index of the Inequability of Curvature. Hence it appears, that the Curvature of this Ellipſis, at the Point D here aſſign'd, is by two times leſs inequable, (or by two times more like to the Curvature of the Circle,) than the Curvature of the Parabola, at that Point of its Curve, from whence an Ordinate let fall upon the Axis is equal to half the Latus rectum.



14. If we have a mind to compare the Concluſions derived in theſe Examples, in the Parabola $2ax = yy$ ariſes $\left(\frac{\dot{v}y}{\dot{t}} =\right) \frac{3y}{a}$ for the Index of Inequability; and in the Ellipſis $2ax - bxx = yy$, ariſes $\left(\frac{\dot{v}y}{\dot{t}} =\right) \frac{3y - 3by}{aa} \times$ BP; and ſo in the Hyperbola $2ax + bxx = yy$, the analogy being obſerved, there ariſes the Index $\left(\frac{\dot{v}y}{\dot{t}} =\right) \frac{3y + 3by}{aa} \times$ BP. Whence it is evident, that at the different Points of any Conic Section conſider'd apart, the Inequability of Curvature is as the Rectangle BD x BP. And that, at the ſeveral Points of the Parabola, it is as the Ordinate BD.

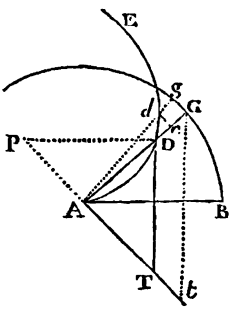
15. Now as the Parabola is the moſt ſimple Figure of thoſe that are curved with inequable Curvature, and as the Inequability of its Curvature is ſo eaſily determined, (for its Index is $6 \times \frac{\text{Ordinate}}{\text{Lat. rect.}}$,) therefore the Curvatures of other Curves may not improperly be compared to the Curvature of this.

16. As if it were inquired, what may be the Curvature of the Ellipſis $2x - 3xx = yy$, at that Point of the Perimeter which is determined by aſſuming $x = \frac{1}{2}$: Becauſe its Index is $\frac{3}{2}$, as before, it might be anſwer'd, that it is like the Curvature of the Parabola

6*x*

$6x = yy$, at that Point of the Curve, between which and the Axis the perpendicular Ordinate is equal to $\frac{1}{2}$.

17. Thus, as the Fluxion of the Spiral ADE is to the Fluxion of the Subtenſe AD, in a certain given Ratio, ſuppoſe as d to e; on its concave ſide erect $AP = \frac{e}{\sqrt{dd-ee}} \times AD$ perpendicular to AD, and P will be the Center of Curvature, and $\frac{AP}{AD}$ or $\frac{e}{\sqrt{dd-ee}}$, will be the Index of Inequability. So that this Spiral has every where its Curvature alike inequable, as the Parabola $6x = yy$ has in that Point of its Curve, from whence to its Abſciſs a perpendicular Ordinate is let fall, which is equal to the quantity $\frac{e}{\sqrt{dd-ee}}$.



18. And thus the Index of Inequability at any Point D of the Trochoid, (ſee Fig. in *Art.* 29. *pag.* 64.) is found to be $\frac{AB}{BL}$. Wherefore its Curvature at the ſame Point D is as inequable, or as unlike to that of a Circle, as the Curvature of any Parabola $ax = yy$ is at the Point where the Ordinate is $\frac{1}{6}a \times \frac{AB}{BL}$.

19. And from theſe Conſiderations the Senſe of the Problem, as I conceive, muſt be plain enough; which being well underſtood, it will not be difficult for any one, who obſerves the Series of the things above deliver'd, to furniſh himſelf with more Examples, and to contrive many other Methods of operation, as occaſion may require. So that he will be able to manage Problems of a like nature, (where he is not diſcouraged by tedious and perplex Calculations,) with little or no difficulty. Such are theſe following;

I. *To find the Point of any Curve, where there is either no Inequability of Curvature, or infinite, or the greateſt, or the leaſt.*

20. Thus at the Vertices of the Conic Sections, there is no Inequability of Curvature; at the Cuſpid of the Trochoid it is infinite; and it is greateſt at thoſe Points of the Ellipſis, where the Rectangle BD × BP is greateſt, that is, where the Diagonal-Lines of the circumſcribed Parallelogram cut the Ellipſis, whoſe Sides touch it in their principal Vertices.

II. *To determine a Curve of ſome definite Species, ſuppoſe a Conic Section, whoſe Curvature at any Point may be equal and ſimilar to the Curvature of any other Curve, at a given Point of it.*

III.

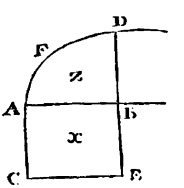
III. *To determine a Conic Section, at any Point of which, the Curvature and Poſition of the Tangent,* (*in reſpect of the Axis,*) *may be like to the Curvature and Poſition of the Tangent, at a Point aſſign'd of any other Curve.*

21. The uſe of which Problem is this, that inſtead of Ellipſes of the ſecond kind, whoſe Properties of refracting Light are explain'd by *Des Cartes* in his Geometry, Conic Sections may be ſubſtituted, which ſhall perform the ſame thing, very nearly, as to their Refractions. And the ſame may be underſtood of other Curves.

PROB. VII.

To find as many Curves as you pleaſe, whoſe Areas may be exhibited by finite Equations.

1. Let AB be the Abſciſs of a Curve, at whoſe Vertex A let the perpendicular AC $=$ 1 be raiſed, and let CE be drawn parallel to AB. Let alſo DB be a rectangular Ordinate, meeting the right Line CE in E, and the Curve AD in D. And conceive theſe Areas ACEB and ADB to be generated by the right Lines BE and BD, as they move along the Line AB. Then their Increments or Fluxions will be always as the deſcribing Lines BE and BD. Wherefore make the Parallelogram ACEB, or AB $\times$ 1, $= \dot{x}$, and the Area of the Curve ADB call z. And the Fluxions $\dot{x}$ and $\dot{z}$ will be as BE and BD; ſo that making $\dot{x} =$ 1 $=$ BE, then $\dot{z} =$ BD.



2. Now if any Equation be aſſumed at pleaſure, for determining the relation of z and x, from thence, (by Prob. 1.) may $\dot{z}$ be derived. And thus there will be two Equations, the latter of which will determine the Curve, and the former its Area.

EXAMPLES.

3. Aſſume $xx = z$, and thence (by Prob. 1.) $2\dot{x}x = \dot{z}$, or $2x = \dot{z}$, becauſe $\dot{x} =$ 1.

4. Aſſume $\frac{x^3}{a} = z$, and thence will ariſe $\frac{3x^2}{a} = \dot{z}$, an Equation to the Parabola.

5. Aſſume $ax^3 = zz$, or $a^{\frac{1}{2}}x^{\frac{3}{2}} = z$, and there will ariſe $\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}} = \dot{z}$, or $\frac{9}{4}ax = \dot{z}z$, an Equation again to the Parabola.

6. Aſſume $a^6x^{-2} = zz$, or $a^3x^{-1} = z$, and there ariſes $-a^3x^{-2} = \dot{z}$, or $a^3 + \dot{z}xx = 0$. Here the negative Value of $\dot{z}$ only inſinuates, that BD is to be taken the contrary way from BE.

7. Again if you aſſume $c^2a^2 + c^2x^2 = z^2$, you will have $2c^2x = 2z\dot{z}$; and z being eliminated, there will ariſe $\frac{cx}{\sqrt{aa+xx}} = \dot{z}$.

8. Or if you aſſume $\frac{aa+xx}{b}\sqrt{aa+xx} = z$, make $\sqrt{aa+xx} = v$, and it will be $\frac{v^3}{b} = z$, and then (by Prob. 1.) $\frac{3\dot{v}vv}{b} = \dot{z}$. Alſo the Equation $aa + xx = vv$ gives $2\dot{x} = 2\dot{v}v$, by the help of which if you exterminate $\dot{v}$, it will become $\frac{3\dot{x}vx}{b} = \dot{z} = \frac{3\dot{x}}{b}\sqrt{aa+xx}$.

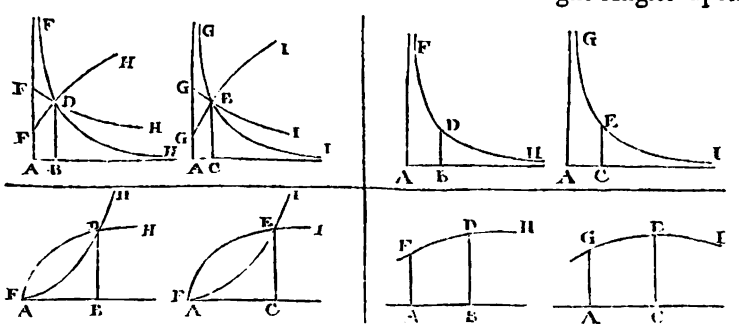
9. Laſtly, if you aſſume $8 - 3xz + \frac{1}{5}z = zz$, you will obtain $-3\dot{z} - 3\dot{x}z + \frac{1}{5}\dot{z} = 2z\dot{z}$. Wherefore by the aſſumed Equation firſt ſeek the Area z, and then the Ordinate $\dot{z}$ by the reſulting Equation.

10. And thus from the Areas, however they may be feign'd, you may always determine the Ordinates to which they belong.

PROB. VIII.

To find as many Curves as you pleaſe, whoſe Areas ſhall have a relation to the Area of any given Curve, aſſignable by finite Equations.

1. Let FDH be a given Curve, and GEI the Curve required, and conceive their Ordinates DB and EC to move at right Angles upon



their Abſciſſes or Baſes AB and AC. Then the Increments or Fluxions of the Areas which they deſcribe, will be as thoſe Ordinates drawn into

into their Velocities of moving, that is, into the Fluxions of their Abſciſſes. Therefore make AB $= x$, BD $= v$, AC $= z$, and CE $= y$, the Area AFDB $= s$, and the Area AGEC $= t$, and let the Fluxions of the Areas be $\dot{s}$ and $\dot{t}$: And it will be $\dot{x}v : \dot{z}y :: \dot{s} : \dot{t}$. Therefore if we ſuppoſe $\dot{x} = 1$, and $v = \dot{s}$, as before; it will be $\dot{z}y = \dot{t}$, and thence $\frac{\dot{t}}{\dot{z}} = y$.

2. Therefore let any two Equations be aſſumed; one of which may expreſs the relation of the Areas s and t, and the other the relation of their Abſciſſes x and z, and thence, (by Prob. 1.) let the Fluxions $\dot{t}$ and $\dot{z}$ be found, and then make $\frac{\dot{t}}{\dot{z}} = y$.

3. Ex. 1. Let the given Curve FDH be a Circle, expreſs'd by the Equation $ax - xx = vv$, and let other Curves be ſought, whoſe Areas may be equal to that of the Circle. Therefore by the Hypotheſis $s = t$, and thence $\dot{s} = \dot{t}$, and $y = \frac{\dot{t}}{\dot{z}} = \frac{v}{\dot{z}}$. It remains to determine z, by aſſuming ſome relation between the Abſciſſes x and z.

4. As if you ſuppoſe $ax = zz$; then (by Prob. 1.) $a = 2\dot{z}z$: So that ſubſtituting $\frac{a}{2z}$ for $\dot{z}$, then $y = \frac{v}{\dot{z}} = \frac{2vz}{a}$. But it is $v = (\sqrt{ax - xx} =) \frac{z}{a}\sqrt{aa - zz}$, therefore $\frac{2zz}{aa}\sqrt{aa - zz} = y$ is the Equation to the Curve, whoſe Area is equal to that of the Circle.

5. After the ſame manner if you ſuppoſe $xx = z$, there will ariſe $2\dot{x} = \dot{z}$, and thence $y = \left(\frac{v}{\dot{z}} =\right) \frac{v}{2\dot{x}}$; whence v and x being exterminated, it will be $y = \frac{\sqrt{az^{\frac{1}{2}} - z}}{2z^{\frac{1}{2}}}$.

6. Or if you ſuppoſe $cc = xz$, there ariſes $o = \dot{z} + x\dot{z}$, and thence $-\frac{v\dot{x}}{z} = y = -\frac{c^3}{z^3}\sqrt{az - cc}$.

7. Again, ſuppoſing $a\dot{x} + \frac{s}{1} = z$, (by Prob. 1.) it is $a + s = z$, and thence $\frac{v}{a+s} = y = \frac{v}{a+v}$, which denotes a mechanical Curve.

8. Ex. 2. Let the Circle $ax - xx = vv$ be given again, and let Curves be ſought, whoſe Areas may have any other aſſumed relation to the Area of the Circle. As if you aſſume $cx + s = t$, and ſuppoſe alſo $ax = zz$. (By Prob. 1.) 'tis $c + \dot{s} = \dot{t}$, and $a = 2\dot{z}z$.

Therefore $y = \frac{\dot{t}}{\dot{z}} = \frac{2\dot{c}z + 2\dot{z}z}{a}$; and ſubſtituting $\sqrt{ax - xx}$ for s, and $\frac{zz}{a}$ for x, 'tis $y = \frac{2\dot{c}z}{a} + \frac{2\dot{z}z}{aa}\sqrt{aa - zz}$.

9. But if you aſſume $s - \frac{2v^3}{3a} = t$, and $x = z$, you will have $\dot{s} - \frac{2\dot{v}v^2}{a} = \dot{t}$, and $1 = \dot{z}$. Therefore $y = \frac{\dot{t}}{\dot{z}} = \dot{s} - \frac{2\dot{v}v^2}{a}$, or $= v - \frac{2\dot{v}v^2}{a}$. Now for exterminating $\dot{v}$, the Equation $ax - xx = vv$, (by Prob. 1.) gives $a - 2x = 2\dot{v}v$, and therefore 'tis $y = \frac{2vx}{a}$. Where if you expunge v and x by ſubſtituting their values $\sqrt{ax - xx}$ and z, there will ariſe $y = \frac{2z}{a}\sqrt{az - zz}$.

10. But if you aſſume $ss = t$, and $x = zz$, there will ariſe $2\dot{s}s = \dot{t}$, and $1 = 2\dot{z}z$; and therefore $y = \frac{\dot{t}}{\dot{z}} = 4s\dot{s}z$. And for $\dot{s}$ and x ſubſtituting $\sqrt{ax - xx}$ and zz, it will become $y = 4szz\sqrt{a - zz}$, which is an Equation to a mechanical Curve.

11. Ex. 3. After the ſame manner Figures may be found, which have an aſſumed relation to any other given Figure. Let the Hyperbola $cc + xx = vv$ be given; then if you aſſume $s = t$, and $xx = cz$, you will have $\dot{s} = \dot{t}$ and $2\dot{x}x = c\dot{z}$; and thence $y = \frac{\dot{t}}{\dot{z}} = \frac{c\dot{x}}{2x}$. Then ſubſtituting $\sqrt{cc + xx}$ for $\dot{s}$, and $c^{\frac{1}{2}}z^{\frac{1}{2}}$ for x, it will be $y = \frac{c}{2z}\sqrt{cz + zz}$.

12. And thus if you aſſume $xv - s = t$, and $xx = cz$, you will have $v\dot{x} + \dot{v}x - \dot{s} = \dot{t}$, and $2\dot{x}x = c\dot{z}$. But $v = \dot{s}$, and thence $\dot{v}x = \dot{t}$. Therefore $y = \frac{\dot{t}}{\dot{z}} = \frac{c\dot{v}}{\dot{z}}$. But now (by Prob. 1.) $cc + xx = vv$ gives $x = \dot{v}v$, and 'tis $y = \frac{cx}{2v}$. Then ſubſtituting $\sqrt{cc + xx}$ for v, and $c^{\frac{1}{2}}z^{\frac{1}{2}}$ for x, it becomes $y = \frac{cz}{2\sqrt{cz + zz}}$.

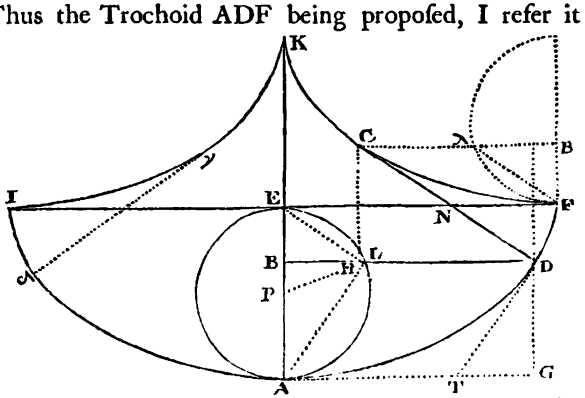
13. Ex. 4. Moreover if the Ciſſoid $\frac{xx}{\sqrt{ax - xx}} = v$ were given, to which other related Figures are to be found, and for that purpoſe you aſſume $\frac{x}{3}\sqrt{ax - xx} + \frac{2}{3}s = t$; ſuppoſe $\frac{x}{3}\sqrt{ax - xx} = b$, and its Fluxion $\dot{b}$; therefore $\dot{b} + \frac{2}{3}\dot{s} = \dot{t}$. But the Equation $\frac{ax^2 - x^4}{9} = bb$

$= hh$ gives $\frac{3ax^2 - 4x^3}{9} = 2\dot{h}h$, where if you exterminate h, it will be $\dot{h} = \frac{3ax - 4xx}{6\sqrt{ax - xx}}\dot{x}$. And beſides ſince it is $\frac{2}{3}\dot{s} = \frac{2}{3}v = \frac{4\dot{x}x}{6\sqrt{ax - xx}}$, 'twill be $\frac{ax}{2\sqrt{ax - xx}} = \dot{t}$. Now to determine z and $\dot{z}$, aſſume $\sqrt{aa - ax} = z$; then (by Prob. 1.) $-a = 2z\dot{z}$, or $\dot{z} = -\frac{a}{2z}$. Wherefore it is $y = \left(\frac{\dot{t}}{\dot{z}} = \frac{-z\dot{x}}{\sqrt{ax - xx}} = \sqrt{\frac{zz\dot{x}}{a - x}} = \sqrt{ax} =\right)$ $\sqrt{aa - zz}$. And as this Equation belongs to the Circle, we ſhall have the relation of the Areas of the Circle and of the Ciſſoid.

14. And thus if you had aſſumed $\frac{2x}{3}\sqrt{ax - xx} + \frac{1}{3}s = t$, and $x = z$, there would have been derived $y = \sqrt{az - zz}$, an Equation again to the Circle.

15. In like manner if any mechanical Curve were given, other mechanical Curves related to it might be found. But to derive geometrical Curves, it will be convenient, that of right Lines depending Geometrically on each other, ſome one may be taken for the Baſe or Abſciſs; and that the Area which compleats the Parallelogram be ſought, by ſuppoſing its Fluxion to be equivalent to the Abſciſs, drawn into the Fluxion of the Ordinate.

16. Ex. 5. Thus the Trochoid ADF being propoſed, I refer it to the Abſciſs AB; and the Parallelogram ABDG being compleated, I ſeek for the complemental Superficies ADG, by ſuppoſing it to be deſcribed by the Motion of the right Line GD, and therefore its Fluxion to be equivalent to the Line GD drawn into the Velocity of the Motion; that is, $x \times v$. Now whereas AL is parallel to the Tangent DT, therefore AB will be to BL as the Fluxion of the ſame AB to the Fluxion of the Ordinate BD, that

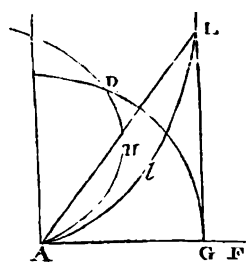


that is, as 1 to $\dot{v}$. So that $\dot{v} = \frac{BL}{AB}$, and therefore $x\dot{v} = BL$. Therefore the Area ADG is deſcribed by the Fluxion BL; ſince therefore the circular Area ALB is deſcribed by the ſame Fluxion, they will be equal.

17. In like manner if you conceive ADF to be a Figure of Arches, or of verſed Sines, that is, whoſe Ordinate BD is equal to the Arch AL; ſince the Fluxion of the Arch AL is to the Fluxion of the Abſciſs AB, as PL to BL, that is, $\dot{v} : 1 :: \frac{1}{2}a : \sqrt{ax - xx}$, then $\dot{v} = \frac{a}{2\sqrt{ax - xx}}$. Then $\dot{v}x$, the Fluxion of the Area ADG, will be $\frac{ax}{2\sqrt{ax - xx}}$. Wherefore if a right Line equal to $\frac{ax}{2\sqrt{ax - xx}}$ be conceived to be apply'd as a rectangular Ordinate at B, a point of the Line AB, it will be terminated at a certain geometrical Curve, whoſe Area, adjoining to the Abſciſs AB, is equal to the Area ADG.

18. And thus geometrical Figures may be found equal to other Figures, made by the application (in any Angle) of Arches of a Circle, of an Hyperbola, or of any other Curve, to the Sines right or verſed of thoſe Arches, or to any other right Lines that may be Geometrically determin'd.

19. As to Spirals, the matter will be very ſhort. For from the Center of Rotation A, the Arch DG being deſcribed, with any Radius AG, cutting the right Line AF in G, and the Spiral in D; ſince that Arch, as a Line moving upon the Abſciſs AG, deſcribes the Area of the Spiral AHDG, ſo that the Fluxion of that Area is to the Fluxion of the Rectangle $1 \times AG$, as the Arch GD to 1; if you raiſe the perpendicular right Line GL equal to that Arch, by moving in like manner upon the ſame Line AG, it will deſcribe the Area A*l*LG equal to the Area of the Spiral AHDG: The Curve A*l*L being a geometrical Curve.



And further, if the Subtenſe AL be drawn, then $\Delta ALG = \frac{1}{2}AG \times GL = \frac{1}{2}AG \times GD =$ Sector AGD; therefore the complemental Segments AL*l* and ADH will alſo be equal. And this not only agrees to the Spiral of *Archimedes*, (in which caſe A*l*L becomes the Parabola of *Apollonius*,) but to any other whatever; ſo that all of them may be converted into equal geometrical Curves with the ſame eaſe.

20.

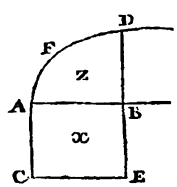
20. I might have produced more Specimens of the Conſtruction of this Problem, but theſe may ſuffice; as being ſo general, that whatever as yet has been found out concerning the Areas of Curves, or (I believe) can be found out, is in ſome manner contain'd herein, and is here determined for the moſt part with leſs trouble, and without the uſual perplexities.

21. But the chief uſe of this and the foregoing Problem is, that aſſuming the Conic Sections, or any other Curves of a known magnitude, other Curves may be found out that may be compared with theſe, and that their defining Equations may be diſpoſed orderly in a Catalogue or Table. And when ſuch a Table is conſtructed, when the Area of any Curve is to be found, if its defining Equation be either immediately found in the Table, or may be transformed into another that is contain'd in the Table, then its Area may be known. Moreover ſuch a Catalogue or Table may be apply'd to the determining of the Lengths of Curves, to the finding of their Centers of Gravity, their Solids generated by their rotation, the Superficies of thoſe Solids, and to the finding of any other flowing quantity produced by a Fluxion analogous to it.

PROB. IX.

To determine the Area of any Curve propoſed.

1. The reſolution of the Problem depends upon this, that from the relation of the Fluxions being given, the relation of the Fluents may be found, (as in Prob. 2.) And firſt, if the right Line BD, by the motion of which the Area required AFDB is deſcribed, move upright upon an Abſciſs AB given in poſition, conceive (as before) the Parallelogram ABEC to be deſcribed in the mean time on the other ſide AB, by a line equal to unity. And BE being ſuppos'd the Fluxion of the Parallelogram, BD will be the Fluxion of the Area required.



2. Therefore make $AB = x$, and then alſo $ABEC = 1 \times x = x$, and $BE = \dot{x}$. Call alſo the Area $AFDB = z$, and it will be $BD = \dot{z}$, as alſo $= \frac{\dot{z}}{\dot{x}}$, becauſe $\dot{x} = 1$. Therefore by the Equation expreſſing BD, at the ſame time the ratio of the Fluions $\frac{\dot{z}}{\dot{x}}$ is

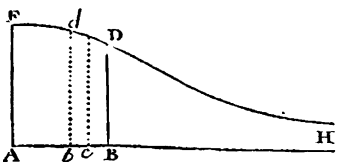
is expreſs'd, and thence (by Prob. 2. Caſe 1.) may be found the relation of the flowing quantities x and z.

3. Ex. 1. When BD, or $\dot{z}$, is equal to ſome ſimple quantity.

4. Let there be given $\frac{xx}{a} = \dot{z}$, or $\frac{\dot{z}}{x}$, (the Equation to the Parabola,) and (Prob. 2.) there will ariſe $\frac{x^3}{3a} = z$. Therefore $\frac{x^3}{3a}$, or $\frac{1}{3}$ AB × BD, = Area of the Parabola AFDB.

5. Let there be given $\frac{x^3}{aa} = \dot{z}$, (an Equation to a Parabola of the ſecond kind,) and there will ariſe $\frac{x^4}{4a^2} = z$, that is, $\frac{1}{4}$ AB × BD = Area AFDB.

6. Let there be given $\frac{a^3}{xx} = \dot{z}$, or $a^3x^{-2} = \dot{z}$, (an Equation to an Hyperbola of the ſecond kind,) and there will ariſe $-a^3\,x^{-1} = z$, or $-\frac{a^3}{x} = z$. That is, AB × BD = Area HDBH, of an infinite length, lying on the other ſide of the Ordinate BD, as its negative value inſinuates.



7. And thus if there were given $\frac{a^4}{x^3} = \dot{z}$, there would ariſe $-\frac{a^4}{2xx} = z$.

8. Moreover, let $ax = z\dot{z}$, or $a^{\frac{1}{2}}x^{\frac{1}{2}} = \dot{z}$, (an Equation again to the Parabola,) and there will ariſe $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = z$, that is, $\frac{2}{3}$AB × BD = Area AFDB.

9. Let $\frac{a^3}{x} = \dot{z}\dot{z}$; then $-2a^{\frac{3}{2}}x^{\frac{1}{2}} = z$, or 2 AB × BD = AFDH.

10. Let $\frac{a^5}{x^3} = \dot{z}\dot{z}$; then $-\frac{2a^{\frac{5}{2}}}{x^{\frac{1}{2}}} = z$, or 2 AB × BD = HDBH.

11. Let $ax^2 = \dot{z}^3$; then $\frac{2}{5}a^{\frac{1}{2}}x^{\frac{5}{2}} = z$, or $\frac{2}{5}$ AB × BD = AFDH. And ſo in others.

12. Ex. 2. Where $\dot{z}$ is equal to an Aggregate of ſuch Quantities.

13. Let $x + \frac{xx}{a} = \dot{z}$; then $\frac{xx}{2} + \frac{xxx}{3a} = z$.

14. Let $a + \frac{a^3}{xx} = \dot{z}$; then $ax - \frac{a^3}{x} = z$.

15. Let $3x^{\frac{1}{2}} - \frac{5}{xx} - \frac{2}{x^{\frac{1}{2}}} = \dot{z}$; then $2x^{\frac{3}{2}} + \frac{5}{x} - 4x^{\frac{1}{2}} = z$.

16. Ex. 3. Where a previous reduction by Diviſion is required.

17. Let there be given $\frac{aa}{b+x}$, $= \dot{z}$ (an Equation to the *Apollonian* Hyperbola,) and the diviſion being performed *in infinitum*, it will be

$\dot{z} = \frac{aa}{b} - \frac{aax}{l^2} + \frac{aax^2}{b^3} - \frac{aax^3}{l^4}$, &c. And thence, (by Prob. 2.) as in the ſecond Set of Examples, you will obtain $z = \frac{a^2x}{b} - \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} - \frac{a^2x^4}{4b^4}$, &c.

18. Let there be given $\frac{1}{1+xx} = \dot{z}$, and by diviſion it will be $\dot{z} = 1 - x^2 + x^4 - x^6$, &c. or elſe $\dot{z} = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6}$, &c. And thence (by Prob. 2.) $z = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7$, &c. = AFDB; or $z = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. = HDBH.

19. Let there be given $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = \dot{z}$, and by diviſion it will be $\dot{z} = 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}$, &c. And thence (by Prob. 2.) $z = \frac{4}{3}x^{\frac{3}{2}} - x^2 + \frac{14}{5}x^{\frac{5}{2}} - \frac{13}{3}x^3 + \frac{68}{7}x^{\frac{7}{2}}$, &c.

20. Ex. 4. Where a previous reduction is required by Extraction of Roots.

21. Let there be given $\dot{z} = \sqrt{aa + xx}$, (an Equation to the Hyperbola,) and the Root being extracted to an infinite multitude of terms, it will be $\dot{z} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{112a^7}$, &c. whence as in the foregoing $z = ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1008a^7}$, &c.

22. In the ſame manner if the Equation $\dot{z} = \sqrt{aa - xx}$ were given, (which is to the Circle,) there would be produced $z = ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1008a^7}$, &c.

23. And ſo if there were given $\dot{z} = \sqrt{x - xx}$, (an Equation alſo to the Circle,) by extracting the Root there would ariſe $\dot{z} = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c. And therefore $z = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$, &c.

24. Thus $\dot{z} = \sqrt{aa + bx - xx}$, (an Equation again to the Circle,) by extraction of the Root it gives $z = a + \frac{bx}{2a} - \frac{xx}{2a} - \frac{b^2x^2}{8a^3}$, &c. whence $z = ax + \frac{bx^2}{4a} - \frac{x^3}{6a} - \frac{b^2x^3}{24a^3}$, &c.

25. And thus $\sqrt{\frac{1+axx}{1-bxx}} = z$, by a due reduction gives

$\dot{z} = 1 + \frac{1}{2}bx^2 + \frac{3}{8}bbx^4$, &c. then $z = x + \frac{1}{6}bx^3 + \frac{3}{40}bbx^5$, &c.
$+ \frac{1}{2}a \quad + \frac{1}{4}ab$ $+ \frac{1}{6}a \quad + \frac{1}{20}ab$
$- \frac{1}{8}aa$ $- \frac{1}{40}aa$

26.

26. Thus finally $\dot{z} = \sqrt[3]{a^3 + x^3}$, by the extraction of the Cubic Root, gives $\dot{z} = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8}$, &c. and then (by Prob. 2.) $z = ax + \frac{x^4}{12a^2} - \frac{x^7}{63a^5} + \frac{x^{10}}{162a^8}$, &c. = AFDB. Or elſe $\dot{z} = \dot{x} + \frac{a^3}{3xx} - \frac{a^6}{9x^5} + \frac{5a^9}{81x^8}$, &c. And thence $z = \frac{x^2}{2} - \frac{a^3}{3x} + \frac{a^6}{36x^4} - \frac{5a^9}{567x^7}$, &c. = HDBH.

27. Ex. 5. Where a previous reduction is required, by the reſolution of an affected Equation.

28. If a Curve be defined by this Equation $\dot{z}^3 + a^2\dot{z} + ax\dot{z} - 2a^3 - x^3 = 0$, extract the Root, and there will ariſe $\dot{z} = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^4}{512aa}$, &c. whence will be obtain'd as before $z = ax - \frac{xx}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$, &c.

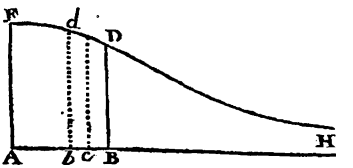
29. But if $\dot{z}^3 - c\dot{z}^2 - 2x^2\dot{z} - c^2\dot{z} + 2x^3 + c^3 = 0$ were the Equation to the Curve, the reſolution will afford a three-fold Root; either $\dot{z} = c + \dot{x} - \frac{xx}{4c} + \frac{x^3}{32c^2}$, &c. or $\dot{z} = c - x + \frac{3x^2}{4c} - \frac{15x^3}{32cc}$, &c. or $\dot{z} = -c - \frac{x^2}{2c} - \frac{x^3}{2cc} + \frac{x^5}{4c^4}$, &c. And hence will ariſe the values of the three correſponding Areas, $z = cx + \frac{1}{2}x^2 - \frac{x^3}{12c} + \frac{x^4}{128c^2}$, &c. $z = cx - \frac{1}{2}x^2 + \frac{x^3}{4c} - \frac{15x^4}{128c^2}$, &c. and $z = -cx - \frac{x^3}{6c} - \frac{x^4}{8c^2} + \frac{x^6}{24c^4}$, &c.

30. I add nothing here concerning mechanical Curves, becauſe their reduction to the form of geometrical Curves will be taught afterwards.

31. But whereas the values of z thus found belong to Areas which are ſituate, ſometimes to a finite part AB of the Abſciſs, ſometimes to a part BH produced infinitely towards H, and ſometimes to both parts, according to their different terms: That the due value of the Area may be aſſign'd, adjacent to any portion of the Abſciſs, that Area is always to be made equal to the difference of the values of z, which belong to the parts of the Abſciſs, that are terminated at the beginning and end of the Area.

32. For Inſtance; to the Curve expreſs'd by the Equation $\frac{1}{1+\dot{x}x} = \dot{z}$,

$= \dot{z}$, it is found that $z = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$, &c. Now that I may determine the quantity of the Area *bd*DB, adjacent to the part of the Abſciſs *b*B; from the value of z which ariſes by putting AB $= x$,

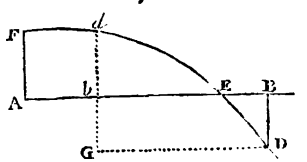


I take the value of z which ariſes by putting A*b* $= x$, and there remains $x - \frac{1}{3}x^3 + \frac{1}{5}x^5$, &c. $- x + \frac{1}{3}x^3 - \frac{1}{5}x^5$, &c. the value of that Area *bd*DB. Whence if A*b*, or x, be put equal to nothing, there will be had the whole Area AFDB $= x - \frac{1}{3}x^3 + \frac{1}{5}x^5$, &c.

33. To the ſame Curve there is alſo found $z = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. Whence again, according to what is before, the Area *bd*DB $= \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5}$, &c. $- \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. Therefore if AB, or x, be ſuppoſed infinite, the adjoining Area *bd*H toward H, which is alſo infinitely long, will be equivalent to $\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5}$, &c. For the latter Series $- \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. will vaniſh, becauſe of its infinite denominators.

34. To the Curve repreſented by the Equation $a + \frac{a^3}{xx} = \dot{z}$, it is found, that $z = ax - \frac{a^3}{x}$. Whence it is that $ax - \frac{a^3}{x} - ax + \frac{a^3}{x} =$ Area *bd*DB. But this becomes infinite, whether x be ſuppoſed nothing, or x infinite; and therefore each Area AFDB and *bd*H is infinitely great, and the intermediate parts alone, ſuch as *bd*DB, can be exhibited. And this always happens when the Abſciſs x is found as well in the numerators of ſome of the terms, as in the denominators of others, of the value of z. But when x is only found in the numerators, as in the firſt Example, the value of z belongs to the Area ſituate at AB, on this ſide the Ordinate. And when it is only in the denominators, as in the ſecond Example, that value, when the ſigns of all the terms are changed, belongs to the whole Area infinitely produced beyond the Ordinate.

35. If at any time the Curve-line cuts the Abſciſs, between the points *b* and B, ſuppoſe in E, inſtead of the Area will be had the difference *bd*E — BDE of the Areas at the different parts of the Abſciſs; to which if there be added the Rectangle BDG*b*, the Area *d*EDG will be obtain'd.



t

36. But it is chiefly to be regarded, that when in the value of z any term is divided by x of only one dimenſion; the Area correſponding to that term belongs to the Conical Hyperbola; and therefore is to be exhibited by it ſelf, in an infinite Series: As is done in what follows.

37. Let $\frac{a^3 - a^2x}{ax + xx} = z$ be an Equation to a Curve; and by diviſion it becomes $z = \frac{aa}{x} - 2a + 2x - \frac{2x^2}{a} + \frac{2x^3}{aa}$, &c. and thence $z = \overline{\left|\frac{aa}{x}\right|} - 2ax + x^2 - \frac{2x^3}{3a} + \frac{x^4}{2a^2}$, &c. And the Area bdDB $= \overline{\left|\frac{aa}{x}\right|} - 2ax + x^2 - \frac{2x^3}{3a}$, &c. $- \overline{\left|\frac{aa}{x}\right|} + 2ax - xx + \frac{2x^3}{3a}$, &c. Where by the Marks $\overline{\left|\frac{aa}{x}\right|}$ and $\overline{\left|\frac{aa}{x}\right|}$ I denote the little Areas belonging to the Terms $\frac{aa}{x}$ and $\frac{aa}{x}$.

38. Now that $\overline{\left|\frac{aa}{x}\right|}$ and $\overline{\left|\frac{aa}{x}\right|}$ may be found, I make Ab, or x, to be definite, and bB indefinite, or a flowing Line, which therefore I call y; ſo that it will be $\overline{\left|\frac{aa}{x+y}\right|}$ = to that Hyperbolical Area adjoining to bB, that is, $\overline{\left|\frac{aa}{x}\right|} - \overline{\left|\frac{aa}{x}\right|}$. But by Diviſion it will be $\frac{aa}{x+y} = \frac{aa}{x} - \frac{a^2y}{x^2} + \frac{a^2y^2}{x^3} - \frac{a^2y^3}{x^4}$, &c. and therefore, $\overline{\left|\frac{aa}{x+y}\right|}$ or $\overline{\left|\frac{aa}{x}\right|} - \overline{\left|\frac{aa}{x}\right|} = \frac{a^2y}{x} - \frac{a^2y^2}{2x^2} + \frac{a^2y^3}{3x^3} - \frac{a^2y^4}{4x^4}$, &c. and therefore the whole Area required bdDB $= \frac{a^2y}{x} - \frac{a^2y^2}{2x^2} + \frac{a^2y^3}{3x^3}$, &c. $- 2ax + x^2 - \frac{2x^3}{3a}$, &c. $+ 2ax - xx + \frac{2x^3}{3a}$, &c.

39. After the ſame manner, AB, or x, might have been uſed for a definite Line, and then it would have been $\overline{\left|\frac{aa}{x}\right|} - \overline{\left|\frac{aa}{x}\right|} = \frac{a^2y}{x} + \frac{a^2y^2}{2x^2} + \frac{a^2y^3}{3x^3} + \frac{a^2y^4}{4x^4}$, &c.

40. Moreover, if bB be biſected in C, and AC be aſſumed to be of a definite length, and Cb and CB indefinite; then making AC $= e$, and Cb or CB $= y$, 'twill be $bd = \frac{aa}{e-y} = \frac{aa}{e} + \frac{e^2y}{e^2} + \frac{a^2y^2}{e^3} + \frac{a^2y^3}{e^4} + \frac{a^2y^4}{e^5}$, &c. and therefore the Hyperbolical Area adjacent

to the Part of the Abſciſs *b*C will be $\frac{a^2y}{e} + \frac{a^2y^2}{2e^2} + \frac{a^2y^3}{3e^3} + \frac{a^2y^4}{4e^4}$, &c. 'Twill be alſo DB $= \frac{aa}{+y} = \frac{aa}{e} - \frac{aay}{e^2} + \frac{aay^2}{e^3} - \frac{aay^3}{e^4} + \frac{aay^4}{e^5}$, &c. And therefore the Area adjacent to the other part of the Abſciſs CB $= \frac{a^2y}{e} - \frac{a^2y^2}{2e^2} + \frac{a^2y^3}{3e^3} - \frac{a^2y^4}{4e^4} + \frac{a^2y^5}{5e^5}$, &c. And the Sum of theſe Areas $\frac{2a^2y}{e} + \frac{2a^2y^3}{3e^3} - \frac{2a^2y^5}{5e^5}$, &c. will be equivalent to $\left|\frac{aa}{x}\right| - \left|\frac{aa}{x}\right|$.

41. Thus in the Equation $\dot{z}^3 + \dot{z}^2 + \dot{z} - \dot{x}^3 = 0$, denoting the nature of a Curve, its Root will be $\dot{z} = \dot{x} - \frac{1}{3} - \frac{2}{9x} + \frac{7}{81xx} + \frac{5}{81x^3}$, &c. Whence there ariſes $z = \frac{1}{2}xx - \frac{1}{3}x - \left|\frac{2}{9x}\right| - \frac{7}{81x} - \frac{5}{162xx}$, &c. And the Area *bd*DB $= \frac{1}{2}x^2 - \frac{1}{3}x - \left|\frac{2}{9x}\right| - \frac{7}{81x}$, &c. $- \frac{1}{2}xx + \frac{1}{3}x + \left|\frac{2}{9x}\right| + \frac{7}{81x}$, &c. that is, $= \frac{1}{2}x^2 - \frac{1}{3}x - \frac{7}{81x}$, &c. $- \frac{1}{2}x^2 + \frac{1}{3}x + \frac{7}{81x}$, &c. $- \frac{4y}{9e} - \frac{4y^3}{27e^3} - \frac{4y^5}{45e^5}$, &c.

42. But this Hyperbolical term, for the moſt part, may be very commodiouſly avoided, by altering the beginning of the Abſciſs, that is, by increaſing or diminiſhing it by ſome given quantity. As in the former Example, where $\frac{a^3 - a^2x}{ax + xx} = \dot{z}$ was the Equation to the Curve, if I ſhould make *b* to be the beginning of the Abſciſs, and ſuppoſing A*b* to be of any determinate length $\frac{1}{2}a$, for the remainder of the Abſciſs *b*B, I ſhall now write x: That is, if I diminiſh the Abſciſs by $\frac{1}{2}a$, by writing $x + \frac{1}{2}a$ inſtead of x, it will become $\frac{\frac{1}{2}a^3 - a^2x}{\frac{3}{4}a^2 + 2ax + x^2} = \dot{z}$, and (by Diviſion) $z = \frac{1}{3}a - \frac{28}{9}x + \frac{200x^2}{27a}$, &c. whence ariſes $z = \frac{4}{3}ax - \frac{14}{9}x^2 + \frac{200x^3}{81a}$, &c. = Area *bd*DB.

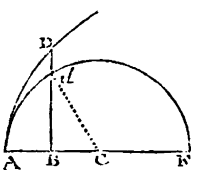
43. And thus by aſſuming another and another point for the beginning of the Abſciſs, the Area of any Curve may be expreſs'd an infinite variety of ways.

44. Alſo the Equation $\frac{a^3 - a^2x}{ax + xx} = \dot{z}$ might have been reſolved into the two infinite Series $z = \frac{a^3}{x^2} - \frac{a^4}{x^3} + \frac{a^5}{x^4}$, &c. $- a + x - \frac{xx}{a} + \frac{x^3}{a^2}$, &c. where there is found no Term divided by the firſt

Power of x. But such kind of Series, where the Powers of x ascend infinitely in the numerators of the one, and in the denominators of the other, are not so proper to derive the value of z from, by Arithmetical computation, when the Species are to be changed into Numbers.

45. Hardly any thing difficult can occur to any one, who is to undertake such a computation in Numbers, after the value of the Area is obtain'd in Species. Yet for the more compleat illustration of the foregoing Doctrine, I shall add an Example or two.

46. Let the Hyperbola AD be proposed, whose Equation is $\sqrt{x+xx} = z$; its Vertex being at A, and each of its Axes is equal to Unity. From what goes before, its Area ADB $= \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{28}x^{\frac{7}{2}} + \frac{2}{72}x^{\frac{9}{2}} - \frac{10}{704}x^{\frac{11}{2}}$, &c. that is $x^{\frac{1}{2}}$ into $\frac{2}{3}x + \frac{2}{5}x^2 - \frac{2}{28}x^3 + \frac{2}{72}x^4 - \frac{10}{704}x^5$, &c. which Series may be infinitely produced by multiplying the last term continually by the succeeding terms of this Progression $\frac{1\cdot3}{2\,5}x$. $\frac{-1\cdot5}{4\,7}x$. $\frac{-3\,7}{6.9}x$. $\frac{-5\cdot9}{8.11}x$. $\frac{-7\cdot11}{10.13}x$. &c. That is, the first term $\frac{2}{3}x^{\frac{3}{2}} \times \frac{1\cdot3}{2\cdot5}x$ makes the second term $\frac{2}{5}x^{\frac{5}{2}}$: Which multiply'd by $\frac{-1\cdot5}{4\,7}x$ makes the third term $-\frac{2}{28}x^{\frac{7}{2}}$: Which multiply'd by $\frac{-3\cdot7}{6.9}x$ makes $\frac{2}{72}x^{\frac{9}{2}}$ the fourth term; and so *ad infinitum*. Now let AB be assumed of any length, suppose $\frac{1}{4}$, and writing this Number for x, and its Root $\frac{1}{2}$ for $x^{\frac{1}{2}}$, and the first term $\frac{2}{3}x^{\frac{3}{2}}$ or $\frac{2}{3} \times \frac{1}{8}$, being reduced to a decimal Fraction, it becomes 0.083333333, &c. This into $\frac{1\cdot3}{2\,5\cdot4}$ makes 0.00625 the second term. This into $\frac{-1\cdot5}{4\cdot7\,4}$ makes $-$ 0.0002790178, &c. the third term. And so on for ever. But the terms, which I thus deduce by degrees, I dispose in two Tables; the affirmative terms in one, and the negative in another, and I add them up as you see here.

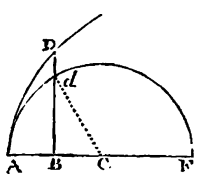


+ 0.

+ 0.0833333333333333	— 0.0002790178571429
62500000000000	3467906605 1
2712673611 11	8344650 27
513516939 6	2628535 4
144628917	96129 6
4954581	3867 6
190948	166 3
7963	7 5
352	4
16	
1	— 0.0002825719389575
+ 0.0896109885646518	+ 0.0896109885646518
+ 0.0896109885646518	0.0893284166257043

Then from the ſum of the Affirmatives I take the ſum of the negatives, and there remains 0.0893284166257043 for the quantity of the Hyperbolic Area ADB; which was to be found.

47. Now let the Circle A*d*F be propoſed, which is expreſſed by the equation $\sqrt{x - xx} = z$; that is, whoſe Diameter is unity, and from what goes before its Area A*d*B will be $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{14}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$, &c. In which Series, ſince the terms do not differ from the terms of the Series, which above expreſs'd the Hyperbolical Area, unleſs in the Signs + and —; nothing elſe remains to be done, than to connect the ſame numeral terms with other ſigns; that is, by ſubtracting the connected ſums of both the afore-mention'd tables, 0.0898935605036193 from the firſt term doubled 0.1666666666666, &c. and the remainder 0.0767731061630473 will be the portion A*d*B of the circular Area, ſuppoſing AB to be a fourth part of the diameter. And hence we may obſerve, that tho' the Areas of the Circle and Hyperbola are not compared in a Geometrical conſideration, yet each of them is diſcover'd by the ſame Arithmetical computation.

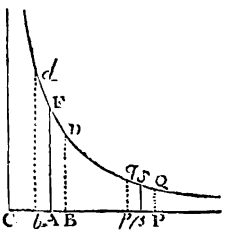


48. The portion of the circle A*d*B being found, from thence the whole Area may be derived. For the Radius *d*C being drawn, multiply B*d*, or $\frac{1}{4}\sqrt{3}$, into BC, or $\frac{1}{4}$, and half of the product $\frac{1}{32}\sqrt{3}$, or 0.0541265877365275 will be the value of the Triangle C*d*B; which added to the Area A*d*B, there will be had the Sector AC*d* = 0.1308996938995747, the ſextuple of which 0.7853981633974482 is the whole Area.

49. And

49. And hence by the way the length of the Circumference will be 3.14159265358979 28, by dividing the Area by a fourth part of the Diameter.

50. To theſe we ſhall add the calculation of the Area comprehended between the Hyperbola *d*FD and its Aſymptote CA. Let C be the Center of the Hyperbola, and putting CA $=a$, AF $=b$, and AB $=$ Ab $=x$; 'twill be $\frac{ab}{a+x}$ $=$BD, and $\frac{ab}{a-x}$ $=bd$; whence the Area AFDB $=bx-\frac{bxx}{2a}+\frac{bx^3}{3a^2}-\frac{bx^4}{4a^3}$, &c. and the Area A$fdb$ $=bx+\frac{bx^2}{2a}+\frac{bx^3}{3a^2}+\frac{bx^4}{4a^3}$, &c. and the ſum bdDB $=2bx+\frac{2bx^3}{3a^2}+\frac{2bx^5}{5a^4}+\frac{2bx^7}{7a^6}$, &c. Now let us ſuppoſe CA $=$ AF $=1$, and Ab or AB $=\frac{1}{10}$, Cb being 0.9, and CB $=1.1$; and ſubſtituting theſe numbers for a, b, and x, the firſt term of the Series becomes 0.2, the ſecond 0.0006666666, &c. the third 0.000004; and ſo on, as you ſee in this Table.



0.2000000000000000
6666666666666
40000000000
2857142 86
2222222
18182
154
1

The ſum 0.2006706954621511 $=$ Area bdDB.

51. If the parts of this Area Ad and AD be deſired ſeparately, ſubtract the leſſer BA from the greater dA, and there will remain $\frac{bx^2}{a}+\frac{bx^4}{2a^2}+\frac{bx^6}{3a^3}+\frac{bx^8}{4a^4}$, &c. Where if 1 be wrote for a and b, and $\frac{1}{10}$ for x, the terms being reduced to decimals will ſtand thus;

0.0100000000000000
500000000000
3333333333
25000000
200000
1667
14

The ſum 0.0100503358535014 $=$ Ad — AD.

52.

52. Now if this difference of the Areas be added to, and ſubtracted from, their ſum before found, half the aggregate 0.1053605156578263 will be the greater Area Ad, and half of the remainder 0.0953101798043248 will be the leſſer Area AD.

53. By the ſame tables thoſe Areas AD and Ad will be obtain'd alſo, when AB and Ab are ſuppos'd $\frac{1}{100}$, or CB = 1.01, and Cb = 0.99, if the numbers are but duly transferr'd to lower places, as may be here ſeen.

0.0200000000000000	0.0001000000000000
6666666666	5000000
400000	3333
28	
Sum 0.0200006667066695 = bD.	Sum 0.0001000050003333 Ad — AD.

$\frac{1}{2}$Aggr.0.0100503358535014=Ad,and $\frac{1}{2}$Reſid.0.0099503308531681 =AD.

54. And ſo putting AB and $Ab = \frac{1}{1000}$, or CB = 1.001, and Cb = 0.999, there will be obtain'd Ad = 0.0010005003335835, and AD = 0.0009995003330835.

55. In the ſame manner (if CA and AF = 1) putting AB and Ab = 0.2, or 0.02, or 0.002, theſe Areas will ariſe,

Ad = 0.2231435513142097, and AD = 0.1823215567939546,
or Ad = 0.0202027073175194, and AD = 0.0198026272961797,
or Ad = 0.002002 and AD = 0.001

56. From theſe Areas thus found it will be eaſy to derive others, by addition and ſubtraction alone. For as it is $\frac{1.2}{0.8}$ into $\frac{1.2}{0.9}$ = 2, the ſum of the Areas 0.6931471805599453 belonging to the Ratio's $\frac{1.2}{0.8}$ and $\frac{1.2}{0.9}$, (that is, inſiſting upon the parts of the Abſciſs 1.2 — 0.8 and 1.2 — 0.9,) will be the Area AF$\delta\beta$, Cβ being = 2, as is known. Again, ſince $\frac{1.2}{0.8}$ into 2 = 3, the ſum 1.0986122886681097 of the Area's belonging to $\frac{1.2}{0.8}$ and 2, will be the Area AF$\delta\beta$, Cβ being 3. Again, as it is $\frac{2\times2}{0.8}$ = 5, and 2 × 5 = 10, by a due addition of Areas will be obtain'd 1.6093379124341004 = AF$\delta\beta$, when Cβ = 5; and 2.3025850929940457 = AF$\delta\beta$, when Cβ = 10. And thus, ſince 10 × 10 = 100, and 10 × 100 = 1000, and $\sqrt{5}$ × 10 × 0.98 = 7, and 10 × 1.1 = 11, and $\frac{1000\times1.001}{7\times11}$ = 13, and $\frac{100\times0.998}{2}$ = 499; it is plain, that the Area AF$\delta\beta$ may be found by the compoſition of the Areas found before, when Cβ = 100; 1000;

73

7; or any other of the above-mention'd numbers, AB = BF being ſtill unity. This I was willing to inſinuate, that a method might be derived from hence, very proper for the conſtruction of a Canon of Logarithms, which determines the Hyperbolical Areas, (from which the Logarithms may eaſily be derived,) correſponding to ſo many Prime numbers, as it were by two operations only, which are not very troubleſome. But whereas that Canon ſeems to be derivable from this fountain more commodiouſly than from any other, what if I ſhould point out its conſtruction here, to compleat the whole?

57. Firſt therefore having aſſumed 0 for the Logarithm of the number 1, and 1 for the Logarithm of the number 10, as is generally done, the Logarithms of the Prime numbers 2, 3, 5, 7, 11, 13, 17, 37, are to be inveſtigated, by dividing the Hyperbolical Areas now found by 2.30258509299404 57, which is the Area correſponding to the number 10: Or which is the ſame thing, by multiplying by its reciprocal 0.43429448 19032518. Thus for Inſtance, if 0.69314718, &c. the Area correſponding to the number 2, were multiply'd by 0.43429, &c. it makes 0.30102999566398 12 the Logarithm of the number 2.

58. Then the Logarithms of all the numbers in the Canon, which are made by the multiplication of theſe, are to be found by the addition of their Logarithms, as is uſual. And the void places are to be interpolated afterwards, by the help of this Theorem.

59. Let n be a Number to which a Logarithm is to be adapted, x the difference between that and the two neareſt numbers equally diſtant on each ſide, whoſe Logarithms are already found, and let d be half the difference of the Logarithms. Then the required Logarithm of the Number n will be obtain'd by adding $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$, &c. to the Logarithm of the leſſer number. For if the numbers are expounded by Cp, Cβ, and CP, the rectangle CBD or C$\beta\delta$ = 1, as before, and the Ordinates pq and PQ being raiſed; if n be wrote for Cβ, and x for βp or βP, the Area pqQP or $\frac{2x}{n} + \frac{2x^3}{3n^3} + \frac{2x^5}{5n^5}$, &c. will be to the Area $pq\delta\beta$ or $\frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3}$, &c. as the difference between the Logarithms of the extream numbers or $2d$, to the difference between the Logarithms of the leſſer and of the middle

one; which therefore will be $\frac{\frac{dx}{n} + \frac{dx^2}{2n^2} + \frac{dx^3}{3n^3} \text{ \&c.}}{\frac{x}{n} + \frac{x^3}{3n^3} + \frac{x^5}{5n^5} \text{ \&c.}}$, that is, when the diviſion is perform'd, $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$ &c.

60. The two firſt terms of this Series $d + \frac{dx}{2n}$ I think to be accurate enough for the conſtruction of a Canon of Logarithms, even tho' they were to be produced to fourteen or fifteen figures; provided the number, whoſe Logarithm is to be found, be not leſs than 1000. And this can give little trouble in the calculation, becauſe x is generally an unit, or the number 2. Yet it is not neceſſary to interpolate all the places by the help of this Rule. For the Logarithms of numbers which are produced by the multiplication or diviſion of the number laſt found, may be obtain'd by the numbers whoſe Logarithms were had before, by the addition or ſubtraction of their Logarithms. Moreover by the differences of the Logarithms, and by their ſecond and third differences, if there be occaſion, the void places may be more expeditiouſly ſupply'd; the foregoing Rule being to be apply'd only, when the continuation of ſome full places is wanted, in order to obtain thoſe differences.

61. By the ſame method rules may be found for the intercalation of Logarithms, when of three numbers the Logarithms of the leſſer and of the middle number are given, or of the middle number and of the greater; and this although the numbers ſhould not be in Arithmetical progreſſion.

62. Alſo by purſuing the ſteps of this method, rules might be eaſily diſcover'd, for the conſtruction of the tables of artificial Sines and Tangents, without the aſſiſtance of the natural Tables. But of theſe things only by the bye.

63. Hitherto we have treated of the Quadrature of Curves, which are expreſs'd by Equations conſiſting of complicate terms; and that by means of their reduction to Equations, which conſiſt of an infinite number of ſimple terms. But whereas ſuch Curves may ſometimes be ſquared by finite Equations alſo, or however may be compared with other Curves, whoſe Areas in a manner may be conſider'd as known; of which kind are the Conic Sections: For this reaſon I thought fit to adjoin the two following catalogues or tables of Theorems, according to my promiſe, conſtructed by the help of the *7th* and *8th* aforegoing Propoſitions.

64.

64. The firſt of theſe exhibits the Areas of ſuch Curves as can be ſquared; and the ſecond contains ſuch Curves, whoſe Areas may be compared with the Areas of the Conic Sections. In each of theſe, the letters *d*, *e*, *f*, *g*, and *h*, denote any given quantities, *x* and *z* the Abſciſſes of Curves, *v* and *y* parallel Ordinates, and *s* and *t* Areas, as before. The letters *n* and θ, annex'd to the quantity *z*, denote the number of the dimenſions of the ſame *z*, whether it be integer or fractional, affirmative or negative. As if $n = 3$, then $z^n = z^3$, $z^{2n} = z^6$, $z^{-n} = z^{-3}$ or $\frac{1}{z^3}$, $z^{n+1} = z^4$, and $z^{n-1} = z^2$.

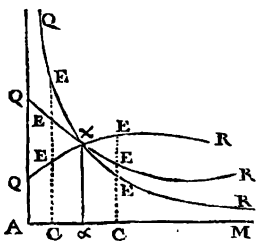
65. Moreover in the values of the Areas, for the ſake of brevity, is written R inſtead of this Radical $\sqrt{e + fz^n}$, or $\sqrt{e + fz^n + gz^{2n}}$, and *p* inſtead of $\sqrt{h + iz^n}$, by which the value of the Ordinate *y* is affected.

66. *The first Table, of some Curvilinear Areas related to Rectilinear Figures, constructed by Prob.* 7.

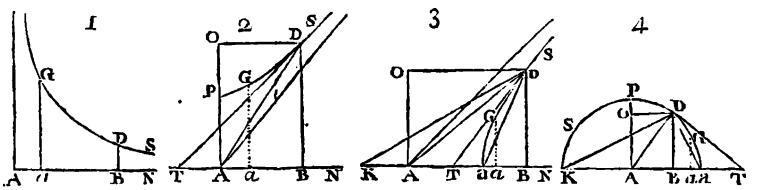
Order of Curves.		Values of their Areas.
I.	$dz^{n-1} = y$	$\frac{d}{n} z^n = t.$
II.	$\frac{dz^{n-1}}{e + 2efz^n + ffz^{2n}} = y$	$\frac{dz^n}{ve^2 + nefz^n} = t$, or $\frac{-d}{nef + nffz^n} = t.$
III. 1	$dz^{n-1}\sqrt{e+fz^n} = y$	$\frac{2d}{3nf} \mathrm{R}^3 = t.$
III. 2	$dz^{2n-1}\sqrt{e+fz^n} = y$	$\frac{-4e+6fz^n}{15nff} d\mathrm{R}^3 = t.$
III. 3	$dz^{3n-1}\sqrt{e+fz^n} = y$	$\frac{16ee-24efz^n+30ffz^{2n}}{105nf^3} d\mathrm{R}^3 = t.$
III. 4	$dz^{4n-1}\sqrt{e+fz^n} = y$	$\frac{-96e^3+144e^2fz^n-180ef^2z^{2n}+210f^3z^{3n}}{945nf^4} d\mathrm{R}^3 = t.$
IV. 1	$\frac{dz^{n-1}}{\sqrt{e+fz^n}} = y$	$\frac{2d}{nf} \mathrm{R} = t.$
IV. 2	$\frac{dz^{2n-1}}{\sqrt{e+fz^n}} = y$	$\frac{-4e+2fz^n}{3nff} d\mathrm{R} = t.$
IV. 3	$\frac{dz^{3n-1}}{\sqrt{e+fz^n}} = y$	$\frac{16e^2-8efz^n+6ffz^{2n}}{15nf^3} d\mathrm{R} = t.$
IV. 4	$\frac{dz^{4n-1}}{\sqrt{e+fz^n}} = y$	$\frac{-96e^3+48e^2fz^n-36ef^2z^{2n}+30f^3z^{3n}}{105nf^4} d\mathrm{R} = t.$

	Order of Curves.	Values of Areas.
V. 1	$2\theta e z^{\theta-1} + 2\theta f z^{\theta+\eta-1} + 3\eta f$ into $\frac{1}{2}\sqrt{e+fz^{\eta}} = y.$	$z^{\theta}R^{2} = t.$
V. 2	$2\theta e z^{\theta-1} + 2\theta f z^{\theta+\eta-1} + 2\theta g z^{\theta+2\eta-1} + 3\eta f + 6\eta g$ into $\frac{1}{2}\sqrt{e+fz^{\eta}+gz^{2\eta}} = y.$	$z^{\theta}R^{3} = t.$
VI. 1	$\frac{2\theta e z^{\theta-1} + \overline{2\theta+\eta}\times f z^{\theta+\eta-1}}{2\sqrt{e+fz^{\eta}}} = y.$	$z^{\theta}R = t.$
VI. 2	$\frac{2\theta e z^{\theta-1} + \overline{2\theta+\eta}\times f z^{\theta+\eta-1} + \overline{2\theta+2\eta}\times g z^{\theta+2\eta-1}}{2\sqrt{e+fz^{\eta}+gz^{2\eta}}} = y.$	$z^{\theta}R = t.$
VII. 1	$\frac{2\theta e z^{\theta-1} + \overline{2\theta-\eta}\times f z^{\theta+\eta-1}}{e+fz^{\eta} \text{ into } 2\sqrt{e+fz^{\eta}}} = y.$	$\frac{z^{\theta}}{R} = t.$
VII. 2	$\frac{2\theta e z^{\theta-1} + \overline{2\theta-\eta}\times f z^{\theta+\eta-1} + \overline{2\theta-2\eta}\times g z^{\theta+2\eta-1}}{e+fz^{\eta}+gz^{2\eta} \text{ into } 2\sqrt{e+fz^{\eta}+gz^{2\eta}}} = y.$	$\frac{z^{\theta}}{R} = t.$
VIII. 1	$\frac{2\theta e z^{\theta-1} + \overline{2\theta-2\eta}\times f z^{\theta+\eta-1}}{ee+2efz^{\eta}+ff z^{2\eta}} = 2y.$	$\frac{z^{\theta}}{R^{2}}\left(\text{or } \frac{z\theta}{e+fz^{\eta}}\right) = t.$
VIII. 2	$\frac{2\theta e z^{\theta-1} + \overline{2\theta-2\eta}\times f z^{\theta+\eta-1} + \overline{2\theta-4\eta}\times g z^{\theta+2\eta-1}}{e^{2}+2efz^{\eta}+ff+2eg\times z^{2\eta}+2fgz^{3\eta}+ggz^{4\eta}} = 2y.$	$\frac{z^{\theta}}{R^{2}}\left(\text{or } \frac{z^{\theta}}{e+fz^{\eta}+gz^{2\eta}}\right) = t.$
IX.	$2\theta e h z^{\theta-1} + \overline{2\theta+3\eta}\times f h z^{\theta+\eta-1} + \overline{2\theta+4\eta}\times f i z^{\theta+2\eta-1} + 2\theta+\eta\times ei$ into $\frac{\sqrt{e+fz^{\eta}}}{2\sqrt{b+iz^{\eta}}} = y.$	$z^{\theta}R^{3}p = t.$
X.	$2\theta e h z^{\theta-1} + \overline{2\theta+3\eta}\times f h z^{\theta+\eta-1} + \overline{2\theta+2\eta}\times f i z^{\theta+2\eta-1} + 2\theta-\eta\times ei$ into $\frac{\sqrt{e+fz^{\eta}}}{b+iz^{\eta} \text{ into } 2\sqrt{b+iz^{\eta}}} = y.$	$\frac{z^{\theta}R^{3}}{p} = t.$

67. Other things of the ſame kind might have been added; but I ſhall now paſs on to another ſort of Curves, which may be compared with the Conic Sections. And in this Table or Catalogue you have the propoſed Curve repreſented by the Line QEχR, the beginning of whoſe Abſciſs is A, the Abſciſs AC, the Ordinate CE, the beginning of the Area $\alpha\chi$, and the Area deſcribed $\alpha\chi$EC. But the beginning of this Area, or the initial term, (which commonly either commences at the beginning of the Abſciſs A, or recedes to an infinite diſtance,) is found by ſeeking the length of the Abſciſs Aα, when the value of the Area is nothing, and by erecting the perpendicular $\alpha\chi$.



68. After the ſame manner you have the Conic Section repreſented by the Line PDG, whoſe Center is A, Vertex a, rectangular



Semidiameters Aa and AP, the beginning of the Abſciſs A, or a, or α, the Abſciſs AB, or aB, or αB, the Ordinate BD, the Tangent DT meeting AB in T, the Subtenſe aD, and the Rectangle inſcribed or adſcribed ABDO.

69. Therefore retaining the letters before defined, it will be AC $=z$, CE $=y$, $\alpha\chi$EC $=t$, AB or aB $=x$, BD $=v$, and ABDP or aGDB $=s$. And beſides, when two Conic Sections are required, for the determination of any Area, the Area of the latter ſhall be call'd σ, the Abſciſs ξ, and the Ordinate Υ. Put p for $\sqrt{ff-4eg}$.

70. *The second Table, of some Curvilinear Areas, related to the Conic Sections, constructed by the help of Prob. 8.*

	Forms of Curves.	Conic Sections. Abscis.	Conic Sections. Ordinate.	Values of the Areas.	
I. 1	$\frac{dz^{n-1}}{e+fz^n}=y$	$z^n=x$	$\frac{d}{e+fx}=v$	$\frac{1}{n}s=t=\frac{\alpha\mathrm{GDB}}{n}$.	*Fig.* 1.
I. 2	$\frac{az^{2n-1}}{e+fz^n}=y$	$z^n=x$	$\frac{d}{e+fx}=v$	$\frac{d}{nf}z^n-\frac{e}{nf}s=t$.	
I. 3	$\frac{dz^{3n-1}}{e+fz^n}=y$	$z^n=x$	$\frac{d}{e+fx}=v$	$\frac{d}{2nf}z^{2n}-\frac{de}{nf^2}z^n+\frac{e^2}{nf^2}s=t$.	
II. 1	$\frac{dz^{\frac{1}{2}n-1}}{+fz^n}=y$	$\sqrt{\frac{d}{e+fz^n}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$	$\frac{2xv-4s}{n}=t=\frac{4}{n}\mathrm{ADGa}$.	*Fig.* 3, 4.
II. 2	$\frac{dz^{\frac{3}{2}n-1}}{e+fz^n}=y$	$\sqrt{\frac{d}{e+fz^n}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$	$\frac{2d}{nf}z^{\frac{1}{2}n}+\frac{4es-2exv}{nf}=t$.	
II. 3	$\frac{dz^{\frac{1}{2}n-1}}{e+fz^n}=y$	$\sqrt{\frac{d}{e+fz^n}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$	$\frac{2d}{3nf}z^{\frac{3}{2}n}-\frac{2de}{nf^2}z^{\frac{1}{2}n}+\frac{2e^2xv-4e^2s}{nf^2}=t$.	
III. 1	$\frac{1}{z}\sqrt{e+fz^n}=y$.	$\frac{1}{z^n}=x^2$	$\sqrt{f+ex^2}=v$	$\frac{4de}{nf}\times\frac{\overline{xv}^3}{2ex}-s=t=\frac{4de}{nf}$ into aGDT, or into APDB $\div$ TDB.	[*Fig.* 2, 3, 4.
	or thus	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{8de^2}{nf^2}\times s-\frac{1}{2}xv-\frac{fv}{4e}+\frac{f^2v}{4e^2x}=t=\frac{8de^2}{nf^2}$ into aGDA $+\frac{f^2v}{4e^2x}$	[*Fig.* 3, 4.
III. 2	$\frac{d}{z^{n+1}}\sqrt{e+fz^n}=y$	$\frac{1}{z^n}=x^2$	$\sqrt{f+ex^2}=v$	$-\frac{2d}{n}s=t=\frac{2d}{n}$ APDB or $\frac{2d}{n}$ aGDB.	*Fig.* 2, 3, 4.
	or thus	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{4de}{nf}\times s-\frac{1}{2}xv-\frac{fv}{2e}=t=\frac{4de}{nf}\times$ aGDK.	*Fig.* 3, 4.
	$\frac{d}{z^{2n+1}}\sqrt{e+fz^n}=y$	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$-\frac{d}{n}s=t=\frac{d}{n}\times-$ aGDB or BDPK.	*Fig.* 4.
	$\frac{d}{z^{3n+1}}\sqrt{e+fz^n}=y$	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{3dfs-2dv^3}{6ne}=t$.	

Forms of Curves.	Conic Sections. Abſciſs.	Ordinate.	Values of the Areas.
IV. 1 $\frac{d}{z\sqrt{e+fz^n}}=y$	$\frac{1}{z^n}=x^2$	$\sqrt{f+ex^2}=v$	$\frac{4d}{nf}\times\overline{\frac{1}{2}xv\div s}=t=\frac{4d}{nf}$ into PAD or into aGDA. *Fig.* 2, 3, 4.
or thus	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{8de}{nf^2}\times\overline{s-\frac{1}{2}xv-\frac{fv}{4e}}=t=\frac{8de}{nf^2}$ into aGDA. *Fig.* 3, 4.
2 $\frac{d}{z^{n+1}\sqrt{e+fz^n}}=y$	$\frac{1}{z^n}=x^2$	$\sqrt{f+ex^2}=v$	$\frac{2d}{ne}\times\overline{s-xv}=t=\frac{2d}{ne}$ into POD or into AODGa. *Fig.* 2, 3, 4
or thus	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{4d}{nf}\times\overline{\frac{1}{2}xv\div s}=t=\frac{4d}{nf}$ into aDG*a*. *Fig.* 3, 4
3 $\frac{d}{z^{2n+1}\sqrt{e+fz^n}}=y$	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{d}{ne}\times\overline{3s\div 2xv}=t=\frac{d}{ne}$ into 3aDGa $\div$ ΔaDB. *Fig.* 3, 4
4 $\frac{d}{z^{3n+1}\sqrt{e+fz^n}}=y$	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{10dfxv-15dfs-2dex^2v}{6ne}=t.$
V. 1 $\frac{dz^{n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\sqrt{\frac{d}{g}+\frac{f^2-4eg}{4g^2}x^2}=v$	$\frac{xv-2s}{n}=t.$
or thus	$\sqrt{\frac{dz^{2n}}{e+fz^n+gz^{2n}}}=x$	$\sqrt{\frac{d}{e}+\frac{f^2-4eg}{4e^2}x^2}=v$	$\frac{2s-xv}{n}=t.$
2 $\frac{dz^{2n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$, $fz^n+gz^{2n}=\xi$	$\sqrt{\frac{d}{g}+\frac{f^2-4eg}{4g^2}x^2}=v$, $\frac{1}{e+\xi}=\Upsilon$	$\frac{d\sigma+2fs-fxv}{2ng}=t.$
VI. 1 $\frac{dz^{\frac{1}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{2dg}{f-p+2gz^n}}=x$, $\sqrt{\frac{2dg}{f+p+2gz^n}}=\xi$	$\sqrt{d+\frac{-f+p}{2g}x^2}=v$, $\sqrt{d+\frac{-f-p}{2g}\xi^2}=\Upsilon$	$\frac{2xv-4s-2\xi\Upsilon+4\sigma}{np}=t.$
2 $\frac{dz^{\frac{3}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{2dez^n}{fz^n-pz^n+2e}}=x$, $\sqrt{\frac{2dez^n}{fz^n+pz^n+2e}}=\xi$	$\sqrt{d+\frac{-f+p}{2e}x^2}=v$, $\sqrt{d+\frac{-f-p}{2e}\xi^2}=\Upsilon$	$\frac{4s-2xv-4\sigma+2\xi\Upsilon}{np}=t.$

Forms of Curves.	Conic Sections. Abſciſs.	Conic Sections. Ordinate.	Values of the Areas.
VII. 1 $\frac{d}{z}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$; $\frac{1}{z^n}=\xi$	$\sqrt{e+fx+gx^2}=v$; $\sqrt{g+f\xi+e\xi^2}=\upsilon$	$\frac{4de^2\xi\upsilon+2defv-2dfgxv-\overset{+4deg}{2ffd}v-8de^2\sigma+4dfgs}{4neg-nff}=t.$
VII. 2 $fz^{n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{d}{n}s=t=\frac{d}{n}\times\alpha\text{GDB}.$ *Fig.* 2, 3, 4.
VII. 3 $dz^{2n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{d}{3ng}v^3-\frac{df}{2ng}s=t.$
VII. 4 $dz^{3n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{6dgx-5df}{24ng^2}v^3+\frac{5df^2-4deg}{16ng^2}s=t.$
VIII. 1 $\frac{dz^{n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{8dgs-4dgxv-2dfv}{4neg-nf^2}=t=\frac{8dz}{4neg-nf^2}\times\alpha\text{GDB}=\triangle\text{DBA}$ [*Fig.* 2, 4
VIII. 2 $\frac{dz^{2n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{-4dfs+2dfxv+4dev}{4neg-nf^2}=t.$
VIII. 3 $\frac{dz^{3n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{3dff\ s-2dff\ xv-2defv}{-4deg\ +4deg}=t.$
VIII. 4 $\frac{dz^{4n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{36defg\ s+8degg\ x^2v+10dfff\ xv+10deff\ v}{-15df^3-2df^2g\ x^2v-28defg\ xv-16de^2g\ v}=t.$
IX. 1 $\frac{dz^{n-1}\sqrt{e+fz^n}}{g+hz^n}=y$	$\sqrt{\frac{d}{g+hz^n}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$	$\frac{4fg\ s-2fg\ xv+2df\frac{v}{x}}{-4eh\ +2eh}=t.$
IX. 2 $\frac{dz^{2n-1}\sqrt{e+fz^n}}{g+hz^n}=y$	$\sqrt{\frac{d}{g+hz^n}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$	$\frac{4egh\ s-2egh\ xv+\frac{2}{3}dh\frac{v^3}{x^3}-2dfg\frac{v}{x}}{-4fgg\ +2fgg}=t.$

	Forms of Curves.	Conic Sections. Abfciſs.	Ordinate.	Values of the Areas.
X. 1	$\frac{dz^{n-1}}{g+bz^n\sqrt{e+fz^n}}=y$	$\sqrt{\frac{d}{g+bz^n}}=x$	$\sqrt{\frac{af}{b}+\frac{eb-fg}{b}x^2}=v$	$\frac{2xv-4s}{nf}=t=\frac{4}{nf}$ ADGa. *Fig.* 3, 4
X. 2	$\frac{dz^{2n-1}}{g+bz^n\sqrt{e+fz^n}}=y$	$\sqrt{\frac{d}{g+bz^n}}=x$	$\sqrt{\frac{df}{b}+\frac{eb-fg}{b}x^2}=v$	$\frac{2gs-2gxv+2d\frac{v}{x}}{nfb}=t.$
XI. 1	$dz^{-1}\sqrt{\frac{e+fz^n}{g+bz^n}}=y$	$\left\{\begin{array}{l}\sqrt{g+bz^n}=x\\ \sqrt{b+gz^{-n}}=\xi\end{array}\right.$	$\left\{\begin{array}{l}\sqrt{\frac{\cdot\cdot}{b}+\frac{f}{b}x^2}=v\\ \sqrt{\frac{f-eb}{g}+\frac{e}{g}\xi^2}=r\end{array}\right.$	$\frac{2dxv^3z^{-n}-4dfs-4de\sigma}{nfg-neb}=t.$
XI. 2	$dz^{n-1}\sqrt{\frac{e+fz^n}{g+bz^n}}=y$	$\sqrt{g+bz^n}=x$	$\sqrt{\frac{eb-fg}{b}+\frac{f}{b}x^2}=v$	$\frac{2d}{nb}s=t.$
XI. 3	$dz^{2n-1}\sqrt{\frac{e+fz^n}{g+bz^n}}=y$	$\sqrt{g+bz^n}=x$	$\sqrt{\frac{\cdot\cdot-fg}{b}+\frac{f}{b}x^2}=v$	$\frac{d^3xv^3-3dfgs-deb\,s}{2nfb^2}=t.$

71. Before I go on to illuſtrate by Examples the Theorems that are deliver'd in theſe claſſes of Curves, I think it proper to obſerve,

72. I. That whereas in the Equations repreſenting Curves, I have all along ſuppoſed all the ſigns of the quantities d, e, f, g, h, and i to be affirmative; whenever it ſhall happen that they are negative, they muſt be changed in the ſubſequent values of the Abſciſs and Ordinate of the Conic Section, and alſo of the Area required.

73. II. Alſo the ſigns of the numeral Symbols n and 0, when they are negative, muſt be changed in the values of the Areas. Moreover their Signs being changed, the Theorems themſelves may acquire a new form. Thus in the 4th Form of Table 2, the Sign of n being changed, the 3d Theorem becomes $\frac{d}{z^{-2n+1}\sqrt{e+fz^{-n}}} = y$, $\frac{1}{z^{-n}} = x$, &c. that is, $\frac{dz^{3n-1}}{\sqrt{ez^{2n}+fz^{n}}} = y$, $z^{n} = x$, $\sqrt{fx+ex^{2}} = v$, $\frac{d}{v^{2}}$ into $2xv - 3s = t$. And the ſame is to be obſerved in others.

74. III. The ſeries of each order, excepting the 2d of the 1ſt Table, may be continued each way *ad infinitum*. For in the Series of the 3d and 4th Order of Table 1, the numeral co-efficients of the initial terms, (2, —4, 16, —96, 768, &c.) are form'd by multiplying the numbers —2, —4, —6, —8, —10, &c. continually into each other; and the co-efficients of the ſubſequent terms are derived from the initials in the 3d Order, by multiplying gradually by $-\frac{3}{2}$, $-\frac{5}{4}$, $-\frac{7}{6}$, $-\frac{9}{8}$, $-\frac{11}{10}$, &c. or in the 4th Order by multiplying by $-\frac{1}{2}$, $-\frac{3}{4}$, $-\frac{5}{6}$, $-\frac{7}{8}$, $-\frac{9}{10}$, &c. But the co-efficients of the denominators 1, 3, 15, 105, &c. ariſe by multiplying the numbers 1, 3, 5, 7, 9, &c. gradually into each other.

75. But in the 2d Table, the Series of the 1ſt, 2d, 3d, 4th, 9th, and 10th Orders are produced *in infinitum* by diviſion alone. Thus having $\frac{dz^{4n-1}}{e+fz^{n}} = y$, in the 1ſt Order, if you perform the diviſion to a convenient period, there will ariſe $\frac{d}{f}z^{3n-1} - \frac{de}{f^{2}}z^{2n-1} + \frac{de^{2}}{f^{3}}z^{n-1} - \frac{de^{3}}{f^{4}}z^{n-1} = y$. The firſt three terms belong to the 1ſt Order of Table 1, and the fourth term belongs to the 1ſt Species of this Order. Whence it appears, that the Area is $\frac{d}{3nf}z^{3n} - \frac{de}{2nf^{2}}z^{2n} + \frac{de^{2}}{nf^{3}}z^{n} - \frac{e^{3}}{nf^{4}}s$; putting s for the Area of the Conic Section, whoſe Abſciſs is $x = z^{n}$, and Ordinate $v = \frac{d}{e+fx}$.

76. But the Series of the 5th and 6th Orders may be infinitely continued, by the help of the two Theorems in the 5th Order of Table 1. by a due addition or ſubtraction: As alſo the 7th and 8th Series, by means of the Theorems in the 6th Order of Table 1. and the Series of the 11th, by the Theorem in the 10th Order of Table 1.

For inſtance, if the Series of the 3d Order of Table 2. be to be farther continued, ſuppoſe $\theta = -4n$, and the 1ſt Theorem of the 5th Order of Table 1. will become $-8nez^{-4n-1} - 5nfz^{-3n-1}$ into $\frac{1}{2}\sqrt{e+fz^n} = y$. $\frac{R^3}{z^{4n}} = t$. But according to the 4th Theorem of this Series to be produced, writing $-\frac{5nf}{2}$ for d, it is $-\frac{5n}{2} fz^{-3n-1}$ $\sqrt{e+fz^n} = y$, $\frac{1}{z^n} = x$, $\sqrt{fx+exx} = v$, and $\frac{10fv^3 - 15f^2x}{12e} = t$. So that ſubtracting the former values of y and t, there will remain $4nez^{-4n-1}\sqrt{e+fz^n} = y$, $\frac{10fv^3 - 15ffx}{12e} - \frac{R^3}{z^{4n}} = t$. Theſe being multiplied by $\frac{d}{4ne}$; and, (if you pleaſe) for $\frac{R^3}{z^{4n}}$ writing xv^3, there will ariſe a 5th Theorem of the Series to be produced, $\frac{d}{z^{4n+1}}\sqrt{e+fz^n} = y$, $\frac{1}{z^n} = x$, $\sqrt{fx+exx} = v$, and $\frac{10dfv^3 - 15dfx}{48ne^2} - \frac{dxv^3}{4ne} = t$.

77. IV. Some of theſe Orders may alſo be otherwiſe derived from others. As in the 2d Table, the 5th, 6th, 7th, and 11th, from the 8th; and the 9th from the 10th: So that I might have omitted them, but that they may be of ſome uſe, tho' not altogether neceſſary. Yet I have omitted ſome Orders, which I might have derived from the 1ſt, and 2d, as alſo from the 9th and 10th, becauſe they were affected by Denominators that were more complicate, and therefore can hardly be of any uſe.

78. V. If the defining Equation of any Curve is compounded of ſeveral Equations of different Orders, or of different Species of the ſame Order, its Area muſt be compounded of the correſponding Areas; taking care however, that they may be rightly connected with their proper Signs. For we muſt not always add or ſubtract at the ſame time Ordinates to or from Ordinates, or correſponding Areas to or from correſponding Areas; but ſometimes the ſum of theſe, and the difference of thoſe, is to be taken for a new Ordinate, or to conſtitute a correſponding Area. And this muſt be done, when the conſtituent Areas are poſited on the contrary ſide of the Ordinate. But that the cautious Geometrician may the more readily avoid this in-

inconveniency, I have prefix'd their proper Signs to the ſeveral Values of the Areas, tho' ſometimes negative, as is done in the 5th and 7th Order of Table 2.

79. VI. It is farther to be obſerved, about the Signs of the Areas, that $+s$ denotes, either that the Area of the Conic Section, adjoining to the Abſciſs, is to be added to the other quantities in the value of t; (ſee the 1ſt Example following;) or that the Area on the other ſide of the Ordinate is to be ſubtracted. And on the contrary, $-s$ denotes ambiguouſly, either that the Area adjacent to the Abſciſs is to be ſubtracted, or that the Area on the other ſide of the Ordinate is to be added, as it may ſeem convenient. Alſo the Value of t, if it comes out affirmative, denotes the Area of the Curve propoſed adjoining to its Abſciſs: And contrariwiſe, if it be negative, it repreſents the Area on the other ſide of the Ordinate.

80. VII. But that this Area may be more certainly defined, we muſt enquire after its Limits. And as to its Limit at the Abſciſs, at the Ordinate, and at the Perimeter of the Curve, there can be no uncertainty: But its initial Limit, or the beginning from whence its deſcription commences, may obtain various poſitions. In the following Examples it is either at the beginning of the Abſciſs, or at an infinite diſtance, or in the concourſe of the Curve with its Abſciſs. But it may be placed elſewhere. And wherever it is, it may be found, by ſeeking that length of the Abſciſs, at which the value of t becomes nothing, and there erecting an Ordinate. For the Ordinate ſo raiſed will be the Limit required.

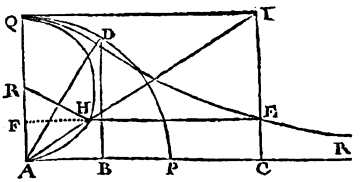
81. VIII. If any part of the Area is poſited below the Abſciſs, t will denote the difference of that, and of the part above the Abſciſs.

82. IX. Whenever the dimenſions of the terms in the values of x, v, and t, ſhall aſcend too high, or deſcend too low, they may be reduced to a juſt degree, by dividing or multiplying ſo often by any given quantity, which may be ſuppos'd to perform the office of Unity, as often as thoſe dimenſions ſhall be either too high or too low.

83. X. Beſides the foregoing Catalogues, or Tables, we might alſo conſtruct Tables of Curves related to other Curves, which may be the moſt ſimple in their kind; as to $\sqrt{a+fx^3}=v$, or to $x\sqrt{e+fx^3}=v$, or to $\sqrt{e+fx^4}=v$, &c. So that we might at all times derive the Area of any propoſed Curve from the ſimpleſt original, and know to what Curves it ſtands related. But now let us illuſtrate by Examples, what has been already delivered.

84.

84. EXAMPLE I. Let QER be a Conchoidal of ſuch a kind, that the Semicircle QHA being deſcribed, and AC being erected perpendicular to the Diameter AQ; if the Parallelogram QACI be compleated, the Diagonal AI be drawn, meeting the Semicircle in H, and from H the perpendicular HE be let fall to IC; then the Point E will deſcribe a Curve, whoſe Area ACEQ is ſought.



85. Therefore make AQ $=a$, AC $=z$, CE $=y$; and becauſe of the continual Proportionals AI, AQ, AH, EC, 'twill be EC or $y = \frac{a^3}{a^2+z^2}$.

86. Now that this may acquire the Form of the Equations in the Tables, make $n=2$, and for z^2 in the denominator write z^n, and $a^3z^{\frac{1}{2}n-1}$ for a^3 or a^3z^{1-1} in the numerator, and there will ariſe $y = \frac{a^3z^{\frac{1}{2}n-1}}{a^2+z^n}$, an Equation of the 1ſt Species of the 2d Order of Table 2. and the Terms being compared, it will be $d=a^3$, $e=a^2$, and $f=1$; ſo that $\sqrt{\frac{a^3}{a^2+z^2}} = x$, $\sqrt{a^3-a^2x^2} = v$, and $xv - 2s = t$.

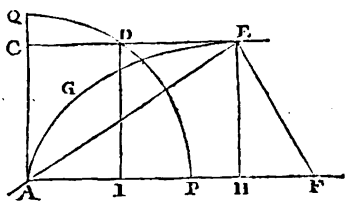
87. Now that the values found of x and v may be reduced to a juſt number of dimenſions, chooſe any given quantity, as a, by which, as unity, a^3 may be multiplied once in the value of x, and in the value of v, a^3 may be divided once, and a^2x^2 twice. And by this means you will obtain $\sqrt{\frac{a^4}{a^2+z^2}} = x$, $\sqrt{a^2-x^2} = v$, and $xv - 2s = t$: of which the conſtruction is thus.

88. Center A, and Radius AQ, deſcribe the Quadrantal Arch QDP; in AC take AB $=$ AH; raiſe the perpendicular BD meeting that Arch in D, and draw AD. Then the double of the Sector ADP will be equal to the Area ſought ACEQ. For $\sqrt{\frac{a^4}{a^2+z^2}} = (\sqrt{AD^2-AB^2} =)$ BD, or v; and $xv - 2s = 2\Delta$ ADB $-$ 2ABDQ, or $= 2\Delta$ ADB $+$ 2BDP, that is, either $= -2$QAD, or $=$ 2DAP: Of which values the affirmative 2DAP belongs to the Area ACEQ on this ſide EC, and the negative -2QAD belongs to the Area RE R extended *ad infinitum* beyond EC.

89. The ſolutions of Problems thus found may ſometimes be made more elegant. Thus in the preſent caſe, drawing RH the ſemidiameter

midiameter of the Circle QHA, becauſe of equal Arches QH and DP, the Sector QRH is half the Sector DAP, and therefore a fourth part of the Surface ACEQ.

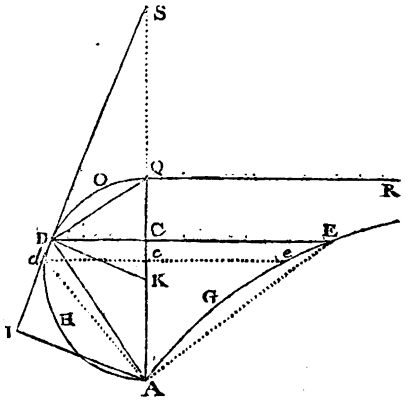
90. EXAMPLE II. Let AGE be a Curve, which is deſcribed by the Angular point E of the *Norma* AEF, whilſt one of the Legs AE, being interminate, paſſes continually through the given point A, and the other CE, of a given length, ſlides upon the right Line AF given in poſition. Let fall EH perpendicular to AF, and compleat the Parallelogram AHEC; and calling AC $= z$, CE $= y$, and EF $= a$, becauſe of HF, HE, HA continual Proportionals, it will be HA or $y = \frac{z^2}{\sqrt{a^2 - z^2}}$.



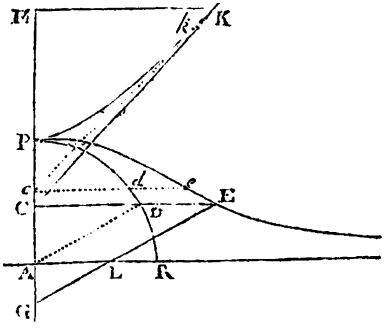
91. Now that the Area AGEC may be known, ſuppoſe $z^2 = z^n$, or $2 = n$, and thence it will be $\frac{z^{\frac{3}{2}n-1}}{\sqrt{a^2 - z^n}} = y$. Here ſince z in the numerator is of a fracted dimenſion, depreſs the value of y by dividing by $z^{\frac{1}{2}n}$, and it will be $\frac{z^{n-1}}{\sqrt{a^2 z^{-n} - 1}} = y$, an Equation of the 2d Species of the 7th Order of Table 2. And the terms being compared, it is $d = 1$, $e = -1$, and $f = a^2$. So that $z^2 = \left(\frac{1}{z^{-n}} =\right) x^2$, $\sqrt{a^2 - x^2} = v$, and $s - xv = t$. Therefore ſince x and z are equal, and ſince $\sqrt{a^2 - x^2} = v$ is an Equation to a Circle, whoſe Diameter is a: with the Center A, and diſtance a or EF, let the Circle PDQ be deſcribed, which CE meets in D, and let the Parallelogram ACDI be compleated; then will AC $= z$, CD $= v$, and the Area ſought AGEC $= s - xv =$ ACDP $-$ ACDI $=$ IDP.

92. Ex-

92. EXAMPLE III. Let AGE be the Ciſſoid belonging to the Circle ADQ, deſcribed with the diameter AQ. Let DCE be drawn perpendicular to the diameter, and meeting the Curves in D and E. And naming AC $= z$, CE $= y$, and AQ $= a$; becauſe of CD, CA, CE continual Proportionals, it will be CE or $y = \frac{zz}{\sqrt{az - zz}}$, and dividing by z, 'tis $y = \frac{z}{\sqrt{az^{-1} - 1}}$. Therefore $z^{-1} = z^n$, or $-1 = n$, and thence $y = \frac{z^{-2n-1}}{\sqrt{az^n - 1}}$, an Equation of the 3d Species of the 4th Order of Table 2. The Terms therefore being compared, 'tis $d = 1$, $e = -1$, and $f = a$. Therefore $z = \frac{1}{z^n} = x$, $\sqrt{ax - xx} = v$, and $3s - 2xv = t$. Wherefore it is AC $= x$, CD $= v$, and thence ACDH $= s$; ſo that $3ACDH - 4\Delta ADC = 3s - 2xv = t$ = Area of the Ciſſoid ACEGA. Or, which is the ſame thing, 3 Segments ADHA = Area ADEGA, or 4 Segments ADHA = Area AHDEGA.



93. EXAMPLE IV. Let PE be the firſt Conchoid of the Ancients, deſcribed from Center G, with the Aſymptote AL, and diſtance LE. Draw its Axis GAP, and let fall the Ordinate EC. Then calling AC $= z$, CE $= y$, GA $= b$, and AP $= c$; becauſe of the Proportionals A C : CE — AL : : GC : CE, it will be CE or $y = \frac{b+z}{z}\sqrt{cc - zz}$.



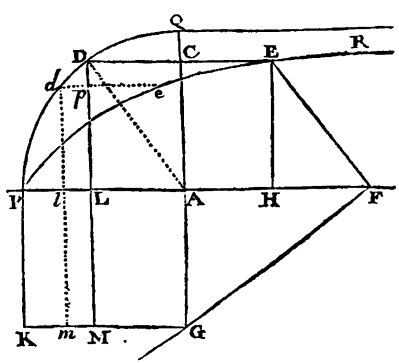
94. Now that its Area PEC may be found from hence, the parts of the Ordinate CE are to be conſider'd ſeparately. And if the Ordinate CE is ſo divided in D, that it is CD $= \sqrt{e^2 - z^2}$, and

and DE $= \frac{b}{z}\sqrt{e^2 - z^2}$; CD will be the Ordinate of a Circle deſcribed from Center A, and with the Radius AP. Therefore the part of the Area PDC is known, and there will remain the other part DPED to be found. Therefore ſince DE, the part of the Ordinate by which it is deſcribed, is equivalent to $\frac{b}{z}\sqrt{e^2 - z^2}$; ſuppoſe $2 = n$, and it becomes $\frac{b}{z}\sqrt{e^n - z^n} =$ DE, an Equation of the 1ſt Species of the 3d Order of Table 2. The terms therefore being compared, it is $d = b$, $e = c^2$, and $f = -1$; and therefore $\frac{1}{z} = \sqrt{\frac{1}{z^n}} = x$, $\sqrt{-1 + c^2x^2} = v$, and $2bc^2s - \frac{bv^3}{x} = t$.

95. Theſe things being found, reduce them to a juſt number of dimenſions, by multiplying the terms that are too depreſs'd, and dividing thoſe that are too high, by ſome given Quantity. If this be done by c, there will ariſe $\frac{c^2}{z} = x$, $\sqrt{-c^2 + x^2} = v$, and $\frac{2bs}{c} - \frac{bv^3}{cx} = t$: The Conſtruction of which is in this manner.

96. With the Center A, principal Vertex P, and Parameter 2AP, deſcribe the Hyperbola PK. Then from the point C draw the right Line CK, that may touch the Parabola in K: And it will be, as AP to 2AG, ſo is the Area CKPC to the Area required DPED.

97. EXAMPLE 5. Let the *Norma* GFE ſo revolve about the Pole G, as that its angular point F may continually ſlide upon the right Line AF given in poſition; then conceive the Curve PE to be deſcribed by any Point E in the other Leg EF. Now that the Area of this Curve may be found, let fall GA and EH perpendicular to the right Line AF, and compleating the Parallelogram AHEC, call AC $= z$, CE $= y$, AG $= b$, and EF $= c$; and becauſe of the Proportionals HF : EH :: AG : AF, we ſhall have AF $= \frac{bz}{\sqrt{cc - zz}}$. Therefore CE or y $= \frac{bz}{\sqrt{c^2 - z^2}} - \sqrt{c^2 - z^2}$. But whereas $\sqrt{cc - zz}$ is the Ordinate of a Circle deſcribed with the Semidiameter c; about the Center A



let ſuch a Circle PDQ be deſcribed, which CE produced meets in D; then it will be DE $= \frac{bz}{\sqrt{cc - zz}}$; By the help of which Equation there remains the Area PDEP or DERQ to be determin'd. Suppoſe therefore $n = 2$, and $\theta = b$, and it will be DE $= \frac{bz^{n-1}}{\sqrt{cc - z^n}}$, an Equation of the 1ſt Species of the 4th Order of Table 1. And the Terms being compared, it will be $b = d$, $cc = e$, and $-1 = f$; ſo that $-b\sqrt{cc - zz} = -b\mathrm{R} = t$.

98. Now as the value of t is negative, and therefore the Area repreſented by t lies beyond the Line DE; that its initial Limit may be found, ſeek for that length of z, at which t becomes nothing, and you will find it to be c. Therefore continue AC to Q, that it may be AQ $= c$, and erect the Ordinate QR; and DQRED will be the Area whoſe value now found is $-b\sqrt{cc - zz}$.

99. If you ſhould deſire to know the quantity of the Area PDE, poſited at the Abſciſs AC, and co-extended with it, without knowing the Limit QR, you may thus determine it.

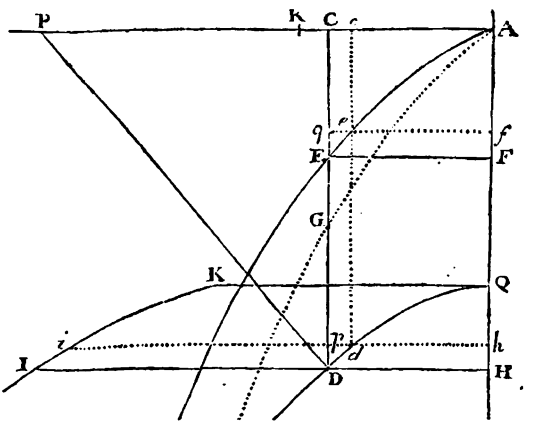
100. From the Value which t obtains at the length of the Abſciſs AC, ſubtract its value at the beginning of the Abſciſs; that is, from $-b\sqrt{cc - zz}$ ſubtract $-bc$, and there will ariſe the deſired quantity $bc - b\sqrt{cc - zz}$. Therefore compleat the Parallelogram PAGK, and let fall DM perpendicular to AP, which meets GK in M; and the Parallelogram PKML will be equal to the Area PDE.

101. Whenever the Equation defining the nature of the Curve cannot be found in the Tables, nor can be reduced to ſimpler terms by diviſion, nor by any other means; it muſt be transform'd into other Equations of Curves related to it, in the manner ſhewn in Prob. 8. till at laſt one is produced, whoſe Area may be known by the Tables. And when all endeavours are uſed, and yet no ſuch can be found, it may be certainly concluded, that the Curve propoſed cannot be compared, either with rectilinear Figures, or with the Conic Sections.

102. In the ſame manner when mechanical Curves are concern'd, they muſt firſt be transform'd into equal Geometrical Figures, as is ſhewn in the ſame Prob. 8. and then the Areas of ſuch Geometrical Curves are to be found from the Tables. Of this matter take the following Example.

103.

103. EXAMPLE 6. Let it be propoſed to determine the Area of the Figure of the Arches of any Conic Section, when they are made Ordinates on their Right Sines. As let A be the Center of the Conic Section, AQ and AR the Semiaxes, CD the Ordinate to the Axis AR, and PD a Perpendicular at the point D. Alſo let AE be the ſaid mechanical Curve meeting CD in E; and from its nature before defined, CE will be equal to the Arch QD. Therefore the Area AEC is ſought, or compleating the parallelogram ACEF, the exceſs AEF is required. To which purpoſe let *a* be the Latus rectum of the Conic Section, and *b* its Latus tranſverſum, or 2AQ. Alſo let AC$=z$, and CD$=y$; then it will be $\sqrt{\frac{1}{4}bb+\frac{b}{a}zz}=y$, an Equation to a Conic Section, as is known. Alſo PC$=\frac{b}{a}z$, and thence PD$=\sqrt{\frac{1}{4}bb+\frac{bb+ab}{aa}zz}$.



104. Now ſince the fluxion of the Arch QD is to the fluxion of the Abſciſs AC, as PD to CD; if the fluxion of the Abſciſs be ſuppos'd 1, the Fluxion of the Arch QD, or of the Ordinate CE, will be $\sqrt{\frac{\frac{1}{4}bb+\frac{bb+ab}{aa}zz}{\frac{1}{4}bb+\frac{b}{a}zz}}$. Draw this into FE, or z, and there will ariſe $z\sqrt{\frac{\frac{1}{4}bb+\frac{bb+ab}{aa}zz}{\frac{1}{4}bb+\frac{b}{a}zz}}$ for the fluxion of the Area AEF. If therefore in the Ordinate CD you take CG$=z\sqrt{\frac{\frac{1}{4}bb+\frac{bb+ab}{aa}zz}{\frac{1}{4}bb+\frac{b}{a}zz}}$, the Area AGC, which is deſcribed by CG moving upon AC, will be equal to the Area AEF, and the Curve

AG will be a Geometrical Curve. Therefore the Area AGC is ſought. To this purpoſe let z^n be ſubſtituted for z^2 in the laſt Equation, and it becomes $z^{n-1}\sqrt{\frac{\frac{1}{4}bb+\frac{bb+ab}{aa}z^n}{\frac{1}{4}bb+\frac{b}{a}z^n}} = \text{CG}$, an Equation of the 2d Species of the 11th Order of Table 2. And from a compariſon of terms it is $d = 1$, $e = \frac{1}{4}bb = g$, $f = \frac{bb+ab}{aa}$, and $h = \frac{b}{a}$; ſo that $\sqrt{\frac{1}{4}bb+\frac{b}{a}zz} = x$, $\sqrt{-\frac{b^3}{4a}+\frac{a+b}{a}xx} = v$, and $\frac{a}{b}s = t$. That is, CD $= x$, DP $= v$, and $\frac{a}{b}s = t$. And this is the Conſtruction of what is now found.

105. At Q erect QK perpendicular and equal to QA, and thro' the point D draw HI parallel to it, but equal to DP. And the Line KI, at which HI is terminated, will be a Conic Section, and the comprehended Area HIKQ will be to the Area ſought AEF, as b to a, or as PC to AC.

106. Here obſerve, that if you change the ſign of b, the Conic Section, to whoſe Arch the right Line CE is equal, will become an Ellipſis; and beſides, if you make $b = -a$, the Ellipſis becomes a Circle. And in this caſe the line KI becomes a right line parallel to AQ.

107. After the Area of any Curve has been thus found and conſtructed, we ſhould conſider about the demonſtration of the conſtruction; that laying aſide all Algebraical calculation, as much as may be, the Theorem may be adorn'd, and made elegant, ſo as to become fit for publick view. And there is a general method of demonſtrating, which I ſhall endeavour to illuſtrate by the following Examples.

Demonſtration of the Conſtruction in Example 5.

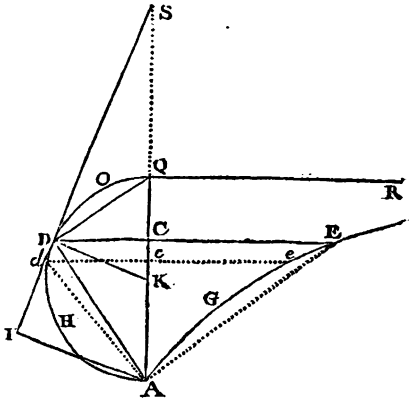
108. In the Arch PQ take a point d indefinitely near to D, (Figure p. 113.) and draw de and dm parallel to DE and DM, meeting DM and AP in p and l. Then will DEed be the moment of the Area PDEP, and LMml will be the moment of the Area LMKP. Draw the ſemidiameter AD, and conceive the indefinitely ſmall arch Dd to be as it were a right line, and the triangles Dpd and ALD will be like, and therefore $Dp : pd :: AL : LD$. But it is $HF : EH :: AG : AF$; that is, $AL : LD :: ML : DE$; and therefore $Dp : pd :: ML : DE$. Wherefore $Dp \times DE = pd \times ML$.

That

That is, the moment DE*ed* is equal to the moment LM*ml*. And ſince this is demonſtrated indeterminately of any contemporaneous moments whatever, it is plain, that all the moments of the Area PDEP are equal to all the contemporaneous moments of the Area PLMK, and therefore the whole Areas compoſed of thoſe moments are equal to each other. Q. E. D.

Demonſtration of the Conſtruction in Example 3.

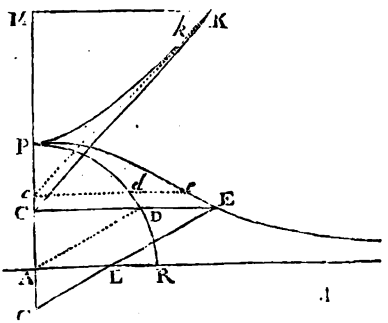
109. Let DE*ed* be the momentum of the ſuperficies AHDE, and A*d*DA be the contemporary moment of the Segment ADH. Draw the ſemidiameter DK, and let *de* meet AK in *c*; and it is C*c* : D*d* :: CD : DK. Beſides it is DC : QA (2DK) :: AC : DE. And therefore C*c* : 2D*d* :: DC : 2DK :: AC : DE, and C*c* × DE = 2D*d* × AC. Now to the moment of the periphery D*d* produced, that is, to the tangent of the Circle, let fall the perpendicular AI, and AI will be equal to AC. So that 2D*d* × AC = 2D*d* × AI = 4 Triangles AD*d*. So that 4 Triangles AD*d* = C*c* × DE = moment DE*ed*. Therefore every moment of the ſpace AHDE is quadruple of the contemporary moment of the Segment ADH, and therefore that whole ſpace is quadruple of the whole Segment. Q. E. D.



Demonſtration

Demonſtration of the Conſtruction in Example 4.

110. Draw *ce* parallel to CE, and at an indefinitely ſmall diſtance from it, and the tangent of the Hyperbola *ck*, and let fall KM perpendicular to AP. Now from the nature of the Hyperbola it will be AC : AP :: AP : AM, and therefore AGq : GLq :: ACq : LEq (or APq) :: APq : AMq ; and *diviſim*, AGq : ALq (DEq) :: APq : AMq — APq (MKq) ; And *inverſè*, AG : AP :: DE : MK. But the little Area DE*ed* is to the Triangle CK*c*, as the altitude DE is to half the altitude KM ; that is, as AG to $\frac{1}{2}$AP. Wherefore all the moments of the Space PDE are to all the contemporaneous moments of the Space PKC, as AG to $\frac{1}{2}$AP. And therefore thoſe whole Spaces are in the ſame ratio. Q. E. D.



Demonſtration of the Conſtruction in Example 6.

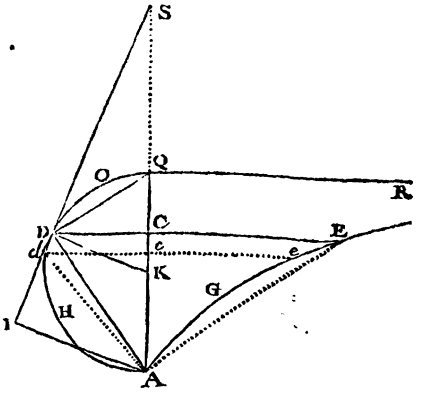
111. Draw *cd* parallel and infinitely near to CD, (Fig. in p. 115.) meeting the Curve AE in *e*, and draw *hi* and *fe* meeting DC in *p* and *q*. Then by the Hypotheſis D*d* = E*q*, and from the ſimilitude of the Triangles D*dp* and DCP, it will be D*p* : (D*d*) E*q* :: CP : (PD) HI, ſo that D*p* × HI = E*q* × CP; and thence D*p* × HI (the moment HI*ih*): E*q* × AC (the moment EF*fe*) :: E*q* × CP : E*q* × AC :: CP : AC. Wherefore ſince PC and AC are in the given ratio of the latus tranſverſum to the latus rectum of the Conic Section QD, and ſince the moments HI*ih* and EF*fe* of the Areas HIKQ and AEF are in that ratio, the Areas themſelves will be in the ſame ratio. Q. E. D.

112. In this kind of demonſtrations it is to be obſerved, that I aſſume ſuch quantities for equal, whoſe ratio is that of equality : And that is to be eſteem'd a ratio of equality, which differs leſs from equality than by any unequal ratio that can be aſſign'd. Thus in the laſt demonſtration I ſuppos'd the rectangle E*q* × AC, or FE*qf*, to be equal to the ſpace FE*ef*, becauſe (by reaſon of the difference E*qe* infinitely leſs than them, or nothing in compariſon of them,) they

they have not a ratio of inequality. And for the ſame reaſon I made DP × HI = HI*ih*; and ſo in others.

113. I have here made uſe of this method of proving the Areas of Curves to be equal, or to have a given ratio, by the equality, or by the given ratio, of their moments; becauſe it has an affinity to the uſual methods in theſe matters. But that ſeems more natural which depends upon the generation of Superficies, by Motion or Fluxion. Thus if the Conſtruction in Example 2. was to be demonſtrated: From the nature of the Circle, the fluxion of the right line ID (Fig. p. 111.) is to the fluxion of the right line IP, as AI to ID; and it is AI : ID :: ID : CE, from the nature of the Curve AGE; and therefore CE × $\dot{\text{ID}}$ = ID × $\dot{\text{IP}}$. But CE × $\dot{\text{ID}}$ = to the fluxion of the Area PDI. And therefore thoſe Areas, being generated by equal fluxion, muſt be equal. Q. E. D.

114. For the ſake of farther illuſtration, I ſhall add the demonſtration of the Conſtruction, by which the Area of the Ciſſoid is determin'd, in Example 3. Let the lines mark'd with points in the ſcheme be expunged; draw the Chord DQ, and the Aſymptote QR of the Ciſſoid. Then, from the nature of the Circle, it is DQ*q* = AQ × CQ, and thence (by Prob. 1.) 2DQ × Fluxion of DQ = AQ × $\dot{\text{CQ}}$. And therefore AQ : DQ :: 2$\dot{\text{DQ}}$: $\dot{\text{CQ}}$. Alſo from the nature of the Ciſſoid it is ED : AD :: AQ : DQ. Therefore ED : AD :: 2$\dot{\text{DQ}}$: $\dot{\text{CQ}}$, and ED × $\dot{\text{CQ}}$ = AD × 2$\dot{\text{DQ}}$, or 4 × ½AD × $\dot{\text{DQ}}$. Now ſince DQ is perpendicular at the end of AD, revolving about A; and ½AD × $\dot{\text{QD}}$ = to the fluxion generating the Area ADOQ; its quadruple alſo ED × $\dot{\text{CQ}}$ = fluxion generating the Ciſſoidal Area QREDO. Wherefore that Area QREDO infinitely long, is generated quadruple of the other ADOQ. Q. E. D.



SCHOLIUM.

SCHOLIUM.

115. By the foregoing Tables not only the Areas of Curves, but quantities of any other kind, that are generated by an analogous way of flowing, may be derived from their Fluxions, and that by the aſſiſtance of this Theorem: That a quantity of any kind is to an unit of the ſame kind, as the Area of a Curve is to a ſuperficial unity; if ſo be that the fluxion generating that quantity be to an unit of its kind, as the fluxion generating the Area is to an unit of its kind alſo; that is, as the right Line moving perpendicularly upon the Abſciſs (or the Ordinate) by which the Area is deſcribed, to a linear Unit. Wherefore if any fluxion whatever is expounded by ſuch a moving Ordinate, the quantity generated by that fluxion will be expounded by the Area deſcribed by ſuch Ordinate; or if the Fluxion be expounded by the ſame Algebraic terms as the Ordinate, the generated quantity will be expounded by the ſame as the deſcribed Area. Therefore the Equation, which exhibits a Fluxion of any kind, is to be ſought for in the firſt Column of the Tables, and the value of t in the laſt Column will ſhow the generated Quantity.

116. As if $\sqrt{1+\frac{9z}{4a}}$ exhibited a Fluxion of any kind, make it equal to y, and that it may be reduced to the form of the Equations in the Tables, ſubſtitute z^n for z, and it will be $z^{n-1}\sqrt{1+\frac{9}{4a}z^n} = y$, an Equation of the firſt Species of the 3d Order of Table 1. And comparing the terms, it will be $d = 1$, $e = 1$, $f = \frac{9}{4a}$, and thence $\frac{8a+18z}{27}\sqrt{1+\frac{9z}{4a}} = \frac{2d}{3nf}R^3 = t$. Therefore it is the quantity $\frac{8a+18z}{27}\sqrt{1+\frac{9z}{4a}}$ which is generated by the Fluxion $\sqrt{1+\frac{9z}{4a}}$.

117. And thus if $\sqrt{1+\frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ repreſents a Fluxion, by a due reduction, (or by extracting $z^{\frac{2}{3}}$ out of the radical, and writing z^n for $z^{-\frac{2}{3}}$,) there will be had $\frac{1}{z^{n+1}}\sqrt{z^n+\frac{16}{9a^{\frac{2}{3}}}} = y$, an Equation of the 2d Species of the 5th Order of Table 2. Then comparing the terms,

terms, it is $d = 1$, $e = \frac{16}{9a^{\frac{2}{3}}}$, and $f = 1$. So that $z^{\frac{3}{3}} = \frac{1}{x^{\frac{1}{4}}} = xx$, $\sqrt{1 + \frac{16xx}{9a^{\frac{2}{3}}}} = v$, and $\frac{3}{2}s = \frac{-2d}{n}s = t$. Which being found, the quantity generated by the fluxion $\sqrt{1 + \frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ will be known, by making it to be to an Unit of its own kind, as the Area $\frac{3}{2}s$ is to ſuperficial unity; or which comes to the ſame, by ſuppoſing the quantity t no longer to repreſent a Superficies, but a quantity of another kind, which is to an unit of its own kind, as that ſuperficies is to ſuperficial unity.

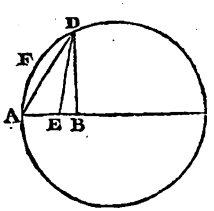
118. Thus ſuppoſing $\sqrt{1 + \frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ to repreſent a linear Fluxion, I imagine t no longer to ſignify a Superficies, but a Line; that Line, for inſtance, which is to a linear unit, as the Area which (according to the Tables) is repreſented by t, is to a ſuperficial unit, or that which is produced by applying that Area to a linear unit. On which account, if that linear unit be made e, the length generated by the foregoing fluxion will be $\frac{3f}{2e}$. And upon this foundation thoſe Tables may be apply'd to the determining the Lengths of Curve-lines, the Contents of their Solids, and any other quantities whatever, as well as the Areas of Curves.

Of Queſtions that are related hereto.

I. *To approximate to the Areas of Curves mechanically.*

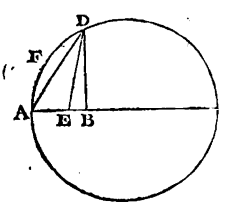
119. The method is this, that the values of two or more right-lined Figures may be ſo compounded together, that they may very nearly conſtitute the value of the Curvilinear Area required.

120. Thus for the Circle AFD which is denoted by the Equation $x - xx = zz$, having found the value of the Area AFDB, *viz.* $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$, &c. the values of ſome Rectangles are to be ſought, ſuch is the value $x\sqrt{x - xx}$, or $x^{\frac{3}{2}} - \frac{1}{4}z^{\frac{5}{2}} - \frac{1}{8}x^{\frac{7}{2}} - \frac{1}{16}x^{\frac{9}{2}}$, &c. of the rectangle BD × AB, and $x\sqrt{x}$, or $x^{\frac{3}{2}}$, the value of AD × AB. Then theſe values are to be multiply'd by any different letters, that ſtand for numbers indefinitely, and then



to be added together, and the terms of the ſum are to be compared with the correſponding terms of the value of the Area AFDB, that as far as is poſſible they may become equal. As if thoſe Parallelograms were multiply'd by e and f, the ſum would be $ex^{\frac{1}{2}} - \frac{1}{2}ex^{\frac{3}{2}} + f - \frac{1}{8}ex^{\frac{5}{2}}$, &c. the terms of which being compared with theſe terms $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}}$, &c. there ariſes $e + f = \frac{2}{3}$, and $-\frac{1}{2}e = -\frac{1}{5}$, or $e = \frac{2}{5}$, and $f = \frac{2}{3} - e = \frac{4}{15}$. So that $\frac{2}{5}$BD × AB + $\frac{4}{15}$AD × AB = Area AFDB very nearly. For $\frac{2}{5}$BD × AB + $\frac{4}{15}$AD × AB is equivalent to $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{20}x^{\frac{7}{2}} - \frac{1}{40}x^{\frac{9}{2}}$, &c. which being ſubtracted from the Area AFDB, leaves the error only $\frac{1}{70}x^{\frac{7}{2}} + \frac{1}{90}x^{\frac{9}{2}}$, &c.

121. Thus if AB were biſected in E, the value of the rectangle AB × DE will be $x\sqrt{x - \frac{3}{4}xx}$, or $x^{\frac{3}{2}} - \frac{3}{8}x^{\frac{5}{2}} - \frac{9}{128}x^{\frac{7}{2}} - \frac{27}{1024}x^{\frac{9}{2}}$, &c. And this compared with the rectangle AD × AB, gives $\frac{8DE + 2AD}{15}$ into AB = Area AFDB, the error being only $\frac{1}{560}x^{\frac{5}{2}} + \frac{1}{5760}x^{\frac{7}{2}}$, &c. which is always leſs than $\frac{1}{1500}$ part of the whole Area, even tho' AFDB were a quadrant of a Circle. But this Theorem may be thus propounded. As 3 to 2, ſo is the rectangle AB into DE, added to a fifth part of the difference between AD and DE, to the Area AFDB, very nearly.



122. And thus by compounding two rectangles AB × ED and AB × BD, or all the three rectangles together, or by taking in ſtill more rectangles, other Rules may be invented, which will be ſo much the more exact, as there are more Rectangles made uſe of. And the ſame is to be underſtood of the Area of the Hyperbola, or of any other Curves. Nay, by one only rectangle the Area may often be very commodiouſly exhibited, as in the foregoing Circle, by taking BE to AB as $\sqrt{10}$ to 5, the rectangle AB × ED will be to the Area AFDB, as 3 to 2, the error being only $\frac{1}{175}x^{\frac{7}{2}} + \frac{1}{2112500}x^{\frac{9}{2}}$, &c.

II. *The Area being given, to determine the Abſciſs and Ordinate.*

123. When the Area is expreſs'd by a finite Equation, there can be no difficulty: But when it is expreſs'd by an infinite Series, the affected root is to be extracted, which denotes the Abſciſs. So for the

the Hyperbola, defined by the Equation $\frac{ab}{a+x} = z$, after we have found $z = bx - \frac{bx^2}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3}$, &c. that from the given Area the Abſciſs x may be known, extract the affected Root, and there will ariſe $x = \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{96a^4b^5}$, &c. And moreover, if the Ordinate $\dot{z}$ were required, divide ab by $a + x$, that is, by $a + \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3}$, &c. and there will ariſe $\dot{z} = \dot{b} - \frac{z}{a} - \frac{z^2}{2a^2b} - \frac{z^3}{6a^3b^2} - \frac{z^4}{24a^4b^3}$, &c.

124. Thus as to the Ellipſis which is expreſs'd by the Equation $ax - \frac{a}{c}xx = \dot{z}\dot{z}$, after the Area is found $z = \frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{a^{\frac{1}{2}}x^{\frac{5}{2}}}{5c} - \frac{a^{\frac{3}{2}}x^{\frac{7}{2}}}{28c^2} - \frac{a^{\frac{1}{2}}x^{\frac{9}{2}}}{72c^3}$, &c. write v^3 for $\frac{3z}{2a^{\frac{1}{2}}}$, and t for $x^{\frac{1}{2}}$, and it becomes $v^3 = t^3 - \frac{3t^5}{10c} - \frac{3t^7}{56c^2} - \frac{t^9}{48c^3}$, &c. and extracting the root $t = v + \frac{v^3}{10c} + \frac{81v^5}{1400c^2} + \frac{1171v^7}{25200c^3}$, &c. whoſe ſquare $v^2 + \frac{v^4}{5c} + \frac{22v^6}{175c^2} + \frac{823v^8}{7875c^3}$, &c. is equal to x. And this value being ſubſtituted inſtead of x in the Equation $ax - \frac{a}{c}xx = \dot{z}\dot{z}$, and the root being extracted, there ariſes $\dot{z} = a^{\frac{1}{2}}v - \frac{2a^{\frac{1}{2}}v^3}{5c} - \frac{38a^{\frac{1}{2}}v^5}{175c^2} - \frac{407a^{\frac{1}{2}}v^7}{2250c^3}$, &c. So that from z, the given Area, and thence v or $\sqrt[3]{\frac{3z}{2a^{\frac{1}{2}}}}$, the Abſciſs x will be given, and the Ordinate $\dot{z}$. All which things may be accommodated to the Hyperbola, if only the ſign of the quantity c be changed, wherever it is found of odd dimenſions.

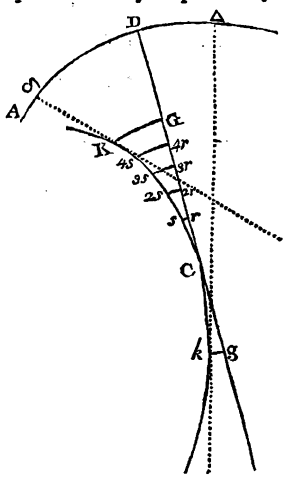
PROB.

PROB. X.

To find as many Curves as we pleaſe, whoſe Lengths may be expreſs'd by finite Equations.

1. The following poſitions prepare the way for the ſolution of this Problem.

2. I. If the right Line DC, ſtanding perpendicularly upon any Curve AD, be conceived thus to move, all its points G, *g*, *r*, &c. will deſcribe other Curves, which are equidiſtant, and perpendicular to that line: As GK, *gk*, *rs*, &c.



3. II. If that right Line is continued indefinitely each way, its extremities will move contrary ways, and therefore there will be a Point between, which will have no motion, but may therefore be call'd the Center of Motion. This Point will be the ſame as the Center of Curvature, which the Curve AD hath at the point D, as is mention'd before. Let that point be C.

4. III. If we ſuppoſe the line AD not to be circular, but unequably curved, ſuppoſe more curved towards δ, and leſs toward Δ; that Center will continually change its place, approaching nearer to the parts more curved, as in K, and going farther off at the parts leſs curved, as in *k*, and by that means will deſcribe ſome line, as K*Ck*.

5. IV. The right Line DC will continually touch the line deſcribed by the Center of Curvature. For if the Point D of this line moves towards δ, its point G, which in the mean time paſſes to K, and is ſituate on the ſame ſide of the Center C, will move the ſame way, by poſition 2. Again, if the ſame point D moves towards Δ, the point *g*, which in the mean time paſſes to *k*, and is ſituate on the contrary ſide of the Center C, will move the contrary way, that is, the ſame way that G moved in the former caſe, while it paſs'd to K. Wherefore K and *k* lie on the ſame ſide of the right Line DC. But as K and *k* are taken indefinitely for any points,

points, it is plain that the whole Curve lies on the ſame ſide of the right line DC, and therefore is not cut, but only touch'd by it.

6. Here it is ſuppos'd, that the line ♌D△ is continually more curved towards ♌, and leſs towards △; for if its greateſt or leaſt Curvature is in D, then the right line DC will cut the Curve KC; but yet in an angle that is leſs than any right-lined angle, which is the ſame thing as if it were ſaid to touch it. Nay, the point C in this caſe is the Limit, or Cuſpid, at which the two parts of the Curve, finiſhing in the moſt oblique concourſe, touch each other; and therefore may more juſtly be ſaid to be touch'd, than to be cut, by the right line DC, which divides the Angle of contact.

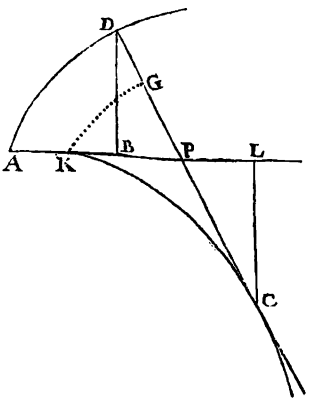
7. V. The right Line CG is equal to the Curve CK. For conceive all the points *r*, 2*r*, 3*r*, 4*r*, &c. of that right Line to deſcribe the arches of Curves *rs*, 2*r*2*s*, 3*r*3*s*, &c. in the mean time that they approach to the Curve CK, by the motion of that right line; and ſince thoſe arches, (by poſition 1.) are perpendicular to the right lines that touch the Curve CK, (by poſition 4.) it follows that they will be alſo perpendicular to that Curve. Wherefore the parts of the line CK, intercepted between thoſe arches, which by reaſon of their infinite ſmallneſs may be conſider'd as right lines, are equal to the intervals of the ſame arches; that is, (by poſition 1.) are equal to ſo many parts of the right line CG. And equals being added to equals, the whole Line CK will be equal to the whole Line CG.

8. The ſame thing would appear by conceiving, that every part of the right Line CG, as it moves along, will apply itſelf ſucceſſively to every part of the Curve CK, and thereby will meaſure them; juſt as the Circumference of a wheel, as it moves forward by revolving upon a Plain, will meaſure the diſtance that the point of Contact continually deſcribes.

9. And hence it appears, that the Problem may be reſolved, by aſſuming any Curve at pleaſure A♌D△, and thence by determining the other Curve KC*k*, in which the Center of Curvature of the aſſumed Curve is always found. Therefore letting fall the perpendiculars DB and CL, to a right Line AB given in poſition, and in AB taking any point A, and calling AB $= x$ and BD $= y$; to define the Curve AD let any relation be aſſumed between x and y, and then by Prob 5. the point C may be found, by which may be determined both the Curve KC, and its Length GC.

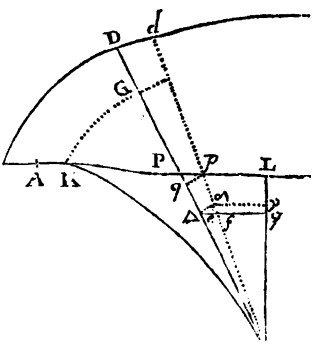
10.

10. EXAMPLE. Let $ax = yy$ be the Equation to the Curve, which therefore will be the *Apollonian* Parabola. And, by Prob. 5. will be found $AL = \frac{1}{3}a + 3x$, $CL = \frac{4y^3}{aa}$, and $DC = \frac{a+4x}{a}\sqrt{\frac{1}{4}aa + ax}$. Which being obtain'd, the Curve KC is determin'd by AL and LC, and its Length by DC. For as we are at liberty to assume the points K and C any where in the Curve KC, let us suppose K to be the Center of Curvature of the Parabola at its Vertex; and putting therefore AB and BD, or x and y, to be nothing, it will be $DC = \frac{1}{2}a$. And this is the Length AK, or DG, which being subtracted from the former indefinite value of DC, leaves GC or $KC = \frac{a+4x}{a}\sqrt{\frac{1}{4}aa + ax} - \frac{1}{2}a$.



11. Now if you desire to know what Curve this is, and what is its Length, without any relation to the Parabola; call $KL = z$, and $LC = v$, and it will be $z = AL - \frac{1}{2}a = 3x$, or $\frac{1}{3}z = x$, and $\frac{ax}{3} = ax = yy$. Therefore $4\sqrt{\frac{z^3}{27a}} = \frac{4y^3}{aa} = CL = v$, or $\frac{16z^3}{27a} = v^2$; which shews the Curve KC to be a Parabola of the second kind. And for its Length there arises $\frac{3a+4z}{3a}\sqrt{\frac{1}{4}aa + \frac{1}{3}az} - \frac{1}{2}a$, by writing $\frac{1}{3}z$ for x in the value of CG.

12. The Problem also may be resolved by taking an Equation, which shall express the relation between AP and PD, supposing P to be the intersection of the Abscis and Perpendicular. For calling $AP = x$, and $PD = y$, conceive CPD to move an infinitely small space, suppose to the place C*pd*, and in CD and C*d* taking CΔ and Cδ both of the same given length, suppose $= 1$, and to CL let fall the perpendiculars Δg and $\delta\gamma$, of which Δg, (which call $= z$) may meet C*d* in *f*. Then compleat the Parallelogram $g\gamma\delta e$, and making $\dot{x}$, $\dot{y}$, and $\dot{z}$ the fluxions of the quantities x, y, and z, as before



it

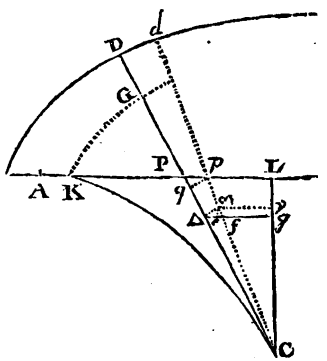
it will be $\Delta e : \Delta f :: \overline{\Delta e}|^2 : \overline{\Delta \delta}|^2 :: \overline{Cg}|^2 : \overline{C\Delta}|^2 :: \frac{\overline{Cg}|^2}{C\Delta} : C\Delta$. And $\Delta f : Pp :: C\Delta : CP$. Then *ex æquo*, $\Delta e : Pp :: \frac{\overline{Cg}|^2}{C\Delta} : CP$. But Pp is the moment of the Abſciſs AP, by the acceſſion of which it becomes Ap; and Δe is the contemporaneous moment of the perpendicular Δg, by the decreaſe of which it becomes $\delta\gamma$. Therefore Δe and Pp are as the fluxions of the lines Δg (z) and AP (x), that is, as $\dot{z}$ and $\dot{x}$. Wherefore $\dot{z} : \dot{x} :: \frac{\overline{Cg}|^2}{C\Delta} : CP$. And ſince it is $\overline{Cg}|^2 = \overline{C\Delta}|^2 - \overline{\Delta g}|^2 = 1 - zz$, and $C\Delta = 1$; it will be $CP = \frac{\dot{x} - \dot{x}z^2}{\dot{z}}$. Moreover ſince we may aſſume any one of the three $\dot{x}, \dot{y}$, and $\dot{z}$ for an uniform fluxion, to which the reſt are to be referr'd, if $\dot{x}$ be that fluxion, and its value is unity, then $CP = \frac{1 - zz}{\dot{z}}$.

13. Beſides it is $C\Delta\ (1) : \Delta g\ (z) :: CP : PL$; alſo $C\Delta\ (1) : Cg\ (\sqrt{1 - zz}) :: CP : CL$; therefore it is $PL = \frac{z - z^3}{\dot{z}}$, and $CL = \frac{1 - zz}{\dot{z}}\sqrt{1 - zz}$. Laſtly, drawing pq parallel to the infinitely ſmall Arch Dd, or perpendicular to DC, Pq will be the momentum of DP, by the acceſſion of which it becomes dp, at the ſame time that AP becomes Ap. Therefore Pp and Pq are as the fluxions of AP (x) and PD (y), that is, as 1 and $\dot{y}$. Therefore becauſe of ſimilar triangles Ppq and $C\Delta g$, ſince $C\Delta$ and Δg, or 1 and z, are in the ſame ratio, it will be $\dot{y} = z$. Whence we have this ſolution of the Problem.

14. From the propoſed Equation, which expreſſes the relation between x and y, find the relation of the fluxions $\dot{x}$ and $\dot{y}$, (by Prob. 1.) and putting $\dot{x} = 1$, there will be had the value of $\dot{y}$, to which z is equal. Then ſubſtituting z for $\dot{y}$, by the help of the laſt Equation find the relation of the Fluxions $\dot{x}, \dot{y}$, and $\dot{z}$, (by Prob. 1.) and again ſubſtituting 1 for $\dot{x}$, there will be had the value of $\dot{z}$. Theſe being found make $\frac{1 - \dot{y}\dot{y}}{\dot{z}} = CP$, $z \times CP = PL$, and $CP \times \sqrt{1 - \dot{y}\dot{y}} = CL$; and C will be a Point in the Curve, any part of which KC is equal to the right Line CG, which is the difference of the tangents, drawn perpendicularly to the Curve Dd from the points C and K.

15.

15. Ex. Let $ax = yy$ be the Equation which expreſſes the relation between AP and PD; and (by Prob. 1.) it will be firſt $a\dot{x} = 2y\dot{y}$, or $a = 2y\dot{z}$. Then $2y\dot{z} + 2\dot{y}z = 0$, or $\frac{-z\dot{z}}{y} = \dot{z}$. Thence it is CP $= \frac{1-\dot{y}}{\dot{z}} = y - \frac{4y^3}{aa}$, PL $= z \times$ CP $= \frac{1}{2}a - \frac{2yy}{a}$, and CL $= \frac{aa-4yy}{2aa}\sqrt{4yy-aa}$. And from CP and PL taking away y and x, there remains CD $= -\frac{4y^3}{aa}$, and AL $= \frac{1}{2}a - \frac{3yy}{a}$. Now I take away y and x, becauſe when CP and PL have affirmative values, they fall on the ſide of the point P towards D and A, and they ought to be diminiſhed, by taking away the affirmative quantities PD and AP. But when they have negative values, they will fall on the contrary ſide of the point P, and then they muſt be encreaſed, which is alſo done by taking away the affirmative quantities PD and AP.



16. Now to know the Length of the Curve, in which the point C is found, between any two of its points K and C; we muſt ſeek the length of the Tangent at the point K, and ſubtract it from CD. As if K were the point, at which the Tangent is terminated, when CΔ and Δg, or 1 and $\dot{z}$, are made equal, which therefore is ſituate in the Abſciſs itſelf AP; write 1 for $\dot{z}$ in the Equation $a = 2y\dot{z}$, whence $a = 2y$. Therefore for y write $\frac{1}{2}a$ in the value of CD, that is in $-\frac{4y^3}{aa}$, and it comes out $-\frac{1}{2}a$. And this is the length of the Tangent at the point K, or of DG; the difference between which and the foregoing indefinite value of CD, is $\frac{4y^3}{aa} - \frac{1}{2}a$, that is GC, to which the part of the Curve KC is equal.

17. Now that it may appear what Curve this is, from AL (having firſt changed its ſign, that it may become affirmative,) take AK, which will be $\frac{1}{4}a$, and there will remain KL $= \frac{3yy}{a} - \frac{1}{4}a$, which call t, and in the value of the line CL, which call v, write $\frac{4at}{3}$ for $4yy - aa$, and there will ariſe $\frac{2t}{3a}\sqrt{\frac{4}{3}at} = v$; or $\frac{16t^3}{27a} = vv$, which is an Equation to a Parabola of the ſecond kind, as was found before.

18.

18. When the relation between t and v cannot conveniently be reduced to an Equation, it may be ſufficient only to find the lengths PC and PL. As if for the relation between AP and PD the Equation $3a^2x + 3a^2y - y^3 = 0$ were aſſumed; from hence (by Prob. 1.) firſt there ariſes $a^2 + a^2z - y^2z = 0$, then $aa\dot{z} - 2y\dot{y}z - y^2\dot{z} = 0$, and therefore it is $z = \frac{aa}{yy - aa}$, and $\dot{z} = \frac{2y\dot{y}z}{aa - yy}$. Whence are given $PC = \frac{1 - yy}{\dot{z}}$, and $PL = z \times PC$, by which the point C is determined, which is in the Curve. And the length of the Curve, between two ſuch points, will be known by the difference of the two correſponding Tangents, DC or $PC - y$.

19. For Example, if we make $a = 1$, and in order to determine ſome point C of the Curve, we take $y = 2$; then AP or x becomes $\frac{y^3 - 3a^2y}{3aa} = \frac{2}{3}$, $z = \frac{1}{3}$, $\dot{z} = -\frac{4}{9}$, $PC = -2$, and $PL = -\frac{2}{3}$. Then to determine another point, if we take $y = 3$, it will be $AP = 6$, $z = \frac{1}{8}$, $\dot{z} = -\frac{3}{256}$, $PC = -84$, and $PL = -10\frac{1}{2}$. Which being had, if y be taken from PC, there will remain -4 in the firſt caſe, and -87 in the ſecond, for the lengths DC; the difference of which 83 is the length of the Curve, between the two points found C and c.

20. Theſe are to be thus underſtood, when the Curve is continued between the two points C and c, or between K and C, without that Term or Limit, which we call'd its Cuſpid. For when one or more ſuch terms come between thoſe points, (which terms are found by the determination of the greateſt or leaſt PC or DC,) the lengths of each of the parts of the Curve, between them and the points C or K, muſt be ſeparately found, and then added together.

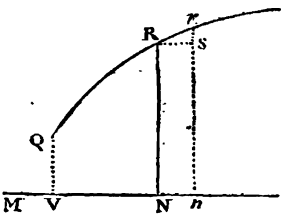
PROB. XI.

To find as many Curves as you pleaſe, whoſe Lengths may be compared with the Length of any Curve propoſed, or with its Area applied to a given Line, by the help of finite Equations.

1. It is performed by involving the Length, or the Area of the propoſed Curve, in the Equation which is aſſumed in the foregoing Problem, to determine the relation between AP and PD (Figure Art. 12. pag. 126.) But that z and $\dot{z}$ may be thence derived, (by

Prob. 1.) the fluxion of the Length, or of the Area, muſt be firſt diſcover'd.

2. The fluxion of the Length is determin'd by putting it equal to the ſquare-root of the ſum of the ſquares of the fluxion of the Abſciſs and of the Ordinate. For let RN be the perpendicular Ordinate, moving upon the Abſciſs MN, and let QR be the propoſed Curve, at which RN is terminated. Then calling MN $= s$, NR $= t$, and QR $= v$, and their Fluxions $\dot{s}$, $\dot{t}$, and $\dot{v}$ reſpectively; conceive the Line NR to move into the place nr infinitely near the former, and letting fall Rs perpendicular to nr, then Rs, sr, and Rr will be the contemporaneous moments of the lines MN, NR, and QR, by the acceſſion of which they become Mn, nr, and Qr. And as theſe are to each other as the fluxions of the ſame lines, and becauſe of the right Angle Rsr, it will be $\sqrt{\dot{R}s^2 + \dot{s}r^2} = Rr$, or $\sqrt{\dot{s}^2 + \dot{t}^2} = \dot{v}$.



3. But to determine the fluxions $\dot{s}$ and $\dot{t}$ there are two Equations required; one of which is to define the relation between MN and NR, or s and t, from whence the relation between the fluxions $\dot{s}$ and $\dot{t}$ is to be derived; and another which may define the relation between MN or NR in the given Figure, and of AP or x in that required, from whence the relation of the fluxion $\dot{s}$ or $\dot{t}$ to the fluxion $\dot{x}$ or 1 may be diſcover'd.

4. Then $\dot{v}$ being found, the fluxions $\dot{y}$ and $\dot{z}$ are to be ſought by a third aſſumed Equation, by which the length PD or y may be defined. Then we are to take PC $= \frac{1 - \dot{y}\dot{y}}{\dot{z}}$, PL $= \dot{y} \times$ PC, and DC $=$ PC $- y$, as in the foregoing Problem.

5. Ex. 1. Let $as - ss = tt$ be an Equation to the given Curve QR, which will be a Circle; $xx = as$ the relation between the lines AP and MN, and $\frac{1}{3}v = y$, the relation between the length of the Curve given QR, and the right Line PD. By the firſt it will be $a\dot{s} - 2s\dot{s} = 2t\dot{t}$, or $\frac{a - 2s}{2t}\dot{s} = \dot{t}$. And thence $\frac{a\dot{s}}{2t} = \sqrt{\dot{s}^2 + \dot{t}^2} = \dot{v}$. By the ſecond it is $2x = a\dot{s}$, and therefore $\frac{x}{t} = \dot{v}$. And by the third $\frac{1}{3}\dot{v} = \dot{y}$, that is, $\frac{2x}{3t} = \dot{z}$, and hence $\frac{2}{3t} - \frac{2x\dot{t}}{3t^2} = \dot{z}$. Which being

being found, you muſt take PC $= \frac{1 - \dot{x}\dot{y}}{\dot{z}}$, PL $= \dot{y} \times$ PC, and DC $=$ PC $- y$, or PC $- \frac{1}{2}$QR. Where it appears, that the length of the given Curve QR cannot be found, but at the ſame time the length of the right Line DC muſt be known, and from thence the length of the Curve, in which the point C is found; and ſo on the contrary.

6. Ex. 2. The Equation $as - ss = tt$ remaining, make $\dot{x} = \dot{s}$, and $vv - 4ax = 4ay$. And by the firſt there will be found $- \frac{a\dot{s}}{2t} = \dot{v}$, as above. But by the ſecond $1 = \dot{s}$, and therefore $\frac{a}{2t} = \dot{v}$. And by the third $2\dot{v}v - 4a = 4a\dot{y}$, or (eliminating $\dot{v}$) $\frac{v}{4t} - 1 = \dot{z}$. Then from hence $\frac{\dot{v}}{4t} - \frac{v\dot{t}}{4tt} = \dot{z}$.

7. Ex. 3. Let there be ſuppos'd three Equations, $aa = st$, $a + 3s = x$, and $x + v = y$. Then by the firſt, which denotes an Hyperbola, it is $0 = \dot{s}t + s\dot{t}$, or $- \frac{\dot{s}t}{s} = \dot{t}$, and therefore $\frac{\dot{s}}{s}\sqrt{ss + tt} = \sqrt{\dot{s}\dot{s} + \dot{t}\dot{t}} = \dot{v}$. By the ſecond it is $3\dot{s} = 1$, and therefore $\frac{1}{3s}\sqrt{ss + tt} = \dot{v}$. And by the third it is $1 + \dot{v} = \dot{y}$, or $1 + \frac{1}{3s}\sqrt{ss + tt} = \dot{z}$; then it is from hence $\dot{w} = \dot{z}$, that is, putting $\dot{w}$ for the Fluxion of the radical $\frac{1}{3s}\sqrt{ss + tt}$, which if it be made equal to w, or $\frac{1}{9} + \frac{tt}{9s} = ww$, there will ariſe from thence $\frac{2t\dot{t}}{9ss} - \frac{2tts}{9s^3} = 2w\dot{w}$. And firſt ſubſtituting $- \frac{\dot{s}t}{s}$ for $\dot{t}$, then $\frac{1}{3}$ for $\dot{s}$, and dividing by $2w$, there will ariſe $\frac{-2tt}{27ws^3} = \dot{w} = \dot{z}$. Now $\dot{y}$ and $\dot{z}$ being found, the reſt is perform'd as in the firſt Example.

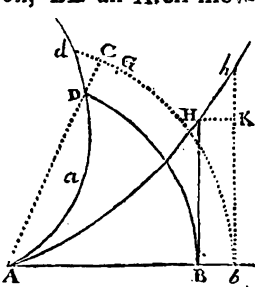
8. Now if from any point Q of a Curve, a perpendicular QV is let fall on MN, and a Curve is to be found whoſe length may be known from the length which ariſes by applying the Area QRNV to any given Line; let that given Line be call'd E, the length $\frac{\text{QRNV}}{\text{E}}$ which is produced by ſuch application be call'd v, and its fluxion $\dot{v}$. And ſince the fluxion of the Area QRNV is to the Fluxion of the Area of a rectangular parallelogram made upon VN, with the height E, as the Ordinate or moving line NR $= t$, by which this is deſcribed, to the moving Line E, by which the other is deſcribed in

the ſame time; and the fluxions $\dot{v}$ and $\dot{s}$ of the lines v and MN, (or s,) or of the lengths which ariſe by applying thoſe Areas to the given Line E, are in the ſame ratio; it will be $\dot{v} = \frac{\dot{s}t}{E}$. Therefore by this Rule the value of $\dot{v}$ is to be inquired, and the reſt to be perform'd as in the Examples aforegoing.

9. Ex. 4. Let QR be an Hyperbola which is defined by this Equation, $aa + \frac{ass}{c} = tt$; and thence ariſes (by Prob. 1.) $\frac{ass}{c} = \dot{tt}$, or $\frac{ass}{ct} = \dot{t}$. Then if for the other two Equations are aſſumed $x = s$ and $y = v$; the firſt will give $1 = \dot{s}$, whence $\dot{v} = \frac{\dot{s}t}{E} = \frac{t}{E}$; and the latter will give $\dot{y} = \dot{v}$, or $\dot{z} = \frac{\dot{t}}{E}$, then from hence $\ddot{z} = \frac{\dot{t}}{E}$, and ſubſtituting $\frac{ass}{ct}$ or $\frac{as}{ct}$ for $\dot{t}$, it becomes $\ddot{z} = \frac{as}{Ect}$. Now $\dot{y}$ and $\dot{z}$ being found, make $\frac{1 - \dot{y}}{\dot{z}} = CP$, and $\dot{y} \times CP = PL$, as before, and thence the Point C will be determin'd, and the Curve in which all ſuch points are ſituated: The length of which Curve will be known from the length DC, which is equivalent to $CP - v$, as is ſufficiently ſhewn before.

10. There is alſo another method, by which the Problem may be reſolved; and that is by finding Curves whoſe fluxions are either equal to the fluxion of the propoſed Curve, or are compounded of the fluxion of that, and of other Lines. And this may ſometimes be of uſe, in converting mechanical Curves into equable Geometrical Curves; of which thing there is a remarkable Example in ſpiral lines.

11. Let AB be a right Line given in poſition, BD an Arch moving upon AB as an Abſciſs, and yet retaining A as its Center, AD*d* a Spiral, at which that arch is continually terminated, *bd* an arch indefinitely near it, or the place into which the arch BD by its motion next arrives, DC a perpendicular to the arch *bd*, *d*G the difference of the arches, AH another Curve equal to the Spiral AD, BH a right Line moving perpendicularly upon AB, and terminated at the Curve AH, *bh* the next place into which that right Line moves, and HK perpendicular to *bh*.



bb. And in the infinitely little triangles DC*d* and HK*b*, ſince DC and HK are equal to the ſame third Line B*b*, and therefore equal to each other, and D*d* and H*b* (by hypotheſis) are correſpondent parts of equal Curves, and therefore equal, as alſo the angles at C and K are right angles; the third ſides *d*C and *b*K will be equal alſo. Moreover ſince it is AB : BD :: A*b* : *b*C :: A*b* — AB (B*b*) : *b*C — BD (CG); therefore $\frac{BD \times Bb}{AB} = CG$. If this be taken away from *d*G, there will remain *d*G — $\frac{BD \times Bb}{AB} = dC = bK$. Call therefore AB = z, BD = v, and BH = y, and their fluxions $\dot{z}$, $\dot{v}$, and $\dot{y}$ reſpectively, ſince B*b*, *d*G, and *b*K are the contemporaneous moments of the ſame, by the acceſſion of which they become A*b*, *bd*, and *bb*, and therefore are to each other as the fluxions. Therefore for the moments in the laſt Equation let the fluxions be ſubſtituted, as alſo the letters for the Lines, and there will ariſe $\dot{v} - \frac{v\dot{z}}{z} = \dot{y}$. Now of theſe fluxions, if $\dot{z}$ be ſuppos'd equable, or the unit to which the reſt are refer'd, the Equation will be $\dot{v} - \frac{v}{z} = \dot{y}$.

12. Wherefore the relation between AB and BD, (or between z and v,) being given by any Equation, by which the Spiral is defined, the fluxion $\dot{v}$ will be given, (by Prob. 1.) and thence alſo the fluxion $\dot{y}$, by putting it equal to $\dot{v} - \frac{v}{z}$. And (by Prob. 2.) this will give the line y, or BH, of which it is the fluxion.

13. Ex. 1. If the Equation $\frac{zz}{a} = v$ were given, which is to the Spiral of *Archimedes*, thence (by Prob. 1.) $\frac{2z\dot{z}}{a} = \dot{v}$. From hence take $\frac{v}{z}$, or $\frac{z}{a}$, and there will remain $\frac{z}{a} = \dot{y}$, and thence (by Prob. 2.) $\frac{zz}{2a} = y$. Which ſhews the Curve AH, to which the Spiral AD is equal, to be the Parabola of *Apollonius*, whoſe Latus rectum is $2a$; or whoſe Ordinate BH is always equal to half the Arch BD.

14. Ex. 2. If the Spiral be propoſed which is defined by the Equation $z^3 = av^2$, or $v = \frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, there ariſes (by Prob. 1.) $\frac{3z^{\frac{1}{2}}\dot{z}}{2a^{\frac{1}{2}}} = \dot{v}$, from which if you take $\frac{v}{z}$, or $\frac{z^{\frac{1}{2}}}{a^{\frac{1}{2}}}$, there will remain $\frac{z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{y}$, and thence (by Prob. 2.) will be produced $\frac{z^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = y$. That is, $\frac{1}{3}$BD = BH, AH being a Parabola of the ſecond kind.

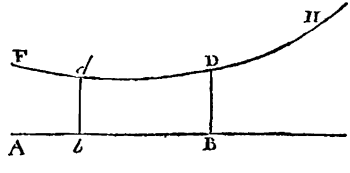
15. Ex. 3. If the Equation to the Spiral be $z\sqrt{\frac{a+z}{c}} = v$, thence (by Prob. 1.) $\frac{2a+3z}{2\sqrt{ac+cz}} = \dot{v}$; from whence if you take away $\frac{v}{z}$ or $\sqrt{\frac{a+z}{c}}$, there will remain $\frac{z}{2\sqrt{ac+cz}} = \dot{y}$. Now ſince the quantity generated by this fluxion $\dot{y}$ cannot be found by Prob. 2. unleſs it be reſolved into an infinite Series; according to the tenor of the Scholium to Prob. 9. I reduce it to the form of the Equations in the firſt column of the Tables, by ſubſtituting z^n for z; then it becomes $\frac{z^{2n-1}}{2\sqrt{ac+cz^n}} = \dot{y}$, which Equation belongs to the 2d Species of the 4th Order of Table 1. And by comparing the terms, it is $d = \frac{1}{2}$, $e = ac$, and $f = c$, ſo that $\frac{z-2a}{3c}\sqrt{ac+cz} = t = y$. Which Equation belongs to a Geometrical Curve AH, which is equal in length to the Spiral AD.

PROB. XII.

To determine the Lengths of Curves.

1. In the foregoing Problem we have ſhewn, that the Fluxion of a Curve-line is equal to the ſquare-root of the ſum of the ſquares of the Fluxions of the Abſciſs and of the perpendicular Ordinate. Wherefore if we take the Fluxion of the Abſciſs for an uniform and determinate meaſure, or for an Unit to which the other Fluxions are to be refer'd, and alſo if from the Equation which defines the Curve, we find the Fluxion of the Ordinate, we ſhall have the Fluxion of the Curve-line, from whence (by Problem 2.) its Length may be deduced.

2. Ex. 1. Let the Curve FDH be propoſed, which is defined by the Equation $\frac{z^3}{aa} + \frac{aa}{12z} = y$; making the Abſciſs AB $= z$, and the moving Ordinate DB $= y$. Then from the Equation will be had, (by Prob. 1.) $\frac{3zz}{aa} - \frac{aa}{12zz} = \dot{y}$, the fluxion of z being 1, and $\dot{y}$ being the fluxion of y. Then adding the ſquares of the fluxions, the ſum will be $\frac{9z^4}{a^4} + \frac{1}{1} + \frac{a^4}{144z^4} = \dot{t}\dot{t}$, and extracting the root, $\frac{3zz}{aa} + \frac{aa}{12zz} = \dot{t}$,



$= \dot{t}$, and thence (by Prob. 2.) $\frac{z^3}{aa} - \frac{aa}{12z} = t$. Here $\dot{t}$ ſtands for the fluxion of the Curve, and t for its Length.

3. Therefore if the length dD of any portion of this Curve were required, from the points d and D let fall the perpendiculars db and DB to AB, and in the value of t ſubſtitute the quantities Ab and AB ſeverally for z, and the difference of the reſults will be dD the Length required. As if $Ab = \frac{1}{2}a$, and AB $= a$, writing $\frac{1}{2}a$ for z, it becomes $t = -\frac{a}{24}$; then writing a for z, it becomes $t = \frac{11a}{12}$, from whence if the firſt value be taken away, there will remain $\frac{23a}{24}$ for the length dD. Or if only Ab be determin'd to be $\frac{1}{2}a$, and AB be look'd upon as indefinite, there will remain $\frac{z^3}{aa} - \frac{aa}{12z} + \frac{a}{24}$ for the value of dD.

4. If you would know the portion of the Curve which is repreſented by t, ſuppoſe the value of t to be equal to nothing, and there ariſes $z^4 = \frac{a^4}{12}$, or $z = \frac{a}{\sqrt[4]{12}}$. Therefore if you take AB $= \frac{a}{\sqrt[4]{12}}$, and erect the perpendicular bd, the length of the Arch dD will be t, or $\frac{z^3}{aa} - \frac{aa}{12z}$. And the ſame is to be underſtood of all Curves in general.

5. After the ſame manner by which we have determin'd the length of this Curve, if the Equation $\frac{z^4}{a^3} + \frac{a^3}{3zz} = y$ be propoſed, for defining the nature of another Curve; there will be deduced $\frac{z^4}{a^3} - \frac{a^3}{3zz} = t$; or if this Equation be propoſed, $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} - \frac{1}{3}a^{\frac{1}{2}}z^{\frac{1}{2}} = y$, there will ariſe $\frac{z^{\frac{1}{2}}}{a^{\frac{1}{2}}} + \frac{1}{3}a^{\frac{1}{2}}z^{\frac{1}{2}} = t$. Or in general, if it is $cz^\theta + \frac{z^{2-\theta}}{49c - 8\theta c} = y$, where θ is uſed for repreſenting any number, either Integer or Fraction, we ſhall have $cz^\theta - \frac{z^{2-\theta}}{49c - 8\theta c} = t$.

6. Ex. 2. Let the Curve be propoſed which is defined by this Equation $\frac{2aa + zz}{3aa}\sqrt{aa + zz} = y$; then (by Prob. 1.) will be had $\dot{y} = \frac{4z^4z + 8z^2z^3 + 4z^5}{3a^4y}$, or exterminating y, $\dot{y} = \frac{2z}{aa}\sqrt{aa + zz}$. To the ſquare of which add 1, and the ſum will be $1 + \frac{4zz}{aa} + \frac{4z^4}{a^4}$, and

and its Root $1 + \frac{2\dot{x}x}{aa} = \dot{t}$. Hence (by Prob. 2.) will be obtain'd $z + \frac{2\dot{x}x^3}{3a^2} = t$.

7. Ex. 3. Let a Parabola of the ſecond kind be propoſed, whoſe Equation is $z^3 = ay^2$, or $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} = y$, and thence by Prob. 1. is derived $\frac{3\dot{x}z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{y}$. Therefore $\sqrt{1 + \frac{9z}{4a}} = \sqrt{1 + \dot{y}\dot{y}} = \dot{t}$. Now ſince the length of the Curve generated by the Fluxion $\dot{t}$ cannot be found by Prob. 2. without a reduction to an infinite Series of ſimple Terms, I conſult the Tables in Prob. 9. and according to the Scholium belonging to it, I have $t = \frac{8a + 18z}{27}\sqrt{1 + \frac{9z}{4a}}$. And thus you may find the lengths of theſe Parabolas $z^5 = ay^4$, $z^7 = ay^6$, $z^9 = ay^8$, &c.

8. Ex. 4. Let the Parabola be propoſed, whoſe Equation is $z^4 = ay^3$, or $\frac{z^{\frac{4}{3}}}{a^{\frac{1}{3}}} = y$; and thence (by Prob. 1.) will ariſe $\frac{4\dot{x}z^{\frac{1}{3}}}{3a^{\frac{1}{3}}} = \dot{y}$. Therefore $\sqrt{1 + \frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}} = \sqrt{\dot{y}\dot{y} + 1} = \dot{t}$. This being found, I conſult the Tables according to the aforeſaid Scholium, and by comparing with the 2d Theorem of the 5th Order of Table 2, I have $z^{\frac{1}{3}} = x$, $\sqrt{1 + \frac{16xx}{9a^{\frac{2}{3}}}} = v$, and $\frac{3}{8}s = t$. Where x denotes the Abſciſs, y the Ordinate, and s the Area of the Hyperbola, and t the length which ariſes by applying the Area $\frac{3}{8}s$ to linear unity.

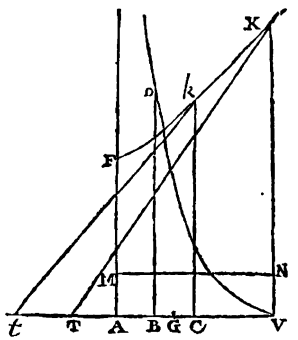
9. After the ſame manner the lengths of the Parabolas $z^6 = ay^5$, $z^8 = ay^7$, $z^{10} = ay^9$, &c. may alſo be reduced to the Area of the Hyperbola.

10. Ex. 5. Let the Ciſſoid of the Ancients be propoſed, whoſe Equation is $\frac{aa - 2az + zz}{\sqrt{az - zz}} = y$, and thence (by Prob. 1.) $\frac{-a - 2z}{2\dot{x}z}\sqrt{az - zz} = y$, and therefore $\frac{a}{2z}\sqrt{\frac{a + 3z}{z}} = \sqrt{\dot{y}\dot{y} + 1} = \dot{t}$; which by writing z^n for $\frac{1}{z}$ or z^{-1}, becomes $\frac{a}{2z}\sqrt{az^n + 3} = \dot{t}$, an Equation of the 1ſt Species of the 3d Order of Table 2; then comparing the Terms, it is $\frac{a}{2} = d$, $3 = e$, and $a = f$; ſo that $z = \frac{1}{z^n} = x^2$, $\sqrt{a + 3xx} = v$, and $6s - \frac{2v^3}{x} = \frac{4de}{nf}$ into $\frac{v^3}{2\dot{x}x} - s = t$.

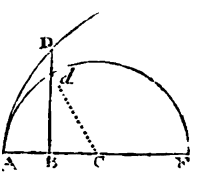
And

And taking a for Unity, by the Multiplication or Diviſion of which, theſe Quantities may be reduced to a juſt number of Dimenſions, it becomes $az = xx$, $\sqrt{aa + 3xx} = v$, and $\frac{6s}{a} - \frac{2v^3}{ax} = t$: Which are thus conſtructed.

11. The Ciſſoid being VD, AV the Diameter of the Circle to which it is adapted, AF its Aſymptote, and DB perpendicular to AV, cutting the Curve in D; with the Semiaxis AF = AV, and the Semiparameter AG = $\frac{1}{3}$AV, let the Hyperbola F*k*K be deſcribed; and taking AC a mean Proportional between AB and AV, at C and V let C*k* and VK drawn perpendicular to AV, cut the Hyperbola in *k*, and K, and let right Lines *kt* and KT touch it in thoſe points, and cut AV in *t* and T; and at AV let the Rectangle AVNM be deſcribed, equal to the Space TK*kt*. Then the length of the Ciſſoid VD will be ſextuple of the Altitude VN.

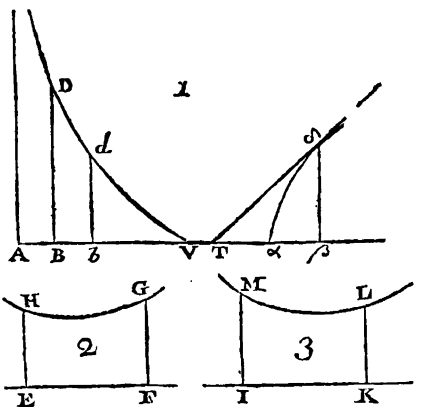


12. Ex. 6. Suppoſing A*d* to be an Ellipſis, which the Equation $\sqrt{az - 2zz} = y$ repreſents; let the mechanical Curve AD be propoſed of ſuch a nature, that if B*d*, or y, be produced till it meets this Curve at D, let BD be equal to the Elliptical Arch A*d*. Now that the length of this may be determin'd, the Equation $\sqrt{az - 2zz} = y$ will give $\frac{a - 4z}{2\sqrt{az - 2zz}} = \dot{y}$, to the ſquare of which if 1 be added, there ariſes $\frac{aa - 4az + 8zz}{4az - 8zz}$, the ſquare of the fluxion of the arch A*d*. To which if 1 be added again, there will ariſe $\frac{aa}{4az - 8zz}$, whoſe ſquare-root $\frac{a}{2\sqrt{az - 2zz}}$ is the fluxion of the Curve-line AD. Where if z be extracted out of the radical, and for z^{-1} be written z^n, there will be had $\frac{a}{2z\sqrt{az^n - 2}}$, a Fluxion of the 1ſt Species of the 4th Order of Table 2. Therefore the terms being collated, there will ariſe $d = \frac{1}{2}a$, $e = -2$, and $f = a$; ſo that $z = \frac{1}{x^n} = x$, $\sqrt{ax - 2xx} = v$, and $\frac{ns}{a} - \frac{4xv}{a} + v = \frac{8de}{vf}$ into $s - \frac{1}{2}xv - \frac{fv}{4e} = t$.



13. The Conſtruction of which is thus; that the right line dC being drawn to the center of the Ellipſis, a parallelogram may be made upon AC, equal to the ſector ACd, and the double of its height will be the length of the Curve AD.

14. Ex. 7. Making $A\beta = \varphi$, (Fig. 1.) and $\alpha\delta$ being an Hyperbola, whoſe Equation is $\sqrt{-a+b\varphi\varphi} = \beta\delta$, and its tangent δT being drawn; let the Curve VdD be propoſed, whoſe Abſciſs is $\frac{1}{\varphi\varphi}$, and its perpendicular Ordinate is the length BD, which ariſes by applying the Area $\alpha\delta T\alpha$ to linear unity. Now that the length of this Curve VD may be determin'd, I ſeek the fluxion of the Area $\alpha\delta T\alpha$, when AB flows uniformly, and I find it to be $\frac{a}{4bz}\sqrt{b-az}$, putting $AB = z$, and its fluxion unity. For



'tis $AT = \frac{a}{b\varphi} = \frac{a}{b}\sqrt{z}$, and its fluxion is $\frac{a}{2b\sqrt{z}}$, whoſe half drawn into the altitude $\beta\delta$, or $\sqrt{-a+\frac{b}{z}}$, is the fluxion of the Area $\alpha\delta T$, deſcribed by the Tangent δT. Therefore that fluxion is $\frac{a}{4bz}\sqrt{b-az}$, and this apply'd to unity becomes the fluxion of the Ordinate BD. To the ſquare of this $\frac{aab-a^3z}{16b^2z^2}$ add 1, the ſquare of the fluxion BD, and there ariſes $\frac{aab-a^3z+16b^2z^2}{16b^2z^2}$, whoſe root $\frac{1}{4bz}\sqrt{a^2b-a^3z+16b^2z^2}$ is the fluxion of the Curve VD. But this is a fluxion of the 1ſt Species of the 7th Order of Table 2: and the terms being collated, there will be $\frac{1}{4b} = \underline{d}$, $\underline{aab} = \underline{e}$, $-a^3 = \underline{f}$, $16b^2 = \underline{g}$, and therefore $z = \underline{x}$, and $\sqrt{a^2b-a^3x+16b^2x^2} = \underline{v}$, (an Equation to one Conic Section, ſuppoſe HG, (Fig. 2.) whoſe Area EFGH is s, where $EF = x$, and $FG = v$;) alſo $\frac{1}{z} = \xi$, and $\sqrt{16bb-a^3\xi+ab\xi^2} = \Upsilon$, (an Equation to another Conic Section,

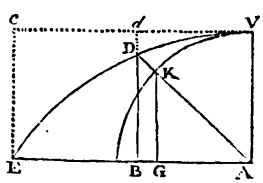
Section, ſuppoſe ML (Fig. 3.) whoſe Area IKLM is σ, where IK $=\xi$, and KL $=\Upsilon$;) Laſtly $\frac{2aabb\xi\Upsilon - a^3b\Upsilon - a^4v - 4aabb\sigma - 32abbs}{64b^4 - a^4} = t$.

15. Wherefore that the length of any portion D*d* of the Curve VD may be known, let fall *db* perpendicular to AB, and make A*b* $= z$; and thence, by what is now found, ſeek the value of *t*. Then make AB $= z$, and thence alſo ſeek for *t*. And the difference of theſe two values of *t* will be the length D*d* required.

16. Ex. 8. Let the Hyperbola be propos'd, whoſe Equation is $\sqrt{aa + bzz} = y$, and thence, (by Prob. 1.) will be had $\dot{y} = \frac{bz}{y}$, or $\frac{bz}{\sqrt{aa + bzz}}$. To the ſquare of this add 1, and the root of the ſum will be $\sqrt{\frac{aa + bzz + bbzz}{aa + bzz}} = \dot{t}$. Now as this fluxion is not to be found in the Tables, I reduce it to an infinite Series; and firſt by diviſion it becomes $\dot{t} = \sqrt{1 + \frac{b^2}{a^2}z^2 - \frac{b^3}{a^4}z^4 + \frac{b^4}{a^6}z^6 - \frac{b^5}{a^8}z^8}$, &c. and extracting the root, $\dot{t} = 1 + \frac{b^2}{2a^2}z^2 - \frac{4b^3 + b^4}{8a^4}z^4 + \frac{8b^4 + 4b^5 + b^6}{16a^6}z^6$, &c. And hence (by Prob. 2.) may be had the length of the Hyperbolical Arch, or $t = z + \frac{b^2}{6a^2}z^3 - \frac{4b^3 + b^4}{40a^4}z^5 + \frac{8b^4 + 4b^5 + b^6}{112a^6}z^7$, &c.

17. If the Ellipſis $\sqrt{aa - bzz} = y$ were propoſed, the Sign of *b* ought to be every where changed, and there will be had $z + \frac{b^2}{6a^2}z^3 + \frac{4b^3 - b^4}{40a^4}z^5 + \frac{8b^4 - 4b^5 + b^6}{112a^6}z^7$, &c. for the length of its Arch. And likewiſe putting Unity for *b*, it will be $z + \frac{z^3}{6a^2} + \frac{3z^5}{40a^4} + \frac{5z^7}{112a^6}$, &c. for the length of the Circular Arch. Now the numeral coefficients of this ſeries may be found *ad infinitum*, by multiplying continually the terms of this Progreſſion $\frac{1\times1}{2\times3}$, $\frac{3\times3}{4\times5}$, $\frac{5\times5}{6\times7}$, $\frac{7\times7}{8\times9}$, $\frac{9\times9}{10\times11}$, &c.

18. Ex. 9. Laſtly, let the Quadratrix VDE be propoſed, whoſe Vertex is V, A being the Center, and AV the ſemidiameter of the interior Circle, to which it is adapted, and the Angle VAE being a right Angle. Now any right Line AKD being drawn through A, cutting the Circle in K, and the Quadratrix in D, and the perpendiculars KG, DB being let fall to AE; call AV $= a$, AG $= z$, VK $= x$, and BD $= y$, and it will



will be as in the foregoing Example, $x = z + \frac{z^3}{6a^2} + \frac{3z^5}{40a^4} + \frac{5z^7}{112a^6}$, &c. Extract the root z, and there will arise $z = x - \frac{x^3}{6a^2} + \frac{x^5}{120a^4} - \frac{x^7}{5040a^6}$, &c. whose Square subtract from AKq, or a^2, and the root of the remainder $a - \frac{x^2}{2a} + \frac{x^4}{24a^3} - \frac{x^6}{720a^5}$, &c. will be GK. Now whereas by the nature of the Quadratrix 'tis $AB = VR = x$, and since it is AG : GK :: AB : BD (y), divide AB × GK by AG, and there will arise $y = a - \frac{xx}{3a} - \frac{x^4}{45a^3} - \frac{2x^6}{945a^5}$, &c. And thence, (by Prob. 1.) $\dot{y} = -\frac{2x}{3a} - \frac{4x^3}{45a^3} - \frac{4x^5}{315a^5}$, &c. to the square of which add 1, and the root of the sum will be $1 + \frac{2xx}{9aa} + \frac{14x^4}{405a^4} + \frac{604x^6}{12,575a^6}$, &c. $= \dot{t}$. Whence (by Prob. 2.) t may be obtain'd, or the Arch of the Quadratrix; *viz.* $VD = x + \frac{2x^3}{27a^2} + \frac{14x^5}{2835a^4} + \frac{604x^7}{893025a^6}$, &c.

THE

THE

CONTENTS of the following Comment.

I. *ANnotations on the Introduction; or the Resolution of Equations by Infinite Series.* —— pag. 143

Sect. I. *Of the nature and construction of infinite or converging Series.* —— —— —— p. 143

Sect. II. *The Resolution of simple Equations, or of pure Powers, by infinite Series.* —— —— p. 159

Sect. III. *The Resolution of Numeral Affected Equations.* p. 186

Sect. IV. *The Resolution of Specious Equations by infinite Series; and first for determining the forms of the Series, and their initial Approximations.* —— p. 191

Sect. V. *The Resolution of Affected specious Equations prosecuted by various Methods of Analysis.* —— p. 209

Sect. VI. *Transition to the Method of Fluxions.* —— p. 235

II. *Annotations on Prob. 1. or, the Relation of the flowing Quantities being given, to determine the Relation of their Fluxions.* —— p. 241

Sect. I. *Concerning Fluxions of the first Order, and to find their Equations.* —— —— p. 241

Sect. II. *Concerning Fluxions of superior Orders, and the method of deriving their Equations.* —— p. 257

Sect. III. *The Geometrical and Mechanical Elements of Fluxions.* —— —— p. 266

III. *Annotations on Prob.* 2. *or, the Relation of the Fluxions being given, to determine the Relation of the Fluents.* p.277

Sect. I. *A particular Solution; with a preparation to the general Solution, by which it is diſtributed into three Caſes.* —— —— —— p.277

Sect. II. *Solution of the firſt Caſe of Equations.* —— p.282

Sect. III. *Solution of the ſecond Caſe of Equations.* —— p.286

Sect. IV. *Solution of the third Caſe of Equations, with ſome neceſſary Demonſtrations.* —— —— p.300

Sect. V. *The Reſolution of Equations, whether Algebraical or Fluxional, by the aſſiſtance of ſuperior Orders of Fluxions.* —— —— —— p.308

Sect. VI. *An Analytical Appendix, explaining ſome Terms and Expreſſions in the foregoing Work.* —— p.321

Sect. VII. *The Concluſion; containing a ſhort recapitulation or review of the whole.* —— —— p.330

THE Reader is desired to correct the following Errors, which I hope will be thought but few, and such as in works of this kind are hardly to be avoided. 'Tis here necessary to take notice of even literal Mistakes, which in subjects of this nature are often very material. That the Errors are so few, is owing to the kind assistance of a skilful Friend or two, who supply'd my frequent absence from the Press; as also to the care of a diligent Printer.

ERRATA.

In the Preface, pag. xiii. *lin.* 3. *read* which is here subjoin'd. *Ibid.* l. 5. *for* matter *read* manner, Pag. xxiii. l. *ult. for* Preface, &c. *read* Conclusion of this Work. P. 7. l. 31. *for* + *read* =. P. 15. l. 9. *read* $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$; &c. P. 17. l. 17. *read* $-\frac{z}{9x}$. P. 32. l. 21. *read* $\frac{\dot{y}}{\dot{x}}$. P. 35. l. 3. *for* $10xy^4$ *read* $10x^4y$. P. 62. l. 4. *read* $\frac{z+z^3}{\dot{z}}$. P. 63. l. 31. *for* y *read* $\dot{y}$. *Ibid.* l. *ult. for* $-\frac{yz}{v}$ *read* $-\frac{y\dot{z}}{v}$. P. 64. l. 9. *for* z *read* z. *Ibid.* l. 30. *read* $\dot{t}$. P. 82. l. 17. *read* $2\dot{z}z$. P. 87. l. 22. *read* $+2a^{\frac{3}{2}}x^{\frac{1}{2}}$. *Ibid.* l. 22, 24. *read* AFDB. P. 88. l. 27. *read* $\dot{x}$. P. 92. l. 5. *read* $+\frac{2a^2y^5}{5e^5}$. *Ibid.* l. 21. *for* z *read* $\dot{z}$. P. 104. l. 8. *read* $6ne^2$. P. 109. l. 33. *dele* as often. P. 110. l. 29. *read* $\sqrt{\frac{a^4}{a^2+z^2}} = \sqrt{AQ \times EC} = HA = AB = x$, and $\sqrt{a^2-x^2} =$. P. 113. l. 17. *for* Parabola *read* Hyperbola. P. 119. l. 12. *read* $CE \times \dot{\overline{ID}}$ = to the Fluxion of the Area ACEG, and $ID \times \dot{\overline{IP}}$ = to. P. 131. l. 8. *read* $+\frac{a\dot{s}}{2t}$. *Ibid.* l. 19. *read* $\frac{tt}{9ss}$. P. 135. l. 15. *read* Ab. P. 138. l. 9. *read* Abscissa AB. P. 145. l. *penult. read* $7X^{-3}$. P. 149. l. 20. *read* which in. P. 157. l. 13. *read* $a\times$. P. 168. l. 5. *read* $\frac{1}{2}ax$. P. 171. l. 17. *for* B^3 *read* β^3. P. 177. l. 15. *read* z^{m+2}. P. 189. l. 30. *for the last* P^{-3} *read* P^{-4}. P. 204. l. 16. *read* to $2m$. P. 213. l. 7. *for* Species *read* Series. P. 229. l. 21. *for* x^{-3} *read* x^{-4}. *Ibid.* l. 24. *for* x^{-4} *read* x^{-5}. P. 234. l. 2. *for* yy *read* y. P. 236. l. 26. *read* generating. P. 243. l. 29. *read* $-ax^2\dot{y}y^{-1}$. P. 284. l. *ult. read* $\frac{\dot{y}}{\dot{x}}$. P. 289. l. 17. *for* right *read* left. P. 295. l. 1, 2. *read* $-\frac{1}{6}x^4 + \frac{5}{12}ax^4$. P. 297. l. 19. *for* $\dot{y}x^{-1}$ *read* yx^{-1}. P. 298. l. 14. *read* $-\dot{y}$. P. 304. l. 20, 21. *dele* $+bc$. P. 309. l. 18. *read* a^{m-3}. P. 317. l. *ult. read* $a^2\dot{y}^2$.

ADVERTISEMENT.

Lately publiſh'd by the Author,

THE BRITISH HEMISPHERE, or a Map of a new contrivance, proper for initiating young Minds in the firſt Rudiments of Geography, and the uſe of the Globes. It is in the form of a Half-Globe, of about 15 Inches Diameter, but comprehends the whole known Surface of our habitable Earth; and ſhews the ſituation of all the remarkable Places, as to their Longitude, Latitude, Bearing and Diſtance from *London*, which is made the Center or Vertex of the Map. It is neatly fitted up, ſo as to ſerve as well for ornament as uſe; and ſufficient Inſtructions are annex'd, to make it intelligible to every Capacity.

Sold by W. REDKNAP, at the *Leg and Dial* near the *Sun* Tavern in *Fleet-ſtreet*; and by J. SENEX, at the *Globe* near St. *Dunſtan*'s Church. Price, Half a Guinea.

THE METHOD of FLUXIONS AND INFINITE SERIES;

OR,

A PERPETUAL COMMENT upon the foregoing TREATISE.

THE METHOD of FLUXIONS AND INFINITE SERIES.

ANNOTATIONS on the Introduction: OR, The Reſolution of Equations by INFINITE SERIES.

SECT. I. *Of the Nature and Conſtruction of Infinite or Converging Series.*

1. THE great Author of the foregoing Work begins it with a ſhort Preface, in which he lays down his main deſign very conciſely. He is not to be here underſtood, as if he would reproach the modern Geometricians with deſerting the Ancients, or with abandoning their Synthetical Method of Demonſtration, much leſs that he intended to diſparage the Analytical Art; for on the contrary he has very much improved both Methods, and particularly in this Treatiſe he wholly applies himſelf to cultivate Analyticks, in which he has ſucceeded to univerſal applauſe and admiration. Not but that we ſhall find here ſome examples of the Synthetical Method likewiſe, which are very maſterly and elegant. Almoſt all that remains of the ancient Geometry is indeed Synthetical, and proceeds by way of demonſtrating truths already known, by ſhewing their dependence upon the Axioms, and other

other firſt Principles, either mediately or immediately. But the buſineſs of Analyticks is to inveſtigate ſuch Mathematical Truths as really are, or may be ſuppos'd at leaſt to be unknown. It aſſumes thoſe Truths as granted, and argues from them in a general manner, till after a ſeries of argumentation, in which the ſeveral ſteps have a neceſſary connexion with each other, it arrives at the knowledge of the propoſition required, by comparing it with ſomething really known or given. This therefore being the Art of Invention, it certainly deſerves to be cultivated with the utmoſt induſtry. Many of our modern Geometricians have been perſuaded, by conſidering the intricate and labour'd Demonſtrations of the Ancients, that they were Maſters of an Analyſis purely Geometrical, which they ſtudiouſly conceal'd, and by the help of which they deduced, in a direct and ſcientifical manner, thoſe abſtruſe Propoſitions we ſo much admire in ſome of their writings, and which they afterwards demonſtrated Synthetically. But however this may be, the loſs of that Analyſis, if any ſuch there were, is amply compenſated, I think, by our preſent Arithmetical or Algebraical Analyſis, eſpecially as it is now improved, I might ſay perfected, by our ſagacious Author in the Method before us. It is not only render'd vaſtly more univerſal and extenſive than that other in all probability could ever be, but is likewiſe a moſt compendious Analyſis for the more abſtruſe Geometrical Speculations, and for deriving Conſtructions and Synthetical Demonſtrations from thence; as may abundantly appear from the enſuing Treatiſe.

2. The conformity or correſpondence, which our Author takes notice of here, between his new-invented Doctrine of infinite Series, and the commonly received Decimal Arithmetick, is a matter of conſiderable importance, and well deſerves, I think, to be ſet in a fuller Light, for the mutual illuſtration of both; which therefore I ſhall here attempt to perform. For Novices in this Doctrine, tho' they may already be well acquainted with the Vulgar Arithmetick, and with the Rudiments of the common Algebra, yet are apt to apprehend ſomething abſtruſe and difficult in infinite Series; whereas indeed they have the ſame general foundation as Decimal Arithmetick, eſpecially Decimal Fractions, and the ſame Notion or Notation is only carry'd ſtill farther, and render'd more univerſal. But to ſhew this in ſome kind of order, I muſt inquire into theſe following particulars. Firſt I muſt ſhew what is the true Nature, and what are the genuine Principles, of our common Scale of Decimal Arithmetick. Secondly what is the nature of other particular Scales, which have been, or may

may be, occaſionally introduced. Thirdly, what is the nature of a general Scale, which lays the foundation for the Doctrine of infinite Series. Laſtly, I ſhall add a word or two concerning that Scale of Arithmetick in which the Root is unknown, and therefore propoſed to be found; which gives occaſion to the Doctrine of Affected Equations.

Firſt then as to the common Scale of Decimal Arithmetick, it is that ingenious Artifice of expreſſing, in a regular manner, all conceivable Numbers, whether Integers or Fractions, Rational or Surd, by the ſeveral Powers of the number *Ten*, and their Reciprocals; with the aſſiſtance of other ſmall Integer Numbers, not exceeding Nine, which are the Coefficients of thoſe Powers. So that Ten is here the Root of the Scale, which if we denote by the Character X, as in the *Roman* Notation and its ſeveral Powers by the help of this Root and Numeral Indexes, ($X^1 = 10$, $X^2 = 100$, $X^3 = 1000$, $X^4 = 10000$, &c.) as is uſual; then by aſſuming the Coefficients 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, as occaſion ſhall require, we may form or expreſs any Number in this Scale. Thus for inſtance $5X^4 + 7X^3 + 4X^2 + 8X^1 + 3X^0$ will be a particular Number expreſs'd by this Scale, and is the ſame as 57483 in the common way of Notation. Where we may obſerve, that this laſt differs from the other way of Notation only in this, that here the ſeveral Powers of X (or Ten) are ſuppreſs'd, together with the Sign of Addition +, and are left to be ſupply'd by the Underſtanding. For as thoſe Powers aſcend regularly from the place of Units, (in which is always X^0, or 1, multiply'd by its Coefficient, which here is 3,) the ſeveral Powers will eaſily be underſtood, and may therefore be omitted, and the Coefficients only need to be ſet down in their proper order. Thus the Number 7906538 will ſtand for $7X^6 + 9X^5 + 0X^4 + 6X^3 + 5X^2 + 3X^1 + 8X^0$, when you ſupply all that is underſtood. And the Number 1736 (by ſuppreſſing what may be eaſily underſtood,) will be equivalent to $X^3 + 7X^2 + 3X + 6$; and the like of all other Integer Numbers whatever, expreſs'd by this Scale, or with this Root X, or Ten.

The ſame Artifice is uniformly carry'd on, for the expreſſing of all Decimal Fractions, by means of the Reciprocals of the ſeveral Powers of Ten, ſuch as $\frac{1}{X} = 0{,}1$; $\frac{1}{X^2} = 0{,}01$; $\frac{1}{X^3} = 0{,}001$; &c. which Reciprocals may be intimated by negative Indices. Thus the Decimal Fraction 0,3172 ſtands for $3X^{-1} + 1X^{-2} + 7X^{-3} + 2X^{-4}$; and the mixt Number 526,384 (by ſupplying what is underſtood)

becomes $5X^2 + 2X^1 + 6X^0 + 3X^{-1} + 8X^{-2} + 4X^{-3}$; and the infinite or interminate Decimal Fraction 0,9999999, &c. ſtands for $9X^{-1} + 9X^{-2} + 9X^{-3} + 9X^{-4} + 9X^{-5} + 9X^{-6}$, &c. which infinite Series is equivalent to Unity. So that by this Decimal Scale, (or by the ſeveral Powers of Ten and their Reciprocals, together with their Coefficients, which are all the whole Numbers below Ten,) all conceivable Numbers may be expreſs'd, whether they are integer or fracted, rational or irrational; at leaſt by admitting of a continual progreſs or approximation *ad infinitum*.

And the like may be done by any other Scale, as well as the Decimal Scale, or by admitting any other Number, beſides Ten, to be the Root of our Arithmetick. For the Root Ten was an arbitrary Number, and was at firſt aſſumed by chance, without any previous conſideration of the nature of the thing. Other Numbers perhaps may be aſſign'd, which would have been more convenient, and which have a better claim for being the Root of the Vulgar Scale of Arithmetick. But however this may prevail in common affairs, Mathematicians make frequent uſe of other Scales; and therefore in the ſecond place I ſhall mention ſome other particular Scales, which have been occaſionally introduced into Computations.

The moſt remarkable of theſe is the Sexagenary or Sexageſimal Scale of Arithmetick, of frequent uſe among Aſtronomers, which expreſſes all poſſible Numbers, Integers or Fractions, Rational or Surd, by the Powers of *Sixty*, and certain numeral Coefficients not exceeding fifty-nine. Theſe Coefficients, for want of peculiar Characters to repreſent them, muſt be expreſs'd in the ordinary Decimal Scale. Thus if ξ ſtands for 60, as in the *Greek* Notation, then one of theſe Numbers will be $53\xi^2 + 9\xi^1 + 34\xi^0$, or in the Sexagenary Scale $53''$, $9'$, $34°$, which is equivalent to 191374° in the Decimal Scale. Again, the Sexageſimal Fraction $53°$, $9'$, $34''$, will be the ſame as $53\xi^0 + 9\xi^{-1} + 34\xi^{-2}$, which in Decimal Numbers will be 53,159444, &c. *ad infinitum*. Whence it appears by the way, that ſome Numbers may be expreſs'd by a finite number of Terms in one Scale, which in another cannot be expreſs'd but by approximation, or by a progreſſion of Terms *in infinitum*.

Another particular Scale that has been conſider'd, and in ſome meaſure has been admitted into practice, is the Duodecimal Scale, which expreſſes all Numbers by the Powers of *Twelve*. So in common affairs we ſay a Dozen, a Dozen of Dozens or a Groſs, a Dozen of Groſſes or a great Groſs, *&c.* And this perhaps would have been the moſt convenient Root of all others, by the Powers of which

to

to conſtruct the popular Scale of Arithmetick; as not being ſo big but that its Multiples, and all below it, might be eaſily committed to memory; and it admits of a greater variety of Diviſors than any Number not much greater than itſelf. Beſides, it is not ſo ſmall, but that Numbers expreſs'd hereby would ſufficiently converge, or by a few figures would arrive near enough to the Number required; the contrary of which is an inconvenience, that muſt neceſſarily attend the taking too ſmall a Number for the Root. And to admit this Scale into practice, only two ſingle Characters would be wanting, to denote the Coefficients Ten and Eleven.

Some have conſider'd the Binary Arithmetick, or that Scale in which *Two* is the Root, and have pretended to make Computations by it, and to find conſiderable advantages in it. But this can never be a convenient Scale to manage and expreſs large Numbers by, becauſe the Root, and conſequently its Powers, are ſo very ſmall, that they make no diſpatch in Computations, or converge exceeding ſlowly. The only Coefficients that are here neceſſary are 0 and 1. Thus $1\times2^5+1\times2^4+0\times2^3+1\times2^2+1\times2^1+0\times2^0$ is one of theſe Numbers, (or compendiouſly 110110,) which in the common Notation is no more than 54. Mr. *Leibnits* imagin'd he had found great Myſteries in this Scale. See the Memoirs of the Royal Academy of *Paris*, *Anno* 1703.

In common affairs we have frequent recourſe, though tacitly, to Millenary Arithmetick, and other Scales, whoſe Roots are certain Powers of Ten. As when a large Number, for the convenience of reading, is diſtinguiſh'd into Periods of three figures: As 382,735,628,490. Here 382, and 735, *&c.* may be conſider'd as Coefficients, and the Root of the Scale is 1000. So when we reckon by Millions, Billions, Trillions, *&c.* a Million may be conceived as the Root of our Arithmetick. Alſo when we divide a Number into pairs of figures, for the Extraction of the Square-root; into ternaries of figures for the Extraction of the Cube-root; *&c.* we take new Scales in effect, whoſe Roots are 100, 1000, *&c.*

Any Number whatever, whether Integer or Fraction, may be made the Root of a particular Scale, and all conceivable Numbers may be expreſs'd or computed by that Scale, admitting only of integral and affirmative Coefficients, whoſe number (including the Cypher 0) need not be greater than the Root. Thus in Quinary Arithmetick, in which the Scale is compoſed of the Powers of the Root 5, the Coefficients need be only the five Numbers 0, 1, 2, 3, 4, and yet all Numbers whatever are expreſſible by this Scale, at leaſt by approxi-

mation, to what accuracy we pleaſe. Thus the common Number 2827,92 in this Arithmetick would be $4\times5^4+2\times5^3+3\times5^2+0\times5^1+2\times5^0+4\times5^{-1}+3\times5^{-2}$; or if we may ſupply the ſeveral Powers of 5 by the Imagination only, as we do thoſe of Ten in the common Scale, this Number will be 42302,43 in Quinary Arithmetick.

All vulgar Fractions and mixt Numbers are, in ſome meaſure, the expreſſing of Numbers by a particular Scale, or making the Denominator of the Fraction to be the Root of a new Scale. Thus $\frac{1}{3}$ is in effect $0\times3^0+2\times3^{-1}$; and $8\frac{1}{5}$ is the ſame as $8\times5^0+3\times5^{-1}$; and $25\frac{4}{9}$ reduced to this Notation will be $25\times9^0+4\times9^{-1}$, or rather $2\times9^1+7\times9^0+4\times9^{-1}$. And ſo of all other Fractions and mixt Numbers.

A Number computed by any one of theſe Scales is eaſily reduced to any other Scale aſſign'd, by ſubſtituting inſtead of the Root in one Scale, what is equivalent to it expreſs'd by the Root of the other Scale. Thus to reduce Sexagenary Numbers to Decimals, becauſe $60=6\times10$, or $\xi=6X$, and therefore $\xi^2=36X^2$, $\xi^3=216X^3$, *&c.* by the ſubſtitution of theſe you will eaſily find the equivalent Decimal Number. And the like in all other Scales.

The Coefficients in theſe Scales are not neceſſarily confin'd to be affirmative integer Numbers leſs than the Root, (tho' they ſhould be ſuch if we would have the Scale to be regular,) but as occaſion may require they may be any Numbers whatever, affirmative or negative, integers or fractions. And indeed they generally come out promiſcuouſly in the Solution of Problems. Nor is it neceſſary that the Indices of the Powers ſhould be always integral Numbers, but may be any regular Arithmetical Progreſſion whatever, and the Powers themſelves either rational or irrational. And thus (thirdly) we are come by degrees to the Notion of what is call'd an univerſal Series, or an indefinite or infinite Series. For ſuppoſing the Root of the Scale to be indefinite, or a general Number, which may therefore be repreſented by x, or y, &c. and aſſuming the general Coefficients a, b, c, d, &c. which are Integers or Fractions, affirmative or negative, as it may happen; we may form ſuch a Series as this, $ax^4+bx^3+cx^2+dx^1+ex^0$, which will repreſent ſome certain Number, expreſs'd by the Scale whoſe Root is x. If ſuch a Number proceeds *in infinitum*, then it is truly and properly call'd an Infinite Series, or a Converging Series, x being then ſuppos'd greater than Unity. Such for example is $x+\frac{1}{2}x^{-1}+\frac{1}{3}x^{-2}+\frac{1}{4}x^{-3}$, &c. where the reſt of the Terms are underſtood *ad infinitum*, and are inſinuated by,

by, *&c.* And it may have any deſcending Arithmetical Progreſſion for its Indices, as $x^m - \frac{1}{3}x^{m-1} + \frac{3}{4}x^{m-2} + \frac{4}{5}x^{m-3}$, *&c.*

And thus we have been led by proper gradations, (that is, by arguing from what is well known and commonly received, to what before appear'd to be difficult and obſcure,) to the knowledge of infinite Series, of which the Learner will find frequent Examples in the ſequel of this Treatiſe. And from hence it will be eaſy to make the following general Inferences, and others of a like nature, which will be of good uſe in the farther knowledge and practice of theſe Series; *viz.* That the firſt Term of every regular Series is always the moſt conſiderable, or that which approaches nearer to the Number intended, (denoted by the Aggregate of the Series,) than any other ſingle Term: That the ſecond is next in value, and ſo on: That therefore the Terms of the Series ought always to be diſpoſed in this regular deſcending order, as is often inculcated by our Author: That when there is a Progreſſion of ſuch Terms-*in infinitum*, a few of the firſt Terms, or thoſe at the beginning of the Series, are or ſhould be a ſufficient Approximation to the whole; and that theſe may come as near to the truth as you pleaſe, by taking in ſtill more Terms: That the ſame Number in which one Scale may be expreſs'd by a finite number of Terms, in another cannot be expreſs'd but by an infinite Series, or by approximation only, and *vice versâ*: That the bigger the Root of the Scale is, by ſo much the faſter, *cæteris paribus*, the Series will converge; for then the Reciprocals of the Powers will be ſo much the leſs, and therefore may the more ſafely be neglected: That if a Series converges by increaſing Powers, ſuch as $ax + bx^2 + cx^3 + dx^4$, &c. the Root x of the Scale muſt be underſtood to be a proper Fraction, the leſſer the better. Yet whenever a Series can be made to converge by the Reciprocals of Ten, or its Compounds, it will be more convenient than a Series that converges faſter; becauſe it will more eaſily acquire the form of the Decimal Scale, to which, in particular Caſes, all Series are to be ultimately reduced. Laſtly, from ſuch general Series as theſe, which are commonly the reſult in the higher Problems, we muſt paſs (by ſubſtitution) to particular Scales or Series, and thoſe are finally to be reduced to the Decimal Scale. And the Art of finding ſuch general Series, and then their Reduction to particular Scales, and laſt of all to the common Scale of Decimal Numbers, is almoſt the whole of the abſtruſer parts of Analyticks, as may be ſeen in a good meaſure by the preſent Treatiſe.

I took notice in the fourth place, that this Doctrine of Scales, and Series, gives us an eaſy notion of the nature of affected Equations, or ſhews us how they ſtand related to ſuch Scales of Numbers. In the other Inſtances of particular Scales, and even of general ones, the Root of the Scale, the Coefficients, and the Indices, are all ſuppos'd to be given, or known, in order to find the Aggregate of the Series, which is here the thing required. But in affected Equations, on the contrary, the Aggregate and the reſt are known, and the Root of the Scale, by which the Number is computed, is unknown and required. Thus in the affected Equation $5x^4 + 3x^3 + 0x^2 + 7x = 53070$, the Aggregate of the Series is given, *viz.* the Number 53070, to find x the Root of the Scale. This is eaſily diſcern'd to be 10, or to be a Number expreſs'd by the common Decimal Scale, eſpecially if we ſupply the ſeveral Powers of 10, where they are underſtood in the Aggregate, thus $5X^4 + 3X^3 + 0X^2 + 7X^1 + 0X^0 = 53070$. Whence by compariſon 'tis $x = X = 10$. But this will not be ſo eaſily perceived in other inſtances. As if I had the Equation $4x^4 + 2x^3 + 3x^2 + 0x^1 + 2x^0 + 4x^{-1} + 3x^{-2} = 2827,92$ I ſhould not ſo eaſily perceive that the Root x was 5, or that this is a Number expreſs'd by Quinary Arithmetick, except I could reduce it to this form, $4 \times 5^4 + 2 \times 5^3 + 3 \times 5^2 + 0 \times 5^1 + 2 \times 5^0 + 4 \times 5^{-1} + 3 \times 5^{-2} = 2827,92$, when by compariſon it would preſently appear, that the Root ſought muſt be 5. So that finding the Root of an affected Equation is nothing elſe, but finding what Scale in Arithmetick that Number is computed by, whoſe Reſult or Aggregate is given in the common Scale; which is a Problem of great uſe and extent in all parts of the Mathematicks. How this is to be done, either in Numeral, Algebraical, or Fluxional Equations, our Author will inſtruct us in its due place.

Before I diſmiſs this copious and uſeful Subject of Arithmetical Scales, I ſhall here make this farther Obſervation; that as all conceivable Numbers whatever may be expreſs'd by any one of theſe Scales, or by help of an Aggregate or Series of Powers derived from any Root; ſo likewiſe any Number whatever may be expreſs'd by ſome ſingle Power of the ſame Root, by aſſuming a proper Index, integer or fracted, affirmative or negative, as occaſion ſhall require. Thus in the Decimal Scale, the Root of which is 10, or X, not only the Numbers 1, 10, 100, 1000, *&c.* or 1, 0.1, 0.01, 0.001, *&c.* that is, the ſeveral integral Powers of 10 and their Reciprocals, may be expreſs'd by the ſingle Powers of X or 10, *viz.* X^0, X^1, X^2, X^3, *&c.* or X^0, X^{-1}, X^{-2}, X^{-3}, *&c.* reſpectively, but alſo all the intermediate

mediate Numbers, as 2, 3, 4, *&c.* 11, 12, 13, *&c.* may be expreſs'd by ſuch ſingle Powers of X or 10, if we aſſume proper Indices. Thus $2 = X^{0,30103,\&c.}$, $3 = X^{0,47712,\&c.}$ $4 = X^{0,60206,\&c.}$ &c. or $11 = X^{1,04139,\&c.}$ $12 = X^{1,07918,\&c.}$ $456 = X^{2,65896,\&c.}$ And the like of all other Numbers. Theſe Indices are uſually call'd the Logarithms of the Numbers (or Powers) to which they belong, and are ſo many Ordinal Numbers, declaring what Power (in order or ſucceſſion) any given Number is, of any Root aſſign'd: And different Scales of Logarithms will be form'd, by aſſuming different Roots of thoſe Scales. But how theſe Indices, Logarithms, or Ordinal Numbers may be conveniently found, our Author will likewiſe inform us hereafter. All that I intended here was to give a general Notion of them, and to ſhew their dependance on, and connexion with, the ſeveral Arithmetical Scales before deſcribed.

It is eaſy to obſerve from the *Arenarius* of *Archimedes*, that he had fully conſider'd and diſcuſs'd this Subject of Arithmetical Scales, in a particular Treatiſe which he there quotes, by the name of his ἀρχαὶ, or Principles; in which (as it there appears) he had laid the foundation of an Arithmetick of a like nature, and of as large an extent, as any of the Scales now in uſe, even the moſt univerſal. It appears likewiſe, that he had acquired a very general notion of the Doctrine and Uſe of Indices alſo. But how far he had accommodated an Algorithm, or Method of Operation, to thoſe his Principles, muſt remain uncertain till that Book can be recover'd, which is a thing more to be wiſh'd than expected. However it may be fairly concluded from his great Genius and Capacity, that ſince he thought fit to treat on this Subject, the progreſs he had made in it was very conſiderable.

But before we proceed to explain our Author's methods of Operation with infinite Series, it may be expedient to enlarge a little farther upon their nature and formation, and to make ſome general Reflexions on their Convergency, and other circumſtances. Now their formation will be beſt explain'd by continual Multiplication after the following manner.

Let the quantity $a + bx + cx^2 + dx^3 + ex^4$, &c. be aſſumed as a Multiplier, conſiſting either of a finite or an infinite number of Terms; and let alſo $\frac{p}{q} + x = 0$ be ſuch a Multiplier, as will give the Root $x = -\frac{p}{q}$. If theſe two are multiply'd together, they will produce $\frac{ap}{q} + \frac{bp+aq}{q}x + \frac{cp+bq}{q}x^2 + \frac{dp+cq}{q}x^3 + \frac{ep+dq}{q}x^4$, &c. $= 0$;

$= 0$; and if inſtead of x we here ſubſtitute its value $-\frac{p}{q}$, the Series will become $\frac{ap}{q} - \frac{bp+aq}{q} \times \frac{p}{q} + \frac{cp+bq}{q} \times \frac{p^2}{q^2} - \frac{dp+cq}{q} \times \frac{p^3}{q^3} + \frac{ep+dq}{q} \times \frac{p^4}{q^4}$, &c. $= 0$; or if we divide by $\frac{p}{q}$, and tranſpoſe, it will be $\frac{bp+aq}{q} - \frac{cp+bq}{q} \times \frac{p}{q} + \frac{dp+cq}{q} \times \frac{p^2}{q^2} - \frac{ep+dq}{q} \times \frac{p^3}{q^3}$, &c. $= a$: which Series, thus derived, may give us a good inſight into the nature of infinite Series in general. For it is plain that this Series, (even though it were continued to infinity,) muſt always be equal to a, whatever may be ſuppoſed to be the values of p, q, a, b, c, d, &c. For $\frac{bp}{q}$, the firſt part of the firſt Term, will always be removed or deſtroy'd by its equal with a contrary Sign, in the ſecond part of the ſecond Term. And $\frac{cp}{q} \times \frac{p}{q}$, the firſt part of the ſecond Term, will be removed by its equal with a contrary Sign, in the ſecond part of the third Term, and ſo on: So as finally to leave $\frac{aq}{q}$, or a, for the Aggregate of the whole Series. And here it is likewiſe to be obſerv'd, that we may ſtop whenever we pleaſe, and yet the Equation will be good, provided we take in the *Supplement*, or a due part of the next Term. And this will always obtain, whatever the nature of the Series may be, or whether it be converging or diverging. If the Series be diverging, or if the Terms continually increaſe in value, then there is a neceſſity of taking in that Supplement, to preſerve the integrity of the Equation. But if the Series be converging, or if the Terms continually decreaſe in any compound Ratio, and therefore finally vaniſh or approach to nothing; the Supplement may be ſafely neglected, as vaniſhing alſo, and any number of Terms may be taken, the more the better, as an Approximation to the Quantity a. And thus from a due conſideration of this fictitious Series, the nature of all converging or diverging Series may eaſily be apprehended. Diverging Series indeed, unleſs when the afore-mention'd increaſing Supplement can be aſſign'd and taken in, will be of no ſervice. And this Supplement, in Series that commonly occur, will be generally ſo entangled and complicated with the Coefficients of the Terms of the Series, that altho' it is always to be underſtood, nevertheleſs, it is often impoſſible to be extricated and aſſign'd. But however, converging Series will always be of excellent uſe, as affording a convenient Approximation to the quantity required, when it cannot be otherwiſe exhibited. In theſe the Supplement aforeſaid, tho'

tho' generally inextricable and unaſſignable, yet continually decreaſes along with the Terms of the Series, and finally becomes leſs than any aſſignable Quantity.

The ſame Quantity may often be exhibited or expreſs'd by ſeveral converging Series; but that Series is to be moſt eſteem'd that has the greateſt Rate of Convergency. The foregoing Series will converge ſo much the faſter, *cæteris paribus*, as p is leſs than q, or as the Fraction $\frac{p}{q}$ is leſs than Unity. For if it be equal to, or greater than Unity, it may become a diverging Series, and will diverge ſo much the faſter, as p is greater than q. The Coefficients will contribute little or nothing to this Convergency or Divergency, if they are ſuppos'd to increaſe or decreaſe (as is generally the caſe) rather in a ſimple and Arithmetical, than a compound and Geometrical Proportion. To make ſome Eſtimate of the Rate of Convergency in this Series, and by analogy in any other of this kind, let k and l repreſent two Terms indefinitely, which immediately ſucceed each other in the progreſſion of the Coefficients of the Multiplier $a + bx + cx^2 + dx^3$, &c. and let the number n repreſent the order or place of k. Then any Term of the Series indefinitely may be repreſented by $\pm \frac{lp+kq}{q^n} t^{n-1}$; where the Sign muſt be $+$ or $-$, according as n is an odd or an even Number. Thus if $n = 1$, then $k = a$, $l = b$, and the firſt Term will be $+ \frac{bp+aq}{q}$. If $n = 2$, then $k = b$, $l = c$, and the ſecond Term will be $- \frac{cp+bq}{q^2} p$. And ſo of the reſt. Alſo if m be the next Term in the aforeſaid progreſſion after l, then $\pm \frac{lp+kq}{q^n} p^{n-1} \mp \frac{mp+lq}{q^{n+1}} p^n$ will be any two ſucceſſive Terms in the ſame Series. Now in order to a due Convergency, the former Term abſolutely conſider'd, that is ſetting aſide the Signs, ſhould be as much greater than the ſucceeding Term, as conveniently may be. Let us ſuppoſe therefore that $\frac{lp+kq}{q^n} p^{n-1}$ is greater than $\frac{mp+lq}{q^{n+1}} t^n$, or (dividing all by the common factor $\frac{t^n}{q^n}$,) that $\frac{lp+kq}{p}$ is greater than $\frac{mp+lq}{q}$, or (multiplying both by pq,) that $lpq + kq^2$ is greater than $mp^2 + lpq$, or (taking away the common lpq,) that kq^2 is greater than mp^2, or (by a farther Diviſion,) that $\frac{k}{m} \times \frac{q^2}{p^2}$ is greater than unity; and as much greater as may be.

This will take effect on a double account; firſt, the greater k is in reſpect of m, and ſecondly, the greater q^2 is in reſpect of p^2. Now in the Multiplier $a + bx + cx^2 + dx^3$, &c. if the Coefficients a, b, c, &c. are in any decreaſing Progreſſion, then k will be greater than l, which is greater than m; ſo that *à fortiori* k will be greater than m. Alſo if q be greater than p, and therefore (in a duplicate ratio) q^2 will be greater than p^2. So that (*cæteris paribus*) the degree of Convergency is here to be eſtimated, from the Rate according to which the Coefficients a, b, c, &c. continually decreaſe, compounded with the Ratio, (or rather its duplicate,) according to which q ſhall be ſuppos'd to be greater than p.

The ſame things obtaining as before, the Term $\pm \frac{lp^n}{q^n}$ will be what was call'd the *Supplement* of the Series. For if the Series be continued to a number of Terms denominated by n, then inſtead of all the reſt of the Terms *in infinitum*, we may introduce this Supplement, and then we ſhall have the accurate value of a, inſtead of an approximation to that value. Here the firſt Sign is to be taken if n is an odd number, and the other when it is even. Thus if $n = 1$, and conſequently $k = a$, and $l = b$, we ſhall have $\frac{bq+aq}{q} - \frac{bp}{q} = a$. Or if $n = 2$, and $l = c$, then $\frac{bp+aq}{q} - \frac{cp+bq}{q} \times \frac{p}{q} + \frac{cp^2}{q^2} = a$. Or if $n = 3$, $l = d$, then $\frac{bp+aq}{q} - \frac{cp+bq}{q} \times \frac{p}{q} + \frac{dp^2+cq}{q} \times \frac{p^2}{q^2} - \frac{dp^3}{q^3} = a$. And ſo on. Here the taking in of the Supplement always compleats the value of a, and makes it perfect, whether the Series be converging or diverging; which will always be the beſt way of proceeding, when that Supplement can readily be known. But as this rarely happens, in ſuch infinite Series as generally occur, we muſt have recourſe to infinite converging Series, wherein this Supplement, as well as the Terms of the Series, are infinitely diminiſh'd; and therefore after a competent number of them are collected, the reſt may be all neglected *in infinitum*.

From this general Series, the better to aſſiſt the Imagination, we will deſcend to a few particular Inſtances of converging Series in pure Numbers. Let the Coefficients a, b, c, d, &c. be expounded by $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c. reſpectively; then $\frac{\frac{1}{2}p+q}{q} - \frac{\frac{1}{2}p+\frac{1}{2}q}{q} \times \frac{p}{q} + \frac{\frac{1}{3}p+\frac{1}{3}q}{q} \times \frac{p^2}{q^2}$, &c. $= 1$, or $\frac{p+2q}{2q} - \frac{2p+3q}{2\times 3q} \times \frac{p}{q} + \frac{3p+4q}{3\times 4q} \times \frac{p^2}{q^2} - \frac{4p+5q}{4\times 5q} \times \frac{p^3}{q^3}$, &c. $= 1$. That the Series hence ariſing may converge, make p leſs than

than q in any given ratio, suppose $\frac{p}{q} = \frac{1}{2}$, or $p = 1$, $q = 2$, then $\frac{5}{4} - \frac{2}{3} \times \frac{1}{2} + \frac{11}{24} \times \frac{1}{4} - \frac{7}{20} \times \frac{1}{8}$, &c. $= 1$. That is, this Series of Fractions, which is computed by Binary Arithmetick, or by the Reciprocals of the Powers of Two, if infinitely continued will finally be equal to Unity. Or if we desire to stop at these four Terms, and instead of the rest *ad infinitum* if we would introduce the Supplement which is equivalent to them, and which is here known to be $\frac{1}{5} \times \frac{1}{16}$, or $\frac{1}{80}$, we shall have $\frac{5}{4} - \frac{1}{8} + \frac{11}{96} - \frac{7}{160} + \frac{1}{80} = 1$, as is easy to prove. Or let the same Coefficients be expounded by $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}$, &c. then it will be $\frac{2q-p}{2q} + \frac{3q-2p}{2 \times 3q} \times \frac{p}{q} + \frac{4q-3p}{3 \times 4q} \times \frac{p^2}{q^2} + \frac{5q-4p}{4 \times 5q} \times \frac{p^3}{q^3}$, &c. $= 1$. This Series may either be continued infinitely, or may be sum'd after any number of Terms express'd by n, by introducing the Supplement $\frac{+p^n}{\overline{n+1} \times q^n}$ instead of all the rest. Or more particularly, if we make $q = 5p$, then $\frac{9}{2 \times 5} + \frac{13}{6 \times 5^2} + \frac{17}{12 \times 5^3} + \frac{21}{20 \times 5^4} + \frac{25}{30 \times 5^5}$, &c. $= 1$, which is a Number express'd by Quinary Arithmetick. And this is easily reduced to the Decimal Scale, by writing $\frac{2}{10}$ for $\frac{1}{5}$, and reducing the Coefficients; for then it will become 0,99999, &c. $= 1$. Now if we take these five Terms, together with the Supplement, we shall have exactly $\frac{9}{2 \times 5} + \frac{13}{6 \times 5^2} + \frac{17}{12 \times 5^3} + \frac{21}{20 \times 5^4} + \frac{25}{30 \times 5^5} + \frac{1}{6 \times 5^5} = 1$. Again, if we make here $3q = 100p$, we shall have the Series $\frac{200-3}{1 \times 2} \times \frac{1}{100} + \frac{300-6}{2 \times 3} \times \frac{3}{10000} + \frac{400-9}{3 \times 4} \times \frac{9}{1000000} + \frac{500-12}{4 \times 5} \times \frac{27}{100000000}$, &c. $= 1$, which converges very fast. And if we would reduce this to the regular Decimal Scale of Arithmetick, (which is always suppos'd to be done, before any particular Problem can be said to be compleatly solved,) we must set the Terms, when decimally reduced, orderly under one another, that their Amount or Aggregate may be discover'd; and then they will stand as in the Margin. Here the Aggregate of the first five Terms is 0,99999999595, which is a near Approximation to the Amount of the whole infinite Series, or to Unity. And if, for proof-sake, we add to this the Supplement $\frac{+p^n}{\overline{n+1}q^n} = \frac{p^5}{6q^5} = 0,00000000405$, the whole will be Unity exactly.

```
0,985
  147
   29325
    6588
     15795
0,99999999595
          405
1,00000000000
```

 There

There are alſo other Methods of forming converging Series, whether general or particular, which ſhall approximate to a known quantity, and therefore will be very proper to explain the nature of Convergency, and to ſhew how the Supplement is to be introduced, when it can be done, in order to make the Series finite; which of late has been call'd the Summing of a Series. Let A, B, C, D, E, *&c.* and *a*, *b*, *c*, *d*, *e*, &c. be any two Progreſſions of Terms, of which A is to be expreſs'd by a Series, either finite or infinite, compos'd of itſelf and the other Terms. Suppoſe therefore the firſt Term of the Series to be *a*, and that *p* is the ſupplement to the value of *a*. Then is $A = a + p$, or $p = \frac{A-a}{1}$. As this is the whole Supplement, in order to form a Series, I ſhall only take ſuch a part of it as is denominated by the Fraction $\frac{b}{B}$, and put *q* for the ſecond Supplement. That is, I will aſſume $\frac{A-a}{1} = (p =) \frac{A-a}{1} \times \frac{b}{B} + q$, or $q = \left(\frac{A-a}{1} \times 1 - \overline{\frac{b}{B}} =\right) \frac{A-a}{B} \times \frac{B-b}{1}$. Again, as this is the whole value of the Supplement *q*, I ſhall only aſſume ſuch a part of it as is denominated by the Fraction $\frac{c}{C}$, and for the next Supplement put *r*. That is, $\frac{A-a}{B} \times \frac{B-b}{1} = (q =) \frac{A-a}{B} \times \frac{B-b}{C} c + r$, or $r = \left(\frac{A-a}{B} \times \frac{B-b}{1} \times 1 - \overline{\frac{c}{C}} = \right) \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{1}$. Now as this is the whole value of the Supplement *r*, I only aſſume ſuch a part of it as is denominated by the Fraction $\frac{d}{D}$, and for the next Supplement put *s*. That is, $\frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{1} = (r =) \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} d + s$, or $s = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{1} \times 1 - \overline{\frac{d}{D}} = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} \times \frac{D-d}{1}$. And ſo on as far as we pleaſe. So that at laſt we have the value of $A = a + p$, where the Supplement $p = \frac{A-a}{B} b + q$, where the ſecond Supplement $q = \frac{A-a}{B} \times \frac{B-b}{C} c + r$, where $r = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} d + s$, where $s = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} \times \frac{D-d}{E} e + t$. And ſo on *ad infinitum*. That is finally $A = a + \frac{A-a}{B} b + \frac{A-a}{B} \times \frac{B-b}{C} c + \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} d + \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} \times \frac{D-d}{E} e$, &c. where A, B, C, D, E, *&c.* and *a*, *b*, *c*, *d*, *e*, &c. may be any two Progreſſions of Numbers whatever, whether regular or deſultory, aſcending or deſcending. And when it

it happens in theſe Progreſſions, that either $A = a$, or $B = b$, or $C = c$, &c. then the Series terminates of itſelf, and exhibits the value of A in a finite number of Terms: But in other caſes it approximates indefinitely to the value of A. But in the caſe of an infinite Approximation, the ſaid Progreſſions ought to proceed regularly, according to ſome ſtated Law. Here it will be eaſy to obſerve, that if K and k are put to repreſent any two Terms indefinitely in the aforeſaid Progreſſions, whoſe places are denoted by the number n, and if L and l are the Terms immediately following; then the Term in the Series denoted by $n + 1$ will be form'd from the preceding Term, by multiplying it by $\frac{K-k}{aL}l$. As if $n = 1$, then $K = A$, $k = a$, $L = B$, $l = b$, and the ſecond Term will be $a + \frac{A-a}{aB}b = \frac{A-a}{B}b$. If $n = 2$, then $K = B$, $k = b$, $L = C$, $l = c$, and the third Term will be $\frac{A-a}{B}b \times \frac{B-b}{bC}c = \frac{A-a}{B} \times \frac{B-b}{C}c$; and ſo of the reſt. And whenever it ſhall happen that $L = l$, then the Series will ſtop at this Term, and proceed no farther. And the Series approximates ſo much the faſter, *cæteris paribus*, as the Numbers A, B, C, D, *&c.* and a, b, c, d, &c. approach nearer to each other reſpectively.

Now to give ſome Examples in pure Numbers. Let A, B, C, D, *&c.* = 2, 2, 2, 2, &c. and a, b, c, d, &c. = 1, 1, 1, 1, *&c.* then we ſhall have $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, *&c.* And ſo always, when the given Progreſſions are Ranks of equals, the Series will be a Geometrical Progreſſion. If we would have this Progreſſion ſtop at the next Term, we may either ſuppoſe the firſt given Progreſſion to be 2, 2, 2, 2, 2, 1, or the ſecond to be 1, 1, 1, 1, 1, 2, 'tis all one which. For in either caſe we ſhall have $L = l$, that is $F = f$, and therefore the laſt Term muſt be multiply'd by $\frac{K-k}{k}$, or $\frac{F-e}{e} = 1$. Then the Progreſſion or Series becomes $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}$. Again, if A, B, C, D, *&c.* = 5, 5, 5, 5, *&c.* and a, b, c, d, &c. = 4, 4, 4, 4, &c. then $5 = 4 + \frac{4}{5} + \frac{4}{25} + \frac{4}{125} + \frac{4}{625}$, &c. or $\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625}$, &c. Or if A, B, C, D, &c. = 4, 4, 4, 4, &c. and a, b, c, d, &c. = 5, 5, 5, 5, &c. then $4 = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \frac{5}{256}$, &c. If A, B, C, D, &c. = 5, 5, 5, 5, &c. and a, b, c, d, &c. = 6, 7, 8, 9, &c. then $5 = 6 - \frac{1}{5}7 + \frac{1}{5} \times \frac{1}{5}8 - \frac{1}{5} \times \frac{1}{5} \times \frac{3}{5}9 + \frac{1}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{4}{5}10$, &c. If we would have the Series ſtop here, or if we would find one more Term, or Supplement, which ſhould be equivalent to all the reſt *ad infinitum*, (which indeed

deed might be desirable here, and in such cases as this, because of the slow Convergency, or rather Divergency of the Series,) suppose $F=f$, and therefore $\frac{F-f}{f} = \frac{5-10}{10} = -\frac{1}{2}$ must be multiply'd by the last Term. So that the Series becomes $5 = 6 - \frac{1}{5}7 + \frac{1}{5}\times\frac{1}{5}8 - \frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}9 + \frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}10 - \frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}5$. If A, B, C, D, &c. $= 2, 3, 4, 5$, &c. and a, b, c, d, &c. $= 1, 2, 3, 4$, &c. then $2 = 1 + \frac{1}{3}2 + \frac{1}{3}\times\frac{1}{4}3 + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}4 + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}\times\frac{1}{6}5$, &c. If A, B, C, D, &c. $= 1, 2, 3, 4$, &c. and a, b, c, d, &c. $= 2, 3, 4, 5$, &c. then $1 = 2 - \frac{1}{2}3 + \frac{1}{2}\times\frac{1}{3}4 - \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}5 + \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}6$, &c. And from this general Series may infinite other particular Series be easily derived, which shall perpetually converge to given Quantities; the chief use of which Speculation, I think, will be, to shew us the nature of Convergency in general.

There are many other such like general Series that may be readily form'd, which shall converge to a given Number. As if I would construct a Series that shall converge to Unity, I set down 1, together with a Rank of Fractions, both negative and affirmative, as here follows.

$$1 - \frac{a}{A} - \frac{b}{B} - \frac{c}{C} - \frac{d}{D} - \frac{e}{E}, \text{ &c.}$$
$$+ \frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \frac{d}{D} + \frac{e}{E}, \text{ &c.}$$

$$\frac{A+a}{A} + \frac{Ab-Ba}{AB} + \frac{B-b}{BC} + \frac{C-c}{CD} + \frac{De-Ed}{DE}, \text{ &c.} = 1.$$

Then proceeding obliquely, I collect the Terms of each Series together, by adding the two first, then the two second, and so on. So that the whole Series thus constructed must necessarily be equal to Unity; which also is manifest by a bare Inspection of the Series. From this Series it is easy to descend to any number of particular Cases. As if we make A, B, C, D, &c. $= 2, 3, 4, 5$, &c. and a, b, c, d, &c. $= 1, 1, 1, 1$, &c. then $\frac{1}{2} - \frac{1}{2\times3} - \frac{1}{3\times4} - \frac{1}{4\times5} - \frac{1}{5\times6}$, &c. $= 1$. Or $\frac{1}{2} = \frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} + \frac{1}{5\times6}$, &c. And so in all other Cases. The Series will stop at a finite number of Terms, whensoever you omit to take in the first part of the Numerator of any Term. As here $\frac{1}{2} - \frac{1}{2\times3} - \frac{1}{3\times4} - \frac{1}{4\times5} - \frac{1}{5\times6} - \frac{1}{6} = 1$.

Lastly, to construct one more Series of this kind, which shall converge to Unity; I set down 1, with a Rank of Fractions along with

with it, both affirmative and negative, ſuch as are ſeen here below; which being added together obliquely as before, will produce the following Series.

$$1 + \frac{a}{A} + \frac{ab}{AB} + \frac{abc}{ABC} + \frac{abcd}{ABCD} + \frac{abcde}{ABCDE}, \text{ \&c.}$$
$$- \frac{a}{A} - \frac{ab}{AB} - \frac{abc}{ABC} - \frac{abcd}{ABCD} - \frac{abcde}{ABCDE}, \text{ \&c.}$$
$$\frac{A-a}{A} + \frac{B-b}{AB}a + \frac{C-c}{ABC}ab + \frac{D-d}{ABCD}abc + \frac{E-e}{ABCDE}abcd, \text{ \&c.} = 1.$$

This Series may be made to ſtop at any finite number of Terms, if you omit to take in the latter part of the Binomial in any Term. Or you may derive particular Series from it, which ſhall have any Rate of Convergency.

For an Example of this Series, make A, B, C, D, &c. $= 3, 3, 3, 3,$ &c. and $a, b, c, d,$ &c. $= 1, 1, 1, 1,$ &c. then $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81}$, &c. $= 1$, or $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$, &c. $= \frac{1}{2}$. And whenever A, B, C, &c. and $a, b, c,$ &c. are Ranks of Equals, the Series will be a Geometrical Progreſſion.

Again, make A, B, C, D, &c. $= 2, 3, 4, 5,$ &c. and $a, b, c, d,$ &c. $= 1, 1, 1, 1,$ &c. then $\frac{1}{2} + \frac{2}{2\times3} + \frac{3}{2\times3\times4} + \frac{4}{2\times3\times4\times5} + \frac{5}{2\times3\times4\times5\times6}$, &c. $= 1$. Or in a finite number of Terms $\frac{1}{2} + \frac{1}{3} + \frac{1}{2\times4} + \frac{1}{2\times3\times5} + \frac{1}{2\times3\times4\times5} = 1$. And the like may be obſerved of others in an infinite variety.

And thus having prepared the way for what follows, by explaining the nature of infinite Series in general, by diſcovering their origin and manner of convergency, and by ſhewing their connexion with our common Arithmetick; I ſhall now return to our Author's Methods of Operation, or to the Reduction of compound Quantities to ſuch infinite Series.

SECT. II. *The Reſolution of ſimple Equations, or pure Powers, by Infinite Series.*

3, 4. THE Author begins his Reduction of compound Quantities, to an equivalent infinite Series of ſimple Terms, firſt by ſhewing how the Proceſs may be perform'd in Diviſion. Now in his Example the manner of the Operation is thus, in imitation

tation of the uſual praxis of Diviſion in Numbers. In order to obtain the Quotient of aa divided by $b+x$, or to reſolve the compound Fraction $\frac{aa}{b+x}$ into a Series of ſimple Terms, firſt find the Quotient of aa divided by b, the firſt Term of the Diviſor. This is $\frac{aa}{b}$, which write in the Quote. Then multiply the Diviſor by this Term, and ſet the Product $aa+\frac{aax}{b}$ under the Dividend, from whence it muſt be ſubtracted, and will leave the Remainder $-\frac{aax}{b}$. Then to find the next Term (or Figure) of the Quotient, divide the Remainder by the firſt Term of the Diviſor, or by b, and put the Quotient $-\frac{aax}{bb}$ for the ſecond Term of the Quote. Multiply the Diviſor by this ſecond Term, and the Product $-\frac{aax}{b}-\frac{aaxx}{bb}$ ſet orderly under the laſt Remainder; from whence it muſt be ſubtracted, to find the new Remainder $+\frac{aaxx}{bb}$. Then to find the next Term of the Quotient, you are to proceed with this new Remainder as with the former; and ſo on *in infinitum*. The Quotient therefore is $\frac{a^2}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{a^2x^3}{b^4}$, &c. (or $\frac{a^2}{b}$ into $1-\frac{x}{b}+\frac{x^2}{b^2}-\frac{x^3}{b^3}$, &c.) So that by this Operation the Number or Quantity $\frac{aa}{b+x}$, (or $a^2\times\overline{b+x}|^{-1}$) is reduced from that Scale in Arithmetick whoſe Root is $b+x$, to an equivalent Number, the Root of whoſe Scale, (or whoſe converging quantity) is $\frac{x}{b}$. And this Number, or infinite Series thus found, will converge ſo much the faſter to the truth, as b is greater than x.

To apply this, by way of illuſtration, to an inſtance or two in common Numbers. Suppoſe we had the Fraction $\frac{1}{7}$, and would reduce it from the ſeptenary Scale, in which it now appears, to an equivalent Series, that ſhall converge by the Powers of 6. Then we ſhall have $\frac{1}{7}=\frac{1}{6+1}$; and therefore in the foregoing general Fraction $\frac{aa}{b+x}$, make $a=1$, $b=6$, and $x=1$, and the Series will become $\frac{1}{6}-\frac{1}{6^2}+\frac{1}{6^3}-\frac{1}{6^4}$, &c. which will be equivalent to $\frac{1}{7}$. Or if we would reduce it to a Series converging by the Powers of 8, becauſe $\frac{1}{7}=\frac{1}{8-1}$, make $a=1$, $b=8$, and $x=-1$, then

then $\frac{1}{7} = \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \frac{1}{8^4}$, &c. which Series will converge faſter than the former. Or if we would reduce it to the common Denary (or Decimal) Scale, becauſe $\frac{1}{7} = \frac{1}{10-3}$, make $a = 1$, $b = 10$, and $x = -3$; then $\frac{1}{7} = \frac{1}{10} + \frac{3}{100} + \frac{9}{1000} + \frac{27}{10000} + \frac{81}{100000}$, &c. $= 0,1428$, &c. as may be eaſily collected. And hence we may obſerve, that this or any other Fraction may be reduced a great variety of ways to infinite Series; but that Series will converge faſteſt to the truth, in which b ſhall be greateſt in reſpect of x. But that Series will be moſt eaſily reduced to the common Arithmetick, which converges by the Powers of 10, or its Multiples. If we ſhould here reſolve 7 into the parts $3 + 4$, or $2 + 5$, or $1 + 6$, &c. inſtead of converging we ſhould have diverging Series, or ſuch as require a Supplement to be taken in.

And we may here farther obſerve, that as in Diviſion of common Numbers, we may ſtop the proceſs of Diviſion whenever we pleaſe, and inſtead of all the reſt of the Figures (or Terms) *ad infinitum*, we may write the Remainder as a Numerator, and the Diviſor as the Denominator of a Fraction, which Fraction will be the Supplement to the Quotient: ſo the ſame will obtain in the Diviſion of Species. Thus in the preſent Example, if we will ſtop at the firſt Term of the Quotient, we ſhall have $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b \times b+x}$. Or if we will ſtop at the ſecond Term, then $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{bb} + \frac{a^2x^2}{b^3+b^2x}$. Or if we will ſtop at the third Term, then $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{bb} + \frac{aax^2}{b^3} - \frac{a^2x^3}{b^4+b^3x}$. And ſo in the ſucceeding Terms, in which theſe Supplements may always be introduced, to make the Quotient compleat. This Obſervation will be found of good uſe in ſome of the following Speculations, when a complicate Fraction is not to be intirely reſolved, but only to be depreſs'd, or to be reduced to a ſimpler and more commodious form.

Or we may hence change Diviſion into Multiplication. For having found the firſt Term of the Quotient, and its Supplement, or the Equation $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{bb+bx}$; multiplying it by $\frac{x}{b}$, we ſhall have $\frac{aax}{bb+bx} = \frac{aax}{bb} - \frac{a^2x^2}{b^3+b^2x}$, ſo that ſubſtituting this value of $\frac{aax}{bb+bx}$ in the firſt Equation, it will become $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{bb} + \frac{a^2x^2}{b^3+b^2x}$, where the two firſt Terms of the Quotient are now known.

Multiply this by $\frac{x^2}{b^2}$, and it will become $\frac{a^2x^2}{b^3+b^2x} = \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5+b^4x}$, which being ſubſtituted in the laſt Equation, it will become $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5+b^4x}$, where the four firſt Terms of the Quotient are now known. Again, multiply this Equation by $\frac{x^4}{b^4}$, and it will become $\frac{aax^4}{b^5+b^4x} = \frac{a^2x^4}{b^5} - \frac{a^2x^5}{b^6} + \frac{a^2x^6}{b^7} - \frac{a^2x^7}{b^8} + \frac{a^2x^8}{b^9+b^8x}$, which being ſubſtituted in the laſt Equation, it will become $\frac{aa}{b+x} = \frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5} - \frac{a^2x^5}{b^6} + \frac{a^2x^6}{b^7} - \frac{a^2x^7}{b^8} + \frac{a^2x^8}{b^9+b^8x}$, where eight of the firſt Terms are now known. And ſo every ſucceeding Operation will double the number of Terms, that were before found in the Quotient.

This method of Reduction may be thus very conveniently imitated in Numbers, or we may thus change Diviſion into Multiplication. Suppoſe (for inſtance) I would find the Reciprocal of the Prime Number 29, or the value of the Fraction $\frac{1}{29}$ in Decimal Numbers. I divide 1,0000, *&c.* by 29, in the common way, ſo far as to find two or three of the firſt Figures, or till the Remainder becomes a ſingle Figure, and then I aſſume the Supplement to compleat the Quotient. Thus I ſhall have $\frac{1}{29} = 0{,}03448\frac{8}{29}$ for the compleat Quotient, which Equation if I multiply by the Numerator 8, it will give $\frac{8}{29} = 0{,}27584\frac{64}{29}$, or rather $\frac{8}{29} = 0{,}27586\frac{6}{29}$. I ſubſtitute this inſtead of the Fraction in the firſt Equation, and I ſhall have $\frac{1}{29} = 0{,}0344827586\frac{6}{29}$. Again, I multiply this Equation by 6, and it will give $\frac{6}{29} = 0{,}206896551\frac{7}{29}$, and then by Subſtitution $\frac{1}{29} = 0{,}0344827586206896551\frac{7}{29}$. Again, I multiply this Equation by 7, and it becomes $\frac{7}{29} = 0{,}241379310344827586\frac{20}{29}$, and then by Subſtitution $\frac{1}{29} = 0{,}0344827586206896551724137931034482758620\frac{20}{29}$, where every Operation will at leaſt double the number of Figures found by the preceding Operation. And this will be an eaſy Expedient for converting Diviſion into Multiplication in all Caſes. For the Reciprocal of the Diviſor being thus found, it may be multiply'd into the Dividend to produce the Quotient.

Now as it is here found, that $\frac{aa}{b+x} = \frac{aa}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4}$, &c. which Series will converge when b is greater than x; ſo when it happens to be otherwiſe, or when x is greater than b, that the Powers of x may be in the Denominators we muſt have recourſe to the

the other Caſe of Diviſion, in which we ſhall find $\frac{aa}{x+b} = \frac{aa}{x} - \frac{a^2b}{x^2} + \frac{a^2b^2}{x^3} - \frac{a^2b^3}{x^4}$, &c. and where the Diviſion is perform'd as before.

5, 6. In theſe Examples of our Author, the Proceſs of Diviſion (for the exerciſe of the Learner) may be thus exhibited:

$1+x^2)\,1+0(1-x^2+x^4-x^6+x^8,\&c.$

$1+x^2$

$0-x^2+0$

$-x^2-x^4$

$0+x^4+0$

$+x^4+x^6$

$0-x^6+0$

$-x^6-x^8$

$0+x^8$

$1+x^{\frac{1}{2}}-3x)2x^{\frac{1}{2}} * -x^{\frac{3}{2}}(2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^2,\&c.$

$2x^{\frac{1}{2}}+2x-6x^{\frac{3}{2}}$

$-2x+5x^{\frac{3}{2}}$

$-2x-2x^{\frac{3}{2}}+6x^2$

$+7x^{\frac{3}{2}}-6x^2$

$+7x^{\frac{3}{2}}+7x^2-21x^{\frac{5}{2}}$

$-13x^2+21x^{\frac{5}{2}}$

Now in order to a due Convergency, in each of theſe Examples, we muſt ſuppoſe x to be leſs than Unity; and if x be greater than Unity, we muſt invert the Terms, and then we ſhall have $\frac{1}{xx+1} = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{x^8}$, &c. and $\frac{-x^{\frac{1}{2}}+2x^{\frac{1}{2}}}{-3x+x^{\frac{1}{2}}+1} = \frac{1}{8}x^{\frac{1}{2}} + \frac{1}{9} - \frac{14}{27x^{\frac{1}{2}}} - \frac{11}{81x}$, &c.

7, 8, 9, 10. This Notation of Powers and Roots by integral and fractional, affirmative and negative, general and particular Indices, was certainly a very happy Thought, and an admirable Improvement of Analyticks, by which the practice is render'd eaſy, regular, and univerſal. It was chiefly owing to our Author, at leaſt he carried on the Analogy, and made it more general. A Learner ſhould be well acquainted with this Notation, and the Rules of its ſeveral Operations ſhould be very familiar to him, or otherwiſe he will often find himſelf involved in difficulties. I ſhall not enter into any farther diſcuſſion of it here, as not properly belonging to this place, or ſubject, but rather to the vulgar Algebra.

11. The Author proceeds to the Extraction of the Roots of pure Equations, which he thus performs, in imitation of the uſual Proceſs in Numbers. To extract the Square-root of $aa + xx$; firſt the Root of aa is a, which muſt be put in the Quote. Then the Square of this, or aa, being ſubtracted from the given Power, leaves $+xx$ for a Reſolvend. Divide this by twice the Root, or $2a$, which is

the firſt part of the Diviſor, and the Quotient $\frac{xx}{2a}$ muſt be made the ſecond Term of the Root, as alſo the ſecond Term of the Diviſor. Multiply the Diviſor thus compleated, or $2a + \frac{xx}{2a}$, by the ſecond Term of the Root, and the Product $xx + \frac{x^4}{4aa}$ muſt be ſubtracted from the Reſolvend. This will leave $-\frac{x^4}{4a^2}$ for a new Reſolvend, which being divided by the firſt Term of the double Root, or $2a$, will give $-\frac{x^4}{8a^3}$ for the third Term of the Root. Twice the Root before found, with this Term added to it, or $2a + \frac{x^2}{a} - \frac{x^4}{8a^3}$, being multiply'd by this Term, the Product $-\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$ muſt be ſubtracted from the laſt Reſolvend, and the Remainder $+\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$ will be a new Reſolvend, to be proceeded with as before, for finding the next Term of the Root; and ſo on as far as you pleaſe. So that we ſhall have $\sqrt{aa + xx} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$, &c.

It is eaſy to obſerve from hence, that in the Operation every new Column will give a new Term in the Quote or Root; and therefore no more Columns need be form'd than it is intended there ſhall be Terms in the Root. Or when any number of Terms are thus extracted, as many more may be found by Diviſion only. Thus having found the three firſt Terms of the Root $a + \frac{x^2}{2a} - \frac{x^4}{8a^3}$, by their double $2a + \frac{x^2}{a} - \frac{x^4}{4a^3}$, dividing the third Remainder or Reſolvend $+\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$, the three firſt Terms of the Quotient $\frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$ will be the three ſucceeding Terms of the Root.

The Series $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$, &c. thus found for the ſquare-root of the irrational quantity $aa + xx$, is to be underſtood in the following manner. In order to a due convergency a is to be ſuppos'd greater than x, that the Root or converging quantity $\frac{x}{a}$ may be leſs than Unity, and that a may be a near approximation to the ſquare-root required. But as this is too little, it is encreaſed by the ſmall quantity $\frac{x^2}{2a}$, which now makes it too big. Then by the next Operation

Operation it is diminiſh'd by the ſtill ſmaller quantity $\frac{x^4}{8a^3}$; which diminution being too much, it is again encreas'd by the very ſmall quantity $\frac{x^6}{16a^5}$, which makes it too great, in order to be farther diminiſh'd by the next Term. And thus it proceeds *in infinitum*, the Augmentations and Diminutions continually correcting one another, till at laſt they become inconſiderable, and till the Series (ſo far continued) is a ſufficiently near Approximation to the Root required.

12. When a is leſs than x, the order of the Terms muſt be inverted, or the ſquare-root of $xx + aa$ muſt be extracted as before; in which caſe it will be $x + \frac{aa}{2x} - \frac{a^4}{8x^3}$, &c. And in this Series the converging quantity, or the Root of the Scale, will be $\frac{a}{x}$. Theſe two Series are by no means to be underſtood as the two different Roots of the quantity $aa + xx$; for each of the two Series will exhibit thoſe two Roots, by only changing the Signs. But they are accommodated to the two Caſes of Convergency, according as a or x may happen to be the greater quantity.

I ſhall here reſolve the foregoing Quantity after another manner, the better to prepare the way for what is to follow. Suppoſe then $yy = aa + xx$, where we may find the value of the Root y by the following Proceſs: $yy = aa + xx = (\text{if } y = a + f)\ aa + 2af + ff$; or $2af + ff = xx = (\text{if } f = \frac{xx}{2a} + g)\ xx + 2ag + \frac{x^4}{4a^2} + \frac{x^2g}{a} + gg$; or $2ag + \frac{x^2g}{a} + gg = -\frac{x^4}{4a^2} = (\text{if } g = -\frac{x^4}{8a^3} + h) - \frac{x^4}{4a^2} + 2ah - \frac{x^6}{8a^4} + \frac{x^2h}{a} + \frac{x^8}{64a^6} - \frac{x^4h}{4a^3} + hh$; or $2ah + \frac{xxh}{a} - \frac{x^4h}{4a^3} + hh = \frac{x^6}{8a^4} - \frac{x^8}{64a^6} = (\text{if } h = \frac{x^6}{16a^5} + k)$ &c. which Proceſs may be thus explain'd in words.

In order to find $\sqrt{aa + xx}$, or the Root y of this Equation $yy = aa + xx$, ſuppoſe $y = a + f$, where a is to be underſtood as a pretty near Approximation to the value of y, (the nearer the better,) and f is the ſmall Supplement to that, or the quantity which makes it compleat. Then by Subſtitution is derived the firſt Supplemental Equation $2af + ff = xx$, whoſe Root f is to be found. Now as $2af$ is much bigger than ff, (for $2a$ is bigger than the Supplement f,) we ſhall have nearly $f = \frac{xx}{2a}$, or at leaſt we ſhall have exactly $f = \frac{xx}{2a} + g$, ſuppoſing g to repreſent the ſecond Supplement

ment of the Root. Then by Subſtitution $2aq + \frac{xx}{a}q + qq = -\frac{x^4}{4a^2}$ will be the ſecond Supplemental Equation, whoſe Root q is the ſecond Supplement. Therefore $\frac{xx}{a}q$ will be a little quantity, and qq much leſs, ſo that we ſhall have nearly $q = -\frac{x^4}{8a^3}$, or accurately $q = -\frac{x^4}{8a^3} + r$, if r be made the third Supplement to the Root. And therefore $2ar + \frac{xx}{a}r - \frac{x^4}{4a^3}r + rr = \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$ will be the third Supplemental Equation, whoſe Root is r. And thus we may go on as far as we pleaſe, to form Reſidual or Supplemental Equations, whoſe Roots will continually grow leſs and leſs, and therefore will make nearer and nearer Approaches to the Root y, to which they always converge. For $y = a + p$, where p is the Root of this Equation $2ap + pp = xx$. Or $y = a + \frac{xx}{2a} + q$, where q is the Root of this Equation $2aq + \frac{xx}{a}q + qq = -\frac{x^4}{4a^2}$. Or $y = a + \frac{xx}{2a} - \frac{x^4}{8a^3} + r$, where r is the Root of this Equation $2ar + \frac{xx}{a}r - \frac{x^4}{4a^3}rr = \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$. And ſo on. The Reſolution of any one of theſe Quadratick Equations, in the ordinary way, will give the reſpective Supplement, which will compleat the value of y.

I took notice before, upon the Article of Diviſion, of what may be call'd a Compariſon of Quotients; or that one Quotient may be exhibited by the help of another, together with a Series of known or ſimple Terms. Here we have an Inſtance of a like Compariſon of Roots; or that the Root of one Equation may be expreſs'd by the Root of another, together with a Series of known or ſimple Terms, which will hold good in all Equations whatever. And to carry on the Analogy, we ſhall hereafter find a like Compariſon of Fluents; where one Fluent, (ſuppoſe, for inſtance, a Curvilinear Area,) will be expreſs'd by another Fluent, together with a Series of ſimple Terms. This I thought fit to inſinuate here, by way of anticipation, that I might ſhew the conſtant uniformity and harmony of Nature, in theſe Speculations, when they are duly and regularly purſued.

But I ſhall here give, *ex abundanti*, another Method for this, and ſuch kind of Extractions, tho' perhaps it may more properly belong to the Reſolution of Affected Equations, which is ſoon to follow; however it may ſerve as an Introduction to their Solution.

The firſt Reſidual or Supplemental Equation in the foregoing Proceſs was $2ap + pp = xx$, which may be reſolved in this manner. Becauſe $p = \frac{xx}{2a+p}$, it will be by Diviſion $p = \frac{xx}{2a} - \frac{x^2p}{4a^2} + \frac{x^2p^2}{8a^3} - \frac{x^2p^3}{16a^4} + \frac{x^2p^4}{32a^5}$, &c. Divide all the Terms of this Series (except the firſt) by p, and then multiply them by the whole Series, or by the value of p, and you will have $p = \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^4p}{8a^4} - \frac{3x^4p^2}{32a^5} + \frac{x^4p^3}{16a^6}$, &c. where the two firſt Terms are clear'd of p. Divide all the Terms of this Series, except the two firſt, by p, and multiply them by the value of p, or by the firſt Series, and you will have a Series for p in which the three firſt Terms are clear'd of p. And by repeating the Operation, you may clear as many Terms of p as you pleaſe. So that at laſt you will have $p = \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$, &c. which will give the ſame value of y as before.

13, 14, 15, 16, 17, 18. The ſeveral Roots of theſe Examples, and of all other pure Powers, whether they are Binomials, Trinomials, or any other Multinomials, may be extracted by purſuing the Method of the foregoing Proceſs, or by imitating the like *Praxes* in Numbers. But they may be perform'd much more readily by general Theorems computed for that purpoſe. And as there will be frequent occaſion, in the enſuing Treatiſe, for certain general Operations to be perform'd with infinite Series, ſuch as Multiplication, Diviſion, raiſing of Powers, and extracting of Roots; I ſhall here derive ſome Theorems for thoſe purpoſes.

I. Let $A + B + C + D + E$, &c. $P + Q + R + S + T$, &c. and $\alpha + \beta + \gamma + \delta + \varepsilon$, &c. repreſent the Terms of three ſeveral Series reſpectively, and let $A+B+C+D+E$, &c. into $P+Q+R+S+T$, &c. $= \alpha + \beta + \gamma + \delta + \varepsilon$, &c. Then by the known Rules of Multiplication, by which every Term of one Factor is to be multiply'd into every Term of the other, it will be $\alpha = AP$, $\beta = AQ + BP$, $\gamma = AR + BQ + CP$, $\delta = AS + BR + CQ + DP$, $\varepsilon = AT + BS + CR + DQ + EP$; and ſo on. Then by Subſtitution it will be

$$\begin{array}{l} A+B+C+D+E, \text{ \&c.} \times P+Q+R+S+T, \text{ \&c.} = AP \\ \qquad +BP \\ \qquad +CP \\ \qquad +DP \\ \qquad +EP, \text{ \&c.} \\ \qquad\quad +AQ \\ \qquad\quad +BQ \\ \qquad\quad +CQ \\ \qquad\quad +DQ \\ \qquad\qquad +AR \\ \qquad\qquad +BR \\ \qquad\qquad +CR \\ \qquad\qquad\quad +AS \\ \qquad\qquad\quad +BS \\ \qquad\qquad\quad +AT \end{array}$$

And

And this will be a ready Theorem for the Multiplication of any infinite Series into each other; as in the following Example.

$$\begin{array}{llllll} (A) & (B) & (C) & (D) & (E) & \\ a + \frac{1}{3}x + \frac{x^2}{3a} + \frac{x^3}{4a^2} + \frac{x^4}{5a^3}, \text{ \&c. into } & & (P) & (Q) & (R) & (S) \quad (T) \\ & & a - \frac{1}{3}x + \frac{x^2}{5a} - \frac{x^3}{7a^2} + \frac{x^4}{9a^3}, \text{ \&c.} & & & \end{array}$$

$$\begin{array}{l} = a^2 + x\tfrac{1}{2}a + \tfrac{1}{3}x^2 + \frac{x^3}{4a} + \frac{x^4}{5a^2}, \text{ \&c.} \\ \quad -\tfrac{1}{3}ax - \tfrac{1}{6}x^2 - \frac{x^3}{9a} - \frac{x^4}{12a^2} \\ \qquad + \tfrac{1}{5}x^2 + \frac{x^3}{10a} + \frac{x^4}{15a^2} \\ \qquad\quad - \frac{x^3}{7a} - \frac{x^4}{14a^2} \\ \qquad\qquad + \frac{x^4}{9a^2} \end{array} \qquad = a^2 + \tfrac{1}{6}ax + \tfrac{1}{30}x^2 + \frac{121x^3}{1200a^2} + \frac{281x^4}{1260a^3}, \text{ \&c.}$$

And so in all other cases.

II. From the same Equations above we shall have $A = \frac{\alpha}{P}$, $B = \frac{\beta - AQ}{P}$, $C = \frac{\gamma - BQ - AR}{P}$, $D = \frac{\delta - CQ - BR - AS}{P}$, $E = \frac{\varepsilon - DQ - CR - BS - AT}{P}$, &c. And then by Substitution $\frac{\alpha+\beta+\gamma+\delta+\varepsilon, \&c.}{P+Q+R+S+T, \&c.} = (A + B + C + D + E, \text{ \&c.} =) \frac{\alpha}{P} + \frac{\beta - AQ}{P} + \frac{\gamma - BQ - AR}{P} + \frac{\delta - CQ - BR - AS}{P} + \frac{\varepsilon - DQ - CR - BS - AT}{P}$, &c. This Theorem will serve commodiously for the Division of one infinite Series by another. Here for conveniency-sake the Capitals A, B, C, D, &c. are retained in the Theorem, to denote the first, second, third, fourth, &c. Terms of the Series respectively.

Thus, for Example, if we would divide the Series $\overset{(\alpha)}{a^2} + \overset{(\beta)}{\tfrac{1}{6}ax} + \overset{(\gamma)}{\tfrac{1}{30}x^2} + \overset{(\delta)}{\frac{121x^3}{1200a}} + \overset{(\varepsilon)}{\frac{281x^4}{1260a^2}}$, &c. by the Series $\overset{(P)}{a} + \overset{(Q)}{\tfrac{1}{2}x} + \overset{(R)}{\frac{x^2}{3a}} + \overset{(S)}{\frac{x^3}{4a^2}} + \overset{(T)}{\frac{x^4}{5a^3}}$, &c. the Quotient will be $a + \frac{\frac{1}{6}ax - \frac{1}{2}aA}{a} + \frac{\frac{1}{30}x^2 - \frac{x^2}{3a}A - \frac{1}{2}xB}{a} + \frac{\frac{121x^3}{1260a^2} - \frac{x^3}{4a^2}A - \frac{x^2}{2a}B - \frac{1}{2}xC}{a}$, &c. Or restoring the Values of A, B, C, D, &c. which represent the several Terms as they stand in order, the Quotient will become $a - \frac{1}{3}x + \frac{x^2}{5a} - \frac{x^3}{7a^2} + \frac{x^4}{9a^3}$, &c. And after the same manner in all other Examples.

III.

III. In the laſt Theorem make $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, &c. then $\frac{1}{P+Q+R+S+T,\&c.} = \frac{1}{P} - \frac{AQ}{P} - \frac{BQ+AR}{P} - \frac{CQ+BR+AS}{P} - \frac{DQ+CR+BS+AT}{P}$, &c. which Theorem will readily find the Reciprocal of any infinite Series. Here A, B, C, D, &c. denote the ſeveral Terms of the Series in order, as before.

Thus if we would know the Reciprocal of the Series $\overset{(P)}{a} + \overset{(Q)}{\tfrac{1}{2}x} + \overset{(R)}{\frac{x^2}{3a}} + \overset{(S)}{\frac{x^3}{4a^2}} + \overset{(T)}{\frac{x^4}{5a^3}}$, &c. we ſhall have by Subſtitution $\frac{1}{a} - \frac{\frac{1}{2}xA}{a} - \frac{\frac{x^2}{3a}A + \frac{1}{2}xB}{a} - \frac{\frac{x^3}{4a^2}A + \frac{x^2}{3a}B + \frac{1}{2}xC}{a} - \frac{\frac{x^4}{5a^3}A + \frac{x^3}{4a^2}B + \frac{x^2}{3a}C + \frac{1}{2}xD}{a}$, &c. And reſtoring the Values of A, B, C, D, &c. it will be $\frac{1}{a} - \frac{x}{2a^2} - \frac{x^2}{12a^3} + \frac{x^3}{8a^4} - \frac{79x^4}{720a^5}$, &c. for the Reciprocal required.

Ex. 2. $\frac{1}{1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{16}x^3,\&c.} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{5}{16}x^3$, &c. And ſo of others.

IV. In the firſt Theorem if we make P = A, Q = B, R = C, S = D, &c. that is, if we make both to be the ſame Series; we ſhall have

$$\begin{aligned}\overline{A+B+C+D+E+F+G,\&c.}^2 = {} & A^2 + 2AB + 2AC + 2AD + 2AE + 2AF + 2AG,\&c.\\ & + B^2 + 2BC + 2BD + 2BE + 2BF\\ & + C^2 + 2CD + 2CE\\ & + D^2\end{aligned}$$

which will be a Theorem for finding the Square of any infinite Series.

Ex. 1. $\overline{\frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9},\&c.}^2 = \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{5x^{10}}{128a^8} + \frac{7x^{12}}{256a^{10}},\&c.$
$+ \frac{x^8}{64a^6} - \frac{x^{10}}{64a^8} + \frac{5x^{12}}{512a^8}$
$+ \frac{x^{12}}{256a^{10}}$
$= \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{5x^8}{64a^6} - \frac{7x^{10}}{128a^8} + \frac{21x^{12}}{512a^{10}}$, &c.

Ex. 2. $-\frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}}$, &c.$\Big|^2 = \frac{1}{4}x^3 + \frac{1}{8}x^4 + \frac{5}{128}x^5$, &c.

Ex. 3. $\frac{bx}{2a} - \frac{x^2}{2a} + \frac{bx^3}{4a^3}$, &c.$\Big|^2 = \frac{b^2x^2}{4a^2} - \frac{bx^3}{2a^2} + \frac{x^4}{4a^2}$, &c.
$- \frac{b^2x^2}{8a^3} + \frac{b^3x^3}{16a^5}$
$- \frac{b^3x^3}{8a^4} + \frac{3b^2x^4}{8a^4}$
$+ \frac{17b^4x^4}{64a^4}$

Ex. 4. $\overline{\frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}},\&c.}^2 = \frac{x^6}{9a^4} - \frac{2x^9}{27a^7} + \frac{13x^{12}}{243a^{10}} - \frac{30x^{15}}{729a^{13}},\&c.$

V. In this laſt Theorem, if we make $A^2 = P$, $2AB = Q$, $2AC + B^2 = R$, $2AD + 2BC = S$, $2AE + 2BD + C^2 = T$, &c. we ſhall have $A = P^{\frac{1}{2}}$, $B = \frac{Q}{2A}$, $C = \frac{R - B^2}{2A}$, $D = \frac{S - 2BC}{2A}$, $E = \frac{T - 2BD - C^2}{2A}$, &c. Or $\overline{P + Q + R + S + T + U, \&c.}|^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{Q}{2A} + \frac{R - B^2}{2A} + \frac{S - 2BC}{2A} + \frac{T - 2BD - C^2}{2A} + \frac{U - 2BE - 2CD}{2A}$, &c. By this Theorem the Square-root of any infinite Series may eaſily be extracted. Here A, B, C, D, &c. will repreſent the ſeveral Terms of the Series as they are in ſucceſſion.

Ex. 1. $\overline{x^2 - 2ax + 2a^2 - \frac{a^3}{x} * + \frac{a^5}{4x^3}, \&c.}|^{\frac{1}{2}} = x - a + \frac{a^2}{2x} * - \frac{a^4}{8x^3} * \&c.$

Ex. 2. $\overline{\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{5x^8}{64a^6} - \frac{7x^{10}}{128a^8} + \frac{21x^{12}}{512a^{10}}, \&c.}|^{\frac{1}{2}} = \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}, \&c.$

VI. Becauſe it is by the fourth Theorem $\overline{\alpha + \beta + \gamma + \delta + \varepsilon, \&c.}|^2 = \alpha^2 + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\alpha\varepsilon$, &c. in the third Theorem for $+ \beta^2 + 2\beta\gamma + 2\beta\delta$ $+ \gamma^2$ P, Q, R, S, T, &c. write α^2, $2\alpha\beta$, $2\alpha\gamma + \beta^2$, $2\alpha\delta + 2\beta\gamma$, $2\alpha\varepsilon + 2\beta\delta + \gamma^2$, &c. reſpectively. Then $\frac{1}{\overline{\alpha + \beta + \gamma + \delta + \varepsilon, \&c.}|^2} = \frac{1}{\alpha^2} - \frac{2\alpha\beta A}{\alpha^2} - \frac{2\alpha\beta B + \overline{2\alpha\gamma + \beta^2} \times A}{\alpha^2} - \frac{2\alpha\beta C + \overline{2\alpha\gamma + \beta^2} \times B + \overline{2\alpha\delta + 2\beta\gamma} \times A}{\alpha^2}$, &c. And this will be a Theorem for finding the Reciprocal of the Square of any infinite Series. Here A, B, C, D, &c. ſtill denote the Terms of the Series in their order.

VII. If in the firſt Theorem for P, Q, R, S, &c. we write A^2, $2AB$, $2AC + B^2$, $2AD + 2BC$, &c. reſpectively, (that is $\overline{A + B + C + D, \&c.}|^2$, by Theor. 4.) we ſhall have $\overline{A + B + C + D + E + F, \&c.}|^3 = A^3 + 3A^2B + 3AB^2 + 3A^2D + 3AC^2 + 3BC^2$, &c. $+ 3A^2C + 6ABC + 3B^2C + 3B^2D$ $+ B^3 + 6ABD + 6ACD$ $+ 3A^2E + 6ABE$ $+ 3AF$ which will readily give the Cube of any infinite Series.

Ex. 1. $\overline{\frac{x^3}{3a^2} - \frac{x^6}{9a^4} + \frac{5x^9}{81a^6} - \frac{10x^{12}}{243a^{8}}, \&c.}|^3 = \frac{x^9}{27a^6} - \frac{x^{12}}{27a^9} + \frac{x^{15}}{81a^{12}} - \frac{10x^{18}}{729a^{15}}, \&c.$ $+ \frac{5x^{15}}{243a^{12}} - \frac{10x^{18}}{729a^{15}}$ $- \frac{x^{18}}{729a^{15}}$

$= \frac{x^9}{27a^6} - \frac{x^{12}}{27a^9} + \frac{8x^{15}}{243a^{12}} - \frac{7x^{18}}{243a^{15}}$, &c.

Ex.

Ex. 2. $\overline{\frac{1}{2}x^2+\frac{1}{16}x^3+\frac{1}{128}x^4, \&c.}\,|^3 = \frac{1}{8}x^6+\frac{1}{64}x^7+\frac{1}{256}x^8, \&c.$

VIII. In the laſt Theorem, if we make $A^3 = P$, $3A^2B = Q$, $AB^2+3A^2C = R$, $B^3+6ABC+3A^2D = S$, &c. then $A = P^{\frac{1}{3}}$, $B = \frac{Q}{3A^2}$, $C = \frac{R-3AB^2}{3A^2}$, $D = \frac{S-6ABC-B^3}{3A^2}$, &c. that is $\overline{P+Q+R+S+T, \&c.}\,|^{\frac{1}{3}} = P^{\frac{1}{3}}+\frac{Q}{3A^2}+\frac{R-3AB^2}{3A^4}+\frac{S-B^3-6ABC}{3A^5}+\frac{T-3AC^2-3B^2C-6ABD}{3A^2}$, &c. And by this Theorem the Cube-root of any infinite Series may be extracted. Here alſo A, B, C, D, &c. will repreſent the Terms as they ſtand in order.

Ex. 1. $\overline{\frac{x^9}{27a^6}-\frac{x^{12}}{27a^9}+\frac{8x^{15}}{243a^{12}}-\frac{7x^{18}}{243a^{15}}, \&c.}\,|^{\frac{1}{3}} = \frac{x^3}{3a^2}-\frac{x^6}{9a^5}+\frac{5x^9}{81a^8}-\frac{10x^{12}}{243a^{11}}, \&c.$

Ex. 2. $\frac{1}{8}x^6+\frac{1}{64}x^7+\frac{1}{128}x^8, \&c.\,|^{\frac{1}{3}} = \frac{1}{2}x^2+\frac{1}{16}x^3+\frac{1}{256}x^4, \&c.$

IX. Becauſe it is by the ſeventh Theorem $\overline{\alpha+\beta+\gamma+\delta, \&c.}\,|^3 = \alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3, \&c.$ in the third Theorem for P, $+3\alpha^2\gamma+6\alpha\beta\gamma$ $+3\alpha^2\delta$ Q, R, S, T, &c. write α^3, $3\alpha^2\beta$, $3\alpha\beta^2+3\alpha^2\gamma$, $\beta^3+6\alpha\beta\gamma+3\alpha^2\delta$, $3\alpha\gamma^2+3\beta^2\gamma+6\alpha\beta\delta+3\alpha^2\varepsilon$, &c. reſpectively; then $\frac{1}{\overline{\alpha+\beta+\gamma+\delta, \&c.}\,|^3} = \frac{1}{\alpha^3}-\frac{3\times 3A}{\alpha^3}+\frac{3\alpha^2\beta B+3\alpha\beta^2+3\alpha^2\gamma\times A}{\alpha^3}-\frac{3\alpha^2\beta C+3\alpha B^2+3\alpha^2\gamma\times B+B^3+6\alpha\beta\gamma+3\alpha\delta\times A\,\&c.}{\alpha^3}$

This Theorem will give the Reciprocal of the Cube of any infinite Series; where A, B, C, D, &c. ſtand for the Terms in order.

X. Laſtly, in the firſt Theorem if we make $P = A^3$, $Q = 3A^2B$, $R = 3AB^2+3A^2C$, $S = B^3+6ABC+3A^2D$, &c. we ſhall have $\overline{A+B+C+D, \&c.}\,|^4 = A^4+4A^3B+6A^2B^2+4AB^3, \&c.$ which $+4A^3C+12A^2BC$ $+4A^3D$ will be a Theorem for finding the Biquadrate of any infinite Series.

And thus we might proceed to find particular Theorems for any other Powers or Roots of any infinite Series, or for their Reciprocals, or any fractional Powers compounded of theſe; all which will be found very convenient to have at hand, continued to a competent number of Terms, in order to facilitate the following Operations. Or it may be ſufficient to lay before you the elegant and general Theorem, contrived for this purpoſe, by that ſkilful Mathematician, and my good Friend, the ingenious Mr. *A. De Moivre*, which was firſt publiſh'd in the Philoſophical Tranſactions, N° 230, and which will readily perform all theſe Operations.

Or we may have recourſe to a kind of Mechanical Artifice, by which all the foregoing Operations may be perform'd in a very eaſy and general manner, as here follows.

When two infinite Series are to be multiply'd together, in order to find a third which is to be their Product, call one of them the Multiplicand, and the other the Multiplier. Write down upon your Paper the Terms of the Multiplicand, with their Signs, in a deſcending order, ſo that the Terms may be at equal diſtances, and juſt under one another. This you may call your fixt or right-hand Paper. Prepare another Paper, at the right-hand Edge of which write down the Terms of the Multiplier, with their proper Signs, in an aſcending Order, ſo that the Terms may be at the ſame equal diſtances from each other as in the Multiplicand, and juſt over one another. This you may call your moveable or left-hand Paper. Apply your moveable Paper to your fixt Paper, ſo that the firſt Term of your Multiplier may ſtand over-againſt the firſt Term of your Multiplicand. Multiply theſe together, and write down the Product in its place, for the firſt Term of the Product required. Move your moveable Paper a ſtep lower, ſo that two of the firſt Terms of the Multiplier may ſtand over-againſt two of the firſt Terms of the Multiplicand. Find the two Products, by multiplying each pair of the Terms together, that ſtand over-againſt one another; abbreviate them if it may be done, and ſet down the Reſult for the ſecond Term of the Product required. Move your moveable Paper a ſtep lower, ſo that three of the firſt Terms of the Multiplier may ſtand over-againſt three of the firſt Terms of the Multiplicand. Find the three Products, by multiplying each pair of the Terms together that ſtand over-againſt one another; abbreviate them, and ſet down the Reſult for the third Term of the Product. And proceed in the ſame manner to find the fourth, and all the following Terms.

I ſhall illuſtrate this Method by an Example of two Series, taken from the common Scale of Denary or Decimal Arithmetick; which will equally explain the Proceſs in all other infinite Series whatever.

Let the Numbers to be multiply'd be 37,528936, &c. and 528,73041, &c. which, by ſupplying X or 10 where it is underſtood, will become the Series $3X + 7X^{0} + 5X^{-1} + 2X^{-2} + 8X^{-3} + 9X^{-4} + 3X^{-5} + 6X^{-6}$ &c. and $5X^{2} + 2X + 8X^{0} + 7X^{-1} + 3X^{-2} + 0X^{-3} + 4X^{-4} + 1X^{-5}$, &c. and call the firſt the Multiplicand, and the ſecond the Multiplier. Theſe being diſpoſed as is preſcribed, will ſtand as follows.

Multiplier,

Multiplier, &c.
$+ 1X^{-5}$
$+ 4X^{-4}$
$+ 0X^{-3}$
$+ 3X^{-2}$
$+ 7X^{-1}$
$+ 8X^{0}$
$+ 2X$
$5X^{2}$

Multiplicand	Product	$1X^{4}$
$3X$	$15X^{3}$ - - - - - - - - -	$9X^{3}$
$+ 7X^{0}$	$+ 41X^{2}$ - - - - - - - - -	$8X^{2}$
$+ 5X^{-1}$	$- + 63X$ - - - - - - - - - -	$4X$
$+ 2X^{-2}$	$- - + 97X^{0}$ - - - - - - - - -	$2X^{0}$
$+ 8X^{-3}$	$- - - + 142X^{-1}$ - - - - - - - -	$6X^{-1}$
$+ 9X^{-4}$	$- - - - + 133X^{-2}$ - - - - - -	$8X^{-2}$
$+ 3X^{-5}$	$- - - - + 138X^{-3}$ - - - - - -	$8X^{-3}$
$+ 6X^{-6}$	$- - - - - + 201X^{-4}$ - - - - -	$1X^{-4}$
&c.	&c.	&c.

Now the firſt Term of the moveable Paper, or Multiplier, being apply'd to the firſt Term of the Multiplicand, will give $5X^{2} \times 3X = 15X^{3}$ for the firſt Term of the Product. Then the two firſt Terms of each being apply'd together, they will give $5X^{2} \times 7X^{0} + 2X \times 3X = 41X^{2}$ for the ſecond Term of the Product. Then the three firſt Terms of each being apply'd together, they will give $5X^{2} \times 5X^{-1} + 2X \times 7X^{0} + 8X^{0} \times 3X = 63X$ for the third Term of the Product. And ſo on. So that the Product required will be $15X^{3} + 41X^{2} + 63X + 97X^{0} + 142X^{-1} + 133X^{-2} + 138X^{-3} + 201X^{-4}$, &c. Now this will be a Number in the Decimal Scale of Arithmetick, becauſe $X = 10$. But in that Scale, when it is regular, the Coefficients muſt always be affirmative Integers, leſs than the Root 10; and therefore to reduce theſe to ſuch, ſet them orderly under one another, as is done here, and beginning at the loweſt, collect them as they ſtand, by adding up each Column. The reaſon of which is this. Becauſe $201X^{-4} = 20X^{-3} + 1X^{-4}$, we muſt ſet down $1X^{-4}$, and add $20X^{-3}$ to the line above. Then becauſe $20X^{-3} + 138X^{-3} = 158X^{-3} = 15X^{-2} + 8X^{-3}$, we muſt ſet down $8X^{-3}$, and add $15X^{-2}$ to the line above. Then becauſe $15X^{-2} + 133X^{-2} = 148X^{-2} = 14X^{-1} + 8X^{-2}$, we muſt ſet down $8X^{-2}$, and add $14X^{-1}$ to the line above. And ſo we muſt proceed through the whole Number. So that at laſt we ſhall find the Product to be $1X^{4} + 9X^{3} + 8X^{2} + 4X + 2X^{0} + 6X^{-1} + 8X^{-2} + 8X^{-3}$, &c. Or by ſuppreſſing X, or 10, and leaving it to be ſupply'd by the Imagination, the Product required will be 19842,688, &c.

When one infinite Series is to be divided by another, write down the Terms of the Dividend, with their proper Signs, in a deſcending order, ſo that the Terms may be at equal diſtances, and juſt under

der one another. This is your fixt or right-hand Paper. Prepare another Paper, at the right-hand Edge of which write down the Terms of the Diviſor in an aſcending order, with all their Signs changed except the firſt, ſo that the Terms may be at the ſame equal diſtances as before, and juſt over one another. This will be your moveable or left-hand Paper. Apply your moveable Paper to your fixt Paper, ſo that the firſt Term of the Diviſor may be over-againſt the firſt Term of the Dividend. Divide the firſt Term of the Dividend by the firſt Term of the Diviſor, and ſet down the Quotient over-againſt them to the right-hand, for the firſt Term of the Quotient required. Move your moveable Paper a ſtep lower, ſo that two of the firſt Terms of the Diviſor may be over-againſt two of the firſt Terms of the Dividend. Collect the ſecond Term of the Dividend, together with the Product of the firſt Term of the Quotient now found, multiply'd by the Terms over-againſt it in the left-hand Paper; theſe divided by the firſt Term of the Diviſor will be the ſecond Term of the Quotient required. Move your moveable Paper a ſtep lower, ſo that three of the firſt Terms of the Diviſor may ſtand over-againſt three of the firſt Terms of the Dividend. Collect the third Term of the Dividend, together with the two Products of the two firſt Terms of the Quotient now found, each being multiply'd into the Term over-againſt it, in the left-hand Paper. Theſe divided by the firſt Term of the Diviſor will be the third Term of the Quotient required. Move your moveable Paper a ſtep lower, ſo that four of the firſt Terms of the Diviſor may ſtand over-againſt four of the firſt Terms of the Dividend. Collect the fourth Term of the Dividend, together with the three Products of the three firſt Terms of the Quotient now found, each being multiply'd by the Term over-againſt it in the left-hand Paper. Theſe divided by the firſt Term of the Diviſor will be the fourth Term of the Quotient required. And ſo on to find the fifth, and the ſucceeding Terms.

For an Example let it be propoſed to divide the infinite Series $a^2 + \frac{1}{8}ax + \frac{1}{30}x^2 + \frac{121x^3}{1260a} + \frac{281x^4}{1260a^2}$, &c. by the Series $a + \frac{1}{2}x + \frac{x^2}{3a} + \frac{x^3}{4a^2} + \frac{x^4}{5a^3}$, &c. Theſe being diſpoſed as is preſcribed, will ſtand as here follows.

Diviſor,

Diviſor, &c.	Dividend		Quotient
	a^2	a -	a
$-\frac{x^4}{5a^3}$	$+\frac{1}{6}ax$	$+\frac{1}{6}ax - \frac{1}{2}ax = -\frac{1}{3}ax$ - - - - - - - -	$-\frac{1}{3}x$
$-\frac{x^3}{4a^2}$	$+\frac{11}{30}x^2$	$+\frac{11}{30}x^2 + \frac{1}{6}x^2 - \frac{1}{5}x^2 = +\frac{1}{5}x^2$ - - - -	$+\frac{x^2}{5a}$
$-\frac{x^2}{3a}$	$+\frac{121x^3}{1260a}$	$+\frac{121x^3}{1260a} - \frac{x^3}{10a} + \frac{x^3}{9a} - \frac{x^3}{4a} = -\frac{x^3}{7a}$ - - -	$-\frac{x^3}{7a^2}$
$-\frac{1}{2}x$	$+\frac{281x^4}{1260a^2}$	$+\frac{281x^4}{1260a^2} + \frac{x^4}{14a^2} - \frac{x^4}{15a^2} + \frac{x^4}{12a^2} - \frac{x^4}{5a^2} = +\frac{x^4}{9a^2}$	$+\frac{x^4}{9a^3}$
a	&c.	&c.	&c.

Here if we apply the firſt Term of the Diviſor a, to the firſt Term of the Dividend a^2, by Diviſion we ſhall have a for the firſt Term of the Quotient. Then applying the two firſt Terms of the Diviſor to the two firſt Terms of the Dividend, we ſhall have $\frac{1}{6}ax$ to be collected with the Product $a \times -\frac{1}{2}x$, or $-\frac{1}{2}ax$, which will make $-\frac{1}{3}ax$; and this divided by a, the firſt Term of the Diviſor, will give $-\frac{1}{3}x$ for the ſecond Term of the Quotient. And ſo of the other Terms; and in like manner for all other Examples.

When an infinite Series is to be raiſed to any Power, or when any Root of it is to be extracted, it may be perform'd in all caſes by a like Artifice. Prepare your fixt or right-hand Paper, by writing down the natural Numbers 0, 1, 2, 3, 4, &c. juſt under one another at equal diſtances, reſerving places to the right-hand for the ſeveral Terms of the Power or Root, as they ſhall be found. The firſt Term of which Series may be immediately known from the firſt Term of the given Series, and from the given Index of the Power or Root, whether that Index be an Integer or a Fraction, affirmative or negative; and that Term therefore may be ſet down in its place, over-againſt the firſt Number 0. Prepare your moveable or left-hand Paper, by writing down, towards the edge of the Paper at the right-hand, all the Terms of the given Series, except the firſt, over one another in order, at the ſame diſtances as the Numbers in the other Paper. After which, nearer the edge of the Paper, write juſt over one another, firſt the Index of the Power or Root to be found, then its double, then its triple, and ſo the reſt of its multiples, with the negative Sign after each, as far as the Terms of the Series extend. And alſo the firſt Term of the given Series may be wrote below. Thus will the moveable Paper be prepared. Theſe multiples, together with the following negative Signs, and the Numbers

0,

0, 1, 2, 3, 4, &c. on the other Paper, when they meet together, will compleat the numeral Coefficients. Apply therefore the ſecond Term of the moveable Paper to the uppermoſt Term of the fixt Paper, and the Product made by the continual Mutiplication of the three Factors that ſtand in a line over-againſt one another, [which are the ſecond Term of the given Series, the numeral Coefficient, (here the given Index,) and the firſt Term of the Series already found,] divided by the firſt Term of the given Series, will be the ſecond Term of the Series required, which is to be ſet down in its place over-againſt 1. Move the moveable Paper a ſtep lower, and the two Products made by the multiplication of the Factors that ſtand over-againſt one another, (in which, and elſewhere, care muſt be had to take the numeral Coefficients compleat,) divided by twice the firſt Term of the given Series, will be the third Term of the Series required, which is to be ſet down in its place over-againſt 2. Move the moveable Paper a ſtep lower, and the three Products made by the multiplication of the Factors that ſtand over-againſt one another, divided by thrice the firſt Term of the given Series, will be the fourth Term of the Series required. And ſo you may proceed to find the next, and the ſubſequent Terms.

It may not be amiſs to give one general Example of this Reduction, which will comprehend all particular Caſes. If the Series $az + bz^2 + cz^3 + dz^4$, &c. be given, of which we are to find any Power, or to extract any Root; let the Index of this Power or Root be m. Then prepare the moveable or left-hand Paper as you ſee below, where the Terms of the given Series are ſet over one another in order, at the edge of the Paper, and at equal diſtances. Alſo after every Term is put a full point, as a Mark of Multiplication, and after every one, (except the firſt or loweſt) are put the ſeveral Multiples of the Index, as m, $2m$, $3m$, $4m$, &c. with the negative Sign — after them. Likewiſe a *vinculum* may be underſtood to be placed over them, to connect them with the other parts of the numeral Coefficients, which are on the other Paper, and which make them compleat. Alſo the firſt Term of the given Series is ſeparated from the reſt by a line, to denote its being a Diviſor, or the Denominator of a Fraction. And thus is the moveable Paper prepared.

To prepare the fixt or right-hand Paper, write down the natural Numbers 0, 1, 2, 3, 4, &c. under one another, at the ſame equal diſtances as the Terms in the other Paper, with a Point after them as a Mark of Multiplication; and over-againſt the firſt Term 0 write

write $a^m z^m$ for the firſt Term of the Series required. The reſt of the Terms are to be wrote down orderly under this, as they ſhall be found, which will be in this manner. To the firſt Term o in the fixt Paper apply the ſecond Term of the moveable Paper, and they will then exhibit this Fraction $\frac{bz^2 . \overline{m-0} . a^m z^m}{az . 1}$, which being reduced to this $ma^{m-1}bz^{m+1}$, muſt be ſet down in its place, for the ſecond Term of the Series required. Move the moveable Paper a ſtep lower, and you will have this Fraction exhibited $+cz^3 . \overline{2m-0} . a^m z^m$ $+bz^2 . \overline{m-1} . ma^{m-1}bz^{m+1}$ over $az . 2$ which being reduced will become $ma^{m-1}c + m \times \frac{m-1}{2} a^{m-2}b^2 \times z^{m+2}$, to be put down for the third Term of the Series required. Bring down the moveable Paper a ſtep lower, and you will have the Fraction $+dz^4 . \overline{3m-0} . a^m z^m$ $+cz^3 . \overline{2m-1} . ma^{m-1}bz^{m+1}$ $+bz^2 . \overline{m-2} . ma^{m-1}c + m \times \frac{m-1}{2} a^{m-2}b^2 \times z^{m+3}$ over $az . 3$ which reduc'd will be $ma^{m-1}d + m \times \frac{m-1}{1} a^{m-2}bc + m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3 \times z^{m+3}$, for the fourth Term of the Series required. And in the ſame manner are all the reſt of the Terms to be found.

Moveable Paper, &c.	Fixt Paper
$+dz^4 . \overline{3m-}$	o. $a^m z^m$
$+cz^3 . \overline{2m-}$	1. $ma^{m-1}bz^{m+1}$
$+bz^2 . \overline{m-}$	2. $m \times \frac{m-1}{2} a^{m-2}b^2 + ma^{m-1}c \times z^{m+2}$
$az.$	3. $m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3 + m \times \frac{m-1}{1} a^{m-2}bc + ma^{m-1}d \times z^{m+3}$
&c.	&c.

N. B. This Operation will produce Mr. *De Moivre*'s Theorem mentioned before, the Inveſtigation of which may be ſeen in the place there quoted, and ſhall be exhibited here in due time and place. And this therefore will ſufficiently prove the truth of the preſent Proceſs. In particular Examples this Method will be found very eaſy and practicable.

But now to ſhew ſomething of the uſe of theſe Theorems, and at the ſame time to prepare the way for the Solution of Affected and Fluxional Equations; we will here make a kind of retroſpect, and reſume our Author's Examples of ſimple Extractions, beginning with Diviſion itſelf, which we ſhall perform after a different and an eaſier manner.

Thus to divide aa by $b+x$, or to reſolve the Fraction $\frac{aa}{b+x}$ into a Series of ſimple Terms; make $\frac{aa}{b+x}=y$, or $by+xy=aa$. Now to find the quantity y diſpoſe the Terms of this Equation after this manner $\left.\begin{matrix}by\\+xy\end{matrix}\right\}=a^2$, and proceed in the Reſolution as you ſee is done here.

$$\left.\begin{matrix}by\\+xy\end{matrix}\right\}\begin{matrix}=a^2-\frac{a^2x}{b}+\frac{a^2x^2}{b^2}-\frac{a^2x^3}{b^3}+\frac{a^2x^4}{b^4}, \text{ \&c.}\\---+\frac{a^2x}{b}-\frac{a^2x^2}{b^2}+\frac{a^2x^3}{b^3}-\frac{a^2x^4}{b^4}, \text{ \&c.}\end{matrix}$$

$$y=\frac{a^2}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{a^2x^3}{b^4}+\frac{a^2x^4}{b^5}, \text{ \&c.}$$

Here by the diſpoſition of the Terms a^2 is made the firſt Term of the Series belonging (or equivalent) to by, and therefore dividing by b, $\frac{a^2}{b}$ will be the firſt Term of the Series equivalent to y, as is ſet down below. Then will $+\frac{a^2x}{b}$ be the firſt Term of the Series $+xy$, which is therefore ſet down over-againſt it; as alſo it is ſet down over-againſt by, but with a contrary Sign, to be the ſecond Term of that Series. Then will $-\frac{a^2x}{b^2}$ be the ſecond Term of y, to be ſet down in its place, which will give $-\frac{a^2x^2}{b^2}$ for the ſecond Term of $+xy$; and this with a contrary Sign muſt be ſet down for the third Term of by. Then will $+\frac{a^2x^2}{b^3}$ be the third Term of y, and therefore $+\frac{a^2x^3}{b^3}$ will be the third Term of $+xy$, which with a contrary Sign muſt be made the fourth Term of by, and therefore $-\frac{a^2x^3}{b^4}$ will be the fourth Term of y. And ſo on for ever.

Now the *Rationale* of this Proceſs, and of all that will here follow of the ſame kind, may be manifeſt from theſe Conſiderations. The unknown Terms of the Equation, or thoſe wherein y is found, are (by the *Hypotheſis*) equal to the known Term aa. And each of thoſe

thoſe unknown Terms is reſolved into its equivalent Series, the Aggregate of which muſt ſtill be equal to the ſame known Term aa; (or perhaps Terms.) Therefore all the ſubſidiary and adventitious Terms, which are introduced into the Equation to aſſiſt the Solution, (or the Supplemental Terms,) muſt mutually deſtroy one another.

Or we may reſolve the ſame Equation in the following manner:

$$\left.\begin{matrix} by \\ +xy \end{matrix}\right\} \begin{matrix} \text{---} + \frac{ba^2}{x} - \frac{b^2a^2}{x^2} + \frac{b^3a^2}{x^3}, \text{ \&c.} \\ = a^2 - \frac{ba^2}{x} + \frac{b^2a^2}{x^2} - \frac{b^3a^2}{x^3}, \text{ \&c.} \end{matrix}$$

$$y = \frac{a^2}{x} - \frac{ba^2}{x^2} + \frac{b^2a^2}{x^3} - \frac{b^3a^2}{x^4}, \text{ \&c.}$$

Here a^2 is made the firſt Term of $+xy$, and therefore $\frac{a^2}{x}$ muſt be put down for the firſt Term of y. This will give $+\frac{ba^2}{x}$ for the firſt Term of by, which with a contrary Sign muſt be the ſecond Term of $+xy$, and therefore $-\frac{ba^2}{x^2}$ muſt be put down for the ſecond Term of y. Then will $-\frac{b^2a^2}{x^2}$ be the ſecond Term of by, which with a contrary Sign will be the third Term of $+xy$, and therefore $+\frac{b^2a^2}{x^3}$ will be the third Term of y. And ſo on. Therefore the Fraction propoſed is reſolved into the ſame two Series as were found above.

If the Fraction $\frac{1}{1+x^2}$ were given to be reſolved, make $\frac{1}{1+x^2} = y$, or $y + x^2y = 1$, the Reſolution of which Equation is little more than writing down the Terms, in the manner following:

$$\left.\begin{matrix} y \\ +x^2y \end{matrix}\right\} \begin{matrix} = 1 - x^2 + x^4 - x^6 + x^8, \text{\&c.} \\ \text{---} + x^2 - x^4 + x^6, \text{ \&c.} \end{matrix} \qquad \left.\begin{matrix} y \\ +x^2y \end{matrix}\right\} \begin{matrix} \text{---} + x^{-2} - x^{-4} + x^{-6} - x^{-8}, \text{\&c.} \\ = 1 - x^{-2} + x^{-4} - x^{-6}, \text{ \&c.} \end{matrix}$$

Here in the firſt Paradigm, as 1 is made the firſt Term of y, ſo will x^2 be the firſt Term of x^2y, and therefore $-x^2$ will be the ſecond Term of y, and therefore $-x^4$ will be the ſecond Term of x^2y, and therefore $+x^4$ will be third Term of y; &c. Alſo in the ſecond Paradigm, as 1 is made the firſt Term of x^2y, ſo will $+x^{-2}$ be the firſt Term of y, and therefore $-x^{-2}$ will be the ſecond Term of x^2y, or $-x^{-4}$ will be the ſecond Term of y; &c.

To reſolve the compound Fraction $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}$ into ſimple Terms, make $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}=y$, or $2x^{\frac{1}{2}}-x^{\frac{3}{2}}=y+x^{\frac{1}{2}}y-3xy$; which Equation may be thus reſolved:

$$= 2x^{\frac{1}{2}} \quad * \quad - x^{\frac{3}{2}}$$

$$\left.\begin{array}{r} y \\ +x^{\frac{1}{2}}y \\ -3xy \end{array}\right\} \begin{array}{l} ---\ 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^{2} + 34x^{\frac{5}{2}} - 73x^{3}, \text{ \&c.} \\ ------\ + 2x - 2x^{\frac{3}{2}} + 7x^{2} - 13x^{\frac{5}{2}} + 34x^{3}, \text{ \&c.} \\ ----------\ - 6x^{\frac{3}{2}} + 6x^{2} - 21x^{\frac{5}{2}} + 39x^{3}, \text{ \&c.} \end{array}$$

Place the Terms of the Equation, in which the unknown quantity y is found, in a regular deſcending order, and the known Terms above, as you ſee is done here. Then bring down $2x^{\frac{1}{2}}$ to be the firſt Term of y, which will give $+2x$ for the firſt Term of the Series $+x^{\frac{1}{2}}y$, which muſt be wrote with a contrary Sign for the ſecond Term of y. Then will the ſecond Term of $+x^{\frac{1}{2}}y$ be $-2x^{\frac{3}{2}}$, and the firſt Term of the Series $-3xy$ will be $-6x^{\frac{3}{2}}$, which together make $-8x^{\frac{3}{2}}$. And this with a contrary Sign would have been wrote for the third Term of y, had not the Term $-x^{\frac{3}{2}}$ been above, which reduces it to $+7x^{\frac{3}{2}}$ for the third Term of y. Then will $+7x^{2}$ be the third Term of $+x^{\frac{1}{2}}y$, and $+6x^{2}$ will be the ſecond Term of $-3ay$, which being collected with a contrary Sign, will make $-13x^{2}$ for the fourth Term of y; and ſo on, as in the Paradigm.

If we would reſolve this Fraction, or this Equation, ſo as to accommodate it to the other caſe of convergency, we may invert the Terms, and proceed thus:

$$= -x^{\frac{3}{2}} \quad * \quad + 2x^{\frac{1}{2}}$$

$$\left.\begin{array}{r} -3xy \\ +x^{\frac{1}{2}}y \\ +y \end{array}\right\} \begin{array}{l} ---\ -x^{\frac{3}{2}} - \frac{1}{3}x + \frac{14}{9}x^{\frac{1}{2}} + \frac{11}{27}, \text{ \&c.} \\ ------\ + \frac{1}{3}x + \frac{5}{9}x^{\frac{1}{2}} - \frac{14}{81}, \text{ \&c.} \\ ----------\ + \frac{5}{3}x^{\frac{1}{2}} + \frac{1}{9}, \text{ \&c.} \end{array}$$

$$y = \tfrac{1}{3}x^{\frac{1}{2}} + \tfrac{1}{9} - \tfrac{14}{27}x^{-\frac{1}{2}} - \tfrac{11}{81}x^{-1}, \text{ \&c.}$$

Bring down $-x^{\frac{3}{2}}$ to be the firſt Term of $-3xy$, whence $+\frac{1}{3}x^{\frac{1}{2}}$ will be the firſt Term of y, to be ſet down in its place. Then the firſt

first Term of $+x^{\frac{1}{3}}y$ will be $+\frac{1}{3}x$, which with a contrary Sign will be the second Term of $-3xy$, and therefore $+\frac{1}{9}$ will be the second Term of y. Then the second Term of $+x^{\frac{1}{3}}y$ will be $+\frac{1}{9}x^{\frac{1}{3}}$, and the first Term of y being $+\frac{1}{3}x^{\frac{1}{3}}$, these two collected with a contrary Sign would have made $-\frac{4}{9}x^{\frac{1}{3}}$ for the third Term of $-3xy$, had not the Term $+2x^{\frac{1}{3}}$ been present above. Therefore uniting these, we shall have $+\frac{14}{9}x^{\frac{1}{3}}$ for the third Term of $-3xy$, which will give $-\frac{1}{2}\frac{4}{7}x^{-\frac{1}{3}}$ for the third Term of y. Then will the third Term of $+x^{\frac{1}{3}}y$ be $-\frac{1}{2}\frac{4}{7}$, and the second Term of y being $+\frac{1}{9}$, these two collected with a contrary Sign will make $+\frac{1}{2}\frac{1}{7}$ for the fourth Term of $-3xy$, and therefore $-\frac{1}{8}\frac{1}{1}x^{-1}$ will be the fourth Term of y; and so on.

And thus much for Division; now to go on to the Author's pure or simple Extractions.

To find the Square-root of $aa+xx$, or to extract the Root y of this Equation $yy=aa+xx$; make $y=a+p$, then we shall have by Substitution $2ap+pp=xx$, of which affected Quadratick Equation we may thus extract the Root p. Dispose the Terms in this manner $\left.\begin{matrix}2ap\\+pp\end{matrix}\right\}=xx$, the unknown Terms in a descending order on one side, and the known Term or Terms on the other side of the Equation, and proceed in the Extraction as is here directed.

$$\left.\begin{matrix}2ap\\+p^2\end{matrix}\right\}\begin{matrix}=x^2-\frac{x^4}{4a^2}+\frac{x^6}{8a^4}-\frac{5x^8}{64a^6}+\frac{7x^{10}}{128a^8},\ \&c.\\---+\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{5x^8}{64a^6}-\frac{7x^{10}}{128a^8},\ \&c.\end{matrix}$$

$$p=\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9},\ \&c.$$

By this Disposition of the Terms, x^2 is made the first Term of the Series belonging to $2ap$; then we shall have $\frac{x^2}{2a}$ for the first Term of the Series p, as here set down underneath. Therefore $\frac{x^4}{4aa}$ will be the first Term of the Series p^2, to be put down in its place over-against p^2. Then, by what is observed before, it must be put down with a contrary Sign as the second Term of $2ap$, which will make the second Term of p to be $-\frac{x^4}{8a^3}$. Having therefore

fore the two firſt Terms of $p = \frac{x^2}{2a} - \frac{x^4}{8a^3}$, we ſhall have, (by any of the foregoing Methods for finding the Square of an infinite Series,) the two firſt Terms of $p^2 = \frac{x^4}{4a^2} - \frac{x^6}{8a^4}$; which laſt Term muſt be wrote with a contrary Sign, as the third Term of $2ap$. Therefore the third Term of p is $\frac{x^6}{16a^5}$, and the third Term of p^2 (by the aforeſaid Methods) will be $\frac{5x^8}{64a^6}$, which is to be wrote with a contrary Sign, as the fourth Term of $2ap$. Then the fourth Term of p will be $-\frac{5x^8}{128a^7}$, and therefore the fourth Term of p^2 is $-\frac{7x^{10}}{128a^8}$, which is to be wrote with a contrary Sign for the fifth Term of $2ap$. This will give $\frac{7x^{10}}{256a^9}$ for the fifth Term of p; and ſo we may proceed in the Extraction as far as we pleaſe.

Or we may diſpoſe the Terms of the Supplemental Equation thus:

$$\left.\begin{array}{r} 2ap \\ +p^2 \end{array}\right\} \text{----} + 2ax - 2a^2 + \frac{a^3}{x} * - \frac{a^5}{4x^3}, \text{ \&c.}$$
$$= x^2 - 2ax + 2a^2 - \frac{a^3}{x} * + \frac{a^5}{4x^3}, \text{ \&c.}$$
$$p = x - a + \frac{a^2}{2x} * - \frac{a^4}{8x^3} *, \text{ \&c.}$$

Here x^2 is made the firſt Term of the Series p^2, and therefore x, (or elſe $-x$,) will be the firſt Term of p. Then $2ax$ will be the firſt Term of $2ap$, and therefore $-2ax$ will be the ſecond Term of p^2. So that becauſe $p^2 = x^2 - 2ax$, &c. by extracting the Square-root of this Series by any of the foregoiug Methods, it will be found $p = x - a$, &c. or $-a$ will be the ſecond Term of the Root p. Therefore the ſecond Term of $2ap$ will be $-2a^2$, which muſt be wrote with a contrary Sign for the third Term of p^2, and thence (by Extraction) the third Term of p will be $\frac{a^2}{2x}$. This will make the third Term of $2ap$ to be $\frac{a^3}{x}$, which makes the fourth Term of p^2 to be $-\frac{a^3}{x}$, and therefore (by Extraction) o will be the fourth Term of p. This makes the fourth Term of $2ap$ to be o, as alſo of p^2. Then $-\frac{a^4}{8x^3}$ will be the fifth Term of p. Then the fifth Term of

$2ap$ will be $-\frac{a^5}{4x^3}$, which will make the ſixth Term of p^2 to be $\frac{a^5}{4x^3}$; and therefore o will be the ſixth Term of p, &c.

Here the Terms will be alternately deficient; ſo that in the given Equation $yy = aa + xx$, the Root will be $y = a + x - a + \frac{a^2}{2x}$, &c. that is $y = x + \frac{a^2}{2x} - \frac{a^4}{8x^3} + \frac{a^6}{16x^5}$, &c. which is the ſame as if we ſhould change the order of the Terms, or if we ſhould change a into x, and x into a.

If we would extract the Square-root of $aa - xx$, or find the Root y of the Equation $yy = aa - xx$; make $y = a + p$, as before; then $2ap + p^2 = -x^2$, which may be reſolved as in the following Paradigm:

$$\left.\begin{array}{l} 2ap \\ +p^2 \end{array}\right\} \begin{array}{l} = -x^2 - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} - \frac{5x^8}{64a^6} - \frac{7x^{10}}{128a^8}, \text{ &c.} \\ ------ + \frac{x^4}{4a^2} + \frac{x^6}{8a^4} + \frac{5x^8}{64a^6} + \frac{7x^{10}}{128a^8}, \text{ &c.} \end{array}$$

$$p = -\frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \frac{7x^{10}}{256a^9} \text{ &c.}$$

Here if we ſhould attempt to make $-x^2$ the firſt Term of $+p^2$, we ſhould have $\sqrt{-x^2}$, or $x\sqrt{-1}$, for the firſt Term of p; which being impoſſible, ſhews no Series can be form'd from that Suppoſition.

To find the Square-root of $x - xx$, or the Root y in this Equation $yy = x - xx$, make $y = x^{\frac{1}{2}} + p$, then $x + 2x^{\frac{1}{2}}p + p^2 = x - xx$, or $2x^{\frac{1}{2}}p + p^2 = -x^2$, which may be reſolved after this manner:

$$\left.\begin{array}{l} 2x^{\frac{1}{2}}p \\ +p^2 \end{array}\right\} \begin{array}{l} = -x^2 - \frac{1}{4}x^3 - \frac{1}{8}x^4, \text{ &c.} \\ ----- + \frac{1}{4}x^3 + \frac{1}{8}x^4, \text{ &c.} \end{array}$$

$$p = -\frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}, \text{ &c.}$$

The Terms being rightly diſpoſed, make $-x^2$ the firſt Term of $2x^{\frac{1}{2}}p$; then will $-\frac{1}{2}x^{\frac{3}{2}}$ be the firſt Term of p. Therefore $+\frac{1}{4}x^3$ will be the firſt Term of p^2, which is alſo to be wrote with a contrary Sign for the ſecond Term of $2x^{\frac{1}{2}}p$, which will give $-\frac{1}{8}x^{\frac{5}{2}}$ for the ſecond Term of p. Then (by ſquaring) the ſecond Term of p^2 will be $\frac{1}{8}x^4$, which will give $-\frac{1}{8}x^4$ for the ſecond Term of $2x^{\frac{1}{2}}p$,

$2x^{\frac{1}{2}}p$, and therefore $-\frac{1}{16}x^{\frac{7}{2}}$ for the third Term of p; and so on. Therefore in this Equation it will be $y = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c.

So to extract the Root y of this Equation $yy = aa + bx - xx$, make $y = a + p$, then $2ap + p^2 = bx - xx$, which may be thus resolved.

$$\left.\begin{array}{r} 2ap \\ \\ \\ +p^2 \end{array}\right\} \begin{array}{l} = bx - x^2 + \frac{bx^3}{2a^2}, \text{ \&c.} \\ \qquad - \frac{b^2x^2}{4a^2} + \frac{b^3x^3}{8a^4} \\ \text{- - - - - - - - - -} + \frac{b^2x^2}{4a^2} - \frac{bx^3}{2a^2}, \text{ \&c.} \\ \qquad\qquad - \frac{b^3x^3}{8a^4} \end{array}$$

$$p = \frac{bx}{2a} - \frac{x^2}{2a} + \frac{bx^3}{4a^3}, \text{ \&c.}$$
$$- \frac{b^2x^2}{8a^3} + \frac{b^3x^3}{16a^5}.$$

Make bx the first Term of $2ap$; then will $\frac{bx}{2a}$ be the first Term of p. Therefore the first Term of p^2 will be $+\frac{b^2x^2}{4a^2}$, which is also to be wrote with a contrary Sign, so that the second Term of $2ap$ will be $-x^2 - \frac{b^2x^2}{4a^2}$, which will make the second Term of p to be $-\frac{x^2}{2a} - \frac{b^2x^2}{8a^3}$. Then by squaring, the second Term of p^2 will be $-\frac{bx^3}{2a^2} - \frac{b^3x^3}{8a^4}$, which must be wrote with a contrary Sign for the third Term of $2ap$. This will give the third Term of p as in the Example; and so on. Therefore the Square-root of the Quantity $a^2 + bx - xx$ will be $a + \frac{bx}{2a} - \frac{x^2}{2a} - \frac{b^2x^2}{8a^3} + \frac{bx^3}{4a^3} + \frac{b^3x^3}{16a^5}$, &c.

Also if we would extract the Square-root of $\frac{1 + ax^2}{1 - bx^2}$, we may extract the Roots of the Numerator, and likewise of the Denominator, and then divide one Series by the other, as before; but more directly thus. Make $\frac{1 + ax^2}{1 - bx^2} = yy$, or $1 + ax^2 = yy - b^2x^2y^2$. Suppose $y = 1 + p$, then $ax^2 = 2p + p^2 - bx^2 - 2bx^2p - bx^2p^2$, which Suppplemental Equation may be thus resolved.

2p

$$\left.\begin{array}{l} 2p \\ \\ \\ -2bx^2p \\ \\ \\ +p^2 \\ \\ \\ -bx^2p^2 \end{array}\right\} \begin{array}{l} = ax^2 + \frac{1}{2}abx^4 + \frac{3}{8}ab^2x^6, \&c. \\ +b \quad +\frac{3}{4}b^2 \quad +\frac{5}{8}b^3 \\ \quad -\frac{1}{4}a^2 \quad -\frac{1}{8}a^2b \\ \quad\quad\quad +\frac{1}{8}a^3 \\ \quad -ab \quad -\frac{1}{2}ab^2, \&c. \\ \quad -b^2 \quad -\frac{3}{4}b^3 \\ \quad\quad\quad +\frac{1}{4}a^2b \\ \quad +\frac{1}{4}a^2 \quad +\frac{1}{8}a^2b, \&c. \\ \quad +\frac{1}{2}ab \quad +\frac{5}{8}ab^2 \\ \quad +\frac{1}{4}b^2 \quad -\frac{1}{8}a^3 \\ \quad\quad\quad +\frac{3}{8}b^3 \\ \quad\quad\quad -\frac{1}{4}a^2b, \&c. \\ \quad\quad\quad -\frac{1}{2}ab^2 \\ \quad\quad\quad -\frac{1}{4}b^3 \end{array} \qquad p = \begin{array}{l} \frac{1}{2}ax^2 + \frac{1}{4}abx^4 + \frac{3}{16}ab^2x^6, \&c. \\ +\frac{1}{4}b \quad +\frac{3}{8}b^2 \quad +\frac{5}{16}b^3 \\ \quad -\frac{1}{8}a^2 \quad -\frac{1}{16}a^2b \\ \quad\quad\quad +\frac{1}{16}a^3 \end{array}$$

Make $ax^2 + bx^2$ the firſt Term of $2p$, then will $\frac{1}{2}ax^2 + \frac{1}{2}bx^2$ be the firſt Term of p. Therefore $-abx^4 - b^2x^4$ will be the firſt Term of $-2bx^2p$, and $\frac{1}{4}a^2x^4 + \frac{1}{2}abx^4 + \frac{1}{4}bx^4$ will be the firſt Term of p^2. Theſe being collected, and their Signs changed, muſt be made the ſecond Term of $2p$, which will give $\frac{1}{4}abx^4 + \frac{3}{8}b^2x^4 - \frac{1}{8}a^2x^4$ for the ſecond Term of p. Then the ſecond Term of $-2bx^2p$ will be $-\frac{1}{2}ab^2x^6 - \frac{3}{4}b^3x^6 + \frac{1}{4}a^2bx^6$, and the ſecond Term of p^2 (by ſquaring) will be found $\frac{1}{8}a^2bx^6 + \frac{5}{8}ab^2x^6 - \frac{1}{8}a^3x^6 + \frac{3}{8}b^3x^6$, and the firſt Term of $-bx^2p^2$ will be $-\frac{1}{4}a^2bx^6 - \frac{1}{2}ab^2x^6 - \frac{1}{4}b^3x^6$; which being collected and the Signs changed, will make the third Term of $2p$, half which will be the third Term of p; and ſo on as far as you pleaſe.

And thus if we were to extract the Cube-root of $a^3 + x^3$, or the Root y of this Equation $y^3 = a^3 + x^3$; make $y = a + p$, then by Subſtitution $a^3 + 3a^2p + 3ap^2 + p^3 = a^3 + x^3$, or $3a^2p + 3ap^2 + p^3 = x^3$; which ſupplemental Equation may be thus reſolved.

$$\left.\begin{array}{l} 3a^2p \\ \\ +3ap^2 \\ \\ +p^3 \end{array}\right\} \begin{array}{l} = x^3 - \frac{x^6}{3a^3} + \frac{5x^9}{27a^6} - \frac{10x^{12}}{81a^9}, \&c. \\ \quad + \frac{x^6}{3a^3} - \frac{2x^9}{9a^6} + \frac{13x^{12}}{81a^9}, \&c. \\ \quad\quad + \frac{x^9}{27a^6} - \frac{x^{12}}{27a^9}, \&c. \end{array}$$

$$p = \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}}, \&c.$$

The Terms being diſpos'd in order, the firſt Term of the Series $3a^2p$ will be x^3, which will make the firſt Term of p to be $\frac{x^3}{3a^2}$. This will make the firſt Term of p^2 to be $\frac{x^6}{9a^4}$. And this will make the firſt Term of $3ap^2$ to be $\frac{x^6}{3a^3}$, which with a contrary Sign muſt be the ſecond Term of $3a^2p$, and therefore the ſecond Term of p will be $-\frac{x^6}{9a^5}$. Then (by ſquaring) the ſecond Term of $3ap^2$ will be $-\frac{2x^9}{9a^6}$, and (by cubing) the firſt Term of p^3 will be $\frac{x^9}{27a^6}$. Theſe being collected make $-\frac{5x^9}{27a^6}$, which with a contrary Sign muſt be the third Term of $3a^2p$, and therefore the third Term of p will be $+\frac{5x^9}{81a^8}$. Then by ſquaring, the third Term of $3ap^2$ will be $\frac{13x^{12}}{81a^9}$, and by cubing, the ſecond Term of p^3 will be $-\frac{x^{12}}{27a^9}$, which being collected will make $\frac{10x^{12}}{81a^9}$; and therefore the fourth Term of $3a^2p$ will be $-\frac{10x^{12}}{81a^9}$, and the fourth Term of p will be $-\frac{10x^{12}}{243a^{11}}$. And ſo on.

And thus may the Roots of all pure Equations be extracted, but in a more direct and ſimple manner by the foregoing Theorems. All that is here intended, is, to prepare the way for the Reſolution of affected Equations, both in Numbers and Species, as alſo of Fluxional Equations, in which this Method will be found to be of very extenſive uſe. And firſt we ſhall proceed with our Author to the Solution of numerical affected Equations.

SECT. III. *The Reſolution of Numeral Affected Equations.*

19. NOW as to the Reſolution of affected Equations, and firſt in Numbers; our Author very juſtly complains, that before his time the *exegeſis numeroſa*, or the Doctrine of the Solution of affected Equations in Numbers, was very intricate, defective, and inartificial. What had been done by *Vieta*, *Harriot*, and *Oughtred* in this matter, tho' very laudable Attempts for the time, yet however was extremely perplex'd and operoſe. So that he had good reaſon to reject their Methods, eſpecially as he has ſubſtituted a much better in their room. They affected too great accuracy in purſuing exact

exact Roots, which led them into tedious perplexities; but he knew very well, that legitimate Approximations would proceed much more regularly and expeditiously, and would answer the same intention much better.

20, 21, 22. His Method may be easily apprehended from this one Instance, as it is contain'd in his Diagram, and the Explanation of it. Yet for farther Illustration I shall venture to give a short *rationale* of it. When a Numeral Equation is propos'd to be resolved, he takes as near an Approximation to the Root as can be readily and conveniently obtain'd. And this may always be had, either by the known Method of Limits, or by a Linear or Mechanical Construction, or by a few easy trials and suppositions. If this be greater or less than the Root, the Excess or Defect, indifferently call'd the Supplement, may be represented by p, and the assumed Approximation, together with this Supplement, are to be substituted in the given Equation instead of the Root. By this means, (expunging what will be superfluous,) a Supplemental Equation will be form'd, whose Root is now p, which will consist of the Powers of the assumed Approximation orderly descending, involved with the Powers of the Supplement regularly ascending, on both which accounts the Terms will be continually decreasing, in a decuple ratio or faster, if the assumed Approximation be suppos'd to be at least ten times greater than the Supplement. Therefore to find a new Approximation, which shall nearly exhaust the Supplement p, it will be sufficient to retain only the two first Terms of this Equation, and to seek the Value of p from the resulting simple Equation. [Or sometimes the three first Terms may be retain'd, and the Value of p may be more accurately found from the resulting Quadratick Equation; &c.] This new Approximation, together with a new Supplement q, must be substituted instead of p in this last supplemental Equation, in order to form a second, whose Root will be q. And the same things may be observed of this second supplemental Equation as of the first; and its Root, or an Approximation to it, may be discover'd after the same manner. And thus the Root of the given Equation may be prosecuted as far as we please, by finding new supplemental Equations, the Root of every one of which will be a correction to the preceding Supplement.

So in the present Example $y^3 - 2y - 5 = 0$, 'tis easy to perceive, that $y = 2$ *ferè*; for $2 \times 2 \times 2 - 2 \times 2 = 4$, which should make 5. Therefore let p be the Supplement of the Root, and it will be $y = 2 + p$, and therefore by substitution $-1 + 10p + 6p^2 + p^3 = 0$. As p is here suppos'd to be much less than the Approximation 2,

by this ſubſtitution an Equation will be form'd, in which the Terms will gradually decreaſe, and ſo much the faſter, *cæteris paribus*, as 2 is greater than p. So taking the two firſt Terms, $-1+10p=0$, *ferè*, or $p=\frac{1}{10}$ *ferè*; or aſſuming a ſecond Supplement q, 'tis $p=\frac{1}{10}+q$ accurately. This being ſubſtituted for p in the laſt Equation, it becomes $0,61+11,23q+6,3q^2+q^3=0$, which is a new Supplemental Equation, in which all the Terms are farther depreſs'd, and in which the Supplement q will be much leſs than the former Supplement p. Therefore it is $0,61+11,23q=0$, *ferè*, or $q=-\frac{0,61}{11,23}$ *ferè*, or $q=-0,0054+r$ *accuratè*, by aſſuming r for the third Supplement. This being ſubſtituted will give $0,00054155+11,162r$, &c. $=0$, and therefore $r=-\frac{0,00054155}{11,162}=-0,00004852$, &c. So that at laſt $y=2+p=$ &c. or $y=2,09455148$, &c.

And thus our Author's Method proceeds, for finding the Roots of affected Equations in Numbers. Long after this was wrote, Mr. *Raphſon* publiſh'd his *Analyſis Æquationum univerſalis*, containing a Method for the Solution of Numeral Equations, not very much different from this of our Author, as may appear by the following Compariſon.

To find the Root of the Equation $y^3-2y=5$, Mr. *Raphſon* would proceed thus. His firſt Approximation he calls g, which he takes as near the true Root as he can, and makes the Supplement x, ſo that he has $y=g+x$. Then by Subſtitution $\begin{matrix}g^3+3g^2x+3gx^2+x^3=5,\\-2g-2\end{matrix}$ or if $g=2$, 'tis $10x+6x^2+x^3=1$, to determine the Supplement x. This being ſuppoſed ſmall, its Powers may be rejected, and therefore $10x=1$, or $x=0,1$ nearly. This added to g or 2, makes a new $g=2,1$, and x being ſtill the Supplement, 'tis $y=2,1+x$, which being ſubſtituted in the original Equation $y^3-2y=5$, produces $11,23x+6,3x^2+x^3=-0,61$, to determine the new Supplement x. He rejects the Powers of x, and thence derives $x=\frac{-0,061}{11,23}=-0,0054$, and conſequently $y=2,0946$, which not being exact, becauſe the Powers of x were rejected, he makes the Supplement again to be x, ſo that $y=2,0946+x$, which being ſubſtituted in the Original Equation, gives $11,162x+$ &c. $=-0,00054155$. Therefore to find the third Supplement x, he has $x=\frac{-0,00054155}{11,162}=-0,00004852$, ſo that $y=2,0946+x=2,09455148$, &c. and ſo on.

By

By this Proceſs we may ſee how nearly theſe two Methods agree, and wherein they differ. For the difference is only this, that our Author conſtantly proſecutes the Reſidual or Supplemental Equations, to find the firſt, ſecond, third, &c. Supplements to the Root: But Mr. *Raphſon* continually corrects the Root itſelf from the ſame ſupplemental Equations, which are formed by ſubſtituting the corrected Roots in the Original Equation. And the Rate of Convergency will be the ſame in both.

In imitation of theſe Methods, we may thus proſecute this Inquiry after a very general manner. Let the given Equation to be reſolved be in this form $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3}$, &c. $= 0$, in which ſuppoſe P to be any near Approximation to the Root y, and the little Supplement to be p. Then is $y = P + p$. Now from what is ſhewn before, concerning the raiſing of Powers and extracting Roots, it will follow that $y^m = \overline{P+p}^m = P^m + mP^{m-1}p$, &c. or that theſe will be the two firſt Terms of y^m; and all the reſt, being multiply'd into the Powers of p, may be rejected. And for the ſame reaſon $y^{m-1} = P^{m-1} + \overline{m-1}P^{m-2}p$, &c. $y^{m-2} = P^{m-2} + \overline{m-2}P^{m-3}p$, &c. and ſo of all the reſt. Therefore theſe being ſubſtituted into the Equation, it will be

$$\left.\begin{array}{l} aP^m + maP^{m-1}p, \text{ \&c.} \\ + bP^{m-1} + \overline{m-1}bP^{m-2}p, \text{ \&c.} \\ + cP^{m-2} + \overline{m-2}cP^{m-3}p, \text{ \&c.} \\ + dP^{m-3} + \overline{m-3}dP^{m-4}p, \text{ \&c.} \\ \text{\&c.} \quad \text{\&c.} \end{array}\right\} = 0 \text{ ; Or dividing by } P^m,$$

$a + bP^{-1} + cP^{-2} + dP^{-3}$, &c. $+ maP^{-1}p + \overline{m-1}bP^{-2}p + \overline{m-2}cP^{-3}p + \overline{m-3}dP^{-4}p$, &c. $= 0$. From whence taking the Value of p, we ſhall have $p = -\frac{a + bP^{-1} + cP^{-2} + dP^{-3}, \text{ \&c.}}{maP^{-1} + \overline{m-1}bP^{-2} + \overline{m-2}cP^{-3} + \overline{m-3}dP^{-4}, \text{ \&c.}}$, and conſequently $y = (P + p = P - \frac{a + bP^{-1} + cP^{-2} + dP^{-3}, \text{ \&c.}}{maP^{-1} + \overline{m-1}bP^{-2} + \overline{m-2}cP^{-3} + \overline{m-3}dP^{-4}, \text{ \&c.}} =)$

$$\frac{\overline{m-1}a + \overline{m-2}bP^{-1} + \overline{m-3}cP^{-2} + \overline{m-4}dP^{-3}, \text{ \&c.}}{maP^{-1} + \overline{m-1}bP^{-2} + \overline{m-2}cP^{-3} + \overline{m-3}dP^{-3}, \text{ \&c.}}$$

To reduce this to a more commodious form, make $P = \frac{A}{B}$, whence $P^{-1} = A^{-1}B$, $P^{-2} = A^{-2}B^2$, &c. which being ſubſtituted, and alſo multiplying the Numerator and Denominator by A^m, it will be

$$y = \frac{\overline{m-1}aA^m + \overline{m-2}bA^{m-1}B + \overline{m-3}cA^{m-2}B^2 + \overline{m-4}dA^{m-3}B^3, \text{ \&c.}}{maA^{m-1}B + \overline{m-1}bA^{m-2}B^2 + \overline{m-2}cA^{m-3}B^3 + \overline{m-3}dA^{m-4}B^4, \text{ \&c.}}$$ which will be a nearer Approach to the Root y, than $\frac{A}{B}$, or P, and ſo much the-

the nearer as $\frac{A}{B}$ is near the Root. And hence we may derive a very convenient and general Theorem for the Extraction of the Roots of Numeral Equations, whether pure or affected, which will be this.

Let the general Equation $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3}$, &c. $= 0$ be propoſed to be ſolved; if the Fraction $\frac{A}{B}$ be aſſumed as near the Root y as conveniently may be, the Fraction $\frac{\overline{m-1}aA^m + \overline{m-2}bA^{m-1}B + \overline{m-3}cA^{m-2}B^2 + \overline{m-4}dA^{m-3}B^3, \&c.}{maA^{m-1}B + \overline{m-1}bA^{m-2}B^2 + \overline{m-2}cA^{m-3}B^3 + \overline{m-3}dA^{m-4}B^4, \&c.}$ will be ſtill a nearer Approximation to the Root. And this Fraction, when computed, may be uſed inſtead of the Fraction $\frac{A}{B}$, by which means a nearer Approximation may again be had; and ſo on, till we approach as near the true Root as we pleaſe.

This general Theorem may be conveniently reſolved into as many particular Theorems as we pleaſe. Thus in the Quadratick Equation $y^2 + by = c$, it will be $y = \frac{A^2 + cB^2}{2A + bB \times B}$, *ferè*. In the Cubick Equation $y^3 + by^2 + cy = d$, it will be $y = \frac{\overline{2A + bB} \times A^2 + dB^3}{3A^2 + 2bAB + cB^2 \times B}$, *ferè*. In the Biquadratick Equation $y^4 + by^3 + cy^2 + dy = e$, it will be $y = \frac{\overline{3A^2 + 2bAB + cB^2} \times A^2 + eB^4}{4A^3 + 3bA^2B + 2cAB^2 + dB^3 \times B}$, *ferè*. And the like of higher Equations.

For an Example of the Solution of a Quadratick Equation, let it be propoſed to extract the Square-root of 12, or let us find the value of y in this Equation $y^2 + 0 = 12$. Then by comparing with the general *formula*, we ſhall have $b = 0$, and $c = 12$. And taking 3 for the firſt approach to the Root, or making $\frac{A}{B} = \frac{3}{1}$, that is, $A = 3$ and $B = 1$, we ſhall have by Subſtitution $y = \frac{9+12}{6} = \frac{7}{2}$, for a nearer Approximation. Again, making $A = 7$ and $B = 2$, we ſhall have $y = \frac{49+48}{14 \times 2} = \frac{97}{28}$ for a nearer Approximation. Again, making $A = 97$ and $B = 28$, we ſhall have $y = \frac{\overline{97}|^2 + 12 \times \overline{28}|^2}{194 \times 28} = \frac{18817}{5432}$ for a nearer Approximation. Again, making $A = 18817$ and $B = 5432$, we ſhall have $y = \frac{\overline{18817}|^2 + 12 \times \overline{5432}|^2}{37634 \times 5432} = \frac{708158977}{204427888}$ for a nearer Approximation. And if we go on in the ſame method, we may find as near an Approximation to the Root as we pleaſe.

This

This Approximation will be exhibited in a vulgar Fraction, which, if it be always kept to its loweſt Terms, will give the Root of the Equation in the ſhorteſt and ſimpleſt manner. That is, it will always be nearer the true Root than any other Fraction whatever, whoſe Numerator and Denominator are not much larger Numbers than its own. If by Diviſion we reduce this laſt Fraction to a Decimal, we ſhall have 3,464101615137754 59 for the Square-root of 12, which exceeds the truth by leſs than an Unit in the laſt place.

For an Example of a Cubick Equation, we will take that of our Author $y^3 * - 2y = 5$, and therefore by Compariſon $b = 0$, $c = -2$, and $d = 5$. And taking 2 for the firſt Approach to the Root, or making $\frac{A}{B} = \frac{2}{1}$, that is, $A = 2$ and $B = 1$, we ſhall have by Subſtitution $y = \frac{16+5}{12-2} = \frac{21}{10}$ for a nearer Approach to the Root. Again, make $A = 21$ and $B = 10$, and then we ſhall have $y = \frac{9261+2500}{6615-1000} = \frac{11761}{5615}$ for a nearer Approximation. Again, make $A = 11761$ and $B = 5615$, and we ſhall have $y = \frac{2 \times \overline{11761}|^3 + 5 \times \overline{5615}|^3}{3 \times \overline{11761}|^2 \times 5615 - 2 \times \overline{5615}|^3} = \frac{4158744325037}{1975957316495}$ for a nearer Approximation. And ſo we might proceed to find as near an Approximation as we think fit. And when we have computed the Root near enough in a Vulgar Fraction, we may then (if we pleaſe) reduce it to a Decimal by Diviſion. Thus in the preſent Example we ſhall have $y = 2,094551481701$, &c. And after the ſame manner we may find the Roots of all other numeral affected Equations, of whatever degree they may be.

SECT. IV. *The Reſolution of Specious Equations by infinite Series; and firſt for determining the forms of the Series, and their initial Approximations.*

23, 24. FROM the Reſolution of numeral affected Equations, our Author proceeds to find the Roots of Literal, Specious, or Algebraical Equations alſo, which Roots are to be exhibited by an infinite converging Series, conſiſting of ſimple Terms. Or they are to be expreſs'd by Numbers belonging to a general Arithmetical Scale, as has been explain'd before, of which the Root is denoted by x or z. The aſſigning or chuſing this Root is what he means here, by diſtinguiſhing one of the literal Coefficients from the reſt, if there are ſeveral. And this is done by ordering or diſpoſing the

the Terms of the given Equation, according to the Dimenſions of that Letter or Coefficient. It is therefore convenient to chuſe ſuch a Root of the Scale, (when choice is allow'd,) as that the Series may converge as faſt as may be. If it be the leaſt, or a Fraction leſs than Unity, its aſcending Powers muſt be in the Numerators of the Terms. If it be the greateſt quantity, then its aſcending Powers muſt be in the Denominators, to make the Series duly converge. If it be very near a given quantity, then that quantity may be conveniently made the firſt Approximation, and that ſmall difference, or Supplement, may be made the Root of the Scale, or the converging quantity. The Examples will make this plain.

25, 26. The Equation to be reſolved, for conveniency-ſake, ſhould always be reduced to the ſimpleſt form it can be, before its Reſolution be attempted; for this will always give the leaſt trouble. But all the Reductions mention'd by the Author, and of which he gives us Examples, are not always neceſſary, tho' they may be often convenient. The Method is general, and will find the Roots of Equations involving fractional or negative Powers, as well as of other Equations, as will plainly appear hereafter.

27, 28. When a literal Equation is given to be reſolved, in diſtinguiſhing or aſſigning a proper quantity, by which its Root is to converge, the Author before has made three caſes or varieties; all which, for the ſake of uniformity, he here reduces to one. For becauſe the Series muſt neceſſarily converge, that quantity muſt be as ſmall as poſſible, in reſpect of the other quantities, that its aſcending Powers may continually diminiſh. If it be thought proper to chuſe the greateſt quantity, inſtead of that its Reciprocal muſt be introduced, which will bring it to the foregoing caſe. And if it approach near to a given quantity, then their ſmall difference may be introduced into the Equation, which again will bring it to the firſt caſe. So that we need only purſue that caſe, becauſe the Equation is always ſuppos'd to be reduced to it.

But before we can conveniently explain our Author's Rule, for finding the firſt Term of the Series in any Equation, we muſt conſider the nature of thoſe Numbers, or Expreſſions, to which theſe literal Equations are reduced, whoſe Roots are required; and in this Inquiry we ſhall be much aſſiſted by what has been already diſcourſed of Arithmetical Scales. In affected Equations that were purely numeral, the Solution of which was juſt now taught, the ſeveral Powers of the Root were orderly diſpoſed, according to a ſingle or ſimple Arithmetical Scale, which proceeded only *in longum*, and was there ſufficient

ſufficient for their Solution. But we muſt enlarge our views in theſe literal affected Equations, in which are found, not only the Powers of the Root to be extracted, but alſo the Powers of the Root of the Scale, or of the converging quantity, by which the Series for the Root of the Equation is to be form'd; on account of each of which circumſtances the Terms of the Equation are to be regularly diſpoſed, and therefore are to conſtitute a double or combined Arithmetical Scale, which muſt proceed both ways, *in latum* as well as *in longum*, as it were in a Table. For the Powers of the Root to be extracted, ſuppoſe y, are to be diſpoſed *in longum*, ſo as that their Indices may conſtitute an Arithmetical Progreſſion, and the vacancies, if any, may be ſupply'd by the Mark $*$. Alſo the Indices of the Powers of the Root, by which the Series is to converge, ſuppoſe x, are to be diſpoſed *in latum*, ſo as to conſtitute an Arithmetical Progreſſion, and the vacancies may likewiſe be fill'd up by the ſame Mark $*$, when it ſhall be thought neceſſary. And both theſe together will make a combined or double Arithmetical Scale. Thus if the Equation $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^3 + 6a^3x^3 + b^2x^4 = 0$, were given, to find the Root y, the Terms may be thus diſpoſed:

	y^6	y^5	y^4	y^3	y^2	y^1	y^0	
x^0	y^6	$*$	$*$	$*$	$*$	$*$	$*$	
x	$*$	$-5xy^5$	$*$	$*$	$*$	$*$	$*$	
x^2	$*$	$*$	$*$	$*$	$-7a^2x^2y^3$	$*$	$*$	$= 0$.
x^3	$*$	$*$	$+\frac{x^3}{a}y^4$	$*$	$*$	$*$	$+6a^3x^3$	
x^4	$*$	$*$	$*$	$*$	$*$	$*$	$+b^2x^4$	

Alſo the Equation $y^3 - by^2 + 9bx^2 - x^3 = 0$ ſhould be thus diſpoſed, in order to its Solution:

y^3	$*$	$*$	$-by^2$	$*$	$*$
					$*$
					$+9bx^2$
					$-x^3$

$= 0$.

And the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$ thus:

y^3	$*$	$+a^2y$	$-2a^3$	
		$+axy$	$*$	
			$*$	
			$-x^3$	

$= 0$.

And the Equation $x^2y^3 - 3c^4xy^2 - c^3x^3 + c^7 = 0$ thus:

$*$	$*$	$*$	$*$	$*$	$+c^7$	
$*$	$*$	$*$	$-3c^4xy^2$	$*$	$*$	$= 0$.
x^2y^3	$*$	$*$	$*$	$*$	$-c^3x^3$	

And the like of all other Equations.

When the Terms of the Equation are thus regularly diſpos'd, it is then ready for Solution; to which the following Speculation will be a farther preparation.

29. This ingenious contrivance of our Author, (which we may call Tabulating the Equation,) for finding the firſt Term of the Root, (which may indeed be extended to the finding all the Terms, or the form of the Series, or of all the Series that may be derived from the given Equation,) cannot be too much admired, or too carefully inquired into: The reaſon and foundation of which may be thus generally explain'd from the following Table, of which the Conſtruction is thus.

$-2a+6b$	$-a+6b$	$+6b$	$a+6b$	$2a+6b$	$3a+6b$	$4a+6b$	$5a+6b$	$6a+6b$	$7a+6b$
$-2a+5b$	$-a+5b$	$+5b$	$a+5b$	$2a+5b$	$3a+5b$	$4a+5b$	$5a+5b$	$6a+5b$	$7a+5b$
$-2a+4b$	$-a+4b$	$+4b$	$a+4b$	$2a+4b$	$3a+4b$	$4a+4b$	$5a+4b$	$6a+4b$	$7a+4b$
$-2a+3b$	$-a+3b$	$+3b$	$a+3b$	$2a+3b$	$3a+3b$	$4a+3b$	$5a+3b$	$6a+3b$	$7a+3b$
$-2a+2b$	$-a+2b$	$+2b$	$a+2b$	$2a+2b$	$3a+2b$	$4a+2b$	$5a+2b$	$6a+2b$	$7a+2b$
$-2a+b$	$-a+b$	$+b$	$a+b$	$2a+b$	$3a+b$	$4a+b$	$5a+b$	$6a+b$	$7a+b$
$-2a$	$-a$	0	a	$2a$	$3a$	$4a$	$5a$	$6a$	$7a$
$-2a-b$	$-a-b$	$-b$	$a-b$	$2a-b$	$3a-b$	$4a-b$	$5a-b$	$6a-b$	$7a-b$
$-2a-2b$	$-a-2b$	$-2b$	$a-2b$	$2a-2b$	$3a-2b$	$4a-2b$	$5a-2b$	$6a-2b$	$7a-2b$
$-2a-3b$	$-a-3b$	$-3b$	$a-3b$	$2a-3b$	$3a-3b$	$4a-3b$	$5a-3b$	$6a-3b$	$7a-3b$

In a Plane draw any number of Lines, parallel and equidiſtant, and others at right Angles to them, ſo as to divide the whole Space, as far as is neceſſary, into little equal Parallelograms. Aſſume any one of theſe, in which write the Term 0, and the Terms a, $2a$, $3a$, $4a$, &c. in the ſucceeding Parallelograms to the right hand, as alſo the Terms $-a$, $-2a$, $-3a$, &c. to the left hand. Over the Term 0, in the ſame Column, write the Terms b, $2b$, $3b$, $4b$, &c. ſucceſſively, and the Terms $-b$, $-2b$, $-3b$, &c. underneath. And theſe we may call primary Terms. Now to inſert its proper Term in any other aſſign'd Parallelogram, add the two primary Terms together, that ſtand over-againſt it each way, and write the Sum in the given Parallelogram. And thus all the Parallelograms being fill'd, as far as there is occaſion every way, the whole Space will

will become a Table, which may be called *a combined Arithmetical Progreſſion in plano*, compoſed of the two general Numbers *a* and *b*, of which theſe following will be the chief properties.

Any Row of Terms, parallel to the primary Series o, *a*, 2*a*, 3*a*, &c. will be an Arithmetical Progreſſion, whoſe common Difference is *a*; and it may be any ſuch Progreſſion at pleaſure. Any Row or Column parallel to the primary Series o, *b*, 2*b*, 3*b*, &c. will be an Arithmetical Progreſſion, whoſe common difference is *b*; and it may be any ſuch Progreſſion. If a ſtrait Ruler be laid on the Table, the Edge of which ſhall paſs thro' the Centers of any two Parallelograms whatever; all the Terms of the Parallelograms, whoſe Centers ſhall at the ſame time touch the Edge of the Ruler, will conſtitute an Arithmetical Progreſſion, whoſe common difference will conſiſt of two parts, the firſt of which will be ſome Multiple of *a*, and the other a Multiple of *b*. If this Progreſſion be ſuppos'd to proceed *inferiora verſus*, or from the upper Term or Parallelogram towards the lower; each part of the common difference may be ſeparately found, by ſubtracting the primary Term belonging to the lower, from the primary Term belonging to the upper Parallelogram. If this common difference, when found, be made equal to nothing, and thereby the Relation of *a* and *b* be determined; the Progreſſion degenerates into a Rank of Equals, or (if you pleaſe) it becomes an Arithmetical Progreſſion, whoſe common difference is infinitely little. In which caſe, if the Ruler be moved by a parallel motion, all the Terms of the Parallelograms, whoſe Centers ſhall at the ſame time be found to touch the Edge of the Ruler, ſhall be equal to each other. And if the motion of the Ruler be continued, ſuch Terms as at equal diſtances from the firſt ſituation are ſucceſſively found to touch the Ruler, ſhall form an Arithmetical Progreſſion. Laſtly, to come nearer to the caſe in hand, if any number of theſe Parallelograms be mark'd out and diſtinguiſh'd from the reſt, or aſſign'd promiſcuouſly and at pleaſure, through whoſe Centers, as before, the Edge of the Ruler ſhall ſucceſſively paſs in its parallel motion, beginning from any two (or more) initial or external Parallelograms, whoſe Terms are made equal; an Arithmetical Progreſſion may be found, which ſhall comprehend and take in all thoſe promiſcuous Terms, without any regard had to the Terms that are to be omitted. Theſe are ſome of the properties of this Table, or of a combined Arithmetical Progreſſion *in plano*, by which we may eaſily underſtand our Author's expedient, of Tabulating the given Equation, and may derive the neceſſary Conſequences from it.

For when the Root y is to be extracted out of a given Equation, conſiſting of the Powers of y and x any how combined together promiſcuouſly, with other known quantities, of which x is to be the Root of the Scale, (or Series,) as explain'd before; ſuch a value of y is to be found, as when ſubſtituted in the Equation inſtead of y, the whole ſhall be deſtroy'd, and become equal to nothing. And firſt the initial Term of the Series, or the firſt Approximation, is to be found, which in all caſes may be Analytically repreſented by Ax^m; or we may always put $y = Ax^m$, &c. So that we ſhall have $y^2 = A^2x^{2m}$, &c. $y^3 = A^3x^{3m}$, &c. $y^4 = A^4x^{4m}$, &c. And ſo of other Powers or Roots. Theſe when ſubſtituted in the Equation, and by that means compounded with the ſeveral Powers of x (or z) already found there, will form ſuch a combined Arithmetical Progreſſion *in plano* as is above deſcribed, or which may be reduced to ſuch, by making $a = m$ and $b = 1$. Theſe Terms therefore, according to the nature of the Equation, will be promiſcuouſly diſperſed in the Table; but the vacancies may always be conceived to be ſupply'd, and then it will have the properties before mention'd. That is, the Ruler being apply'd to two (or perhaps more) initial or external Terms, (for if they were not external, they could not be at the beginning of an Arithmetical Progreſſion, as is neceſſarily required,) and thoſe Terms being made equal, the general Index m will thereby be determined, and the general Coefficient A will alſo be known. If the external Terms made choice of are the loweſt in the Table, which is the caſe our Author purſues, the Powers of x will proceed by increaſing. But the higheſt may be choſen, and then a Series will be found, in which the Powers of x will proceed by decreaſing. And there may be other caſes of external Terms, each of which will commonly afford a Series. The initial Index being thus found, the other compound Indices belonging to the Equation will be known alſo, and an Arithmetical Progreſſion may be found, in which they are all comprehended, and conſequently the form of the Series will be known.

Or inſtead of Tabulating the Indices of the Equation, as above, it will be the ſame thing in effect, if we reduce the Terms themſelves to the form of a combined Arithmetical Progreſſion, as was ſhewn before. But then due care muſt be taken, that the Terms may be rightly placed at equal diſtances; otherwiſe the Ruler cannot be actually apply'd, to diſcover the Progreſſions of the Indices, as may be done in the Parallelogram.

For

For the ſake of greater perſpicuity, we will reduce our general Table, or combined Arithmetical Progreſſion *in plano*, to the particular caſe, in which $a = m$ and $b = 1$; which will then appear thus:

$-2m+6$	$-m+6$	$+6$	$m+6$	$2m+6$	$3m+6$	$4m+6$	$5m+6$	$6m+6$	$7m+6$
$-2m+5$	$-m+5$	$+5$	$m+5$	$2m+5$	$3m+5$	$4m+5$	$5m+5$	$6m+5$	$7m+5$
$-2m+4$	$-m+4$	$+4$	$m+4$	$2m+4$	$3m+4$	$4m+4$	$5m+4$	$6m+4$	$7m+4$
$-2m+3$	$-m+3$	$+3$	$m+3$	$2m+3$	$3m+3$	$4m+3$	$5m+3$	$6m+3$	$7m+3$
$-2m+2$	$-m+2$	$+2$	$m+2$	$2m+2$	$3m+2$	$4m+2$	$5m+2$	$6m+2$	$7m+2$
$-2m+1$	$-m+1$	$+1$	$m+1$	$2m+1$	$3m+1$	$4m+1$	$5m+1$	$6m+1$	$7m+1$
$-2m$	$-m$	0	m	$2m$	$3m$	$4m$	$5m$	$6m$	$7m$
$-2m-1$	$-m-1$	-1	$m-1$	$2m-1$	$3m-1$	$4m-1$	$5m-1$	$6m-1$	$7m-1$
$-2m-2$	$-m-2$	-2	$m-2$	$2m-2$	$3m-2$	$4m-2$	$5m-2$	$6m-2$	$7m-2$
$-2m-3$	$-m-3$	-3	$m-3$	$2m-3$	$3m-3$	$4m-3$	$5m-3$	$6m-3$	$7m-3$

Now the chief properties of this Table, ſubſervient to the preſent purpoſe, will be theſe. If any Parallelogram be ſelected, and another any how below it towards the right hand, and if their included Numbers be made equal, by determining the general Number m, which in this caſe will always be affirmative; alſo if the Edge of the Ruler be apply'd to the Centers of theſe two Parallelograms; all the Numbers of the other Parallelograms, whoſe Centers at the ſame time touch the Ruler, will likewiſe be equal to each other. Thus if the Parallelogram denoted by $m+4$ be ſelected, as alſo the Parallelogram $3m+2$; and if we make $m+4 = 3m+2$, we ſhall have $m = 1$. Alſo the Parallelograms $-m+6$, $m+4$, $3m+2$, $5m$, $7m-2$, &c. will at the ſame time be found to touch the Edge of the Ruler, every one of which will make 5, when $m = 1$.

And the ſame things will obtain if any Parallelogram be ſelected, and another any how below it towards the left-hand, if their included Numbers be made equal, by determining the general Number m, which in this caſe will be always negative. Thus if the Parallelogram denoted by $5m+4$ be ſelected, as alſo the Parallelogram $4m+2$; and if we make $5m+4 = 4m+2$, we ſhall have $m = -2$. Alſo the Parallelograms $6m+6$, $5m+4$, $4m+2$, $3m$, $2m-2$, &c. will

will be found at the ſame time to touch the Ruler, every one of which will make — 6, when $m = -2$.

The ſame things remaining as before, if from the firſt ſituation of the Ruler it ſhall move towards the right-hand by a parallel motion, it will continually arrive at greater and greater Numbers, which at equal diſtances will form an aſcending Arithmetical Progreſſion. Thus if the two firſt ſelected Parallelograms be $2m - 1 = 5m - 3$, whence $m = \frac{2}{3}$, the Numbers in all the correſponding Parallelograms will be $\frac{1}{3}$. Then if the Ruler moves towards the right-hand, into the parallel ſituation $3m + 1$, $6m - 1$, &c. theſe Numbers will each be 3. If it moves forwards to the ſame diſtance, it will arrive at $4m + 3$, $7m + 1$, &c. which will each be $5\frac{2}{3}$. If it moves forward again to the ſame diſtance, it will arrive at $5m + 5$, $8m + 3$, &c. which will each be $8\frac{1}{3}$. And ſo on. But the Numbers $\frac{1}{3}$, 3, $5\frac{2}{3}$, $8\frac{1}{3}$, &c. are in an Arithmetical Progreſſion whoſe common difference is $2\frac{2}{3}$. And the like, *mutatis mutandis*, in other circumſtances.

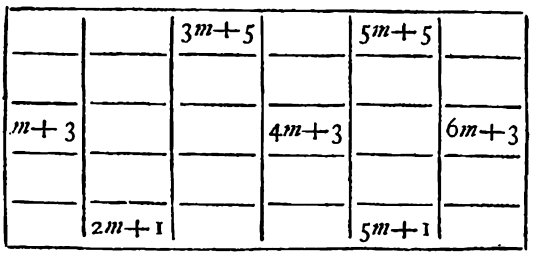
And hence it will follow *è contrà*, that if from the firſt ſituation of the Ruler, it moves towards the left-hand by a parallel motion, it will continually arrive at leſſer and leſſer Numbers, which at equal diſtances will form a decreaſing Arithmetical Progreſſion.

But in the other ſituation of the Ruler, in which it inclines downwards towards the left-hand, if it be moved towards the right-hand by a parallel motion, it will continually arrive at greater and greater Numbers, which at equal diſtances will form an increaſing Arithmetical Progreſſion. Thus if the two firſt ſelected Numbers or Parallelograms be $8m + 1 = 5m - 1$, whence $m = -\frac{2}{3}$, and the Numbers in all the correſponding Parallelograms will be $-4\frac{1}{3}$. If the Ruler moves upwards into the parallel ſituation $5m + 2$, $2m$, &c. theſe Numbers will each be $-1\frac{2}{3}$. If it move on at the ſame diſtance, it will arrive at $2m + 3$, $-m + 1$, &c. which will each be $1\frac{1}{3}$. If it move forward again to the ſame diſtance, it will arrive at $-m + 4$, $-4m + 2$, &c. which will each be $4\frac{2}{3}$. And ſo on. But the Numbers $-4\frac{1}{3}$, $-1\frac{1}{3}$, $1\frac{2}{3}$, $4\frac{2}{3}$, &c. or $-\frac{13}{3}$, $-\frac{4}{3}$, $\frac{5}{3}$, $\frac{14}{3}$, &c. are in an increaſing Arithmetical Progreſſion, whoſe common difference is $\frac{9}{3}$, or 3.

And hence it will follow alſo, if in this laſt ſituation of the Ruler it moves the contrary way, or towards the left-hand, it will continually arrive at leſſer and leſſer Numbers, which at equal diſtances will form a decreaſing Arithmetical Progreſſion.

Now if out of this Table we ſhould take promiſcuouſly any number of Parallelograms, in their proper places, with their reſpective Num-

Numbers included, neglecting all the reſt; we ſhould form ſome certain Figure, ſuch as this, of which theſe would be the properties.



The Ruler being apply'd to any two (or perhaps more) of the Parallelograms which are in the Ambit or Perimeter of the Figure, that is, to two of the external Parallelograms, and their Numbers being made equal, by determining the general Number m; if the Ruler paſſes over all the reſt of the Parallelograms by a parallel motion, thoſe Numbers which at the ſame time come to the Edge of the Ruler will be equal, and thoſe that come to it ſucceſſively will form an Arithmetical Progreſſion, if the Terms ſhould lie at equal diſtances; or at leaſt they may be reduced to ſuch, by ſupplying any Terms that may happen to be wanting.

Thus if the Ruler ſhould be apply'd to the two uppermoſt and external Parallelograms, which include the Numbers $3m + 5$ and $5m + 5$, and if they be made equal, we ſhall have $m = 0$, ſo that each of theſe Numbers will be 5. The next Numbers that the Ruler will arrive at will be $m + 3$, $4m + 3$, $6m + 3$, of which each will be 3. The laſt are $2m + 1$, $5m + 1$, of which each is 1. So that here $m = 0$, and the Numbers ariſing are 5, 3, 1, which form a decreaſing Arithmetical Progreſſion, the common difference of which is 2. And if there had been more Parallelograms, any how diſpoſed, their Numbers would have been comprehended by this Arithmetical Progreſſion, or at leaſt it might have been interpolated with other Terms, ſo as to comprehend them all, however promiſcuouſly and irregularly they might have been taken.

Thus ſecondly, if the Ruler be apply'd to the two external Parallelograms $5m + 5$ and $6m + 3$, and if theſe Numbers be made equal, we ſhall have $m = 2$, and the Numbers themſelves will be each 15. The three next Numbers which the Ruler will arrive at will

will be each 11, and the two laſt will be each 5. But the Numbers 15, 11, 5, will be comprehended in the decreaſing Arithmetical Progreſſion 15, 13, 11, 9, 7, 5, whoſe common difference is 2.

Thirdly, if the Ruler be apply'd to the two external Parallelograms $6m + 3$ and $5m + 1$, and if theſe Numbers be made equal, we ſhall have $m = -2$, and the Numbers will be each -9. The two next Numbers that the Ruler will arrive at will be each -5, the next will be -3, the next -1, and the laſt $+1$. All which will be comprehended in the aſcending Arithmetical Progreſſion -9, -7, -5, -3, -1, $+1$, whoſe common difference is 2.

Fourthly, if the Ruler be apply'd to the two loweſt and external Parallelograms $2m + 1$ and $5m + 1$, and if they be made equal, we ſhall have again $m = 0$, ſo that each of theſe Numbers will be 1. The next three Numbers that the Ruler will approach to, will each be 3, and the laſt 5. But the Numbers 1, 3, 5, will be comprehended in an aſcending Arithmetical Progreſſion, whoſe common difference is 2.

Fifthly, if the Ruler be apply'd to the two external Parallelograms $m + 3$ and $2m + 1$, and if theſe Numbers be made equal, we ſhall have $m = 2$, and the Numbers themſelves will be each 5. The three next Numbers that the Ruler will approach to will each be 11, and the two next will be each 15. But the Numbers 5, 11, 15, will be comprehended in the aſcending Arithmetical Progreſſion 5, 7, 9, 11, 13, 15, of which the common difference is 2.

Laſtly, if the Ruler be apply'd to the two external Parallelograms $3m + 5$ and $m + 3$, and if theſe Numbers be made equal, we ſhall have $m = -1$, and the Numbers themſelves will each be 2. The next Number to which the Ruler approaches will be 0, the two next are each -1, the next -3, the laſt -4. All which Numbers will be found in the deſcending Arithmetical Progreſſion 2, 1, 0, -1, -2, -3, -4, whoſe common difference is 1. And theſe ſix are all the poſſible caſes of external Terms.

Now to find the Arithmetical Progreſſion, in which all theſe reſulting Terms ſhall be comprehended; find their differences, and the greateſt common Diviſor of thoſe differences ſhall be the common difference of the Progreſſion. Thus in the fifth caſe before, the reſulting Numbers were 5, 11, 15, whoſe differences are 6, 4, and their greateſt common Diviſor is 2. Therefore 2 will be the common difference of the Arithmetical Progreſſion, which will include all the reſulting Numbers 5, 11, 15, without any ſuperfluous Terms. But the application of all this will be beſt apprehended from the Examples that are to follow.

30,

30. We have before given the form of this Equation, $y^6 - 5xy^5 + \frac{x^5}{a}y^4 - 7a^2x^2y^3 + 6a^3x^3 + b^4x^4 = 0$, when the Terms are diſpoſed according to a double or combined Arithmetical Scale, in order to its Solution. Or obſerving the ſame diſpoſition of the Terms, they may be inſerted in their reſpective Parallelograms, as the Table requires. Or rather, it may be ſufficient to tabulate the ſeveral Indices of x only, when they are derived as follows. Let Ax^m repreſent the firſt Term of the Series to be form'd for y, as before, or let $y = Ax^m$, &c. Then by ſubſtituting this for y in the given Equation, we ſhall have $A^6x^{6m} - 5A^5x^{5m+1} + \frac{A^4}{a}x^{4m+3} - 7a^2A^3x^{3m+2} + 6a^3x^3 + b^4x^4$, &c. $= 0$. Theſe Indices of x, when ſelected from the general Table, with their reſpective Parallelograms, will ſtand thus:

4						
3				$4m+3$		
		$2m+2$				
					$5m+1$	
						$6m$

Here if we would have an aſcending Series for the Root y, we may apply the Ruler to the three external Terms 3, $2m+2$, $6m$, which being made equal to each other, will give $m = \frac{1}{2}$, and each of the Numbers will be 3. The Ruler in its parallel motion will next arrive at $5m + 1$, or $3\frac{1}{2}$; then at 4; then at $4m + 3$, or 5; which Numbers will be comprehended in the Arithmetical Progreſſion 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, whoſe common difference is $\frac{1}{2}$. This therefore will be the common difference of the Progreſſion of the Indices, in the Series to be derived for y. So that now we intirely know the form of the Series, which will reſult from this Caſe. For if A, B, C, D, &c. be put to repreſent the ſeveral Coefficients of the Series in order, and as the firſt Index m is found to be $\frac{1}{2}$, and the common difference of the aſcending Series is alſo $\frac{1}{2}$, we ſhall have here $y = Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2$, &c.

As to the Value of the firſt Coefficient A, this is found by putting the initial or external Terms of the Parallelogram equal to nothing.

This here will give the Equation $A^6 - 7a^2A^2 + 6a^3 = 0$, which has theſe ſix Roots, $A = \pm\sqrt{a}$, $A = \pm\sqrt{2a}$, $A = \pm\sqrt{-3a}$, of which the two laſt are impoſſible, and to be rejected. Of the others any one may be taken for A, according as we would proſecute this or that Root of the Equation.

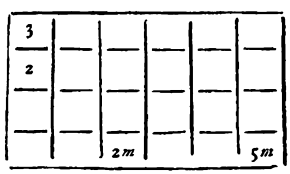
Now that this is a legitimate Method for finding the firſt Approximation Ax^m, may appear from conſidering, that when the Terms of the Equation are thus ranged, according to a double Arithmetical Scale, the initial or external Terms, (each Caſe in its turn,) become the moſt conſiderable of the Series, and the reſt continually decreaſe, or become of leſs and leſs value, according as they recede more and more from thoſe initial Terms. Conſequently they may be all rejected, as leaſt conſiderable, which will make thoſe initial or external Terms to be (nearly) equal to nothing; which Suppoſition gives the Value of A, or of Ax^m, for the firſt Approximation. And this Suppoſition is afterwards regularly purſued in the ſubſequent Operations, and proper Supplements are found, by means of which the remaining Terms of the Root are extracted.

We may try here likewiſe, if we can obtain a deſcending Series for the Root y, by applying the Ruler to the two external Terms $4m + 3$ and $6m$; which being made equal to each other, will give $m = \frac{3}{2}$, and hence each of the Numbers will be 9. The Ruler in its motion will next arrive at $5m + 1$, or $8\frac{1}{2}$. Then at $2m + 2$, or 5. Then at 4. And laſtly at 3. But theſe Numbers 9, $8\frac{1}{2}$, 5, 4, 3, will be comprehended in an Arithmetical Progreſſion, of which the common difference is $\frac{1}{2}$. So that the form of the Series here will be $y = Ax^{\frac{3}{2}} + Bx + Cx^{\frac{1}{2}} + Dx^0$, &c. But if we put the two external Terms equal to nothing, in order to obtain the firſt Approximation, we ſhall have $A^6 + \frac{A^4}{a} = 0$, or $A^2 + \frac{1}{a} = 0$, which will afford none but impoſſible Roots. So that we can have no initial Approximation from this ſuppoſition, and conſequently no Series.

But laſtly, to try the third and laſt caſe of external Parallelograms, we may apply the Ruler to 4 and $4m + 3$, which being made equal, will give $m = \frac{1}{4}$, and each of the Numbers will be 4. The next Number will be 3; the next $2m + 2$, or $2\frac{1}{2}$; the next $5m + 1$, or $2\frac{1}{4}$; the laſt will be $6m$, or $1\frac{1}{2}$. But the Numbers 4, 3, $2\frac{1}{2}$, $2\frac{1}{4}$, $1\frac{1}{2}$, will all be found in a decreaſing Arithmetical Progreſſion, whoſe common difference will be $\frac{1}{4}$. So that $Ax^{\frac{1}{4}} + Bx^0 + Cx^{-\frac{1}{4}} + Dx^{-\frac{1}{2}}$, &c. may repreſent the form of this Series, if the circumſtances of the

the Coefficients will allow of an Approximation from hence. But if we make the initial Terms equal to nothing, we ſhall have $\frac{A^4}{a} + b^2 = 0$, which will give none but impoſſible Roots. So that we can have no initial Approximation from hence, and conſequently no Series for the Root in this form.

31. The Equation $y^5 - by^2 + 9bx^3 - x^5 = 0$, when the Terms are diſpoſed according to a double Arithmetical Scale, will have the form as was ſhewn before; from whence it may be known, what caſes of external Terms there are to be try'd, and what will be the circumſtances of the ſeveral Series for the Root y, which may be derived from hence. Or otherwiſe more explicitely thus. Putting Ax^m for the firſt Term of the Series y, this Equation will become by Subſtitution $A^5x^{5m} - bA^2x^{2m} + 9bx^3 - x^5$, &c. $= 0$. So that if we take theſe Indices of x out of the general Table, they will ſtand as in the following Diagram.



Now in order to have an aſcending Series for y, we may apply the Ruler to the two external Parallelograms 2 and $2m$, which therefore being made equal, will give $m = 1$, and each of the Numbers will be 2. The Ruler then in its parallel progreſs will firſt come to 3, and then to $5m$, or 5. But the Numbers 2, 3, 5, are all contain'd in an aſcending Arithmetical Progreſſion, whoſe common difference is 1. Therefore the form of the Series will here be $y = Ax + Bx^2 + Cx^3$, &c. And to determine the firſt Coefficient A, we ſhall have the Equation $-bA^2x^2 + 9bx^2 = 0$, or $A^2 = 9$, that is $A = \pm 3$. So that either $+3x$, or $-3x$ may be the initial Approximation, according as we intend to extract the affirmative or the negative Root.

We ſhall have another caſe of external Terms, and perhaps another aſcending Series for y, by applying the Ruler to the Parallelograms $2m$ and $5m$, which Numbers being made equal, will give $m = 0$. (For by the way, when we put $2m = 5m$, we are not at liberty to argue by Diviſion, that $2 = 5$, becauſe this would bring us to an abſurdity. And the laws of Argumentation require, that no Abſurdities muſt be admitted, but when they are inevitable, and when they are of uſe to ſhew the falſity of ſome Suppoſition. We ſhould therefore here argue by Subtraction, thus: Becauſe $5m = 2m$, then $5m - 2m = 0$, or $3m = 0$, and therefore $m = 0$. This Caution I thought the more neceſſary, becauſe I have obſerved ſome,

who would lay the blame of their own Abſurdities upon the Analytical Art. But theſe Abſurdities are not to be imputed to the Art, but rather to the unſkilfulneſs of the Artiſt, who thus abſurdly applies the Principles of his Art.) Having therefore $m = 0$, we ſhall alſo have the Numbers $2m = 5m = 0$. The Ruler in its parallel motion will next arrive at 2; and then at 3. But the Numbers 0, 2, 3, will be comprehended in the Arithmetical Progreſſion 0, 1, 2, 3, whoſe common difference is 1. Therefore $y = A + Bx + Cx^2$, &c. will be the form of this Series. Now from the exterior Terms $A^3 - bA^2 = 0$, or $A^3 = b$, or $A = b^{\frac{1}{3}}$, we ſhall have the firſt Term of the Series.

There is another caſe of external Terms to be try'd, which poſſibly may afford a deſcending Series for y. For applying the Ruler to the Parallelograms 3 and $5m$, and making theſe equal, we ſhall have $m = \frac{3}{5}$, and each of theſe Numbers will be 3. Then the Ruler will come to 2; and laſtly $2m$, or $\frac{6}{5}$. But the Numbers 3, 2, $1\frac{1}{5}$, will be comprehended in a deſcending Progreſſion, whoſe common difference is $\frac{1}{5}$. Therefore the form of the Series will be $y = Ax^{\frac{3}{5}} + Bx^{\frac{2}{5}} + Cx^{\frac{1}{5}} + D$, &c. And the external Terms $A^5x^3 - x^3 = 0$ will give $A = 1$ for the firſt Coefficient. Now as the two former caſes will each give a converging Series for y in this Equation, when x is leſs than Unity; ſo this caſe will afford us a Series when x is greater than Unity; which will converge ſo much the faſter, the greater x is ſuppoſed to be.

32. We have already ſeen the form of this Equation $y^3 + axy + aay - x^3 - 2a^3 = 0$, when the Terms are diſpoſed according to a double Arithmetical Scale. And if we take the fictitious quantity Ax^m to repreſent the firſt Approximation to the Root y, we ſhall have by ſubſtitution $A^3x^{3m} + aAx^{m+1} + a^2Ax^m - x^3 - 2a^3$, &c. $= 0$. Theſe Terms, or at leaſt theſe Indices of x, being ſelected out of the general Table, will appear thus.

3			
	$m+1$		
0	m		$3m$

Now to obtain an aſcending Series for the Root y, we may apply the Ruler to the three external Terms 0, m, $3m$, which being made equal, will give $m = 0$. Therefore theſe Numbers are each 0. In the next place the Ruler will come to $m + 1$, or 1; and laſtly to 3. But the Numbers 0, 1, 3, are contain'd in the Arithmetical Progreſſion 0, 1, 2, 3, whoſe common difference is 1. Therefore the form of the Root is $y = A + Bx + Cx^2 + Dx^3$, &c. Now if the Equation $A^3 + a^2A - 2a^3 = 0$, (which is derived from the initial

initial Terms,) is divided by the factor $A^2 + aA + 2a^2$, it will give the Quotient $A - a = 0$, or $A = a$ for the initial Term of the Root y.

If we would alſo derive a deſcending Series for this Equation, we may apply the Ruler to the external Parallelograms 3, $3m$, which being made equal to each other, will give $m = 1$; alſo theſe Numbers will each be 3. Then the Ruler will approach to $m + 1$, or 2; then to m, or 1; laſtly to 0. But the Numbers 3, 2, 1, 0, are a decreaſing Arithmetical Progreſſion, of which the common difference is 1. So that the form of the Series will here be $y = Ax + B + Cx^{-1} + Dx^{-2}$, &c. And the Equation form'd by the external Terms will be $A^3x^3 - x^3 = 0$, or $A = 1$.

33. The form of the Equation $x^2y^5 - 3c^4xy^2 - c^5x^2 + c^7 = 0$, as expreſs'd by a combined Arithmetical Scale, we have already ſeen, which will eaſily ſhew us all the varieties of external Terms, with their other Circumſtances. But for farther illuſtration, putting Ax^m for the firſt Term of the Root y, we ſhall have by ſubſtitution $A^5x^{5m+2} - 3c^4A^2x^{2m+1} - c^5x^2 + c^7$, &c. $= 0$. Theſe Indices of x being tabulated, will ſtand thus.

2					$5m+2$
		$2m+1$			
0					

Now to have an aſcending Series, we muſt apply the Ruler to the two external Terms 0 and $5m + 2$, which being made equal, will give $m = -\frac{1}{5}$, and the two Numbers ariſing will be each 0. The next Number that the Ruler arrives at is $2m + 1$, or $\frac{1}{5}$; and the laſt is 2. But the Numbers 0, $\frac{1}{5}$, 2, will be found in an aſcending Arithmetical Progreſſion, whoſe common difference is $\frac{1}{5}$. Therefore $y = Ax^{-\frac{1}{5}} + Bx^{\frac{1}{5}} + C + Dx^{\frac{2}{5}}$, &c. will be the form of the Root. To determine the firſt Coefficient A, we ſhall have from the exterior Terms $A^5 + c^7 = 0$, which will give $A = -\sqrt[5]{c^7} = -c^{\frac{7}{5}}$. Therefore the firſt Term or Approximation to the Root will be $y = -\sqrt[5]{\frac{c^7}{x^2}}$, &c.

We may try if we can obtain a deſcending Series, by applying the Ruler to the two external Parallelograms, whoſe Numbers are 2 and $5m + 2$, which being made equal, will give $m = 0$, and theſe Numbers will each be 2. The Ruler will next arrive at $2m + 1$, or 1; and laſtly at 0. But the Numbers 2, 1, 0, form a deſcending Progreſſion, whoſe common difference is 1. So that the form of the Series will here be $y = A + Bx^{-1} + Cx^{-2}$, &c. And putting the initial

initial Terms equal to nothing, as they ſtand in the Equation, we ſhall have $A^3x^2 - c^3x^2 = 0$, or $A = c$, for the firſt Approximation to the Root. And this Series will be accommodated to the caſe of Convergency, when x is greater than c; as the other Series is accommodated to the other caſe, when x is leſs than c.

34. If the propoſed Equation be $8z^6y^3 + az^6y^2 - 27a^9 = 0$, it may be thus reſolved without any preparation. When reduced to our form, it will ſtand thus, $8z^6y^3 + az^6y^2 * \quad * \quad * - 27a^9 \} = 0$; and by putting $y = Az^m$, &c. it will become $8A^3z^{3m+6} + aA^2z^{2m+6} * \quad * \&c. * - 27a^9 \} = 0$.

The firſt caſe of external Terms will give $8A^3z^{3m+6} - 27a^9 = 0$, whence $3m + 6 = 0$, or $m = -2$. Theſe Indices or Numbers therefore will be each 0; and the other $2m + 6$ will be 2. But 0,2, will be in an aſcending Arithmetical Progreſſion, of which the common difference is 2. So that the form of the Series will be $y = Az^{-2} + B + Cz^2 + Dz^4$, &c. And becauſe $8A^3 = 27a^9$, or $2A = 3a^3$, it will be $A = \frac{3}{2}a^3$. Therefore the firſt Term or Approximation to the Root will be $\frac{3a^3}{2z^2}$.

But another caſe of external Terms will give $aA^2z^{2m+6} - 27a^9 = 0$, whence $2m + 6 = 0$, or $m = -3$. Theſe Indices or Numbers therefore will be each 0; and the other $3m + 6$ will be -3. But 0, -3, will be found in a deſcending Arithmetical Progreſſion, whoſe common difference is 3. So that the form of the Series will be $y = Az^{-3} + Bz^{-6} + Cz^{-9}$, &c. And becauſe $aA^2 = 27a^9$, 'tis $A = \pm 3\sqrt{3} \times a^4$, for the firſt Coefficient.

Laſtly, there is another caſe of external Terms, which may poſſibly afford us a deſcending Series, by making $8A^3z^{3m+6} + aA^2z^{2m+6} = 0$; whence $m = 0$. And the Numbers will be each equal to 6; the other Number, or Index of z, is 0. But 6, 0, will be in a deſcending Arithmetical Progreſſion, of which the common difference is 6. Therefore the form of the Series will be $y = A + Bz^{-6} + Cz^{-12}$, &c. Alſo becauſe $8A^3 + aA^2 = 0$, it is $A = -\frac{1}{8}a$ for the firſt Coefficient.

I ſhall produce one Example more, in order to ſhew what variety of Series may be derived from the Root in ſome Equations; as alſo to ſhew all the caſes, and all the varieties that can be derived, in the preſent ſtate of the Equation. Let us therefore aſſume this Equation, $y^3 - \frac{1^2x^2}{a} + x^3 - \frac{a^3x^2}{2} + \frac{c^6}{y^3} - \frac{a^7}{1^2x^2} + \frac{c^6}{x^3} - \frac{a^3y^2}{x^2} + a^3 = 0$, or rather $y^3 - a^{-1}y^2x^2 + x^3 - a^3y^{-2}x^2 + a^6y^{-3} - a^7y^{-2}x^{-2} + a^6x^{-3} - a^3y^2x^{-2} + a^3 = 0$. Which if we make $y = Ax^m$, &c. and diſpoſe

diſpoſe the Terms according to a combined Arithmetical Progreſſion, will appear thus:

$$\left.\begin{array}{ccccc} & & +a^6x^{-3} & & \\ -a^3A^2x^{2m-2} & * & * & * & -a^7A^{-2}x^{-2m-2} \\ * & * & * & * & * \\ A^3x^{3m} \quad * & * & +a^3 & * & * \quad +a^6A^{-3}x^{-3m}, \&c. \\ * & * & * & * & * \\ -a^{-1}A^2x^{2m+2} & * & * & * & -a^3A^{-2}x^{-2m+2} \\ & & +x^3 & & \end{array}\right\} = 0.$$

Now here it is plain by the diſpoſition of the Terms, that the Ruler can be apply'd eight times, and no oftner, or that there are eight caſes of external Terms to be try'd, each of which may give a Series for the Root, if the Coefficients will allow it, of which four will be aſcending, and four deſcending. And firſt for the four caſes of aſcending Series, in which the Root will converge by the aſcending Powers of x; and afterwards for the other four caſes, when the Series converges by the deſcending Powers of x.

I. Apply the Ruler, or, (which is the ſame thing,) aſſume the Equation $a^6A^{-3}x^{-3m} - a^7A^{-2}x^{-2m-2} = 0$, which will give $-3m = -2m-2$, or $m = 2$; alſo $A = \frac{1}{a}$. The Number reſulting from theſe Indices is -6. But the Ruler in its parallel motion will next come to the Index -3; then to $-2m+2$, or -2; then to 0; then to $2m-2$, or 2; then to 3; and laſtly to $3m$ and $2m+2$, or 6. But the Numbers $-6, -3, -2, 0, 2, 3, 6$, are in an aſcending Arithmetical Progreſſion, of which the common difference is 1; and therefore the form of the Series will be $y = Ax^2 + Bx^3 + Cx^4$, &c. and its firſt Term will be $\frac{x^2}{a}$.

II. Aſſume the Equation $a^6x^{-3} - a^7A^{-2}x^{-2m-2} = 0$, which will give $-3 = -2m-2$, or $m = \frac{1}{2}$; alſo $A = \pm a^{\frac{1}{2}}$. The Number reſulting hence is -3; the next will be $-3m$, or $-1\frac{1}{2}$; the next $2m-2$, or -1; the next 0; the next $-2m+2$, or 1; the next $3m$, or $1\frac{1}{2}$; the two laſt $2m+2$ and 3, are each 3. But the Numbers $-3, -1\frac{1}{2}, -1, 0, 1, 1\frac{1}{2}, 3$, will be found in an aſcending Arithmetical Progreſſion, of which the common difference is $\frac{1}{2}$; and therefore the form of the Series will be $y = Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2$, &c. and its firſt Term will be $\pm\sqrt{ax}$.

III.

III. Aſſume the Equation $a^6x^{-3} - a^3A^2x^{2m-2} = 0$, which will give $-3 = 2m-2$, or $m = -\frac{1}{2}$; alſo $A = \pm a^{\frac{3}{2}}$. The Number reſulting is -3; the next $3m$, or $-1\frac{1}{2}$; the next $-2m-2$, or -1; the next 0; the next $2m+2$, or 1; the next $-3m$, or $1\frac{1}{2}$; the two laſt 3 and $-2m+2$, which are each 3. But the Numbers -3, $-1\frac{1}{2}$, -1, 0, 1, $1\frac{1}{2}$, 3, will be all comprehended in an aſcending Arithmetical Progreſſion, of which the common difference is $\frac{1}{2}$; and therefore the form of the Series will be $y = Ax^{-\frac{1}{2}} + B + Cx^{\frac{1}{2}} + Dx$, &c. and the firſt Term will be $\pm a^{\frac{3}{2}}x^{-\frac{1}{2}}$, or $\pm a\sqrt{\frac{a}{x}}$.

IV. Aſſume the Equation $A^3x^{3m} - a^3A^2x^{2m-2} = 0$, which will give $3m = 2m-2$, or $m = -2$; alſo $A = a^3$. The Number reſulting is -6; the next will be -3; the next $2m+2$, or -2; the next 0; the next $-2m-2$, or 2; the next 3; the two laſt $-3m$ and $-2m+2$, each of which is 6. But the Numbers -6, -3, -2, 0, 2, 3, 6, belong to an aſcending Arithmetical Progreſſion, of which the common difference is 1. Therefore the form of the Series will be $y = Ax^{-2} + Bx^{-1} + C + Dx$, &c. and its firſt Term will be $\frac{a^3}{x^2}$.

The four deſcending Series are thus derived.

I. Aſſume the Equation $A^3x^{3m} - a^{-1}A^2x^{2m+2} = 0$, which will give $3m = 2m+2$, or $m = 2$; alſo $A = \frac{1}{a}$. The Number reſulting is 6; the next will be 3; the next $2m-2$, or 2; the next 0; the next $-2m+2$, or -2; the next -3; the two laſt $-3m$ and $-2m-2$, each of which is -6. But the Numbers 6, 3, 2, 0, -2, -3, -6, belong to a deſcending Arithmetical Progreſſion, of which the common difference is 1. Therefore the form of the Series will be $y = Ax^2 + Bx + C + Dx^{-1}$, &c. and the firſt Term will be $\frac{x^2}{a}$.

II. Aſſume the Equation $x^5 - a^{-1}A^2x^{2m+2} = 0$, which will give $2m+2 = 3$, or $m = \frac{1}{2}$; alſo $A = \pm a^{\frac{1}{2}}$. The Number reſulting is 3; the next will be $3m$, or $1\frac{1}{2}$; the next $-2m+2$, or 1; the next 0; the next $2m-2$, or -1; the next $-3m$, or $-1\frac{1}{2}$; the two laſt -3 and $-2m-2$ are each -3. But the Numbers 3, $1\frac{1}{2}$, 1, 0, -1, $-1\frac{1}{2}$, -3, belong to a deſcending Arithmetical Progreſſion, of which the common difference is $\frac{1}{2}$. Therefore the form of the Series will be $y = Ax^{\frac{1}{2}} + Bx^0 + Cx^{-\frac{1}{2}} + Dx^{-1}$, &c. and the firſt Term will be $\pm\sqrt{ax}$.

III.

III. Aſſume the Equation $x^3 - a^1A^{-2}x^{-2m+2} = 0$, which will give $3 = -2m+2$, or $m = -\frac{1}{2}$; alſo $A = \pm a^{\frac{1}{2}}$. The Number reſulting from hence is 3; the next will be $-3m$, or $1\frac{1}{2}$; the next $2m+2$, or 1; the next 0; the next $-2m-2$, or -1; the next $3m$, or $-1\frac{1}{2}$; the two laſt -3 and $2m-2$, each of which are -3. But the Numbers $3, 1\frac{1}{2}, 1, 0, -1, -1\frac{1}{2}, -3$, are comprehended in a deſcending Arithmetical Progreſſion, of which the common difference is $\frac{1}{2}$. Therefore the form of the Series will be $y = Ax^{-\frac{1}{2}} + Bx^{-1} + Cx^{-\frac{3}{2}} + Dx^{-2}$, &c. and the firſt Term will be $\pm a^{\frac{1}{2}}x^{-\frac{1}{2}}$, or $\pm a\sqrt{\frac{a}{x}}$.

IV. Laſtly, aſſume the Equation $a^6A^{-3}x^{-3m} - a^5A^{-2}x^{-2m+2} = 0$, which will give $-3m = -2m+2$, or $m = -2$; alſo $A = a^1$. The Number reſulting is 6; the next will be 3; the next $-2m-2$, or 2; the next 0; the next $2m+2$, or -2; the next -3; the two next $3m$ and $2m-2$, are each -6. But the Numbers $6, 3, 2, 0, -2, -3, -6$, belong to a deſcending Arithmetical Progreſſion, of which the common difference is 1. Therefore the form of the Series will be $y = Ax^{-2} + Bx^{-3} + Cx^{-4} + Dx^{-5}$, &c. and the firſt Term is $\frac{a^3}{x^2}$.

And this may ſuffice in all Equations of this kind, for finding the forms of the ſeveral Series, and their firſt Approximations. Now we muſt proceed to their farther Reſolution, or to the Method of finding all the reſt of the Terms ſucceſſively.

SECT. V. *The Reſolution of Affected Specious Equations, proſecuted by various Methods of Analyſis.*

35. HITHERTO it has been ſhewn, when an Equation is propoſed, in order to find its Root, how the Terms of the Equation are to be diſpoſed in a two-fold regular ſucceſſion, ſo as thereby to find the initial Approximations, and the ſeveral forms of the Series in all their various circumſtances. Now the Author proceeds in like manner to diſcover the ſubſequent Terms of the Series, which may be done with much eaſe and certainty, when the form of the Series is known. For this end he finds Reſidual or Supplemental Equations, in a regular ſucceſſion alſo, the Roots of which are a continued Series of Supplements to the Root required. In every one of which Supplemental Equations the Approximation is

found, by rejecting the more remote or leſs conſiderable Terms, and ſo reducing it to a ſimple Equation, which will give a near Value of the Root. And thus the whole affair is reduced to a kind of Compariſon of the Roots of Equations, as has been hinted already. The Root of an Equation is nearly found, and its Supplement, which ſhould make it compleat, is the Root of an inferior Equation, the Supplement of which is again the Root of an inferior Equation; and ſo on for ever. Or retaining that Supplement, we may ſtop where we pleaſe.

36. The Author's Diagram, or his Proceſs of Reſolution, is very eaſy to be underſtood; yet however it may be thus farther explain'd. Having inſerted the Terms of the given Equation in the left-hand Column, (which therefore are equal to nothing, as are alſo all the ſubſequent Columns,) and having already found the firſt Approximation to the Root to be a; inſtead of the Root y he ſubſtitutes its equivalent $a+p$ in the ſeveral Terms of the Equation, and writes the Reſult over-againſt them reſpectively, in the right-hand Margin. Theſe he collects and abbreviates, writing the Reſult below, in the left-hand Column; of which rejecting all the Terms of too high a compoſition, he retains only the two loweſt Terms $4a^2p+a^2x=0$, which give $p=-\frac{1}{4}x$ for the ſecond Term of the Root. Then aſſuming $p=-\frac{1}{4}x+q$, he ſubſtitutes this in the deſcending Terms to the left-hand, and writes the Reſult in the Column to the right-hand. Theſe he collects and abbreviates, writing the Reſult below in the left-hand Column. Of which rejecting again all the higher Terms, he retains only the two loweſt $4a^2q-\frac{1}{16}ax^2=0$, which give $q=\frac{x^2}{64a}$ for the third Term of the Root. And ſo on.

Or in imitation of a former Proceſs, (which may be ſeen, pag. 165.) the Reſolution of this, and all ſuch like Equations, may be thus perform'd.

$$\begin{array}{l} y^3+a^2y=2a^3 \\ \quad +axy+x^3 \end{array} = \left(\text{if } y=a+p\right) \begin{array}{l} a^3+3a^2p+3ap^2+p^3 \\ \quad +a^3+a^2p \\ \quad +a^2x+axp \end{array} \Big\} \text{Or collecting and expunging,}$$

$$\begin{array}{l} 4a^2p+3ap^2+p^3=-a^2x \\ \quad +axp \quad +x^3 \end{array} = \left(\text{if } p=-\tfrac{1}{4}x+q\right) \begin{array}{l} -a^2x+4a^2q \\ \quad -\frac{1}{4}ax^2+axq \\ \quad +\frac{3}{16}ax^2-\frac{3}{2}axq+3aq^2 \\ \quad -\frac{1}{64}x^3+\frac{3}{16}x^2q-\frac{3}{4}xq^2+q^3 \end{array} \Big\},$$

Or collecting and expunging,

$$\begin{array}{l} 4a^2q+3aq^2+q^3 \\ -\frac{1}{2}axq-\frac{3}{4}xq^2 \\ +\frac{3}{16}x^2q \end{array} = \frac{1}{16}ax^2 = \left(\text{if } q=\frac{x^2}{64a}+r\right) \text{ &c.} \quad +\frac{65}{64}x^5$$

By which Proceſs the Root will be found $y=a-\frac{1}{4}x+\frac{x^2}{64a}$, &c.

Or

Or in imitation of the Method before taught, (pag. 178, &c.) we may thus reſolve the firſt Supplemental Equation of this Example; *viz.* $4a^2p + axp + 3ap^2 + p^3 = -a^2x + x^3$; where the Terms muſt be diſpos'd in the following manner. But to avoid a great deal of unneceſſary prolixity, it may be here obſerved, that $y = a$, &c. briefly denotes, that a is the firſt Term of the Series, to be derived for the Value of y. Alſo $y = * - \frac{1}{4}x$, &c. inſinuates, that $-\frac{1}{4}x$ is the ſecond Term of the ſame Series y. Alſo $y = * * + \frac{x^2}{64a}$, &c. inſinuates, that $+\frac{x^2}{64a}$ is the third Term of the Series y, without any regard to the other Terms. And ſo for all the ſucceeding Terms; and the like is to be underſtood of all other Series whatever.

$$\left.\begin{array}{rl} 4a^2p & = -a^2x \quad * \quad + x^3 \\ & \quad + \frac{1}{16}ax^2 + \frac{3x^3}{128} + \frac{509x^4}{4096a}, \text{ &c.} \\ +axp & ---------- \quad \frac{1}{4}ax^2 + \frac{1}{64}x^3 + \frac{131x^4}{512a}, \text{ &c.} \\ +3ap^2 & ---------+ \frac{3}{16}ax^2 - \frac{3x^3}{128} - \frac{1569x^4}{4096a}, \text{ &c.} \\ +p^3 & ---------- - \frac{1}{64}x^3 + \frac{3x^4}{1024a}, \text{ &c.} \end{array}\right\}$$

$$p = -\tfrac{1}{4}x + \tfrac{x^2}{64a} + \tfrac{131x^3}{512a^2} + \tfrac{509x^4}{16384a^3}, \text{ &c.}$$

To explain this Proceſs, it may be obſerved, that here $-a^2x$ is made the firſt Term of the Series, into which $4a^2p$ is to be reſolved; or $4a^2p = -a^2x$, &c. and therefore $p = -\frac{1}{4}x$, &c. which is ſet down below. Then is $+axp = -\frac{1}{4}ax^2$, &c. and (by ſquaring) $+3ap^2 = +\frac{3}{16}ax^2$, &c. each of which are ſet down in their proper Places. Theſe Terms being collected, will make $-\frac{1}{16}ax^2$, which with a contrary Sign muſt be ſet down for the ſecond Term of $4a^2p$; or $4a^2p = * + \frac{1}{16}ax^2$, &c. and therefore $p = * + \frac{x^2}{64a}$, &c. Then $axp = * + \frac{x^3}{64}$, &c. and (by ſquaring) $3ap^2 = * - \frac{3x^3}{128}$, &c. and (by cubing) $p^3 = -\frac{1}{64}x^3$, &c. Theſe being collected will make $-\frac{3x^3}{128}$, to be wrote down with a contrary Sign; and this, together with x^3, one of the Terms of the given Equation, will make $4a^2p = * * + \frac{131}{128}x^3$, &c. and therefore $p = * * + \frac{131x^3}{512a^2}$, &c. Then $axp = * * + \frac{131x^4}{512a}$, &c. and (by ſquaring) $3ap^2 = * *$

$-\frac{1569x^4}{4096a}$, &c. and (by cubing) $p^3 = * + \frac{3x^4}{1024a}$, &c. all which being collected with a contrary Sign, will make $4a^2p = *** + \frac{509x^4}{4096a}$, &c. and therefore $p = *** + \frac{509x^4}{16384a}$, &c. And by the ſame Method we may continue the Extraction as far as we pleaſe.

The *Rationale* of this Proceſs has been already deliver'd, but as it will be of frequent uſe, I ſhall here mention it again, in ſomewhat a different manner. The Terms of the Equation being duly order'd, ſo as that the Terms involving the Root, (which are to be reſolved into their reſpective Series,) being all in a Column on one ſide, and the known Terms on the other ſide; any adventitious Terms may be introduced, ſuch as will be neceſſary for forming the ſeveral Series, provided they are made mutually to deſtroy one another, that the integrity of the Equation may be thereby preſerved. Theſe adventitious Terms will be ſupply'd by a kind of Circulation, which will make the work eaſy and pleaſant enough; and the neceſſary Terms of the ſimple Powers or Roots, of ſuch Series as compoſe the Equation, muſt be derived one by one, by any of the foregoing Theorems.

Or if we are willing to avoid too many, and too high Powers in theſe Extractions, we may proceed in the following manner. The Example ſhall be the ſame Supplemental Equation as before, which may be reduced to this form, $4a^2 + ax + 3ap + pp \times p = -a^2x * + x^3$, of which the Reſolution may be thus:

$$\begin{array}{l}
4a^2 + ax \\
+ 3ap \text{ - - - - } - \frac{3}{4}ax + \frac{3}{64}x^2 + \frac{393x^3}{512a}, \text{ \&c.} \\
+ p^2 \text{ - - - - - - - - - } + \frac{1}{16}x^2 - \frac{x^3}{128a}, \text{ \&c.} \\
\hline
4a^2 + \frac{1}{4}ax + \frac{7}{64}x^2 + \frac{389x^3}{512a}, \text{ \&c.} \\
p = -\frac{1}{8}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}, \text{ \&c.} \\
\hline
-a^2x \quad * \quad + \quad x^3 \quad *
\end{array}$$

The Terms $4a^2 + ax + 3ap + pp$ I call the aggregate Factor, of which I place the known part or parts $4a^2 + ax$ above, and the unknown parts $3ap + pp$ in a Column to the left-hand, ſo as that their reſpective Series, as they come to be known, may be placed regularly over-againſt them. Under theſe a Line is drawn, to receive

the aggregate Series beneath it, which is form'd by the Terms of the aggregate Factor, as they become known. Under this aggregate Series comes the ſimple Factor p, or the ſymbol of the Root to be extracted, as its Terms become known alſo. Laſtly, under all are the known Terms of the Equation in their proper places. Now as theſe laſt Terms (becauſe of the Equation) are equivalent to the Product of the two Species above them; from this conſideration the Terms of the Series p are gradually derived, as follows.

Firſt, the initial Term $4a^2$ (of the aggregate Series) is brought down into its place, as having no other Term to be collected with it. Then becauſe this Term, multiply'd by the firſt Term of p, ſuppoſe q, is equal to the firſt Term of the Product, that is, $4a^2q = -a^2x$, it will be $q = -\frac{1}{4}x$, or $p = -\frac{1}{4}x$, &c. to be put down in its place. Thence we ſhall have $3ap = -\frac{3}{4}ax$, &c. which together with $+ax$ above, will make $+\frac{1}{4}ax$ for the ſecond Term of the aggregate Series. Now if we ſuppoſe r to repreſent the ſecond Term of p, and to be wrote in its place accordingly; by croſs-multiplication we ſhall have $4a^2r - \frac{1}{16}ax^2 = 0$, becauſe the ſecond Term of the Product is abſent, or $= 0$. Therefore $r = \frac{x^2}{64a}$, which may now be ſet down in its place. And hence $3ap = * + \frac{3}{64}x^2$, &c. and $p^2 = \frac{1}{16}x^2$, &c. which being collected will make $\frac{7}{64}x^2$, for the third Term of the aggregate Factor. Now if we ſuppoſe s to repreſent the third Term of p, then by croſs-multiplication, (or by our Theorem for Multiplication of infinite Series,) $4a^2s + \frac{x^3}{256} - \frac{7x^3}{256} = x^3$; (for x^3 is the third Term of the Product.) Therefore $s = \frac{131x^3}{512a^2}$, to be ſet down in its place. Then $3ap = ** + \frac{393x^3}{512a}$, &c. and $p^2 = * - \frac{x^3}{128a}$, &c. which together will make $+\frac{389x^3}{512a}$ for the fourth Term of the aggregate Series. Then putting t to repreſent the fourth Term of p, by multiplication we ſhall have $4a^2t + \frac{131x^4}{2048a} + \frac{7x^4}{4096a} - \frac{389x^4}{2048a} = 0$, whence $t = \frac{509x^4}{16384a^3}$, to be ſet down in its place. If we would proceed any farther in the Extraction, we muſt find in like manner the fourth Term of the Series $3ap$, and the third Term of p^2, in order to find the fifth Term of the aggregate Series. And thus we may eaſily and ſurely carry on the Root to what degree of accuracy we pleaſe, without any danger of computing any ſuperfluous Terms; which will be no mean advantage of theſe Methods.

Or-

Or we may proceed in the following manner, by which we ſhall avoid the trouble of raiſing any ſubſidiary Powers at all. The Supplemental Equation of the ſame Example, $4a^2p + axp + 3ap^2 + p^3 = -a^2x + x^3$, (and all others in imitation of this,) may be reduced to this form, $\overline{4a^2 + ax + \overline{3a + p} \times p} \times p = -a^2x + x^3$, which may be thus reſolved.

$$
\begin{array}{rl}
 & 4a^2 + ax \\
+\overline{3a+p} & \text{-----} + 3a - \frac{1}{4}x + \frac{x^2}{64a}, \text{ \&c.} \\
\times p & \text{-----} - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}, \text{ \&c.} \\
\hline
 & 4a^2 + \frac{1}{4}ax + \frac{7}{64}x^2 + \frac{389x^3}{512a}, \text{ \&c.} \\
\times p = & -\frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}, \text{ \&c.} \\
\hline
 & -a^2x \quad * \quad + x^3 \quad *
\end{array}
$$

The Terms being diſpoſed as in this Paradigm, bring down $4a^2$ for the firſt Term of the aggregate Series, as it may ſtill be call'd, and ſuppoſe q to repreſent the firſt Term of the Series p. Then will $4a^2q = -a^2x$, or $q = -\frac{1}{4}x$, which is to be wrote every where for the firſt Term of p. Multiply $+3a$ by $-\frac{1}{4}x$ for the firſt Term of $\overline{3a+p} \times p$, with which product $-\frac{3}{4}ax$ collect the Term above, or $+ax$; the Reſult $\frac{1}{4}ax$ will be the ſecond Term of the aggregate Series. Then let r repreſent the ſecond Term of p, and we ſhall have by Multiplication $4a^2r - \frac{1}{16}ax^2 = 0$, or $r = \frac{x^2}{64a}$, to be wrote every where for the ſecond Term of p. Then as above, by croſs-multiplication we ſhall have $3a \times \frac{x^2}{64a} + \frac{1}{16}x^2 = \frac{7}{64}x^2$ for the third Term of the aggregate Series. Again, ſuppoſing s to repreſent the third Term of p, we ſhall have by Multiplication, (ſee the Theorem for that purpoſe,) $4a^2s + \frac{x^3}{256} - \frac{7x^3}{256} = x^3$, that is, $s = \frac{131x^3}{512a^2}$, to be wrote every where for the third Term of p. And by the ſame way of Multiplication the fourth Term of the aggregate Series will be found to be $\frac{389x^3}{512a}$, which will make the fourth Term of p to be $\frac{509x^4}{16384a^3}$. And ſo on.

Among all this variety of Methods for theſe Extractions, we muſt not omit to ſupply the Learner with one more, which is common

mon and obvious enough, but which ſuppoſes the form of the Series required to be already known, and only the Coefficients to be unknown. This we may the better do here, becauſe we have already ſhewn how to determine the form and number of ſuch Series, in any caſe propoſed. This Method conſiſts in the aſſumption of a general Series for the Root, ſuch as may conveniently repreſent it, by the ſubſtitution of which in the given Equation, the general Coefficients may be determined. Thus in the preſent Equation $y^3 + axy + aay - x^3 - 2a^3 = 0$, having already found (pag. 204.) the form of the Root or Series to be $y = A + Bx + Cx^2$, &c. by the help of any of the Methods for Cubing an infinite Series, we may eaſily ſubſtitute this Series inſtead of y in this Equation, which will then become

$$\left.\begin{array}{llllll} A^3 + 3A^2Bx + 3AB^2x^2 + B^3x^3 + 3AC^2x^4, \text{ \&c.} & & & & & \\ \quad + 3A^2C + 6ABC + 3B^2C & & & & & \\ \quad\quad + 3A^2D + 6ABD & & & & & \\ \quad\quad\quad + 3A^2E & & & & & \\ + aAx + aBx^2 + aCx^3 + aDx^4, \text{ \&c.} & & & & & \\ + a^2A + a^2Bx + a^2Cx^2 + a^2Dx^3 + a^2Ex^4, \text{ \&c.} & & & & & \\ - 2a^3 \quad * \quad * \quad - x^3 \quad * & & & & & \end{array}\right\} = 0.$$

Now becauſe x is an indeterminate quantity, and muſt continue ſo to be, every Term of this Equation may be ſeparately put equal to nothing, by which the general Coefficients A, B, C, D, &c. will be determined to congruous Values; and by this means the Root y will be known. Thus, (1.) $A^3 + a^2A - 2a^3 = 0$, which will give $A = a$, as before. (2.) $3A^2B + aA + a^2B = 0$, or $B = \frac{-aA}{3A^2+a^2} = -\frac{1}{4}$. (3.) $3AB^2 + 3A^2C + aB + a^2C = 0$, or $C = -\frac{3AB^2+aB}{3A^2+a^2} = \frac{1}{64a}$. (4.) $B^3 + 6ABC + 3A^2D + aC + a^2D - 1 = 0$, or $D = \frac{131}{512a^2}$. (5.) $3AC^2 + 3B^2C + 6ABD + 3A^2E + aD + a^2E = 0$, or $E = \frac{509}{16384a^3}$. And ſo on, to determine F, G, H, &c. Then by ſubſtituting theſe Values of A, B, C, D, &c. in the aſſumed Root, we ſhall have the former Series $y = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$, &c.

Or laſtly, we may conveniently enough reſolve this Equation, or any other of the ſame kind, by applying it to the general Theorem, pag. 190. for extracting the Roots of any affected Equations in Numbers. For this Equation being reduced to this form $y^3 * + a^2 + ax$

$\times y - 2a^3 + x^3 \times y^0 = 0$, we ſhall have there $m = 3$. And inſtead of the firſt, ſecond, third, fourth, fifth, &c. Coefficients of the Powers of y in the Theorem, if we write $1, 0, aa + ax, -2a^3 - x^3, 0$, &c. reſpectively; and if we make the firſt Approximation $\frac{A}{B} = \frac{a}{1}$, or $A = a$ and $B = 1$; we ſhall have $\frac{4a^3 + x^3}{4a^2 + ax}$ for a nearer Approximation to the Root. Again, if we make $A = 4a^3 + x^3$, and $B = 4a^2 + ax$, by Subſtitution we ſhall have the Fraction $\frac{256a^9 + 96a^8x + 24a^7x^2 + 114a^6x^3 + 48a^5x^4 + 12a^4x^5 + 25a^3x^6 \ldots + 2x^9}{256a^8 + 160a^7x + 60a^6x^2 + 109a^5x^3 + 25a^4x^4 \ldots + 12a^2x^6 + 3ax^7}$ for a nearer Approximation to the Root. And taking this Numerator for A, and the Denominator for B, we ſhall approach nearer ſtill. But this laſt Approximation is ſo near, that if we only take the firſt five Terms of the Numerator, and divide them by the firſt five Terms of the Denominator, (which, if rightly managed, will be no troubleſome Operation,) we ſhall have the ſame five Terms of the Series, ſo often found already.

And the Theorem will converge ſo faſt on this, and ſuch like occaſions, that if we here take the firſt Approximation $A = a$, (making $B = 1$,) we ſhall have $y = \frac{4a^3 + x^3}{4a^2 + ax}$, &c. $= a - \frac{1}{4}x$, &c. And if again we make this the ſecond Approximation, or $A = a - \frac{1}{4}x$, (making $B = 1$,) we ſhall have $y = \frac{2A^3 + 2a^3 + x^3}{3A^3 + a^2 + ax}$, &c. $= \frac{4a^3 - \frac{3}{2}a^2x + \frac{3}{8}ax^2 + \frac{11}{32}x^3}{4a^2 - \frac{1}{2}ax + \frac{3}{16}x^2}$, &c. $= a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c. And if again we make this the third Approximation, or $A = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c. (making $B = 1$,) we ſhall have the Value of the true Root to eight Terms at this Operation. For every new Operation will double the number of Terms, that were found true by the laſt Operation.

To proceed ſtill with the ſame Equation; we have found before, pag. 205, that we might likewiſe have a deſcending Series in this form, $y = Ax + B + Cx^{-1}$, &c. for the Root y, which we ſhall extract two or three ways, for the more abundant exemplification of this Doctrine. It has been already found, that $A = 1$, or that x is the firſt Approximation to the Root. Make therefore $y = x + p$, and ſubſtitute this in the given Equation $y^3 + axy + aay - x^3 - 2a^3 = 0$, which will then become $3x^2p + axp + a^2p + 3xp^2 + p^3 + ax^2 + a^2x - 2a^3 = 0$. This may be reduced to this form $3x^2 + ax + a^2 + 3xp + p^2 \times p = -ax^2 - a^2x + 2a^3$, and may be reſolved as follows.

$3x^2$

$$
\begin{array}{rl}
 & 3x^2 + ax + a^2 \quad * \\
+ 3xp & \text{- - - - -} - ax - a^2 + \frac{55a^3}{27x}\ , \text{ \&c.} \\
+ \quad p^2 & \text{- - - - - - -} + \frac{1}{9}a^2 + \frac{2a^3}{9x}\ , \text{ \&c.} \\
\hline
 & 3x^2 \quad * + \frac{1}{9}a^2 + \frac{61a^3}{27x}\ , \text{ \&c.} \\
p = & -\frac{1}{3}a - \frac{a^2}{3x} + \frac{55a^3}{81x^2} + \frac{64a^4}{243x^3}\ , \text{ \&c.} \\
\hline
 & -ax^2 - a^2x + 2a^3 \quad * \\
\end{array}
$$

The Terms of the aggregate Factor, as alſo the known Terms of the Equation, being diſpoſed as in the Paradigm, bring down $3x^2$ for the firſt Term of the aggregate Series; and ſuppoſing q to repreſent the firſt Term of the Series p, it will be $3x^2q = -ax^2$, or $q = -\frac{1}{3}a$, for the firſt Term of p. Therefore $-ax$ will be the firſt Term of $3xp$, to be put down in its place. This will make the ſecond Term of the aggregate Series to be nothing; ſo that if r repreſent the ſecond Term of p, we ſhall have by multiplication $3x^2r = -a^2x$, or $r = -\frac{a^2}{3x}$ for the ſecond Term of p, to be put down in its place. Then will $-a^2$ be the ſecond Term of $3xp$, as alſo $\frac{1}{9}a^2$ will be the firſt Term of p^2, to be ſet down each in their places. The Reſult of this Column will be $\frac{1}{9}a^2$, which is to be made the third Term of the aggregate Series. Then putting s for the third Term of p, we ſhall have by Multiplication $3x^2s - \frac{1}{27}a^3 = 2a^3$, or $s = \frac{55a^3}{81x^2}$. And thus by the next Operation we ſhall have $t = \frac{64a^4}{243x^2}$, and ſo on.

Or if we would reſolve this reſidual Equation by one of the foregoing Methods, by which the raiſing of Powers was avoided, and wherein the whole was perform'd by Multiplication alone; we may reduce it to this form, $3x^2 + ax + a^2 + 3x + p \times p \times p = -ax^2 - a^2x + 2a^3$, the Reſolution of which will be thus:

$$\begin{array}{rl}
 & 3x^2 + ax + a^2 \quad * \\
+3x+p \text{ ------} + & 3x - \frac{1}{3}a - \frac{a^2}{3x}, \text{ \&c.} \\
\times p \text{ ------} - & \frac{1}{3}a + \frac{a^2}{3x} + \frac{55a^3}{81x^2}, \text{ \&c.} \\
\hline
 & 3x^2 \quad * + \frac{1}{9}a^2 + \frac{61a^3}{27x}, \text{ \\&c.} \\
\times p = - & \frac{1}{3}a - \frac{a^2}{3x} + \frac{55a^3}{81x^2} + \frac{64a^4}{243x^3}, \text{ \&c.} \\
\hline
 & -ax^2 - a^2x + 2a^3 \quad *
\end{array}$$

The Terms being diſpos'd as in the Example, bring down $3x^2$ for the firſt Term of the aggregate Series, and ſuppoſing q to repreſent the firſt Term of the Series p, it will be $3x^2q = -ax^2$, or $q = -\frac{1}{3}a$. Put down $+3x$ in its proper place, and under it (as alſo after it) put down the firſt Term of p, or $-\frac{1}{3}a$, which being multiply'd, and collected with $+ax$ above, will make o for the ſecond Term of the aggregate Series. If the ſecond Term of p is now repreſented by r, we ſhall have $3x^2r * = -a^2x$, or $r = -\frac{a^2}{3x}$, to be put down in its ſeveral places. Then by multiplying and collecting we ſhall have $+\frac{1}{9}a^2$ for the third Term of the aggregate Series. And putting s for the third Term of p, we ſhall have by Multiplication $3x^2s - \frac{1}{27}a^3 = 2a^3$, or $s = \frac{55a^3}{81x^2}$. And ſo on as far as we pleaſe.

Laſtly, inſtead of the Supplemental Equation, we may reſolve the given Equation itſelf in the following manner:

$$\left.\begin{array}{rl}
y^3 & = x^3 \quad * \quad * + 2a^3 \quad * \\
 & \quad -ax^2 - \frac{2}{3}a^2x + \frac{1}{3}a^3 - \frac{28a^4}{81x}, \text{ \&c.} \\
+axy & - \text{ ----} + ax^2 - \frac{1}{3}a^2x - \frac{1}{3}a^3 + \frac{55a^4}{81x}, \text{ \&c.} \\
+a^2y & \text{ ---------} + a^2x - \frac{1}{3}a^3 - \frac{a^4}{3x}, \text{ \&c.}
\end{array}\right\}$$

$$y = x - \frac{1}{3}a - \frac{a^2}{3x} + \frac{55a^3}{81x^2} + \frac{64a^4}{243x^3}, \text{ \&c.}$$

Here becauſe it is $y^3 = x^3$, &c. it will be $y = x$, &c. and therefore $+axy = +ax^2$, &c. which muſt be ſet down in its place. Then it muſt be wrote again with a contrary ſign, that it may be $y^3 = * - ax^2$, &c. and therefore (extracting the cube-root,) $y = * - \frac{1}{3}a$, &c. Then $+a^2y = +a^2x$, &c. and $+axy = * - \frac{1}{3}a^2x$, &c. which

which being collected with a contrary ſign, will make $y^3 = * * - \frac{1}{3}a^2x$, &c. and (by Extraction) $y = * * - \frac{a^2}{3x}$, &c. Hence $+ a^2y = * - \frac{1}{3}a^3$, &c. and $+ axy = * * - \frac{1}{3}a^3$, &c. which being collected with a contrary ſign, and united with $+ 2a^3$ above, will make $y^3 = * * * \frac{5}{3}a^3$, &c. whence (by Extraction) $y = * * * \frac{5a^3}{81x^2}$, &c. Then $+ a^2y = * * - \frac{a^4}{3x}$, &c. and $+ axy = * * * + \frac{5a^4}{81x}$, &c. which being collected with a contrary ſign, will make $y^3 = * * * * - \frac{28a^4}{81x}$, &c. and then (by Extraction) $y = * * * * + \frac{61a^4}{243x^3}$, &c. And ſo on.

37, 38. I think I need not trouble the Learner, or myſelf, with giving any particular Explication (or Application) of the Author's Rules, for continuing the Quote only to ſuch a certain period as ſhall be before determined, and for preventing the computation of ſuperfluous Terms; becauſe moſt of the Methods of Analyſis here deliver'd require no Rules at all, nor is there the leaſt danger of making any unneceſſary Computations.

39. When we are to find the Root y of ſuch an Equation as this, $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$, &c. $= z$, this is uſually call'd *the Reverſion of a Series*. For as here the Aggregate z is expreſs'd by the Powers of y; ſo when the Series is reverted, the Aggregate y will be expreſs'd by the Powers of z. This Equation, as now it ſtands, ſuppoſes z (or the Aggregate of the Series) to be unknown, and that we are to approximate to it indefinitely, by means of the known Number y and its Powers. Or otherwiſe; the unknown Number z is equivalent to an infinite Series of decreaſing Terms, expreſs'd by an Arithmetical Scale, of which the known Number y is the Root. This Root therefore muſt be ſuppoſed to be leſs than Unity, that the Series may duly converge. And thence it will follow, that z alſo will be much leſs than Unity. This is uſually called a Logarithmick Series, becauſe in certain circumſtances it expreſſes the Relation between the Logarithms and their Numbers, as will appear hereafter. If we look upon z as known, and therefore y as unknown, the Series muſt be reverted; or the Value of y muſt be expreſs'd by a Series of Terms compos'd of the known Number z and its Powers. The Author's Method for reverting this Series will be very obvious from the conſideration of his Diagram; and we ſhall meet with another Method hereafter, in another part of his Works. It will be ſufficient therefore in this place, to perform it after the manner of ſome of the foregoing Extractions.

$$\left.\begin{array}{rl} y & = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5, \text{ \&c.} \\ -\frac{1}{2}y^2 & - - - - \frac{1}{2}z^2 - \frac{1}{2}z^3 - \frac{7}{24}z^4 - \frac{1}{8}z^5, \text{ \&c.} \\ +\frac{1}{3}y^3 & - - - - - - - - + \frac{1}{3}z^3 + \frac{1}{2}z^4 + \frac{5}{24}z^5, \text{ \&c.} \\ -\frac{1}{4}y^4 & - - - - - - - - - - - - - \frac{1}{4}z^4 - \frac{1}{2}z^5, \text{ \&c.} \\ +\frac{1}{5}y^5 & - - - - - - - - - - - - - - - - - - + \frac{1}{5}z^5, \text{ \&c.} \\ \text{\&c.} & \end{array}\right\}$$

In this Paradigm the unknown parts of the Equation are ſet down in a deſcending order to the left-hand, and the known Number z is ſet down over-againſt y to the right-hand. Then is $y = z$, &c. and therefore $-\frac{1}{2}y^2 = -\frac{1}{2}z^2$, &c. which is to be ſet down in its place, and alſo with a contrary ſign, ſo that $y = * + \frac{1}{2}z^2$, &c. And therefore (ſquaring) $-\frac{1}{2}y^2 = * - \frac{1}{2}z^3$, &c. and (cubing) $+\frac{1}{3}y^3 = +\frac{1}{3}z^3$, &c. which Terms collected with a contrary ſign, make $y = * * + \frac{1}{6}z^3$, &c. And therefore (ſquaring) $-\frac{1}{2}y^2 = * * - \frac{7}{24}z^4$, &c. and (cubing) $+\frac{1}{3}y^3 = * + \frac{1}{2}z^4$, &c. and $-\frac{1}{4}y^4 = -\frac{1}{4}z^4$, &c. which Terms collected with a contrary ſign, make $y = * * * + \frac{1}{24}z^4$, &c. Therefore $-\frac{1}{2}y^2 = * * * - \frac{1}{8}z^5$, &c. and $+\frac{1}{3}y^3 = * * + \frac{5}{24}z^5$, &c. and $-\frac{1}{4}y^4 = * - \frac{1}{2}z^5$, &c. and $+\frac{1}{5}y^5 = +\frac{1}{5}z^5$, &c. which Terms collected with a contrary ſign, make $y = * * * * * + \frac{1}{120}z^5$, &c. And ſo of the reſt.

40. Thus if we were to revert the Series $y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \frac{35}{1152}y^9 + \frac{63}{2816}y^{11}$, &c. $= z$, (where the Aggregate of the Series, or the unknown Number z, will repreſent the Arch of a Circle, whoſe Radius is 1, if its right Sine is repreſented by the known Number y,) or if we were to find the value of y, conſider'd as unknown, to be expreſsd by the Powers of z, now conſider'd as known; we may proceed thus:

$$\left.\begin{array}{rl} y & = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9, \text{ \&c.} \\ +\frac{1}{6}y^3 & - - - + \frac{1}{6}z^3 - \frac{1}{12}z^5 + \frac{13}{720}z^7 - \frac{41}{14144}z^9, \text{ \&c.} \\ +\frac{3}{40}y^5 & - - - - - - - + \frac{3}{40}z^5 - \frac{1}{10}z^7 + \frac{23}{960}z^9, \text{ \&c.} \\ +\frac{5}{112}y^7 & - - - - - - - - - - - - + \frac{5}{112}z^7 - \frac{35}{672}z^9, \text{ \&c.} \\ +\frac{35}{1152}y^9 & - - - - - - - - - - - - - - - - - - - + \frac{35}{1152}z^9, \text{ \&c.} \\ \text{\&c.} & \end{array}\right\}$$

The Terms being diſpoſed as you ſee here, we ſhall have $y = z$, &c. and therefore (cubing) $\frac{1}{6}y^3 = \frac{1}{6}z^3$, &c. which makes $y = * - \frac{1}{6}z^3$, &c. ſo that (cubing) we ſhall have $+\frac{1}{6}y^3 = * - \frac{1}{12}z^5$, &c. and alſo $\frac{3}{40}y^5 = \frac{3}{40}z^5$, &c. and collecting with a contrary ſign,

y

$y = * * + \frac{1}{120}z^5$, &c. Hence $\frac{1}{6}y^3 = * * \frac{1}{720}z^7$, &c. and $\frac{3}{40}y^5 = * - \frac{1}{160}z^7$, &c. and $\frac{5}{112}y^7 = \frac{5}{112}z^7$, &c. and collecting with a contrary ſign, $y = * * * - \frac{1}{5040}z^7$, &c. And ſo on.

If we ſhould deſire to perform this Extraction by another of the foregoing Methods, that is, by ſuppoſing the Equation to be reduced to this form $\overline{1 + \frac{1}{6}y^2 + \frac{3}{40}y^4 + \frac{5}{112}y^6 + \frac{35}{1152}y^8}$, &c. $\times y = z$, it may be ſufficient to ſet down the Praxis, as here follows.

	1	*	*	*	*
+	$\frac{1}{6}y^2$ - - - -	$+ \frac{1}{6}z^2$	$- \frac{1}{18}z^4$	$+ \frac{1}{135}z^6$	$- \frac{1}{1890}z^8$, &c.
+	$\frac{3}{40}y^4$ - - - - - - - - - -	$+ \frac{3}{40}z^4$	$- \frac{1}{20}z^6$	$+ \frac{3}{200}z^8$, &c.	
+	$\frac{5}{112}y^6$ - - - - - - - - - - - - - -	$+ \frac{5}{112}z^6$	$- \frac{5}{112}z^8$, &c.		
+	$\frac{35}{1152}y^8$ -	$+ \frac{35}{1152}z^8$, &c.			
	1	$+ \frac{1}{6}z^2 + \frac{7}{360}z^4 + \frac{31}{15120}z^6 + \frac{127}{604800}z^8$, &c.			
$y =$	z	$- \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c.			
	z	*	*	*	*

41. The affected Cubick Equation, which the Author here aſſumes to be ſolved, has infinite Series for the Coefficients of the Powers of y; and therefore its Terms being diſpoſed (as is taught before) according to a double Arithmetical Scale, the Roots of each of which are y and z, it will ſtand as is repreſented here below. Or taking Az^m for the firſt Approximation to the Root y, and ſubſtituting it in the firſt Table, it will appear as is here ſet down in the ſecond Table.

	*	*	*	− 8		*	*	*	− 8
	z^3y^3	$-z^3y^2$	$+ z^3y$	$+ z^3$		A^3z^{3m+3}	$-A^2z^{2m+3}$	$+ Az^{m+3}$	$+ z^3$
	$-\frac{1}{2}z^4$	$+z^4$	$-2z^4$	$- 4z^4$		$-\frac{1}{2}A^3z^{3m+4}$	$+A^2z^{2m+4}$	$-2Az^{m+4}$	$-4z^4$
	$+\frac{1}{36}z^6$	$-z^6$	$+3z^6$	$+ 9z^6$		$+\frac{1}{36}A^3z^{3m+6}$	$-A^2z^{2m+6}$	$+3Az^{m+6}$	$+9z^6$
	$-\frac{1}{576}z^8$	$+z^8$	$-4z^8$	$-16z^8$					
	&c.	&c.	&c.	&c.		&c.	&c.	&c.	&c.

Now the only caſe of external Terms, to be diſcover'd by applying the Ruler, will give the Equation $A^3z^{3m+3} - 8 = 0$, whence $3m + 2 = 0$, or $m = -\frac{2}{3}$, and the Coefficient $A = 2$. The next Number or Index, to which the Ruler in its parallel motion will apply itſelf, will be $2m + 2$, or $\frac{2}{3}$; the next will be $m + 2$, or $\frac{4}{3}$; and ſo on. Which aſcending Arithmetical Progreſſion 0, $\frac{2}{3}$, $\frac{4}{3}$, &c. will have $\frac{2}{3}$ for its common difference. Therefore $y = Az^{-\frac{2}{3}} + B + Cz^{\frac{2}{3}} + Dz^{\frac{4}{3}} + Ez^2$, &c. will be the form of the Root in this Equation. It may be reſolved by any of the foregoing Methods, but

but perhaps moſt readily by ſubſtituting the Value of y now found in the given Equation, and thence determining the general Coefficients as before. By which the Root will be found to be $y = 2z^{-\frac{2}{3}} + \frac{1}{3} - \frac{1}{9}z^{\frac{2}{3}} + \frac{5}{162}z^{\frac{4}{3}} - \frac{14}{81}z^{2} + \frac{299}{1458}z^{\frac{8}{3}} + \frac{6721}{26244}z^{\frac{10}{3}}$, &c.

42. To reſolve this affected Quadratick Equation, in which one of the Coefficients is an infinite Series; if we ſuppoſe $y = Ax^m$, &c. we ſhall have (by Subſtitution) the Equation as it ſtands here below. Then by applying the Ruler, we ſhall have $-aAx^m + \frac{x^4}{4a^2} = 0$, whence $m = 4$, and $A = \frac{1}{4a^3}$. The next Index, that the Ruler in its parallel motion will arrive at, is $m + 1$, or 5; the next is $m + 2$, or 6; &c. ſo that the common difference of the Progreſſion is 1, and the Root may be repreſented by $y = Ax^4 + Bx^5 + Cx^6$, &c. which may be extracted as here follows.

$$\left.\begin{array}{lll} A^2x^{2m} - aAx^m & \&c. \\ * - Ax^{m+1} & * \\ * - \frac{A}{a}x^{m+2} & * \\ * - \frac{A}{a^2}x^{m+3} & * \\ * - \frac{A}{a^3}x^{m+4} & + \frac{x^4}{4a^2} \\ \&c. & \end{array}\right\} = 0.$$

$$\left.\begin{array}{ll} -ay & = -\frac{x^4}{4a^2} + \frac{x^5}{4a^3} \quad * \quad * - \frac{x^8}{16a^6}, \&c. \\ -xy & ---- - \frac{x^5}{4a^3} + \frac{x^6}{4a^4} \quad * \quad * \\ -\frac{x^2}{a}y & ------ - \frac{x^6}{4a^4} + \frac{x^7}{4a^5} \quad * \quad \&c. \\ -\frac{x^3}{a^2}y & ---------- - \frac{x^7}{4a^5} + \frac{x^8}{4a^6}, \&c. \\ -\frac{x^4}{a^3}y & ---------- - \frac{x^8}{4a^6}, \&c. \\ \&c. & \\ +y^2 & ---------- + \frac{x^8}{16a^6}, \&c. \end{array}\right\}$$

$$y = \frac{x^4}{4a^3} - \frac{x^5}{4a^4} * * + \frac{x^8}{16a^7}, \&c.$$

Here becauſe it is $-ay = -\frac{x^4}{4a^2}$, &c. it will be $y = \frac{x^4}{4a^3}$, &c. Therefore $-xy = -\frac{x^5}{4a^3}$, &c. which wrote with a contrary Sign will make $-ay = * + \frac{x^5}{4a^3}$, and therefore $y = * - \frac{x^5}{4a^4}$, &c. Then $-xy = * + \frac{x^6}{4a^4}$, &c. and $-\frac{x^2}{a}y = -\frac{x^6}{4a^4}$, &c. which collected will deſtroy each other, and therefore $-ay = * * + 0$, &c. and conſequently $y = * * + 0$, &c. &c.

But there is another caſe of external Terms, which will be diſcover'd by the Ruler, and which will give $A^2x^{2m} - aAx^m = 0$, whence $m = 0$, and $A = a$. Here the Progreſſion of the Indices will be 0, 1, 2, &c. ſo that $y = A + Bx + Cx^2$, &c. will be the form of the Series. And if this Root be proſecuted by any of the Methods

Methods taught before, it will be found $y = a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{3x^4}{4a^3}$, &c.

Now in the given Equation, becauſe the infinite Series $a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3}$, &c. is a Geometrical Progreſſion, and therefore is equal to $\frac{a^2}{a-x}$, as may be proved by Diviſion; if we ſubſtitute this, the Equation will become $y^2 - \frac{a^2}{a-x}y + \frac{x^4}{4a^2} = 0$. And if we extract the ſquare-root in the ordinary way, it will give $y = \frac{a^3 \pm \sqrt{a^6 - a^4x^4 + \ldots - x^6}}{2a^2 - 2ax}$ for the exact Root. And if this Radical be reſolved, and then divided by this Denominator, the ſame two Series will ariſe as before, for the two Roots of this Equation. And this ſufficiently verifies the whole Proceſs.

43. In Series that are very remarkable, and of general uſe, the Law of Continuation (if not obvious) ſhould be always aſſign'd, when that can be conveniently done; which renders a Series ſtill more uſeful and elegant. This may commonly be diſcover'd in the Computation, by attending to the formation of the Coefficients, eſpecially if we put Letters to repreſent them, and thereby keep them as general as may be, deſcending to particulars by degrees. In the Logarithmic Series, for inſtance, $z = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c. the Law of Conſecution is very obvious, ſo that any Term, tho' ever ſo remote, may eaſily be aſſign'd at pleaſure. For if we put T to repreſent any Term indefinitely, whoſe order in the Series is expreſs'd by the natural Number m, then will $T = \pm\frac{1}{m}y^m$, where the Sign muſt be $+$ or $-$ according as m is an odd or an even Number. So that the hundredth Term is $-\frac{1}{100}y^{100}$, the next is $+\frac{1}{101}y^{101}$, &c. In the Reverſe of this Series, or $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. the Law of Continuation is thus. Let T repreſent any Term indefinitely, whoſe order in the Series is expreſs'd by m; then is $T = \frac{z^m}{1 \times 2 \times 3 \times 4 \times \&c.}$, which Series in the Denominator muſt be continued to as many Terms as there are Units in m. Or if c ſtands for the Coefficient of the Term immediately preceding, then is $T = \frac{c}{m}z^m$.

Again, in the Series $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c. (by which the Relation between the Circular Arch and its right Sine is expreſs'd,) the Law of Continuation will be thus.

If

If T be any Term of the Series, whoſe order is expreſs'd by m, and if c be the Coefficient immediately before; then $T = \frac{-cz^{2m-1}}{2m-1 \times 2m-2}$. And in the Reverſe of this Series, or $z = y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \frac{35}{1152}y^9$, &c. the Law of Conſecution will be thus. If T repreſents any Term, the Index of whoſe place in the Series is m, and if c be the preceding Coefficient; then $T = \frac{2m-3 \times 2m-3 \times c}{2m-1 \times 2m-2} y^{2m-1}$. And the like of others.

44, 45, 46. If we would perform theſe Extractions after a more indefinite and general manner, we may proceed thus. Let the given Equation be $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, the Terms of which ſhould be diſpoſed as in the Margin. Suppoſe $y = b + p$, where b is to be conceived as a near Approximation to the Root y, and p as its ſmall Supplement. When this is ſubſtituted, the Equation will ſtand as it does here. Now becauſe x and p are both ſmall quantities, the moſt conſiderable quantities are at the beginning of the Equation, from whence they proceed gradually diminiſhing, both downwards and towards the right-hand; as ought always to be ſuppos'd, when the Terms of an Equation are diſpos'd according to a double Arithmetical Scale. And becauſe inſtead of one unknown quantity y, we have here introduced two, b and p, we may determine one of them b, as the neceſſity of the Reſolution ſhall require. To remove therefore the moſt conſiderable Quantities out of the Equation, and to leave only a Supplemental Equation, whoſe Root is p; we may put $b^3 + a^2b - 2a^3 = 0$, which Equation will determine b, and which therefore henceforward we are to look upon as known. And for brevity ſake, if we put $a^2 + 3b^2 = c$, we ſhall have the Equation in the Margin.

Now here, becauſe the two initial Terms $+cp + abx$ are the moſt conſiderable of the Equation, which might be removed, if for the firſt Approximation to p we ſhould aſſume $-\frac{abx}{c}$, and the reſulting Supplemental Equation would be depreſs'd lower; therefore make $p = -\frac{abx}{c} + q$, and by ſubſtitution we ſhall have this Equation following.

Or

Or in this Equation, if we make $\frac{3a^2b^3}{c^2} - \frac{a^2b}{c} = d$, $a - \frac{6ab^2}{c} = e$, and $\frac{a^3b^3}{c^3} + 1 = f$; it will aſſume this form.

$$\left.\begin{array}{llll} \cdot & +cq & +3bq^2 & +q^3 \\ \cdot & -\frac{6ab^2}{c}xq & -\frac{3ab}{c}xq^2 & \cdot \\ +\frac{3a^2b^3}{c^2}x^2 & +axq & \cdot & \cdot \\ -\frac{a^2b}{c}x^2 & +\frac{3a^2b^2}{c^2}x^2q & \cdot & \cdot \\ -\frac{a^3b^3}{c^3}x^3 & \cdot & \cdot & \cdot \\ -x^3 & \cdot & \cdot & \cdot \end{array}\right\} = 0.$$

Here becauſe the Terms to be next removed are $+cq + dx^2$, we may put $q = -\frac{d}{c}x^2 + r$, and by Subſtitution we ſhall have another Supplemental Equation, which will be farther depreſs'd, and ſo on as far as we pleaſe. Therefore we ſhall have the Root $y = b - \frac{ab}{c}x - \frac{d}{c}x^2$, &c. where b will be the Root of this Equation $b^3 + a^2b - 2a^3 = 0$, $c = a^2 + 3b^2$, $d = \frac{3a^2b^3}{c^2} - \frac{a^2b}{c}$, $e = a - \frac{6ab^2}{c}$, $f = \frac{a^3b^3}{c^3} + 1$, &c.

$$\left.\begin{array}{llll} \cdot & +cq & +3bq^2 & +q^3 \\ \cdot & +exq & -\frac{3ab}{c}xq^2 & \cdot \\ +dx^2 & +\frac{3a^2b^2}{c^2}x^2q & \cdot & \cdot \\ -fx^3 & \cdot & \cdot & \cdot \end{array}\right\} = 0.$$

Or by another Method of Solution, if in this Equation we aſſume (as before) $y = A + Bx + Cx^2 + Dx^3$, &c. and ſubſtitute this in the Equation, to determine the general Coefficients, we ſhall have $y = A - \frac{aA}{c^2}x + \frac{a^4A}{c^6}x^2 + \frac{c^8 + 7a^5A^3 - a^7A}{c^{10}}x^3$, &c. wherein A is the Root of the Equation $A^3 + a^2A - 2a^3 = 0$, and $c^2 = 3A^2 + a^2$.

47. All Equations cannot be thus immediately reſolved, or their Roots cannot always be exhibited by an Arithmetical Scale, whoſe Root is one of the Quantities in the given Equation. But to perform the Analyſis it is ſometimes required, that a new Symbol or Quantity ſhould be introduced into the Equation, by the Powers of which the Root to be extracted may be expreſs'd in a converging Series. And the Relation between this new Symbol, and the Quantities of the Equation, muſt be exhibited by another Equation. Thus if it were propoſed to extract the Root y of this Equation, $x = a + y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c. it would be in vain to expect, that it might be expreſs'd by the ſimple Powers of either x or a. For the Series itſelf ſuppoſes, in order to its converging, that y is ſome ſmall Number leſs than Unity; but x and a are under no ſuch limitations. And therefore a Series, compoſed of the aſcending Powers of x, may be a diverging Series. It is therefore neceſſary to introduce a new Symbol, which ſhall alſo be ſmall, that a Series

form'd of its Powers may converge to y. Now it is plain, that x and a, tho' ever ſo great, muſt always be near each other, becauſe their difference $y - \frac{1}{2}y^2$, &c. is a ſmall quantity. Aſſume therefore the Equation $x - a = z$, and z will be a ſmall quantity as required; and being introduced inſtead of $x - a$, will give $z = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c. whoſe Root being extracted will be $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$, &c. as before.

48. Thus if we had the Equation $y^3 + y^2 + y - x^3 = 0$, to find the Root y; we might have a Series for y compoſed of the aſcending Powers of x, which would converge if x were a ſmall quantity, leſs than Unity, but would diverge in contrary Circumſtances. Suppoſing then that x was known to be a large Quantity; in this caſe the Author's Expedient is this. Making z the Reciprocal of x, or ſuppoſing the Equation $x = \frac{1}{z}$, inſtead of x he introduces z into the Equation, by which means he obtains a converging Series, conſiſting of the Powers of z aſcending in the Numerators, that is in reality, of the Powers of x aſcending in the Denominators. This he does, to keep within the Caſe he propoſed to himſelf; but in the Method here purſued, there is no occaſion to have recourſe to this Expedient, it being an indifferent matter, whether the Powers of the converging quantity aſcend in the Numerators or the Denominators.

Thus in the given Equation $y^3 + y^2 + y \left.\begin{matrix} \\ \\ - x^3 \end{matrix}\right\} = 0$, or (making $y = Ax^m$, &c.) $A^3x^{3m} + A^2x^{2m} + Ax^m \left.\begin{matrix} * \\ - x^3 \end{matrix}\right\}$, &c. $= 0$, by applying the Ruler we ſhall have the exterior Terms $A^3x^{3m} - x^3 = 0$, or $m = 1$, and $A = 1$. Alſo the reſulting Number or Index is 3. The next Term to which the Ruler approaches will give $2m$, or 2; the laſt m, or 1. But 3, 2, 1, make a deſcending Progreſſion, of which the common difference is 1. Therefore the form of the Root will be $y = Ax + B + Cx^{-1} + Dx^{-2}$, &c. which we may thus extract.

$$\left.\begin{matrix} y^3 \\ + y^2 \\ + y \end{matrix}\right\} \begin{matrix} = x^3 - x^2 - \frac{1}{3}x + \frac{2}{3}x^0 + \frac{7}{81}x^{-1}, \text{ &c.} \\ ----+ x^2 - \frac{1}{3}x - \frac{1}{3}x^0 + \frac{1}{81}x^{-1}, \text{ &c.} \\ ------- + x - \frac{1}{3}x^0 - \frac{2}{9}x^{-1} + \frac{7}{81}x^{-2} + \frac{5}{81}x^{-3}, \text{ &c.} \end{matrix}$$

Becauſe $y^3 = x^3$, &c. it will be $y = x$, &c. and therefore $y^2 = x^2$, &c. which will make $y^3 = * - x^2$, &c. and (by Extraction) $y = * - \frac{1}{3}$, &c. Then (by ſquaring) $y^2 = * - \frac{2}{3}x$, &c. which with x below, and changing the Sign, makes $y^3 = * * - \frac{1}{3}x$, &c. and therefore

$y = ** - \frac{1}{9}x^{-1}$, &c. Then $y^2 = ** - \frac{1}{3}$, &c. and $y = * - \frac{1}{3}$, &c. which together, changing the Sign, make $y^3 = *** + \frac{2}{3}$, &c. and $y = *** + \frac{7}{81}x^{-2}$, &c. Then $y^2 = *** + \frac{14}{81}x^{-1}$, &c. and $y = ** - \frac{2}{9}x^{-1}$, &c. and therefore $y^3 = **** + \frac{7}{81}x^{-1}$, &c. and $y = **** + \frac{5}{81}x^{-3}$, &c.

Now as this Series is accommodated to the caſe of convergency when x is a large Quantity, ſo we may derive another Series from hence, which will be accommodated to the caſe when x is a ſmall quantity. For the Ruler will direct us to the external Terms $Ax^m - x^3 = 0$, whence $m = 3$, and $A = 1$; and the reſulting Number is 3. The next Term will give $2m$, or 6; and the laſt is $3m$, or 9. But 3, 6, 9 will form an aſcending Progreſſion, of which the common difference is 3. Therefore $y = Ax^3 + Bx^6 + Cx^9$, &c. will be the form of the Series in this caſe, which may be thus derived.

$$\left.\begin{array}{l} y \\ +y^2 \\ +y^3 \end{array}\right\} \begin{array}{l} = x^3 - x^6 + x^9 \quad * - 4x^{15} + 14x^{18}, \text{ &c.} \\ ----+ x^6 - 2x^9 + 3x^{12} - 2x^{15} - 7x^{18}, \text{ &c.} \\ --------+ x^9 - 3x^{12} + 6x^{15} - 7x^{18}, \text{ &c.} \end{array}$$

Here becauſe $y = x^3$, &c. it will be $y^2 = x^6$, &c. and therefore $y = * - x^6$, &c. Then $y^2 = * - 2x^9$, &c. and $y^3 = x^9$, &c. and therefore $y = ** + x^9$, &c. Then $y^2 = ** + 3x^{12}$, &c. and $y^3 = * - 3x^{12}$, &c. and therefore $y = *** + 0$, &c.

The Expedient of the Ruler will indicate a third caſe of external Terms, which may be try'd alſo. For we may put $A^3x^{3m} + A^2x^{2m} + Ax^m = 0$, whence $m = 0$, and the Number reſulting from the other Term is 3. Therefore 3 will be the common difference of the Progreſſion, and the form of the Root will be $y = A + Bx^3 + Cx^6$, &c. But the Equation $A^3 + A^2 + A = 0$, will give $A = 0$, which will reduce this to the former Series. And the other two Roots of the Equation will be impoſſible.

If the Equation of this Example $y^3 + y^2 + y - x^3 = 0$ be multiply'd by the factor $y - 1$, we ſhall have the Equation $y^4 - y - x^3y + x^3 = 0$, or $\left.\begin{array}{l} y^4 \\ -x^3y \\ +x^3 \end{array}\right\} \begin{array}{l} ** - y \\ \quad\quad * \end{array} = 0$, which when reſolved, will only afford the ſame Series for the Root y as before.

49. This Equation $y^4 - x^2y^2 + xy^2 + 2y^2 - 2y + 1 = 0$, when reduced to the form of a double Arithmetical Scale, will ſtand as in the Margin.

Now the firſt Caſe of external Terms, ſhewn by the Ruler, in order for an aſcending Series, will make $A^4x^{4m} + 2A^2x^{2m} - 2Ax^m + 1 = 0$, or $m = 0$; where the reſulting Number is alſo 0. The ſecond is $2m + 1$, or 1; the third $2m + 2$, or 2. Therefore the Arithmetical Progreſſion will be 0, 1, 2, whoſe common difference is 1; and conſequently it will be $y = A + Bx + Cx^2 + Dx^3$, &c. But the Equation $A^4 + 2A^2 - 2A + 1 = 0$, which ſhould give the Value of the firſt Coefficient, will ſupply us with none but impoſſible Roots; ſo that y, the Root of this Equation, cannot be expreſs'd by an Arithmetical Scale whoſe Root is x, or by an aſcending Series that converges by the Powers of x, when x is a ſmall quantity.

$$\left.\begin{array}{l} y^4 * + 2y^2 - 2y + 1 \\ + xy^2 \\ - x^2y^2 \end{array}\right\} = 0.$$

Or making $y = Ax^m$, &c.

$$\left.\begin{array}{l} A^4x^{4m} * + 2A^2x^{2m} \quad - 2Ax^{m+1} \\ + A^2x^{2m+1} \quad \&c. \\ - A^2x^{2m+2} \end{array}\right\} = 0.$$

As for deſcending Series, there are two caſes to be try'd; firſt the Ruler will give us $A^4x^{4m} - A^2x^{2m+2} = 0$, whence $4m = 2m + 2$, or $m = 1$, and $A = \pm 1$. The Number ariſing is 4; the next will be $2m + 1$, or 3; the next $2m$, or 2; the next m, or 1; the laſt 0. But the Arithmetical Progreſſion 4, 3, 2, 1, 0, has 1 for its common difference, and therefore the form of the Series will be $y = Ax + B + Cx^{-1}$, &c. But to extract this Series by our uſual Method, it will be beſt to reduce the Equation to this form, $y^2 - x^2 + x + 2 - 2y^{-1} + y^{-2} = 0$, and then to proceed thus:

$$\left.\begin{array}{l} y^2 \\ -2y^{-1} \\ +y^{-2} \end{array}\right\} \begin{array}{l} = x^2 - x - 2 + 2x^{-1} - \frac{3}{2}x^{-2}, \text{ \&c.} \\ \text{----------} - 2x^{-1} + \frac{1}{2}x^{-2}, \text{ \&c.} \\ \text{----------} + x^{-2}, \text{ \&c.} \end{array}$$

$$y = x - \tfrac{1}{2} - \frac{9}{8x} + \frac{7}{16x^2} - \frac{177}{128x^3}, \text{ \&c.}$$

Becauſe $y^2 = x^2 - x - 2$, &c. 'tis therefore (by Extraction) $y = x - \frac{1}{2} - \frac{9}{8}x^{-1}$, &c. Then (by Diviſion) $-2y^{-1} = -2x^{-1}$, &c. ſo that $y^2 = * * * + 2x^{-1}$, &c. and (by Extraction) $y = * * * + \frac{7}{16}x^{-2}$, &c. Then $-2y^{-1} = * + \frac{1}{2}x^{-2}$, &c. and $y^{-2} = x^{-2}$, &c. which being united with a contrary ſign, make $y^2 = * * * * - \frac{3}{2}x^{-2}$, &c. and therefore by Extraction $y = * * * * * - \frac{177}{128}x^{-3}$, &c.

In the other caſe of a deſcending Series we ſhall have the Equation $-A^2x^{2m+2} + 1 = 0$, whence $2m + 2 = 0$, or $m = -1$, and $A = \pm 1$. The Number hence ariſing is 0; the next will be $2m + 1$,

or

or -1; the next $2m$, or -2; and the laſt $4m$, or -4. But the Numbers $0, -1, -2, -4$, will be found in a deſcending Arithmetical Progreſſion, the common difference of which is 1. Therefore the form of the Root is $y = Ax^{-1} + Bx^{-2} + Cx^{-3}$, &c. and the Terms of the Equation muſt be thus diſpoſed for Reſolution.

$$\left.\begin{array}{rl} y^{-2} & = x^2 - x - 2 - \frac{5}{4}x^{-1} - \frac{1\,3}{8}x^{-2}, \text{ \&c.} \\ & \quad + 2x + 1 \\ -2y^{-1} & \quad - 2x - 1 + \frac{5}{4}x^{-1} + \frac{5}{8}x^{-2}, \text{ \&c.} \\ + y^2 & \quad + x^{-2}, \text{ \&c.} \end{array}\right\}$$

$$y = \frac{1}{x} - \frac{1}{2x^2} + \frac{7}{8x^3} - \frac{7}{16x^4} + \frac{187}{128x^5}, \text{ \&c.}$$

Here becauſe it is $y^{-2} = x^2$, &c. it will be by Extraction of the Square-root $y^{-1} = x$, &c. and by finding the Reciprocal, $y = x^{-1}$, &c. Then becauſe $-2y^{-1} = -2x$, &c. this with a contrary Sign, and collected with $-x$ above, will make $y^{-2} = * + x$, &c. which (by Extraction) makes $y^{-1} = * + \frac{1}{2}$, &c. and by taking the Reciprocal, $y = * - \frac{1}{2}x^{-2}$, &c. Then becauſe $-2y^{-1} = * - 1$, &c. this with a contrary ſign, and collected with -2 above, will make $y^{-2} = * * - 1$, &c. and therefore (by Extraction) $y^{-1} = * * - \frac{5}{8}x^{-}$, &c. and (by Diviſion) $y = * * + \frac{7}{8}x^{-3}$, &c. Then becauſe $-2y^{-1} = * * + \frac{5}{4}x^{-1}$, it will be $y^{-2} = * * * - \frac{5}{4}x^{-1}$, &c. and $y^{-1} = * * * - \frac{5}{8}x^{-1}$, &c. and $y = * * * - \frac{7}{16}x^{-3}$, &c. Then becauſe $-2y^{-1} = * * * + \frac{5}{8}x^{-1}$, &c. and $y^2 = x^{-2}$, &c. theſe collected with a contrary ſign will make $y^{-2} = * * * * - \frac{1\,3}{8}x^{-2}$, &c. and $y^{-1} = * * * * - \frac{5}{16}x^{-2}$, &c. and $y = * * * * + \frac{1\,8\,7}{1\,2\,8}x^{-4}$, &c.

Theſe are the two deſcending Series, which may be derived for the Root of this Equation, and which will converge by the Powers of x, when it is a large quantity. But if x ſhould happen to be ſmall, then in order to obtain a converging Series, we much change the Root of the Scale. As if it were known that x differs but little from 2, we may conveniently put z for that ſmall difference, or we may aſſume the Equation $x - 2 = z$. That is, inſtead of x in this Equation ſubſtitute $z + 2$, and we ſhall have a new Equation $y^4 - z^2y^2 - 3zy^2 - 2y + 1 = 0$, which will appear as in the Margin.

Here

Here to have an aſcending Series, we muſt put $A^4z^{4m} - 2Az^m + 1 = 0$, whence $m = 0$, and $A = 1$. The Number hence ariſing is 0; the next is $2m + 1$, or 1; and the laſt $2m + 2$, or 2. But 0, 1, 2, are in an aſcending Progreſſion, whoſe common difference is 1. Therefore the form of the Series is $y = A + Bz + Cz^2 + Dz^3$, &c. And if the Root y be extracted by any of the foregoing Methods, it will be found $y = 1 + \frac{3}{2}z - \frac{7}{4}z^2$, &c. Alſo we may hence find two deſcending Series, which would converge by the Root of the Scale z, if it were a large quantity.

$$\left.\begin{array}{lll} y^4 & * \quad * & -2y+1 \\ & -3zy^2 & \\ & -z^2y^2 & \end{array}\right\} = 0$$

Or making $y = Az^m$, &c.

$$\left.\begin{array}{lll} A^4z^{4m} & * \quad * & -2Az^m+1 \\ & -3A^2z^{2m+1} & \\ & -A^2z^{2m+2} & \&c. \end{array}\right\} = 0.$$

50, 51. Our Author has here opened a large field for the Solution of theſe Equations, by ſhewing, that the indeterminate quantity, or what we call the Root of the Scale, or the converging quantity, may be changed a great variety of ways, and thence new Series will be derived for the Root of the Equation, which in different circumſtances will converge differently, ſo that the moſt commodious for the preſent occaſion may always be choſe. And when one Series does not ſufficiently converge, we may be able to change it for another that ſhall converge faſter. But that we may not be left to uncertain interpretations of the indeterminate quantity, or be obliged to make Suppoſitions at random; he gives us this Rule for finding initial Approximations, that may come at once pretty near the Root required, and therefore the Series will converge apace to it. Which Rule amounts to this: We are to find what quantities, when ſubſtituted for the indefinite Species in the propoſed Equation, will make it diviſible by the radical Species, increaſed or diminiſhed by another quantity, or by the radical Species alone. The ſmall difference that will be found between any one of thoſe quantities, and the indeterminate quantity of the Equation, may be introduced inſtead of that indeterminate quantity, as a convenient Root of the Scale, by which the Series is to converge.

Thus if the Equation propoſed be $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, and if for x we here ſubſtitute a, we ſhall have the Terms $y^3 + 2a^2y - 3a^3$, which are diviſible by $y - a$, the Quotient being $y^2 + ay + 3a^2$. Therefore we may ſuppoſe, by the foregoing Rule, that $a - x = z$ is but a ſmall quantity, or inſtead of x we may ſubſtitute $a - z$ in the propoſed Equation, which will then become $y^3 + 2a^2y - azy + 3a^2z - 3az^2 + z^3 - 2a^3 = 0$. A Series

Series derived from hence, compoſed of the aſcending Powers of z, muſt converge faſt, *cæteris paribus*, becauſe the Root of the Scale z is a ſmall quantity.

Or in the ſame Equation, if for x we ſubſtitute $-a$, we ſhall have the Terms $y^3 - a^3$, which are diviſible by $y - a$, the Quotient being $y^2 + ay + a^2$. Therefore we may ſuppoſe the difference between $-a$ and x to be but little, or that $-a - x = z$ is a ſmall quantity, and therefore inſtead of x we may ſubſtitute its equal $-a - z$ in the given Equation. This will then become $y^3 - azy + 3a^2z + 3az^2 - a^3 = 0$, where the Root y will converge by the Powers of the ſmall quantity z.

Or if for x we ſubſtitute $-2a$, we ſhall have the Terms $y^3 - a^2y + 6a^3$, which are diviſible by $y + 2a$, the Quotient being $y^2 - 2ay + 3a^2$. Wherefore we may ſuppoſe there is but a ſmall difference between $-2a$ and x, or that $-2a - x = z$ is a ſmall quantity; and therefore inſtead of x we may introduce its equal $-2a - z$ into the Equation, which will then become $y^3 - a^2y - azy + 6a^3 + 12a^2z + 6az^2 + z^3 = 0$.

Laſtly, if for x we ſubſtitute $-2\frac{1}{3}a$, we ſhall have the Terms $y^3 - 2\frac{1}{3}a^2y + a^2y$, which are diviſible by y, the Radical Species alone. Wherefore we may ſuppoſe there is but a ſmall difference between $-2\frac{1}{3}a$ and x, or that $-2\frac{1}{3}a - x = z$ is a ſmall quantity; and therefore inſtead of x we may ſubſtitute its equal $-2\frac{1}{3}a - z$, which will reduce the Equation to $y^3 + \overline{1 - \sqrt[3]{2}} \times a^2y - azy + 3\sqrt[3]{4} \times a^2z + 3\sqrt[3]{2} \times az^2 + z^3 = 0$, wherein the Series for the Root y may converge by the Powers of the ſmall quantity z.

But the reaſon of this Operation ſtill remains to be inquired into, which I ſhall endeavour to explain from the preſent Example. In the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, the indeterminate quantity x, of its own nature, muſt be ſuſceptible of all poſſible Values; at leaſt, if it had any limitations, they would be ſhew'd by impoſſible Roots. Among other values, it will receive theſe, a, $-a$, $-2a$, $-2\frac{1}{3}a$, &c. in which caſes the Equation would become $y^3 + 2a^2y - 3a^3 = 0$, $y^3 - a^3 = 0$, $y^3 - a^2y + 6a^3 = 0$, $y^3 - 2\frac{1}{3}a^2y + a^2y = 0$, &c. reſpectively. Now as theſe Equations admit of juſt Roots, as appears by their being diviſible by $y +$ or $-$ another quantity, and the laſt by y alone; ſo that in the Reſolution, the whole Equation (in thoſe caſes) would be immediately exhauſted: And in other caſes, when x does not much recede from one of thoſe Values,

Values, the Equation would be nearly exhausted. Therefore the introducing of z, which is the small difference between x and any one of those Values, must depress the Equation; and z itself must be a convenient quantity to be made the Root of the Scale, or the converging Quantity.

I shall give the Solution of one of the Equations of these Examples, which shall be this, $y^3 - azy + 3a^2z + 3az^2 - a^3 = 0$, or

$$\left.\begin{array}{r} y^3 \quad * \quad * - a^3 \\ - azy + 3a^2z \\ + 3az^2 \end{array}\right\} = 0. \qquad \left.\begin{array}{r} y^3 \\ \\ -azy \end{array}\right\} \begin{array}{l} = a^3 - 3a^2z - 3az^2. \\ \quad + a^2z - \frac{2}{3}az^2 - \frac{5}{3}z^3, \text{ \&c.} \\ \text{- - - -} \; a^2z + \frac{2}{3}az^2 + \frac{5}{3}z^3, \text{ \&c.} \end{array}$$

$$y = a - \tfrac{2}{3}z - \frac{5z^2}{3a} - \frac{217z^3}{81a^2}, \text{ \&c.}$$

Here because $y^3 = a^3$, &c. it will be $y = a$, &c. Then $-azy = -a^2z$, &c. which must be wrote again with a contrary sign, and united with $-3a^2z$ above, to make $y^3 = * - 2a^2z$, &c. and therefore $y = * - \frac{2}{3}z$, &c. Then $-azy = * + \frac{2}{3}az^2$, &c. and $y^3 = * * - \frac{11}{3}az^2$, &c. and $y = * * - \frac{5z^2}{3a}$, &c. Then $-azy = * * + \frac{5}{3}z^3$, &c. and $y^3 = * * * - \frac{5}{3}z^3$, &c. and $y = * * * - \frac{217z^3}{81a^2}$, &c.

The Author hints at many other ways of deriving a variety of Series from the same Equation; as when we suppose the afore-mention'd difference z to be indefinitely great, and from that Supposition we find Series, in which the Powers of z shall ascend in the Denominators. This Case we have all along pursued indiscriminately with the other Case, in which the Powers of the converging quantity ascend in the Numerators, and therefore we need add nothing here about it. Another Expedient is, to assume for the converging quantity some other quantity of the Equation, which then may be consider'd as indeterminate. So here, for instance, we may change a into x, and x into a. Or lastly, to assume any Relation at pleasure, (suppose $x = az + bz^2$, $x = \frac{a}{b+z}$, $x = \frac{a+cz}{b+z}$, &c.) between the indeterminate quantity of the Equation x, and the quantity z we would introduce into its room, by which new equivalent Equations may be form'd, and then their Roots may be extracted. And afterwards the value of z may be express'd by x, by means of the assumed Equation.

52. The

52. The Author here, in a ſummary way, gives us a *Rationale* of his whole Method of Extractions, proving *à priori*, that the Series thus form'd, and continued *in infinitum*, will then be the juſt Roots of the propoſed Equation. And if they are only continued to a competent number of Terms, (the more the better,) yet then will they be a very near Approximation to the juſt and compleat Roots. For, when an Equation is propoſed to be reſolved, as near an Approach is made to the Root, ſuppoſe y, as can be had in a ſingle Term, compoſed of the quantities given by the Equation; and becauſe there is a Remainder, a Reſidual or Secondary Equation is thence form'd, whoſe Root p is the Supplement to the Root of the given Equation, whatever that may be. Then as near an approach is made to p, as can be done by a ſingle Term, and a new Reſidual Equation is form'd from the Remainder, wherein the Root q is the Supplement to p. And by proceeding thus, the Reſidual Equations are continually depreſs'd, and the Supplements grow perpetually leſs and leſs, till the Terms at laſt are leſs than any aſſignable quantities. We may illuſtrate this by a familiar Example, taken from the uſual Method of Diviſion of Decimal Fractions. At every Operation we put as large a Figure in the Quotient, as the Dividend and Diviſor will permit, ſo as to leave the leaſt Remainder poſſible. Then this Remainder ſupplies the place of a new Dividend, which we are to exhauſt as far as can be done by one Figure, and therefore we put the greateſt number we can for the next Figure of the Quotient, and thereby leave the leaſt Remainder we can. And ſo we go on, either till the whole Dividend is exhauſted, if that can be done, or till we have obtain'd a ſufficient Approximation in decimal places or figures. And the ſame way of Argumentation, that proves our Author's Method of Extraction, may eaſily be apply'd to the other ways of Analyſis that are here found.

53, 54. Here it is ſeaſonably obſerved, that tho' the indefinite Quantity ſhould not be taken ſo ſmall, as to make the Series converge very faſt, yet it would however converge to the true Root, tho' by more ſteps and ſlower degrees. And this would obtain in proportion, even if it were taken never ſo large, provided we do not exceed the due Limits of the Roots, which may be diſcover'd, either from the given Equation, or from the Root when exhibited by a Series, or may be farther deduced and illuſtrated by ſome Geometrical Figure, to which the Equation is accommodated.

So if the given Equation were $yy = ax - xx$, it is eaſy to obſerve, that neither y nor x can be infinite, but they are both liable to

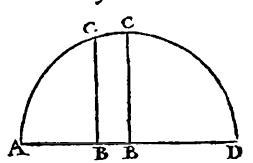
ſeveral Limitations. For if x be ſuppos'd infinite, the Term ax would vaniſh in reſpect of $-xx$, which would give the Value of yy impoſſible on this Suppoſition. Nor can x be negative; for then the Value of yy would be negative, and therefore the Value of y would again become impoſſible. If $x = 0$, then is $y = 0$ alſo; which is one Limitation of both quantities. As yy is the difference between ax and xx, when that difference is greateſt, then will yy, and conſequently y, be greateſt alſo. But this happens when $x = \frac{1}{2}a$, as alſo $y = \frac{1}{2}a$, as may appear from the following Prob. 3. And in general, when y is expreſs'd by any number of Terms, whether finite or infinite, it will then come to its Limit when the difference is greateſt between the affirmative and negative Terms; as may appear from the ſame Problem. This laſt will be a Limitation for y, but not for x. Laſtly, when $x = a$, then $y = 0$; which will limit both x and y. For if we ſuppoſe x to be greater than a, the negative Term will prevail over the affirmative, and give the Value of yy negative, which will make the Value of y impoſſible. So that upon the whole, the Limitations of x in this Equation will be theſe, that it cannot be leſs than 0, nor greater than a, but may be of any intermediate magnitude between thoſe Limits.

Now if we reſolve this Equation, and find the Value of y in an infinite Series, we may ſtill diſcover the ſame Limitations from thence. For from the Equation $yy = ax - xx$, by extracting the ſquare-root, as before, we ſhall have $y = a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{16a^{\frac{5}{2}}}$, &c. that is, $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$ into $1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{x^3}{16a^3}$, &c. Here x cannot be negative; for then $x^{\frac{1}{2}}$ would be an impoſſible quantity. Nor can x be greater than a; for then the converging quantity $\frac{x}{a}$, or the Root of the Scale by which the Series is expreſs'd, would be greater than Unity, and conſequently the Series would diverge, and not converge as it ought to do. The Limit between converging and diverging will be found, by putting $x = a$, and therefore $y = 0$; in which caſe we ſhall have the identical Numeral Series $1 = \frac{1}{2} + \frac{1}{8} + \frac{1}{16}$, &c. of the ſame nature with ſome of thoſe, which we have elſewhere taken notice of. So that we may take x of any intermediate Value between 0 and a, in order to have a converging Series. But the nearer it is taken to the Limit 0, ſo much faſter the Series will converge to the true Root; and the nearer it is taken to the Limit a, it will converge ſo much the ſlower. But it will however

however converge, if x be taken never ſo little leſs than a. And by Analogy, a like Judgment is to be made in all other caſes.

The Limits and other affections of y are likewiſe diſcoverable from this Series. When $x = 0$, then $y = 0$. When x is a naſcent quantity, or but juſt beginning to be poſitive, all the Terms but the firſt may be neglected, and y will be a mean proportional between a and x. Alſo $y = 0$, when the affirmative Term is equal to all the negative Terms, or when $1 = \frac{x}{2a} + \frac{x^2}{8a^2} + \frac{x^3}{16a^3}$, &c. that is, when $x = a$. For then $1 = \frac{1}{2} + \frac{1}{8} + \frac{1}{16}$, &c. as above. Laſtly, y will be a *Maximum* when the difference between the affirmative Term and all the negative Terms is greateſt, which by Prob. 3. will be found when $x = \frac{1}{3}a$.

Now the Figure or Curve that may be adapted to this Equation, and to this Series, and which will have the ſame Limitations that they have, is the Circle ACD, whoſe Diameter is $AD = a$, its Abſciſs $AB = x$, and its perpendicular Ordinate $BC = y$. For as the Ordinate $BC = y$ is a mean proportional between the Segments of the Diameter $AB = x$ and $BD = a - x$, it will be $yy = ax - xx$. And therefore the Ordinate $BC = y$ will be expreſs'd by the foregoing Series. But it is plain from the nature of the Circle, that the Abſciſs AB cannot be extended backwards, ſo as to become negative; neither can it be continued forwards beyond the end of the Diameter D. And that at A and D, where the Diameter begins and ends, the Ordinate is nothing. And the greateſt Ordinate is at the Center, or when $AB = \frac{1}{2}AD$.



SECT. VI. *Tranſition to the Method of Fluxions.*

55. THE learned and ſagacious Author having thus accompliſh'd one part of his deſign, which was, to teach the Method of converting all kinds of Algebraic Quantities into ſimple Terms, by reducing them to infinite Series: He now goes on to ſhew the uſe and application of this Reduction, or of theſe Series, in the Method of Fluxions, which is indeed the principal deſign of this Treatiſe. For this Method has ſo near a connexion with, and dependence upon the foregoing, that it would be very lame and defective without it. He lays down the fundamental Principles of

this Method in a very general and ſcientifick manner, deducing them from the received and known laws of local Motion. Nor is this inverting the natural order of Science, as ſome have pretended, by introducing the Doctrine of Motion into pure Geometrical Speculations. For Geometrical and Analytical Quantities are beſt conceived as generated by local Motion; and their properties may as well be derived from them while they are generating, as when their generation is ſuppos'd to be already accompliſh'd, in any other way. A right line, or a curve line, is deſcribed by the motion of a point, a ſurface by the motion of a line, a ſolid by the motion of a ſurface, an angle by the rotation of a radius; all which motions we may conceive to be perform'd according to any ſtated law, as occaſion ſhall require. Theſe generations of quantities we daily ſee to obtain *in rerum naturâ*, and is the manner the ancient Geometricians had often recourſe to, in conſidering their production, and then deducing their properties from ſuch actual deſcriptions. And by analogy, all other quantities, as well as theſe continued geometrical quantities, may be conceived as generated by a kind of motion or progreſs of the Mind.

The Method of Fluxions then ſuppoſes quantities to be generated by local Motion, or ſomething analogous thereto, tho' ſuch generations indeed may not be eſſentially neceſſary to the nature of the thing ſo generated. They might have an exiſtence independent of theſe motions, and may be conceived as produced many other ways, and yet will be endued with the ſame properties. But this conception, of their being now generated by local Motion, is a very fertile notion, and an exceeding uſeful artifice for diſcovering their properties, and a great help to the Mind for a clear, diſtinct, and methodical perception of them. For local Motion ſuppoſes a notion of time, and time implies a ſucceſſion of Ideas. We eaſily diſtinguiſh it into what was, what is, and what will be, in theſe generations of quantities; and ſo we commodiouſly conſider thoſe things by parts, which would be too much for our faculties, and extream difficult for the Mind to take in the whole together, without ſuch artificial partitions and diſtributions.

Our Author therefore makes this eaſy ſuppoſition, that a Line may be conceived as now deſcribing by a Point, which moves either equably or inequably, either with an uniform motion, or elſe according to any rate of continual Acceleration or Retardation. Velocity is a Mathematical Quantity, and like all ſuch, it is ſuſceptible of infinite gradations, may be intended or remitted, may be increaſed

or

or diminiſh'd in different parts of the ſpace deſcribed, according to an infinite variety of ſtated Laws. Now it is plain, that the ſpace thus deſcribed, and the law of acceleration or retardation, (that is, the velocity at every point of time,) muſt have a mutual relation to each other, and muſt mutually determine each other; ſo that one of them being aſſign'd, the other by neceſſary inference may be derived from it. And therefore this is ſtrictly a Geometrical Problem, and capable of a full Determination. And all Geometrical Propoſitions once demonſtrated, or duly inveſtigated, may be ſafely made uſe of, to derive other Propoſitions from them. This will divide the preſent Problem into two Caſes, according as either the Space or Velocity is aſſign'd, at any given time, in order to find the other. And this has given occaſion to that diſtinction which has ſince obtain'd, of the *direct and inverſe Method of Fluxions*, each of which we ſhall now conſider apart.

56. In the direct Method the Problem is thus abſtractedly propoſed. *From the Space deſcribed, being continually given, or aſſumed, or being known at any point of Time aſſign'd; to find the Velocity of the Motion at that Time.* Now in equable Motions it is well known, that the Space deſcribed is always as the Velocity and the Time of deſcription conjunctly; or the Velocity is directly as the Space deſcribed, and reciprocally as the Time of deſcription. And even in inequable Motions, or ſuch as are continually accelerated or retarded, according to ſome ſtated Law, if we take the Spaces and Times very ſmall, they will make a near approach to the nature of equable Motions; and ſtill the nearer, the ſmaller thoſe are taken. But if we may ſuppoſe the Times and Spaces to be indefinitely ſmall, or if they are naſcent or evaneſcent quantities, then we ſhall have the Velocity in any infinitely little Space, as that Space directly, and as the *tempuſculum* inverſely. This property therefore of all inequable Motions being thus deduced, will afford us a medium for ſolving the preſent Problem, as will be ſhewn afterwards. So that the Space deſcribed being thus continually given, and the whole time of its deſcription, the Velocity at the end of that time will be thence determinable.

57. The general abſtract Mechanical Problem, which amounts to the ſame as what is call'd the inverſe Method of Fluxions, will be this. *From the Velocity of the Motion being continually given, to determine the Space deſcribed, at any point of Time aſſign'd.* For the Solution of which we ſhall have the aſſiſtance of this Mechanical Theorem, that in inequable Motions, or when a Point deſcribes a Line

Line according to any rate of acceleration or retardation, the indefinitely little Space deſcribed in any indefinitely little Time, will be in a compound ratio of the Time and the Velocity; or the *ſpatiolum* will be as the velocity and the *tempuſculum* conjunctly. This being the Law of all equable Motions, when the Space and Time are any finite quantities, it will obtain alſo in all inequable Motions, when the Space and Time are diminiſh'd *in infinitum*. For by this means all inequable Motions are reduced, as it were, to equability. Hence the Time and the Velocity being continually known, the Space deſcribed may be known alſo; as will more fully appear from what follows. This Problem, in all its caſes, will be capable of a juſt determination; tho' taking it in its full extent, we muſt acknowledge it to be a very difficult and operoſe Problem. So that our Author had good reaſon for calling it *moleſtiſſimum & omnium difficillimum problema.*

58. To fix the Ideas of his Reader, our Author illuſtrates his general Problems by a particular Example. If two Spaces x and y are deſcribed by two points in ſuch manner, that the Space x being uniformly increaſed, in the nature of Time, and its equable velocity being repreſented by the Symbol $\dot{x}$; and if the Space y increaſes inequably, but after ſuch a rate, as that the Equation $y = xx$ ſhall always determine the relation between thoſe Spaces; (or x being continually given, y will be thence known;) then the velocity of the increaſe of y ſhall always be repreſented by $2x\dot{x}$. That is, if the ſymbol $\dot{y}$ be put to repreſent the velocity of the increaſe of y, then will the Equation $\dot{y} = 2x\dot{x}$ always obtain, as will be ſhewn hereafter. Now from the given Equation $y = xx$, or from the relation of the Spaces y and x, (that is, the Space and Time, or its repreſentative,) being continually given, the relation of the Velocities $\dot{y} = 2x\dot{x}$ is found, or the relation of the Velocity $\dot{y}$, by which the Space increaſes, to the Velocity $\dot{x}$, by which the repreſentative of the Time increaſes. And this is an inſtance of the Solution of the firſt general Problem, or of a particular Queſtion in the direct Method of Fluxions. But *vice versâ*, if the laſt Equation $\dot{y} = 2x\dot{x}$ were given, or if the Velocity $\dot{y}$, by which the Space y is deſcribed, were continually known from the Time x being given, and its Velocity $\dot{x}$; and if from thence we ſhould derive the Equation $y = xx$, or the relation of the Space and Time: This would be an inſtance of the Solution of the ſecond general Problem, or of a particular Queſtion of the inverſe Method of Fluxions. And in analogy to this deſcription of Spaces by moving points, our Author conſiders all other quantities whatever as generated

nerated and produced by continual augmentation, or by the perpetual acceſſion and accretion of new particles of the ſame kind.

59. In ſettling the Laws of his Calculus of Fluxions, our Author very ſkilfully and judiciouſly diſengages himſelf from all conſideration of Time, as being a thing of too Phyſical or Metaphyſical a nature to be admitted here, eſpecially when there was no abſolute neceſſity for it. For tho' all Motions, and Velocities of Motion, when they come to be compared or meaſured, may ſeem neceſſarily to include a notion of Time; yet Time, like all other quantities, may be repreſented by Lines and Symbols, as in the foregoing example, eſpecially when we conceive them to increaſe uniformly. And theſe repreſentatives or proxies of Time, which in ſome meaſure may be made the objects of Senſe, will anſwer the preſent purpoſe as well as the thing itſelf. So that Time, in ſome ſenſe, may be ſaid to be eliminated and excluded out of the inquiry. By this means the Problem is no longer Phyſical, but becomes much more ſimple and Geometrical, as being wholly confined to the deſcription of Lines and Spaces, with their comparative Velocities of increaſe and decreaſe. Now from the equable Flux of Time, which we conceive to be generated by the continual acceſſion of new particles, or Moments, our Author has thought fit to call his Calculus *the Method of Fluxions.*

60, 61. Here the Author premiſes ſome Definitions, and other neceſſary preliminaries to his Method. Thus Quantities, which in any Problem or Equation are ſuppos'd to be ſuſceptible of continual increaſe or decreaſe, he calls *Fluents*, or *flowing Quantities*; which are ſometimes call'd *variable* or *indeterminate quantities*, becauſe they are capable of receiving an infinite number of particular values, in a regular order of ſucceſſion. The Velocities of the increaſe or decreaſe of ſuch quantities are call'd their *Fluxions*; and quantities in the ſame Problem, not liable to increaſe or decreaſe, or whoſe Fluxions are nothing, are call'd *conſtant*, *given*, *invariable*, and *determinate quantities*. This diſtinction of quantities, when once made, is carefully obſerved through the whole Problem, and inſinuated by proper Symbols. For the firſt Letters of the Alphabet are generally appropriated for denoting conſtant quantities, and the laſt Letters commonly ſignify variable quantities, and the ſame Letters, being pointed, repreſent the Fluxions of thoſe variable quantities or Fluents reſpectively. This diſtinction between theſe quantities is not altogether arbitrary, but has ſome foundation in the nature of the thing, at leaſt during the Solution of the preſent Problem. For the flowing

or

or variable quantities may be conceived as *now generating* by Motion, and the conſtant or invariable quantities as ſome how o other *already generated.* Thus in any given Circle or Parabola, the Diameter or Parameter are conſtant lines, or already generated; but the Abſciſs, Ordinate, Area, Curve-line, *&c.* are flowing and variable quantities, becauſe they are to be underſtood as now deſcribing by local Motion, while their properties are derived. Another diſtinction of theſe quantities may be this. A conſtant or given line in any Problem is *linea quædam*, but an indeterminate line is *linea quævis vel quæcunque*, becauſe it may admit of infinite values. Or laſtly, conſtant quantities in a Problem are thoſe, whoſe ratio to a common Unit, of their own kind, is ſuppos'd to be known; but in variable quantities that ratio cannot be known, becauſe it is varying perpetually. This diſtinction of quantities however, into determinate and indeterminate, ſubſiſts no longer than the preſent Calculation requires; for as it is a diſtinction form'd by the Imagination only, for its own conveniency, it has a power of aboliſhing it, and of converting determinate quantities into indeterminate, and *vice versâ*, as occaſion may require; of which we ſhall ſee Inſtances in what follows. In a Problem, or Equation, there may be any number of conſtant quantities, but there muſt be at leaſt two that are flowing and indeterminate; for one cannot increaſe or diminiſh, while all the reſt continue the ſame. If there are more than two variable quantities in a Problem, their relation ought to be exhibited by more than one Equation.

ANNO-

ANNOTATIONS on Prob. I.

OR,

The relation of the flowing Quantities being given, to determine the relation of their Fluxions.

SECT. I. *Concerning Fluxions of the firſt order, and to find their Equations.*

1. THE Author having thus propoſed his fundamental Problems, in an abſtract and general manner, and gradually brought them down to the form moſt convenient for his Method; he now proceeds to deliver the Precepts of Solution, which he illuſtrates by a ſufficient variety of Examples, reſerving the Demonſtration to be given afterwards, when his Readers will be better prepared to apprehend the force of it, and when their notions will be better ſettled and confirm'd. Theſe Precepts of Solution, or the Rules for finding the Fluxions of any given Equation, are very ſhort, elegant, and comprehenſive; and appear to have but little affinity with the Rules uſually given for this purpoſe: But that is owing to their great degree of univerſality. We are to form, as it were, ſo many different Tables for the Equation, as there are flowing quantities in it, by diſpoſing the Terms according to the Powers of each quantity, ſo as that their Indices may form an Arithmetical Progreſſion. Then the Terms are to be multiply'd in each caſe, either by the Progreſſion of the Indices, or by the Terms of any other Arithmetical Progreſſion, (which yet ſhould have the ſame common difference with the Progreſſion of the Indices;)

as alſo by the Fluxion of that Fluent, and then to be divided by the Fluent itſelf. Laſt of all, theſe Terms are to be collected, according to their proper Signs, and to be made equal to nothing; which will be a new Equation, exhibiting the relation of the Fluxions. This proceſs indeed is not ſo ſhort as the Method for taking Fluxions, (to be given preſently,) which he elſewhere delivers, and which is commonly follow'd; but it makes ſufficient amends by the univerſality of it, and by the great variety of Solutions which it will afford. For we may derive as many different Fluxional Equations from the ſame given Equation, as we ſhall think fit to aſſume different Arithmetical Progreſſions. Yet all theſe Equations will agree in the main, and tho' differing in form, yet each will truly give the relation of the Fluxions, as will appear from the following Examples.

2. In the firſt Example we are to take the Fluxions of the Equation $x^3 - ax^2 + axy - y^3 = 0$, where the Terms are always brought over to one ſide. Theſe Terms being diſpoſed according to the powers of the Fluent x, or being conſider'd as a Number expreſs'd by the Scale whoſe Root is x, will ſtand thus $x^3 - ax^2 + ayx^1 - y^3x^0 = 0$; and aſſuming the Arithmetical Progreſſion 3, 2, 1, 0, which is here that of the Indices of x, and multiplying each Term by each reſpectively, we ſhall have the Terms $3x^3 - 2ax^2 + ayx$ *; which again multiply'd by $\frac{\dot{x}}{x}$, or $\dot{x}x^{-1}$, according to the Rule, will make $3\dot{x}x^2 - 2a\dot{x}x + ay\dot{x}$. Then in the ſame Equation making the other Fluent y the Root of the Scale, it will ſtand thus, $-y^3 + 0y^2 + axy^1 - ax^2y^0 = 0$; and aſſuming the Arithmetical Progreſſion 3, 2, 1, 0, which alſo is the Progreſſion of the Indices of y, and multiplying as before, we ſhall have the Terms $-3y^3$ * $+ axy$ *, which multiply'd by $\frac{\dot{y}}{y}$, or $\dot{y}y^{-1}$, will make $-3\dot{y}y^2 + ax\dot{y}$. Then collecting the Terms, the Equation $3\dot{x}x^2 - 2ax\dot{x} + ay\dot{x} - 3\dot{y}y^2 + ax\dot{y} = 0$ will give the required relation of the Fluxions. For if we reſolve this Equation into an Analogy, we ſhall have $\dot{x} : \dot{y} :: 3y^2 - ax : 3x^2 - 2ax + ay$; which, in all the values that x and y can aſſume, will give the ratio of their Fluxions, or the comparative velocity of their increaſe or decreaſe, when they flow according to the given Equation.

Or to find this ratio of the Fluxions more immediately, or the value of the Fraction $\frac{\dot{y}}{\dot{x}}$ by fewer ſteps, we may proceed thus. Write down the Fraction $\frac{\dot{y}}{\dot{x}}$ with the note of equality after it, and in the Numerator

Numerator of the equivalent Fraction write the Terms of the Equation, diſpos'd according to x, with their reſpective ſigns; each being multiply'd by the Index of x in that Term, (increaſed or diminiſh'd, if you pleaſe, by any common Number,) as alſo divided by x. In the Denominator do the ſame by the Terms, when diſpoſed according to y, only changing the ſigns. Thus in the preſent Equation $x^3 - ax^2 + axy - y^3 = 0$, we ſhall have at once $\frac{\dot{y}}{\dot{x}} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}$.

Let us now apply the Solution another way. The Equation $x^3 - ax^2 + axy - y^3 = 0$ being order'd according to x as before, will be $x^3 - ax^2 + ayx^1 - y^3x^0 = 0$; and ſuppoſing the Indices of x to be increas'd by an unit, or aſſuming the Arithmetical Progreſſion $\frac{4\dot{x}}{x}$, $\frac{3\dot{x}}{x}$, $\frac{2\dot{x}}{x}$, $\frac{\dot{x}}{x}$, and multiplying the Terms reſpectively, we ſhall have theſe Terms $4\dot{x}x^2 - 3a\dot{x}x + 2ay\dot{x} - y^3\dot{x}x^{-1}$. Then ordering the Terms according to y, they will become $-y^3 + 0y^2 + axy^1 + x^3y^0 = 0$; and ſuppoſing the Indices of y to be diminiſh'd by an unit, or aſſuming the Arithmetical Progreſſion $\frac{2\dot{y}}{y}$, $\frac{\dot{y}}{y}$, $\frac{0\dot{y}}{y}$, $\frac{-\dot{y}}{y}$, and multiplying the Terms reſpectively, we ſhall have theſe Terms $-2\dot{y}y^2 - x^3\dot{y}y^{-1} + ax^2\dot{y}y^{-1}$. So that collecting the Terms, we ſhall have $4\dot{x}x^2 - 3a\dot{x}x + 2ay\dot{x} - y^3\dot{x}x^{-1} - 2\dot{y}y^2 - x^3\dot{y}y^{-1} + ax^2\dot{y}y^{-1} = 0$, for the Fluxional Equation required. Or the ratio of the Fluxions will be $\frac{\dot{y}}{\dot{x}} = \frac{4x^2 - 3ax + 2ay - y^3x^{-1}}{2y^2 + x^3y^{-1} - ax^2y^{-1}}$; which ratio may be found immediately by applying the foregoing Rule.

Or contrary-wiſe, if we multiply the Equation in the firſt form by the Progreſſion $\frac{2\dot{x}}{x}$, $\frac{\dot{x}}{x}$, $\frac{0\dot{x}}{x}$, $\frac{-\dot{x}}{x}$, we ſhall have the Terms $2\dot{x}x^2 - a\dot{x}x + y^3\dot{x}x^{-1}$. And if we multiply the Equation in the ſecond form by $\frac{4\dot{y}}{y}$, $\frac{3\dot{y}}{y}$, $\frac{2\dot{y}}{y}$, $\frac{\dot{y}}{y}$, we ſhall have the Terms $-4\dot{y}y^2 + 2ax\dot{y} + x^3\dot{y}y^{-1} - ax^2\dot{y}y^{-1}$. Therefore collecting 'tis $2\dot{x}x^2 - a\dot{x}x + y^3\dot{x}x^{-1} - 4\dot{y}y^2 + 2ax\dot{y} + x^3\dot{y}y^{-1} - ax^2\dot{y}y^{-1} = 0$. Or the ratio of the Fluxions will be $\frac{\dot{y}}{\dot{x}} = \frac{2x^2 - ax + y^3x^{-1}}{4y^2 - 2ax + x^3y^{-1} + ax^2y^{-1}}$, which might have been found at once by the foregoing Rule.

And in general, if the Equation $x^3 - ax^2 + axy - y^3 = 0$, in the form $x^3 - ax^2 + axy - y^3x^0 = 0$, be multiply'd by the Terms of this Arithmetical Progreſſion $\frac{m+3}{x}\dot{x}$, $\frac{m+2}{x}\dot{x}$, $\frac{m+1}{x}\dot{x}$, $\frac{m}{x}\dot{x}$; it will produce the Terms $\overline{m+3}\dot{x}x^2 - \overline{m+2}a\dot{x}x + \overline{m+1}ax\dot{y} - my^3\dot{x}x^{-1}$;

and if the ſame Equation, reduced to the form $-y^3+0y^2+axy^1+x^3y^0=0$, be multiply'd by the Terms of this Arithmetical Progreſſion $\frac{n+3}{y}\dot{y}$, $\frac{n+2}{y}\dot{y}$, $\frac{n+1}{y}\dot{y}$, $\frac{n}{y}\dot{y}$, it will produce the Terms $-\overline{n+3}\dot{y}y^2$, $*+\overline{n+1}a\dot{x}y+nx^3\dot{y}y^{-1}-nax^2\dot{y}y^{-1}$. Then collecting the Terms, we ſhall have $\overline{m+3}\dot{x}x^2-\overline{m+2}a\dot{x}x+\overline{m+1}a\dot{x}y-my^3\dot{x}x^{-1}-\overline{n+3}\dot{y}y^2 *+\overline{n+1}a\dot{x}y+nx^3\dot{y}y^{-1}-nax^2\dot{y}y^{-1}=0$, for the Fluxional Equation required. Or the ratio of the Fluxions will be $\frac{\dot{y}}{\dot{x}}=\frac{\overline{m+3}x^2-\overline{m+2}ax+\overline{m+1}ay-my^3x^{-1}}{\overline{n+3}y^2 *-\overline{n+1}ax-nx^3y^{-1}+nax^2y^{-1}}$; which might have been found immediately from the given Equation, by the foregoing Rule.

Here the general Numbers m and n may be determined *pro lubitu*, by which means we may obtain as many Fluxional Equations as we pleaſe, which will all belong to the given Equation. And thus we may always find the ſimpleſt Expreſſion, or that which is beſt accommodated to the preſent exigence. Thus if we make $m=0$, and $n=0$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{3x^2-2ax+ay}{3y^2-ax}$, as found before. Or if we make $m=1$, and $n=-1$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{4x^2-3ax+2ay-y^3x^{-1}}{2y^2+x^3y^{-1}-ax^2y^{-1}}$, as before. Or if we make $m=-1$, and $n=1$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{2x^2-ax+y^3x^{-1}}{4y^2-2ax-x^3y^{-1}+ax^2y^{-1}}$, as before. Or if we make $m=-3$, and $n=-3$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{*ax-2ay+3y^3x^{-1}}{**2ax+3x^3y^{-1}-3ax^2y^{-1}}=\frac{ax^2y-2axy^2+3y^4}{2ax^2y+3x^4-3ax^3}$. And ſo of others. Now this variety of Solutions will beget no ambiguity in the Concluſion, as poſſibly might have been ſuſpected; for it is no other than what ought neceſſarily to ariſe, from the different forms the given Equation may acquire, as will appear afterwards. If we confine ourſelves to the Progreſſion of the Indices, it will bring the Solution to the common Method of taking Fluxions, which our Author has taught elſewhere, and which, becauſe it is eaſy and expeditious, and requires no certain order of the Terms, I ſhall here ſubjoin.

For every Term of the given Equation, ſo many Terms muſt be form'd in the Fluxional Equation, as there are flowing Quantities in that Term. And this muſt be done, (1.) by multiplying the Term by the Index of each flowing Quantity contain'd in it. (2.) By dividing it by the quantity itſelf; and, (3.) by multiplying by its Fluxion. Thus in the foregoing Equation $x^3-ax^2+ayx-y^3=0$, the Fluxion belonging to the Term x^3 is $\frac{3^1x^3\dot{x}}{x}$, or $3x^2\dot{x}$.

The

The Fluxion belonging to $-ax^2$ is $-\frac{2ax^2\dot{x}}{x}$, or $-2ax\dot{x}$. The Fluxion belonging to ayx is $\frac{ayx\dot{y}}{y}+\frac{ayx\dot{x}}{x}$, or $ax\dot{y}+ay\dot{x}$. And the Fluxion belonging to $-y^3$ is $-\frac{3y^3\dot{y}}{y}$, or $-3y^2\dot{y}$. So that the Fluxion of the whole Equation, or the whole Fluxional Equation, is $3x^2\dot{x}-2ax\dot{x}+a\dot{y}x+ay\dot{x}-3y^2\dot{y}=0$. Thus the Equation $x^m=y$, will give $m\dot{x}x^{m-1}=\dot{y}$; and the Equation $x^mz^n=y$, will give $m\dot{x}x^{m-1}z^n+nx^m\dot{z}z^{n-1}=\dot{y}$ for its Fluxional Equation. And the like of other Examples.

If we take the Author's ſimple Example, in pag. 19, or the Equation $y=xx$, or rather $ay-x^2=0$, that is $ayx^0-x^2y^0=0$, in order to find its moſt general Fluxional Equation; it may be perform'd by the Rule before given, ſuppoſing the Index of x to be encreas'd by m, and the Index of y by n. For then we ſhall have directly $\frac{\dot{y}}{\dot{x}}=\frac{mayx^{-1}-\overline{m+2x}}{nx^2y^{-1}-\overline{n+1a}}$. For the firſt Term of the given Equation being ayx^0, this multiply'd by the Index of x increas'd by m, that is by m, and divided by x, will give $mayx^{-1}$ for the firſt Term of the Numerator. Alſo the ſecond Term being $-x^2y^0$, this multiply'd by the Index of x increas'd by m, that is by $m+2$, and divided by x, will give $-\overline{m+2x}$ for the ſecond Term of the Numerator. Again, the firſt Term of the given Equation may be now $-x^2y^0$, which multiply'd by the Index of y increas'd by n, that is by n, and divided by y, will give (changing the ſign) nx^2y^{-1} for the firſt Term of the Denominator. Alſo the ſecond Term will then be ayx^0, which multiply'd by the Index of y increas'd by n, that is by $n+1$, and divided by y, will give (changing the Sign) $-\overline{n+1a}$ for the ſecond Term of the Denominator, as found above. Now from this general relation of the Fluxions, we may deduce as many particular ones as we pleaſe. Thus if we make $m=0$, and $n=0$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{2x}{a}$, or $a\dot{y}=2x\dot{x}$, agreeable to our Author's Solution in the place before cited. Or if we make $m=-2$, and $n=-1$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{2ayx^{-1}}{x^2y^{-1}}=\frac{2ay^2}{x^3}$. Or if we make $m=0$, and $n=-1$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{2x}{x^2y^{-1}}=\frac{2y}{x}$. Or if we make $n=0$, and $m=-2$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{2ax^{-1}}{a}=\frac{2y}{x}$, as before. All which, and innumerable other caſes, may be eaſily proved by a ſubſtitution of equivalents. Or we may prove it generally

rally thus. Becauſe by the given Equation it is $y = x^{\frac{2}{3}}a^{-\frac{1}{3}}$, in the value of the ratio $\frac{\dot{y}}{\dot{x}} = \frac{\frac{2}{3}ayx^{-\frac{1}{3}}}{\frac{1}{3}a^{\frac{2}{3}}y^{-\frac{1}{3}}}$ for y ſubſtitute its value, and we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{\frac{2}{3}x}{\frac{1}{3}a} = \frac{2x}{a}$, as above.

3. The Equation of the ſecond Example is $2y^3 + x^2y - 2cyz + 3yz^2 - z^3 = 0$, in which there are three flowing quantities y, x, and z, and therefore there muſt be three operations, or three Tables muſt be form'd. Firſt diſpoſe the Terms according to y, thus; $2y^3 + 0y^2 + x^2y - z^3y^0 = 0$, and multiply by the Terms of the Progreſſion $2 \times \dot{y}y^{-1}$, $1 \times \dot{y}y^{-1}$, $0 \times \dot{y}y^{-1}$, $-1 \times \dot{y}y^{-1}$, reſpectively, (where the Coefficients are form'd by diminiſhing the Indices of y by the common Number 1,) and the reſulting Terms will be $4\dot{y}y^2 * * + z^3\dot{y}y^{-1}$. Secondly, diſpoſe the Terms according to x, thus; $yx^2 + 0x + 2y^3x^0 = 0$, $-2cyz$, $+3yz^2$, $-z^3$, and multiply by the Terms of the Progreſſion $2 \times \dot{x}x^{-1}$, $1 \times \dot{x}x^{-1}$, $0 \times \dot{x}x^{-1}$, (where the Coefficients are the ſame as the Indices of x,) and the only reſulting Term here is $+ 2y\dot{x}x * *$. Laſtly, diſpoſe the Terms according to z, thus; $-z^3 + 3yz^2 - 2cyz + x^2yz^0 = 0$, $+2y^3$, and multiply by the Progreſſion $3 \times \dot{z}z^{-1}$, $2 \times \dot{z}z^{-1}$, $1 \times \dot{z}z^{-1}$, $0 \times \dot{z}z^{-1}$, (where the Coefficients are alſo the ſame as the Indices of z,) and the Terms will be $-3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z} *$. Then collecting all theſe Terms together, we ſhall have the Fluxional Equation $4\dot{y}y^2 + z^3\dot{y}y^{-1} + 2y\dot{x}x - 3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z} = 0$.

Here we have a notable inſtance of our Author's dexterity, at finding expedients for abbreviating. For in every one of theſe Operations ſuch a Progreſſion is choſe, as by multiplication will make the greateſt deſtruction of the Terms. By which means he arrives at the ſhorteſt Expreſſion, that the nature of the Problem will allow. If we ſhould ſeek the Fluxions of this Equation by the uſual method, which is taught above, that is, if we always aſſume the Progreſſions of the Indices, we ſhall have $6\dot{y}y^2 + 2\dot{x}xy + x^2\dot{y} - 2c\dot{y}z - 2cy\dot{z} + 3\dot{y}z^2 + 6yz\dot{z} - 3\dot{z}z^2 = 0$; which has two Terms more than the other form. And if the Progreſſions of the Indices are increas'd, in each caſe, by any common general Numbers, we may form the moſt general Expreſſion for the Fluxional Equation, that the Problem will admit of.

4. On occaſion of the laſt Example, in which are three Fluents and their Fluxions, our Author makes an uſeful Obſervation, for the Reduction and compleat Determination of ſuch Equations, tho' it be derived from the Rules of the vulgar Algebra; which matter may be conſider'd thus. Every Equation, conſiſting of two flowing or variable Quantities, is what correſponds to an indetermin'd Problem, admitting of an infinite number of Anſwers. Therefore one of thoſe quantities being aſſumed at pleaſure, or a particular value being aſſign'd to it, the other will alſo be compleatly determined. And in the Fluxional Equation derived from thence, thoſe particular values being ſubſtituted, the Ratio of the Fluxions will be given in Numbers, in any particular caſe. And one of the Fluxions being taken for Unity, or of any determinate value, the value of the other may be exhibited by a Number, which will be a compleat Determination.

But if the given Equation involve three flowing or indeterminate Quantities, two of them muſt be aſſumed to determine the third; or, which is the ſame thing, ſome other Equation muſt be either given or aſſumed, involving ſome or all the Fluents, in order to a compleat Determination. For then, by means of the two Equations, one of the Fluents may be eliminated, which will bring this to the former caſe. Alſo two Fluxional Equations may be derived, involving the three Fluxions, by means of which one of them may be eliminated. And ſo if the given Equation ſhould involve four Fluents, two other Equations ſhould be either given or aſſumed, in order to a compleat Determination. This will be ſufficiently explain'd by the two following Examples, which will alſo teach us how complicate Terms, ſuch as compound Fractions and Surds, are to be managed in this Method.

5, 6. Let the given Equation be $y^2 - a^2 - x\sqrt{a^2 - x^2} = 0$, of which we are to take the Fluxions. To the two Fluents y and x we may introduce a third z, if we aſſume another Equation. Let that be $z = x\sqrt{a^2 - x^2}$, and we ſhall have the two Equations $y^2 - a^2 - z = 0$, and $a^2x^2 - x^4 - z^2 = 0$. Then by the foregoing Solution their Fluxional Equations (at leaſt in one caſe) will be $2y\dot{y} - \dot{z} = 0$, and $a^2\dot{x}x - 2\dot{x}x^3 - \dot{z}z = 0$. Theſe two Fluential Equations, and their Fluxional Equations, may be reduced to one Fluential and one Fluxional Equation, by the uſual methods of Reduction: that is, we may eliminate z and $\dot{z}$ by ſubſtituting their values $yy - aa$ and $2y\dot{y}$. Then we ſhall have $y^2 - a^2 - x\sqrt{a^2 - x^2} = 0$,

$= 0$, and $2\dot{y}y - \frac{a^2\dot{x} - 2\dot{x}x^2}{\sqrt{a^2 - x^2}} = 0$. Or by taking away the ſurds, 'tis $a^2x^2 - x^4 - y^4 + 2a^2y^2 - a^4 = 0$, and then $a^2\dot{x}x - 2\dot{x}x^3 - 2\dot{y}y^3 + 2a^2\dot{y}y = 0$.

7. Or if the given Equation be $x^3 - ay^2 + \frac{by^3}{a+y} - x^2\sqrt{ay + x^2} = 0$, to find its correſponding Fluxional Equation; to the two flowing quantities x and y we may introduce two others z and v, and thereby remove the Fraction and the Radical, if we aſſume the two Equations $\frac{by^3}{a+y} = z$, and $x^2\sqrt{ay + xx} = v$. Then we ſhall have the three Equations $x^3 - ay^2 + z - v = 0$, $az + yz - by^3 = 0$, and $ayx^4 + x^6 - v^2 = 0$, which will give the three Fluxional Equations $3\dot{x}x^2 - 2a\dot{y}y + \dot{z} - \dot{v} = 0$, $a\dot{z} + y\dot{z} + \dot{y}z - 3b\dot{y}y^2 = 0$, and $a\dot{y}x^4 + 4ay\dot{x}x^3 + 6\dot{x}x^5 - 2\dot{v}v = 0$. Theſe by known Methods of the common Algebra may be reduced to one Fluential and one Fluxional Equation, involving x and y, and their Fluxions, as is required.

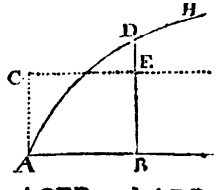
8. And by the ſame Method we may take the Fluxions of Binomial or other Radicals, of any kind, any how involved or complicated with one another. As for inſtance, if we were to find the Fluxion of $\sqrt{ax + \sqrt{aa - xx}}$, put it equal to y, or make $ax + \sqrt{aa - xx} = yy$. Alſo make $\sqrt{aa - xx} = z$. Then we ſhall have the two Fluential Equations $ax + z - y^2 = 0$, and $a^2 - x^2 - z^2 = 0$, from whence we ſhall have the two Fluxional Equations $a\dot{x} + \dot{z} - 2\dot{y}y = 0$, and $-2\dot{x}x - 2\dot{z}z = 0$, or $\dot{x}x + \dot{z}z = 0$. This laſt Equation, if for z and $\dot{z}$ we ſubſtitute their values $yy - ax$ and $2\dot{y}y - a\dot{x}$, will become $x\dot{x} + 2\dot{y}y^3 - 2ax\dot{y}y - a\dot{x}y^2 + a^2\dot{x}x = 0$; whence $\dot{y} = \frac{a\dot{x}y^2 - a^2\dot{x} - \dot{x}x}{2y^3 - 2axy}$. And here if for y we ſubſtitute its value $\sqrt{ax + \sqrt{aa - xx}}$, we ſhall have the Fluxion required $\dot{y} = \frac{a\dot{x}\sqrt{aa - xx} - \dot{x}x}{2\sqrt{aa - xx} \times \sqrt{ax + \sqrt{aa - xx}}}$. And many other Examples of a like kind will be found in the ſequel of this Work.

9, 10, 11, 12. In Examp. 5. the propoſed Equation is $zz + axz - y^4 = 0$, in which there are three variable quantities x, y, and z, and therefore the relation of the Fluxions will be $2\dot{z}z + a\dot{x}z + ax\dot{z} - 4\dot{y}y^3 = 0$. But as there wants another Fluential Equation, and thence another Fluxional Equation, to make a compleat determination; if only another Fluxional Equation were given or aſſumed, we ſhould have the required relation of the Fluxions $\dot{x}$ and $\dot{y}$.

Suppoſe

Suppoſe this Fluxional Equation were $\dot{z} = \dot{x}\sqrt{ax - xx}$; then by ſubſtitution we ſhould have the Equation $2z + ax \times \dot{x}\sqrt{ax - xx} + a\dot{x}z - 4\dot{y}y^3 = 0$, or the Analogy $\dot{x} : \dot{y} :: 4y^3 : 2z + ax \times \sqrt{ax - xx} + az$, which can be reduced no farther, (or z cannot be eliminated,) till we have the Fluential Equation, from which the Fluxional Equation $\dot{z} = \dot{x}\sqrt{ax - xx}$ is ſuppos'd to be derived. And thus we may have the relation of the Fluxions, even in ſuch caſes as we have not, or perhaps cannot have, the relation of the Fluents.

But tho' this Reduction may not perhaps be conveniently perform'd Analytically, or by Calculation, yet it may poſſibly be perform'd Geometrically, as it were, and by the Quadrature of Curves; as we may learn from our Author's preparatory Propoſition, and from the following general Conſiderations. Let the right Line AC, perpendicular to the right Line AB, be conceived to move always parallel to itſelf, ſo as that its extremity A may deſcribe the line AB. Let the point C be fixt, or always at the ſame diſtance from A, and let another point move from A towards C, with a velocity any how accelerated or retarded. The parallel motion of the line AC does not at all affect the progreſſive motion of the point moving from A towards C, but from a combination of theſe two independent motions, it will deſcribe the Curve ADH; while at the ſame time the fixt point C will deſcribe the right line CE, parallel to AB. Let the line AC be conceived to move thus, till it comes into the place BE, or BD. Then the line AC is conſtant, and remains the ſame, while the indefinite or flowing line becomes BD. Alſo the Areas deſcribed at the ſame time, ACEB and ADB, are likewiſe flowing quantities, and their velocities of deſcription, or their Fluxions, muſt neceſſarily be as their reſpective deſcribing lines, or Ordinates, BE and BD. Let AC or BE be Linear Unity, or a conſtant known right line, to which all the other lines are to be compared or refer'd; juſt as in Numbers, all other Numbers are tacitely refer'd to 1, or to Numeral Unity, as being the ſimpleſt of all Numbers. And let the Area ADB be ſuppos'd to be apply'd to BE, or Linear Unity, by which it will be reduced from the order of Surfaces to that of Lines; and let the reſulting line be call'd z. That is, make the Area ADB $= z \times$ BE; and if AB be call'd x, then is the Area ACEB $= x \times$ BE. Therefore the



Fluxions of theſe Areas will be $\dot{z} \times$ BE and $\dot{x} \times$ BE, which are as $\dot{z}$ and $\dot{x}$. But the Fluxions of the Areas were found before to be as BD to BE. So that it is $\dot{z} : \dot{x} ::$ BD : BE $= 1$, or $\dot{z} = \dot{x} \times$ BD. Conſequently in any Curve, the Fluxion of the Area will be as the Ordinate of the Curve, drawn into the Fluxion of the Abſciſs.

Now to apply this to the preſent caſe. In the Fluxional Equation before aſſumed $\dot{z} = \dot{x}\sqrt{ax - xx}$, if x repreſents the Abſciſs of a Curve, and $\sqrt{ax - xx}$ be the Ordinate; then will this Curve be a Circle, and z will repreſent the correſponding Area. So that we ſee from hence, whether the Area of a Circle can be exhibited or no, or, in general Terms, tho' in the Equation propoſed there ſhould be quantities involved, which cannot be determined or expreſs'd by any Geometrical Method, ſuch as the Areas or Lengths of Curve-lines; yet the relation of their Fluxions may nevertheleſs be found.

13. We now come to the Author's Demonſtration of his Solution, or to the proof of the Principles of the Method of Fluxions, here laid down, which certainly deſerves to engage our moſt ſerious attention. And more eſpecially, becauſe theſe Principles have been lately drawn into debate, without being well conſider'd or underſtood; poſſibly becauſe this Treatiſe of our Author's, expreſſly wrote on the ſubject, had not yet ſeen the light. As theſe Principles therefore have been treated as precarious at leaſt, if not wholly inſufficient to ſupport the Doctrine derived from them; I ſhall endeavour to examine into every the moſt minute circumſtance of this Demonſtration, and that with the utmoſt circumſpection and impartiality.

We have here in the firſt place a Definition and a Theorem together. *Moments* are defined to be *the indefinitely ſmall parts of flowing quantities, by the acceſſion of which, in indefinitely ſmall portions of time, thoſe quantities are continually increaſed.* The word *Moment (momentum, movimentum, à moveo,)* by analogy ſeems to have been borrow'd from Time. For as Time is conceived to be in continual flux, or motion, and as a greater and a greater Time is generated by the acceſſion of more and more Moments, which are conceived as the ſmalleſt particles of Time: So all other flowing Quantities may be underſtood as perpetually increaſing, by the acceſſion of their ſmalleſt particles, which therefore may not improperly be call'd their Moments. But what are here call'd their *ſmalleſt particles*, are not to be underſtood as if they were Atoms, or of any definite and determinate magnitude, as in the Method of Indiviſibles; but to be indefinitely ſmall, or continually decreaſing, till they are leſs than

than any aſſignable quantities, and yet may then retain all poſſible varieties of proportion to one another. That theſe Moments are not chimerical, viſionary, or merely imaginary things, but have an exiſtence *ſui generis*, at leaſt Mathematically and in the Underſtanding, is a neceſſary conſequence from the infinite Diviſibility of Quantity, which I think hardly any body now conteſts *. For all continued quantity whatever, tho' not indeed actually, yet mentally may be conceived to be divided *in infinitum*. Perhaps this may be beſt illuſtrated by a comparative gradation or progreſs of Magnitudes. Every finite and limited *Quantity* may be conceived as divided into any finite number of ſmaller *parts*. This Diviſion may proceed, and thoſe parts may be conceived to be farther divided in very little, but ſtill finite parts, or *particles*, which yet are not Moments. But when theſe particles are farther conceived to be divided, not actually but mentally, ſo far as to become of a magnitude leſs than any aſſignable, (and what can ſtop the progreſs of the Mind?) then are they properly the *Moments* which are to be underſtood here. As this gradation of diminution certainly includes no abſurdity or contradiction, the Mind has the privilege of forming a Conception of theſe Moments, a poſſible Notion at leaſt, though perhaps not an adequate one; and then Mathematicians have a right of applying them to uſe, and of making ſuch Inferences from them, as by any ſtrict way of reaſoning may be derived.

It is objected, that we cannot form an intelligible and adequate Notion of theſe Moments, becauſe ſo obſcure and incomprehenſible an Idea, as that of Infinity is, muſt needs enter that Notion; and therefore they ought to be excluded from all Geometrical Diſquiſitions. It may indeed be allowed, that we have not an adequate Notion of them on that account, ſuch as exhauſts the whole nature of the thing, neither is it at all neceſſary; for a partial Notion, which is that of their Diviſibility *ſine fine*, without any regard to their magnitude, is ſufficient in the preſent caſe. There are many other Speculations in the Mathematicks, in which a Notion of Infinity is a neceſſary ingredient, which however are admitted by all Geometricians, as uſeful and demonſtrable Truths. The Doctrine of commenſurable and incommenſurable magnitudes includes a Notion of Infinity, and yet is received as a very demonſtrable Doctrine. We have a perfect Idea of a Square and its Diagonal, and yet we

* Perhaps the ingenious Author of the Diſcourſe call'd *The Analyſt* muſt be excepted, who is pleas'd to ask, in his fifth Query, *Whether it be not unneceſſary, as well as abſurd, to ſuppoſe that finite Extenſion is infinitely diviſible?* See alſo Query 19, 20, 21, &c.

know they will admit of no finite common meaſure, or that their proportion cannot be exhibited in rational Numbers, tho' ever ſo ſmall, but may by a ſeries of decimal or other parts continued *ad infinitum*. In common Arithmetick we know, that the vulgar Fraction $\frac{1}{3}$, and the decimal Fraction 0,666666, &c. continued *ad infinitum*, are one and the ſame thing; and therefore if we have a ſcientifick notion of the one, we have likewiſe of the other. When I deſcribe a right line with my Pen, ſuppoſe of an Inch long, I deſcribe firſt one half of the line, then one half of the remainder, then one half of the next remainder, and ſo on. That is, I actually run over all thoſe infinite diviſions and ſubdiviſions, before I have compleated the Line, tho' I do not attend to them, or cannot diſtinguiſh them. And by this I am indubitably certain, that this Series of Fractions $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, &c. continued *ad infinitum*, is preciſely equal to Unity. *Euclid* has demonſtrated in his Elements, that the Circular Angle of Contact is leſs than any aſſignable right-lined Angle, or, which is the ſame thing, is an infinitely little Angle in compariſon with any finite Angle: And our Author ſhews us ſtill greater Myſteries, about the infinite gradations of Angles of Contact. In Geometry we know, that Curves may continually approach towards their Aſymptotes, and yet will not actually meet with them; till both are continued to an infinite diſtance. We know likewiſe, that many of their included Areas, or Solids, will be but of a finite and determinable magnitude, even tho' their lengths ſhould be actually continued *ad infinitum*. We know that ſome Spirals make infinite Circumvolutions about a Pole, or Center, and yet the whole Line, thus infinitely involved, is but of a finite, determinable, and aſſignable length. The Methods of computing Logarithms ſuppoſe, that between any two given Numbers, an infinite number of mean Proportionals may be interpoſed; and without ſome Notion of Infinity their nature and properties are hardly intelligible or diſcoverable. And in general, many of the moſt ſublime and uſeful parts of knowledge muſt be baniſh'd out of the Mathematicks, if we are ſo ſcrupulous as to admit of no Speculations, in which a Notion of Infinity will be neceſſarily included. We may therefore as ſafely admit of Moments, and the Principles upon which the Method of Fluxions is here built, as any of the fore-mention'd Speculations.

The nature and notion of Moments being thus eſtabliſh'd, we may paſs on to the afore-mention'd Theorem, which is this.

The

The (contemporary) Moments of flowing quantities are as the Velocities of flowing or increaſing; that is, as their Fluxions. Now if this be proved of Lines, it will equally obtain in all flowing quantities whatever, which may always be adequately repreſented and expounded by Lines. But in equable Motions, the Times being given, the Spaces deſcribed will be as the Velocities of Deſcription, as is known in Mechanicks. And if this be true of any finite Spaces whatever, or of all Spaces in general, it muſt alſo obtain in infinitely little Spaces, which we call Moments. And even in Motions continually accelerated or retarded, the Motions in infinitely little Spaces, or Moments, muſt degenerate into equability. So that the Velocities of increaſe or decreaſe, or the Fluxions, will be always as the contemporary Moments. Therefore the Ratio of the Fluxions of Quantities, and the Ratio of their contemporary Moments, will always be the ſame, and may be uſed promiſcuouſly for each other.

14. The next thing to be ſettled is a convenient Notation for theſe Moments, by which they may be diſtinguiſh'd, repreſented, compared, and readily ſuggeſted to the Imagination. It has been appointed already, that when x, y, z, v, &c. ſtand for variable or flowing quantities, then their Velocities of increaſe, or their Fluxions, ſhall be repreſented by $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{v}$, &c. which therefore will be proportional to the contemporary Moments. But as theſe are only Velocities, or magnitudes of another Species, they cannot be the Moments themſelves, which we conceive as indefinitely little Spaces, or other analogous quantities. We may therefore here aptly introduce the Symbol o, not to ſtand for abſolute nothing, as in Arithmetick, but a vaniſhing Space or Quantity, which was juſt now finite, but by continually decreaſing, in order preſently to terminate in mere nothing, is now become leſs than any aſſignable Quantity. And we have certainly a right ſo to do. For if the notion is intelligible, and implies no contradiction as was argued before, it may ſurely be inſinuated by a Character appropriate to it. This is not aſſigning the quantity, which would be contrary to the *hypotheſis*, but is only appointing a mark to repreſent it. Then multiplying the Fluxions by the vaniſhing quantity o, we ſhall have the ſeveral quantities $\dot{x}o$, $\dot{y}o$, $\dot{z}o$, $\dot{v}o$, &c. which are vaniſhing likewiſe, and proportional to the Fluxions reſpectively. Theſe therefore may now repreſent the contemporary Moments of x, y, z, v, &c. And in general, whatever other flowing quantities, as well as Lines and

Spaces, are repreſented by x, y, z, v, &c. as o may ſtand for a vaniſhing quantity of the ſame kind, and as $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{v}$, &c. may ſtand for their Velocities of increaſe or decreaſe, (or, if you pleaſe, for Numbers proportional to thoſe Velocities,) then may $\dot{x}o$, $\dot{y}o$, $\dot{z}o$, $\dot{v}o$, &c. always denote their reſpective ſynchronal Moments, or momentary acceſſions, and may be admitted into Computations accordingly. And this we come now to apply.

15. We muſt now have recourſe to a very notable, uſeful, and extenſive property, belonging to all Equations that involve flowing Quantities. Which property is, that in the progreſs of flowing, the Fluents will continually acquire new values, by the acceſſion of contemporary parts of thoſe Fluents, and yet the Equation will be equally true in all theſe caſes. This is a neceſſary reſult from the Nature and Definition of variable Quantities. Conſequently theſe Fluents may be any how increaſed or diminiſh'd by their contemporary Increments or Decrements; which Fluents, ſo increaſed or diminiſhed, may be ſubſtituted for the others in the Equation. As if an Equation ſhould involve the Fluents x and y, together with any given quantities, and X and Y are ſuppoſed to be any of their contemporary Augments reſpectively. Then in the given Equation we may ſubſtitute $x + X$ for x, and $y + Y$ for y, and yet the Equation will be good, or the equality of the Terms will be preſerved. So if X and Y were contemporary Decrements, inſtead of x and y we might ſubſtitute $x - X$ and $y - Y$ reſpectively. And as this muſt hold good of all contemporary Increments or Decrements whatever, whether finitely great or infinitely little, it will be true likewiſe of contemporary Moments. That is, inſtead of x and y in any Equation, we may ſubſtitute $x + \dot{x}o$ and $y + \dot{y}o$, and yet we ſhall ſtill have a good Equation. The tendency of this will appear from what immediately follows.

16. The Author's ſingle Example is a kind of Induction, and the proof of this may ſerve for all caſes. Let the Equation $x^3 - ax^2 + axy - y^3 = o$ be given as before, including the variable quantities x and y, inſtead of which we may ſubſtitute theſe quantities increas'd by their contemporary Moments, or $x + \dot{x}o$ and $y + \dot{y}o$ reſpectively. Then we ſhall have the Equation $\overline{x + \dot{x}o}|^3 - a \times \overline{x + \dot{x}o}|^2 + a \times \overline{x + \dot{x}o} \times \overline{y + \dot{y}o} - \overline{y + \dot{y}o}|^3 = o$. Theſe Terms being expanded, and reduced to three orders or columns, according as the vaniſhing quantity o is of none, one, or of more dimenſions, will ſtand as in the Margin.

17.

17, 18. Here the Terms of the firſt order, or column, remove or deſtroy one another, as being abſolutely equal to nothing by the given Equation. They being therefore expunged, the remaining Terms may all be divided by the common Multiplier o, whatever it is. This being done, all the Terms of the third order will ſtill be affected by o, of one or more dimenſions, and may therefore be expunged, as infinitely leſs than the others. Laſtly, there will only remain thoſe of the ſecond order or column, that is $3\dot{x}\dot{x}^2 - 2a\dot{x}\dot{x} + a\dot{x}\dot{y} + a\dot{y}\dot{x} - 3\dot{y}\dot{y}^2 = 0$, which will be the Fluxional Equation required. Q. E. D.

$$\left.\begin{array}{llll} x^3 + 3\dot{x}ox^2 & + 3\dot{x}^2o^2x & & \\ & + \dot{x}^3o^3 & & \\ -ax^2 - 2a\dot{x}ox & - a\dot{x}^2o^2 & & \\ +axy + a\dot{x}oy & + a\dot{x}\dot{y}o^2 & & \\ + a\dot{y}\dot{x}o & & & \\ -y^3 - 3\dot{y}\dot{y}^2o & - 3\dot{y}^2\dot{y}o^2 & & \\ & - \dot{y}^3o^3 & & \end{array}\right\} = 0.$$

The ſame Concluſions may be thus derived, in ſomething a different manner. Let X and Y be any ſynchronal Augments of the variable quantities x and y, as before, the relation of which quantities is exhibited by any Equation. Then may $x + X$ and $y + Y$ be ſubſtituted for x and y in that Equation. Suppoſe for inſtance that $x^3 - ax^2 + axy - y^3 = 0$; then by ſubſtitution we ſhall have $\overline{x + X}|^3 - a \times \overline{x + X}|^2 + a \times \overline{x + X} \times \overline{y + Y} - \overline{y + Y}|^3 = 0$; or *in terminis expanſis* $x^3 + 3x^2X + 3xX^2 + X^3 - ax^2 - 2axX - aX^2 + axy + axY + aXy + aXY - y^3 - 3y^2Y - 3yY^2 - Y^3 = 0$. But the Terms $x^3 - ax^2 + axy - y^3 = 0$ will vaniſh out of the Equation, and leave $3x^2X + 3xX^2 + X^3 - 2axX - aX^2 + axY + aXy + aXY - 3y^2Y - 3yY^2 - Y^3 = 0$, for the relation of the contemporary Augments, let their magnitude be what it will. Or reſolving this Equation into an Analogy, the ratio of theſe Augments may be this, $\frac{Y}{X} = \frac{3x^2 + 3xX + X^2 - 2ax - aX + ay}{-ax - aX + y^2 + 3yY + Y^2}$. Now to find the *ultimate ratio* of theſe Augments, or their ratio when they become Moments, ſuppoſe X and Y to diminiſh till they become vaniſhing quantities, and then they may be expunged out of this value of the ratio. Or in thoſe circumſtances it will be $\frac{Y}{X} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}$, which is now the ratio of the Moments. And this is the ſame ratio as that of the Fluxions, or it will be $\frac{\dot{y}}{\dot{x}} = \frac{3x^2 - 2ax + ay}{y^2 - ax}$, or $3y^2\dot{y} - ax\dot{y} = 3x^2\dot{x} - 2ax\dot{x} + ay\dot{x}$, as was found before.

In this way of arguing there is no aſſumption made, but what is juſtifiable by the received Methods both of the ancient and modern Geometricians. We only deſcend from a general Propoſition, which is undeniable, to a particular caſe which is certainly included in it.

That

That is, having the relation of the variable Quantities, we thence directly deduce the relation or ratio of their contemporary Augments; and having this, we directly deduce the relation or ratio of thoſe contemporary Augments when they are naſcent or evaneſcent, juſt beginning or juſt ceaſing to be; in a word, when they are Moments, or vaniſhing Quantities. To evade this reaſoning, it ought to be proved, that no Quantities can be conceived leſs than aſſignable Quantities; that the Mind has not the privilege of conceiving Quantity as perpetually diminiſhing *ſine fine*; that the Conception of a vaniſhing Quantity, a Moment, an Infiniteſimal, &c. includes a contradiction: In ſhort, that Quantity is not (even mentally) diviſible *ad infinitum*; for to that the Controverſy muſt be reduced at laſt. But I believe it will be a very difficult matter to extort this Principle from the Mathematicians of our days, who have been ſo long in quiet poſſeſſion of it, who are indubitably convinced of the evidence and certainty of it, who continually and ſucceſsfully apply it, and who are ready to acknowledge the extreme fertility and uſefulneſs of it, upon ſo many important occaſions.

19. Nothing remains, I think, but to account for theſe two circumſtances, belonging to the Method of Fluxions, which our Author briefly mentions here. Firſt that the given Equation, whoſe Fluxional Equation is to be found, may involve any number of flowing quantities. This has been ſufficiently proved already, and we have ſeen ſeveral Examples of it. Secondly, that in taking Fluxions we need not always confine ourſelves to the progreſſion of the Indices, but may aſſume infinite other Arithmetical Progreſſions, as conveniency may require. This will deſerve a little farther illuſtration, tho' it is no other than what muſt neceſſarily reſult from the different forms, which any given Equation may aſſume, in an infinite variety. Thus the Equation $x^3 - ax^2 + axy - y^3 = 0$, being multiply'd by the general quantity $x^m y^n$, will become $x^{m+3}y^n - ax^{m+2}y^n + ax^{m+1}y^{n+1} - x^m y^{n+3} = 0$, which is virtually the ſame Equation as it was before, tho' it may aſſume infinite forms, according as we pleaſe to interpret m and n. And if we take the Fluxions of this Equation, in the uſual way, we ſhall have $m + 3\dot{x}x^{m+2}y^n + nx^{m+3}\dot{y}y^{n-1} - m + 2a\dot{x}x^{m+1}y^n - nax^{m+2}\dot{y}y^{n-1} + m + 1a\dot{x}x^m y^{n+1} + n + 1ax^{m+1}\dot{y}y^n - m\dot{x}x^{m-1}y^{n+3} - n + 3x^m\dot{y}y^{n+2} = 0$. Now if we divide this again by $x^m y^n$, we ſhall have $m + 3\dot{x}x^2 + nx^3\dot{y}y^{-1} - m + 2a\dot{x}x - nax^2\dot{y}y^{-1} + m + 1a\dot{x}y + n + 1ax\dot{y} - m\dot{x}x^{-1}y^3 - n + 3\dot{y}y^2 = 0$, which is the ſame general Equation as was derived before. And the like may be underſtood of all other Examples.

SECT.

SECT. II. *Concerning Fluxions of ſuperior orders, and the method of deriving their Equations.*

IN this Treatiſe our Author conſiders only firſt Fluxions, and has not thought fit to extend his Method to ſuperior orders, as not directly falling within his preſent purpoſe. For tho' he here purſues Speculations which require the uſe of ſecond Fluxions, or higher orders, yet he has very artfully contrived to reduce them to firſt Fluxions, and to avoid the neceſſity of introducing Fluxions of ſuperior orders. In his other excellent Works of this kind, which have been publiſh'd by himſelf, he makes expreſs mention of them, he diſcovers their nature and properties, and gives Rules for deriving their Equations. Therefore that this Work may be the more ſerviceable to Learners, and may fulfil the deſign of being an Inſtitution, I ſhall here make ſome inquiry into the nature of ſuperior Fluxions, and give ſome Rules for finding their Equations. And afterwards, in its proper place, I ſhall endeavour to ſhew ſomething of their application and uſe.

Now as the Fluxions of quantities which have been hitherto conſider'd, or their comparative Velocities of increaſe and decreaſe, are themſelves, and of their own nature, variable and flowing quantities alſo, and as ſuch are themſelves capable of perpetual increaſe and decreaſe, or of perpetual acceleration and retardation; they may be treated as other flowing quantities, and the relation of their Fluxions may be inquired and diſcover'd. In order to which we will adopt our Author's Notation already publiſh'd, in which we are to conceive, that as x, y, z, &c. have their Fluxions $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. ſo theſe likewiſe have their Fluxions $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, &c. which are the ſecond Fluxions of x, y, z, &c. And theſe again, being ſtill variable quantities, have their Fluxions denoted by $\dddot{x}$, $\dddot{y}$, $\dddot{z}$, &c. which are the third Fluxions of x, y, z, &c. And theſe again, being ſtill flowing quantities, have their Fluxions $\ddddot{x}$, $\ddddot{y}$, $\ddddot{z}$, &c. which are the fourth Fluxions of x, y, z, &c. And ſo we may proceed to ſuperior orders, as far as there ſhall be occaſion. Then, when any Equation is propoſed, conſiſting of variable quantities, as the relation of its Fluxions may be found by what has been taught before; ſo by repeating only the ſame operation, and conſidering the Fluxions as flowing Quantities, the

relation of the ſecond Fluxions may be found. And the like for all higher orders of Fluxions.

Thus if we have the Equation $y^2 - ax = 0$, in which are the two Fluents y and x, we ſhall have the firſt Fluxional Equation $2\dot{y}y - a\dot{x} = 0$. And here, as we have the three Fluents y, $\dot{y}$, and $\dot{x}$, if we take the Fluxions again, we ſhall have the ſecond Fluxional Equation $2\ddot{y}y + 2\dot{y}^2 - a\ddot{x} = 0$. And here, as there are four Fluents y, $\dot{y}$, $\ddot{y}$, and $\ddot{x}$, if we take the Fluxions again, we ſhall have the third Fluxional Equation $2\dot{\ddot{y}}y + 2\ddot{y}\dot{y} + 4\dot{y}\ddot{y} - a\dot{\ddot{x}} = 0$, or $2\dot{\ddot{y}}y + 6\ddot{y}\dot{y} - a\dot{\ddot{x}} = 0$. And here, as there are five Fluents y, $\dot{y}$, $\ddot{y}$, $\dot{\ddot{y}}$, and $\dot{\ddot{x}}$, if we take the Fluxions again, we ſhall have the fourth Fluxional Equation $2\ddot{\ddot{y}}y + 2\dot{\ddot{y}}\dot{y} + 6\dot{\ddot{y}}\ddot{y} + 6\ddot{y}^2 - a\ddot{\ddot{x}} = 0$, or $2\ddot{\ddot{y}}y + 8\dot{\ddot{y}}\dot{y} + 6\ddot{y}^2 - a\ddot{\ddot{x}} = 0$. And here, as there are ſix Fluents y, $\dot{y}$, $\ddot{y}$, $\dot{\ddot{y}}$, $\ddot{\ddot{y}}$, and $\dot{\ddot{\ddot{x}}}$, if we take the Fluxions again, we ſhall have $2\dot{\ddot{\ddot{y}}}y + 2\ddot{\ddot{y}}\dot{y} + 8\dot{\ddot{\ddot{y}}}\ddot{y} + 8\ddot{\dot{\ddot{y}}}\dot{\ddot{y}} + 12\dot{\ddot{y}}\ddot{\ddot{y}} - a\dot{\ddot{\ddot{x}}} = 0$, or $2\dot{\ddot{\ddot{y}}}y + 10\ddot{\ddot{y}}\dot{y} + 20\dot{\ddot{y}}\ddot{y} - a\dot{\ddot{\ddot{x}}} = 0$, for the fifth Fluxional Equation. And ſo on to the ſixth, ſeventh, &c.

Now the Demonſtration of this will proceed much after the manner as our Author's Demonſtration of firſt Fluxions, and is indeed virtually included in it. For in the given Equation $y^2 - ax = 0$, if we ſuppoſe y and x to become at the ſame time $y + \dot{y}o$ and $x + \dot{x}o$, (that is, if we ſuppoſe $\dot{y}o$ and $\dot{x}o$ to denote the ſynchronal Moments of the Fluents y and x,) then by ſubſtitution we ſhall have $\overline{y + \dot{y}o}|^2 - a \times \overline{x + \dot{x}o} = 0$, or *in terminis expanſis*, $y^2 + 2y\dot{y}o + \dot{y}^2o^2 - ax - a\dot{x}o = 0$. Where expunging $y^2 - ax = 0$, and $\dot{y}^2o^2$, and dividing the reſt by o, it will be $2y\dot{y} - a\dot{x} = 0$ for the firſt fluxional Equation. Now in this Equation, if we ſuppoſe the ſynchronal Moments of the Fluents y, $\dot{y}$, and $\dot{x}$, to be $\dot{y}o$, $\ddot{y}o$, and $\ddot{x}o$ reſpectively; for thoſe Fluents we may ſubſtitute $\overline{y + \dot{y}o}$, $\overline{\dot{y} + \ddot{y}o}$, and $\overline{\dot{x} + \ddot{x}o}$ in the laſt Equation, and it will become $2\overline{y + \dot{y}o} \times \overline{\dot{y} + \ddot{y}o} - a \times \overline{\dot{x} + \ddot{x}o} = 0$, or expanding, $2y\dot{y} + 2\dot{y}\dot{y}o + 2y\ddot{y}o + 2\dot{y}\ddot{y}oo - a\dot{x} - a\dot{x}o = 0$. Here becauſe $2y\dot{y} - a\dot{x} = 0$ by the given Equation, and becauſe $2\dot{y}\ddot{y}oo$ vaniſhes; divide the reſt by o, and we ſhall have $2\dot{y}^2 + 2y\ddot{y} - a\dot{x} = 0$ for the ſecond fluxional Equation. Again in this Equation, if we ſuppoſe the Synchronal Moments of the Fluents y, $\dot{y}$, $\ddot{y}$, and $\dot{\ddot{x}}$, to be $\dot{y}o$, $\ddot{y}o$, $\dot{\ddot{y}}o$, and $\dot{\ddot{x}}o$ reſpectively; for thoſe Fluents we

we may ſubſtitute $y + \dot{y}o$, $\dot{y} + \ddot{y}o$, $\ddot{y} + \dddot{y}o$, and $\ddot{x} + \dddot{x}o$ in the laſt Equation, and it will become $2 \times \dot{y} + \ddot{y}o|^2 + 2y + 2\dot{y}o \times \ddot{y} + \dddot{y}o - a \times \ddot{x} + \dddot{x}o = o$, or expanding and collecting, $2\dot{y}^2 + 6\dot{y}\ddot{y}o + 2\ddot{y}^2o^2 + 2y\dddot{y} + 2\dddot{y}o + 2\dot{y}\dddot{y}o^2 - a\dddot{x} - a\dddot{x}o = o$. But here becauſe $2\dot{y}^2 + 2y\ddot{y} - a\ddot{x} = o$ by the laſt Equation; dividing the reſt by o, and expunging all the Terms in which o will ſtill be found, we ſhall have $6\dot{y}\ddot{y} + 2y\dddot{y} - a\dddot{x} = o$ for the third fluxional Equation. And in like manner for all other orders of Fluxions, and for all other Examples. Q. E. D.

To illuſtrate the method of finding ſuperior Fluxions by another Example, let us take our Author's Equation $x^3 - ax^2 + axy - y^3 = o$, in which he has found the ſimpleſt relation of the Fluxions to be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + axy - 3\dot{y}y^2 = o$. Here we have the flowing quantities x, y, $\dot{x}$, $\dot{y}$; and by the ſame Rules the Fluxion of this Equation, when contracted, will be $3\ddot{x}x^2 + 6\dot{x}^2x - 2a\ddot{x}x - 2a\dot{x}^2 + a\ddot{x}y + 2a\dot{x}\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = o$. And in this Equation we have the flowing quantities x, y, $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, ſo that taking the Fluxions again by the ſame Rules, we ſhall have the Equation, when contracted, $3\dddot{x}x^2 + 18\ddot{x}\dot{x}x + 6\dot{x}^3 - 2a\dddot{x}x - 6a\ddot{x}\dot{x} + a\dddot{x}y + 3a\ddot{x}\dot{y} + 3a\dot{x}\ddot{y} + ax\dddot{y} - 3\dddot{y}y^2 - 18\ddot{y}\dot{y}y - 6\dot{y}^3 = o$. And as in this Equation there are found the flowing quantities x, y, $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, $\dddot{x}$, $\dddot{y}$, we might proceed in like manner to find the relations of the fourth Fluxions belonging to this Equation, and all the following orders of Fluxions.

And here it may not be amiſs to obſerve, that as the propoſed Equation expreſſes the conſtant relation of the variable quantities x and y; and as the firſt fluxional Equation expreſſes the conſtant relation of the variable (but finite and aſſignable) quantities $\dot{x}$ and $\dot{y}$, which denote the comparative Velocity of increaſe or decreaſe of x and y in the propoſed Equation: So the ſecond fluxional Equation will expreſs the conſtant relation of the variable (but finite and aſſignable) quantities $\ddot{x}$ and $\ddot{y}$, which denote the comparative Velocity of the increaſe or decreaſe of $\dot{x}$ and $\dot{y}$ in the foregoing Equation. And in the third fluxional Equation we have the conſtant relation of the variable (but finite and aſſignable) quantities $\dddot{x}$ and $\dddot{y}$, which will denote the com-

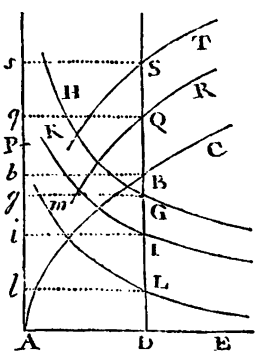
comparative Velocity of the increaſe or decreaſe of $\ddot{x}$ and $\ddot{y}$ in the foregoing Equation. And ſo on for ever. Here the Velocity of a Velocity, however uncouth it may ſound, will be no abſurd Idea when rightly conceived, but on the contrary will be a very rational and intelligible Notion. If there be ſuch a thing as Motion any how continually accelerated, that continual Acceleration will be the Velocity of a Velocity; and as that variation may be continually varied, that is, accelerated or retarded, there will be in nature, or at leaſt in the Underſtanding, the Velocity of a Velocity of a Velocity. Or in other words, the Notion of ſecond, third, and higher Fluxions, muſt be admitted as ſound and genuine. But to proceed:

We may much abbreviate the Equations now derived, by the known Laws of Analyticks. From the given Equation $x^3 - ax^2 + axy - y^3 = 0$ there is found a new Equation, wherein, becauſe of two new Symbols $\dot{x}$ and $\dot{y}$ introduced, we are at liberty to aſſume another Equation, beſides this now found, in order to a juſt Determination. For ſimplicity-ſake we may make $\dot{x}$ Unity, or any other conſtant quantity; that is, we may ſuppoſe x to flow equably, and therefore its Velocity is uniform. Make therefore $\dot{x} = 1$, and the firſt fluxional Equation will become $3x^2 - 2ax + ay + ax\dot{y} - 3\dot{y}y^2 = 0$. So in the Equation $3\ddot{x}x^2 + 6\dot{x}^2x - 2a\ddot{x}x - 2a\dot{x}^2 + a\ddot{x}y + 2a\dot{x}\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = 0$ there are four new Symbols introduced, $\dot{x}$, $\dot{y}$, $\ddot{x}$, and $\ddot{y}$, and therefore we may aſſume two other congruous Equations, which together with the two now found, will amount to a compleat Determination. Thus if for the ſake of ſimplicity we make one to be $\dot{x} = 1$, the other will neceſſarily be $\ddot{x} = 0$; and theſe being ſubſtituted, will reduce the ſecond fluxional Equation to this, $6x - 2a + 2a\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = 0$. And thus in the next Equation, wherein there are ſix new Symbols $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, $\dddot{x}$, $\dddot{y}$; beſides the three Equations now found, we may take $\dot{x} = 1$, and thence $\ddot{x} = 0$, $\dddot{x} = 0$, which will reduce it to $6 + 3a\ddot{y} + ax\dddot{y} - 3\dddot{y}y^2 - 18\ddot{y}\dot{y}y - 6\dot{y}^3 = 0$. And the like of Equations of ſucceeding orders.

But all theſe Reductions and Abbreviations will be beſt made as the Equations are derived. Thus the propoſed Equation being $x^3 - ax^2 + axy - y^3 = 0$, taking the Fluxions, and at the ſame time making $\dot{x} = 1$, (and conſequently $\ddot{x}$, $\dddot{x}$, &c. $= 0$,) we ſhall have $3x^2 - 2ax + ay + ax\dot{y} - 3\dot{y}y^2 = 0$. And taking the Fluxions again,

again, it will be $6x - 2a + 2a\dot{y} + a\ddot{x}\dot{y} - 3\ddot{y}\dot{y}^2 - 6\dot{y}^2 y = 0$. And taking the Fluxions again, it will be $6 + 3a\ddot{y} + a\dddot{x}\dot{y} - 3\dddot{y}\dot{y}^3 - 18\ddot{y}\dot{y}\dot{y} - 6\dot{y}^3 = 0$. And taking the Fluxions again, it will be $4a\dddot{y} + a\dddot{x}\dot{y} - 3\dddot{y}\dot{y}^2 - 24\dddot{y}\dot{y}\dot{y} - 18\ddot{y}^2\dot{y} - 36\ddot{y}\dot{y}^2 = 0$. And ſo on, as far as there is occaſion.

But now for the clearer apprehenſion of theſe ſeveral orders of Fluxions, I ſhall endeavour to illuſtrate them by a Geometrical Figure, adapted to a ſimple and a particular caſe. Let us aſſume the Equation $y^2 = ax$, or $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$, which will therefore belong to the Parabola ABC, whoſe Parameter is AP $= a$, Abſciſs AD $= x$, and Ordinate BD $= y$; where AP is a Tangent at the Vertex A. Then taking the Fluxions, we ſhall have $\dot{y} = \frac{1}{2}a^{\frac{1}{2}}\dot{x}x^{-\frac{1}{2}}$. And ſuppoſing the Parabola to be deſcribed by the equable motion of the Ordinate upon the Abſciſs, that equable Velocity may be expounded by the given Line or Parameter a, that is, we may put $\dot{x} = a$. Then it will be $\dot{y} = (\frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{2x^{\frac{1}{2}}} = \frac{a^{\frac{1}{2}}x^{\frac{1}{2}}}{2x} =) \frac{ay}{2x}$, which will give us this Conſtruction. Make x (AD) : y (BD) :: $\frac{1}{2}a$ ($\frac{1}{2}$AP) : DG $= \frac{ay}{2x} = \dot{y}$, and the Line DG will therefore repreſent the Fluxion of y or BD. And if this be done every where upon AE, (or if the Ordinate DG be ſuppos'd to move upon AE with a parallel motion,) a Curve GH will be conſtructed or deſcribed, whoſe Ordinates will every where expound the Fluxions of the correſponding Ordinates of the Parabola ABC. This Curve will be one of the Hyperbola's between the Aſymptotes AE and AP; for its Equation is $\dot{y} = \frac{a^{\frac{1}{2}}}{2x^{\frac{1}{2}}}$, or $\dot{y}\dot{y} = \frac{a^3}{4x}$.



Again, from the Equation $\dot{y} = \frac{ay}{2x}$, or $2x\dot{y} = ay$, by taking the Fluxions again, and putting $\dot{x} = a$ as before, we ſhall have $2a\dot{y} + 2x\ddot{y} = a\dot{y}$, or $-\ddot{y} = \frac{a\dot{y}}{2x}$; where the negative ſign ſhews only, that $\ddot{y}$ is to be conſider'd rather as a retardation than an acceleration, or an acceleration the contrary way. Now this will give us the following

following Conſtruction. Make x (AD) : $\dot{y}$ (DG) :: $\frac{1}{4}a$ ($\frac{1}{2}$AP) : DI $= \ddot{y}$, and the Line DI will therefore repreſent the Fluxion of DG, or of $\dot{y}$, and therefore the ſecond Fluxion of BD, or of y. And if this be done every where upon AE, a Curve IK will be conſtructed, whoſe Ordinates will always expound the ſecond Fluxions of the correſponding Ordinates of the Parabola ABC. This Curve likewiſe will be one of the Hyperbola's, for its Equation is $-\ddot{y} = \frac{a\dot{y}}{2x} = \frac{a^{\frac{5}{2}}}{4x^{\frac{3}{2}}}$, or $\dot{y}\ddot{y} = \frac{a^5}{16x^3}$.

Again, from the Equation $-\ddot{y} = \frac{a\dot{y}}{2x}$, or $-2x\ddot{y} = a\dot{y}$, by taking the Fluxions we ſhall have $-2a\ddot{y} - 2x\dot{\ddot{y}} = a\ddot{y}$, or $-\dot{\ddot{y}} = \frac{3a\ddot{y}}{2x}$, which will give us this Conſtruction. Make x (AD) : $\ddot{y}$ (DI) :: $\frac{1}{2}a$ ($\frac{1}{2}$AP) : DL $= \dot{\ddot{y}}$, and the Line DL will therefore repreſent the Fluxion of DI, or of $\ddot{y}$, the ſecond Fluxion of DG, or of $\dot{y}$, and the third Fluxion of BD, or of y. And if this be done every where upon AE, a Curve LM will be conſtructed, whoſe Ordinates will always expound the third Fluxions of the correſponding Ordinates of the Parabola ABC. This Curve will be an Hyperbola, and its Equation will be $-\dot{\ddot{y}} = \frac{3a\ddot{y}}{2x} = \frac{3a^{\frac{7}{2}}}{8x^{\frac{5}{2}}}$, or $\dot{y}\dot{\ddot{y}} = \frac{9a^7}{64x^5}$. And ſo we might proceed to conſtruct Curves, the Ordinates of which (in the preſent Example) would expound or repreſent the fourth, fifth, and other orders of Fluxions.

We might likewiſe proceed in a retrograde order, to find the Curves whoſe Ordinates ſhall repreſent the Fluents of any of theſe Fluxions, when given. As if we had $\dot{y} = \frac{a^{\frac{3}{2}}}{2x^{\frac{1}{2}}} = \frac{1}{2}a^{\frac{3}{2}}\dot{x}x^{-\frac{1}{2}}$, or if the Curve GH were given; by taking the Fluents, (as will be taught in the next Problem,) it would be $y = (a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{a^{\frac{3}{2}}x}{x^{\frac{1}{2}}} =) \frac{2x\dot{y}}{a}$, which will give us this Conſtruction. Make $\frac{1}{2}a$ ($\frac{1}{2}$AP) : x (AD) :: $\dot{y}$ (DG) : DB $= \frac{2x\dot{y}}{a}$, and the Line DB will repreſent the Fluent of DG, or of $\dot{y}$. And if this be done every where upon the Line AE, a Curve AB will be conſtructed, whoſe Ordinates will always expound the Fluents of the correſponding Ordinates of the Curve GH. This Curve will be the common Parabola, whoſe

Parameter is the Line AP $= a$. For its Equation is $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$, or $yy = ax$.

So if we had the Parabola ABC, we might conceive its Ordinates to repreſent Fluxions, of which the correſponding Ordinates DQ of ſome other Curve, ſuppoſe QR, would repreſent the Fluents. To find which Curve, put $\dot{y}$ for the Fluent of y, $\ddot{y}$ for the Fluent of $\dot{y}$, &c. (That is, let, &c. $\dddot{y}$, $\ddot{y}$, $\dot{y}$, y, $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. be a Series of Terms proceeding both ways indefinitely, of which every ſucceeding Term repreſents the Fluxion of the preceding, and *vice verſâ*; according to a Notation of our Author's, deliver'd elſewhere.) Then becauſe it is $y = (a^{\frac{1}{2}}x^{\frac{1}{2}} = a^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{\dot{x}}{a} =) \frac{\dot{x}x^{\frac{1}{2}}}{a^{\frac{1}{2}}}$, taking the Fluents it will be $\dot{y} = \left(\frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = \frac{2a^{\frac{1}{2}}x^{\frac{3}{2}}}{3a} =\right) \frac{2xy}{3a}$; which will give us this Conſtruction. Make $\frac{3}{2}a$ ($\frac{3}{2}$AP) : x (AD) :: y (BD) : $\frac{2xy}{3a} = \dot{y} =$ DQ, and the Line DQ will repreſent the Fluent of DB, or of y. And if the ſame be done at every point of the Line AE, a Curve QR will be form'd, the Ordinates of which will always expound the Fluents of the correſponding Ordinates of the Parabola ABC. This Curve alſo will be a Parabola, but of a higher order, the Equation of which is $\dot{y} = \frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}}$, or $\dot{y}\dot{y} = \frac{4x^3}{9a}$.

Again, becauſe $\dot{y} = \left(\frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = \frac{2x^{\frac{3}{2}} \times \dot{x}}{3a^{\frac{1}{2}} \times a} =\right) \frac{2\dot{x}x^{\frac{3}{2}}}{3a^{\frac{3}{2}}}$; taking the Fluents it will be $\ddot{y} = \left(\frac{4x^{\frac{5}{2}}}{15a^{\frac{3}{2}}} = \frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} \times \frac{2x}{5a} =\right) \frac{2xy}{5a}$, which will give us this Conſtruction. Make $\frac{5}{2}a$ ($\frac{5}{2}$AP) : x (AD) :: $\dot{y}$ (DQ) : $\frac{2xy}{5a} = \ddot{y} = \dot{y}$ = DS, and the Line DS will repreſent the Fluent of DQ, or of $\dot{y}$. And if the ſame be done at every point of the Line AE, a Curve ST will thereby be form'd, the Ordinates of which will expound the Fluents of the correſponding Ordinates of the Curve QR. This Curve will be a Parabola, whoſe Equation is $\ddot{y} = \frac{4x^{\frac{5}{2}}}{15a^{\frac{1}{2}}}$, or $\dddot{y}\dddot{y} = \frac{16x^5}{225a^3}$. And ſo we might go on as far as we pleaſe.

Laſtly,

Laſtly, if we conceive DB, the common Ordinate of all theſe Curves, to be any where thus conſtructed upon AD, that is, to be thus divided in the points S, Q, B, G, I, L, &c. from whence to AP are drawn S*s*, Q*q*, B*b*, G*g*, I*i*, L*l*, &c. parallel to AE; and if this Ordinate be farther conceived to move either backwards or forwards upon AE, with an equable Velocity, (repreſented by AP $= a = \dot{x}$,) and as it deſcribes theſe Curves, to carry the aforeſaid Parallels along with it in its motion: Then the points *s*, *q*, *b*, *g*, *i*, *l*, &c. will likewiſe move in ſuch a manner, in the Line AP, as that the Velocity of each point will be repreſented by the diſtance of the next from the point A. Thus the Velocity of *s* will be repreſented by A*q*, the Velocity of *q* by A*b*, of *b* by A*g*, of *g* by A*i*, of *i* by A*l*, &c. Or in other words, A*q* will be the Fluxion of A*s*; A*b* will be the Fluxion of A*q*, or the ſecond Fluxion of A*s*; A*g* will be the Fluxion of A*b*, or the ſecond Fluxion of A*q*, or the third Fluxion of A*s*; A*i* will be the Fluxion of A*g*, or the ſecond Fluxion of A*b*, or the third Fluxion of A*q*, or the fourth Fluxion of A*s*; and ſo on. Now in this inſtance the ſeveral orders of Fluxions, or Velocities, are not only expounded by their Proxies and Repreſentatives, but alſo are themſelves actually exhibited, as far as may be done by Geometrical Figures. And the like obtains wherever elſe we make a beginning; which ſufficiently ſhews the relative nature of all theſe orders of Fluxions and Fluents, and that they differ from each other by mere relation only, and in the manner of conceiving. And in general, what has been obſerved from this Example, may be eaſily accommodated to any other caſes whatſoever.

Or theſe different orders of Fluents and Fluxions may be thus explain'd abſtractedly and Analytically, without the aſſiſtance of Curve-lines, by the following general Example. Let any conſtant and known quantity be denoted by a, and let a^m be any given Power or Root of the ſame. And let x^m be the like Power or Root of the variable and indefinite quantity x. Make $a^m : x^m :: a : y$, or $y = \frac{ax^m}{a^m} = a^{1-m}x^m$. Here y alſo will be an indefinite quantity, which will become known as ſoon as the value of x is aſſign'd. Then taking the Fluxions, it will be $\dot{y} = ma^{1-m}\dot{x}x^{m-1}$; and ſuppoſing x to flow or increaſe uniformly, and making its conſtant Velocity or Fluxion $\dot{x} = a$, it will be $\dot{y} = ma^{2-m}x^{m-1}$. Here if for $a^{1-m}x^m$ we write its value y, it will be $\dot{y} = \frac{may}{x}$, that is, $x : ma :: y : \dot{y}$. So that $\dot{y}$ will be alſo a known and aſſignable Quantity,

tity, whenever x (and therefore y) is assign'd. Then taking the Fluxions again, we shall have $\ddot{y} = m \times \overline{m-1}a^{2-m}\dot{x}x^{m-2} = m \times \overline{m-1}a^{2-m}x^{m-2}$; or for $ma^{2-m}x^{m-1}$ writing its value $\dot{y}$, it will be $\ddot{y} = \frac{\overline{m-1}a\dot{y}}{x}$, that is, $x : \overline{m-1}a :: \dot{y} : \ddot{y}$. So that $\ddot{y}$ will become a known quantity, when x (and therefore y and $\dot{y}$) is assign'd. Then taking the Fluxions again, we shall have $\dddot{y} = m \times \overline{m-1} \times \overline{m-2}a^{3-m}x^{m-3}$, or $\dddot{y} = \frac{\overline{m-2}a\ddot{y}}{x}$, that is, $x : \overline{m-2}a :: \ddot{y} : \dddot{y}$; where also $\dddot{y}$ will be known, when x is given. And taking the Fluxions again, we shall have $\ddddot{y} = m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3}a^{4-m}x^{m-4} = \frac{\overline{m-3}a\dddot{y}}{x}$; that is, $x : \overline{m-3}a :: \dddot{y} : \ddddot{y}$. So that $\ddddot{y}$ will also be known, whenever x is given. And from this Induction we may conclude in general, that if the order of Fluxions be denoted by any integer number n, or if n be put for the number of points over the Letter y, it will always be $x : \overline{m-n}a :: \overset{n}{y} : \overset{n+1}{y}$; or from the Fluxion of any order being given, the Fluxion of the next immediate order may be hence found.

Or we may thus invert the proportion $\overline{m-n}a : x :: \overset{n+1}{y} : \overset{n}{y}$, and then from the Fluxion given, we shall find its next immediate Fluent. As if $n = 2$, 'tis $\overline{m-2}a : x :: \dddot{y} : \ddot{y}$. If $n = 1$, 'tis $\overline{m-1}a : x :: \ddot{y} : \dot{y}$. If $n = 0$, 'tis $ma : x :: \dot{y} : y$. And observing the same analogy, if $n = -1$, 'tis $\overline{m+1}a : x :: y : \overset{1}{y}$; where $\overset{1}{y}$ is put for the Fluent of y, or for y with a negative point. And here because $y = a^{1-m}x^m$, it will be $\overline{m+1}a : x :: a^{1-m}x^m : \overset{1}{y}$, or $\overset{1}{y} = \frac{a^{1-m}x^{m+1}}{\overline{m+1}a} = \frac{x^{m+1}}{\overline{m+1}a^m}$: which also may thus appear. Because $y = (a^{1-m}x^m = \frac{a^{1-m}\dot{x}x^m}{a} =) \frac{\dot{x}x^m}{a^m}$, taking the Fluents, (see the next Problem,) it will be $\overset{1}{y} = \frac{x^{m+1}}{\overline{m+1}a^m}$. Again, if we make $n = -2$, 'tis $\overline{m+2}a : x :: \overset{1}{y} : y$, or $\overset{2}{y} = \frac{\overset{1}{xy}}{\overline{m+2}a} = \frac{x^{m+2}}{\overline{m+1} \times \overline{m+2}a^{m+1}}$. For

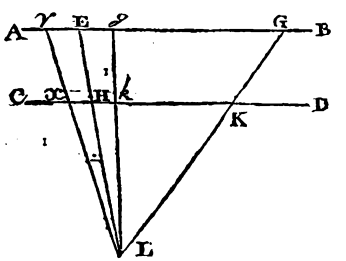
 because

becauſe $\dot{y} = \frac{x^{m+1}}{\overline{m+1}a^{m}} \times \frac{\dot{x}}{a} = \frac{\dot{x}x^{m+1}}{\overline{m+1}a^{m+1}}$, taking the Fluents it will be $y = \frac{x^{m+2}}{\overline{m+1} \times \overline{m+2}a^{m+1}}$. Again, if we make $n = -3$, 'tis $\overline{m+3}a$: $x :: y : y$, or $y = \frac{xy}{\overline{m+3}a} = \frac{x^{m+3}}{\overline{m+1} \times \overline{m+2} \times \overline{m+3}a^{m+2}}$. And ſo for all other ſuperior orders of Fluents.

And this may ſuffice in general, to ſhew the comparative nature and properties of theſe ſeveral orders of Fluxions and Fluents, and to teach the operations by which they are produced, or to find their reſpective fluxional Equations. As to the uſes they may be apply'd to, when found, that will come more properly to be conſider'd in another place.

SECT. III. *The Geometrical and Mechanical Elements of Fluxions.*

THE foregoing Principles of the Doctrine of Fluxions being chiefly abſtracted and Analytical, I ſhall here endeavour, after a general manner, to ſhew ſomething analogous to them in Geometry and Mechanicks; by which they may become, not only the object of the Underſtanding, and of the Imagination, (which will only prove their poſſible exiſtence,) but even of Senſe too, by making them actually to exiſt in a viſible and ſenſible form. For it is now become neceſſary to exhibit them all manner of ways, in order to give a ſatisfactory proof, that they have indeed any real exiſtence at all.

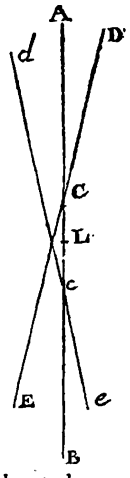


And firſt, by way of preparation, it will be convenient to conſider uniform and equable motions, as alſo ſuch as are alike inequable. Let the right Line AB be deſcribed by the equable motion of a point, which is now at E, and will preſently be at G. Alſo let the Line CD, parallel to the former, be deſcribed by the equable motion of a point, which is in H and K, at the ſame times as the former is in E and G. Then will EG and HK be contemporaneous Lines, and therefore will be proportional to the

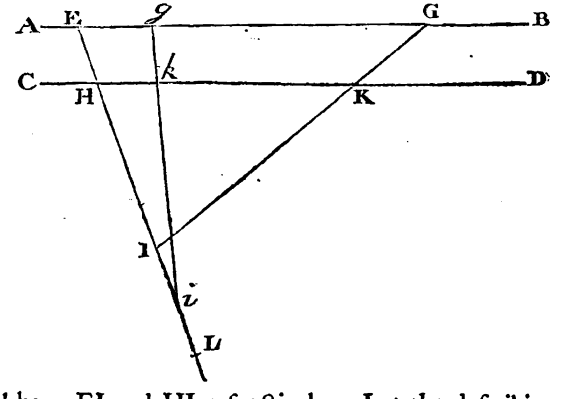
the Velocity of each moving point reſpectively. Draw the indefinite Lines EH and GK, meeting in L; then becauſe of like Triangles ELG and HLK, the Velocities of the points E and H, which were before as EG and HK, will be now as EL and HL. Let the deſcribing points G and K be conceived to move back again, with the ſame Velocities, towards A and C, and before they approach to E and H let them be found in *g* and *k*, at any ſmall diſtance from E and H, and draw *gk*, which will paſs through L; then ſtill their Velocities will be in the ratio of E*g* and H*k*, be thoſe Lines ever ſo little, that is, in the ratio of EL and HL. Let the moving points *g* and *k* continue to move till they coincide with E and H; in which caſe the decreaſing Lines E*g* and H*k* will paſs through all poſſible magnitudes that are leſs and leſs, and will finally become vaniſhing Lines. For they muſt intirely vaniſh at the ſame moment, when the points *g* and *k* ſhall coincide with E and H. In all which ſtates and circumſtances they will ſtill retain the ratio of EL to HL, with which at laſt they will finally vaniſh. Let thoſe points ſtill continue to move, after they have coincided with E and H, and let them be found again at the ſame time in γ and $\varkappa$, at any diſtance beyond E and H. Still the Velocities, which are now as Eγ and H$\varkappa$, and may be eſteemed negative, will be as EL and HL, whether thoſe Lines Eγ and H$\varkappa$ are of any finite magnitude, or are only naſcent Lines; that is, if the Line $\gamma\varkappa$L, by its angular motion, be but juſt beginning to emerge and divaricate from EHL. And thus it will be when both theſe motions are equable motions, as alſo when they are alike inequable; in both which caſes the common interſection of all the Lines EHL, GKL, *gk*L, &c. will be the fixt point L. But when either or both theſe motions are ſuppos'd to be inequable motions, or to be any how continually accelerated or retarded, theſe Symptoms will be ſomething different; for then the point L, which will ſtill be the common interſection of thoſe Lines when they firſt begin to coincide, or to divaricate, will no longer be a fixt but a moveable point, and an account muſt be had of its motion. For this purpoſe we may have recourſe to the following Lemma.

Let AB be an indefinite and fixt right Line, along which another indefinite but moveable right Line DE may be conceived to move or roll in ſuch a manner, as to have both a progreſſive motion, as alſo an angular motion about a moveable Center C. That is, the common interſection C of the two Lines AB and DE may be ſuppoſed to move with any progreſſive motion from A towards B, while at the

ſame time the moveable Line DE revolves about the ſame point C, with any angular motion. Then as the Angle ACD continually decreaſes, and at laſt vaniſhes when the two Lines ACB and DCE coincide; yet even then the point of interſection C, (as it may be ſtill call'd,) will not be loſt and annihilated, but will appear again, as ſoon as the Lines begin to divaricate, or to ſeparate from each other. That is, if C be the point of interſection before the coincidence, and *c* the point of interſection after the coincidence, when the Line *dce* ſhall again emerge out of AB; there will be ſome intermediate point L, in which C and *c* were united in the ſame point, at the moment of coincidence. This point, for diſtinction-ſake, may be call'd the *Node*, or *the point of no divarication*. Now to apply this to inequable Motions:

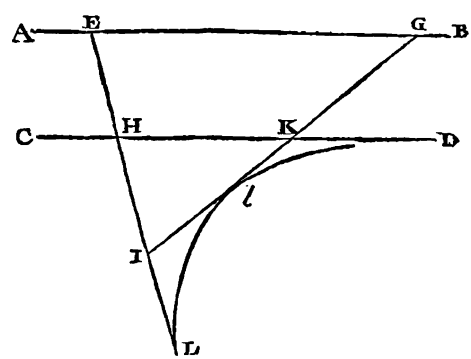


Let the Line AB be deſcribed by the continually accelerated motion of a point, which is now in E, and will be preſently found in G. Alſo let the Line CD, parallel to the former, be deſcribed by the equable motion of a point, which is found in H and K, at the ſame times as the other point is in E and G. Then will EG and HK be contemporaneous Lines; and producing EH and GK till they meet in I, thoſe contemporaneous Lines will be as EI and HI reſpectively. Let the deſcribing points G and K be conceived to move back again towards A and C, each with the ſame degrees of Velocity, in every point of their motion, as they had before acquired; and let them arrive at the ſame time at *g* and *k*, at ſome ſmall diſtance from E and H, and draw *gki* meeting EH in *i*. Then E*g* and H*k*, being contemporary Lines alſo, and very little by ſuppoſition, they will be nearly as the Velocities



locities at *g* and *k*, that is, at E and H; which contemporary Lines will be now as E*i* and H*i*. Let the points *g* and *k* continue their motion till they coincide with E and H, or let the Line GKI or *gki* continue its progreſſive and angular motion in this manner, till it coincides with EHL, and let L be the Node, or point of no divarication, as in the foregoing Lemma. Then will the laſt ratio of the vaniſhing Lines E*g* and H*k*, which is the ratio of the Velocities at E and H, be as EL and HL reſpectively.

Hence we have this Corollary. If the point E (in the foregoing figure,) be ſuppos'd to move from A towards B, with a Velocity any how accelerated, and at the ſame time the point H moves from C towards D with an equable Velocity, (or inequable, if you pleaſe;) thoſe Velocities in E and H will be reſpectively as the Lines EL and HL, which point L is to be found, by ſuppoſing the contemporary Lines EG and HK continually to diminiſh, and finally to vaniſh. Or by ſuppoſing the moveable indefinite Line GKI to move with a progreſſive and angular motion, in ſuch manner, as that EG and HK ſhall always be contemporary Lines, till at laſt GKI ſhall coincide with the Line EHL, at which time it will determine the Node L, or the point of no divarication. So that if the Lines AE and CH repreſent two Fluents, any how related, their Velocities of deſcription at E and H, or their reſpective Fluxions, will be in the ratio of EL and HL.



And hence it will follow alſo, that the Locus of the moveable point or Node L, that is, of all the points of no divarication, will be ſome Curve-line L*l*, to which the Lines EHL and GK*l* will always be Tangents in L and *l*. And the nature of this Curve L*l* may be determined by the given relation of the Fluents or Lines AE and CH; and *vice verſâ*. Or however the relation of its intercepted Tangents EL and HL may be determined in all caſes; that is, the ratio of the Fluxions of the given Fluents.

For

For illuſtration-ſake, let us apply this to an Example. Make the Fluents AE $= y$ and CH $= x$, and let the relation of theſe be always expreſs'd by this Equation $y = x^n$. Make the contemporary Lines EG $=$ Y and HK $=$ X; and becauſe AE and CH are contemporary by ſuppoſition, we ſhall have the whole Lines AG and CK contemporary alſo, and thence the Equation $y + Y = \overline{x + X}|^n$. This by our Author's Binomial Theorem will produce $y + Y = x^n + nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2$, &c. which (becauſe $y = x^n$) will become $Y = nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2$, &c. or in an Analogy, X : Y :: 1 : $nx^{n-1} + n \times \frac{n-1}{2}x^{n-2}X$, &c. which will be the general relation of the contemporary Lines or Increments EG and HK. Now let us ſuppoſe the indefinite Line GKI, which limits theſe contemporary Lines, to return back by a progreſſive and angular motion, ſo as always to intercept contemporary Lines EG and HK, and finally to coincide with EHL, and by that means to determine the Node L; that is, we may ſuppoſe EG $=$ Y and HK $=$ X, to diminiſh *in infinitum*, and to become vaniſhing Lines, in which caſe we ſhall have X : Y :: 1 : nx^{n-1}. But then it will be likewiſe X : Y :: HK : EG :: HL : EL :: $\dot{x} : \dot{y}$, or 1 : nx^{n-1} :: $\dot{x} : \dot{y}$, or $\dot{y} = n\dot{x}x^{n-1}$.

And hence we may have an expedient for exhibiting Fluxions and Fluents Geometrically and Mechanically, in all circumſtances, ſo as to make them the objects of Senſe and ocular Demonſtration. Thus in the laſt figure, let the two parallel lines AB and CD be deſcribed by the motion of two points E and H, of which E moves any how inequably, and (if you pleaſe) H may be ſuppos'd to move equably and uniformly; and let the points H and K correſpond to E and G. Alſo let the relation of the Fluents AE $= y$ and CH $= x$ be defined by any Equation whatever. Suppoſe now the deſcribing points E and H to carry along with them the indefinite Line EHL, in all their motion, by which means the point or Node L will deſcribe ſome Curve L*l*, to which EL will always be a Tangent in L. Or ſuppoſe EHL to be the Edge of a Ruler, of an indefinite length, which moves with a progreſſive and angular motion thus combined together; the moveable point or Node L in this Line, which will have the leaſt angular motion, and which is always the point of no divarication, will deſcribe the Curve, and the Line or Edge itſelf will be a Tangent to it in L. Then will the ſegments EL and HL be proportional to the Velocity of the points E and H reſpectively; or will exhibit the ratio of the Fluxions $\dot{y}$ and $\dot{x}$, belonging to the Fluents AE $= y$ and CF $= x$.

Or if we ſuppoſe the Curve L*l* to be given, or already conſtructed, we may conceive the indefinite Line EHIL to revolve or roll about it, and by continually applying itſelf to it, as a Tangent, to move from the ſituation EHIL to GK*l*I. Then will AE and CH be the Fluents, the ſenſible velocities of the deſcribing points E and H will be their Fluxions, and the intercepted Tangents EL and HL will be the rectilinear meaſures of thoſe Fluxions or Velocities. Or it may be repreſented thus: If L*l* be any rigid obſtacle in form of a Curve, about which a flexible Line, or Thread, is conceived to be wound, part of which is ſtretch'd out into a right Line LE, which will therefore touch the Curve in L; if the Thread be conceived to be farther wound about the Curve, till it comes into the ſituation L*l*KG; by this motion it will exhibit, even to the Eye, the ſame increaſing Fluents as before, their Velocities of increaſe, or their Fluxions, as alſo the Tangents or rectilinear repreſentatives of thoſe Fluxions. And the ſame may be done by unwinding the Thread, in the manner of an Evolute. Or inſtead of the Thread we may make uſe of a Ruler, by applying its Edge continually to the curved Obſtacle L*l*, and making it any how revolve about the moveable point of Contact L or *l*. In all which manners the Fluents, Fluxions, and their rectilinear meaſures, will be ſenſibly and mechanically exhibited, and therefore they muſt be allowed to have a place *in rerum naturâ*. And if they are in nature, even tho' they were but barely poſſible and conceiveable, much more if they are ſenſible and viſible, it is the province of the Mathematicks, by ſome method or other, to inveſtigate and determine their properties and proportions.

Or as by one Thread EHL, perpetually winding about the curved obſtacle L*l*, of a due figure, we ſhall ſee the Fluents AE and CH continually to increaſe or decreaſe, at any rate aſſign'd, by the motion of the Thread EHL either backwards or forwards; and as we ſhall thereby ſee the comparative Velocities of the points E and H, that is, the Fluxions of the Fluents AE and CH, and alſo the Lines EL and HL, whoſe variable ratio is always the rectilinear meaſure of thoſe Fluxions: So by the help of another Thread GK*l*L, winding about the obſtacle in its part *l*L, and then ſtretching out into a right Line or Tangent *l*KG, and made to move backwards or forwards, as before; if the firſt Thread be at reſt in any given ſituation EHL, we may ſee the ſecond Thread deſcribe the contemporary Lines or Increments EG and HK, by which the Fluents AE and CH are continually increaſed; and if GK*l* is made to approach

proach towards EHL, we may ſee thoſe contemporary Lines continually to diminiſh, and their ratio continually approaching towards the ratio of EL to HL; and continuing the motion, we may preſently ſee thoſe two Lines actually to coincide, or to unite as one Line, and then we may ſee the contemporary Lines actually to vaniſh at the ſame time, and their ultimate ratio actually to become that of EL to HL. And if the motion be ſtill continued, we ſhall ſee the Line GK*l* to emerge again out of EHL, and begin to deſcribe other contemporary Lines, whoſe naſcent proportion will be that of EL to HL. And ſo we may go on till the Fluents are exhauſted. All theſe particulars may be thus eaſily made the objects of ſight, or of Ocular Demonſtration.

This may ſtill be added, that as we have here exhibited and repreſented firſt Fluxions geometrically and mechanically, we may do the ſame thing, *mutatis mutandis*, by any higher orders of Fluxions. Thus if we conceive a ſecond figure, in which the Fluential Lines ſhall increaſe after the rate of the ratio of the intercepted Tangents (or the Fluxions) of the firſt figure; then its intercepted Tangents will expound the ratio of the ſecond Fluxions of the Fluents in the firſt figure. Alſo if we conceive a third figure, in which the Fluential Lines ſhall increaſe after the rate of the intercepted Tangents of the ſecond figure; then its intercepted Tangents will expound the third Fluxions of the Fluents in the firſt figure. And ſo on as far as we pleaſe. This is a neceſſary conſequence from the relative nature of theſe ſeveral orders of Fluxions, which has been ſhewn before.

And farther to ſhew the univerſality of this Speculation, and how well it is accommodated to explain and repreſent all the circumſtances of Fluxions and Fluents; we may here take notice, that it may be alſo adapted to thoſe caſes, in which there are more than two Fluents, which have a mutual relation to each other, expreſs'd by one or more Equations. For we need but introduce a third parallel Line, and ſuppoſe it to be deſcribed by a third point any how moving, and that any two of theſe deſcribing points carry an indefinite Line along with them, which by revolving as a Tangent, deſcribes the Curve whoſe Tangents every where determine the Fluxions. As alſo that any other two of thoſe three points are connected by another indefinite Line, which by revolving in like manner deſcribes another ſuch Curve. And ſo there may be four or more parallel Lines. All but one of theſe Curves may be aſſumed at pleaſure, when they are not given by the ſtate of the Queſtion. Or Analytically,

tically, ſo many Equations may be aſſumed, except one, (if not given by the Problem,) as is the number of the Fluents concern'd.

But laſtly, I believe it may not be difficult to give a pretty good notion of Fluents and Fluxions, even to ſuch Perſons as are not much verſed in Mathematical Speculations, if they are willing to be inform'd, and have but a tolerable readineſs of apprehenſion. This I ſhall here attempt to perform, in a familiar way, by the inſtance of a Fowler, who is aiming to ſhoot two Birds at once, as is repreſented in the Frontiſpiece. Let us ſuppoſe the right Line AB to be parallel to the Horizon, or level with the Ground, in which a Bird is now flying at G, which was lately at F, and a little before at E. And let this Bird be conceived to fly, not with an equable or uniform ſwiftneſs, but with a ſwiftneſs that always increaſes, (or with a Velocity that is continually accelerated,) according to ſome known rate. Let there alſo be another right Line CD, parallel to the former, at the ſame or any other convenient diſtance from the Ground, in which another Bird is now flying at K, which was lately at I, and a little before at H; juſt at the ſame points of time as the firſt Bird was at G, F, E, reſpectively. But to fix our Ideas, and to make our Conceptions the more ſimple and eaſy, let us imagine this ſecond Bird to fly equably, or always to deſcribe equal parts of the Line CD in equal times. Then may the equable Velocity of this Bird be uſed as a known meaſure, or ſtandard, to which we may always compare the inequable Velocity of the firſt Bird. Let us now ſuppoſe the right Line EH to be drawn, and continued to the point L, ſo that the proportion (or ratio) of the two Lines EL and HL may be the ſame as that of the Velocities of the two Birds, when they were at E and H reſpectively. And let us farther ſuppoſe, that the Eye of a Fowler was at the ſame time at the point L, and that he directed his Gun, or Fowling-piece, according to the right Line LHE, in hopes to ſhoot both the Birds at once. But not thinking himſelf then to be ſufficiently near, he forbears to diſcharge his Piece, but ſtill pointing it at the two Birds, he continually advances towards them according to the direction of his Piece, till his Eye is preſently at M, and the Birds at the ſame time in F and I, in the ſame right Line FIM. And not being yet near enough, we may ſuppoſe him to advance farther in the ſame manner, his Piece being always directed or level'd at the two Birds, while he himſelf walks forward according to the direction of his Piece, till his Eye is now at N, and the Birds in the ſame right Line with his Eye, at K and G. The Path of his Eye, deſcribed by this

double motion, (or compounded of a progreſſive and angular motion,) will be ſome Curve-line LMN, in the ſame Plain as the reſt of the figure, which will have this property, that the proportion of the diſtances of his Eye from each Bird, will be the ſame every where as that of their reſpective Velocities. That is, when his Eye was at L, and the Birds at E and H, their Velocities were then as EL and HL, by the Conſtruction. And when his Eye was at M, and the Birds at F and I, their Velocities were in the ſame proportion as the Lines FM and IM, by the nature of the Curve LMN. And when his Eye is at N, and the Birds at G and K, their Velocities are in the proportion of GN to KN, by the nature of the ſame Curve. And ſo univerſally, of all other ſituations. So that the Ratio of thoſe two Lines will always be the ſenſible meaſure of the ratio of thoſe two ſenſible Velocities. Now if theſe Velocities, or the ſwiftneſſes of the flight of the two Birds in this inſtance, are call'd *Fluxions*; then the Lines deſcribed by the Birds in the ſame time, may be call'd their *contemporaneous Fluents*; and all inſtances whatever of Fluents and Fluxions, may be reduced to this Example, and may be illuſtrated by it.

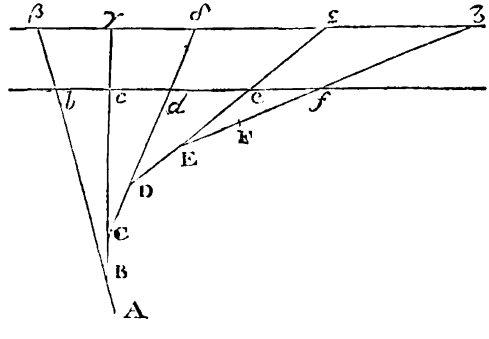
And thus I would endeavour to give ſome notion of Fluents and Fluxions, to Perſons not much converſant in the Mathematicks; but ſuch as had acquired ſome ſkill in theſe Sciences, I would thus proceed farther to inſtruct, and to apply what has been now deliver'd. The contemporaneous Fluents being EF $= y$, and HI $= x$, and their rate of flowing or increaſing being ſuppos'd to be given or known; their relation may always be expreſs'd by an Equation, which will be compos'd of the variable quantities x and y, together with any known quantities. And that Equation will have this property, becauſe of thoſe variable quantities, that as FG and IK, EG and HK, and infinite others, are alſo contemporaneous Fluents; it will indifferently exhibit the relation of thoſe Lines alſo, as well as of EF and HI; or they may be ſubſtituted in the Equation, inſtead of x and y. And hence we may derive a Method for determining the Velocities themſelves, or for finding Lines proportional to them. For making FG $=$ Y, and IK $=$ X; in the given Equation I may ſubſtitute $y + $ Y inſtead of y, and $x + $ X inſtead of x, by which I ſhall obtain an Equation, which in all circumſtances will exhibit the relation of thoſe Quantities or Increments. Now it may be plainly perceived, that if the Line MIF is conceived continually to approach nearer and nearer to the Line NKG, (as juſt now, in the inſtance of the Fowler,) till it finally coincides with it; the Lines FG $=$ Y,

and

and IK = X, will continually decreaſe, and by decreaſing will approach nearer and nearer to the Ratio of the Velocities at G and K, and will finally vaniſh at the ſame time, and in the proportion of thoſe Velocities, that is, in the Ratio of GN to KN. Conſequently in the Equation now form'd, if we ſuppoſe Y and X to decreaſe continually, and at laſt to vaniſh, that we may obtain their ultimate Ratio; we ſhall thereby obtain the Ratio of GN to KN. But when Y and X vaniſh, or when the point F coincides with G, and I with H, then it will be EG = y, and HK = x; ſo that we ſhall have $\dot{y} : \dot{x}$ :: GM : KN. And hence we ſhall obtain a Fluxional Equation, which will always exhibit the relation of the Fluxions, or Velocities, belonging to the given Algebraical or Fluential Equation.

Thus, for Example, if EF = y, and HI = x, and the indefinite Lines y and x are ſuppoſed to increaſe at ſuch a rate, as that their relation may always be expreſs'd by this Equation $x^3 - ax^2 + axy - y^3 = 0$; then making FG = Y, and IK = X, by ſubſtituting $y + Y$ for y, and $x + X$ for x, and reducing the Equation that will ariſe, (ſee before, pag. 255.) we ſhall have $3x^2X + 3xX^2 + X^3 - 2axX - aX^2 + axY + aXy + aXY - 3y^2Y - 3yY^2 - Y^3 = 0$, which may be thus expreſs'd in an Analogy, $Y : X :: 3x^2 - 2ax + ay + 3xX + X^2 - aX : 3y^2 - ax - aX + 3yY + Y^2$. This Analogy, when Y and X are vaniſhing quantities, or their ultimate Ratio, will become $Y : X :: 3x^2 - 2ax + ay : 3y^2 - ax$. And becauſe it is then Y : X :: GN : KN :: $\dot{y} : \dot{x}$, it will be $\dot{y} : \dot{x} :: 3x^2 - 2ax + ay : 3y^2 - ax$. Which gives the proportion of the Fluxions. And the like in all other caſes. Q. E. I.

We might alſo lay a foundation for theſe Speculations in the following manner. Let ABCDEF, &c. be the Periphery of a Polygon, or any part of it, and let the Sides AB, BC, CD, DE, &c. be of any magnitude whatever. In the ſame Plane, and at any diſtance, draw the two parallel Lines $\beta\zeta$, and bf, to which continue the right Lines AB$b\beta$, BC$c\gamma$, CD$d\delta$, DE$e\varepsilon$, &c. meeting the parallels as in the figure. Now if we ſuppoſe



poſe two moving points, or bodies, to be at β and b, and to move in the ſame time to γ and c, with any equable Velocities; thoſe Velocities will be to each other as $\beta\gamma$ and bc, that is, becauſe of the parallels, as βB and bB. Let them ſet out again from γ and c, and arrive at the ſame time at δ and d, with any equable Velocities; thoſe Velocities will be as $\gamma\delta$ and cd, that is, as γC and cC. Let them depart again from δ and d, and arrive in the ſame time at ε and e, with any equable Velocities; thoſe Velocities will be as $\delta\varepsilon$ and de, that is, as δD and dD. And it will be the ſame thing every where, how many ſoever, and how ſmall ſoever, the Sides of the Polygon may be. Let their number be increaſed, and their magnitude be diminiſh'd *in infinitum*, and then the Periphery of the Polygon will continually approach towards a Curve-line, to which the Lines AB$b\beta$, BC$c\gamma$, CD$d\delta$, &c. will become Tangents; as alſo the Motions may be conceived to degenerate into ſuch as are accelerated or retarded continually. Then in any two points, ſuppoſe δ and d, where the deſcribing points are found at the ſame time, their Velocities (or Fluxions) will be as the Segments of the reſpective Tangents δD and dD; and the Lines $\beta\delta$ and bd, intercepted by any two Tangents δD and βB, will be the contemporaneous Lines, or Fluents. Now from the nature of the Curve being given, or from the property of its Tangents, the contemporaneous Lines may be found, or the relation of the Fluents. And *vice versâ*, from the Rate of flowing being given, the correſponding Curve may be found.

ANNO-

ANNOTATIONS on Prob. 2.

OR,

The Relation of the Fluxions being given, to find the Relation of the Fluents.

SECT. I. *A particular Solution; with a preparation for the general Solution, by which it is diſtributed into three Caſes.*

1, 2. WE are now come to the Solution of the Author's ſecond fundamental Problem, borrow'd from the Science of Rational Mechanicks: Which is, from the Velocities of the Motion at all times given, to find the quantities of the Spaces deſcribed; or to find the Fluents from the given Fluxions. In diſcuſſing which important Problem, there will be occaſion to expatiate ſomething more at large. And firſt it may not be amiſs to take notice, that in the Science of Computation all the Operations are of two kinds, either Compoſitive or Reſolutative. The Compoſitive or Synthetic Operations proceed neceſſarily and directly, in computing their ſeveral *quæſita*, and not tentatively or by way of tryal. Such are Addition, Multiplication, Raiſing of Powers, and taking of Fluxions. But the Reſolutative or Analytical Operations, as Subtraction, Diviſion, Extraction of Roots, and finding of Fluents, are forced to proceed indirectly and tentatively, by long deductions, to arrive at their ſeveral *quæſita*; and ſuppoſe or require the contrary Synthetic Operations, to prove and confirm every ſtep of the Proceſs. The Compoſitive Operations, always when the *data* are finite and terminated, and often when they are interminate

or infinite, will produce finite concluſions; whereas very often in the Reſolutative Operations, tho' the *data* are in finite Terms, yet the *quæſita* cannot be obtain'd without an infinite Series of Terms. Of this we ſhall ſee frequent Inſtances in the ſubſequent Operation, of returning to the Fluents from the Fluxions given.

The Author's particular Solution of this Problem extends to ſuch caſes only, wherein the Fluxional Equation propoſed either has been, or at leaſt might have been, derived from ſome finite Algebraical Equation, which is now required. Here all the neceſſary Terms being preſent, and no more than what are neceſſary, it will not be difficult, by a Proceſs juſt contrary to the former, to return back again to the original Equation. But it will moſt commonly happen, either if we aſſume a Fluxional Equation at pleaſure, or if we arrive at one as the reſult of ſome Calculation, that ſuch an Equation is to be reſolved, as could not be derived from any previous finite Algebraical Equation, but will have Terms either redundant or deficient; and conſequently the Algebraic Equation required, or its Root, muſt be had by Approximation only, or by an infinite Series. In all which caſes we muſt have recourſe to the general Solution of this Problem, which we ſhall find afterwards.

The Precepts for this particular Solution are theſe. (1.) All ſuch Terms of the given Equation as are multiply'd (ſuppoſe) by $\dot{x}$, muſt be diſpoſed according to the Powers of x, or muſt be made a Number belonging to the Arithmetical Scale whoſe Root is x. (2.) Then they muſt be divided by $\dot{x}$, and multiply'd by x; or $\dot{x}$ muſt be changed into x, by expunging the point. (3.) And laſtly, the Terms muſt be ſeverally divided by the Progreſſion of the Indices of the Powers of x, or by ſome other Arithmetical Progreſſion, as need ſhall require. And the ſame things muſt be repeated for every one of the flowing quantities in the given Equation.

Thus in the Equation $3\ddot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, the Terms $3\ddot{x}x^2 - 2a\dot{x}x + a\dot{x}y$ by expunging the points become $3x^3 - 2ax^2 + axy$, which divided by the Progreſſion of the Indices 3, 2, 1, reſpectively, will give $x^3 - ax^2 + axy$. Alſo the Terms $-3\dot{y}y^2 * + a\dot{y}x$ by expunging the points become $-3y^3 * + ayx$, which divided by the Progreſſion of the Indices 3, 2, 1, reſpectively, will give $-y^3 * + ayx$. The aggregate of theſe, neglecting the redundant Term ayx, is $x^3 - ax^2 + axy - y^3 = 0$, the Equation required. Where it muſt be noted, that every Term, which occurs more than once, muſt be accounted a redundant Term.

So

So if the propoſed Equation were $\overline{m+3}yx\dot{x}\dot{x}^3 - \overline{m+2}ay\dot{x}\dot{x}^2 + \overline{m+1}ay^2\dot{x}\dot{x} - my^4\dot{x} - \overline{n+3}x\dot{y}\dot{y}^3 + \overline{n+1}ax^2\dot{y}\dot{y} + nx^4\dot{y} - nax^3\dot{y} = 0$, whatever values the general Numbers m and n may acquire; if thoſe Terms in which $\dot{x}$ is found are reduced to the Scale whoſe Root is x, they will ſtand thus: $\overline{m+3}yx\dot{x}^3 - \overline{m+2}ay\dot{x}\dot{x}^2 + \overline{m+1}ay^2\dot{x}\dot{x} - my^4\dot{x}$; or expunging the points they will become $\overline{m+3}yx^4 - \overline{m+2}ayx^3 + \overline{m+1}ay^2x^2 - my^4x$. Theſe being divided reſpectively by the Arithmetical Progreſſion $m+3$, $m+2$, $m+1$, m, will give the Terms $yx^4 - ayx^3 + ay^2x^2 - y^4x$. Alſo the Terms in which $\dot{y}$ is found; being reduced to the Scale whoſe Root is y, will ſtand thus: $-\overline{n+3}x\dot{y}\dot{y}^3 * + \overline{n+1}ax^2\dot{y}\dot{y} + nx^4\dot{y}$; $- nax^3\dot{y}$ or expunging the points they will become $-\overline{n+3}xy^4 * + \overline{n+1}ax^2y^2 + nx^4y$. Theſe being divided reſpectively by the Arithmetical Pro- $- nax^3y$ greſſion $\overline{n+3}$, $\overline{n+2}$, $\overline{n+1}$, n, will give the Terms $-xy^4 + ax^2y^2 + x^4y - ax^3y$. But theſe Terms, being the ſame as the former, muſt all be conſider'd as redundant, and therefore are to be rejected. So that $yx^4 - ayx^3 + ay^2x^2 - y^4x = 0$, or dividing by yx, the Equation $x^3 - ax^2 + ayx - y^3 = 0$ will ariſe as before.

Thus if we had this Fluxional Equation $may\dot{x}x^{-1} - \overline{m+2}x\dot{x}x - nx^2\dot{y}y^{-1} + \overline{n+1}a\dot{y} = 0$, to find the Fluential Equation to which it belongs; the Terms $may\dot{x}x^{-1} * - \overline{m+2}x\dot{x}x$, by expunging the points, and dividing by the Terms of the Progreſſion m, $m+1$, $m+2$, will give the Terms $ay - x^2$. Alſo the Terms $-nx^2\dot{y}y^{-1} + \overline{n+1}a\dot{y}$, by expunging the points, and dividing by n, $n+1$, will give the Terms $-x^2 + ay$. Now as theſe are the ſame as the former, they are to be eſteem'd as redundant, and the Equation required will be $ay - x^2 = 0$. And when the given Fluxional Equation is a general one, and adapted to all the forms of the Fluential Equation, as is the caſe of the two laſt Examples; then all the Terms ariſing from the ſecond Operation will be always redundant, ſo that it will be ſufficient to make only one Operation.

Thus if the given Equation were $4\dot{y}y^2 + z^3\dot{y}y^{-1} + 2y\dot{x}x - 3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z} = 0$, in which there are found three flowing quantities; the only Term in which $\dot{x}$ is found is $2y\dot{x}x$, in which expunging the point, and then dividing by the Index 2, it will become yx^2. Then the Terms in which $\dot{y}$ is found are $4\dot{y}y^2 + z^3\dot{y}y^{-1}$, which expunging the points become $4y^3 * * + z^3$, and dividing by

by the Progreſſion 2, 1, 0, — 1, give the Terms $2y^3 - z^3$. Laſtly the Terms in which $\dot{z}$ is found are $-3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z}$, which expunging the points become $-3z^3 + 6yz^2 - 2cyz$, and dividing by the Progreſſion 3, 2, 1, give the Terms $-z^3 + 3yz^2 - 2cyz$. Now if we collect theſe Terms, and omit the redundant Term $-z^3$, we ſhall have $yx^2 + 2y^3 - z^3 + 3yz^2 - 2cyz = 0$ for the Equation required.

3, 4. But theſe deductions are not to be too much rely'd upon, till they are verify'd by a proof; and we have here a ſure method of proof, whether we have proceeded rightly or not, in returning from the relation of the Fluxions to the relation of the Fluents. For every reſolutative Operation ſhould be proved by its contrary compoſitive Operation. So if the Fluxional Equation $\dot{x}x - \dot{x}y - x\dot{y} + a\dot{y} = 0$ were given, to return to the Equation involving the Fluents; by the foregoing Rule we ſhall firſt have the Terms $\dot{x}x - \dot{x}y$, which by expunging the points will become $x^2 - xy$, and dividing by the Progreſſion 2, 1, will give the Terms $\frac{1}{2}x^2 - xy$. Alſo the Terms, or rather Term, $-x\dot{y} + a\dot{y}$, by expunging the points will become $-xy + ay$, which are only to be divided by Unity. So that leaving out the redundant Term $-xy$, we ſhall have the Fluential Equation $\frac{1}{2}x^2 - xy + ay = 0$. Now if we take the Fluxions of this Equation, we ſhall find by the foregoing Problem $x\dot{x} - \dot{x}y - x\dot{y} + a\dot{y} = 0$, which being the ſame as the Equation given, we are to conclude our work is true. But if either of the Fluxional Equations $x\dot{x} - \dot{x}y + a\dot{y} = 0$, or $x\dot{x} - x\dot{y} + a\dot{y} = 0$ had been propoſed, tho' by purſuing the foregoing method we ſhould arrive at the Equation $\frac{1}{2}x^2 - xy + ay = 0$, for the relation of the Fluents; yet as this concluſion would not ſtand the teſt of this proof, we muſt reject it as erroneous, and have recourſe to the following general Method; which will give the value of y in either of thoſe Equations by an infinite Series, and therefore for uſe and practice will be the moſt commodious Solution.

5. As Velocities can be compared only with Velocities, and all other quantities with others of the ſame Species only; therefore in every Term of an Equation, the Fluxions muſt always aſcend to the ſame number of Dimenſions, that the homogeneity may not be deſtroy'd. Whenever it happens otherwiſe, 'tis becauſe ſome Fluxion, taken for Unity, is there underſtood, and therefore muſt be ſupply'd when occaſion requires. The Equation $\dot{x}\dot{z} + \dot{x}\dot{y}x - a\dot{z}^2x^2 = 0$, by making $\dot{z} = 1$, may become $\dot{x} + \dot{x}\dot{y}x - ax^2 = 0$, and likewiſe *vice verſâ*. And as this Equation virtually involves three variable quantities,

quantities, it will require another Equation, either Fluential or Fluxional, for a compleat determination, as has been already obſerved. So as the Equation $\dot{y}x = \dot{x}yy$, by putting $\dot{x} = 1$ becomes $\dot{y}x = yy$; in like manner this Equation requires and ſuppoſes the other.

6, 7, 8, 9, 10, 11. Here we are taught ſome uſeful Reductions, in order to prepare the Equation for Solution. As when the Equation contains only two flowing Quantities with their Fluxions, the ratio of the Fluxions may always be reduced to ſimple Algebraic Terms. The Antecedent of the Ratio, or its Fluent, will be the quantity to be extracted; and the Conſequent, for the greater ſimplicity, may be made Unity. Thus the Equation $2\dot{x} + 2x\dot{x} - y\dot{x} - \dot{y} = 0$ is reduced to this, $\frac{\dot{y}}{\dot{x}} = 2 + 2x - y$, or making $\dot{x} = 1$, 'tis $\dot{y} = 2 + 2x - y$. So the Equation $\dot{y}a - \dot{y}x - \dot{x}a + \dot{x}x - \dot{x}y = 0$, making $\dot{x} = 1$, will become $\dot{y} = \left(\frac{a-x+y}{a-x} = 1 + \frac{y}{a-x} =\right) 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, &c. by Diviſion. But we may apply the particular Solution to this Example, by which we ſhall have $\frac{1}{2}x^2 - xy - ax + ay = 0$, and thence $y = \frac{ax - \frac{1}{2}x^2}{a-x}$. Thus the Equation $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}xx$, making $\dot{x} = 1$, becomes $\dot{y}\dot{y} = \dot{y} + xx$, and extracting the ſquare-root, 'tis $\dot{y} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + xx} = \frac{1}{2} \pm$ the Series $\frac{1}{2} + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. that is, either $\dot{y} = 1 + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. or $\dot{y} = -x^2 + x^4 - 2x^6 + 5x^8 - 14x^{10}$, &c. Again, the Equation $\dot{y}^3 + ax\dot{x}^2\dot{y} + a^2\dot{x}^2\dot{y} - x^3\dot{x}^3 - 2\dot{x}^3a^3 = 0$, putting $\dot{x} = 1$, becomes $\dot{y}^3 + ax\dot{y} + a^2\dot{y} - x^3 - 2a^3 = 0$. Now an affected Cubic Equation of this form has been reſolved before, (pag. 12.) by which we ſhall have $\dot{y} = a - \frac{1}{4}x + \frac{xx}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$, &c.

12. For the ſake of perſpicuity, and to fix the Imagination, our Author here introduces a diſtinction of Fluents and Fluxions into *Relate* and *Correlate*. The Correlate is that flowing Quantity which he ſuppoſes to flow equably, which is given, or may be aſſumed, at any point of time, as the known meaſure or ſtandard, to which the Relate Quantity may be always compared. It may therefore very properly denote Time; and its Velocity or Fluxion, being an uniform and conſtant quantity, may be made the Fluxional Unit, or the known meaſure of the Fluxion (or of the rate of flowing) of the Relate Quantity. The Relate Quantity, (or Quantities if ſeve-

ral are concern'd,) is that which is ſuppos'd to flow inequably, with any degrees of acceleration or retardation; and ts inequability may be meaſured, or reduced as it were to equability, by conſtantly comparing it with its correſponding Correlate or equable Quantity. This therefore is the Quantity to be found by the Problem, or whoſe Root is to be extracted from the given Equation. And it may be conceived as a Space deſcribed by the inequable Velocity of a Body or Point in motion, while the equable Quantity, or the Correlate, repreſents or meaſures the time of deſcription. This may be illuſtrated by our common Mathematical Tables, of Logarithms, Sines, Tangents, Secants, &c. In the Table of Logarithms, for inſtance, the Numbers are the Correlate Quantity, as proceeding equably, or by equal differences, while their Logarithms, as a Relate Quantity, proceed inequably and by unequal differences. And this reſemblance would more nearly obtain, if we ſhould ſuppoſe infinite other Numbers and their Logarithms to be interpolated, (if that infinite Number be every where the ſame,) ſo as that in a manner they may become continuous. So the Arches or Angles may be conſider'd as the Correlate Quantity, becauſe they proceed by equal differences, while the Sines, Tangents, Secants, &c. are as ſo many Relate Quantities, whoſe rate of increaſe is exhibited by the Tables.

13, 14, 15, 16, 17. This Diſtribution of Equations into Orders, or Claſſes, according to the number of the flowing Quantities and their Fluxions, tho' it be not of abſolute neceſſity for the Solution, may yet ſerve to make it more expedite and methodical, and may ſupply us with convenient places to reſt at.

SECT. II. *Solution of the firſt Caſe of Equations.*

18, 19, 20, 21, 22, 23. THE firſt Caſe of Equations is, when the Quantity $\frac{\dot{y}}{\dot{x}}$, or what ſupplies its place, can always be found in Terms compoſed of the Powers of x, and known Quantities or Numbers. Theſe Terms are to be multiply'd by x, and to be divided by the Index of x in each Term, which will then exhibit the Value of y. Thus in the Equation $\dot{y}^2 = \dot{x}\dot{y} + \dot{x}^2x^2$, it has been found that $\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. Therefore $\frac{\dot{y}x}{\dot{x}} = x + x^3 - x^5 + 2x^7 - 5x^9 + 14x^{11}$, &c. and conſequently $y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 - \frac{5}{9}x^9 + \frac{14}{11}x^{11}$, &c. as may eaſily be proved by the direct Method.

But

But this, and the like Equations, may be reſolved more readily by a Method form'd in imitation of ſome of the foregoing Analyſes, after this manner. In the given Equation make $\dot{x} = 1$; then it will be $\dot{y}^2 = \dot{y} + x^2$, which is thus reſolved:

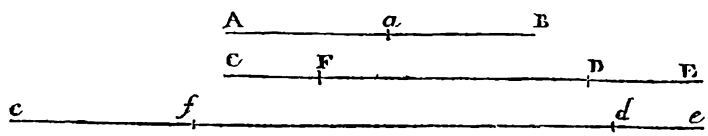
$$\left.\begin{array}{r} \dot{y} \\ -\dot{y}^2 \end{array}\right\} = \begin{array}{l} -x^2 + x^4 - 2x^6 + 5x^8, \text{ \&c.} \\ \text{- - - - - -} -x^4 + 2x^6 - 5x^8, \text{ \&c.} \end{array}$$

Make $-x^2$ the firſt Term of $\dot{y}$; then will $-x^4$ be the firſt Term of $-\dot{y}^2$, which is to be put with a contrary Sign for the ſecond Term of $\dot{y}$. Then by ſquaring, $+2x^6$ will be the ſecond Term of $-\dot{y}^2$, and $-2x^6$ will be the third Term of $\dot{y}$. Therefore $-5x^8$ will be the third Term of $-\dot{y}^2$, and $+5x^8$ will be the fourth Term of $\dot{y}$; and ſo on. Therefore taking the Fluents, $y = -\frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{2}{7}x^7 + \frac{5}{9}x^9$, &c. which will be one Root of the Equation. And if we ſubtract this from x, we ſhall have $y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 - \frac{5}{9}x^9$, &c. for the other Root.

So if $\frac{\dot{y}}{\dot{x}} = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c. that is, if $\frac{\dot{y}x}{\dot{x}} = ax - \frac{1}{4}x^2 + \frac{x^3}{64a} + \frac{131x^4}{512a^2}$, &c. then $y = ax - \frac{1}{8}x^2 + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$, &c. If $\frac{\dot{y}}{\dot{x}} = \frac{1}{x^3} - \frac{1}{x^2} + \frac{a}{x^{\frac{1}{2}}} - x^{\frac{1}{2}} + x^{\frac{3}{2}}$, &c. or $\frac{\dot{y}x}{\dot{x}} = x^{-2} - x^{-1} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}$, &c. then $y = -\frac{1}{2}x^{-2} + x^{-1} + 2ax^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}$, &c. If $\frac{\dot{x}}{\dot{y}} = \frac{2b^2c}{\sqrt{ay^3}} + \frac{3y^2}{a+b} + \sqrt{by + cy}$, or $\frac{\dot{x}y}{\dot{y}} = \frac{2b^2cy^{-\frac{1}{2}}}{a^{\frac{1}{2}}} + \frac{3y^3}{a+b} + y^{\frac{3}{2}}\sqrt{b+c}$, then $x = -\frac{4b^2c}{\sqrt{ay}} + \frac{y^3}{a+b} + \frac{2}{3}y^{\frac{3}{2}}\sqrt{b+c}$. If $\frac{\dot{y}}{\dot{z}} = z^{\frac{2}{3}}$, or $\frac{\dot{y}z}{\dot{z}} = z^{\frac{5}{3}}$, then $y = \frac{3}{5}z^{\frac{5}{3}}$. If $\frac{\dot{y}}{\dot{x}} = \frac{ab}{cx^{\frac{1}{3}}} = \frac{ab}{c}x^{-\frac{1}{3}}$, or $\frac{\dot{y}x}{\dot{x}} = \frac{ab}{c}x^{\frac{2}{3}}$, then $y = \frac{3ab}{2c}x^{\frac{2}{3}}$.

Laſtly, if $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, or $\frac{\dot{y}x}{\dot{x}} = a = ax^0$; dividing by the Index 0, it will be $y = \frac{a}{0}$, or y is infinite. That this Expreſſion, or value of y, muſt be infinite, is very plain. For as 0 is a vaniſhing quantity, or leſs than any aſſignable quantity, its Reciprocal $\frac{1}{0}$ or $\frac{a}{0}$ muſt be bigger than any aſſignable quantity, that is, infinite.

Now that this quantity ought to be infinite, may be thus proved. In the Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, let AB repreſent the conſtant quantity a, and in CE let a point move equably from C towards E, and deſcribe the Line CDE, of which let any indefinite part CD be x, and its equable Velocity in D, (and every where elſe,) is repreſented



by $\dot{x}$. Alſo let a point move from a diſtant point c along the Line cde, with an inequable Velocity, and let the Line deſcribed in the ſame time, or the indefinite part of it cd, be call'd y, and let the Velocity in d be call'd $\dot{y}$. The Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$ muſt always obtain, whatever the contemporaneous values of x and y may be; or in the whole Motion the conſtant Line AB (a) muſt be to the variable Line CD (x), as the Velocity in d ($\dot{y}$) is to the Velocity in D ($\dot{x}$). But at the beginning of the Motion, or when CD (x) was indefinitely little, as the ratio of AB to CD was then greater than any aſſignable ratio, ſo alſo was the ratio $\frac{\dot{y}}{\dot{x}}$ of the Velocities, or the Velocity $\dot{y}$ was infinitely greater than the Velocity $\dot{x}$. But an infinite Velocity muſt deſcribe an infinite Space in a finite time, or the point c is at an infinite diſtance from the point d, that is, y is an infinite quantity.

24, 25. But to avoid ſuch infinite Expreſſions, from whence we can conclude nothing; we are at liberty to change the initial points of the Fluents, by which their Rate of flowing, (the only thing to be here regarded,) will not at all be affected. Thus in the foregoing Figure, we ſuppoſed the points D and d to be ſuch, as limited the contemporaneous Fluents, or in which the two deſcribing points were found at the ſame time. Let F and f be any other two ſuch points, and then the finite Line $CF = b$ will be contemporaneous to, or will correſpond with, the infinite Line $cf = c$; and FD, which may be made the new x, will correſpond to fd, which will be the new y. So that in the given Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, inſtead of

x we may write $b + x$, and we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{a}{b+x}$, and then by Multiplication and Diviſion it is $\frac{\dot{y}x}{\dot{x}} = \left(\frac{ax}{b+x} =\right) \frac{ax}{b} - \frac{ax^2}{b^2} + \frac{ax^3}{b^3} - \frac{ax^4}{b^4}$, &c. and therefore $y = \frac{ax}{b} - \frac{ax^2}{2b^2} + \frac{ax^3}{3b^3} - \frac{ax^4}{4b^4}$, &c.

26. So if $\frac{\dot{y}}{\dot{x}} = \frac{2}{x} + 3 - xx$, becauſe of the Term $\frac{2}{x}$, which would give an infinite value for y, we may write $1 + x$ inſtead of x, and we ſhall then have $\frac{\dot{y}}{\dot{x}} = \frac{2}{1+x} + 2 - 2x - xx$, or $\frac{\dot{y}x}{\dot{x}} = \frac{2x}{1+x} + 2x - 2x^2 - x^3$, or by Diviſion $\frac{\dot{y}x}{\dot{x}} = 4x - 4x^2 + x^3 - 2x^4 + 2x^5$, &c. and therefore $y = 4x - 2x^2 + \frac{4}{3}x^3 - \frac{1}{2}x^4 + \frac{2}{5}x^5$, &c.

Or the Equation $\frac{\dot{y}}{\dot{x}} = \frac{2}{1+x} + 2 - 2x - x^2$, that is $\dot{y} + x\dot{y} = 4 - 3x^2 - x^3$, may be thus reſolved:

$$\left.\begin{array}{r} \dot{y} \\ + x\dot{y} \end{array}\right\} \begin{array}{l} = 4 \quad * \quad - 3x^2 - x^3 \\ \quad - 4x + 4x^2 - x^3 + 2x^4, \text{ &c.} \\ \text{- - - - } + 4x - 4x^2 + x^3 - 2x^4, \text{ &c.} \end{array}$$

$$\dot{y} = 4 - 4x + x^2 - 2x^3 + 2x^4, \text{ &c.}$$

$$y = 4x - 2x^2 + \tfrac{4}{3}x^3 - \tfrac{1}{2}x^4 + \tfrac{4}{5}x^5, \text{ &c.}$$

Make 4 the firſt Term of $\dot{y}$, then $4x$ will be the firſt Term of $x\dot{y}$, and conſequently $-4x$ will be the ſecond Term of $\dot{y}$. Then $-4x^2$ will be the ſecond Term of $x\dot{y}$, and therefore $+4x^2 - 3x^2$, or x^2, will be the third Term of $\dot{y}$; and ſo on.

27. So if $\frac{\dot{y}}{\dot{x}} = x^{-\frac{1}{2}} + x^{-1} - x^{\frac{1}{2}}$, becauſe of the Term x^{-1} change x into $1 - x$, then $\frac{\dot{y}}{\dot{x}} = \frac{1}{\sqrt{1-x}} + \frac{1}{1-x} - \sqrt{1-x}$. But by the foregoing Methods of Reduction 'tis $\frac{1}{1-x} = 1 + x + x^2 + x^3$, &c. and $\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$, &c. and $\frac{1}{\sqrt{1-x}} = \frac{1}{1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3, \text{ &c.}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3$, &c. Therefore collecting theſe according to their Signs, 'tis $\frac{\dot{y}}{\dot{x}} = 1 + 2x + \frac{3}{2}x^2 + \frac{7}{16}x^3$, &c. that is $\frac{\dot{y}x}{\dot{x}} = x + 2x^2 + \frac{3}{2}x^3 + \frac{7}{16}x^4$, &c. and therefore $y = x + x^2 + \frac{1}{2}x^3 + \frac{7}{64}x^4$, &c.

28. So if the given Equation were $\frac{\dot{y}}{\dot{x}} = \frac{c^2x}{c^3 - 3c^2x + 3cx^2 - x^3} = \frac{c^2x}{\overline{c-x}^3}$; change the beginning of x, that is, inſtead of x write

$c - x$,

$c - x$, then $\frac{\dot{y}}{\dot{x}} = \frac{c^3 - c^2x}{x^3} = c^3x^{-3} - c^2x^{-2}$, or $\frac{\dot{y}\dot{x}}{\dot{x}} = c^3x^{-2} - c^2x^{-1}$, and therefore $y = -\frac{1}{2}c^3x^{-2} + c^2x^{-1}$.

SECT. III. *Solution of the ſecond Caſe of Equations.*

29, 30. EQuations belonging to this ſecond caſe are thoſe, wherein the two Fluents and their Fluxions, ſuppoſe x and y, $\dot{x}$ and $\dot{y}$, or any Powers of them, are promiſcuouſly involved. As our Author's Analyſes are very intelligible, and ſeem to want but little explication, I ſhall endeavour to reſolve his Examples in ſomething an eaſier and ſimpler manner, than is done here; by applying to them his own artifice of the Parallelogram, when needful, or the properties of a combined Arithmetical Progreſſion *in plano*, as explain'd before: As alſo the Methods before made uſe of, in the Solution of affected Equations.

31. The Equation $\dot{y}ax - \dot{x}xy - aa\dot{x} = 0$ by a due Reduction becomes $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{x}$, in which, becauſe of the Term $\frac{a}{x}$ there is occaſion for a Tranſmutation, or to change the beginning of the Correlate Quantity x. Aſſuming therefore the conſtant quantity b, we may put $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b+x}$, whence by Diviſion will be had $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4}$, &c. which Equation is then prepared for the Author's Method of Solution.

But without this previous Reduction to an infinite Series, and the Reſolution of an infinite Equation conſequent thereon, we may perform the Solution thus, in a general manner. The given Equation is now $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b+x}$, or putting $\dot{x} = 1$, it is $ab\dot{y} + ax\dot{y} = by + yx + a^2$, which may be thus reſolved:

$$\left.\begin{array}{l} ab\dot{y} \\ +ax\dot{y} \\ -by \\ -xy \end{array}\right\} = a^2 + \frac{ab-a^2}{b}x + \frac{2a^2+b^2-ab}{2b^2}x^2 + \frac{b^3+2a^2b-ab^2-6a^3}{6ab^3}x^3, \text{ \&c.}$$
$$\left.\begin{array}{l} \\ +ax\dot{y} \\ -by \\ -xy \end{array}\right\} = \quad + \frac{a^2}{b}x + \frac{ab-aa}{bb}x^2 + \frac{2a^2+b^2-ab}{2b^3}x^3, \text{ \&c.}$$
$$\left.\begin{array}{l} \\ \\ -by \\ -xy \end{array}\right\} = \quad - ax + \frac{a-b}{2b}x^2 + \frac{ab-2a^2-b^2}{6ab^2}x^3, \text{ \&c.}$$
$$\left.\begin{array}{l} \\ \\ \\ -xy \end{array}\right\} = \quad - \frac{a}{b}x^2 + \frac{a-b}{2b^2}x^3, \text{ \&c.}$$
$$\dot{y} = \frac{a}{b} + \frac{b-a}{b^2}x + \frac{2a^2+b^2-ab}{2ab^3}x^2 + \frac{b^3+2a^2b-ab^2-6a^3}{6a^2b^4}x^3, \text{ \&c.}$$
$$y = \frac{a}{b}x + \frac{b-a}{2b^2}x^2 + \frac{2a^2+b^2-ab}{6ab^3}x^3 + \frac{b^3+2a^2b-ab^2-6a^3}{24a^2b^4}x^4, \text{ \&c.}$$

Diſpoſing the Terms as you ſee is done here, make a^2 the firſt Term of $ab\dot{y}$, then $\frac{a}{b}$ will be the firſt Term of $\dot{y}$, and thence $\frac{a}{b}x$ will be the firſt Term of y. So that $\frac{a^2}{b}x$ will be the firſt Term of $ax\dot{y}$, and $-ax$ will be the firſt Term of $-by$. Theſe two together, or $\frac{a^2}{b}x - ax = \frac{a^2-ab}{b}x$, with a contrary Sign, muſt be put down for the ſecond Term of $ab\dot{y}$. Therefore the ſecond Term of $\dot{y}$ will be $\frac{b-a}{b^2}x$, and the like Term of y will be $\frac{b-a}{2b^2}x^2$. Then the ſecond Term of $ax\dot{y}$ will be $\frac{ab-a^2}{b^2}x^2$, and the ſecond Term of $-by$ will be $\frac{a-b}{2b}x^2$, and the firſt Term of $-xy$ will be $-\frac{a}{b}x^2$. Theſe three together make $\frac{ab-2a^2-b^2}{2b^2}x^2$, which with a contrary Sign muſt be made the third Term of $ab\dot{y}$. Therefore the third Term of $\dot{y}$ will be $\frac{2a^2+b^2-ab}{2ab^3}x^2$, and the third Term of y will be $\frac{2a^2+b^2-ab}{6ab^3}x^3$. And ſo on. Here in a particular caſe if we make $b = a$, we ſhall have the ſimple Series $y = x * + \frac{x^3}{3a^2} - \frac{x^4}{6a^3}$, &c.

Or if we would have a deſcending Series for the Root y of this Equation, we may proceed as follows:

$$\left.\begin{array}{l} -xy \\ -by \\ +ax\dot{y} \\ +ab\dot{y} \end{array}\right\} \begin{array}{l} = a^2 - \overline{a+b} \times a^2x^{-1} + \overline{2a^2+2ab+b^2} \times a^2x^{-2}, \text{ \&c.} \\ \text{---} + a^2bx^{-1} - \overline{a+b} \times a^2bx^{-2}, \text{ \&c.} \\ \text{---} + a^3x^{-1} - \overline{a+b} \times 2a^3x^{-2}, \text{ \&c.} \\ \text{----------} + a^3bx^{-2}, \text{ \&c.} \end{array}$$

$$y = -a^2x^{-1} + \overline{a+b} \times a^2x^{-2} - \overline{2a^2+2ab+b^2} \times a^2x^{-3}, \text{ \&c.}$$

$$\dot{y} = a^2x^{-2} - \overline{a+b} \times 2a^2x^{-3}, \text{ \&c.}$$

Diſpoſe the Terms as you ſee, and make a^2 the firſt Term of the Series $-xy$; then will $-\frac{a^2}{x}$ be the firſt Term of y, and a^2x^{-2} will be the firſt Term of y. Then will $+a^2bx^{-1}$ be the firſt Term of $-by$, and a^3x^{-1} will be the firſt Term of $ax\dot{y}$, which together make $\overline{a+b} \times a^2x^{-1}$; this therefore with a contrary Sign muſt be the ſecond Term of $-xy$. Then the ſecond Term of y will be $\overline{a+b} \times a^2x^{-2}$, and the ſecond Term of $\dot{y}$ will be $-\overline{a+b} \times 2a^2x^{-3}$. Therefore the ſecond Term of $-by$ will be $-\overline{a+b} \times a^2bx^{-2}$, and

and the ſecond Term of $axy\dot{}$ will be $-\overline{a+b}\times 2a^{3}x^{-2}$, and the firſt Term of $aby\dot{}$ will be $a^{3}bx^{-2}$; which three together make $-\overline{2a^{2}+2ab+b^{2}}\times a^{2}x^{-2}$. This with a contrary Sign muſt be the third Term of $-xy$, which will give $-\overline{2a^{2}+2ab+b^{2}}\times a^{2}x^{-3}$ for the third Term of y; and ſo on. Here if we make $b=a$, then $y=-\frac{a^{2}}{x}+\frac{2a^{3}}{x^{2}}-\frac{5a^{4}}{x^{3}}$, &c.

And theſe are all the Series, by which the value of y can be exhibited in this Equation, as may be proved by the Parallelogram. For that Method may be extended to theſe Fluxional Equations, as well as to Algebraical or Fluential Equations. To reduce theſe Equations within the Limits of that Rule, we are to conſider, that as Ax^{m} may repreſent the initial Term of the Root y, in both theſe kinds of Equations, or becauſe it may be $y=Ax^{m}$, &c. ſo in Fluxional Equations (making $\dot{x}=1$, we ſhall have alſo $\dot{y}=mAx^{m-1}$, &c. or writing y for Ax^{m}, &c. 'tis $\dot{y}=myx^{-1}$, &c. So that in every Term of the given Equation, in which $\dot{y}$ occurs, or the Fluxion of the Relate Quantity, we may conceive it to take away one Dimenſion from the Correlate Quantity, ſuppoſe x, and to add it to the Relate Quantity, ſuppoſe y; according to which Reduction we may inſert the Terms in the Parallelogram. And we are to make a like Reduction for all the Powers of the Fluxion of the Relate Quantity. This will bring all Fluxional Equations to the Caſe of Algebraic Equations, the Reſolution of which has been ſo amply treated of before.

Thus in the preſent Equation $aby\dot{}+axy\dot{}=by+yx+aa$, the Terms muſt be inſerted in the Parallelogram, as if yx^{-1} were ſubſtituted inſtead of $\dot{y}$; ſo that the Indices will ſtand as in the Margin, and the Ruler will give only two Caſes of external Terms. Or rather, if we would reduce this Equation to the form of a double Arithmetical Scale, as explain'd before, we ſhould have it in this form. Here in the firſt Column are contain'd thoſe Terms which have y of one Dimenſion, or what is equivalent to it. In the ſecond Column is $-a^{2}$, or y of no Dimenſions. Alſo in the firſt Line is $-xy$, or ſuch Terms in which x is of one Dimenſion. In the ſecond Line are the Terms $\left.\begin{matrix}-by\\+axy\end{matrix}\right\}-a^{2}$, which have no Dimenſions of x, becauſe $+axy\dot{}$ is regarded as if it were ay. Laſtly, in the third line is $aby\dot{}$, or the Term in which x is of one negative Dimenſion,

	$m+1$
0	m
	$m-1$

$$\left.\begin{matrix}-xy\\-by\\+axy\\+aby\end{matrix}\right\}\left.\begin{matrix}*\\-a^{2}\\*\end{matrix}\right\}=0.$$

Dimenſion, becauſe $+aby$ is conſider'd as if it were $+abx^{-1}y$. And theſe Terms being thus diſpos'd, it is plain there can be but two Caſes of external Terms, which we have already diſcuſs'd.

32. If the propoſed Equation be $\frac{\dot{y}}{\dot{x}} = 3y - 2x + \frac{x}{y} - \frac{2v}{x\dot{x}}$, or making $\dot{x} = 1$, 'tis $-\dot{y} + 3y - 2x + xy^{-1} - 2yx^{-2} = 0$; the Solution of which we ſhall attempt without any preparation, or without any new interpretation of the Quantities. Firſt, the Terms are to be diſpos'd according to a double Arithmetical Scale, the Roots of which are y and x, and then they will ſtand as in the Margin. The Method of doing this with certainty in all caſes is as follows. I obſerve in the Equation there are three powers of y, which are y^1, y^0, and y^{-1}; therefore I place theſe in order at the top of the Table. I obſerve likewiſe that there are four Powers of x, which are x^1, x^0, x^{-1}, and x^{-2}, which I place in order in a Column at the right hand; or it will be enough to conceive this to be done. Then I inſert every Term of the Equation in its proper place, according to its Dimenſions of y and x in that Term; filling up the vacancies with Aſteriſms, to denote the abſence of the Terms belonging to them. The Term $-\dot{y}$ I inſert as if it were $-yx^{-1}$, as is explain'd before. Then we may perceive, that if we apply the Ruler to the exterior Terms, we ſhall have three caſes that may produce Series; for the fourth caſe, which is that of direct aſcent or deſcent, is always to be omitted, as never affording any Series. To begin with the deſcending Series, which will ariſe from the two external Terms $-2x$ and $+xy^{-1}$. The Terms are to be diſpos'd, and the Analyſis to be perform'd, as here follows:

	y^1	y^0	y^{-1}
x^1	*	$-2x$	$+x$
x^0	$+3y$	*	*
x^{-1}	$-\dot{y}$	*	*
x^{-2}	$-2yx^{-2}$	*	*

$= 0.$

$$\left.\begin{array}{l} xy^{-1} \\ +3y \\ -\dot{y} \\ -2yx^{-2} \end{array}\right\} \begin{array}{l} = 2x - \frac{3}{4} - \frac{9}{8}x^{-1} - \frac{1}{16}\frac{7}{8}x^{-2}, \text{ \&c.} \\ \text{- - - - } + \frac{3}{4} + \frac{9}{8}x^{-1} + \frac{1}{16}\frac{7}{8}x^{-2}, \text{ \&c.} \\ \text{- - - - - - - - - - } * + \frac{3}{8}x^{-2}, \text{ \&c.} \\ \text{- - - - - - - - - - - - } - x^{-2}, \text{ \&c.} \end{array}$$

$$y = \frac{1}{2} + \frac{3}{8}x^{-1} + \frac{9}{16}x^{-2} + \frac{1}{12}\frac{5}{8}x^{-3}, \text{ \&c.}$$

Make $xy^{-1} = 2x$, &c. then $y^{-1} = 2$, &c. and by Diviſion $y = \frac{1}{2}$, &c. Therefore $3y = \frac{3}{2}$, &c. and conſequently $xy^{-1} = * - \frac{3}{2}$, &c. or $y^{-1} = * - \frac{3}{4}x^{-1}$, &c. and by Diviſion $y = * + \frac{3}{8}x^{-1}$, &c. Therefore $3y = * \frac{9}{8}x^{-1}$, &c. and conſequently $xy^{-1} = ** - \frac{9}{8}x^{-1}$, &c. So that $y^{-1} = ** - \frac{9}{8}x^{-2}$, &c. and by Diviſion $y = ** + \frac{9}{16}x^{-2}$&c. Then $3y = ** + \frac{1}{16}\frac{7}{8}x^{-2}$, &c. and

$-\dot{y} = * + \frac{3}{8}x^{-2}$, &c. and $-2yx^{-2} = -x^{-2}$, &c. These three together make $+\frac{17}{16}x^{-2}$, and therefore $xy^{-1} = * * * - \frac{17}{16}x^{-2}$, &c. so that $y = *. * * + \frac{115}{128}x^{-3}$, &c. And so on.

Another descending Series will arise from the two external Terms $+3y$ and $-2x$, which may be thus extracted:

$$\left.\begin{array}{l} 3y \\ +xy^{-1} \\ -\dot{y} \\ -2yx^{-2} \end{array}\right\} \begin{array}{l} = 2x - \frac{5}{6} + \frac{17}{24}x^{-1} - \frac{55}{48}x^{-2}, \text{ &c.} \\ \text{- - - - } + \frac{3}{2} + \frac{5}{8}x^{-1} - \frac{33}{48}x^{-2}, \text{ &c.} \\ \text{- - - - } - \frac{2}{3} \quad * \quad + \frac{17}{72}x^{-2}, \text{ &c.} \\ \text{- - - - - - - } - \frac{4}{3}x^{-1} + \frac{5}{9}x^{-2}, \text{ &c.} \end{array}$$

$$y = \frac{2}{3}x - \frac{5}{12}8 + \frac{17}{72}x^{-1} - \frac{255}{864}x^{-2}, \text{ &c.}$$

$$y^{-1} = \frac{3}{4}x^{-1} + \frac{5}{8}x^{-2} - \frac{33}{48}x^{-3}, \text{ &c.}$$

$$\dot{y} = \frac{2}{3} * - \frac{17}{72}x^{-2}, \text{ &c.}$$

Make $3y = 2x$, &c. then $y = \frac{2}{3}x$, &c. and (by Division) $y^{-1} = \frac{3}{2}x^{-1}$, &c. and $xy^{-1} = \frac{3}{2}$, &c. and $-\dot{y} = -\frac{2}{3}$, &c. Therefore $3y = * - \frac{5}{6}$, &c. and $y = * - \frac{5}{18}$, &c. and (by Division) $xy^{-1} = * \frac{5}{8}x^{-1}$, &c. and $-\dot{y} = *$ o, &c. and $-2yx^{-2} = -\frac{4}{3}x^{-1}$, &c. Therefore $3y = * * + \frac{17}{24}x^{-1}$, &c. and $y = * * + \frac{17}{72}x^{-1}$, &c. &c.

The ascending Series in this Equation will arise from the two external Terms $-2yx^{-2}$ and xy^{-1}; or multiplying the whole Equation by $-y$, (that one of the external Terms may be clear'd from y,) we shall have $y\dot{y} - 3y^2 + 2xy - x + 2y^2x^{-2} = 0$, of which the Resolution is thus:

$$\left.\begin{array}{l} 2y^2x^{-2} \\ +y\dot{y} \\ +2xy \\ -3y^2 \end{array}\right\} \begin{array}{l} = x^{\frac{3}{2}} * - \frac{3}{4}x^{\frac{4}{2}} - \frac{2}{\sqrt{2}}x^{\frac{5}{2}} + \frac{9}{4}x^3, \text{ &c.} \\ \text{- - - - - } + \frac{3}{4}x^2 \quad * \quad - \frac{3}{4}x^3, \text{ &c.} \\ \text{- - - - - - - - - - } + \frac{2}{\sqrt{2}}x^{\frac{5}{2}} \quad * \text{ , &c.} \\ \text{- - - - - - - - - - - - - - - } - \frac{3}{2}x^3, \text{ &c.} \end{array}$$

$$y = \frac{1}{\sqrt{2}}x^{\frac{3}{2}} * - \frac{3}{8\sqrt{2}}x^{\frac{5}{2}} - \frac{1}{2}x^3 + \frac{135}{128\sqrt{2}}x^{\frac{7}{2}}, \text{ &c.}$$

$$\dot{y} = \frac{3}{2\sqrt{2}}x^{\frac{1}{2}} * - \frac{15}{16\sqrt{2}}x^{\frac{3}{2}} - \frac{3}{2}x^2, \text{ &c.}$$

Make $2y^2x^{-2} = x$, &c. then $y^2 = \frac{1}{2}x^3$, &c. and $y = \frac{1}{\sqrt{2}}x^{\frac{3}{2}}$, &c. Here because of the fractional Indices, and that the first Term of $+\frac{1}{2}xy$, or $+\frac{2}{\sqrt{2}}x^{\frac{3}{2}}$, may be afterwards admitted, we must take o for the second Term of $2y^2x^{-2}$; and therefore for the second Term

of y. Then $\dot{y}\dot{y} = \frac{1}{4}x^2$, &c. and conſequently $2y^2x^{-2} = * * - \frac{1}{4}x^2$, &c. and $y^2 = * * - \frac{1}{8}x^4$, &c. and by extracting the ſquare-root, $y = * * - \frac{3}{8\sqrt{2}}x^{\frac{5}{2}}$, &c. Then $\dot{y}\dot{y} = * + 0$, &c. and $2xy = \frac{2}{\sqrt{2}}x^{\frac{5}{2}}$, &c. and therefore $2y^2x^{-2} = * * * - \frac{2}{\sqrt{2}}x^{\frac{1}{2}}$, &c. and $y = * * * - \frac{1}{2}x^3$, &c. &c.

33, 34. The Author's Proceſs of Reſolution, in this and the following Examples, is very natural, ſimple, and intelligible; it proceeds *ſeriatim & terminatim*, by paſſing from Series to Series, and by gathering Term after Term, in a kind of circulating manner, of which Method we have had frequent inſtances before. By this means he collects into a Series what he calls the Sum, which Sum is the value of $\frac{\dot{y}}{\dot{x}}$ or of the Ratio of the Fluxions of the Relate and Correlate in the given Equation; and then by the former Problem he obtains the value of y. When I firſt obſerved this Method of Solution, in this Treatiſe of our Author's, I confeſs I was not a little pleaſed; it being nearly the ſame, and differing only in a few circumſtances that are not material, from the Method I had happen'd to fall into ſeveral years before, for the Solution of Algebraical and Fluxional Equations. This Method I have generally purſued in the courſe of this work, and ſhall continue to explain it farther by the following Examples.

The Equation of this Example $1 - 3x + y + x^2 + xy - \dot{y} = 0$ being reduced to the form of a double Arithmetical Scale, will ſtand as here in the Margin; and the Ruler will diſcover two caſes to be try'd, of which one may give us an aſcending, and the other a deſcending Series for the Root y. And firſt for the aſcending Series.

	y^1	y^0	
x^2	*	$+ x^2$	} = 0.
x^1	$+ xy$	$- 3x$	
x^0	$+ y$	$+ 1$	
x^{-1}	$- \dot{y}$	*	

$$\left.\begin{array}{r} \dot{y} \\ -y \\ -xy \\ \dot{y} \\ y \end{array}\right\} \begin{array}{l} = 1 - 3x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5, \text{ \&c.} \\ \quad + x \quad * \\ \text{---} - x + x^2 - \frac{1}{3}x^3 + \frac{1}{6}x^4 - \frac{1}{30}x^5, \text{ \&c.} \\ \text{-------} - x^2 + x^3 - \frac{1}{3}x^4 + \frac{1}{6}x^5, \text{ \&c.} \\ = 1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5, \text{ \&c.} \\ = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6, \text{ \&c.} \end{array}$$

The Terms being diſpoſed as you ſee, make $\dot{y} = 1$, &c. then $y = x$, &c. Therefore $-y = -x$, &c. the Sign of which Term being changed, it will be $\dot{y} = * + x - 3x$, &c. $= * - 2x$, &c.

and therefore $y = * - x^2$, &c. Then $-y = * + x^2$, &c. and $-xy = -x^2$, &c. theſe deſtroying each other, 'tis $\dot{y} = * * + x^2$, &c. and therefore $y = * * + \frac{1}{3}x^3$, &c. Then $-y = * * - \frac{1}{2}x^3$, &c. and $-xy = * + x^3$, &c. it will be $\dot{y} = * * * - \frac{1}{3}x^3$, &c. and therefore $y = * * * - \frac{1}{6}x^4$, &c. &c.

The Analyſis in the ſecond caſe will be thus:

$$\left.\begin{array}{l} -xy \\ -\ y \\ +\ \dot{y} \end{array}\right\} \begin{array}{l} = x^2 - 3x + 1 \\ \quad - x + 5 - 6x^{-1} \quad * \quad + 12x^{-3}, \text{ \&c.} \\ - - - - + x - 4 + 6x^{-1} - 6x^{-2} \quad * \quad , \text{ \&c.} \\ - - - - - - - - - 1 \quad * \quad + 6x^{-2} - 12x^{-3}, \text{ \&c.} \end{array}$$

$$y = -x + 4 - 6x^{-1} + 6x^{-2} * - 12x^{-4}, \text{ \&c.}$$

Make $-xy = x^2$, &c. then $y = -x$, &c. Therefore $-y = x$, &c. and changing the Sign, 'tis $-xy = * - x - 3x$, &c. $= * - 4x$, &c. and therefore $y = * + 4$, &c. Then $-y = * - 4$, &c. and $\dot{y} = -1$, &c. and changing the Signs, 'tis $-xy = * * + 5 + 1$, &c. $= * * + 6$, &c. and $y = * * - 6x^{-1}$, &c. &c.

35, 36. If the given Equation were $\frac{\dot{y}}{x} = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, &c. its Reſolution may be thus perform'd:

$$\left.\begin{array}{l} \dot{y} \\ -\frac{y}{a} \\ -\frac{xy}{a^2} \\ -\frac{x^2y}{a^3} \\ -\frac{x^3y}{a^4} \\ \text{\&c.} \end{array}\right\} \begin{array}{l} = 1 + \frac{x}{a} + \frac{3x^2}{2a^2} + \frac{2x^3}{a^3} + \frac{5x^4}{2a^4}, \text{ \&c.} \\ - - - - \frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{2a^3} - \frac{x^4}{2a^4}, \text{ \&c.} \\ - - - - - - - \frac{x^2}{a^2} - \frac{x^3}{2a^3} - \frac{x^4}{2a^4}, \text{ \&c.} \\ - - - - - - - - - - - \frac{x^3}{a^3} - \frac{x^4}{2a^4}, \text{ \&c.} \\ - - - - - - - - - - - - - \frac{x^4}{a^4}, \text{ \&c.} \end{array}$$

$$y = x + \frac{x^2}{2a} + \frac{x^3}{2a^2} + \frac{x^4}{2a^3} + \frac{x^5}{2a^4}, \text{ \&c.}$$

Make $\dot{y} = 1$, &c. then $y = x$, &c. Therefore $-\frac{y}{a} = -\frac{x}{a}$, &c. and $\dot{y} = * + \frac{x}{a}$, &c. and therefore $y = * + \frac{x^2}{2a}$, &c. Then $-\frac{y}{a} = * - \frac{x^2}{2a^2}$, &c. and $-\frac{xy}{a^2} = -\frac{x^2}{a^2}$, &c. and therefore $\dot{y} = * * + \frac{3x^2}{2a^2}$, &c. and $y = * * + \frac{x^3}{2a^2}$, &c. And ſo on.

Now

Now in this Example, becauſe the Series $\frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, &c. is equal to $\frac{y}{a-x}$, it will be $\dot{y} = \frac{y}{a-x} + 1$, or $a\dot{y} - x\dot{y} = y + a - x$, that is, $y\dot{x} + a\dot{x} - x\dot{x} - a\dot{y} + x\dot{y} = 0$; which Equation, by the particular Solution before deliver'd, will give the relation of the Fluents $yx - ay + ax - \frac{1}{2}x^2 = 0$. Hence $y = \frac{ax - \frac{1}{2}xx}{a-x}$, and by Diviſion $y = x + \frac{x^2}{2a} + \frac{x^3}{2a^2} + \frac{x^4}{2a^3}$, &c. as found above.

37. The Equation of this Example being tabulated, or reduced to a double Arithmetical Scale, will ſtand as here in the Margin. Where it may be obſerved, that becauſe of the Series proceeding both ways *ad infinitum*, there can be but one caſe of exterior Terms, of which the Solution here follows:

	y^0	y^1	y^2	y^3	y^4, &c.	
x^{-1}	*	$-\dot{y}$	*			
x^0	*	*	$+y^2$	$+y^3$	$+y^4$, &c.	
x^1	$-3x$	$+3xy$	$-xy^2$	$-xy^3$	$-xy^4$, &c.	
x^2	$-6x^2$	$+6x^2y$	*	*	*	
x^3	$-8x^3$	$+8x^3y$	*	*	*	
x^4	$-10x^4$	$+10x^4y$	*	*	*	$= 0.$
x^5	$-12x^5$	$+12x^5y$	*	*	*	
x^6	$-14x^6$	&c.				
&c.	&c.					

$$
\left.\begin{array}{r}
\dot{y} \\ -3xy \\ -6x^2y \\ -y^2 \\ -8x^3y \\ +xy^2 \\ -10x^4y \\ -y^3 \\ \text{\&c.}
\end{array}\right\}
\begin{array}{l}
= -3x - 6x^2 - 8x^3 - 10x^4 - 12x^5 - 14x^6, \text{ \&c.} \\
\qquad\qquad - \frac{9}{2}x^3 - 12\frac{3}{4}x^4 - 29\frac{5}{8}x^5 - 59\frac{2}{5}x^6, \text{ \&c.} \\
\text{----------} + \frac{9}{2}x^3 + 6x^4 + \frac{75}{8}x^5 + \frac{273}{20}x^6, \text{ \&c.} \\
\text{----------} + 9x^4 + 12x^5 + \frac{75}{4}x^6, \text{ \&c.} \\
\text{----------} - \frac{9}{4}x^4 - 6x^5 - \frac{107}{8}x^6, \text{ \&c.} \\
\text{----------} + 12x^5 + 16x^6, \text{ \&c.} \\
\text{----------} + \frac{9}{4}x^5 + 6x^6, \text{ \&c.} \\
\text{----------} + 15x^6, \text{ \&c.} \\
\text{----------} + \frac{27}{8}x^6, \text{ \&c.}
\end{array}
$$

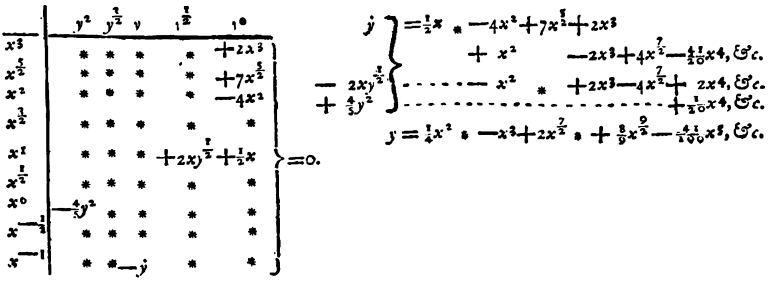
$$y = -\frac{3}{2}x^2 - 2x^3 - \frac{25}{8}x^4 - \frac{91}{20}x^5 - \frac{111}{16}x^6 - \frac{367}{35}x^7, \text{ \&c.}$$

Make $\dot{y} = -3x$, &c. then $y = -\frac{3}{2}x^2$, &c. Then $\dot{y} = * - 6x^2$, &c. and $y = * - 2x^3$, &c. Then $-3xy = +\frac{9}{2}x^3$, &c. and therefore $\dot{y} = * * - \frac{9}{2}x^3 - 8x^3$, &c. $= * * - \frac{25}{2}x^3$, &c. and $y = * * - \frac{25}{8}x^4$, &c. And ſo of the reſt.

The Author here takes notice, that as the value of $\dot{y}$ is negative, and therefore contrary to that of $\dot{x}$, it ſhews that as x increaſes, y muſt decreaſe, and on the contrary. For a negative Velocity is a Velocity backwarks, or whoſe direction is contrary to that which was

was ſuppos'd to be an affirmative Velocity. This Remark muſt take place hereafter, as often as there is occaſion for it.

38. In this Example the Author puts x to repreſent the Relate Quantity, or the Root to be extracted, and y to repreſent the Correlate. But to prevent the confuſion of Ideas, we ſhall here change x into y, and y into x; ſo that y ſhall denote the Relate, and x the Correlate Quantity, as uſual. Let the given Equation therefore be $\frac{\dot{y}}{\dot{x}} = \frac{1}{2}x - 4x^2 + 2xy^{\frac{1}{2}} - \frac{4}{5}y^2 + 7x^{\frac{5}{2}} + 2x^3$, whoſe Root y is to be extracted. Theſe Terms being diſpoſed in a Table, will ſtand thus: And the Reſolution will be as follows, taking $-\dot{y}$ and $+\frac{1}{2}x$ for the two external Terms.



Make $\dot{y} = \frac{1}{2}x$, &c. then $y = \frac{1}{4}x^2$, &c. Now becauſe it is $\dot{y} = * 0$, &c. it will be alſo $y = * 0$, &c. And whereas it is $y^{\frac{1}{2}} = \frac{1}{2}x$, &c. it will be $-2xy^{\frac{1}{2}} = -x^2$, &c. and therefore $\dot{y} = * * + x^2 - 4x^2$, &c. $= * * - 3x^2$, &c. then $y = * * - x^3$, &c. Now becauſe it is $y = * + 0$, &c. it will be alſo $y^{\frac{1}{2}} = * + 0$, &c. and $-2xy^{\frac{1}{2}} = * + 0$, &c. and conſequently $\dot{y} = * * * + 7x^{\frac{5}{2}}$, &c. and therefore $y = * * * + 2x^{\frac{7}{2}}$, &c. And ſo on.

There are two other caſes of external Terms, which will ſupply us with two other Series for the Root y, but they will run too much into Surds. This may be ſufficient to ſhew the univerſality of the Method, and how we are to proceed in like caſes.

39. The Author ſhews here, that the ſame Fluxional Equation may often afford a great variety of Series for the Root, according as we ſhall introduce any conſtant quantity at pleaſure. Thus the Equation of Art. 34. or $\dot{y} = 1 - 3x + y + x^2 + xy$, may be reſolved after the following general manner:

$$\left.\begin{array}{l} \dot{y} \\ \\ \\ -y \\ \\ \\ -xy \end{array}\right\} \left\{\begin{array}{l} = 1 - 3x + x^2 \\ +a + x + 2ax^2 + \frac{4}{3}ax^3, \&c. \\ \quad + 2ax - \frac{4}{3}x^3 \\ -a - x + x^2 - \frac{1}{3}x^3, \&c. \\ \quad - ax - ax^2 - \frac{1}{3}ax^3 \\ \quad - ax - x^2 - ax^3, \&c. \\ \quad\quad - ax^2 + x^3 \end{array}\right. \qquad \left\{\begin{array}{l} y = a + x - x^2 + \frac{4}{3}x^3 - \frac{1}{2}x^3, \&c. \\ \quad + ax + ax^2 + \frac{1}{3}ax^3 + \frac{1}{12}ax^3 \end{array}\right.$$

Here inſtead of making $\dot{y} = 1$, &c. we may make $\dot{y} = 0$, &c. and therefore $y = a$, &c. becauſe then $\dot{y} = 0$, &c. then $-y = -a$, &c. and conſequently $\dot{y} = * + a + 1$, &c. and therefore $y = * + ax + x$, &c. Then $-y = * - ax - x$, &c. and $-xy = -ax$, &c. and therefore $\dot{y} = * * + 2ax + x - 3x$, &c. $= * * + 2ax - 2x$, &c. and then $y = * * + ax^2 - x^2$, &c. Therefore $-y = * * - ax^2 + x^2$, &c. and $-xy = * - ax^2 - x^2$, &c. and conſequently $\dot{y} = * * * + 2ax^2 + x^2$, &c. and $y = * * * + \frac{2}{3}ax^3 + \frac{1}{3}x^3$, &c. &c. Here if we make $a = 0$, we ſhall have the ſame value of y as was extracted before. And by whatever Number a is interpreted, ſo many different Series we ſhall obtain for y.

40. The Author here enumerates three caſes, when an arbitrary Number ſhould be aſſumed, if it can be done, for the firſt Term of the Root. Firſt, when in the given Equation the Root is affected with a Fractional Dimenſion, or when ſome Root of it is to be extracted; for then it is convenient to have Unity for the firſt Term, or ſome other Number whoſe Root may be extracted without a Surd, if ſuch Number does not offer itſelf of its own accord. As in the fourth Example 'tis $x = \frac{1}{4}y^2$, &c. and therefore we may eaſily have $x^{\frac{1}{2}} = \frac{1}{2}y$, &c. Secondly, it muſt be done, when by reaſon of the ſquare-root of a negative Quantity, we ſhould otherwiſe fall upon impoſſible Numbers. Laſtly, we muſt aſſume ſuch a Number, when otherwiſe there would be no initial Quantity, from whence to begin the computation of the Root; that is, when the Relate Quantity, or its Fluxion, affects all the Terms of the Equation.

41, 42, 43. The Author's Compendiums of Extraction are very curious, and ſhew the univerſality of his Method. As his ſeveral Proceſſes want no explanation, I ſhall proceed to reſolve his Examples by the foregoing general Method. As if the given Equation were $\dot{y} = \frac{1}{y} - x^2$, or $\dot{y} - y^{-1} = -x^2$, the Reſolution might be thus:

$\dot{y}$

$$\left.\begin{array}{l} \dot{y} \\ -y^{-1} \end{array}\right\} \begin{array}{l} = 0 \quad * \quad * \quad - \quad x^2 - \frac{5}{2}a^{-7}\dot{x}^3, \text{ \&c.} \\ \quad + a^{-1} - a^{-3}x + \frac{3}{2}a^{-5}x^2 + \frac{1}{2}a^{-2}x^3 \\ \quad - a^{-1} + a^{-3}x - \frac{3}{2}a^{-5}x^2 + \frac{5}{2}a^{-7}x^3, \text{ \&c.} \\ \qquad\qquad\qquad - \frac{1}{2}a^{-2}x^3 \end{array}$$

$$y = a + \frac{x}{a} - \frac{x^2}{2a^3} + \frac{x^3}{2a^5} - \frac{5x^4}{8a^7}, \text{ \&c.}$$
$$- \frac{1}{3}x^3 + \frac{x^4}{12a^2}$$

Make $\dot{y} = 0$, &c. then aſſuming any conſtant quantity a, it may be $y = a$, &c. Then by Diviſion $-y^{-1} = -a^{-1}$, &c. and therefore $\dot{y} = * + a^{-1}$, &c. and conſequently $y = * + a^{-1}x$, &c. Then by Diviſion $-y^{-1} = * + a^{-3}x$, &c. and therefore $\dot{y} = * * - a^{-3}x$, &c. and conſequently $y = * * - \frac{1}{2}a^{-3}x^2$, &c. Then again by Diviſion $-y^{-1} = * * - \frac{3}{2}a^{-5}x^2$, &c. and therefore $\dot{y} = * * * + \frac{3}{2}a^{-5}x^2 - x^2$, &c. and conſequently $y = * * * \frac{1}{2}a^{-5}x^3 - \frac{1}{3}x^3$, &c. And ſo of the reſt. Here if we make $a = 1$, we ſhall have $y = 1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4$, &c.

Or the ſame Equation may be thus reſolved:

$$\left.\begin{array}{l} -y^{-1} \\ +\dot{y} \end{array}\right\} \begin{array}{l} = -x^2 + 2x^{-3} + 14x^{-8} + 216x^{-13}, \text{ \&c.} \\ \quad - 2x^{-3} - 14x^{-8} - 216x^{-13}, \text{ \&c.} \end{array}$$
$$y = x^{-2} + 2x^{-7} + 18x^{-12} + 280x^{-17}, \text{ \&c.}$$

Make $-y^{-1} = -x^2$, &c. or $y = x^{-2}$, &c. Then $\dot{y} = -2x^{-3}$, &c. and therefore $-y^{-1} = * + 2x^{-3}$, &c. and conſequently by Diviſion $y = * + 2x^{-7}$, &c. Then $\dot{y} = * - 14x^{-8}$, &c. and therefore $-y^{-1} = * * + 14x^{-8}$, &c. and by Diviſion $y = * * + 18x^{-12}$, &c. Then $\dot{y} = * * - 216x^{-13}$, &c. and therefore $-y^{-1} = * * * + 216x^{-13}$, &c. and by Diviſion $y = * * * + 280x^{-17}$, &c. And ſo on.

Another aſcending Series may be had from this Equation, *viz.* $y = \sqrt{2x} - \frac{2}{7}x^3 + \frac{x^{\frac{11}{2}}}{147\sqrt{2}} + \frac{10x^8}{17493}$, &c. by multipying it by y, and then making 1 the firſt Term of $y\dot{y}$.

44. The Equation $\dot{y} = 3 + 2y - x^{-1}y^2$ may be thus reſolved:

$$\left.\begin{array}{l} \dot{y} \\ -2y \\ +x^{-1}y^2 \end{array}\right\} \begin{array}{l} = 3 - 3x + 6x^2, \text{ \&c.} \\ \quad - 6x + 3x^2, \text{ \&c.} \\ \quad + 9x - 9x^2, \text{ \&c.} \\ y = 3x - \frac{3}{2}x^2 + 2x^3, \text{ \&c.} \end{array} \qquad \begin{array}{c|c} & y^2 \quad y^1 \quad y^0 \\ \hline x^0 & * \quad +2y+3 \\ x^{-1} & -x^{-1}y^2 - \dot{y} \quad * \end{array} = 0.$$

Make

Make $\dot{y} = 3$, &c. then $y = 3x$, &c. Therefore $-2y = -6x$, &c. and $x^{-1}y^2 = 9x$, &c. and conſequently $\dot{y} = * - 3x$, &c. Therefore $y = * - \frac{3}{2}x^2$, &c. Then $-2y = * + 3x^2$, &c. and $x^{-1}y^2 = * - 9x^2$, &c. Therefore $\dot{y} = * * + 6x^2$, &c. and $y = * * + 2x^3$, &c. &c.

Or the Reſolution may be perform'd after theſe two following manners:

$$\left.\begin{array}{l} -2y \\ +\dot{y} \\ +x^{-1}y^2 \end{array}\right\} = 3 - \tfrac{9}{4}x^{-1} + \tfrac{9}{2}x^{-2}, \&c. \qquad \left.\begin{array}{l} yx^{-1} \\ +\dot{y}y^{-1} \\ -3y^{-1} \end{array}\right\} = 2 + \tfrac{1}{2}x^{-1} - \tfrac{3}{8}x^{-2}, \&c.$$

$$y = -\tfrac{3}{2} + \tfrac{9}{8}x^{-1} - \tfrac{9}{4}x^{-2}, \&c. \qquad y = 2x + \tfrac{1}{2} - \tfrac{3}{8}x^{-1}, \&c.$$

Make $-2y = 3$, &c. or $y = -\frac{3}{2}$, &c. then $\dot{y} = 0$, &c. and $x^{-1}y^2 = +\frac{9}{4}x^{-1}$, &c. Therefore $-2y = * - \frac{9}{4}x^{-1}$, &c. or $y = * + \frac{9}{8}x^{-1}$, &c. and $\dot{y} = * - \frac{9}{8}x^{-2}$, &c. and by ſquaring $x^{-1}y^2 = * - \frac{27}{8}x^{-2}$, &c. and therefore $-2y = * * + \frac{9}{2}x^{-2}$, &c. and $y = * * - \frac{9}{4}x^{-2}$, &c. And ſo on.

Again, divide the whole Equation by y, and make $x^{-1}y = 2$, &c. then $y = 2x$, &c. And becauſe $\dot{y} = 2$, &c. and $y^{-1} = \frac{1}{2}x^{-1}$, &c. 'tis $\dot{y}y^{-1} = x^{-1}$, &c. and $-3y^{-1} = -\frac{3}{2}x^{-1}$, &c. therefore $\dot{y}x^{-1} = * + \frac{1}{2}x^{-1}$, &c. and $y = * + \frac{1}{2}$, &c. Then becauſe $\dot{y}y^{-1} = * + 0$, &c. and $-3y^{-1} = * + \frac{3}{8}x^{-2}$, &c. 'tis $yx^{-1} = * * - \frac{3}{8}x^{-2}$, &c. and $y = * * - \frac{3}{8}x^{-1}$, &c. &c.

45, 46. If the propoſed Equation be $\dot{y} = -y + x^{-1} - x^{-2}$, its Solution may be thus:

$$\left.\begin{array}{l} \dot{y} \\ +y \end{array}\right\} \begin{array}{r} = -x^{-2} + x^{-1} \\ -x^{-1} \\ +x^{-1} \\ y = x^{-1} \end{array} \qquad \left.\begin{array}{l} y \\ +\dot{y} \end{array}\right\} \begin{array}{r} = x^{-1} - x^{-2} \\ +x^{-2} \\ -x^{-2} \\ y = x^{-1} \end{array}$$

	y^1	y^0
x^0	$-y$	$*$
x^{-1}	$-\dot{y}$	$+x^{-1}$
x^{-2}	$*$	$-x^{-2}$

$= 0.$

Make $\dot{y} = -x^{-2}$, &c. then $y = x^{-1}$, &c. Conſequently $\dot{y} = * 0$, &c. and therefore $y = * 0$, &c. that is, $y = x^{-1}$.

Again, make $y = x^{-1}$, &c. then $\dot{y} = -x^{-2}$, &c. and conſequently $y = * + 0$, &c. that is, $y = x^{-1}$.

That this ſhould be ſo, may appear by the direct Method. For if $y = x^{-1}$, 'tis $\dot{y} = -\dot{x}x^{-2}$; alſo $y\dot{x} = \dot{x}x^{-1}$. Then adding theſe two Equations together, 'tis $y\dot{x} + \dot{y} = \dot{x}x^{-1} - \dot{x}x^{-2}$, or $\dot{y} = -y + x^{-1} - x^{-2}$. Thus may we form as many Fluxional Equations

as we pleaſe, of which the Fluents may be expreſs'd in finite Terms; but to return to theſe again may ſometimes require particular Expedients. Thus if we aſſume the Equation $y = 2x - \frac{4}{3}x^2 + \frac{1}{5}x^3$, taking the Fluxions, and putting $\dot{x} = 1$, we ſhall have $\dot{y} = 2 - \frac{8}{3}x + \frac{3}{5}x^2$, as alſo $\frac{y}{2x} = 1 - \frac{1}{3}x + \frac{1}{10}x^2$. Subtract this laſt from the foregoing Equation, and we ſhall have $\dot{y} - \frac{y}{2x} = 1 - 2x + \frac{1}{2}x^2$, the Solution of which here follows.

47. Let the propos'd Equation be $\dot{y} = \frac{y}{2x} + 1 - 2x + \frac{1}{2}x^2$, of which the Solution may be thus:

$$\left.\begin{array}{l}\dot{y}\\ -\frac{y}{2x}\\ \\ y = 2ex + 2fx^2 + 2gx^3\\ y = 2x - \frac{4}{3}x^2 + \frac{1}{5}x^3\end{array}\right\}\begin{array}{l}= 1 - 2x + \frac{1}{2}x^2\\ \quad + e + fx + gx^2\\ = -e - fx - gx^2\end{array}\qquad \left.\begin{array}{l}\dot{y}\\ -\frac{y}{2x}\\ \\ y = 2ex^3 + 2fx^2 + 2g\\ y = \frac{1}{5}x^3 - \frac{4}{3}x^2 + 2x\end{array}\right\}\begin{array}{l}= \frac{1}{2}x^2 - 2x + 1\\ \quad + ex^2 + fx + g\\ = -ex^2 - fx - g\end{array}$$

	y^1	y^0
x^2	*	$+\frac{1}{2}x^2$
x^1	*	$-2x$
x^0	*	$+1$
x^{-1}	$\dot{y}$	*
	$+\frac{1}{2}yx^{-1}$	*

$= 0.$

By tabulating the Terms of this Equation, as uſual, it may be obſerved, that one of the external Terms $-\dot{y} + \frac{1}{2}yx^{-1}$ is a double Term, to which the other external Term 1 belongs in common. Therefore to ſeparate theſe, aſſume $y = 2ex$, &c. then $-\frac{y}{2x} = -e$, &c. and conſequently $\dot{y} = 1 + e$, &c. and therefore $y = x + ex$, &c. That is, becauſe $2ex = x + ex$, or $2e = 1 + e$, 'tis $e = 1$, or $y = 2x$, &c. So if we make $y = * + 2fx^2$, &c. then $-\frac{y}{2x} = * - fx$, &c. therefore $\dot{y} = * + fx - 2x$, &c. and $y = * + \frac{1}{2}fx^2 - x^2$, &c. that is, $2f = \frac{1}{2}f - 1$, or $f = -\frac{2}{3}$. So that $y = * - \frac{4}{3}x^2$, &c. So if we make $y = * * + 2gx^3$, &c. then $-\frac{y}{2x} = * * - gx^2$, &c. and therefore $\dot{y} = * * + gx^2 + \frac{1}{2}x^2$, &c. and $y = * * + \frac{1}{3}gx^3 + \frac{1}{6}x^3$, &c. or $2g = \frac{1}{3}g + \frac{1}{6}$, or $g = \frac{1}{10}$, ſo that $y = * * \frac{1}{5}x^3$, &c. So if we make $y = * * * 2bx^4$, &c. then $-\frac{y}{2x} = * * * - bx^3$, &c. and therefore $\dot{y} = * * * + bx^3$, &c. and $y = * * * + \frac{1}{4}bx^4$, &c. But becauſe here $2b = \frac{1}{4}b$, this Equation would be abſurd except $b = 0$. And ſo all the ſubſequent Terms will vaniſh *in infinitum*, and this will be the exact value of y. And the ſame may be done from the other caſe of external Terms, as will appear from the Paradigm.

48. Nothing can be added to illuſtrate this Inveſtigation, unleſs we would demonſtrate it ſynthetically. Becauſe $y = ex^1$, as is here found,

found, therefore $\dot{y} = \frac{3}{4}ex^{\frac{1}{4}-1}$, or $\dot{y} = \frac{3ex^{\frac{1}{4}}}{4x}$. Here inſtead of $ex^{\frac{1}{4}}$ ſubſtitute y, and we ſhall have $\dot{y} = \frac{3y}{4x}$, as given at firſt.

49, 50. The given Equation $\dot{y} = yx^{-2} + x^{-3} + 3 + 2x - 4x^{-1}$ may be thus reſolved after a general manner.

$$\left.\begin{array}{r} \dot{y} \\ -x^{-2}y \end{array}\right\} \begin{array}{l} = 2x + 3 - 4x^{-1} + x^{-2} - x^{-3} + \frac{1}{2}x^{-4}, \text{ \&c.} \\ \quad + 1 + 4x^{-1} + ax^{-2} - ax^{-3} + \frac{1}{2}ax^{-4} \\ \quad - 1 - 4x^{-1} - ax^{-2} + x^{-3} - \frac{1}{2}x^{-4}, \text{ \&c.} \\ \quad + ax^{-3} - \frac{1}{2}ax^{-4} \end{array}$$

$$y = x^2 + 4x + a - x^{-1} + \frac{1}{2}x^{-2} - \frac{1}{6}x^{-3}, \text{ \&c.}$$
$$- ax^{-1} + \frac{1}{2}ax^{-2} - \frac{1}{6}ax^{-3}.$$

Make $\dot{y} = 2x$, &c. then $y = x^2$, &c. Therefore $-x^{-2}y = -1$, &c. conſequently $\dot{y} = * + 1 + 3$, &c. $= * 4$, &c. and therefore $y = * + 4x$, &c. Then $-x^{-2}y = * - 4x^{-1}$, &c. and conſequently $\dot{y} = * * + 0$, &c. and therefore aſſuming any conſtant quantity a, it may be $y = * * + a$, &c. Then $-x^{-2}y = * * - ax^{-2}$, &c. and therefore $\dot{y} = * * * + ax^{-2} + x^{-2}$, &c. and $y = * * * - ax^{-1} - x^{-1}$, &c. And ſo on. Here if we make $a = 0$, 'tis $y = x^2 + 4x * - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3}$, &c.

51, 52. The Equation of this Example is $\dot{y} = 3xy^{\frac{2}{3}} + y$, which we ſhall reſolve by our uſual Method, without any other preparation than dividing the whole by $y^{\frac{2}{3}}$, that one of the Terms may be clear'd from the Relate Quantity; which will reduce it $\dot{y}y^{-\frac{2}{3}} - y^{\frac{1}{3}} = 3x$, of which the Reſolution may be thus:

$$\left.\begin{array}{r} \dot{y}y^{-\frac{2}{3}} \\ -y^{\frac{1}{3}} \end{array}\right\} \begin{array}{l} = 3x + \frac{1}{1}x^2 + \frac{1}{1\,8}x^3 + \frac{1}{2\,1\,6}x^4 + \frac{1}{3\,2\,4\,0}x^5, \text{ \&c.} \\ \quad - \frac{1}{2}x^2 - \frac{1}{1\,8}x^3 - \frac{1}{2\,1\,6}x^4 - \frac{1}{3\,2\,4\,0}x^5, \text{ \&c.} \end{array}$$

$$y = \frac{1}{8}x^6 + \frac{1}{2\,4}x^7 + \frac{1}{2\,8\,8}x^8, \text{ \&c.}$$

Make $\dot{y}y^{-\frac{2}{3}} = 3x$, &c. or taking the Fluents, $3y^{\frac{1}{3}} = \frac{3}{2}x^2$, &c. or $y^{\frac{1}{3}} = \frac{1}{2}x^2$, &c. or $y = \frac{1}{8}x^6$, &c. And becauſe $-y^{\frac{1}{3}} = -\frac{1}{2}x^2$, &c. it will be $\dot{y}y^{-\frac{2}{3}} = * + \frac{1}{2}x^2$, &c. and therefore $3y^{\frac{1}{3}} = * + \frac{1}{6}x^3$, &c. and $y^{\frac{1}{3}} = * + \frac{1}{1\,8}x^3$, &c. and by cubing $y = * + \frac{1}{2\,4}x^7$, &c. Then becauſe $-y^{\frac{1}{3}} = * - \frac{1}{1\,8}x^3$, &c. 'tis $\dot{y}y^{-\frac{2}{3}} = * * + \frac{1}{1\,8}x^3$, &c. and therefore $3y^{\frac{1}{3}} = * * + \frac{1}{7\,2}x^4$, &c. and $y^{\frac{1}{3}} = * * + \frac{1}{2\,1\,6}x^4$, &c. and by cubing $y = * * + \frac{1}{2\,8\,8}x^8$, &c. And ſo on.

53. Laſtly, in the Equation $\dot{y} = 2y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}}$, or $\dot{y}y^{-\frac{1}{2}} = 2\dot{x} + \dot{x}x^{\frac{1}{2}}$, aſſuming c for a conſtant quantity, whoſe Fluxion therefore is 0, and taking the Fluents, it will be $2y^{\frac{1}{2}} = 2c + 2x + \frac{2}{3}x^{\frac{3}{2}}$, or $y^{\frac{1}{2}} = c + x + \frac{1}{3}x^{\frac{3}{2}}$. Then by ſquaring, $y = c^2 + 2cx + x^2 + \frac{2}{3}cx^{\frac{3}{2}} + \frac{c}{3}x^{\frac{1}{2}} + \frac{1}{9}x^3$. Here the Root y may receive as many different values, while x remains the ſame, as c can be interpreted different ways. Make $c = 0$, then $y = x^2 + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$.

The Author is pleas'd here to make an Excuſe for his being ſo minute and particular, in diſcuſſing matters which, as he ſays, will but ſeldom come into practice; but I think any Apology of this kind is needleſs, and we cannot be too minute, when the perfection of a Method is concern'd. We are rather much obliged to him for giving us his whole Method, for applying it to all the caſes that may happen, and for obviating every difficulty that may ariſe. The uſe of theſe Extractions is certainly very extenſive; for there are no Problems in the inverſe Method of Fluxions, and eſpecially ſuch as are to be anſwer'd by infinite Series, but what may be reduced to ſuch Fluxional Equations, and may therefore receive their Solutions from hence. But this will appear more fully hereafter.

SECT. IV. *Solution of the third Caſe of Equations, with ſome neceſſary Demonſtrations.*

54. FOR the more methodical Solution of what our Author calls *a moſt troubleſome and difficult Problem*, (and ſurely the Inverſe Method of Fluxions, in its full extent, deſerves to be call'd ſuch a Problem,) he has before diſtributed it into three Caſes. The firſt Caſe, in which two Fluxions and only one flowing Quantity occur in the given Equation, he has diſpatch'd without much difficulty, by the aſſiſtance of his Method of infinite Series. The ſecond Caſe, in which two flowing Quantities and their Fluxions are any how involved in the given Equation, even with the ſame aſſiſtance is ſtill an operoſe Problem, but yet is diſcuſs'd in all its varieties, by a ſufficient number of appoſite Examples. The third Caſe, in which occur more than two Fluxions with their Fluents, is here very artfully managed, and all the difficulties of it are reduced to the other two Caſes. For if the Equation involves (for inſtance) three Fluxions, with ſome or all of their Fluents, another Equation ought to be given by the Queſtion, in order to a full Determination,

termination, as has been already argued in another place; or if not, the Queſtion is left indetermined, and then another Equation may be aſſumed *ad libitum*, ſuch as will afford a proper Solution to the Queſtion. And the reſt of the work will only require the two former Caſes, with ſome common Algebraic Reductions, as we ſhall ſee in the Author's Example.

55. Now to conſider the Author's Example, belonging to this third Caſe of finding Fluents from their Fluxions given, or when there are more than two variable Quantities, and their Fluxions, either expreſs'd or underſtood in the given Equation. This Example is $2\dot{x} - \dot{z} + \dot{y}x = 0$, in which becauſe there are three Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$, (and therefore virtually three Fluents x, y, and z,) and but one Equation given; I may aſſume (for inſtance) $x = y$, whence $\dot{x} = \dot{y}$, and by ſubſtitution $2\dot{y} - \dot{z} + \dot{y}y = 0$, and therefore $2y - z + \frac{1}{2}y^2 = 0$. Now as here are only two Equations $x - y = 0$ and $2y - z + \frac{1}{2}y^2 = 0$, the Quantities x, y, and z are ſtill variable Quantities, and ſuſceptible of infinite values, as they ought to be. Indeed a third Equation may be had, as $2x - z + \frac{1}{2}x^2 = 0$; but as this is only derived from the other two, it brings no new limitation with it, but leaves the quantities ſtill flowing and indeterminate quantities. Thus if I ſhould aſſume $2y = a + z$ for the ſecond Equation, then $2\dot{y} = \dot{z}$, and by ſubſtitution $2\dot{x} - 2\dot{y} + \dot{y}x = 0$, or $\dot{y} = \frac{2\dot{x}}{2-x} = \dot{x} + \frac{1}{2}x\dot{x} + \frac{1}{4}x^2\dot{x}$, &c. and therefore $y = x + \frac{1}{4}x^2 + \frac{1}{12}x^3$, &c. which two Equations are a compleat Determination. Again, if we aſſume with the Author $x = y^2$, and thence $\dot{x} = 2y\dot{y}$, we ſhall have by ſubſtitution $4y\dot{y} - \dot{z} + \dot{y}y^2 = 0$, and thence $2y^2 - z + \frac{1}{3}y^3 = 0$, which two Equations are a ſufficient Determination. We may indeed have a third, $2x - z + \frac{1}{3}x^{\frac{3}{2}} = 0$; but as this is included in the other two, and introduces no new limitation, the quantities will ſtill remain fluent. And thus an infinite variety of ſecond Equations may be aſſumed, tho' it is always convenient, that the aſſumed Equation ſhould be as ſimple as may be. Yet ſome caution muſt be uſed in the choice, that it may not introduce ſuch a limitation, as ſhall be inconſiſtent with the Solution. Thus if I ſhould aſſume $2x - z = 0$ for the ſecond Equation, I ſhould have $2\dot{x} - \dot{z} = 0$ to be ſubſtituted, which would make $\dot{y}x = 0$, and therefore would afford no Solution of the Equation.

'Tis eaſy to extend this reaſoning to Equations, that involve four or more Fluxions, and their flowing Quantities; but it would be needleſs here to multiply Examples. And thus our Author has compleatly ſolved this Caſe alſo, which at firſt view might appear formidable

midable enough, by reducing all its difficulties to the two former Cases.

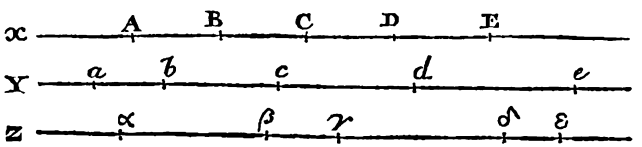
56, 57. The Author's way of demonstrating the Inverse Method of Fluxions is short, but satisfactory enough. We have argued elsewhere, that from the Fluents given to find the Fluxions, is a direct and synthetical Operation; and on the contrary, from the Fluxions given to find the Fluents, is indirect and analytical. And in the order of nature Synthesis should always precede Analysis, or Composition should go before Resolution. But the Terms Synthesis and Analysis are often used in a vague sense, and taken only relatively, as in this place. For the direct Method of Fluxions being already demonstrated synthetically, the Author declines (for the reasons he gives) to demonstrate the Inverse Method synthetically also, that is, primarily, and independently of the direct Method. He contents himself to prove it analytically, that is, by supposing the direct Method, as sufficiently demonstrated already, and shewing the necessary connexion between this and the inverse Method. And this will always be a full proof of the truth of the conclusions, as Multiplication is a good proof of Division. Thus in the first Example we found, that if the given Equation is $y + xy - \dot{y} = 3x - x^2 - 1$, we shall have the Root $y = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, &c. To prove the truth of which conclusion, we may hence find, by the direct Method, $\dot{y} = 1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5$, &c. and then substitute these two Series in the given Equation, as follows:

$$\begin{array}{rrrrrrrrrrrrrrr}
y & - & - & - & - & - & + x & - x^2 & + \frac{1}{3}x^3 & - \frac{1}{6}x^4 & + \frac{1}{30}x^5 & - \frac{1}{45}x^6, & \&c. \\
+ xy & - & - & - & - & - & - & + x^2 & - x^3 & + \frac{1}{3}x^4 & - \frac{1}{6}x^5 & + \frac{1}{30}x^6, & \&c. \\
- \dot{y} & - & - & 1 & + & 2x & - x^2 & + \frac{2}{3}x^3 & - \frac{1}{6}x^4 & + \frac{2}{15}x^5 & - \frac{1}{90}x^6, & \&c. \\
\hline
= & - & 1 & + & 3x & - x^2 & & * & * & * & * &
\end{array}$$

Now by collecting these Series, we shall find the result to produce the given Equation, and therefore the preceding Operation will be sufficiently proved.

58. In this and the subsequent paragraphs, our Author comes to open and explain some of the chief Mysteries of Fluxions and Fluents, and to give us a Key for the clearer apprehension of their nature and properties. Therefore for the Learners better instruction, I shall not think much to inquire something more circumstantially into this matter. In order to which let us conceive any number of right Lines, AE, *ae*, αε, &c. indefinitely extended both ways, along which a Body, or a describing Point, may be supposed to move in each Line,

Line, from the left-hand towards the right, according to any Law or Rate of Acceleration or Retardation whatever. Now the Motion of every one of theſe Points, at all times, is to be eſtimated by its diſtance from ſome fixt point in the ſame Line; and any ſuch Points may be choſen for this purpoſe, in each Line, ſuppoſe B, *b*, *β*, in which all the Bodies have been, are, or will be, in the ſame Moment of Time, from whence to compute their contemporaneous Augments, Differences, or flowing Quantities. Theſe Fluents may be conceived as negative before the Body arrives at that point, as nothing when in it, and as affirmative when they are got beyond it. In the firſt Line AE, whoſe Fluent we denominate by *x*, we may ſuppoſe the Body to move uniformly, or with any equable Velocity; then may the Fluent *x*, or the Line which is continually deſcribed,



repreſent Time, or ſtand for the Correlate Quantity, to which the ſeveral Relate Quantities are to be conſtantly refer'd and compared. For in the ſecond Line *ae*, whoſe Fluent we call *y*, if we ſuppoſe the Body to move with a Motion continually accelerated or retarded, according to any conſtant Rate or Law, (which Law is expreſs'd by any Equation compos'd of *x* and *y* and known quantities;) then will there always be contemporaneous parts or augments, deſcribed in the two Lines, which parts will make the whole Fluents to be contemporaneous alſo, and accommodate themſelves to the Equation in all its Circumſtances. So that whatever value is aſſumed for the Correlate *x*, the correſponding or contemporaneous value of the Relate *y* may be known from the Equation, and *vice verſâ*. Or from the Time being given, here repreſented by *x*, the Space repreſented by *y* may always be known. The Origin (as we may call it) of the Fluent *x* is mark'd by the point B, and the Origin of the Fluent *y* by the point *b*. If the Bodies at the ſame time are found in A and *a*, then will the contemporaneous Fluents be —BA and —*ba*. If at the ſame time, as was ſuppoſed, they are found in their reſpective Origins B and *b*, then will each Fluent be nothing. If at the ſame time they are found in C and *c*, then will their Fluents be +BC and +*bc*. And the like of all other points, in which the

moving Bodies either have been, or ſhall be found, at the ſame time.

As to the Origins of theſe Fluents, or the points from whence we begin to compute them, (for tho' they muſt be conceived to be variable and indetermined in reſpect of one of their Limits, where the deſcribing points are at preſent, yet they are fixt and determined as to their other Limit, which is their Origin,) tho' before we appointed the Origin of each Fluent to be in B and *b*, yet it is not of abſolute neceſſity that they ſhould begin together, or at the ſame Moment of Time. All that is neceſſary is this, that the Motions may continue as before, or that they may obſerve the ſame rate of flowing, and have the ſame contemporaneous Increments or Decrements, which will not be at all affected by changing the beginnings of the Fluents. The Origins of the Fluents are intirely arbitrary things, and we may remove them to what other points we pleaſe. If we remove them from B and *b* to A and *c*, for inſtance, the contemporaneous Lines will ſtill be AB and *ab*, BC and *bc*, &c. tho' they will change their names. Inſtead of $-$ AB we ſhall have o, inſtead of B or o we ſhall have $+$ AB, inſtead of $+$ BC we ſhall have $+$ AC; &c. So inſtead of $-\ ab$ we ſhall have $-\ ac + bc$, inſtead of *b* or o we ſhall have $-\ bc$, inſtead of $+\ bd$ we ſhall have $+\ bc + cd$, &c. That is, in the Equation which determines the general Law of flowing or increaſing, we may always increaſe or diminiſh x, or y, or both, by any given quantity, as occaſion may require, and yet the Equation that ariſes will ſtill expreſs the rate of flowing; which is all that is neceſſary here. Of the uſe and conveniency of which Reduction we have ſeen ſeveral inſtances before.

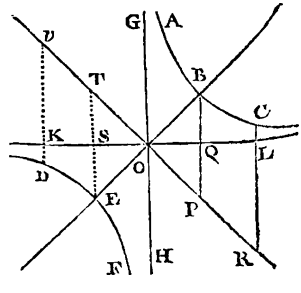
If there be a third Line $\alpha\varepsilon$, deſcribed in like manner, whoſe Fluent may be z, having its parts correſponding with the others, as $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, &c. there muſt be another Equation, either given or aſſumed, to aſcertain the rate of flowing, or the relation of z to the Correlate x. Or it will be the ſame thing, if in the two Equations the Fluents x, y, z, are any how promiſcuouſly involved. For theſe two Equations will limit and determine the Law of flowing in each Line. And we may likewiſe remove the Origin of the Fluent z to what point we pleaſe of the Line $\alpha\varepsilon$. And ſo if there were more Lines, or more Fluents.

59. To exemplify what has been ſaid by an eaſy inſtance. Thus inſtead of the Equation $\dot{y} = \dot{x}xy$, we may aſſume $\dot{y} = \dot{x}y + \dot{x}xy$, where the Origin of x is changed, or x is diminiſh'd by Unity; for $1 + x$ is ſubſtituted inſtead of x. The lawfulneſs of which Reduction

duction may be thus proved from the Principles of Analyticks. Make $x = 1 + z$, whence $\dot{x} = \dot{z}$, which ſhews, that x and z flow or increaſe alike. Subſtitute theſe inſtead of x and $\dot{x}$ in the Equation $\dot{y} = \dot{x}xy$, and it will become $\dot{y} = \dot{z}y + \dot{z}zy$. This differs in nothing elſe from the aſſumed Equation $\dot{y} = \dot{x}y + \dot{x}xy$, only that the Symbol x is changed into the Symbol z, which can make no real change in the argumentation. So that we may as well retain the ſame Symbols as were given at firſt, and, becauſe $z = x - 1$, we may as well ſuppoſe x to be diminiſh'd by Unity.

60, 61. The Equation expreſſing the Relation of the Fluents will at all times give any of their contemporaneous parts; for aſſuming different values of the Correlate Quantity, we ſhall thence have the correſponding different values of the Relate, and then by ſubtraction we ſhall obtain the contemporary differences of each. Thus if the given Equation were $y = x + \frac{1}{x}$, where x is ſuppos'd to be a quantity equably increaſing or decreaſing; make $x = 0, 1, 2, 3, 4, 5$, &c. ſucceſſively, then $y =$ infinite, $2, 2\frac{1}{2}, 3\frac{1}{3}, 4\frac{1}{4}, 5\frac{1}{5}$, &c. reſpectively. And taking their differences, while x flows from 0 to 1, from 1 to 2, from 2 to 3, &c. y will flow from infinite to 2, from 2 to $2\frac{1}{2}$, from $2\frac{1}{2}$ to $3\frac{1}{3}$, &c. that is, their contemporaneous parts will be 1, 1, 1, 1, &c. and infinite, $\frac{1}{2}$, $\frac{5}{6}$, $\frac{11}{12}$, &c. reſpectively. Likewiſe, if we go backwards, or if we make x negative, we ſhall have $x = 0, -1, -2$, &c. which will make $y =$ infinite, -2, $-2\frac{1}{2}$, &c. ſo that the contemporaneous differences will be as before.

Perhaps it may make a ſtronger impreſſion upon the Imagination, to repreſent this by a Figure. To the rectangular Aſymptotes GOH and KOL let ABC and DEF be oppoſite Hyperbola's; biſect the Angle GOK by the indefinite right Line vOR, perpendicular to which draw the Diameter BOE, meeting the Hyperbola's in B and E, from whence draw BQP and EST, as alſo CLR and DKU parallel to GOH. Now if OL is made to repreſent the indefinite and equable quantity x in the Equation $y = x + \frac{1}{x}$,



then CR may repreſent y. For $CL = \frac{1}{OL} = \frac{1}{x}$, (ſuppoſing $BQ = OQ = 1$,) and $LR = OL = x$; therefore $CR = LR + CL$,

or $y = x + \frac{1}{x}$. Now the Origin of OL, or x, being in O; if $x = 0$, then CR, or y, will coincide with the Aſymptote OG, and therefore will be infinite. If $x = 1 =$ OQ, then $y =$ BP $= 2$. If $x = 2 =$ OL, then $y =$ CR $= 2\frac{1}{2}$. And ſo of the reſt. Alſo proceeding the contrary way, if $x = 0$, then y may be ſuppoſed to coincide with the Aſymptote OH, and therefore will be negative and infinite. If $x =$ OS $= -1$, then $y =$ ET $= -2$. If $x =$ OK $= -2$, then $y =$ D$v = -2\frac{1}{2}$, &c. And thus we may purſue, at leaſt by Imagination, the correſpondent values of the flowing quantities x and y, as alſo their contemporary differences, through all their poſſible varieties; according to their relation to each other, as exhibited by the Equation $y = x + \frac{1}{x}$.

The Tranſition from hence to Fluxions is ſo very eaſy, that it may be worth while to proceed a little farther. As the Equation expreſſing the relation of the Fluents will give (as now obſerved) any of their contemporary parts or differences; ſo if theſe differences are taken very ſmall, they will be nearly as the Velocities of the moving Bodies, or points, by which they are deſcribed. For Motions continually accelerated or retarded, when perform'd in very ſmall ſpaces, become nearly equable Motions. But if thoſe differences are conceived to be diminiſhed *in infinitum*, ſo as from finite differences to become Moments, or vaniſhing Quantities, the Motions in them will be perfectly equable, and therefore the Velocities of their Deſcription, or the Fluxions of the Fluents, will be accurately as thoſe Moments. Suppoſe then x, y, z, &c. to repreſent Fluents in any Equation, or Equations, and their Fluxions, or Velocities of increaſe or decreaſe, to be repreſented by $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. and their reſpective contemporary Moments to be op, oq, or, &c. where p, q, r, &c. will be the Exponents of the Proportions of the Moments, and o denotes a vaniſhing quantity, as the nature of Moments requires. Then $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. will be as op, oq, or, &c. that is, as p, q, r, &c. So that $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. may be uſed inſtead of p, q, r, &c. in the deſignation of the Moments. That is, the ſynchronous Moments of x, y, z, &c. may be repreſented by $o\dot{x}$, $o\dot{y}$, $o\dot{z}$, &c. Therefore in any Equation the Fluent x may be ſuppoſed to be increaſed by its Moment $o\dot{x}$, and the Fluent y by its Moment $o\dot{y}$, &c. or $x + o\dot{x}$, $y + o\dot{y}$, &c. may be ſubſtituted in the Equation inſtead of x, y, &c. and yet the Equation will ſtill be true, becauſe the Moments are ſuppoſed to be ſynchronous. From which Operation

ration an Equation will be form'd, which, by due Reduction, muſt neceſſarily exhibit the relation of the Fluxions.

Thus, for example, if the Equation $y = x + z$ be given, by Subſtitution we ſhall have $y + o\dot{y} = x + o\dot{x} + z + o\dot{z}$, which, becauſe $y = x + z$, will become $o\dot{y} = o\dot{x} + o\dot{z}$, or $\dot{y} = \dot{x} + \dot{z}$, which is the relation of the Fluxions. Here again, if we aſſume $z = \frac{1}{x}$, or $zx = 1$, by increaſing the Fluents by their contemporary Moments, we ſhall have $\overline{z + o\dot{z}} \times \overline{x + o\dot{x}} = 1$, or $zx + o\dot{z}x + o\dot{x}z + oo\dot{z}\dot{x} = 1$. Here becauſe $zx = 1$, 'tis $o\dot{z}x + o\dot{x}z + oo\dot{z}\dot{x} = 0$, or $\dot{z}x + \dot{x}z + o\dot{z}\dot{x} = 0$. But becauſe $o\dot{z}\dot{x}$ is a vaniſhing Term in reſpect of the others, 'tis $\dot{z}x + \dot{x}z = 0$, or $\dot{z} = -\frac{\dot{x}z}{x} = -\frac{\dot{x}}{x^2}$. Now as the Fluxion of z comes out negative, 'tis an indication that as x increaſes z will decreaſe, and the contrary. Therefore in the Equation $y = x + z$, if $z = \frac{1}{x}$, or if the relation of the Fluents be $y = x + \frac{1}{x}$, then the relation of the Fluxions will be $\dot{y} = \dot{x} - \frac{\dot{x}}{x^2}$.

And as before, from the Equation $y = x + \frac{1}{x}$ we derived the contemporaneous parts, or differences of the Fluents; ſo from the Fluxional Equation $\dot{y} = \dot{x} - \frac{\dot{x}}{x^2}$ now found, we may obſerve the rate of flowing, or the proportion of the Fluxions at different values of the Fluents.

For becauſe it is $\dot{x} : \dot{y} :: 1 : 1 - \frac{1}{x^2} :: x^2 : x^2 - 1$; when $x = 0$, or when the Fluent is but beginning to flow, (conſequently when y is infinite,) it will be $\dot{x} : \dot{y} :: 0 : -1$. That is, the Velocity wherewith x is deſcribed is infinitely little in compariſon of the velocity wherewith y is deſcribed; and moreover it is inſinuated, (becauſe of -1,) that while x increaſes by any finite quantity, tho' never ſo little, y will decreaſe by an infinite quantity at the ſame time. This will appear from the inſpection of the foregoing Figure. When $x = 1$, (and conſequently $y = 2$,) then $\dot{x} : \dot{y} :: 1 : 0$. That is, x will then flow infinitely faſter than y. The reaſon of which is, that y is then at its Limit, or the leaſt that it can poſſibly be, and therefore in that place it is ſtationary for a moment, or its Fluxion is nothing in compariſon of that of x. So in the foregoing Figure, BP is the leaſt of all ſuch Lines as are repreſented by CR. When $x = 2$, (and therefore $y = 2\frac{1}{2}$,) it will be $\dot{x} : \dot{y} :: 4 : 3$. Or

the Velocity of x is there greater than that of y, in the ratio of 4 to 3. When $x = 3$, then $\dot{x} : \dot{y} :: 9 : 8$. And ſo on. So that the Velocities or Fluxions conſtantly tend towards equality, which they do not attain till $\frac{1}{x}$ (or CL) finally vaniſhing, x and y become equal. And the like may be obſerved of the negative values of x and y.

SECT. V. *The Reſolution of Equations, whether Algebraical or Fluxional, by the aſſiſtance of ſuperior orders of Fluxions.*

ALL the foregoing Extractions (according to a hint of our Author's,) may be perform'd ſomething more expeditiouſly, and without the help of ſubſidiary Operations, if we have recourſe to ſuperior orders of Fluxions. To ſhew this firſt by an eaſy Inſtance.

Let it be required to extract the Cube-root of the Binomial $a^3 + x^3$, or to find the Root y of this Equation $y^3 = a^3 + x^3$; or rather, for ſimplicity-ſake, let it be $y^3 = a^3 + z$. Then $y = a$, &c. or the initial Term of y will be a. Taking the Fluxions of this Equation, we ſhall have $3\dot{y}y^2 = \dot{z} = 1$, or $\dot{y} = \frac{1}{3}y^{-2}$. But as it is $y = a$, &c. by ſubſtitution it will be $\dot{y} = \frac{1}{3}a^{-2}$, &c. and taking the Fluents, 'tis $y = * + \frac{1}{3}a^{-2}z$, &c. Here a vacancy is left for the firſt Term of y, which we already know to be a. For another Operation take the Fluxions of the Equation $\dot{y} = \frac{1}{3}y^{-2}$; whence $\ddot{y} = -\frac{2}{3}\dot{y}y^{-3} = -\frac{1}{9}y^{-5}$. Then becauſe $y = a$, &c. 'tis $\ddot{y} = -\frac{1}{9}a^{-5}$, &c. and taking the Fluents, 'tis $\dot{y} = * - \frac{2}{9}a^{-5}z$, &c. and taking the Fluents again, 'tis $y = * * - \frac{1}{9}a^{-5}z^2$, &c. Here two vacancies are to be left for the two firſt Terms of y, which are already known. For the next Operation take the Fluxions of the Equation $\ddot{y} = -\frac{2}{9}y^{-5}$, that is, $\dddot{y} = +\frac{10}{9}\dot{y}y^{-6} = +\frac{10}{27}y^{-8}$. Or becauſe $y = a$, &c. 'tis $\dddot{y} = \frac{10}{27}a^{-8}$, &c. Then taking the Fluents, 'tis $\ddot{y} = * \frac{10}{27}a^{-8}z$, &c. $\dot{y} = * * \frac{5}{27}a^{-8}z^2$, &c. and $y = * * * \frac{5}{81}a^{-8}z^3$, &c. Again, for another Operation take the Fluxions of the Equation $\dddot{y} = \frac{10}{27}y^{-8}$; whence $\ddddot{y} = -\frac{80}{27}\dot{y}y^{-9} = -\frac{80}{81}y^{-11}$. Or becauſe $y = a$, &c. 'tis $\ddddot{y} = -\frac{80}{81}a^{-11}$, &c. Then taking the Fluents, $\dddot{y} = * - \frac{80}{81}a^{-11}z$, &c. $\ddot{y} = * * - \frac{40}{81}a^{-11}z^2$,

$\frac{4\cdot 0}{1\cdot 1\cdot 2}a^{-11}x^2$, &c. $\dot{y} = * * * - \frac{4\cdot 0}{1\cdot 2\cdot 3}a^{-11}x^3$, &c. and $y = * * * * - \frac{4\cdot 0}{1\cdot 2\cdot 3\cdot 4}a^{-11}x^4$, &c. And ſo we may go on as far as we pleaſe. We have therefore found at laſt, that $y = a + \frac{x}{3a^2} - \frac{x^2}{9a^5} + \frac{5x^3}{81a^8} - \frac{10x^4}{243a^{11}}$, &c. or for x writing x^3, 'tis $\sqrt[3]{a^3 + x^3} = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}}$, &c.

Or univerſally, if we would reſolve $\overline{a+x}|^m$ into an equivalent infinite Series, make $y = \overline{a+x}|^m$, and we ſhall have a^m for the firſt Term of the Series y, or it will be $y = a^m$, &c. Then becauſe $\dot{y}^{\frac{1}{m}} = a + x$, taking the Fluxions we ſhall have $\frac{1}{m}\dot{y}\dot{y}^{\frac{1}{m}-1} = \dot{x} = 1$, or $\dot{y} = m\dot{y}^{1-\frac{1}{m}}$. But becauſe it is $y = a^m$, &c. it will be $\dot{y} = ma^{m-1}$, &c. and now taking the Fluents, 'tis $y = * ma^{m-1}x$, &c. Again, becauſe it is $\dot{y} = m\dot{y}^{1-\frac{1}{m}}$, taking the Fluxions it will be $\ddot{y} = \overline{m-1}\dot{y}\dot{y}^{-\frac{1}{m}} = m \times \overline{m-1}\dot{y}^{1-\frac{2}{m}}$; and becauſe $y = a^m$, &c. 'tis $\ddot{y} = m \times \overline{m-1}a^{m-2}$, &c. And taking the Fluents, 'tis $\dot{y} = * m \times \overline{m-1}a^{m-2}x$, &c. and therefore $y = * * m \times \frac{m-1}{2}a^{m-2}x^2$, &c. Again, becauſe it is $\ddot{y} = m \times \overline{m-1}\dot{y}^{1-\frac{2}{m}}$, taking the Fluxions it will be $\dddot{y} = \overline{m-1} \times \overline{m-2}\dot{y}\dot{y}^{-\frac{2}{m}} = m \times \overline{m-1} \times \overline{m-2}\dot{y}^{1-\frac{3}{m}}$; and becauſe $y = a^m$, &c. 'tis $\dddot{y} = m \times \overline{m-1} \times \overline{m-2}a^{m-3}$, &c. And taking the Fluents, 'tis $\ddot{y} = * m \times \overline{m-1} \times \overline{m-2}a^{m-3}x$, &c. $\dot{y} = * * m \times \frac{m-1}{2} \times \overline{m-2}a^{m-3}x^2$, &c. and $y = * * * m \times \frac{m-1}{2} \times \frac{m-2}{3}a^{m-3}x^3$, &c. And ſo we might proceed as far as we pleaſe, if the Law of Continuation had not already been ſufficiently manifeſt. So that we ſhall have here $\overline{a+x}|^m = a^m + ma^{m-1}x + m \times \frac{m-1}{2}a^{m-2}x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3}a^{m-3}x^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}a^{m-4}x^4$, &c.

[4] This is a famous Theorem of our Author's, tho' diſcover'd by him after a very different manner of Inveſtigation, or rather by Induction. It is commonly known by the name of his Binomial Theorem, becauſe by its aſſiſtance any Binomial, as $a+x$, may be raiſed to any Power at pleaſure, or any Root of it may be extracted. And it is obvious, that when m is interpreted by any integer

teger affirmative Number, the Series will break off, and become finite, at a number of Terms denominated by m. But in all other caſes it will be an infinite Series, which will converge when x is leſs than a.

Indeed it can hardly be ſaid, that this, or any other that is derived from the Method of Fluxions, is a ſtrict Inveſtigation of this Theorem. Becauſe that Method itſelf is originally derived from the Method of raiſing Powers, at leaſt integral Powers, and previouſly ſuppoſes the knowledge of the *Unciæ*, or the numeral Coefficients. However it may anſwer the intention, of being a proper Example of this Method of Extraction, which is all that is neceſſary here.

There is another Theorem for this purpoſe, which I found many years ago, and then communicated it to my ingenious Friend Mr. *A. de Moivre*, who liked it ſo well as to inſert it in a Mathematical Treatiſe he was then publiſhing. I ſhall here give the Reader its Inveſtigation, in the ſame manner it was found.

Let us ſuppoſe $\overline{a+x}\,|^m = a^m + p$, and that $a+x = z$, and therefore $\dot{z} = \dot{x} = 1$. Now becauſe $z^m = a^m + p$, it will be $\dot{p} = mz^{m-1} = \frac{mz^m}{z}$; where for z^m writing its value $a^m + p$, we ſhall have $\dot{p} = \frac{ma^m}{z} + \frac{mp}{z}$. Now if we make $p = \frac{ma^m x}{z} + q$, it will be $\dot{p} = \frac{ma^m}{z} - \frac{ma^m x}{z^2} + \dot{q}$. And comparing theſe two values of $\dot{p}$, we ſhall have $\dot{q} = \frac{ma^m x}{z^2} + \frac{mp}{z}$; where if for p we write its value as above, it will be $\dot{q} = \frac{ma^m x}{z^2} + \frac{m^2 a^m x}{z^2} + \frac{mq}{z}$, or $\dot{q} = m \times \overline{m+1} \times \frac{a^m x}{z^2} + \frac{mq}{z}$; make $q = m \times \frac{m+1}{2} \times \frac{a^m x^2}{z^2} + r$; therefore $\dot{q} = m \times \overline{m+1} \times \frac{a^m x}{z^2} - m \times \overline{m+1} \times \frac{a^m x^2}{z^3} + \dot{r}$. From which two values of $\dot{q}$ we ſhall have $\dot{r} = m \times \overline{m+1} \times \frac{a^m x^2}{z^3} + \frac{mq}{z}$. And for q ſubſtituting its value, it will be $\dot{r} = m \times \overline{m+1} \times \frac{a^m x^2}{z^3} + m^2 \times \frac{m+1}{2} \times \frac{a^m x^2}{z^3} + \frac{mr}{z}$. Or $\dot{r} = m \times \frac{m+1}{2} \times \frac{m+2}{1} \times \frac{a^m x^2}{z^3} + \frac{mr}{z}$. Make $r = m \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{a^m x^3}{z^3} + s$; then, &c. So that we

ſhall

ſhall have $\overline{a+x|}^{m} = a^m + m \times \frac{a^m x}{a+x} + m \times \frac{m+1}{2} \frac{a^m x^2}{\overline{a+x|^2}} + m \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{a^m x^3}{\overline{a+x|^3}}$, &c.

Now this Series will ſtop of its own accord, at a finite number of Terms, when m is any integer and negative Number; that is, when the Reciprocal of any Power of a Binomial is to be found. But in all other caſes we ſhall have an infinite converging Series for the Power or Root required, which will always converge when a and x have the ſame Sign; becauſe the Root of the Scale, or the converging quantity, is $\frac{x}{a+x}$, which is always leſs than Unity.

By comparing theſe two Series together, or by collecting from each the common quantity $\frac{\overline{a+x|}^{m} - a^m}{ma^m x}$, we ſhall have the two equivalent Series $\frac{1}{a} + \frac{m-1}{2} \times \frac{x}{a^2} + \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{x^2}{a^3}$, &c. $= \frac{1}{a+x} + \frac{m+1}{2} \times \frac{x}{\overline{a+x|^2}} + \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{x^2}{\overline{a+x|^3}}$, &c. from whence we might derive an infinite number of Numeral Converging Series, not inelegant, which would be proper to explain and illuſtrate the nature of Convergency in general, as has been attempted in the former part of this work. For if we aſſume ſuch a value of m as will make either of the Series become finite, the other Series will exhibit the quantity that ariſes by an Approximation *ad infinitum*. And then a and x may be afterwards determined at pleaſure.

As another Example of this Method, we ſhall ſhew (according to promiſe) how to derive Mr. *de Moivre*'s elegant Theorem; for raiſing an Infinitinomial to any indeterminate Power, or for extracting any Root of the ſame. The way how it was derived from the abſtract conſideration of the nature and geneſis of Powers, (which indeed is the only legitimate method of Inveſtigation in the preſent caſe,) and the Law of Continuation, have been long ago communicated and demonſtrated by the Author, in the Philoſophical Tranſactions, N° 230. Yet for the dignity of the Problem, and the better to illuſtrate the preſent Method of Extraction of Roots, I ſhall deduce it here as follows.

Let us aſſume the Equation $\overline{a + bz + cz^2 + dz^3 + ez^4, \&c.|}^{m} = y$, where the value of y is to be found by an infinite Series, of which the firſt Term is already known to be a^m, or it is $y = a^m$, &c. Make $v = a + bz + cz^2 + dz^3 + ez^4$, &c. and putting $\dot{z} = 1$, and taking the Fluxions, we ſhall have $\dot{v} = b + 2cz + 3dz^2$.

$3dz^2 + 4ez^3$, &c. Then becauſe $y = v^m$, it is $\dot{y} = m\dot{v}v^{m-1}$, where if we make $v = a$, &c. and $\dot{v} = b$, &c. we ſhall have $\dot{y} = ma^{m-1}b$, &c. and taking the Fluents, it will be $y = * ma^{m-1}bz$, &c.

For another Operation, becauſe $\dot{y} = m\dot{v}v^{m-1}$, it is $\ddot{y} = m\ddot{v}v^{m-1} + m \times \overline{m-1}\dot{v}^2v^{m-2}$. And becauſe $\ddot{v} = 2c + 6dz + 12ez^2$, &c. for v, $\dot{v}$, and $\ddot{v}$ ſubſtituting their values a, &c. b, &c. and $2c$, &c. reſpectively, we ſhall have $\ddot{y} = 2mca^{m-1} + m \times \overline{m-1}b^2a^{m-2}$, &c. and taking the Fluents $\dot{y} = * 2mca^{m-1}z + m \times \overline{m-1}b^2a^{m-2}z$, &c. and taking the Fluents again, $y = * * mca^{m-1}z^2 + m \times \frac{m-1}{2}b^2a^{m-2}z^2$, &c.

For another Operation, becauſe $\ddot{y} = m\ddot{v}v^{m-1} + m \times \overline{m-1}\dot{v}^2v^{m-2}$, 'tis $\dddot{y} = m\dddot{v}v^{m-1} + 3m \times \overline{m-1}v^{m-2}\dot{v}\ddot{v} + m \times \overline{m-1} \times \overline{m-2}v^{m-3}\dot{v}^3$. And becauſe $\dddot{v} = 6d + 24ez$, &c. for v, $\dot{v}$, $\ddot{v}$, $\dddot{v}$, ſubſtituting a, &c. b, &c. $2c$, &c. $6d$, &c. we ſhall have $\dddot{y} = 6mda^{m-1} + 6m \times \overline{m-1}bca^{m-2} + m \times \overline{m-1} \times \overline{m-2}b^3a^{m-3}$, &c. And taking the Fluents it will be $\ddot{y} = * 6mda^{m-1}z + 6m \times \overline{m-1}bca^{m-2}z + m \times \overline{m-1} \times \overline{m-2}b^3a^{m-3}z$, &c. $\dot{y} = * * 3mda^{m-1}z^2 + 3m \times \overline{m-1}bca^{m-2}z^2 + m \times \frac{m-1}{2} \times \overline{m-2}b^3a^{m-3}z^2$, &c. and $y = * * * mda^{m-1}z^3 + m \times \frac{m-1}{1}bca^{m-2}z^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3}b^3a^{m-3}z^3$, &c. And ſo on *in infinitum*. We ſhall therefore have $\overline{a + bz + cz^2 + dz^3 + ez^4, \&c.}|^m = (y =) a^m + ma^{m-1}bz + m \times \frac{m-1}{2}a^{m-2}b^2 \times z^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3}a^{m-3}b^3 \times z^3, \&c.$

$+ ma^{m-1}c \qquad + m \times \frac{m-1}{1}a^{m-2}bc$

$+ ma^{m-1}d$

And if the whole be multiply'd by z^m, and continued to a due length, it will have the form of Mr. *de Moivre*'s Theorem.

The Roots of all Algebraical or Fluential Equations may be extracted by this Method. For an Example let us take the Cubick Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, ſo often before reſolved, in which $y = a$, &c. Then taking the Fluxions, and making $\dot{x} = 1$, we ſhall have $3\dot{y}y^2 + ay + ax\dot{y} + a^2\dot{y} - 3x^2 = 0$. Here if for y we ſubſtitute a, &c. we ſhall have $4a^2\dot{y} + a^2 + ax\dot{y} - 3x^2$, &c. $= 0$, or $\dot{y} = \frac{-a^2 + 3x^2, \&c.}{4a^2 + ax, \&c.} = \frac{-a^2}{4a^2}$, &c. $= -\frac{1}{4}$, &c. And taking the Fluents, $y = * - \frac{1}{4}x$, &c. Then taking the Fluxions

again of the laſt Equation, we ſhall have $3\ddot{y}y^2 + 6\dot{y}^2y + 2a\dot{y} + ax\dot{y} + a^2\ddot{y} - 6x = 0$. Where if we make $y = a$, &c. and $\dot{y} = -\frac{1}{4}$, &c. we ſhall have $\ddot{y} = \frac{-\frac{3}{8}a + \frac{1}{4}a, \&c.}{4a^2, \&c.} = \frac{1}{32a}$, &c. and therefore $\dot{y} = * + \frac{x}{32a}$, &c. and $y = * * + \frac{x^2}{64a}$, &c. Again, $3\dot{y}y^2 + 18\ddot{y}\dot{y}y + 6\dot{y}^3 + 3a\ddot{y} + ax\dot{y} + a^2\dot{y} - 6 = 0$. Make $y = a$, &c. $\dot{y} = -\frac{1}{4}$, &c. and $\ddot{y} = \frac{1}{32a}$, &c. then $\dot{y} = \frac{\frac{9}{256} + \frac{9}{256} - \frac{1}{16} + 6}{4a^2}$, &c. $= \frac{393}{256a^2}$, &c. and therefore $\ddot{y} = * \frac{393x}{256a^2}$, &c. $\dot{y} = * * \frac{393x^2}{512a^2}$, &c. and $y = * * * \frac{131x^3}{512a^2}$, &c. Again, $3\ddot{y}y^2 + 24\dot{y}\ddot{y}y + 18\ddot{y}^2y + 36\dot{y}^2y + 4a\dot{y} + ax\dot{y} + a^2\ddot{y} = 0$. Make $y = a$, &c. $\dot{y} = -\frac{1}{4}$, &c. $\ddot{y} = \frac{1}{32a}$, &c. and $\dot{y} = \frac{393}{256a^2}$, &c. then $\ddot{y} = -\frac{24\dot{y}\ddot{y}y + 18\ddot{y}^2y + 36\dot{y}^2y + 4a\dot{y}}{3y^2 + a^2, \&c.}$ $= \frac{1527}{2048a^3}$, &c. and $\dot{y} = * \frac{1527x}{2048a^3}$, &c. $\ddot{y} = * * \frac{1527x^2}{4096a^3}$, &c. $\dot{y} = * * * \frac{509x^3}{4096a^3}$, &c. and $y = * * * * \frac{509x^4}{16384a^3}$, &c. And ſo on as far as we pleaſe. Therefore the Root is $y = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$, &c.

The Series for the Root, when found by this Method, muſt always have its Powers aſcending; but if we deſire likewiſe to find a Series with deſcending Powers, it may be done by this eaſy artifice. As in the preſent Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, we may conceive x to be a conſtant quantity, and a to be a flowing quantity; or rather, to prevent a confuſion of Ideas, we may change a into x, and x into a, and then the Equation will be $y^3 + axy + x^2y - a^3 - 2x^3 = 0$. In this we ſhall have $y = a$, &c. and taking the Fluxions, 'tis $3\dot{y}y^2 + ay + ax\dot{y} + 2xy + x^2\dot{y} - 6x^2 = 0$, or $\dot{y} = \frac{-ay - 2xy + 6x^2}{3y^2 + ax + x^2}$. But becauſe $y = a$, &c. 'tis $\dot{y} = \frac{-aa}{3aa}$, &c. $= -\frac{1}{3}$, &c. and therefore $y = * - \frac{1}{3}x$, &c. Again taking the Fluxions 'tis $3\ddot{y}y^2 + 6\dot{y}^2y + 2a\dot{y} + ax\ddot{y} + 2y + 4x\dot{y} + x^2\ddot{y} - 12x = 0$, or $\ddot{y} = \frac{-6\dot{y}^2y - 2a\dot{y} - 2y - 4x\dot{y} + 12x}{3y^2 + ax + x^2} = \frac{-6\dot{y}^2y - 2a\dot{y} - 2y}{3y^2}$, &c. Or making $y = a$, &c. and $\dot{y} = -\frac{1}{3}$, &c. 'tis $\ddot{y} = \frac{-\frac{1}{3}a + \frac{2}{3}a - 2a}{3a^2}$, &c. $= -\frac{2}{3a}$, &c. and $\dot{y} = * - \frac{2x}{3a}$, &c. and $y = * * - \frac{x^2}{3a}$, &c. Again it is $3\dot{y}y^2 + 18\ddot{y}\dot{y}y + 6\dot{y}^3 + 3a\ddot{y} + ax\dot{y} + 6\dot{y} + 6x\ddot{y} + x^2\dot{y}$

$-12=0$, or $\dddot{y}=\frac{-18\ddot{y}\dot{y}-6\dot{y}^3-3a\ddot{y}-6\dot{y}+12}{3y^2}$, &c. $=$ (by making $y=a$, &c. $\dot{y}=-\frac{1}{3}$, &c. and $\ddot{y}=-\frac{2}{3a}$, &c.) $\frac{-4+\frac{2}{3}+2+2+12}{3a^2}$, &c. $=\frac{110}{27a^2}$, &c. Then taking the Fluents, $\ddot{y}=*\frac{110x}{27a^2}$, &c. $\dot{y}=**\frac{55x^2}{27a^2}$, &c. and $y=***\frac{55x^3}{81a^2}$, &c. And ſo on. Therefore we ſhall have $y=a-\frac{1}{3}x-\frac{x^2}{3a}+\frac{55x^3}{81a^2}$, &c. Or now we may again change x into a, and a into x; then it will be $y=x-\frac{1}{3}a-\frac{a^2}{3x}+\frac{55a^3}{81x^2}$, &c. for the Root of the given Equation, as was found before, pag. 216, &c.

Alſo in the Solution of Fluxional Equations, we may proceed in the ſame manner. As if the given Equation were $a^2\dot{y}-a^2\dot{x}+x^2\dot{y}=0$, (in which, if the Radius of a Circle be repreſented by a, and if y be any Arch of the ſame, the correſponding Tangent will be repreſented by x;) let it be required to extract the Root y out of this Equation, or to expreſs it by a Series compoſed of the Powers of a and x. Make $\dot{x}=1$, then the Equation will be $a^2\dot{y}-a^2+x^2\dot{y}=0$. Here becauſe $\dot{y}=\frac{a^2}{a^2+x^2}=1$, &c. taking the Fluents it will be $y=*x$, &c. Then taking the Fluxions of this Equation, we ſhall have $a^2\ddot{y}+2x\dot{y}+x^2\ddot{y}=0$, or $\ddot{y}=-\frac{2x\dot{y}}{a^2+x^2}$. But becauſe we are to have a conſtant quantity for the firſt Term of $\ddot{y}$, we may ſuppoſe $\ddot{y}=\frac{0-2x\dot{y}}{a^2+x^2}=0$, &c. Then taking the Fluents 'tis $\dot{y}=*0$, &c. and $y=**0$, &c. Then taking the Fluxions again, 'tis $\dddot{y}+2\dot{y}+4x\ddot{y}+x^2\dddot{y}=0$, or $\dddot{y}=\frac{-2\dot{y}-4x\ddot{y}}{a^2+x^2}$. Here if for $\dot{y}$ and $\ddot{y}$ we write their values 1, &c. and 0, &c. we ſhall have $\dddot{y}=-\frac{2}{a^2}$, &c. whence $\ddot{y}=*-\frac{2x}{a^2}$, &c. $\dot{y}=**-\frac{x^2}{a^2}$, &c. and $y=***-\frac{x^3}{3a^2}$, &c. Taking the Fluxions again, 'tis $\ddddot{y}+6\ddot{y}+6x\dddot{y}+x^2\ddot{y}=0$, or $\ddot{y}=\frac{-6\ddot{y}-6x\dddot{y}}{a^2+x^2}=0$, &c. Therefore $\dddot{y}=*0$, &c. $\ddot{y}=**0$, &c. $\dot{y}=***0$, &c. $y=****0$, &c. Again, $a^2\overset{5}{\dot{y}}+12\dddot{y}+8x\ddot{y}+x^2\overset{5}{\dot{y}}=0$, or $\overset{5}{\dot{y}}=\frac{-12\dddot{y}-8x\ddot{y}}{a^2+x^2}$

$=$

$= + \frac{24}{a^4}$, &c. Then $\ddot{y} = * + \frac{24x}{a^4}$, &c. $\dddot{y} = * * + \frac{12x^2}{a^4}$, &c. $\ddot{y} = * * * + \frac{4x^3}{a^4}$, &c. $\dot{y} = * * * * + \frac{x^4}{a^4}$, &c. and $y = * * * * * + \frac{x^5}{5a^4}$, &c. Again, $a^2\overset{6}{y} + 20\overset{4}{y} + 10x\overset{5}{y} + x^2\overset{6}{y} = 0$, whence $y = * * * * * * 0$, &c. Again, $a^2\overset{7}{y} + 30\overset{5}{y} + 12x\overset{6}{y} + x^2\overset{7}{y} = 0$, or $\overset{7}{y} = \frac{-30\overset{5}{y} - 12x\overset{6}{y}}{a^2 + x^2} = -30 \times 24a^{-6}$, &c. Then $\overset{6}{y} = * - \frac{24 \times 30x}{a^6}$, &c. $\overset{5}{y} = * * - \frac{12 \times 30x^2}{a^6}$, &c. $\overset{4}{y} = * * * - \frac{4 \times 30x^3}{a^6}$, &c. $\dddot{y} = * * * * - \frac{30x^4}{a^6}$, &c. $\ddot{y} = * * * * * - \frac{6x^5}{a^6}$, &c. $\dot{y} = * * * * * * - \frac{x^6}{a^6}$, &c. and $y = * * * * * * * - \frac{x^7}{7a^6}$, &c. And so on. So that we have here $y = * x + 0x^2 - \frac{x^3}{3a^2} + 0x^4 + \frac{x^5}{5a^4}$, &c. that is, $y = x - \frac{1x^3}{3a^2} + \frac{x^5}{5a^4} - \frac{x^7}{7a^6}$, &c.

This Example is only to ſhew the univerſality of this Method, and how we are to proceed in other like caſes; for as to the Equation itſelf, it might have been reſolved much more ſimply and expeditiouſly, in the following manner. Becauſe $\dot{y} = \frac{a^2}{a^2 + x^2}$, by Diviſion it will be $\dot{y} = 1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6} + \frac{x^8}{a^8}$, &c. And taking the Fluents, $y = x - \frac{x^3}{3a^2} + \frac{x^5}{5a^4} - \frac{x^7}{7a^6} + \frac{x^9}{9a^8}$, &c.

In the ſame Equation $a^2\dot{y} - a^2\dot{x} + x^2\dot{y} = 0$, if it were requir'd to expreſs x by y, (the Tangent by the Arch,) or if x were made the Relate, and y the Correlate, we might proceed thus. Make $\dot{y} = 1$, then $a^2 - a^2\dot{x} + x^2 = 0$, or $\dot{x} = 1 + \frac{x^2}{a^2} = 1$, &c. Then $x = * y$, &c. And taking the Fluxions, 'tis $\ddot{x} = \frac{2x\dot{x}}{a^2} = \frac{2x}{a^2}$, &c. $= 0 + \frac{2x}{a^2}$, &c. whence $\dot{x} = * 0$, &c. and $x = * * 0$, &c. So that the Terms of this Series will be alternately deficient, and therefore we need not compute them. Taking the Fluxions again, 'tis $\dddot{x} = \frac{2\dot{x}^2}{a^2} + \frac{2x\ddot{x}}{a^2} = \frac{2}{a^2}$, &c. Therefore $\ddot{x} = * \frac{2y}{a^2}$, &c. $\dot{x} = * * \frac{y^2}{a^2}$, &c. and $x = * * * \frac{y^3}{3a^2}$, &c. Again, $\overset{\text{iv}}{x} = \frac{6\dot{x}\ddot{x}}{a^2} + \frac{2x\dddot{x}}{a^2}$,

and again, $\overset{5}{\dot{x}} = \frac{6\ddot{x}^2}{a^2} + \frac{8\dot{x}\dddot{x}}{a^2} + \frac{2x\overset{4}{x}}{a^2}$. Subſtituting 1, &c. and $\frac{2}{a^2}$, &c. for $\dot{x}$ and $\ddot{x}$, and alſo 0, &c. for $\dddot{x}$ and $\overset{4}{x}$, it will be $\overset{5}{\dot{x}} = \frac{16}{a^4}$, &c. whence $\overset{4}{\dot{x}} = * \frac{16y}{a^4}$, &c. $\dddot{x} = * * \frac{8y^2}{a^4}$, &c. $\ddot{x} = * * * \frac{8y^3}{3a^2}$, &c. $\dot{x} = * * * * \frac{2y^4}{3a^4}$, &c. and $x = * * * * * \frac{2y^5}{15a^4}$, &c. Again, $\overset{6}{\dot{x}} = \frac{20\ddot{x}\dddot{x} + 10\dot{x}\overset{4}{x} + 2x\overset{5}{x}}{a^2}$, and again, $\overset{7}{\dot{x}} = \frac{20\dddot{x}^2 + 30\ddot{x}\overset{4}{x} + 12\dot{x}\overset{5}{x} + 2x\overset{6}{x}}{a^2}$. Here for $\dot{x}$, $\dddot{x}$, and $\overset{5}{\dot{x}}$ writing 1, &c. $\frac{2}{a^2}$, &c. and $\frac{16}{a^4}$, &c. reſpectively, 'tis $\overset{7}{\dot{x}} = \frac{80 + 12 \times 16}{a^6}$, &c. $= \frac{272}{a^6}$, &c. Then $\overset{6}{\dot{x}} = * \frac{272y}{a^6}$, &c. $\overset{5}{\dot{x}} = * * \frac{136y^2}{a^6}$, &c. $\overset{4}{\dot{x}} = * * * \frac{136y^3}{3a^6}$, &c. $\dddot{x} = * * * * \frac{34y^4}{3a^6}$, &c. $\ddot{x} = * * * * * \frac{34y^5}{15a^6}$, &c. $\dot{x} = * * * * * * \frac{17y^6}{45a^6}$, &c. and $\dot{x} = * * * * * * * \frac{17y^7}{315a^6}$, &c. That is, $x = y + \frac{y^3}{3a^2} + \frac{2y^5}{15a^4} + \frac{17y^7}{315a^6}$, &c.

For another Example, let us take the Equation $a^2\dot{y}^2 - x^2\dot{y}^2 - a^2\dot{x}^2 = 0$, (in which, if the Radius of a Circle be denoted by a, and if y be any Arch of the ſame, then the correſponding right Sine will be denoted by x;) from which we are to extract the Root y. Make $\dot{x} = 1$, then it will be $a^2\dot{y}^2 - x^2\dot{y}^2 = a^2$, or $\dot{y}^2 = \frac{a^2}{a^2 - x^2} = 1$, &c. or $\dot{y} = 1$, &c. and therefore $y = * x$, &c. Taking the Fluxions we ſhall have $2a^2\dot{y}\ddot{y} - 2x\dot{y}^2 - 2x^2\dot{y}\ddot{y} = 0$, or $a^2\ddot{y} - x\dot{y} - x^2\ddot{y} = 0$, or $\ddot{y} = \frac{x\dot{y}}{a^2 - x^2} = 0$, &c. And taking the Fluxions again, 'tis $a^2\dddot{y} - \dot{y} - 3x\ddot{y} - x^2\dddot{y} = 0$, or $\dddot{y} = \frac{\dot{y} + 3x\ddot{y}}{a^2 - x^2} = \frac{1}{a^2}$, &c. Therefore $\ddot{y} = * \frac{x}{a^2}$, &c. $\dot{y} = * * \frac{x^2}{2a^2}$, &c. and $y = * * * \frac{x^3}{6a^2}$, &c. Then $a^2\ddot{y} - 4\ddot{y} - 5x\dddot{y} - x^2\ddot{y} = 0$, and again $a^2\overset{5}{\dot{y}} - 9\dddot{y} - 7x\ddot{y} - x^2\overset{5}{y} = 0$, or $\overset{5}{\dot{y}} = \frac{9\dddot{y} + 7x\ddot{y}}{a^2 - x^2} = \frac{9\dddot{y}}{a^2}$, &c. $= \frac{9}{a^4}$, &c. Therefore $\ddot{y} = * \frac{9x}{a^4}$, &c. $\dddot{y} = * * \frac{9x^2}{2a^4}$, &c. $\ddot{y} = * * * \frac{3x^3}{2a^4}$, &c. $\dot{y} =$

* * * *

$* * * * \frac{3x^4}{8a^4}$, &c. and $y = * * * * * \frac{3x^5}{40a^4}$, &c. Taking the Fluxions again, 'tis $a^2\overset{6}{y} - 16\overset{\cdot\cdot}{y} - 9x\overset{5}{y} - x^2\overset{6}{y} = 0$, and again, $a^2\overset{7}{y} - 25\overset{5}{y} - 11x\overset{6}{y} - x^2\overset{7}{y} = 0$, or $\overset{7}{y} = \frac{25\overset{5}{y} + 11x\overset{6}{y}}{a^2 - x^2} = \frac{25}{a^2}\dot{y}^5$, &c. $= \frac{25 \times 9}{a^6}$, &c. Therefore $\overset{6}{y} = * \frac{25 \times 9}{a^6}x$, &c. $\overset{5}{y} = * * \frac{25 \times 9}{2a^6}x^2$, &c. $\overset{4}{y} = * * * \frac{25 \times 3}{2a^6}x^3$, &c. $\dddot{y} = * * * * \frac{25 \times 3}{8a^6}x^4$, &c. $\ddot{y} = * * * * * \frac{5 \times 3}{8a^6}x^5$, &c. $\dot{y} = * * * * * * \frac{5x^6}{16a^6}$, &c. and $y = * * * * * * * \frac{5x^7}{112a^6}$, &c. Or $y = x + \frac{x^3}{6a^2} + \frac{3x^5}{40a^4} + \frac{5x^7}{112a^6}$, &c.

If we were required to extract the Root x out of the same Equation, $a^2\dot{y}^2 - x^2\dot{y}^2 - a^2\dot{x}^2 = 0$, (or to express the Sine by the Arch,) put $\dot{y} = 1$, then $a^2 - x^2 - a^2\dot{x}^2 = 0$, or $\dot{x}^2 = 1 - \frac{x^2}{a^2}$, and therefore $\dot{x} = 1$, &c. and $x = * y$, &c. Taking the Fluxions 'tis $-2\dot{x}x - 2a^2\dot{x}\ddot{x} = 0$, or $\ddot{x} = -\frac{x}{a^2} = 0$, &c. Therefore $\dot{x} = * 0$, &c. $x = * * 0$, &c. Taking the Fluxions again, 'tis $\dddot{x} = -\frac{\dot{x}}{a^2} = -\frac{1}{a^2}$, &c. Thence $\ddot{x} = * - \frac{y}{a^2}$, &c. $\dot{x} = * * - \frac{y^2}{2a^2}$, &c. and $x = * * * - \frac{y^3}{6a^2}$, &c. Again, $\overset{\cdot\cdot}{x} = -\frac{\ddot{x}}{a^2}$, and $\overset{5}{x} = -\frac{\dddot{x}}{a^2} = +\frac{1}{a^4}$, &c. Therefore $\ddot{x} = * \frac{y}{a^4}$, &c. $\dddot{x} = * * \frac{y^2}{2a^4}$, &c. $\ddot{x} = * * * \frac{y^3}{6a^4}$, &c. $\dot{x} = * * * * \frac{y^4}{24a^4}$, &c. and $x = * * * * * \frac{y^5}{120a^4}$, &c. Again, $\overset{6}{x} = -\frac{\overset{\cdot\cdot}{x}}{a^2}$, and $\overset{7}{x} = -\frac{\overset{5}{x}}{a^2} = -\frac{1}{a^6}$, &c. Therefore $\overset{6}{x} = * - \frac{y}{a^6}$, $\overset{5}{x} = * * - \frac{y^2}{2a^6}$, &c. $\overset{\cdot\cdot}{x} = * * * - \frac{y^3}{6a^6}$, &c. $\dddot{x} = * * * * - \frac{y^4}{24a^6}$, &c. $\ddot{x} = * * * * * - \frac{y^5}{120a^6}$, &c. $\dot{x} = * * * * * * - \frac{y^6}{720a^6}$, &c. and $x = * * * * * * * - \frac{y^7}{5040a^6}$, &c. And therefore $x = y - \frac{y^3}{6a^2} + \frac{y^5}{120a^4} - \frac{y^7}{5040a^6}$, &c.

If it were required to extract the Root y out of this Equation, $a^2\dot{y}^2 - x^2\dot{y}^2 + m^2y^2 - m^2a^2 = 0$, (where $\dot{x} = 1$,) we might proceed

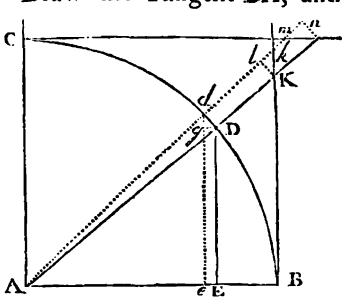
ceed thus. Becauſe $\dot{y}^2 = \frac{m^2a^2 - m^2y^2}{a^2 - x^2} = m^2$, &c. 'tis $\dot{y} = m$, &c. and $y = * mx$, &c. Taking the Fluxions, we ſhall have $2a^2\dot{y}\ddot{y} - 2x\dot{y}^2 - 2x^2\dot{y}\ddot{y} + 2m^2\dot{y}\dot{y} = 0$, or $a^2\ddot{y} - x\dot{y} - x^2\ddot{y} + m^2y = 0$, or $\ddot{y} = \frac{x\dot{y} - m^2y}{a^2 - x^2} = 0$, &c. Therefore taking the Fluxions again, 'tis $a^2\dddot{y} - \dot{y} - 3x\ddot{y} - x^2\dddot{y} + m^2\dot{y} = 0$, that is, $a^2\dddot{y} + \overline{m^2 - 1} \times \dot{y} - 3x\ddot{y} - x^2\dddot{y} = 0$, or $\dddot{y} = \frac{\overline{1 - m^2} \times \dot{y} + 3x\ddot{y}}{a^2 - x^2}$; and making $\dot{y} = m$, &c. 'tis $\dddot{y} = \frac{m \times \overline{1 - m^2}}{a^2}$, &c. and therefore $\ddot{y} = * \frac{m \times \overline{1 - m^2}}{a^4}x$, &c. $\dot{y} = * * \frac{m \times \overline{1 - m^2}}{2a^4}x^2$, &c. and $y = * * * \frac{m \times \overline{1 - m^2}}{2 \times 3a^2}x^3$, &c. Taking the Fluxions again, 'tis $a^2\ddot{y} + \overline{m^2 - 4} \times \ddot{y} - 5x\dddot{y} - x^2\ddot{y} = 0$; and again, $a^2\overset{5}{y} + \overline{m^2 - 9} \times \dddot{y} - 7x\ddot{y} - x^2\dddot{y} = 0$, or $\overset{5}{y} = \frac{\overline{9 - m^2} \times \dddot{y} + 7x\ddot{y}}{a^2 - x^2} = \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{a^4}$, &c. Therefore $\ddot{y} = * \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{a4}x$, &c. $\dddot{y} = * * \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{2a4}x^2$, &c. $\dot{y} = * * * \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{2 \times 3a4}x^3$, &c. $\dot{y} = * * * * \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{2 \times 3 \times 4a4}x^4$, &c. and $y = * * * * * \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{2 \times 3 \times 4 \times 5a4}x^5$, &c. And ſo on. Therefore we ſhall have $y = mx + m \times \frac{1 - m^2}{2 \times 3a^2}x^3 + m \times \frac{1 - m^2}{2 \times 3} \times \frac{9 - m^2}{4 \times 5a^4}x^5 + m \times \frac{1 - m^2}{2 \times 3} \times \frac{9 - m^2}{4 \times 5} \times \frac{25 - m^2}{6 \times 7a^6}x^7$, &c.

This Series is equivalent to a Theorem of our Author's, which (in another place) he gives us for Angular Sections. For if x be the Sine of any given Arch, to Radius a; then will y be the Sine of another Arch, which is to the firſt Arch in the given Ratio of m to 1. Here if m be any odd Number, the Series will become finite; and in other caſes it will be a converging Series.

And theſe Examples may be ſufficient to explain this Method of Extraction of Roots; which, tho' it carries its own Demonſtration along with it, yet for greater evidence may be thus farther illuſtrated. In Equations whoſe Roots (for example) may be repreſented by the general Series $y = A + Bx + Cx^2 + Dx^3$, &c. (which by due Reduction may be all Equations whatever,) the firſt Term A of the Root will be a given quantity, or perhaps $= 0$, which is to be known from the circumſtances of the Queſtion, or from the given Equation,

Equation, by Methods that have been abundantly explain'd already. Then making $\dot{x} = 1$, we ſhall have have $y = B + 2Cx + 3Dx^2$, &c. where B likewiſe is a conſtant quantity, or perhaps $= 0$, and repreſents the firſt Term of the Series $\dot{y}$. This therefore is to be derived from the firſt Fluxional Equation, either given or elſe to be found; and then, becauſe it is $\dot{y} = B$, &c. by taking the Fluents it will be $y = * Bx$, &c. whence the ſecond Term of the Root will be known. Then becauſe it is $\ddot{y} = 2C + 6Dx$, &c. or becauſe the conſtant quantity 2C will repreſent the firſt Term of $\ddot{y}$; this is to be derived from the ſecond Fluxional Equation, either given or to be found. And then, becauſe it is $\ddot{y} = 2C$, &c. by taking the Fluents it will be $\dot{y} = * 2Cx$, &c. and again $y = * * Cx^2$, &c. by which the third Term of the Root will be known. Then becauſe it is $\dddot{y} = 6D$, &c. or becauſe the conſtant quantity 6D will repreſent the firſt Term of the Series $\dddot{y}$; this is to be derived from the third Fluxional Equation. And then, becauſe it is $\dddot{y} = 6D$, &c. by taking the Fluents it will be $\ddot{y} = * 6Dx$, &c. $\dot{y} = * * 3Dx^2$, &c. and $y = * * * Dx^3$, &c. by which the fourth Term of the Root will be known. And ſo for all the ſubſequent Terms. And hence it will not be difficult to obſerve the compoſition of the Co-efficients in moſt caſes, and thereby diſcover the Law of Continuation, in ſuch Series as are notable and of general uſe.

If you ſhould deſire to know how the foregoing Trigonometrical Equations are derived from the Circle, it may be ſhewn thus: on the Center A, with Radius $AB = a$, let the Quadrantal Arch BC be deſcribed, and draw the Radius AC. Draw the Tangent BK, and through any point of the Circumference D, draw the Secant ADK, meeting the Tangent in K. At any other point *d* of the Circumference, but as near to D as may be, draw the Secant A*dk*, meeting BK in *k*; on Center A, with Radius AK, deſcribe the Arch K*l*, meeting A*k* in *l*. Then ſuppoſing the point *d* continually to approach towards D, till it finally coincides with it, the Trilineum K*lk* will continually approach to a right-lined Triangle, and to ſimilitude with the Triangle ABK: So that when D*d* is a Moment



Moment of the Circumference, it will be $\frac{Kk}{Dd} = \frac{Kk}{Kl} \times \frac{Kl}{Dd} = \frac{AK}{AB} \times \frac{AK}{AB}$. Make $AB = a$, the Tangent $BK = x$, and the Arch $BD = y$; and inſtead of the Moments Kk and Dd, ſubſtitute the proportional Fluxions $\dot{x}$ and $\dot{y}$, and it will be $\frac{\dot{x}}{\dot{y}} = \frac{a^2+x^2}{a^2}$, or $a^2\dot{y} + x^2\dot{y} - a^2\dot{x} = 0$.

From D to AB and *de* let fall the Perpendiculars DE and D*g*, which D*g* meets *de*, parallel to DE, in *g*. Then the ultimate form of the Trilineum D*dg* will be that of a right-lined Triangle ſimilar to DAE. Whence $Dd : dg :: AD : AE = \sqrt{AD^2 - DE^2}$. Make $AD = a$, $BD = y$, and $DE = x$; and for the Moments Dd, dg, ſubſtitute their proportional Fluxions $\dot{y}$ and $\dot{x}$, and it will be $\dot{y} : \dot{x} :: a : \sqrt{a^2 - x^2}$. Or $\dot{y}^2 : \dot{x}^2 :: a^2 : a^2 - x^2$, or $a^2\dot{y}^2 - x^2\dot{y}^2 - a^2\dot{x}^2 = 0$.

Hence the Fluxion of an Arch, whoſe right Sine is x, being expreſs'd by $\frac{a\dot{x}}{\sqrt{a^2-x^2}}$; and likewiſe the Fluxion of an Arch, whoſe right Sine is y, being expreſs'd by $\frac{a\dot{y}}{\sqrt{a^2-y^2}}$; if theſe Arches are to each other as 1 to m, their Fluxions will be in the ſame proportion, and *vice verſâ*. Therefore $\frac{a\dot{x}}{\sqrt{a^2-x^2}} : \frac{a\dot{y}}{\sqrt{a^2-y^2}} :: 1 : m$, or $\frac{m\dot{x}}{\sqrt{a^2-x^2}} = \frac{\dot{y}}{\sqrt{a^2-y^2}}$, or $\frac{m^2\dot{x}^2}{a^2-x^2} = \frac{\dot{y}^2}{a^2-y^2}$, or putting $\dot{x} = 1$, 'tis $a^2\dot{y}^2 - x^2\dot{y}^2 - m^2a^2 + m^2y^2 = 0$; the ſame Equation as before reſolved.

We might derive other Fluxional Equations, of a like nature with theſe, which would be accommodated to Trigonometrical uſes. As if y were the Circular Arch, and x its verſed Sine, we ſhould have the Equation $2ax\dot{y}^2 - x^2\dot{y}^2 - a^2\dot{x}^2 = 0$. Or if y were the Arch, and x the correſponding Secant, it would be $x^4\dot{y}^2 - a^2x^2\dot{y}^2 - a^4\dot{x}^2 = 0$. Or inſtead of the natural, we might derive Equations for the artificial Sines, Tangents, Secants, &c. But I ſhall leave theſe Diſquiſitions, and many ſuch others that might be propoſed, to exerciſe the Induſtry and Sagacity of the Learner.

SECT.

SECT. VI. *An Analytical Appendix, explaining ſome Terms and Expreſſions in the foregoing work.*

BEcauſe mention has been frequently made of *given Equations*, and others *aſſumed ad libitum*, and the like; I ſhall take occaſion from hence, by way of Appendix, to attempt ſome kind of explanation of this Mathematical Language, or of the Terms *given*, *aſſign'd*, *aſſumed*, and *required* Quantities or Equations, which may give light to ſome things that may otherwiſe ſeem obſcure, and may remove ſome doubts and ſcruples, which are apt to ariſe in the Mind of a Learner. Now the origin of ſuch kind of Expreſſions in all probability ſeems to be this. The whole affair of purſuing Mathematical Inquiries, or of reſolving Problems, is ſuppoſed (tho' tacitely) to be tranſacted between two Perſons, or Parties, the Propoſer and the Reſolver of the Problem, or (if you pleaſe) between the Maſter (or Inſtructor) and his Scholar. Hence this, and ſuch like Phraſes, *datam rectam*, *vel datum angulum*, *in imperatâ ratione ſecare*. As Examples inſtruct better than Precepts, or perhaps when both are join'd together they inſtruct beſt, the Maſter is ſuppos'd to propoſe a Queſtion or Problem to his Scholar, and to chuſe ſuch Terms and Conditions as he thinks fit; and the Scholar is obliged to ſolve the Problem with thoſe limitations and reſtrictions, with thoſe Terms and Conditions, and no other. Indeed it is required on the part of the Maſter, that the Conditions he propoſes may be conſiſtent with one another; for if they involve any inconſiſtency or contradiction, the Problem will be unfair, or will become abſurd and impoſſible, as the Solution will afterwards diſcover. Now theſe Conditions, theſe Points, Lines, Angles, Numbers, Equations, *&c.* that at firſt enter the ſtate of the Queſtion, or are ſuppoſed to be choſen or given by the Maſter, are the *data* of the Problem, and the Anſwers he expects to receive are the *quæſita*. As it may ſometimes happen, that the *data* may be more than are neceſſary for determining the Queſtion, and ſo perhaps may interfere with one another, and the Problem (as now propoſed) may become impoſſible; ſo they may be fewer than are neceſſary, and the Problem thence will be indetermin'd, and may require other Conditions to be given, in order to a compleat Determination, or perfectly to fulfil the *quæſita*. In this caſe the Scholar is to ſupply what is wanting, and at his diſcretion may *aſſume* ſuch and ſo many other Terms and Conditions, Equations and Limitations, as he finds

will be neceſſary to his purpoſe, and will beſt conduce to the ſimpleſt, the eaſieſt, and neateſt Solution that may be had, and yet in the moſt general manner. For it is convenient the Problem ſhould be propoſed as particular as may be, the better to fix the Imagination; and yet the Solution ſhould be made as general as poſſible, that it may be the more inſtructive, and extend to all caſes of a like nature.

Indeed the word *datum* is often uſed in a ſenſe which is ſomething different from this, but which ultimately centers in it. As that is call'd a *datum*, when one quantity is not immediately given, but however is neceſſarily infer'd from another, which other perhaps is neceſſarily infer'd from a third, and ſo on in a continued Series, till it is neceſſarily infer'd from a quantity, which is known or given in the ſenſe before explain'd. This is the Notion of *Euclid*'s *data*, and other Analytical Argumentations of that kind. Again, that is often call'd a *given quantity*, which always remains conſtant and invariable, while other quantities or circumſtances vary; becauſe ſuch as theſe only can be the given quantities in a Problem, when taken in the foregoing ſenſe.

To make all this the more ſenſible and intelligible, I ſhall have recourſe to a few practical inſtances, by way of Dialogue, (which was the old didactic method,) between Maſter and Scholar; and this only in the common Algebra or Analyticks, in which I ſhall borrow my Examples from our Author's admirable Treatiſe of Univerſal Arithmetick. The chief artifice of this manner of Solution will conſiſt in this, that as faſt as the Maſter propoſes the Conditions of his Queſtion, the Scholar applies thoſe Conditions to uſe, argues from them Analytically, makes all the neceſſary deductions, and derives ſuch conſequences from them, in the ſame order they are propoſed, as he apprehends will be moſt ſubſervient to the Solution. And he that can do this in all caſes, after the ſureſt, ſimpleſt, and readieſt manner, will be the beſt *ex-tempore* Mathematician. But this method will be beſt explain'd from the following Examples.

I. M. *A Gentleman being willing to diſtribute Alms* - - - *S.* Let the Sum he intended to diſtribute be repreſented by x. M. *Among ſome poor people.* *S.* Let the number of poor be y, then $\frac{x}{y}$ would have been the ſhare of each. M. *He wanted* 3 *ſhillings* - - - *S.* Make $3 = a$, for the ſake of univerſality, and let the pecuniary Unit be one Shilling; then the Sum to be diſtributed would have been $x + a$, and

and the ſhare of each would have been $\frac{x+a}{y}$. M. *So that each might receive* 5 *ſhillings.* S. Make $5 = b$, then $\frac{x+a}{y} = b$, whence $x = by - a$. M. *Therefore he gave every one* 4 *ſhillings.* S. Make $4 = c$, then the Money diſtributed will be cy. M. *And he has* 10 *ſhillings remaining.* S. Make $10 = d$, then $cy + d$ was the Money he intended at firſt to diſtribute; or $cy + d = (x =) by - a$, or $y = \frac{a+d}{b-c}$. M. *What was the number of poor people?* S. The number was $y = \frac{a+d}{b-c} = \frac{3+10}{5-4} = 13$. M. *And how much Alms did he at firſt intend to diſtribute?* S. He had at firſt $x = by - a = 5 \times 13 - 3 = 62$ ſhillings. M. *How do you prove your Solution?* S. His Money was at firſt 62 ſhillings, and the number of poor people was 13. But if his Money had been $62 + 3 = 65 = 13 \times 5$ ſhillings, then each poor perſon might have received 5 ſhillings. But as he gives to each 4 ſhillings, that will be $13 \times 4 = 52$ ſhillings diſtributed in all, which will leave him a Remainder of $62 - 52 = 10$ ſhillings.

II. M. *A young Merchant, at his firſt entrance npon buſineſs, began the World with a certain Sum of Money.* S. Let that Sum be x, the pecuniary Unit being one Pound. M. *Out of which, to maintain himſelf the firſt year, he expended* 100 *pounds.* S. Make the given number $100 = a$; then he had to trade with $x - a$. M. *He traded with the reſt, and at the end of the year had improved it by a third part.* S. For univerſality-ſake I will aſſume the general number n, and will make $\frac{1}{3} = n - 1$, (or $n = \frac{4}{3}$;) then the Improvement was $n - 1 \times x - a = nx - na - x + a$, and the Trading-ſtock and Improvement together, at the end of the firſt year, was $nx - na$. M. *He did the ſame thing the ſecond year.* S. That is, his whole Stock being now $nx - na$, deducting a, his Expences for this year, he would have $nx - na - a$ for a Trading-ſtock, and $n - 1 \times nx - na - a$, or $n^2x - n^2a - nx + a$ for this year's Improvement, which together make $n^2x - n^2a - na$ for his Eſtate at the end of the ſecond year. M. *As alſo the third year.* S. His whole Stock being now $n^2x - n^2a - na$, taking out his Expences for the third year, his Trading-ſtock will be $n^2x - n^2a - na - a$, and the Improvement this year will be $n - 1 \times n^2x - n^2a - na - a$, or $n^3x - n^3a - n^2x + a$, and the Stock and Improvement together, or his whole Eſtate at the end of the third year will be $n^3x - n^3a - n^2a - na$, or in a better form $n^3x + \frac{n^3-1}{1-n}na$. In like manner

if he proceeded thus the fourth year, his Eſtate being now $n^3x - n^3a - n^2a - na$, taking out this year's Expence, his Trading-ſtock will be $n^3x - n^3a - n^2a - na - a$, and this year's Improvement is $n - 1 \times n^3x - n^3a - n^2a - na - a$, or $n^4x - n^4a - n^3x + a$, which added to his Trading-ſtock will be $n^4x - n^4a - n^3a - n^2a - na$, or $n^4x + \frac{n^4-1}{1-n}na$, for his Eſtate at the end of the fourth year. And ſo, by Induction, his Eſtate will be found $n^5x + \frac{n^5-1}{1-n}na$ at the end of the fifth year. And univerſally, if I aſſume the general Number m, his Eſtate will be $n^mx + \frac{n^m-1}{1-n}na$ at the end of any number of years denoted by m. M. *But he made his Eſtate double to what it was at firſt.* S. Make $2 = b$, then $n^mx + \frac{n^m-1}{1-n}na = bx$, or $x = \frac{n^m-1}{n-1 \times n^m-b}na$. M. *At the end of* 3 *years.* S. Then $m = 3$, $a = 100$, $b = 2$, $n = \frac{4}{3}$, and therefore $x = \frac{\frac{4^3}{3^3}-1}{\frac{1}{3}\times\frac{4^3}{3^3}-2} \times \frac{4}{3} \times 100 = \frac{4^3-3^3}{4^3-2\times3^3} \times 400 = \frac{64-27}{64-54} \times 400 = \frac{37}{10} \times 400 = 1480$. M. *What was his Eſtate at firſt?* S. It was 1480 pounds.

III. M. *Two Bodies* A *and* B *are at a given diſtance from each other.* S. As their diſtance is ſaid to be given, though it is not ſo actually, I may therefore aſſume it. Let the initial diſtance of the Bodies be $59 = e$, and let the Linear Unit be one Mile. M. *And move equably towards one another.* S. Let x repreſent the whole ſpace deſcribed by A before they meet; then will $e - x$ be the whole ſpace deſcribed by B. M. *With given Velocities.* S. I will aſſume the Velocity of A to be ſuch, that it will move $7 = c$ Miles in $2 = f$ Hours, the Unit of Time being one Hour. Then becauſe it is $c : f :: x : \frac{fx}{c}$, A will move his whole ſpace x in the time $\frac{fx}{c}$. Alſo I will aſſume the Velocity of B to be ſuch, that it will move $8 = d$ Miles in $3 = g$ Hours. Then becauſe it is $d : g :: e - x : \frac{e-x}{d}g$, B will move his whole ſpace $e - x$ in the time $\frac{e-x}{d}g$. M. *But* A *moves a given time* - - - S. Let that time be $1 = h$ Hour. M. *Before* B *begins to move.* S. Then A's time is equal to B's time added to the time h, or $\frac{fx}{c} = \frac{e-x}{d}g + h$.

M 2

M. *Where will they meet, or what will be the ſpace that each will have deſcribed?* S. From this Equation we ſhall have $x = \frac{eg+db}{df+cg}c$ $= \frac{59\times 3+8\times 1}{8\times 2+7\times 3}\times 7 = \frac{185}{37}\times 7 = 5\times 7 = 35$ Miles, which will be the whole ſpace deſcribed by *A*. Then $e - x = 59 - 35 = 24$ Miles will be the whole ſpace deſcribed by *B*.

IV. M. *If* 12 *Oxen can be maintained by the Paſture of* $3\frac{1}{3}$ *Acres of Meadow-ground for* 4 *weeks,* S. Make $12 = a$, $3\frac{1}{3} = b$, $4 = c$; then aſſuming the general Numbers e, f, h, to be determin'd afterwards as occaſion ſhall require, we ſhall have by analogy

		Oxen		Paſture		Time
If	Reciprocally Directly	a	require	b	during	c
Then		ae		be		c
Alſo		$\frac{ae}{b}$		e		c
And		$\frac{ace}{b}$		e		1
Alſo		$\frac{ace}{bf}$		e		f
Alſo		$\frac{ace}{bh}$		e		h

M. *And if, becauſe of the continual growth of the Graſs after the four weeks, it be found that* 21 *Oxen can be maintain'd by the paſture of* 10 *ſuch Acres for* 9 *weeks,* S. Make $21 = d$, $e = 10$, $f = 9$; then becauſe on this ſuppoſition, the Oxen d require the paſture e during the time f; and in the former caſe the Oxen $\frac{ace}{bf}$ required the ſame paſture during the ſame time: Therefore the growth of the Graſs of the quantity of paſture e, (commencing after 4 or c weeks, and continuing to the end of the Time f, or during the whole time $f - c$,) is ſuch, as alone was ſufficient to maintain the difference of the Oxen, or the number $d - \frac{ace}{bf}$, for the whole time f. Then reciprocally that growth would be ſufficient to maintain the number of Oxen $df - \frac{ace}{b}$ for the time 1, or the number of Oxen $\frac{df}{b} - \frac{ace}{bh}$ for the time h. And becauſe this growth will be proportional to the time, and will maintain a greater number of Oxen in proportion as the time is greater; we ſhall have

Time

$$\text{Time } f-c \;.\; \text{Oxen } \frac{df}{b}-\frac{ace}{bb} :: \text{Time } b-c \;.\; \text{Oxen } \frac{b-c}{f-c} \text{ into } \frac{df}{b}-\frac{ace}{bb},$$

which will be the number of Oxen that may be maintain'd by the growth only of the paſture e, during the whole time b. But it was found before, that without this growth of the Graſs, the Oxen $\frac{ace}{bb}$ might be maintain'd by the paſture e for the time b. Therefore theſe two together, or $\frac{ace}{bb}+\frac{b-c}{f-c}\times\frac{bdf-ace}{bb}$, will be the number of Oxen that may be maintain'd by the paſture e, and its growth together, during the time b. M. *How many Oxen may be maintain'd by* 24 *Acres of ſuch paſture for* 18 *weeks?* S. Suppoſe x to be that number of Oxen, and make $24=g$, and $b=18$. Then by analogy

$$\begin{array}{lll} & \text{Oxen} & \text{Paſture} \\ \text{If} & x & g \\ \text{Then} & ex & eg \\ \text{And} & \frac{ex}{g} & e \end{array} \text{ require during the time } b.$$

And conſequently $\frac{ex}{g}=\frac{ace}{bb}+\frac{b-c}{f-c}\times\frac{bdf-ace}{bb}$, or $x=\frac{acg}{bb}+\frac{b-c}{f-c}\times\overline{\frac{dfg}{eb}-\frac{acg}{bb}}=\frac{ac}{b}+\frac{b-c}{f-c}\times\overline{\frac{df}{e}-\frac{ac}{b}}$ into $\frac{g}{b}=\frac{12\times4}{3\frac{1}{2}}+\frac{18-4}{9-4}\times\overline{\frac{21\times9}{10}-\frac{12\times4}{3\frac{1}{2}}}$ into $\frac{4}{18}=36$.

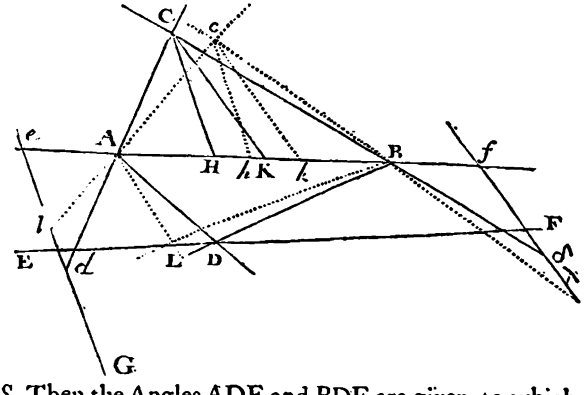
V. M. *If I have an Annuity* - - - S. Let x be the preſent value of 1 pound to be received 1 year hence, then (by analogy) x^2 will be the preſent value of 1 pound to be received 2 years hence, &c. and in general, x^m will be the preſent value of 1 pound to be received m years hence. Therefore, in the caſe of an Annuity, the Series $x+x^2+x^3+x^4$, &c. to be continued to ſo many Terms as there are Units in m, will be the preſent value of the whole Annuity of 1 pound, to be continued for m years. But becauſe $\frac{x-x^{m+1}}{1-x}=x+x^2+x^3+x^4$, &c. continued to ſo many Terms as there are Units in m, (as may appear by Diviſion;) therefore $\frac{x-x^{m+1}}{1-x}$ will repreſent the Amount of an Annuity of 1 pound, to be continued for m years. M. *Of Pounds.* S. Make

$=a$,

$= a$, then the Amount of this Annuity for m years will be $\frac{x - x^{m+1}}{1-x} a$. M. *To be continued for* 5 *years ſucceſſively.* S. Then $m = 5$. M. *Which I ſell for* *pounds in ready Money.* S. Make $= c$, then $\frac{x - x^{m+1}}{1-x} a = c$, or $x^{m+1} - \frac{c}{a} + 1 \times x + \frac{c}{a} = 0$. In any particular caſe the value of x may be found by the Reſolution of this affected Equation. M. *What Intereſt am I allow'd* per centum per annum? S. Make $100 = b$; then becauſe x is the preſent value of 1 pound to be received 1 year hence, or (which is the ſame thing) becauſe the preſent Money x, if put out to uſe, in 1 year will produce 1 pound; the Intereſt alone of 1 pound for 1 year will be $1 - x$, and therefore the Intereſt of 100 (or b) pounds for 1 year will be $b - bx$, which will be known when x is known.

And this might be ſufficient to ſhew the conveniency of this Method; but I ſhall farther illuſtrate it by one Geometrical Problem, which ſhall be our Author's LVII.

VI. M. *In the right Line* AB *I give you the two points* A *and* B. S. Then their diſtance AB $= m$ is given alſo. M. *As likewiſe the two points* C *and* D *out of the Line* AB. S. Then conſequently the figure ACBD is given in magnitude and ſpecie; and producing CA and CB towards *d* and *δ*, I can take A*d* = AD, and B*δ* = BD. M. *Alſo I give you the indefinite right Line* EF *in poſition, paſſing thro' the given point* D.



S. Then the Angles ADE and BDF are given, to which (producing AB both ways, if need be, to *e* and *f*,) I can make the Angles A*de* and B*δf* equal reſpectively, and that will determine the points *e* and *f*, or the Lines A*e* = *a*, and B*f* = *c*. And becauſe *de* and *δf* are thereby known, I can continue *de* to G, ſo that *d*G = *δf*, and make the given line *e*G = *b*. Likewiſe I can draw CH and

and CK parallel to *ed* and *fδ* reſpectively, meeting AB in H and K; and becauſe the Triangle CHK will be given in magnitude and ſpecie, I will make CK $= d$, CH $= e$, and HK $= f$. M. *Now let the given Angles* CAD *and* CBD *be conceived to revolve about the given points or Poles* A *and* B. *S.* Then the Lines AD and CA*d* will move into another ſituation AL and *c*A*l*, ſo as that the Angles DAL, *d*A*l*, and CA*c* will be equal. Alſo the Lines BD and CB*δ* will obtain a new ſituation BL and *c*Bλ, ſo as that the Angles DBL, *δ*Bλ and CB*c* will be equal. M. *And let* D, *the Interſection of the Lines* AD *and* BD, *always move in the right Line* EF. *S.* Then the new point of Interſection L is in EF; then the Triangles DAL and *d*A*l*, as alſo DBL and *δ*Bλ, are equal and ſimilar; then $dl = DL = \delta\lambda$, and therefore $Gl = f\lambda$. M. *What will be the nature of the Curve deſcribed by the other point of Interſection* C? *S.* From the new point of Interſection *c* to AB, I will draw the Lines *ch* and *ck*, parallel to CH and CK reſpectively. Then will the Triangle *chk* be given in ſpecie, though not in magnitude, for it will be ſimilar to CHK. Alſo the Triangle B*ck* will be ſimilar to Bλ*f*. And the indefinite Line B*k* $= x$ may be aſſumed for an Abſciſs, and *ck* $= y$ may be the correſponding Ordinate to the Curve C*c*. Then becauſe it is B*k* (x) : *ck* (y) :: B*f* (e) : $f\lambda = \frac{ey}{x} = Gl$. Subtract this from G*e* $= b$, and there will remain $le = b - \frac{ey}{x}$. Then becauſe of the ſimilar Triangles *chk* and CHK, it will be CK (d) : CH (e) :: *ck* (y) : *ch* $= \frac{ey}{d}$. And CK (d) : HK (f) :: *ck* (y) : *hk* $= \frac{fy}{d}$. Therefore A*h* $=$ AB $-$ B*k* $-$ *hk* $= m - x - \frac{fy}{d}$. But it is A*h* $\left(m - x - \frac{fy}{d}\right)$: *ch* $\left(\frac{ey}{d}\right)$ :: A*e* (a) : *le* $\left(b - \frac{cy}{x}\right)$. Therefore $\overline{m - x - \frac{fy}{x}} \times \overline{b - \frac{cy}{d}} = \frac{aey}{d}$, or $fcy^2 + \overline{dc - ae - bf} \times xy - dcmy - bdx^2 + bdmx = 0$. In which Equation, becauſe the indeterminate quantities x and y ariſe only to two Dimenſions, it ſhews that the Curve deſcribed by the point C is a Conic Section.

M. *You have therefore ſolved the Problem in general, but you ſhould now apply your Solution to the ſeveral ſpecies of Conic Sections in particular.* *S.* That may eaſily be done in the following manner: Make $\frac{ae + bf - cd}{c} = 2p$, and then the foregoing Equation will become $fcy^2 - 2pcxy - dcmy - bdx^2 + bdmx = 0$, and by extracting

tracting the Square-root it will be $y = \frac{p}{f}x + \frac{dm}{2f} \pm \sqrt{\frac{pp}{ff} + \frac{bd}{fc} \times x^2 + \frac{pdm}{ff} - \frac{bd \cdot n}{fc} \times x + \frac{d^2m^2}{4ff}}$. Now here it is plain, that if the Term $\frac{pp}{ff} + \frac{bd}{fc} \times x^2$ were abſent, or if $\frac{pp}{ff} + \frac{bd}{fc} = 0$, or $\frac{pp}{ff} = -\frac{bd}{fc}$; that is, if the quantity $\frac{bd}{fc}$ (changing its ſign) ſhould be equal to $\frac{pp}{ff}$, then the Curve would be a Parabola. But if the ſame Term were preſent, and equal to ſome affirmative quantity, that is, if $\frac{pp}{ff} + \frac{bd}{fc}$ be affirmative, (which will always be when $\frac{bd}{fc}$ is affirmative, or if it be negative and leſs than $\frac{pp}{ff}$,) the Curve will be an Hyperbola. Laſtly, if the ſame Term were preſent and negative, (which can only be when $\frac{bd}{fc}$ is negative, and greater than $\frac{pp}{ff}$,) the Curve will be an Ellipſis or a Circle.

I ſhould make an apology to the Reader, for this Digreſſion from the Method of Fluxions, if I did not hope it might contribute to his entertainment at leaſt, if not to his improvement. And I am fully convinced by experience, that whoever ſhall go through the reſt of our Author's curious Problems, in the ſame manner, (wherein, according to his uſual brevity, he has left many things to be ſupply'd by the ſagacity of his Reader,) or ſuch other Queſtions and Mathematical Diſquiſitions, whether Arithmetical, Algebraical, Geometrical, &c. as may eaſily be collected from Books treating on theſe Subjects; I ſay, whoever ſhall do this after the foregoing manner, will find it a very agreeable as well as profitable exerciſe: As being the proper means to acquire a habit of Inveſtigation, or of arguing ſurely, methodically, and Analytically, even in other Sciences as well as ſuch as are purely Mathematical; which is the great end to be aim'd at by theſe Studies.

SECT. VII. *The Concluſion; containing a ſhort recapitulation or review of the whole.*

WE are now arrived at a period, which may properly enough be call'd *the concluſion of the Method of Fluxions and Infinite Series*; for the deſign of this Method is to teach the nature of Series in general, and of Fluxions and Fluents, what they are, how they are derived, and what Operations they may undergo; which deſign (I think) may now be ſaid to be accompliſh'd. As to the application of this Method, and the uſes of theſe Operations, which is all that now remains, we ſhall find them inſiſted on at large by the Author in the curious Geometrical Problems that follow. For the whole that can be done, either by Series or by Fluxions, may eaſily be reduced to the Reſolution of Equations, either Algebraical or Fluxional, as it has been already deliver'd, and will be farther apply'd and purſued in the ſequel. I have continued my Annotations in a like manner upon that part of the Work, and intended to have added them here; but finding the matter to grow ſo faſt under my hands, and ſeeing how impoſſible it was to do it juſtice within ſuch narrow limits, and alſo perceiving this work was already grown to a competent ſize; I reſolved to lay it before the Mathematical Reader unfiniſh'd as it is, reſerving the completion of it to a future opportunity, if I ſhall find my preſent attempts to prove acceptable. Therefore all that remains to be done here is this, to make a kind of review of what has been hitherto deliver'd, and to give a ſummary account of it, in order to acquit myſelf of a Promiſe I made in the Preface. And having there done this already, as to the Author's part of the work, I ſhall now only make a ſhort recapitulation of what is contain'd in my own Comment upon it.

And firſt in my Annotations upon what I call the Introduction, or the Reſolution of Equations by infinite Series, I have amply purſued a uſeful hint given us by the Author, that Arithmetick and Algebra are but one and the ſame Science, and bear a ſtrict analogy to each other, both in their Notation and Operations; the firſt computing after a definite and particular manner, the latter after a general and indefinite manner: So that both together compoſe but one uniform Science of Computation. For as in common Arithmetick we reckon by the Root *Ten*, and the ſeveral Powers of that Root; ſo in Algebra, or Analyticks, when the Terms are orderly diſpos'd

diſpos'd as is preſcribed, we reckon by any other Root and its Powers, or we may take any general Number for the Root of our Arithmetical Scale, by which to expreſs and compute any Numbers required. And as in common Arithmetick we approximate continually to the truth, by admitting Decimal Parts *in infinitum*, or by the uſe of Decimal Fractions, which are compoſed of the reciprocal Powers of the Root *Ten*; ſo in our Author's improved Algebra, or in the Method of infinite converging Series, we may continually approximate to the Number or Quantity required, by an orderly ſucceſſion of Fractions, which are compoſed of the reciprocal Powers of any Root in general. And the known Operations in common Arithmetick, having a due regard to Analogy, will generally afford us proper patterns and ſpecimens, for performing the like Operations in this Univerſal Arithmetick.

Hence I proceed to make ſome Inquiries into the nature and formation of infinite Series in general, and particularly into their two principal circumſtances of Convergency and Divergency; wherein I attempt to ſhew, that in all ſuch Series, whether converging or diverging, there is always a Supplement, which if not expreſs'd is however to be underſtood; which Supplement, when it can be aſcertained and admitted, will render the Series finite, perfect, and accurate. That in diverging Series this Supplement muſt indiſpenſably be admitted and exhibited, or otherwiſe the Concluſion will be imperfect and erroneous. But in converging Series this Supplement may be neglected, becauſe it continually diminiſhes with the Terms of the Series, and finally becomes leſs than any aſſignable quantity. And hence ariſes the benefit and conveniency of infinite converging Series; that whereas that Supplement is commonly ſo implicated and entangled with the Terms of the Series, as often to be impoſſible to be extricated and exhibited; in converging Series it may ſafely be neglected, and yet we ſhall continually approximate to the quantity required. And of this I produce a variety of Inſtances, in numerical and other Series.

I then go on to ſhew the Operations, by which infinite Series are either produced, or which, when produced, they may occaſionally undergo. As firſt when ſimple ſpecious Equations, or pure Powers, are to be reſolved into ſuch Series, whether by Diviſion, or by Extraction of Roots; where I take notice of the uſe of the afore-mention'd Supplement, by which Series may be render'd finite, that is, may be compared with other quantities, which are conſider'd as given. I then deduce ſeveral uſeful Theorems, or other Artifices,

for the more expeditious Multiplication, Diviſion, Involution, and Evolution of infinite Series, by which they may be eaſily and readily managed in all caſes. Then I ſhew the uſe of theſe in pure Equations, or Extractions; from whence I take occaſion to introduce a new praxis of Reſolution, which I believe will be found to be very eaſy, natural, and general, and which is afterwards apply'd to all ſpecies of Equations.

Then I go on with our Author to the *Exegeſis numeroſa*, or to the Solution of affected Equations in Numbers; where we ſhall find his Method to be the ſame that has been publiſh'd more than once in other of his pieces, to be very ſhort, neat, and elegant, and was a great Improvement at the time of its firſt publication. This Method is here farther explain'd, and upon the ſame Principles a general Theorem is form'd, and diſtributed into ſeveral ſubordinate Caſes, by which the Root of any Numerical Equation, whether pure or affected, may be computed with great exactneſs and facility.

From Numeral we paſs on to the Reſolution of Literal or Specious affected Equations by infinite Series; in which the firſt and chief difficulty to be overcome, conſiſts in determining the forms of the ſeveral Series that will ariſe, and in finding their initial Approximations. Theſe circumſtances will depend upon ſuch Powers of the Relate and Correlate Quantities, with their Coefficients, as may happen to be found promiſcuouſly in the given Equation. Therefore the Terms of this Equation are to be diſpoſed *in longum & in latum*, or at leaſt the Indices of thoſe Powers, according to a combined Arithmetical Progreſſion *in plano*, as is there explain'd; or according to our Author's ingenious Artifice of the Parallelogram and Ruler, the reaſon and foundation of which are here fully laid open. This will determine all the caſes of exterior Terms, together with the Progreſſions of the Indices; and therefore all the forms of the ſeveral Series that may be derived for the Root, as alſo their initial Coefficients, Terms, or Approximations.

We then farther proſecute the Reſolution of Specious Equations, by diverſe Methods of Analyſis; or we give a great variety of Proceſſes, by which the Series for the Roots are eaſily produced to any number of Terms required. Theſe Proceſſes are generally very ſimple, and depend chiefly upon the Theorems before deliver'd, for finding the Terms of any Power or Root of an infinite Series. And the whole is illuſtrated and exemplify'd by a great variety of Inſtances, which are chiefly thoſe of our Author.

The

The Method of infinite Series being thus ſufficiently diſcuſs'd, we make a Tranſition to the Method of Fluxions, wherein the nature and foundation of that Method is explain'd at large. And ſome general Obſervations are made, chiefly from the Science of Rational Mechanicks, by which the whole Method is divided and diſtinguiſh'd into its two grand Branches or Problems, which are the Direct and Inverſe Methods of Fluxions. And ſome preparatory Notations are deliver'd and explain'd, which equally concern both theſe Methods.

I then proceed with my Annotations upon the Author's firſt Problem, or the Relation of the flowing Quantities being given, to determine the Relation of their Fluxions. I treat here concerning Fluxions of the firſt order, and the method of deducing their Equations in all caſes. I explain our Author's way of taking the Fluxions of any given Equation, which is much more general and ſcientifick than that which is uſually follow'd, and extends to all the varieties of Solutions. This is alſo apply'd to Equations involving ſeveral flowing Quantities, by which means it likewiſe comprehends thoſe caſes, in which either compound, irrational, or mechanical Quantities may be included. But the Demonſtration of Fluxions, and of the Method of taking them, is the chief thing to be conſider'd here; which I have endeavour'd to make as clear, explicite, and ſatisfactory as I was able, and to remove the difficulties and objections that have been raiſed againſt it: But with what ſucceſs I muſt leave to the judgment of others.

I then treat concerning Fluxions of ſuperior orders, and give the Method of deriving their Equations, with its Demonſtration. For tho' our Author, in this Treatiſe, does not expreſſly mention theſe orders of Fluxions, yet he has ſometimes recourſe to them, tho' tacitely and indirectly. I have here ſhewn, that they are a neceſſary reſult from the nature and notion of firſt Fluxions; and that all theſe ſeveral orders differ from each other, not abſolutely and eſſentially, but only relatively and by way of compariſon. And this I prove as well from Geometry as from Analyticks; and I actually exhibit and make ſenſible theſe ſeveral orders of Fluxions.

But more eſpecially in what I call the Geometrical and Mechanical Elements of Fluxions, I lay open a general Method, by the help of Curve-lines and their Tangents, to repreſent and exhibit Fluxions and Fluents in all caſes, with all their concomitant Symptoms and Affections,

Affections, after a plain and familiar manner, and that even to ocular view and inſpection. And thus I make them the Objects of Senſe, by which not only their exiſtence is proved beyond all poſſible contradiction, but alſo the Method of deriving them is at the ſame time fully evinced, verified, and illuſtrated.

Then follow my Annotations upon our Author's ſecond Problem, or the Relation of the Fluxions being given, to determine the Relation of the flowing Quantities or Fluents; which is the ſame thing as the Inverſe Method of Fluxions. And firſt I explain (what our Author calls) a particular Solution of this Problem, becauſe it cannot be generally apply'd, but takes place only in ſuch Fluxional Equations as have been, or at leaſt might have been, previouſly derived from ſome finite Algebraical or Fluential Equations. Whereas the Fluxional Equations that uſually occur, and whoſe Fluents or Roots are required, are commonly ſuch as, by reaſon of Terms either redundant or deficient, cannot be reſolved by this particular Solution; but muſt be refer'd to the following general Solution, which is here diſtributed into theſe three Caſes of Equations.

The firſt Caſe of Equations is, when the Ratio of the Fluxions of the Relate and Correlate Quantities, (which Terms are here explain'd,) can be expreſs'd by the Terms of the Correlate Quantity alone; in which Caſe the Root will be obtain'd by an eaſy proceſs: In finite Terms, when it may be done, or at leaſt by an infinite Series. And here a uſeful Rule is explain'd, by which an infinite Expreſſion may be always avoided in the Concluſion, which otherwiſe would often occur, and render the Solution inexplicable.

The ſecond Caſe of Equations comprehends ſuch Fluxional Equations, wherein the Powers of the Relate and Correlate Quantities, with their Fluxions, are any how involved. Tho' this Caſe is much more operoſe than the former, yet it is ſolved by a variety of eaſy and ſimple Analyſes, (more ſimple and expeditious, I think, than thoſe of our Author,) and is illuſtrated by a numerous collection of Examples.

The third and laſt Caſe of Fluxional Equations is, when there are more than two Fluents and their Fluxions involved; which Caſe, without much trouble, is reduced to the two former. But here are alſo explain'd ſome other matters, farther to illuſtrate this Doctrine; as the Author's Demonſtration of the Inverſe Method of Fluxions, the Rationale of the Tranſmutation of the Origin of Fluents to other

places at pleaſure, the way of finding the contemporaneous Increments of Fluents, and ſuch like.

Then to conclude the Method of Fluxions, a very convenient and general Method is propoſed and explain'd, for the Reſolution of all kinds of Equations, Algebraical or Fluxional, by having recourſe to ſuperior orders of Fluxions. This Method indeed is not contain'd in our Author's preſent Work, but is contrived in purſuance of a notable hint he gives us, in another part of his Writings. And this Method is exemplify'd by ſeveral curious and uſeful Problems.

Laſtly, by way of Supplement or Appendix, ſome Terms in the Mathematical Language are farther explain'd, which frequently occur in the foregoing work, and which it is very neceſſary to apprehend rightly. And a ſort of Analytical Praxis is adjoin'd to this Explanation, to make it the more plain and intelligible; in which is exhibited a more direct and methodical way of reſolving ſuch Algebraical or Geometrical Problems as are uſually propoſed; or an attempt is made, to teach us to argue more cloſely, diſtinctly, and Analytically.

And this is chiefly the ſubſtance of my Comment upon this part of our Author's work, in which my conduct has always been, to endeavour to digeſt and explain every thing in the moſt direct and natural order, and to derive it from the moſt immediate and genuine Principles. I have always put myſelf in the place of a Learner, and have endeavour'd to make ſuch Explanations, or to form this into ſuch an Inſtitution of Fluxions and infinite Series, as I imagined would have been uſeful and acceptable to myſelf, at the time when I firſt enter'd upon theſe Speculations. Matters of a trite and eaſy nature I have paſs'd over with a ſlight animadverſion: But in things of more novelty, or greater difficulty, I have always thought myſelf obliged to be more copious and explicite; and am conſcious to myſelf, that I have every where proceeded *cum ſincero animo docendi.* Wherever I have fallen ſhort of this deſign, it ſhould not be imputed to any want of care or good intentions, but rather to the want of ſkill, or to the abſtruſe nature of the ſubject. I ſhall be glad to ſee my defects ſupply'd by abler hands, and ſhall always be willing and thankful to be better inſtructed.

What perhaps will give the greateſt difficulty, and may furniſh moſt matter of objection, as I apprehend, will be the Explanations before given, of *Moments, vaniſhing quantities, infinitely little quantities,*

tities, and the like, which our Author makes uſe of in this Treatiſe, and elſewhere, for deducing and demonſtrating his Method of Fluxions. I ſhall therefore here add a word or two to my foregoing Explanations, in hopes farther to clear up this matter. And this ſeems to be the more neceſſary, becauſe many difficulties have been already ſtarted about the abſtract nature of theſe quantities, and by what name they ought to be call'd. It has even been pretended, that they are utterly impoſſible, inconceiveable, and unintelligible, and it may therefore be thought to follow, that the Concluſions derived by their means muſt be precarious at leaſt, if not erroneous and impoſſible.

Now to remove this difficulty it ſhould be obſerved, that the only Symbol made uſe of by our Author to denote theſe quantities, is the letter *o*, either by itſelf, or affected by ſome Coefficient. But this Symbol *o* at firſt repreſents a finite and ordinary quantity, which muſt be underſtood to diminiſh continually, and as it were by local Motion; till after ſome certain time it is quite exhauſted, and terminates in mere nothing. This is ſurely a very intelligible Notion. But to go on. In its approach towards nothing, and juſt before it becomes abſolute nothing, or is quite exhauſted, it muſt neceſſarily paſs through vaniſhing quantities of all proportions. For it cannot paſs from being an aſſignable quantity to nothing at once; that were to proceed *per ſaltum*, and not continually, which is contrary to the Suppoſition. While it is an aſſignable quantity, tho' ever ſo little, it is not yet the exact truth, in geometrical rigor, but only an Approximation to it; and to be accurately true, it muſt be leſs than any aſſignable quantity whatſoever, that is, it muſt be a vaniſhing quantity. Therefore the Conception of a Moment, or vaniſhing quantity, muſt be admitted as a rational Notion.

But it has been pretended, that the Mind cannot conceive quantity to be ſo far diminiſh'd, and ſuch quantities as theſe are repreſented as impoſſible. Now I cannot perceive, even if this impoſſibility were granted, that the Argumentation would be at all affected by it, or that the Concluſions would be the leſs certain. The impoſſibility of Conception may ariſe from the narrowneſs and imperfection of our Faculties, and not from any inconſiſtency in the nature of the thing. So that we need not be very ſolicitious about the poſitive nature of theſe quantities, which are ſo volatile, ſubtile, and fugitive, as to eſcape our Imagination; nor need we be much in pain, by what name they are to be call'd; but we may confine ourſelves wholly to the uſe of them, and to diſcover their properties.

properties. They are not introduced for their own ſakes, but only as ſo many intermediate ſteps, to bring us to the knowledge of other quantities, which are real, intelligible, and required to be known. It is ſufficient that we arrive at them by a regular progreſs of diminution, and by a juſt and neceſſary way of reaſoning; and that they are afterwards duly eliminated, and leave us intelligible and indubitable Concluſions. For this will always be the conſequence, let the *media* of ratiocination be what they will, when we argue according to the ſtrict Rules of Art. And it is a very common thing in Geometry, to make impoſſible and abſurd Suppoſitions, which is the ſame thing as to introduce impoſſible quantities, and by their means to diſcover truth.

We have an inſtance ſimilar to this, in another ſpecies of Quantities, which, though as inconceiveable and as impoſſible as theſe can be, yet when they ariſe in Computations, they do not affect the Concluſion with their impoſſibility, except when they ought ſo to do; but when they are duly eliminated, by juſt Methods of Reduction, the Concluſion always remains ſound and good. Theſe Quantities are thoſe Quadratick Surds, which are diſtinguiſh'd by the name of impoſſible and imaginary Quantities; ſuch as $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$, &c. For they import, that a quantity or number is to be found, which multiply'd by itſelf ſhall produce a negative quantity; which is manifeſtly impoſſible. And yet theſe quantities have all varieties of proportion to one another, as thoſe aforegoing are proportional to the poſſible and intelligible numbers 1, $\sqrt{2}$, $\sqrt{3}$, 2, &c. reſpectively; and when they ariſe in Computations, and are regularly eliminated and excluded, they always leave a juſt and good Concluſion.

Thus, for Example, if we had the Cubick Equation $x^3 - 12x^2 + 41x - 42 = 0$, from whence we were to extract the Root x; by proceeding according to Rule, we ſhould have this ſurd Expreſſion for the Root, $x = 4 + \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{27}}}$, in which the impoſſible quantity $\sqrt{-\frac{100}{27}}$ is involved; and yet this Expreſſion ought not to be rejected as abſurd and uſeleſs, becauſe, by a due Reduction, we may derive the true Roots of the Equation from it. For when the Cubick Root of the firſt *vinculum* is rightly extracted, it will be found to be the impoſſible Number $-1 + \sqrt{-\frac{4}{3}}$, as may appear by cubing; and when the Cubick Root of the ſecond *vinculum* is extracted, it will be found to be $-1 - \sqrt{-\frac{4}{3}}$. Then by collecting theſe Numbers, the

impoſſibie Number $\surd -\frac{4}{3}$ will be eliminated, and the Root of the Equation will be found $x = 4 - 1 - 1 = 2$.

Or the Cubick Root of the firſt *vinculum* will alſo be $\frac{3}{2} + \surd -\frac{1}{12}$, as may likewiſe appear by Involution; and of the ſecond *vinculum* it will be $\frac{3}{2} - \surd -\frac{1}{12}$. So that another of the Roots of the given Equation will be $x = 4 + \frac{3}{2} + \frac{3}{2} = 7$. Or the Cubick Root of the ſame firſt *vinculum* will be $-\frac{1}{2} - \surd -\frac{5}{12}$; and of the ſecond will be $-\frac{1}{2} + \surd -\frac{5}{12}$. So that the third Root of the given Equation will be $x = 4 - \frac{1}{2} - \frac{1}{2} = 3$. And in like manner in all other Cubick Equations, when the ſurd *vincula* include an impoſſible quantity, by extracting the Cubick Roots, and then by collecting, the impoſſible parts will be excluded, and the three Roots of the Equation will be found, which will always be poſſible. But when the aforeſaid ſurd *vincula* do not include an impoſſible quantity, then by Extraction one poſſible Root only will be found, and an impoſſibility will affect the other two Roots, or will remain (as it ought) in the Concluſion. And a like judgment may be made of higher degrees of Equations.

So that theſe impoſſible quantities, in all theſe and many other inſtances that might be produced, are ſo far from infecting or deſtroying the truth of theſe Concluſions, that they are the neceſſary means and helps of diſcovering it. And why may we not conclude the ſame of that other ſpecies of impoſſible quantities, if they muſt needs be thought and call'd ſo? Surely it may be allow'd, that if theſe Moments and infinitely little Quantities are to be eſteem'd a kind of impoſſible Quantities, yet nevertheleſs they may be made uſeful, they may aſſiſt us, by a juſt way of Argumentation, in finding the Relations of Velocities, or Fluxions, or other poſſible Quantities required. And finally, being themſelves duly eliminated and excluded, they may leave us finite, poſſible, and intelligible Equations, or Relations of Quantities.

Therefore the admitting and retaining theſe Quantities, however impoſſible they may ſeem to be, the inveſtigating their Properties with our utmoſt induſtry, and applying thoſe Properties to uſe whenever occaſion offers, is only keeping within the Rules of Reaſon and Analogy; and is alſo following the Example of our ſagacious aud illuſtrious Author, who of all others has the greateſt right to be our Precedent in theſe matters. 'Tis enlarging the number of general Principles and Methods, which will always greatly

contribute to the Advancement of true Science. In ſhort, it will enable us to make a much greater progreſs and proficience, than we otherwiſe can do, in cultivating and improving what I have elſewhere call'd *The Philoſophy of Quantity*.

F I N I S.

A

COMMENTARY

ON

NEWTON'S PRINCIPIA.

WITH

A SUPPLEMENTARY VOLUME.

DESIGNED FOR THE USE OF STUDENTS AT THE UNIVERSITIES.

BY

J. M. F. WRIGHT, A. B.

LATE SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE, AUTHOR OF SOLUTIONS OF THE CAMBRIDGE PROBLEMS, &c. &c.

IN TWO VOLUMES.

VOL. I.

LONDON:

PRINTED FOR T. T. & J. TEGG, 73, CHEAPSIDE;

AND RICHARD GRIFFIN & CO., GLASGOW.

MDCCCXXXIII.

GLASGOW:
GEORGE BROOKMAN, PRINTER, VILLAFIELD.

TO THE TUTORS

OF THE SEVERAL COLLEGES AT CAMBRIDGE,

THESE PAGES,

WHICH WERE COMPOSED WITH THE VIEW

OF PROMOTING THE STUDIES

OVER WHICH THEY SO ABLY PRESIDE,

ARE RESPECTFULLY INSCRIBED

BY THEIR DEVOTED SERVANT,

THE AUTHOR.

PREFACE.

THE flattering manner in which the *Glasgow Edition* of Newton's Principia has been received, a second impression being already on the verge of publication, has induced the projectors and editor of that work, to render, as they humbly conceive, their labours still more acceptable, by presenting these additional volumes to the public. From amongst the several testimonies of the esteem in which their former endeavours have been held, it may suffice, to avoid the charge of self-eulogy, to select the following, which, coming from the high authority of *French* mathematical criticism, must be considered at once as the more decisive and impartial. It has been said by one of the first geometers of France, that "L'edition de Glasgow fait honneur aux presses de cette ville industrieuse. On peut affirmer que jamais l'art typographique ne rendit un plus bel hommage à la memoire de Newton. Le mérite de l'impression, quoique très-remarquable, n'est pas ce que les editeurs ont recherché avec le plus de soin, pour tant le materiel de leur travail, ils pouvaient s'en rapporter à l'habilité de leur artistes: mais le choix des meilleures éditions, la revision la plus scrupuleuse du texte et des épreuves, la recherche attentive des fautes qui pourraient échapper même au lecteur studieux, et passer inaperçues ce travail consciencieux de l'intelligence et du savoir, voilà ce qui élève cette edition au-dessus de toutes celles qui l'ont précédée.

"Les editeurs de Glasgow ne s'etaient chargés que d'un travail de revision. *S'ils avaient conçu le projet d'améliorer et compléter l'œuvre des*

commentateurs, ils auraient sans doute employé, comme eux, les travaux des successeurs de Newton sur les questions traitées dans le livre des Principes.

" Les descendans de Newton sont nombreux, et leur genealogie est prouvée par des titres incontestibles; ceux qui vivent aujourd'hui verraient sans doute avec satisfaction que l'on formât un tableau de leur famille, en reunissant les productions les plus remarquables dont l'ouvrage de Newton a fourni le germe: que ce livre immortel soit entouré de tout ce l'on peut regarder comme ses developpemens: voilà son meilleur commentaire. *L'editiòn de Glasgow pourrait donc être continuée, et prodigieusement enrichie.*"

The same philosopher takes occasion again to remark, that " Le plus beau monument que l'on puisse élever à la gloire de Newton, c'est une bonne edition de ses ouvrages: et il est étonnant que les Anglais en aient laissé ce soin aux nations etrangères. Les presses de Glasgow viennent de reparer, en partie, le tort de la nation Anglaise: la nouvelle edition des *Principes* est effectivement *la plus belle, la plus correcte et la plus commode qui ait paru jusqu'ici.* La collation des anciennes editions, la revision des calculs, &c. ont été confiees á un habile mathematicien et rien n'a été négligé pour éviter toutes les erreurs et toutes les omissions.

" Il faut esperer que *les editeurs continueront leur belle entreprise, et qu'ils y seront assez encouragés pour nous donner, non seulement tous les ouvrages de Newton, mais ceux des savans qui ont complété ses travaux.*"

The encouragement here anticipated has not been withheld, nor has the idea of improving and completing the comments of " The Jesuits", contained in the Glasgow Newton, escaped us, inasmuch as long before these hints were promulgated, had the following work, which is composed principally as a succedaneum to the former, been planned, and partly written. It is at least, however, a pleasing confirmation of the justness of our own conceptions, to have encountered even at any time with these after-suggestions. The plan of the work is, nevertheless, in several respects, a deviation from that here so forcibly recommended.

The object of the first volume is, to make the text of the Principia, by

supplying numerous steps in the very concise demonstrations of the propositions, and illustrating them by every conceivable device, as easy as can be desired by students even of but moderate capacities. It is universally known, that Newton composed this wonderful work in a very hasty manner, merely selecting from a huge mass of papers such discoveries as would succeed each other as the connecting links of one vast chain, but without giving himself the trouble of explaining to the world the mode of fabricating those links. His comprehensive mind could, by the feeblest exertion of its powers, condense into one view many syllogisms of a proposition even heretofore uncontemplated. What difficulties, then, *to him* would seem his own discoveries? Surely none; and the modesty for which he is proverbially remarkable, gave him in his own estimation so little the advantage of the rest of created beings, that he deemed these difficulties as easy to others as to himself: the lamentable consequence of which humility has been, that he himself is scarcely comprehended at this day—a century from the birth of the Principia.

We have had, in the first place, the Lectures of Whiston, who descants not even respectably in his lectures delivered at Cambridge, upon the discoveries of his master. Then there follow even lower and less competent interpreters of this great prophet of science—for such Newton must have been held in those dark days of knowledge—whom it would be time mis-spent to dwell upon. But the first, it would seem, who properly estimated the Principia, was Clairaut. After a lapse of nearly half a century, this distinguished geometer not only acknowledged the truths of the Principia, but even extended the domain of Newton and of Mathematical Science. But even Clairaut did not condescend to explain his views and perceptions to the rest of mankind, farther than by publishing his own discoveries. For these we owe a vast debt of gratitude, but should have been still more highly benefited, had he bestowed upon us a sort of running Commentary on the Principia. It is generally supposed, indeed, that the greater portion of the Commentary called Madame Chastellet's, was due to Clairaut. The best things, however, of that work are alto-

gether unworthy of so great a master; at the most, showing the performance was not one of his own seeking. At any rate, this work does not deserve the name of a Commentary on the Principia. The same may safely be affirmed of many other productions intended to facilitate Newton. Pemberton's View, although a bulky tome, is little more than a eulogy. Maclaurin's speculations also do but little, elucidate the dark passages of the Principia, although written more immediately for that purpose. This is also a heavy unreadable performance, and not worthy a place on the same shelf with the other works of that great geometer. Another great mathematician, scarcely inferior to Maclaurin, has also laboured unprofitably in the same field. Emerson's Comments is a book as small in value as it is in bulk, affording no helps worth the perusal to the student. Thorpe's notes to the First Book of the Principia, however, are of a higher character, and in many instances do really facilitate the reading of Newton. Jebb's notes upon certain sections deserve the same commendation; and praise ought not to be withheld from several other commentators, who have more or less succeeded in making small portions of the Principia more accessible to the student—such as the Rev. Mr. Newton's work, Mr. Carr's, Mr. Wilkinson's, Mr. Lardner's, &c. It must be confessed, however, that all these fall far short in value of the very learned labours, contained in the Glasgow Newton, of the Jesuits Le Seur and Jacquier, and their great coadjutor. Much remained, however, to be added even to this erudite production, and subsequently to its first appearance much has been excogitated, principally by the mathematicians of Cambridge, that focus of science, and native land of the Principia, of which, in the composition of the following pages, the author has liberally availed himself. The most valuable matter thus afforded are the Tutorial MSS. in circulation at Cambridge. Of these, which are used in explaining Newton to the students by the Private Tutors there, the author confesses to have had abundance, and also to have used them so far as seemed auxiliary to his own resources. But at the same time it must be remarked, that little has been the assistance hence derived, or, indeed, from all

other known sources, which from the first have been constantly at command.

The plan of the work being to make those parts of Newton easy which are required to be read at Cambridge and Dublin, that portion of the Principia which is better read in the elementary works on Mechanics, viz. the preliminary Definitions, Laws of Motion, and their Corollaries, has been disregarded. For like reasons the fourth and fifth sections have been but little dwelt upon. The eleventh section and third book have not met with the attention their importance and intricacy would seem to demand, partly from the circumstance of an excellent Treatise on Physics, by Mr. Airey, having superseded the necessity of such labours; and partly because in the second volume the reader will find the same subjects treated after the easier and more comprehensive methods of Laplace.

The first section of the first book has been explained at great length, and it is presumed that, for the first time, the true principles of what has been so long a subject of contention in the scientific world, have there been fully established. It is humbly thought (for in these intricate speculations it is folly to be proudly confident), that what has been considered in so many lights and so variously denominated Fluxions, Ultimate Ratios, Differential Calculus, Calculus of Derivations, &c. &c. is here laid down on a basis too firm to be shaken by future controversy. It is also hoped that the text of this section, hitherto held almost impenetrably obscure, is now laid open to the view of most students. The same merit it is with some confidence anticipated will be awarded to the illustrations of the 2nd, 3rd, 6th, 7th, 8th, and 9th sections, which, although not so recondite, require much explanation, and many of the steps to be supplied in the demonstration of almost every proposition. Many of the things in the first volume are new to the author, but very probably not original in reality—so vast and various are the results of science already accumulated. Suffice it to observe, that if they prove useful in unlocking the treasures of the Principia, the author will rest satisfied with the meed of approbation, which he will to that extent have earned from a discriminating and impartial public.

The second volume is designed to form a sort of Appendix or Supplement to the Principia. It gives the principal discoveries of Laplace, and, indeed, will be found of great service, as an introduction to the entire perusal of the immortal work of that author—the Mecanique Celeste. This volume is prefaced by much useful matter relative to the Integration of Partial Differences and other difficult branches of Abstract Mathematics, those powerful auxiliaries in the higher departments of Physical Astronomy, and which appear in almost every page of the Mecanique Celeste. These and other preparations, designed to facilitate the comprehension of the Newton of these days, will, it is presumed, be found fully acceptable to the more advanced readers, who may be prosecuting researches even in the remotest and most hidden receptacles of science; and, indeed, the author trusts he is by no means unreasonably exorbitant in his expectations, when he predicates of himself that throughout the undertaking he has proved himself a labourer not unworthy of reward.

THE AUTHOR.

A COMMENTARY

ON

NEWTON'S PRINCIPIA.

SECTION I. BOOK I.

1. This section is introductory to the succeeding part of the work. It comprehends the substance of the method of Exhaustions of the Ancients, and also of the Modern Theories, variously denominated *Fluxions*, *Differential Calculus*, *Calculus of Derivations*, *Functions*, &c. &c. Like them it treats of the relations which Indefinite quantities bear to one another, and conducts in general by a nearer route to precisely the same results.

2. In what precedes this section, *finite* quantities only are considered, such as the spaces described by bodies moving uniformly in *finite* times with *finite* velocities; or at most, those described by bodies whose motions are *uniformly accelerated*. But what follows relates to the motions of bodies accelerated according to various hypotheses, and requires the consideration of quantities indefinitely small or great, or of such whose Ratios, by their decrease or increase, continually approximate to certain Limiting Values, but which they cannot reach be the quantities ever so much diminished or augmented. These *Limiting Ratios* are called by Newton, "Prime and Ultimate Ratios," *Prime Ratio* meaning the Limit from which the Ratio of two quantities *diverges*, and *Ultimate Ratio* that towards which the Ratio *converges*. To prevent ambiguity, the term *Limiting Ratio* will subsequently be used throughout this Commentary.

LEMMA I.

3. QUANTITIES AND THE RATIOS OF QUANTITIES.] Hereby Newton would infer the truth of the Lemma not only for quantities mensurable by *Integers*, but also for such as may be denoted by *Vulgar Fractions*. The necessity or use of the distinction is none; there being just as much reason for specifying all other sorts of quantities. The truth of the LEMMA does not depend upon the *species* of quantities, but upon their conformity with the following conditions, viz.

4. That they *tend continually to equality, and approach nearer to each other than by any given difference.* They must tend *continually* to equality, that is, every Ratio of their successive corresponding values must be nearer and nearer a Ratio of Equality, the number of these convergencies being without end. By *given difference* is merely meant any that can be assigned or proposed.

5. FINITE TIME.] Newton obviously introduces the idea of time in this enunciation, to show illustratively that he supposes the quantities to converge continually to equality, without ever actually *reaching or passing* that state; and since to fix such an idea, he says, "before the end of that time," it was moreover necessary to consider the time *Finite*. Hence our author would avoid the charge of "*Fallacia Suppositionis*," or of "*shifting the hypothesis.*" For it is contended that if you frame certain relations between actual quantities, and afterwards deduce conclusions from such relations on the supposition of the quantities having vanished, such conclusions are illogically deduced, and ought no more to subsist than the quantities themselves.

In the Scholium at the end of this Section he is more explicit. He says, *The ultimate Ratios, in which quantities vanish, are not in reality the Ratios of Ultimate quantities; but the Limits to which the Ratios of quantities continually decreasing always approach; which they never can pass beyond or arrive at, unless the quantities are continually and indefinitely diminished.* After all, however, neither our Author himself nor any of his Commentators, though much has been advanced upon the subject, has obviated this objection. Bishop Berkeley's ingenious criticisms in the Analyst remain to this day unanswered. He therein facetiously denominates the results, obtained from the supposition that the quantities, before

considered finite and real, have vanished, the "*Ghosts of Departed Quantities;*" and it must be admitted there is reason as well as wit in the appellation. The fact is, Newton himself, if we may judge from his own words in the above cited Scholium, where he says, "If two quantities, whose DIFFERENCE IS GIVEN are augmented continually, their Ultimate Ratio will be a Ratio of Equality," had no knowledge of the *true nature* of his Method of Prime and Ultimate Ratios. If there be meaning in words, he plainly supposes in this passage, a mere Approximation to be the same with an Ultimate Ratio. He loses sight of the condition expressed in Lemma I. namely, *that the quantities tend to equality nearer than by any assignable difference*, by supposing the difference of the quantities continually augmented to be *given*, or always the same. In this sense the whole Earth, compared with the whole Earth minus a grain of sand, would constitute an Ultimate Ratio of equality; whereas so long as any, the minutest difference exists between two quantities, they cannot be said to be more than *nearly* equal. But it is now to be shown, that

6. *If two quantities tend continually to equality, and approach to one another nearer than by any assignable difference, their Ratio is* ULTIMATELY *a Ratio of* ABSOLUTE *equality.* This may be demonstrated as follows, even without supposing the quantities ultimately evanescent.

It is acknowledged by all writers on Algebra, and indeed self-evident, that if in any equation put = 0, there be quantities absolutely different in kind, the aggregate of each species is separately equal to 0. For example, if

$$A + a + B\sqrt{2} + b\sqrt{2} + C\sqrt{-1} = 0,$$

since A + a is *rational*, (B + b) $\sqrt{2}$ *surd* and C $\sqrt{-1}$ *imaginary*, they cannot in any way destroy one another by the opposition of signs, and therefore

$$A + a = 0,\ B + b = 0,\ C = 0.$$

In the same manner, if *logarithms*, *exponentials*, or any other quantities differing essentially from one another constitute an equation like the above, they must separately be equal to 0. This being premised, let L, L′ denote the Limits, whatever they are, towards which the quantities L + l, L′ + l′ continually converge, and suppose their difference, in any state of the convergence, to be D. Then

$$L + l - L' - l' = D,$$
$$\text{or } L - L' + l - l' - D = 0,$$

and since L, L′ are fixed and definite, and l, l′, D always variable, the former are independent of the latter, and we have

$$L - L' = 0, \text{ or } \frac{L}{L'} = 1, \text{ accurately.} \quad \text{Q. e. d.}$$

This way of considering the question, it is presumed, will be deemed free from every objection. The principle upon which it rests depending upon the *nature* of the variable quantities, and not upon their evanescence, (as it is equally true even for constant quantities provided they be of different natures), it is hoped we have at length hit upon the true and logical method of expounding the doctrine of *Prime and Ultimate Ratios*, or of *Fluxions*, or of the *Differential Calculus*, &c.

It may be here remarked, in passing, that the Method of *Indeterminate Coefficients*, which is at bottom the same as that of Prime and Ultimate Ratios, is treated illogically in most books of Algebra. Instead of "shifting the hypothesis," as is done in Wood, Bonnycastle and others, by making $x = 0$, in the equation

$$a + b x + c x^2 + d x^3 + \ldots\ldots = 0,$$

it is sufficient to know that each term x being indefinitely *variable*, is heterogeneous compared with the rest, and consequently that each term must equal 0.

7. Having established the truth of LEMMA I. on incontestable principles, we proceed to make such applications as may produce results useful to our subsequent comments. As these applications relate to the Limits of the Ratios of the *Differences* of Quantities, we shall term, after Leibnitz, the Method of Prime and Ultimate Ratios,

THE DIFFERENTIAL CALCULUS.

8. According to the established notation, let a, b, c, &c., denote constant quantities, and z, y, x, &c., variable ones. Also let Δz, Δy, Δx, &c., represent the difference between any two values of z, y, x, &c., respectively.

9. *Required the Limiting or Ultimate Ratio of* $\Delta (a x)$ and Δx, i. e. *the Limit of the Difference of a Rectangle having one side* (a) *constant, and the other* (x) *variable, and of the Difference of the variable side.*

Let L be the Limit sought, and $L + l$ any value whatever of the varying Ratio. Then

$$L + l = \frac{\Delta (a x)}{\Delta x} = \frac{a (x + \Delta x) - a x}{\Delta x} = a. \quad \therefore \text{ by No. 7}$$

$$L = a.$$

In this instance the Ratio is the same for all values of x. But if in the Limit we change the characteristic Δ into d, we have

$$\left.\begin{array}{l} \dfrac{d\,(a\,x)}{d\,x} = a \\ \text{or} \\ d\,(a\,x) = a\,d\,x \end{array}\right\} \quad \ldots\ldots \quad (b)$$

d (a x), d x being called the *Differentials* of a x and x respectively.

10. *Required the Limit of* $\dfrac{\Delta\,(x^2)}{\Delta\,x}$.

Let L be the Limit required, and L + l the value of the Ratio generally. Then

$$L + l = \frac{\Delta\,(x^2)}{\Delta\,x} = \frac{(x + \Delta\,x)^2 - x^2}{\Delta\,x} =$$

$$\frac{2\,x\,\Delta\,x + \Delta\,x^2}{\Delta\,x} = 2\,x + \Delta\,x.$$

$$\therefore L - 2\,x + l - \Delta\,x = 0$$

and since L — 2 x and l — Δ x are heterogeneous

$$L - 2\,x = 0,$$

or

$$L = 2x.$$

and $\therefore$

$$\frac{d\,(x^2)}{d\,x} = 2\,x.$$

or

$$d\,(x^2) = 2\,x\,d\,x \quad \ldots\ldots\ldots \quad (c)$$

11. *Generally, required the Limit of* $\dfrac{\Delta\,(x^n)}{\Delta\,x}$.

Let L and L + l be the Limit of the Ratio and the Ratio itself respectively. Then

$$L + l = \frac{\Delta\,(x^n)}{\Delta\,x} = \frac{(x + \Delta\,x)^n - x^n}{\Delta\,x}$$

$$= n\,x^{n-1} + \frac{n.\,(n-1)}{2}.\,x^{n-2}\,\Delta\,x + \&c.$$

and L — n x^{n-1} being essentially different from the other terms of the series and from l, we have

$$\frac{d\,(x^n)}{d\,x} = L = n\,x^{n-1} \text{ or } d\,(x^n) = n\,x^{n-1}\,d\,x \quad \ldots\ldots \quad (d)$$

or in words,

The Differential of any power or root of a variable quantity is equal to the product of the Differential of the quantity itself, the same power or root MINUS *one of the quantity, and the index of the power or root.*

We have here supposed the Binomial Theorem as fully established by Algebra. It may, however, easily be demonstrated by the general principle explained in (7).

12. From 9 and 11 we get

$$d(a x^n) = n a x^{n-1} d x \quad \ldots\ldots\ldots \text{(e)}$$

13. *Required the Limit of* $\dfrac{\Delta (a + b x^n + c x^m + e x^p + \&c.)}{\Delta x}$

Let L be the Limit sought, and L + 1 the variable Ratio of the finite differences; then

$$L + 1 = \frac{\Delta (a + b x^n + c x^m + \&c.)}{\Delta x}$$

$$= \frac{a + b(x + \Delta x)^n + c(x + \Delta x)^m + \&c. - a - b x^n - c x^m - \&c.}{\Delta x}$$

$$= n b x^{n-1} + m c x^{m-1} + \&c. + P \Delta x + Q (\Delta x)^2 + \&c.$$

P, Q, &c. being the coefficients of Δx, Δx^2 + &c. And equating the homogeneous determinate quantities, we have

$$\frac{d(a + b x^n + c x^m + \&c.)}{d x} = L = n b x^{n-1} + m c x^{m-1} + p e x^{p-1} + \&c. \ldots \text{(f)}$$

14. *Required the Limit of* $\dfrac{\Delta (a + b x^n + c x^m + \&c.)^r}{\Delta x}$.

By 11 we have

$$\frac{d.\ (a + b x^n + c x^m + \&c.)^r}{d\,(a + b x^n + \&c.)} = r (a + b x^n + c x^m + \&c.)^{r-1}$$

and by 13

$$d (a + b x^n + c x^m + \&c.) = (n b x^{n-1} + m c x^{m-1} + \&c.)\, d x$$

$$\therefore \frac{d(a + b x^n + c x^m + \&c.)^r}{d x} = r (n b x^{n-1} + m c x^{m-1} + \&c.)(a + b x^n + \&c.)^{r-1} \ldots \text{(g)}$$

the Limiting Ratio of the Finite Differences $\Delta (a + b x^n + c x^m + \&c.)$, Δx, that is the Ratio of the Differentials of $a + b x^n + c x^m$ + &c., and x.

15. *Required the Ratio of the Differentials of* $\dfrac{A + B x^n + C x^m + \&c.}{a + b x^r + C x^\mu + \&c.}$ and x, *or the Limiting Ratio of their Finite Differences.*

Let L be the Limit required, and L + 1 the varying Ratio. Then

$$L + 1 = \frac{A + B(x + \Delta x)^n + C(x + \Delta x)^m + \&c.}{a + b(x + \Delta x)^r + c(x + \Delta x)^\mu + \&c.} - \frac{A + B x^n + \&c.}{a + b x^r + \&c.}$$

which being expanded by the Binomial Theorem, and properly reduced gives

$$L \times (a + b x^{\nu} + \&c.)^2 + L \times \{P.\, \Delta x + Q (\Delta x)^2 + \&c. + l \times \{a + b x^{\nu} + \&c. + P.\, \Delta x + Q (\Delta x)^2 + \&c.\} = (a + b x^{\nu} + c x^{\mu} + \&c.) \times (n B x^{n-1} + m C x^{m-1} + \&c.) - (A + B x^{n} + C x^{m} + \&c.) \times (\nu b x^{\nu-1} + \mu c x^{\mu-1} + \&c.) + P'.\, \Delta x + Q' (\Delta x)^2 + \&c.$$

P, Q, P', Q' &c. being coefficients of Δx, $(\Delta x)^2$ &c. and independent of them.

Now equating those *homogeneous* terms which are independent of the powers of Δx, we get

$$L(a + b x^{\nu} + \&c.)^2 = (a + b x^{\nu} + \&c.) - (n B x^{n-1} + m C x^{m-1} + \&c.) - (A + B x^{n} + C x^{m} + \&c.) - (\nu b x^{\nu-1} + \mu c x^{\mu-1} + \&c.)$$

and putting $u = \dfrac{A + B x^{n} + C x^{m} + \&c.}{a + b x^{\nu} + c x^{\mu} + \&c.}$ we have finally

$$\frac{du}{dx} = L, \text{ and therefore } \frac{du}{dx} =$$

$$\frac{(a + b x^{\nu} + \&c.)(n B x^{n-1} + m C x^{m-1} + \&c.) - (A + B x^{n} + \&c.)(\nu b x^{\nu-1} + \mu c x^{\mu-1} + \&c.)}{(a + b x^{\nu} + c x^{\mu} + \&c.)^2}$$

the Ratio required.

16. Hence and from 11 we have the Ratio of the Differentials of $\dfrac{(A + B x^{n} + C x^{m} + \&c.)^{p}}{(a + b x^{\nu} + c x^{\mu} + \&c.)^{q}}$ and x; and in short, from what has already been delivered it is easy to obtain the Ratio of the Differentials of any *Algebraic Function whatever of one variable* and of that variable.

N. B. By *Function of a variable* is meant a quantity anyhow involving that variable. The term was first used to denote the Powers of a quantity, as x^2, x^3, &c. But it is now used in the general sense.

The quantities next to Algebraical ones, in point of simplicity, are Exponential Functions; and we therefore proceed to the investigation of their Differentials.

17. *Required the Ratio of the Differentials of* a^x and x; *or the Limiting Ratio of their Differences.*

Let L be the required Limit and L + l the varying Ratio; then

$$L + l = \frac{\Delta (a^x)}{\Delta x} = \frac{a^{x + \Delta x} - a^x}{\Delta x}$$

$$= a^x \times \frac{a^{\Delta x} - 1}{\Delta x}.$$

But since

$$a^y = (1 + a - 1)^y$$

$$= 1 + y(a-1) + \frac{y.(y-1)}{2}.(a-1)^2 +$$

$$\frac{y.(y-1)(y-2)}{2.3}(a-1)^3 + \&c.,$$

it is easily seen that the coefficient of y in the expansion is

$$a - 1 - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.$$

Hence

$$L + l = \frac{a^x}{\Delta x}\{(a-1-\frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.)\Delta x + P(\Delta x)^2 + \&c.\}$$

and equating homogeneous quantities, we have

$$\frac{d.(a^x)}{dx} = L = \{a - 1 - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.\}\, a^x$$

$$= A\, a^x \quad \ldots\ldots\ldots \quad (h)$$

or the Ratio of the Differentials of any Exponential and its exponent is equal to the product of the Exponential and a constant Quantity.

Hence and from the preceding articles, *the Ratio of the Differentials of any Algebraic Function of Exponentials having the same variable index, may be found.* The Student may find abundance of practice in the Collection of Examples of the Differential and Integral Calculus, by Messrs. Peacock, Herschel and Babbage.

Before we proceed farther in Differentiation of quantities, let us investigate the nature of the constant A which enters the equation (h).

For that purpose, let (the two first terms have been already found)

$$a^x = 1 + Ax + Px^2 + Qx^3 + \&c.$$

Then, by 13,

$$\frac{d(a^x)}{dx} = A + 2Px + 3Qx^2 + 4Rx^3 + \&c.$$

But by equation (h)

$$\frac{d(a^x)}{dx} \text{ also } = A\, a^x$$

$$= A + A^2x + APx^2 + AQx^3 + \&c.$$

$$\therefore A + 2Px + 3Qx^2 + 4Rx^3 + \&c. = A + A^2x + APx^2 + \&c.$$

and equating *homogeneous quantities*, we get

$$2P = A^2,\ 3Q = AP,\ 4R = AQ,\ \&c. = \&c.$$

whence

$$P = \frac{A^2}{2}, Q = \frac{AP}{3} = \frac{A^3}{2.3}, R = \frac{AQ}{4} = \frac{A^4}{2.3.4} \text{ &c. &c.}$$

Therefore,

$$a^x = 1 + Ax + \frac{A^2}{2}x^2 + \frac{A^3}{2.3}x^3 + \frac{A^4}{2.3.4}x^4 + \text{&c.}$$

Again, put $Ax = 1$, then

$$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \text{&c.}$$

$= 2.718281828459$ as is easily calculated

$= e$

by supposition. Hence

$$A = \frac{\log. a}{\log. e} \quad \ldots\ldots\ldots \quad (k)$$

$$\therefore a - 1 - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \text{&c.} = \frac{\log. a}{\log. e} = \text{l. a}$$

for the system whose base is e, l being the characteristic of that system.

This system being that which gives

$$e - 1 - \frac{(e-1)^2}{2} + \frac{(e-1)^3}{3} - \text{&c.} = 1$$

is called Natural from being the most simple.

Hence the equation (h) becomes

$$\frac{d(a^x)}{dx} = l\,a \times a^x \quad \ldots\ldots\ldots \quad (l)$$

17 a. *Required the Ratio of the Differentials of* l (x) and x.

Let $l\,x = u$. Then $e^u = x$

$\therefore dx = d(e^u) = l\,e \times e^u\,du = e^u\,du$, by 16

$$\therefore \frac{d(l\,x)}{dx} = \frac{1}{e^u} = \frac{1}{x} \quad \ldots\ldots\ldots \quad (m)$$

In any other system whose base is a, we have $\log. (x) = \frac{l\,x}{l\,a}$.

$$\therefore \frac{d \log. x}{dx} = \frac{1}{l\,a} \times \frac{1}{x} \quad \ldots\ldots\ldots \quad (n)$$

We are now prepared to differentiate any Algebraic, or Exponential Functions of Logarithmic Functions, provided there be involved but one variable.

Before we differentiate circular functions, viz. the sines, cosines, tangents, &c., of circular arcs, we shall proceed with our comments on the text as far as Lemma VIII.

LEMMA II.

18. In No. 6, calling L and L′ Limits of the circumscribed and inscribed rectilinear figures, and L + l, L′ + l′ any other values of them, whose variable difference is D, the *absolute* equality of L and L′ is clearly demonstrated, without the supposition of the bases A B, B C, C D, D E, being infinitely diminished in number and augmented in magnitude. In the view there taken of the subject, it is necessary merely to suppose them *variable.*

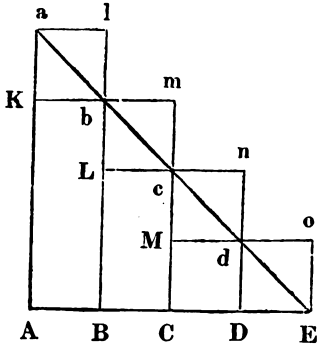
LEMMA III.

19. This LEMMA is also demonstrable by the same process in No. 6, as LEMMA II.

Cor. 1. The rectilinear figures cannot possibly *coincide* with the curvilinear figure, because the rectilinear boundaries a l b m c n d o E, a K b L c M d D E cut the curve a b E in the points a, b, c, d, E in finite angles. The learned Jesuits, Jacquier and Le Seur, in endeavouring to remove this difficulty, suppose the four points a, l, b, K to coincide, and thus to form a small element of the curve. But this is the language of Indivisibles, and quite inadmissible. It is plain that no *straight line,* or combination of straight lines, can form a *curve line,* so long as we understand by a *straight line* "that which lies evenly between its extreme points," and by a *curve line,* "that which does not lie evenly between its extreme points;" for otherwise it would be possible for a line to be straight and not straight at the same time. The truth is manifestly this. The Limiting Ratio of the inscribed and circumscribed figures is that of equality, because they continually tend to a fixed *area,* viz. that of the given intermediate curve. But although this intermediate curvilinear area, is the Limit towards which the rectilinear areas continually tend and approach nearer than by any difference; yet it does not follow that the rectilinear boundaries also tend to the curvilinear one as a limit. The rectilinear boundaries are, in fact, entirely *heterogeneous* with the intermediate one, and consequently cannot be equal to it, nor coincide therewith. We will now clear up the above, and at the same time introduce a striking illustration of the necessity there exists, of taking into consideration the *nature* of quantities, rather than their evanescence or infinitesimality.

Take the simplest example of Lemma II., in the case of the right-angled triangle a E A, having its two legs A a, A E equal.

The figure being constructed as in the text of Lemma II, it follows from that Lemma, that the Ultimate Ratio of the inscribed and circumscribed figures is a ratio of equality; and moreover it would also follow from *Cor.* 1. that either of these coincided ultimately with the triangle a E A. Hence then the exterior boundary a l b m c n d o E coincides exactly with a E *ultimately*, and they are consequently equal in the Limit. As we have only straight lines to deal with in this example, let us try to ascertain the exact ratio of a E to the exterior boundary.



If n be the indefinite number of equal bases A B, B C, &c., it is evident, since A a = A E, that the whole length of a l b m c n d o E = 2 n × A B. Also since a b = b c = &c. = $\sqrt{a\,l^2 + b\,l^2}$ = $\sqrt{2}$. A B, we have a E = n $\sqrt{2}$. A B.

Consequently,

a l b m c n d o E : a E : : 2 : $\sqrt{2}$: : $\sqrt{2}$: 1.

Hence it is plain the exterior boundary cannot possibly coincide with a E. Other examples might be adduced, but it must now be sufficiently clear, that Newton confounded the *ultimate equality* of the inscribed and circumscribed figures, to the intermediate one, with their actual coincidence, merely from deducing their Ratios on principles of approximation or rather of Exhaustion, instead of those, as explained in No. 6; which relate to the *homogeneity* of the quantities. In the above example the boundaries being *heterogeneous* inasmuch as they are *incommensurable*, cannot be compared as to magnitude, and unless lines are absolutely equal, it is not easy to believe in their coincidence.

Profound as our veneration is, and ought to be, for the Great Father of Mathematical Science, we must occasionally perhaps find fault with his obscurities. But it shall be done with great caution, and only with the view of removing them, in order to render accessible to students in general, the comprehension of "This greatest monument of human genius."

20. *Cor.* 2. 3. and 4. will be explained under Lemma VII, which relates to the Limits of the Ratios of the chord, tangent and the arc.

LEMMA IV.

21. Let the areas of the parallelograms inscribed in the two figures be denoted by

P, Q, R, &c.

p, q, r, &c.

respectively; and let them be such that

$$P : p :: Q : q :: R : r, \text{ \&c.} :: m : n.$$

Then by compounding these equal ratios, we get

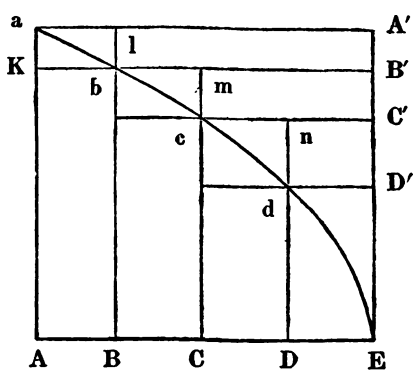
$$P + Q + R + \ldots . : p + q + r + \ldots . :: m : n$$

But $P + Q + R \ldots .$ and $p + q + r + \ldots .$ have with the curvilinear areas an ultimate ratio of equality. Consequently these curvilinear areas are in the given ratio of m : n.

Hence may be found the areas of certain curves, by comparing their incremental rectangles with those of a known area.

Ex. 1. *Required the area of the common Apollonian parabola comprised between its vertex and a given ordinate.*

Let a c E be the parabola, whose vertex is E, axis E A and *Latus-Rectum* = a. Then A A′ being its circumscribing rectangle, let any number of rectangles vertically opposite to one another be inscribed in the areas a E A, a E A′, viz. A b, b A′; B c, c B′, &c.



And since

$$A\,b = A\,K.\,A\,B$$

$$A'\,b = A'\,l.\,A'\,B' = \frac{A\,K^2}{a}.\,A'\,B'$$

from the equation to the parabola.

$$\therefore \frac{A\,b}{A'\,b} = \frac{a.\,A\,B}{A\,K.\,A'\,B'}$$

Also

$$(A\,a)^2 - B\,b^2 = a \times A\,E - a \times B\,E = a \times A\,B$$

or

$$(A\,a + B\,b) \times A'\,B' = a \times A\,B$$

$$\therefore \frac{a \times AB}{A'B'} = Aa + Bb$$

$$\therefore \frac{Ab}{A'b} = \frac{Aa + Bb}{Bb} = \frac{2Bb + Ka}{Bb} = 2 + \frac{Ka}{Bb}$$

Hence, since in the Limit $\frac{Ab}{A'b}$ becomes fixed or of the same nature with the first term, we have

$$\frac{Ab}{A'b} = 2$$

ultimately.

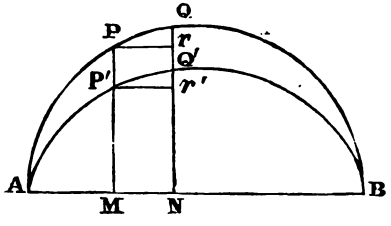
And the same may be shown of all other corresponding pairs of rectangles; consequently by Lemma IV.

$$aEA : aEA' :: 2 : 1$$

$$\therefore aEA : \text{rectangle } AA' :: 2 : 3.$$

or the area of a parabola is equal to two thirds of its circumscribing rectangle.

Ex. 2. *To compare the area of a semiellipse with that of a semicircle described on the same diameter.*



Taking any two corresponding inscribed rectangles P N, P′ N; we have

$$PN : P'N :: PM : P'M :: a : b$$

a and b being the semiaxes major and minor of the ellipse; and all other corresponding pairs of inscribed rectangles have the same constant ratio; consequently by Lemma IV, *the semicircle has to the semiellipse the ratio of the major to the minor axis.*

As another example, the student may compare the area of a cycloid with that of its circumscribing rectangle, in a manner very similar to Ex. 1.

This method of squaring curves is very limited in its application. In the progress of our remarks upon this section, we shall have to exhibit a general way of attaining that object.

LEMMA V.

22. For the definition of similar rectilinear figures, and the truth of this LEMMA as it applies to them, see Euclid's Elements B. VI, Prop. 4, 19 and 20.

The farther consideration of this LEMMA must be deferred to the explanation of LEMMA VII.

LEMMA VI.

23. In the demonstration of this LEMMA, "*Continued Curvature*" at any point, is tacitly defined to be *such, that the arc does not make with the tangent at that point, an angle equal to a finite rectilinear angle.*

In a Commentary on this LEMMA if the demonstration be admitted, any other definition than this is plainly inadmissible, and yet several of the Annotators have stretched their ingenuity to substitute notions of continued curvature, wholly inconsistent with the above. The fact is, this LEMMA is so exceedingly obscure, that it is difficult to make any thing of it. In the enunciation, Newton speaks of the *angle between the chord and tangent ultimately vanishing*, and in the demonstration, it is the angle between the *arc and tangent* that must vanish ultimately. So that in the Limit, it would seem, the *arc and chord* actually coincide. This has not yet been established. In LEMMA III, *Cor.* 2, the cointidence ultimately of a chord and its arc is implied; but this conclusion by no means follows from the LEMMA itself, as may easily be gathered from No. 19. The very thing to be proved by aid of this LEMMA is, that the Ultimate Ratio of the chord to the arc is a ratio of equality, it being merely subsidiary to LEMMA VII. But if it be already considered that they *coincide*, of course they are equal, and LEMMA VII becomes nothing less than "*argumentum in circulo.*"

Newton introduces the idea of curves of "*continued curvature,*" *or such as make no angle with the tangent,* to intimate that this LEMMA does not apply to curves of *non-continued curvature*, or *to such as do make a finite angle with the tangent.* At least this is the plain meaning of his words. But it may be asked, are there any curves whose tangents are inclined to them? The question can only be resolved, by again admitting

the arc to be ultimately coincident with the chord; and by then showing, that curves may be imagined whose chord and tangent ultimately shall be inclined at a finite angle. The Ellipse, for instance, whose minor axis is *indefinitely less* than its major axis, is a curve of that kind; for taking the tangent at the vertex, and putting a, b, for the semiaxes, and y, x, for the ordinate and abscissa, we have

$$y^2 = \frac{b^2}{a^2} \times (2ax - x^2)$$

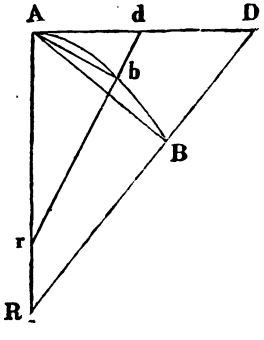
and

$$\frac{y}{x} = \frac{b}{a}\sqrt{\frac{2a}{x} \times 1} = \frac{b}{a\sqrt{x}}\sqrt{2a - x}$$

$\therefore$ since b is indefinitely smaller than $a\sqrt{x}$, x is indefinitely greater than y, and supposing y to be the tangent cut off by the secant x parallel to the axis, x and y are sides of a right angled $\triangle$, whose hypothenuse is the chord. Hence it is plain the $\angle$ opposite x is ultimately indefinitely greater than the $\angle$ opposite to y. But they are together equal to a right angle. Consequently the angle opposite x, or that between the chord and tangent, is ultimately finite. Other cases might be adduced, but enough has been said upon what it appears impossible to explain and establish as logical and direct demonstration. We confess our inability to do this, and feel pretty confident the critics will not accomplish it.

24. Having exposed the fallacy of Newton's reasoning in the proof of this Lemma, we shall now attempt something by way of substitute.

Let A D *be the tangent to the curve at the point* A, *and* A B *its chord. Then if* B *be supposed to move indefinitely near to* A, *the angle* B A D *shall indefinitely decrease, provided the curvature be not indefinitely great.*



Draw R D passing through B at right angles to A B, and meeting the tangent A D and normal A R in the points D and R respectively. Then since the angle B A D equals the angle A R B, if A R B decrease indefinitely when B approaches A; that is, if A R become indefinitely greater than A B; or which is the same thing, if the curvature at A, be not indefinitely great; the angle B A D also decreases indefinitely. Q. e. d.

We have already explained, by an example in the last article, what is

meant by curvature indefinitely great. It is the same with Newton's expression "continued curvature." The subject will be discussed at length under LEMMA XI.

As vanishing quantities are objectionable on account of their nothingness as it has already been hinted, and it being sufficient to consider variable quantities, to get their limiting ratios, *as capable of indefinite diminution*, the above enunciation has been somewhat modified to suit those views.

LEMMA VII.

25. This LEMMA, *supposing the two preceding ones to have been fully established*, would have been a masterpiece of ingenuity and elegance. By the aid of the proportionality of the homologous sides of similar curves, our author has exhibited quantities evanescent by others of any finite magnitude whatever, apparently a most ingenious device, and calculated to obviate all objections. But in the course of our remarks, it will be shown that LEMMA V cannot be demonstrated without the aid of this LEMMA.

First, by supposing A d, A b always finite, the angles at d and b and therefore those at D and B which are equal to the former are virtually considered finite, or R D cuts the chord and tangent at finite angles.

Hence the elaborate note upon this subject of Le Seur and Jacquier is rendered valueless as a direct comment.

Secondly. In the construction of the figure in this LEMMA, the description of a figure *similar* to any given one, is taken for granted. But the student would perhaps like to know how this can be effected.

LEMMA V, which is only enunciated, from being supposed to be a mere corollary to LEMMA III and LEMMA IV, would afford the means immediately, were it thence legitimately deduced. But we have clearly shown (Art. 19.) that rectilinear boundaries, consisting of lines cutting the intermediate curve ultimately *at finite angles*, cannot be equal ultimately to the curvilinear one, and thence we show that the boundaries formed by the chords or tangents, as stated in LEMMA III, *Cor.* 2 and 3, are not ultimately equal, by *consequence of that LEMMA*, to the curvilinear one.

Newton in *Cor.* 1, LEMMA III, asserts the ultimate coincidence, and therefore equality of the rectilinear boundary whose component lines cut the curve at finite angles, and thence would establish the succeeding cor-

ollaries *a fortiori*. But the truth is that the curvilinear boundary is the limit, as to magnitude, or length, of the tangential and chordal boundaries; although in the other case, it is a limit merely in respect of area. Yet, we repeat it, that LEMMA V cannot be made to follow from the LEMMAS preceding it. According to Newton's *implied* definition of similar curves, as explained in the note of Le Seur and Jacquier, *they are the curvilinear limits of similar rectilinear figures.* So they might be considered, if it were already demonstrated that the limiting ratio of the chord and arc is a ratio of equality; but this belongs to LEMMA VII. Newton himself and all the commentators whom we have perused, have thus committed a solecism. Even the best Cambridge MSS. and we have seen many belonging to the most celebrated private as well as college tutors in that learned university, have the same error. Nay most of them are still more inconsistent. They give definitions of similar curves wholly different from Newton's notion of them, and yet endeavour to prove LEMMA V, by aid of LEMMA VII. For the verification of these assertions, which may else appear presumptuously gratuitous, let the Cantabs peruse their MSS. The origin of all this may be traced to the falsely deduced *ultimate coincidence* of the curvilinear and rectilinear boundaries, in the corollaries of LEMMA III. See Art. 19.

We now give a demonstration of the LEMMA without the assistance of similar curves, and yet independently of quantities actually evanescent.

By hypothesis the secant R D cuts the chord and tangent at finite angles. Hence, since

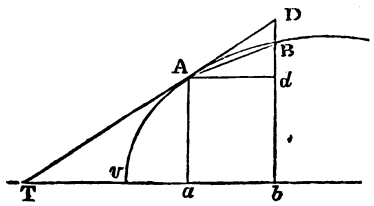
$$\begin{aligned} A + B + D &= 180^\circ \\ \therefore B + D &= 180^\circ - A \\ \text{or } L + l + L' + l' &= 180^\circ - A \end{aligned}$$

L and L′ being the limits of B and D and l, l′ their variable parts as in Art. 6; and since by LEMMA VI, or rather by Art. 24, A is indefinitely diminutive, we have, by collecting homogeneous quantities

$$L + L' = 180^\circ$$

But A B, A D being ultimately *not indefinitely great*, it might easily be shown from Euclid that L = L′, and ∴ A B = A D ultimately, (see Art. 6) and the intermediate arc is equal to either of them.

OTHERWISE,



If we refer the curve to its axis, A a, B b being ordinates, &c. as in the annexed diagram. Then, by Euclid, we have

$$AD^2 = AB^2 + BD^2 + 2\,B\,D.\,B\,d$$

$$\therefore \frac{AD^2}{AB^2} = 1 + BD.\frac{BD + 2\,Bd}{AB^2}$$

Now, since by Art. 24 or LEMMA VI, the $\angle$ B A D is indefinitely less than either of the angles B or D, $\therefore$ B D is indefinite compared with A B or A D. Hence L being the limit of $\frac{AD}{AB}$ and l its variable part, if we extract the root of both sides of the equation and compare *homogeneous* terms, we get,

$$L = 1 \text{ or \&c. \&c.}$$

26. Having thus demonstrated that *the limiting Ratio of the chord, arc and tangent, is a ratio of equality, when the secant cuts the chord and tangent at FINITE angles*, we must again digress from the main object of this work, to take up the subject of Article 17. By thus deriving the limits of the ratios of the finite differences of functions and their variables, directly from the LEMMAS of this Section, and giving to such limits a convenient algorithm or notation, we shall not only clear up the doctrine of limits by numerous examples, but also prepare the way for understanding the abstruser parts of the Principia. This has been before observed.

Required to find the Limit of the Finite Differences of the sine of a circular arc and of the arc itself, or the Ratio of their Differentials.

Let x be the arc, and Δ x its finite variable increment. Then L being the limit required and L + l the variable ratio, we have

$$L + l = \frac{\Delta \sin. x}{\Delta x} = \frac{\sin.(x + \Delta x) - \sin. x}{\Delta x}$$

$$= \frac{\sin. x.\ \cos.(\Delta x) + \cos. x.\ \sin.(\Delta x) - \sin. x}{\Delta x}$$

$$= \cos. x.\frac{\sin.(\Delta x)}{\Delta x} + \frac{\sin. x.\ \cos. \Delta x}{\Delta x} - \frac{\sin. x}{\Delta x}$$

Now by LEMMA VII, as demonstrated in the preceding Article, the limit of $\frac{\sin. \Delta x}{\Delta x}$ is 1, and $\frac{\cos.(\Delta x)}{\Delta x}, -\frac{\sin. x}{\Delta x}$ have no *definite* limits.

Consequently putting

$$\cos. x. \frac{\sin. (\triangle x)}{\triangle x} = \cos. x + l',$$

we have

$$L + l = \cos. x + l' + \frac{\sin. x. \cos. \triangle x}{\triangle x} - \frac{\sin. x}{\triangle x}$$

and equating homogeneous terms

$$L = \cos. x$$

or adopting the differential symbols

$$\left.\begin{array}{l} \dfrac{d. \sin. x}{d x} = \cos x \\ \text{or} \\ d \sin. x = d x. \cos. x \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (a)$$

27. Hence and from the rules for the differentiation of algebraic, exponential, &c. functions, we can differentiate all other circular functions of one variable, viz. cosines, tangents, cotangents, secants, &c.

Thus,

$$\frac{d \sin. \left(\frac{\pi}{2} - x\right)}{d. \left(\frac{\pi}{2} - x\right)} = \cos. \left(\frac{\pi}{2} - x\right) = \sin. x$$

or

$$\frac{d. \cos. x}{-d x} = \sin. x$$

or

$$\left.\begin{array}{l} \dfrac{d. \cos. x}{d x} = -\sin. x \\ \text{or} \\ d. \cos. x = -d x. \sin. x \end{array}\right\} \quad \ldots\ldots \quad (b)$$

Again, since for radius 1, which is generally used as being the most simple,

$$1 + \tan.^2 x = \sec.^2 x = \frac{1}{\cos.^2 x}$$

$$\therefore 2 \tan. x. d. \tan. x = d. \frac{1}{\cos.^2 x} = \frac{-2 \cos. x. d. \cos. x}{\cos.^4 x}$$

See 12 (d). Hence and from (b) immediately above, we have

$$\tan. x. d. \tan. x = \frac{d x. \sin. x}{\cos.^3 x}$$

$$\therefore d. \tan. x = d x. \frac{1}{\cos.^2 x} \quad \ldots\ldots \quad (c)$$

Again,

$$\cot. x = \frac{1}{\tan. x}$$

Therefore,

$$\text{d. cot. } x = \text{d.} \frac{1}{\text{tan. } x} = \frac{-\text{ d. tan. } x}{\text{tan.}^2 x} \quad (12.\ \text{d})$$

$$= \frac{-\text{ d } x}{\text{tan.}^2 x. \text{ cos.}^2 x} = \frac{-\text{ d } x}{\text{sin.}^2 x} \quad \ldots\ldots \quad (\text{d})$$

Again,

$$\text{sec. } x = \frac{1}{\text{cos. } x}$$

$$\therefore \text{ d. sec. } x = \text{d.} \frac{1}{\text{cos. } x} = \frac{-\text{ d cos. } x}{\text{cos.}^2 x} \quad (12.\ \text{d})$$

$$= \frac{\text{d } x. \text{ sin. } x}{\text{cos.}^2 x} \quad \ldots\ldots\ldots \quad (\text{e})$$

and lastly since cosec. x = sec. $\left(\frac{\pi}{2} - x\right)$

we have

$$\text{d. cosec. } x = \text{d. sec.} \left(\frac{\pi}{2} - x\right) = \frac{\text{d.} \left(\frac{\pi}{2} - x\right) \text{ sin.} \left(\frac{\pi}{2} - x\right)}{\text{cos.}^2 \left(\frac{\pi}{2} - x\right)}$$

$$= \frac{-\text{ d } x. \text{ cos. } x}{\text{sin.}^2 x} \quad \ldots\ldots\ldots \quad (\text{f})$$

Any function of sines, cosines, &c. may hence be differentiated.

28. In articles 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 26 and 27, are to be found forms for the differentiation of any function of one variable, whether it be algebraic, exponential, logarithmic, or circular.

In those Articles we have found in short, *the limit of the* ratio of the first difference of a function, and of the first difference of its variable. Now suppose in this first difference of the function, the variable x should be increased again by Δx, then taking the difference between the first difference and what it becomes when x is thus increased, we have the difference of the first difference of a function, or the second difference of a function, and so on through all the orders of differences, making Δx always the same, merely for the sake of simplicity. Thus,

$$\Delta (x^3) = (x + \Delta x)^3 - x^3$$

$$= 3 x^2 \Delta x + 3 x \Delta x^2 + \Delta x^3$$

and $\Delta^2 (x)^3 = 3 (x + \Delta x)^2 \Delta x + 3 (x + \Delta x) \Delta x^2 + \Delta x^3 - 3 x^2 \Delta x$

$$- 3 x \Delta x^2 - \Delta x^3$$

$$= 3.\ 2 x \Delta x^2 + 3 \Delta x^3$$

denoting by Δ^2 the second difference.

Hence,

$$\frac{\Delta^2 (x^3)}{\Delta x^2} = 3.\ 2.\ x + 3\,\Delta x$$

and if the limiting ratio of $\Delta^2 (x^3)$ and Δx^2, or the ratio of the second differential of x^3, and the square of the differential of its variable x, be required, we should have

$$L + l = 3.\ 2.\ x + 3\,\Delta x$$

and equating homogeneous terms

$$\frac{d^2 (x^3)}{d x^2} = L = 3.\ 2.\ x$$

In a word, without considering the difference, we may obtain the second, third, &c. differentials $d^2 u$, $d^3 u$, &c. of any function u of x immediately, if we observe that $\frac{d u}{d x}$ is always a function itself of x, and make d x constant. For example, let

$$u = a x^n + b x^m + \&c.$$

Then, from Art. 13. we have

$$\frac{d u}{d x} = n a x^{n-1} + m b x^{m-1} + \&c.$$

$$\frac{d.\left(\frac{d u}{d x}\right)}{d x} = \frac{d (d u)}{d x^2} = \frac{d^2 u}{d x^2} \text{ (by notation)}$$

$$= n.\ (n - 1)\, a x^{n-2} + m\,(m - 1)\, b x^{m-2} + \&c.$$

Similarly,

$$\frac{d^3 u}{d x^3} = n.\ (n - 1).\ (n - 2)\ a x^{n-3} + \&c.$$

$$\&c. = \&c.$$

Having thus explained the method of ascertaining the limits of the ratios of all orders of finite differences of a function, and the corresponding powers of the invariable first difference of the variable, or the ratios of the differentials of all orders of a function, and of the corresponding power of the first differential of its variable, we proceed to explain the use of these limiting ratios, or ratios of differentials, by the following

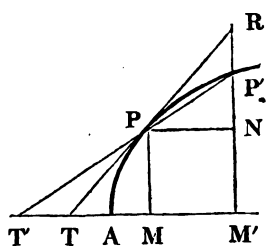
APPLICATIONS

OF THE

DIFFERENTIAL CALCULUS.

29. *Let it be required to draw a tangent to a given curve at any given point of it.*

Let P be the given point, and A M′ being the axis of the curve, let P M = y, A M = x, be the ordinate and abscissa. Also let P′ be any other point; draw P N meeting the ordinate P′ M′ in N, and join P P′. Now let T P R meeting M′ P′ and M A in R and T be the tangent required.



Then since by similar triangles

$$P'N : PN :: PM : MT'$$

$$\therefore MT' = MT + TT' = y.\frac{\Delta x}{\Delta y}$$

Now y being supposed, as it always is in curves, a function of x, we have seen that whether that function be algebraic, exponential, &c. $\frac{\Delta x}{\Delta y}$ in the limit, or $\frac{dx}{dy}$ is always a *definite function* of x. Hence putting

$$\frac{\Delta x}{\Delta y} = \frac{dx}{dy} + 1$$

we have

$$MT + TT' = y\left(\frac{dx}{dy} + 1\right)$$

and equating *homogeneous* terms,

$$MT = \frac{y\,dx}{dy} \quad \ldots\ldots\ldots \text{(e)}$$

which being found from the equation to the curve, the point T will be known, and therefore the position of the tangent P T. M T is called the subtangent.

Ex. 1. In the common parabola,

$$y^2 = ax$$

Therefore,

$$\frac{dx}{dy} = \frac{2y}{a}$$

and

$$MT = \frac{2y^2}{a} = 2x$$

or the subtangent M T is equal to twice the abscissa.

Ex. 2. In the ellipse,

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

and it will be found by differentiating, &c. that

$$MT = \frac{-(a^2 - x^2)}{x}$$

Ex. 3. In the logarithmic curve,

$$y = a^x$$

$$\therefore \frac{dy}{dx} = l.\ a \times y \qquad \text{(see 17.)}$$

$$\therefore MT = \frac{1}{l\,a}$$

which is therefore the same for all points.

The above method of deducing the expression for the subtangent is strictly logical, and obviates at once the objections of Bishop Berkeley relative to the compensation of errors in the denominator. The fact is, these supposed errors being different in their very essence or nature from the other quantities with which they are connected, must in their aggregate be equal to nothing, as it has been shown in Art. 6. This ingenious critic calls P′ R = z; then, says he, (see fig. above)

$$MT = \frac{y.\,dx}{dy + z} \text{ accurately;}$$

whereas it ought to have been

$$MT = \frac{y\,\Delta x}{\Delta y + z} = \frac{y}{\frac{\Delta y}{\Delta x} + \frac{z}{\Delta x}}$$

the finite differences being here considered. Now in the limit, $\frac{\Delta y}{\Delta x}$ becomes a definite function of x represented by $\frac{dy}{dx}$. Consequently if l be put for the variable part of $\frac{\Delta y}{\Delta x}$, we have

$$M\,T = \frac{y}{\frac{d\,y}{d\,x} + 1 + \frac{z}{\Delta\,x}}$$

and it is evident from LEMMA VII and Art. 25, that z is indefinite compared with $\Delta\,x$. $\therefore \frac{z}{\Delta\,x}$ is indefinite compared with M T, $\frac{d\,y}{d\,x}$, and y; and 1 is also so; hence

$$M\,T.\,\frac{d\,y}{d\,x} + \left(1 + \frac{z}{\Delta\,x}\right) M\,T = y$$

gives

$$M\,T = \frac{y.\,d\,x}{d\,y}, \text{ and } 1 + \frac{z}{\Delta\,x} = 0$$

which proves generally for all curves, what Berkeley established in the case of the common parabola; and at the same time demonstrates, as had been already done by using T T′ instead of P′ R, incontestably the accuracy of the equation for the subtangent.

30. If it were required to draw a tangent to any point of a curve, referred to a center by a *radius-vector* ϱ and the $\angle\,\theta$ which ϱ describes by revolving round the fixed point, instead of the rectangular coordinates x, y; then the mode of getting the subtangent will be somewhat different.

Supposing x to originate in this center, it is plain that

$$\left.\begin{aligned} x &= \varrho\ \cos.\,\theta \\ y &= \varrho\ \sin.\,\theta \end{aligned}\right\}$$

and substituting for x, y, d x, d y, hence derived in the expression (29. e.) we have

$$M\,T = \varrho\,\sin.\,\theta \times \frac{d\,\varrho\,\cos.\,\theta - \varrho\,d\,\theta\,\sin.\,\theta}{d\,\varrho\,\sin.\,\theta + \varrho\,d\,\theta\,\cos.\,\theta} \quad \ldots \quad (f)$$

Ex. In the parabola

$$\varrho = \frac{2\,a}{1 - \cos.\,\theta},$$

where a is the distance between the focus and vertex, or the value of ϱ at the vertex. Then substituting we get, after proper reductions

$$M\,T = 2\,a \times \frac{1 + \cos.\,\theta}{1 - \cos.\,\theta}$$

and the distance from the focus to the extremity of the subtangent is

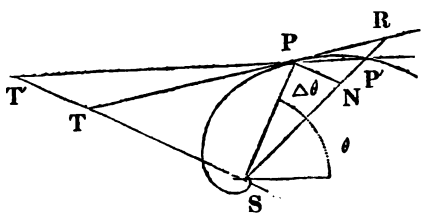
$$M\,T - \varrho\,\cos.\,\theta = 2\,a\left(\frac{1 + \cos.\,\theta}{1 - \cos.\,\theta} - \frac{\cos.\,\theta}{1 - \cos.\,\theta}\right)$$

$$= \frac{2a}{1 - \cos.\theta} = \rho,$$

as is well known.

30. a. The expression (f) being too complicated in practice, the following one may be substituted for it.

Let P T be a tangent to the curve, referred to the center S, at the point P, meeting S T drawn at right angles to S P, in T; and let P′ be any other point. Join P P′ and produce it to T′, and let T P be produced to meet S P′ produced in R, &c. Then drawing P N parallel to S T, we have



$$S T' = S T + T T' = \frac{P N}{N P'} \times S P'$$

But

$$P N = \rho \tan. \Delta\theta, \quad S P' = \rho + \Delta\rho$$

and

$$N P' = S P' - S N = \rho + \Delta\rho - \frac{\rho}{\cos.(\Delta\theta)}.$$

Therefore, substituting and equating homogeneous terms, after having applied Lemma VII to ascertain their limits, we get

$$S T = \frac{\rho^2 d\theta}{d\rho} \quad \ldots\ldots\ldots \quad (g)$$

Ex. 1. In the spiral of Archimedes we have

$$\rho = a\theta;$$

$$\therefore S T = \frac{\rho^2}{a}.$$

Ex. 2. In the hyperbolic spiral

$$\rho = \frac{a}{\theta};$$

$$\therefore S T = -a$$

31. It is sometimes useful to know the angle between the tangent and axis.

$$\text{Tan. } T = \frac{P M}{M T} = \frac{dy}{dx} \quad \ldots\ldots\ldots \quad (h)$$

See fig. to Art. 29.

Again, in fig. Art. 30 a.

$$\text{Tan. } T = \frac{S\,P}{S\,T} = \frac{d\,\varrho}{\varrho\,d\,\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (k)$$

32. It is frequently of great use, in the theory of curves and in many other collateral subjects, to be able to expand or develope any given function of a variable into an infinite series, proceeding according to the powers of that variable. We have already seen one use of such developments in Art. 17. This may be effected in a general manner by aid of successive differentiations, as follows.

If $u = f(x)$ where $f(x)$ means any function of x, or any expression involving x and constants; then, as it has been seen,

$$d\,u = u'\,d\,x$$

(u' being a new function of x)

Similarly

$$\begin{aligned} d\,u' &= u''\,d\,x \\ d\,u'' &= u'''\,d\,x \\ \&c. &= \&c. \end{aligned}$$

But

$$d\,u' = d.\left(\frac{d\,u}{d\,x}\right) = \frac{d^2u \times d\,x - d^2x \times d\,u}{d\,x^2} \quad (6\,k)$$

$$\&c. = \&c.$$

denoting d. (d u), d. (d x) by d^2u, d^2x, and $(d\,x)^2$ by $d\,x^2$, according to the received notation;

Or, (to abridge these expressions) supposing $d\,x$ constant, and $\therefore d^2x = 0$,

$$d\,u' = \frac{d^2u}{d\,x}$$

$$\left.\begin{aligned} \therefore \frac{d\,u}{d\,x} &= u' \\ \frac{d^2u}{d\,x^2} &= u'' \\ \frac{d^3u}{d\,x^3} &= u''' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$\&c. = \&c.$$

which give the various orders of fluxions required.

Ex. 1. Let $u = x^n$

Then

$$\frac{d\,u}{d\,x} = n\,x^{n-1}$$

$$\frac{d^2u}{d\,x^2} = n.\ (n-1)\,x^{n-2}$$

$$\frac{d^3 u}{d x^3} = n.\ (n - 1).\ (n - 2)\ x^{n-3}$$

$$\&c. = \&c.$$

$$\frac{d^n u}{d x^n} = n.\ (n - 1).\ (n - 2)\ \ldots\ldots\ 3.\ 2.\ 1.$$

Ex. 2. Let $u = A + B x + C x^2 + D x^3 + E x^4 + \&c.$

Then,

$$\frac{d u}{d x} = B + 2 C x + 3 D x^2 + 4 E x^3 + \&c.$$

$$\frac{d^2 u}{d x^2} = 2 C + 2.\ 3 D x + 3.\ 4 E x^2 + \&c.$$

$$\frac{d^3 u}{d x^3} = 2.\ 3 D + 2.\ 3.\ 4 E x + \&c.$$

$$\&c. = \&c.$$

Hence, if u be known, and the coefficients A, B, C, D, &c. be unknown, the latter may be found; for if U, U′, U″, U‴, &c. denote the values of u, $\frac{d u}{d x}, \frac{d^2 u}{d x^2}, \frac{d^3 u}{d x^3}$, &c.

when $x = 0$, then

$$A = U,\ B = U',\ C = \frac{1}{2} \cdot U'',\ D = \frac{1}{2.\ 3.} \cdot U''',\ E = \frac{1}{2.\ 3.\ 4} \cdot U'''',$$

&c. = &c.

and by substitution,

$$u = U + U' x + U'' \frac{x^2}{2} + U''' \frac{x^3}{2.\ 3} + \&c. \quad \ldots\ldots \quad (b)$$

This method of discovering the coefficients is named (after its inventor),

MACLAURIN'S THEOREM.

The uses of this Theorem in the expansion of functions into series are many and obvious.

For instance, let it be required to develope sin. x, or cos. x, or tan. x, or l. (1 + x) into series according to the powers of x. Here

$$u = \text{sin. } x, \text{ or } = \text{cos. } x, \text{ or } = \text{tan. } x, \text{ or } = \text{l. } (1 + x),$$

$$\therefore \frac{d u}{d x} = \text{cos. } x, \text{ or } = -\text{ sin. } x, \text{ or } = \frac{1}{\text{cos.}^2 x}, \text{ or } = \frac{1}{1 + x}$$

$$\frac{d^2 u}{d x^2} = -\text{ sin. } x, \text{ or } = -\text{ cos. } x, \text{ or } = \frac{2 \text{ sin. } x}{\text{cos.}^3 x}, \text{ or } = -\frac{1}{(1 + x)^2}$$

$$\frac{d^3 u}{dx^3} = -\cos. x, \text{ or } = \sin. x, \text{ or } = \frac{2 + 4 \sin.^2 x}{\cos.^4 x}, \text{ or } = \frac{2}{(1+x)^3}$$

&c. = &c.

$$\therefore \begin{aligned} U &= 0, \quad \text{or} = 1, \quad \text{or} = 0, \text{ or} = 0 \\ U' &= 1, \quad \text{or} = 0, \quad \text{or} = 1, \text{ or} = 1 \\ U'' &= 0, \quad \text{or} = -1, \text{ or} = 0, \text{ or} = -1 \\ U''' &= -1, \text{ or} = 0, \quad \text{or} = 2, \text{ or} = 2 \\ &\&c. = \&c. \end{aligned}$$

Hence

$$\sin. x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c.$$

$$\cos. x = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \&c.$$

$$\tan. x = x + \frac{x^3}{3} + \frac{2x^5}{3.5} + \frac{17x^7}{3^2 5.7} + \&c.$$

$$l.(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.$$

Hence may also be derived

TAYLOR'S THEOREM.

For let

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$$

Then

$$\begin{aligned} f(x+h) &= A + B.(x+h) + C.(x+h)^2 + D.(x+h)^3 + \&c. \\ &= A + Bx + Cx^2 + Dx^3 + \&c. \\ &\quad + (B + 2Cx + 3Dx^2)h \\ &\quad + (C + 3Dx + 6Ex^2)h^2 \\ &\quad + (D + 4Ex + 10Fx^2)h^3 \\ &\quad + \&c. \\ &= f(x) + \frac{d.f(x)}{dx}h + \frac{d.^2 f(x)}{dx^2}\cdot\frac{h^2}{2} \\ &\quad + \frac{d^3 f(x)}{dx^3}\cdot\frac{h^3}{2.3} + \&c. \quad \ldots\ldots\ldots \quad (c) \end{aligned}$$

the theorem in question, which is also of use in the expansion of series.

For the extension of these theorems to functions of two or more *variables*, and for the still more effective theorems of Lagrange and Laplace, the reader is referred to the elaborate work of Lacroix. 4to.

Having shown the method of finding the differentials of any quanti-

ties, and moreover, entered in a small degree upon the practical application of such differentials, we shall continue for a short space to explain their farther utility.

33. *To find the* MAXIMA *and* MINIMA *of quantities.*

If a quantity increase to a certain magnitude and then decrease, the state between its increase and decrease is its *maximum.* If it decrease to a certain limit, and then increase, the intermediate state is its *minimum.* Now it is evident that in the change from increasing to decreasing, or *vice versa*, which the quantity undergoes, its differential must have changed signs from positive to negative, or *vice versa*, and therefore (since moreover this change is *continued*) have passed through *zero.* Hence *When a quantity is a* MAXIMUM *or* MINIMUM, *its differential* = 0. . . (a)

Since a quantity may have several different maxima and minima, (as for instance the ordinate of an undulating kind of curve) it is useful to have some means of distinguishing between them.

34. *To distinguish between* MAXIMA *and* MINIMA.

LEMMA. To show that in Taylor's Theorem (32. c.) any one term can be rendered greater than the sum of the succeeding ones, supposing the coefficients of the powers of h to be finite.

Let $Q\,h^{n-1}$ be any term of the theorem, and P the greatest coefficient of the succeeding terms. Then, supposing h less than unity,

$$P\,h^n\,(1 + h + h^2 + \ldots \text{. in infin.}) = P\,h^n \times \frac{1}{1 - h}$$

is greater than the sum (S) of the succeeding terms. But supposing h to decrease *in infin.*

$P\,h.^n \frac{1}{1-h} = P\,h^n$ *ultimately.* Hence *ultimately*

$$P\,h^n > S$$

Now

$$Q\,h^{n-1} : P\,h^n :: Q : P\,h,$$

and since Q and P are finite, and h infinitely small; therefore Q is $>$ P h, Hence $Q\,h^{n-1}$ is $>$ $P\,h^n$, and a fortiori $>$ S.

Having established this point, let

$$u = f(x)$$

be the function whose *maxima* and *minima* are to be determined; also when u = max. or min. let x = a. Then by Taylor's Theorem

$$f(a - h) = f(a) - \frac{du}{da}h + \frac{d^2u}{da^2}\cdot\frac{h^2}{1.2} - \frac{d^3u}{da^3}\cdot\frac{h^3}{2.3} + \&c.$$

and

$$f(a+h) = f(a) + \frac{du}{da}h + \frac{d^2u}{da^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{da^3}\cdot\frac{h^3}{2.3} + \&c.$$

and since by the LEMMA, the sign of each term is the sign of the sum of that and the subsequent terms,

$$\therefore f(a-h) = f(a) - \frac{du}{da}\cdot M$$

$$f(a+h) = f(a) + \frac{du}{da}\cdot N$$

Now since f (a) = max. or min. f (a) is > or < than both f (a — h) and f (a + h), which cannot be unless

$$\frac{du}{da} = 0.$$

Hence

$$\left.\begin{aligned} f(a-h) &= f(a) + \frac{d^2u}{da^2}\cdot M' \\ f(a+h) &= f(a) + \frac{d^2u}{da^2}\cdot N' \end{aligned}\right\}$$

and f (a) is *max.* or *min.* or *neither*, according as f (a) is >, < or = to both f (a — h) and f (a + h), or according as

$\frac{d^2u}{da^2}$ is *negative*, *positive*, or *zero*.....

If it be *zero* as well as $\frac{du}{da}$, we have

$$\left.\begin{aligned} f(a-h) &= f(a) - \frac{d^3u}{da^3}\cdot M'' \\ f(a+h) &= f(a) + \frac{d^3u}{da^3}\cdot N'' \end{aligned}\right\}$$

and f (a) cannot = max. or min. unless

$$\frac{d^3u}{da^3} = 0,$$

which being the case we have

$$\left.\begin{aligned} f(a-h) &= fa + \frac{d^4u}{da^4}\cdot M''' \\ f(a+h) &= fa + \frac{d^4u}{da^4}\cdot N''' \end{aligned}\right\}$$

and as before,

f (a) is *max.* or *min.* or *neither*, according as $\frac{d^4 u}{d a^4}$ is *negative*, *positive*, or *zero*, and so on continually.

Hence the following *criterion.*

If in $u = f(x)$, $\frac{d u}{d x} = 0$, *the resulting value of* x *shall give* $u =$ MAX. or MIN. or NEITHER, *according as* $\frac{d^2 u}{d x^2}$ *is negative, positive,* or *zero.*

If $\frac{d u}{d x} = 0$, $\frac{d^2 u}{d x^2} = 0$, and $\frac{d^3 u}{d x^3} = 0$, *then the resulting value of u shall be a* MAX., MIN. *or* NEITHER *according as* $\frac{d^4 u}{d x^4}$ *is* NEGATIVE, POSITIVE, *or* ZERO ; *and so on continually.*

Ex. 1. *To find the* MAX. *and* MIN. *of the ordinate of a common parabola.*

$$y = \sqrt{a x}$$

$$\therefore \frac{d y}{d x} = \frac{1}{2} \cdot \frac{\sqrt{a}}{\sqrt{x}}$$

which cannot $= 0$, unless $x = \infty$.

Hence the parabola has no maxima or minima ordinates.

Ex. 2. *To find the* MAXIMA *and* MINIMA *of y in the equation*

$$y^2 - 2 a x y + x^2 = b^2.$$

Here

$$\frac{d y}{d x} = \frac{a y - x}{y - a x}, \quad \frac{d^2 y}{d x^2} = \frac{2 a \frac{d y}{d x} - \left(\frac{d y}{d x}\right)^2 - 1}{y - a x}$$

and putting $\frac{d y}{d x} = 0$, we get

$$x = \frac{\pm a b}{\sqrt{(1 - a^2)}}, \quad y = \frac{\pm b}{\sqrt{(1 - a^2)}}, \quad \frac{d}{d x^2} = \frac{\mp 1}{b \sqrt{(1 - a^2)}}$$

which indicate and determine both a maximum and a minimum.

Ex. 3. *To divide* a *in such a manner that the product of the* m^{th} *power of the one part, and the* n^{th} *power of the other shall be a maximum.*

Let x be one part, then $a - x =$ the other, and by the question

$$u = x^m \cdot (a - x)^n = \text{max.}$$

$$\therefore \frac{d u}{d x} = x^{m-1} \cdot (a - x)^{n-1} \times (m a - x. \overline{m + n})$$

and

$$\frac{d^2 u}{d x^2} = x^{m-2}(a - x)^{n-2} \times (\overline{m + n - 1}.\ \overline{m + n}.\ x^2 - \&c.)$$

Put $\frac{d u}{d x} = 0$; then

$$x = 0, \text{ or } x = a, \text{ or } x = \frac{m a}{m + n},$$

the two former of which when *m* and *n* are *even* numbers give *minima*, and the last the required *maximum*.

Ex. 4. Let $u = x^{\frac{1}{x}}$.

Here

$$\frac{d u}{d x} = u.\ \frac{1 - l.\ x}{x^2} = 0, \ \therefore\ l.\ x = 1, \text{ and } x = e \text{ the } \textit{hyperbolic} \text{ base} = 2.71828, \&c.$$

Innumerable other examples occur in researches in the doctrine of curves, optics, astronomy, and in short, every branch of both abstract and applied mathematics. Enough has been said, however, fully to demonstrate the general principle, when applied to functions of *one independent variable only*.

For the MAXIMA and MINIMA of functions of two or more variables, see *Lacroix*, 4to.

35. If in the expression (30 a. g) S T should be finite when ϱ is infinite, then the corresponding tangent is called an *Asymptote* to the curve, and since ϱ and this Asymptote are both infinite they are parallel. Hence

To find the Asymptotes to a curve,

In $S T = \varrho^2 \frac{d \theta}{d \varrho}$, make $\varrho = \alpha$, then each *finite* value of S T gives an Asymptote; which may be drawn, by finding from the equation to the curve the values of θ for $\varrho = \alpha$, (which will determine the positions of ϱ), then by drawing through S at right angles to ϱ, S T, S T′, S T″, &c. the several values of the subtangent of the asymptotes, and finally through T, T′, T″, &c. perpendiculars to S T, S T′, S T″, &c. These perpendiculars will be the asymptotes required.

Ex. In the hyperbola

$$\varrho = \frac{b^2}{a(1 - e \cos. \theta)}.$$

Here $\varrho = \alpha$, gives $1 - e \cos. \theta = 0$, $\therefore \cos. \theta = \frac{1}{e}$

$\therefore \pm \theta = \angle$ whose cos. is $\frac{1}{e}$.

Also $S T = \frac{b^2}{a e \sin. \theta} = \frac{b^2}{a \sqrt{e^2 - 1}} = b$; whence it will be seen that the asymptotes are equally inclined (viz. by $\angle \theta$) to the axis, and pass through the center.

The expression (29. e) will also lead to the discovery and construction of *asymptotes.*

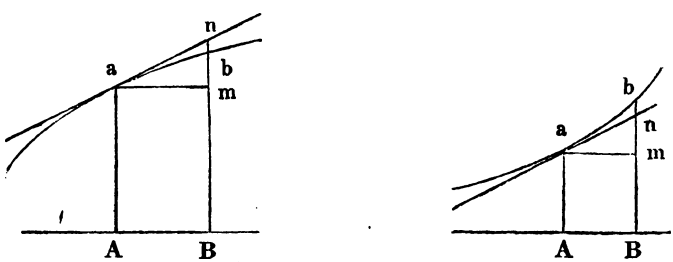
Since the tangent is the nearest straight line that can be drawn to the curve at the point of contact, it affords the means of ascertaining the inclination of the curve to any line given in position; also whether at any point the curve be *inflected,* or from *concave* become *convex* and vice versa; also whether at any point two or more branches of the curve meet, i. e. whether that *point* be *double, triple,* &c.

36. *To find the inclination of a curve at any point of it to a given line; find that of the tangent at that given point, which will be the inclination required.*

Hence if the inclination of the tangent to the axis of a curve be zero, the ordinate will then be a *maximum* or *minimum*; for then

$$\text{tan. } T = \frac{dy}{dx} = 0 \qquad (31. h)$$

37. *To find the points of Inflexion of a curve.*



Let $y = f(x)$ be the equation to the curve a b; then A a, B b being any two ordinates, and a n a tangent at the point a, if we put A a = y, and A B = h, we get

$$A a = f x$$

$$B b = f(x + h) = y + \frac{dy}{dx} h + \frac{d^2 y}{dx^2} \frac{h^2}{1.2} + \&c. \qquad (32. c)$$

$$= y + \frac{dy}{dx} h + \frac{d^2 y}{dx^2} M \qquad (34)$$

But $B n = y + m n = y. + \frac{dy}{dx}. h.$ Consequently B b is $<$ or $>$ B n according as $\frac{d^2 y}{dx^2}$ is negative or positive, i. e. *the curve is concave or con-*

vex towards its axis according as $\frac{d^2y}{dx^2}$ *is negative or positive.*

Hence also, since a quantity in passing from positive to negative, and *vice versa*, must become zero or infinity, *at a point of inflexion*

$$\frac{d^2y}{dx^2} = 0 \text{ or } \alpha$$

Ex. *In the Conchoid of Nicomedes*

$$xy = (a + y)\sqrt{(b^2 - y^2)}$$

which gives, by making d y constant,

$$\frac{d^2x}{dy^2} = \frac{2b^4a - b^2y^3 - 3b^2ay^2}{(b^2y^3 - y^5)\sqrt{(b^2 - y^2)}}$$

and putting this = 0, and reducing, there results

$$y^3 + 3ay^2 = 2b^2a$$

which will give y and then x.

These points of *inflexion* are those which the Theory of (34) indicates as belonging to neither *maxima* nor *minima*; and pursuing this subject still farther, it will be found, in like manner, that in some curves

$$\frac{d^4y}{dx^4} = 0 \text{ or } \alpha, \quad \frac{d^6y}{dx^6} = 0 \text{ or } \alpha, \text{ \&c.} = \text{\&c.}$$

also determine *Points* of *Inflexion.*

38. *To find* DOUBLE, TRIPLE, *&c. points of a curve.*

If the branches of the curve cut one another, there will evidently be as many tangents as branches, and consequently either of the expressions,

$$\text{Tan. T} = \frac{dy}{dx} \qquad (31.\ h)$$

$$\text{M T} = \frac{y\,dx}{dy} \qquad (29.\ e)$$

as derived from the equation of the curve, will have as many values as there are branches, and thus the nature and position of the point will be ascertained.

If the branches of the curve touch, then the tangents coincide, and the method of discovering such *multiple points* becomes too intricate to be introduced in a brief sketch like the present. For the entire Theory of Curves the reader is referred to Cramer's express treatise on that subject, or to Lacroix's Different. and Integ. Calculus, 4to. edit.

39. We once more return to the text, and resume our comments. We pass by LEMMA VIII as containing no difficulty which has not been already explained.

As similar figures and their properties are required for the demonstra-

tion of Lemma IX, we shall now use Lemma VII in establishing Lemma V, and shall thence proceed to show what figures are similar and how to construct them.

According to Newton's notion of similar curvilinear figures, we may define *two curvilinear figures to be similar when any rectilinear polygon being inscribed in one of them, a rectilinear polygon similar to the former, may always be inscribed in the other.*

Hence, increasing the number of the sides of the polygons, and diminishing their lengths indefinitely, the lengths and areas of the curvilinear figures will be the limits by Lemmas VII and III, of those of the rectilinear polygons, and we shall, therefore, have by Euclid these lengths and areas in direct and duplicate proportions of the homologous sides respectively.

40. *To construct curves similar to given ones.*

If y, x be the ordinate and abscissa, and x′ the corresponding abscissa of the required curve, we have

$$x : y :: x' : \frac{y}{x} \times x' = y' \quad . \quad . \quad . \quad . \quad . \quad (a)$$

the ordinate of the required curve, which gives that point in it which corresponds to the point in the given curve whose coordinates are x, y; and in the same manner may as many other points as we please be determined.

In such curves, however, as admit a practical or mechanical construction, it will frequently be sufficient to determine but one or two values of y′.

Ex. 1. In the circle let x, measured along the diameter from its extremity, be r (the radius); then y = r, and we have

$$y' = \frac{y}{x} \times x' = x'$$

where x′ may be of any magnitude whatever. Hence, *all semicircles, and therefore circles, are similar figures.*

Ex. 2. In a circular arc (2 α) let x be measured along the chord (2 b), and suppose $x = r' \sin. \alpha$; then $y = r' . \text{vers. } \alpha$

$$y' = \frac{\text{vers. } \alpha}{\sin. \alpha} \times x'$$

which gives the greatest ordinate to any semichord as an abscissa, of the required arc, and thence since

$$y' = r' - \sqrt{r'^2 - x'^2}$$

it will be easy to find the radius r′ and centre, and to describe the arc required.

But since

$$\frac{y'}{x'} = \frac{r' \text{ vers. } \alpha'}{r' \sin. \alpha'} = \frac{\text{vers. } \alpha'}{\sin. \alpha'} = \frac{\text{vers. } \alpha}{\sin. \alpha}$$

therefore

$$\frac{1 - \cos. \alpha}{\sin. \alpha} = \frac{2 \sin.^2 \frac{\alpha}{2}}{2 \cos. \frac{\alpha}{2} \sin. \frac{\alpha}{2}} = \frac{1 - \cos. \alpha'}{\sin. \alpha'} = \frac{2 \sin.^2 \frac{\alpha'}{2}}{2 \cos. \frac{\alpha'}{2} \sin. \frac{\alpha'}{2}}$$

or

$$\tan. \frac{\alpha}{2} = \tan. \frac{\alpha'}{2},$$

and

$$\therefore \alpha = \alpha'$$

which accords with Euclid, and shows *that similar arcs of circles subtend equal angles.*

Ex. 3. *Given an arc of a parabola, whose latus-rectum is* p, *to find a similar one, whose latus-rectum shall be* p'.

In the first place, since the arc is given, the coordinates at its extremities are; whence may be determined its axis and vertex; and by the usual mode of describing the parabola it may be completed to the vertex. Now, since

$$y^2 = p x$$

x, x' being measured along the axis, and when

$$x = \frac{p}{4}, \; y = \frac{p}{2}$$

$$\therefore y' = \frac{y}{x} . x' = \frac{p}{y} . x' = 2 x'$$

which shows *that all semi-parabolas, and therefore parabolas, are similar figures.* Hence, having described upon the axis of the given parabola, any other having the same vertex, the arc of this latter intercepted between the points whose coordinates correspond to those of the extremities of the given arc will be the arc required.

Ex. 4. In the ellipse whose semi-diameters are a, b, if x be measured along the axis, when x = a, y = b. Hence

$$y' = \frac{b}{a} . x'$$

and x' or the *semi-axis major* being assumed any whatever, this value of y' will give the *semi-axis minor*, whence the ellipse may be described.

This being accomplished, let (α, β) (α', β') be the coordinates at the

extremities of any *given arc* of the given ellipse, then the similar one of the ellipse described will be that intercepted between the points whose coordinates, (x', y') (x'', y'') are given by

$$\left.\begin{array}{l}\alpha : \beta :: x' : y' \\ \alpha' : \beta' :: x'' : y''\end{array}\right\} \text{ and } \begin{array}{l} y' = \frac{b'}{a'}\sqrt{(2\,a'\,x' - x'^2)} \\ y'' = \frac{b''}{a''}\sqrt{(2\,a''\,x'' - x''^2)}\end{array}$$

In like manner it may be found, that

All cycloids are similar.

Epicycloids are so, when the radii of their wheels $\propto$ *radii of the spheres.*

Catenaries are similar when the bases $\propto$ *tensions, &c. &c.*

40. If it were required to describe the curve A c b (fig. to LEMMA VII) not only similar to A C B, but also such that its chord should be of the given length (c); then having found, as in the last example, the coordinates (x', y') (x'', y'') in terms of the assumed value of the abscissa (as a' in Ex. 4), and (α, β), (α', β') the coordinates at the extremities of the given arc, we have

$$c = \sqrt{(x' - x'')^2 + (y' - y'')^2} = f(\alpha')$$

a function of a': whence a' may be found.

Ex. In the case of a parabola whose equation is $y^2 = a\,x$, it will be found that ($y'^2 = a'\,x'$ being the equation of the required parabola)

$$c = \frac{a'}{a^2}\cdot(\beta - \beta')\sqrt{(\beta + \beta')^2 + \alpha^2}$$

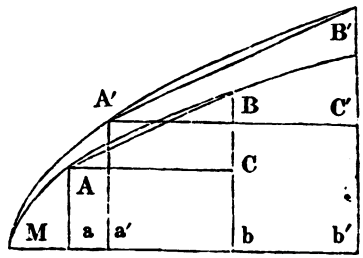
whence (a') is known, or the latus-rectum of the required parabola is so determined, that the arc similar to the given one shall have a chord $= c$.

41. It is also assumed in the construction both to LEMMA VII and LEMMA IX, that, *If in similar figures, originating in the same point, the chords or axes coincide, the tangents at that origin will coincide also.*

Since the chords A B, A b (fig. to LEMMA VII), the *parallel* secants B D, b d, and the tangents A D, A d are corresponding sides, each to each, to the similar figures, we have (by LEMMA V)

$$A\,B : B\,D :: A\,b : b\,d$$

and $\angle B = \angle b$. Consequently, by Euclid the $\angle B\,A\,D = \angle b\,A\,d$, or the tangents coincide.



To make this still clearer. Let M B, M B′ be two similar curves, and A B, A′ B′ similar parts of them. Let fall from A, B, A′, B′, the ordinates A a, B b, A′ a′, B′ b′ cutting off the corresponding abscissæ M a, M b, M a′, M b′, and draw the chords A B, A′ B′; also draw A C, A′ C′ at right angles to B b, B′ C′.

Then, since (by LEMMA V)

$$\left.\begin{array}{l} M\,a : M\,b :: A\,a : B\,b \\ M\,a' : M\,b' :: A'\,a' : B'\,b' \end{array}\right\}$$

$$\therefore \left.\begin{array}{l} M\,a : a\,b :: A\,a : B\,C \\ M\,a' : a'\,b' :: A'\,a' : B'\,C' \end{array}\right\}$$

$$\therefore \left.\begin{array}{l} A\,C : B\,C :: M\,a : A\,a \\ A'\,C' : B'\,C' :: M\,a' : A'\,a' \end{array}\right\}$$

But

$$M\,a : A\,a :: M\,a' : A'\,a'$$

$$\therefore A\,C : B\,C :: A'\,C' : B'\,C'$$

and the $\angle C = \angle C'$

$\therefore$ the triangles A B C, A′ B′ C′ are similar, and the $\angle$ B A C = $\angle$ B′ A′ C′, i. e. A B is parallel to A′ B′.

Hence if B, B′ move up to A, the chords A B, A′ B′ shall *ultimately* be parallel, i. e. the tangents (see LEMMA III, Cor. 2 and 3, or LEMMA VI,) at A, A′ are parallel.

Hence, if the chords coincide, as in fig. to LEMMA VII, the tangents coincide also.

The student is now prepared for the demonstration of the LEMMA. He will perceive that as B approaches A, new curves, or parts of curves, A c b similar to the parts A C B are supposed continually to be described, the point *b* also approaching *d*, which may not only be at a *finite* distance from A, but absolutely fixed. It is also apparent, that as the ratio between A B and A b decreases, the curve A c b approaches to the straight line A b as its limit.

42. LEMMA XI. The construction will be better understood when thus effected.

Take A e of any given magnitude and draw the ordinate e c meeting A C produced in c, and upon A c describe the curve A b c (see 39)

similar to A B C. Take $A\,d = A\,e \times \frac{A\,D}{A\,E}$ and erect the ordinate d b meeting A b c in b. Then, since A d, A e are the abscissæ corresponding to A D, A E, the ordinates d b, e c also correspond to the ordinates D B, E C, and by Lemma V we have

$$d\,b : D\,B :: e\,c : E\,C :: A\,e : A\,E$$
$$:: A\,d : A\,C \text{ (by construction)}$$

and the $\angle D = \angle d$. Hence

b is in the straight line A B produced, &c. &c.

43. This Lemma may be proved, without the aid of similar curves, as follows:

$$A\,B\,D = \frac{A\,D}{2}.\,(D\,F + F\,B)$$
$$= A\,D^2.\frac{\tan.\,\alpha}{2} + \frac{A\,D.\,B\,F}{2}$$

and

$$A\,C\,E = A\,E^2.\frac{\tan.\,\alpha}{2} + \frac{A\,E.\,C\,G}{2},$$

where $\alpha = \angle D\,A\,F$.

$$\therefore \frac{A\,B\,D}{A\,C\,E} = \frac{A\,D^2.\tan.\,\alpha + A\,D.\,B\,F}{A\,E^2.\tan.\,\alpha + A\,E.\,C\,G}$$

Now by Lemma VII, since $\angle B\,A\,F$ is indefinite compared with F or B; therefore B F, C G are indefinite compared with A D or A E. Hence if L be the limit of $\frac{A\,B\,D}{A\,C\,E}$, and L + l its varying value, we have

$$L + l = \frac{A\,D^2.\tan.\,\alpha + A\,D.\,B\,F}{A\,E^2.\tan.\,\alpha + A\,E.\,C\,G}$$

and multiplying by the denominator and equating homogeneous terms we get

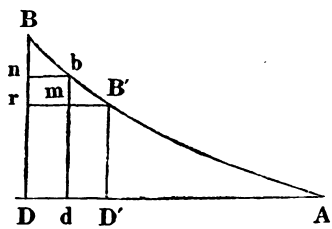
$$L.\,A\,E^2.\tan.\,\alpha = A\,D^2.\tan.\,\alpha$$

$$\therefore \text{Limit of } \frac{A\,B\,D}{A\,C\,E} = \frac{A\,D^2}{A\,E^2}.$$

44. Lemma X. "*Continually* increased or diminished." The word "*continually*" is here introduced for the same reason as "*continued curvature*" in Lemma VI.

If the force, moreover, be not "*finite*," neither will its effects be; or the velocity, space described, and time will not admit of comparison.

45. Let the time A D be divided into several portions, such as D d, A b B being the *locus* of the extremities of the ordinates which D represent, the velocities acquired D B, d b, &c. Then upon these lines D d, &c. as bases, there being inscribed rectangles in the figure A D B, and when their number is increased and bases diminished indefinitely, their ultimate sum shall = the curvilinear area A B D (LEMMA III.) But each of these rectangles represents the space described in the time denoted by its base; for during an *instant* the velocity may be considered constant, and by mechanics we have for constant velocities S = T × V. Hence the area A B D represents the whole space described in the time A D.



In the same manner, A C E (see fig. LEMMA X) represents the time A E. But by LEMMA IX these areas are "ipso motus initio," as $A D^2$ and $A E^2$ Hence, in the very beginning of the motion, the spaces described are also in the duplicate ratio of the times.

46. Hence may be derived the differential expressions for the *space described*, *velocity acquired*, &c.

Let the velocity B D acquired in the time t (A D) be denoted by v, and the space described, by s.

Then, *ultimately*, we have

$$D d = d t, B n = d v,$$

and

$$D n b d = d s = D d \times d b = d t \times v.$$

Hence

$$v = \frac{d s}{d t}, \quad d s = v \, d t, d t = \frac{d s}{v} \quad \ldots\ldots \text{(a)}$$

Again, if D d = d D′, the spaces described in these successive *instants*, are D b, D′ m, and therefore *ultimately* the fluxion of the space represented by the ultimate state of D′ m is b n r m or 2 b m B′. Hence

$$d (d s) = 2 \times b m B' \text{ ultimately,}$$

and supposing B′ to move up to A, since in the limit at A, B′ coincides with A, and B′ m with A D, and therefore b m B′ or d (d s) represents the space described "in the very beginning of the motion."

Hence by the LEMMA,

$$d (d s) \propto 2 d t^2 \propto d t^2$$

or with the same accelerating force

$$d^2 s \propto d t^2 \quad \ldots\ldots\ldots \text{(b)}$$

With different accelerating forces $d^2 s$ must be proportionably increased or diminished, and $\therefore$ (see Wood's Mechanics)

$$d^2 s \propto F d t^2$$

Hence we have, after properly adjusting the units of force, &c.

$$\left.\begin{aligned} d^2 s &= F d t^2 \\ \text{and } \therefore \quad F &= \frac{d^2 s}{d t^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(c)}$$

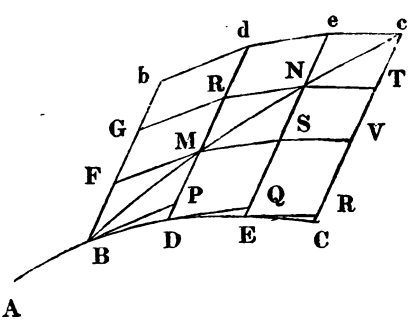
Hence also and by means of (a) considering d t constant,

$$F = \frac{d v}{d t}, \quad v d v = F d s \quad \ldots\ldots \text{(d)}$$

all of which expressions will be of the utmost use in our subsequent comments.

47. Lemma X. Cor. I. To make this corollary intelligible it will be useful to prove the general principle, that

If a body, moving in a curve, be acted upon by any new accelerating force, the distance between the points at which it would arrive WITHOUT *and* WITH *the new force in the same time, or "error," is equal to the space that the new force, acting solely, would cause it to describe in that same time.*



Let a body move in the curve A B C, and when at B, let an additional force act upon it in the direction B b. Also let B D, D E, E C; B F, F G, G b be spaces that would be described in equal times by the body moving in the curve, and when moved by the sole action of the new force. Then draw tangents at the points B, D, E meeting D d, E e, C c, each parallel to B b, in P, Q, R. Also draw F M, G R, b d parallel to B P; M S, R N, d e parallel to D Q; and S V, N T, e c parallel to E R.

Now since the body at B is acted upon by forces which separately would cause it to move through B D, B F, or, when the number of the spaces is increased and their magnitude diminished *in infinitum*, through B P, B F in same time, therefore by LAW III, Cor. 1, when these forces act together, the body will move in that time through the diagonal up to M. In the same manner it may be shown to move from M to N, and from N to C in the succeeding times. Hence, if the number of the times be increased and their duration indefinitely diminished, the body will have moved through an indefinite number of points M, N, &c. up to C, describing a curve B C. Also since b d, d e, e c are each parallel to the tangents at B, D, E, or ultimately to the curve B D E C; ∴ b d e c ultimately assimilates itself to a curve equal and parallel to B D E C; moreover C c is parallel to B b. Hence C c is also equal to B b.

Hence, then, *The Error caused by any disturbing force acting upon a body moving in a curve, is equal to the space that would be described by means of the sole action of that force, and moreover it is parallel to the direction of that force.* Wherefore, if the disturbing force be *constant*, it is easily inferred from LEMMAS X and IX, and indeed is shown in all books on Mechanics, that *the errors are as the squares of the times in which they are generated.* Also, if the disturbing forces be *nearly constant*, then *the errors are as the squares of the times quam proxime.* But these conclusions, the same as those which Note 118 of the Jesuits, Le Seur and Jacquier, (see Glasgow edit. 1822.) leads to, do not prove the assertion of Newton in the corollary under consideration, inasmuch as they are general for all curves, and apply not to *similar* curves in particular.

48. Now let a curve similar to the above be constructed, and completing the figure, let the points corresponding to A, B, &c. be denoted by A′, B′, &c. and let the times in which the similar parts of these curves, viz. B D, B′ D′; D E, D′ E′; E C, E′ C′ are described, be in the ratio $t : t'$. Then the times in which, by the *same disturbing force*, the spaces B F, B′ F′; F G, F′ G′; G b, G′ b′ are described, are in the ratio of $t : t'$. Hence, "in ipso motus initio" (by LEMMA X) we have

$$B\,F : B'\,F' :: t^2 : t'^2$$
$$F\,G : F'\,G' :: t^2 : t'^2$$
$$\&c.\ \&c.$$

and therefore,

$$B\,F + F\,G + \&c. : B'\,F' + F'\,G' + \&c. :: t^2 : t'^2$$

But, (by 15,)

$$B\ F + F\ G + \&c. = \text{the error } C\ c,$$

and

$$B'\ F' + F'\ G' + \&c. = \text{the error } C'\ c',$$

and the times in which B C, B′ C′ are described, are in the ratio t : t′. Hence then

$$C\ c : C'\ c' : : t^2 : t'^2$$

or *The* ERRORS *arising from equal forces, applied at corresponding points, disturbing the motions of bodies in similar curves, which describe similar parts of those curves in proportional times, are as the squares of the times in which they are generated* EXACTLY, *and not "quam proxime."*

Hence Newton appears to have neglected to investigate this corollary. The corollary indeed did not merit any great attention, being limited by several restrictions to very particular cases.

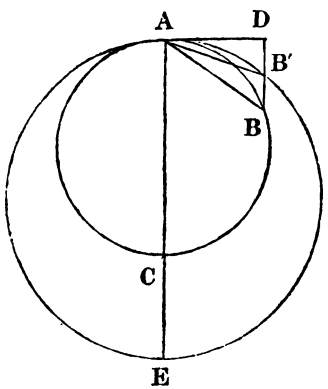
It would seem from this and the last No. that Newton's meaning in the forces being "similarly applied," is merely that they are to be applied at corresponding points, and do not *necessarily* act in directions *similarly* situated with respect to the curves.

For explanation with regard to the other corollaries, see 46.

49. LEMMA XI. "*Finite Curvature.*" Before we can form any precise notion as to the curvature at any point of a curve's being *Finite*, *Infinite* or *Infinitesimal*, some method of measuring curvature in general must be devised. This measure evidently depends on the *ultimate* angle contained by the chord and tangent (A B, A D) or on the *angle of contact*. Now, although this angle can have no finite value when singly considered, yet when two such angles are compared, their ratio may be finite, and if any known curvature be assumed of a standard magnitude, we shall have, by the equality between the ratios of the angles of contact and the curvatures, the curvature at any point in any curve whatever. In practice, however, it is more commodious to compare the subtenses of the angles of contact (which may be considered circular arcs, see LEMMA VII, having radii in a ratio of equality, and therefore are accurate measures of them), than the angles themselves.

50. Ex. 1. Let the circumference of a circle be divided into any number of equal parts and the points of division being joined, let there be a tangent drawn at every such point meeting a perpendicular let fall from the next point; then it may easily be shown that these perpendiculars or subtenses are all equal, and if the number of parts be increased, and their

magnitude diminished, *in infinitum*, they will have a ratio of equality. Hence, *the CIRCLE has the same curvature at every point, or it is a curve of uniform curvature.*



51. Ex. 2. Let two circles touch one another in the point A, having the common tangent A D. Also let B D be perpendicular to A D and cut the circle A D in B'. Join A B, A B'. Then since A B, A B' are ultimately equal to A D (LEMMA VII) they are equal to one another, and consequently the limiting ratio of B D and B' D, is that of the curvatures of the respective circles A C, A D (by 17.)

But, by the nature of the circle,

$$A\,D^2 = 2\,R \times D\,B' - D\,B'^2 = 2\,r \times D\,B - D\,B^2$$

R and r being the radii of the circles.

Therefore

$$L + l = \frac{D\,B}{D\,B'} = \frac{2\,R - D\,B'}{2\,r - D\,B}$$

and equating homogeneous terms we have

$$L = \frac{R}{r},$$

i. e. *The curvatures of circles are inversely as their radii.*

52. Hence, if the curvature of the circle whose radius is 1, (inch, foot, or any other measure,) be denoted by C, that of any other circle whose radius is r, is

$$\frac{C}{r}.$$

53. Hence, if the radius r of a circle compared with 1, be *finite*, its curvature compared with C, is *finite*; if r be *infinite* the curvature is *infinitesimal*; if r be *infinitesimal* the curvature is *infinite*, and so on through all the higher orders of *infinites* and *infinitesimals*. By infinites and infinitesimals are understood quantities indefinitely great or small.

The above sufficiently explains why curvature, compared with a given standard (as C), can be said to be *finite* or *indefinite*. We are yet to show the reason of the restriction to curves of *finite curvature*, in the enunciation of the LEMMA.

54. The circles which pass through A, B, G; a, b, g, (fig. LEMMA XI)

have the same tangent A D with the curve and the same subtenses. Hence (49. and 52.) these circles *ultimately* have the same *curvature* as the curve, i. e. A I is the diameter of that circle which has the same curvature as the curve at A. Hence, according as A I is finite or indefinite, the curvature at A is so likewise, compared with that of circles of finite radius.

Now A G ultimately, or

$$A I = \frac{A B^2}{B D}$$

whether A I be finite or not. If finite, B D $\propto$ A B^2, as we also learn from the text.

55. If the curvature be infinitesimal or A I infinite; then since $\frac{A B^2}{B D}$ is infinite, B D must be infinitely less than A B^2, or, A B being always considered in its ultimate state an infinitesimal of the first order, B D is that of the third order, i. e. B D $\propto$ A B^3. The converse is also true.

Ex. In the cubical parabola, the abscissa $\propto$ as the cube of the ordinate; hence at its vertex the curvature is infinitely small. At other points, however, of this curve, as we shall see hereafter, the curvature is finite.

To show at once the different proportions between the subtenses of the angles of contact and the conterminous arcs, corresponding to the different orders of infinitesimal or infinite curvatures, and to make intelligible this intricate subject, let A B ultimately considered be indefinitely small compared with 1; then since $\frac{A B^2}{A B} = A B$, A B^2 is infinitesimal compared with A B; and generally $\frac{A B^n}{A B^{n-1}} = A B$, shows that A B^n is infinitely small compared with A B^{n-1} so that *the different orders of infinitesimals may be correctly denoted by*

$$A B, A B^2, A B^3, A B^4, \&c.$$

Also since 1 is infinite compared with the infinitesimal A B, and A B compared with A B^2, &c. *the different orders of infinites* may be represented by

$$\frac{1}{A B}, \frac{1}{A B^2}, \frac{1}{A B^3}, \frac{1}{A B^4}, \&c.$$

56. Hence if the curvature at any point of a curve be infinitesimal in the second degree

$$\frac{AB^2}{BD} \propto \frac{1}{AB^2}, \text{ and } BD \propto AB^4, \text{ and conversely.}$$

And generally, if the curvature be infinitesimal in the nth degree,

$$\frac{AB^2}{BD} \propto \frac{1}{AB^n}, \text{ and } BD \propto AB^{n+2}, \text{ and conversely.}$$

Again, if the curvature be infinite in the nth degree,

$$\frac{AB^2}{BD} \propto AB^n, \text{ and } BD \propto AB^{2-n}, \text{ and conversely.}$$

The parabolas of the different orders will afford examples to the above conclusions.

57. The above is sufficient to explain the first case of the LEMMA.

Case 2. presents no difficulty; for b d, B D being inclined at any equal angles to A D, they will be parallel and form, with the perpendiculars let fall from b, B upon A D, similar triangles, whose sides being proportional, the ratio between B D, b d will be the same as in Case 1.

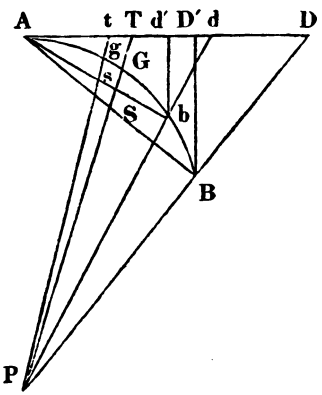
Case 3. If B D converge, i. e. pass through when produced to a given point, b d will also, and ultimately when d and D move up to A, the difference between the angles A d b, A D B will be less than any that can be assigned, i. e. B D and b d will be ultimately parallel; which reduces this case to Case 2. (See Note 125. of PP. Le Seur and Jacquier.)

Instead of passing through a given point, B D, b d may be supposed to touch perpetually any given curve, as a circle for instance, and B D will still $\propto AD^2$; for the angles D, d are ultimately equal, inasmuch as from the same point A there can evidently be but one line drawn touching the circle or curve.

Many other laws determining B D might be devised, but the above will be sufficient to illustrate Newton's expression, "or let B D be determined by any other law whatever." It may, however, be farther observed that this law must be definite or such as will *fix* B D. For instance, the LEMMA would not be true if this law were that B D should *cut* instead of *touch* the given circle.

58. LEMMA XI. COR. II. It may be thus explained. Let P be the given point towards which the sagittæ S G, s g, bisecting the chords A B, A b, converge. S G, s g shall ultimately be as the squares of A B, A b, &c.

For join P B, P b and produce them, as also P G, P g, to meet the tangent in D, d, T, t. Then if B and b move up to A, the angles T P D, t P d, or the differences between the angles A T P and A D P, and between A t P and A d P, may be diminished without limit; that is, (Lemma I), the angles at T, D and at t, d are ultimately equal. Hence the triangles A T S, A D B are similar, as likewise are A t s, A d b.



Consequently

$$S\,T : D\,B : : A\,S : A\,B : : 1 : 2$$

and

$$s\,t : d\,b : : A\,s : A\,b : : 1 : 2$$

and

$$\therefore S\,T : s\,t : : D\,B : d\,b$$

Also by Lemma VII,

$$S\,T : s\,t : : S\,G : s\,g$$

and by Lemma XI, Case 3,

$$D\,B : d\,b : : A\,B^2 : A\,b^2$$

$$\therefore S\,G : s\,g : : A\,B^2 : A\,b^2$$

Q. e. d.

Moreover, it hence appears, that *the sagittæ which cut the chords, in ANY GIVEN RATIO WHATEVER, and tend to a given point, have ultimately the same ratio as the subtenses of the angles of contact, and are as the squares of the corresponding arcs, chords, or tangents.*

59. Lemma XI. Cor. III. If the velocity of a body be constant or "given," the space described is proportional to the time t. Hence $A\,B \propto t$, and $\therefore S\,G \propto A\,B^2 \propto t^2$.

60. Lemma XI. Cor. IV. Supposing B D, b d at right angles to A D (and they have the same proportion when inclined at a given angle to A D, and also when tending to a given point, &c.) we have

$$\triangle A D B : \triangle A d b :: \frac{A D \times D B}{2} : \frac{A d \times d b}{2}$$

$$:: \frac{D B}{d b} \times A D : A d$$

$$:: \frac{A D^2}{A d^2} \times A D : A d$$

$$:: A D^3 : A d^3.$$

Also

$$\triangle A D B : \triangle A d b :: \frac{A D}{A d} \times D B : d b$$

$$:: \frac{\surd D B}{\surd d b} \times D B : d b$$

$$:: (D B)^{\frac{3}{2}} : (d b)^{\frac{3}{2}}.$$

It may be observed here, that the tyro, on reverting to LEMMA IX, usually infers from it that

$$\triangle A D B \propto A D^2 \text{ and does not } \propto A D^3,$$

but then he does not consider that A D, in LEMMA IX, cuts or makes a *finite* angle with the curve, whereas in LEMMA XI it touches the curve.

61. LEMMA XI. COR. V. Since in the common parabola the abscissa $\propto$ square of the ordinate, and likewise B D or A C $\propto A D^2$ or $C D^2$, it is evident that the curve may ultimately be considered a parabola.

This being admitted, we learn from Ex. 1, No. 4, that the curvilinear area A C B $= \frac{2}{3}$ of the rectangle C D. Whence the curvilinear area A B D $= \frac{1}{3}$ of C D $= \frac{2}{3}$ of the triangle A B D, or the area A B D $\propto$ triangle A B D $\propto A D^3$, &c. (by Cor. 4.) So far B D, b d have been considered at right angles to A D. Let them now be inclined to it at a given angle, or let them tend to a given point, or "be determined by any other law;" then (LEMMA, Case 3, and No. 25) B D, b d will ultimately be parallel. Hence, B D′, b d′ (fig. No. 26) being the corresponding subtenses perpendicular to A D, it is plain enough that the ultimate differences between the curvilinear areas A B D, A B D′ and between A b d, A b d′ are the similar triangles B D D′, b d d′, which differences are therefore as $B D^2$, $b d^2$, or as $A B^4$, $A b^4$, i. e. B D D′ $\propto A B^4$.

But we have shown that A B D $\propto A B^3$.

Consequently

$$A\,B\,D' = A\,B\,D \mp B\,D\,D' = a \times A\,B^3 \mp b \times A\,B^4 = A\,B^3(a \mp b \times A\,B)$$

and $b \times A\,B$ being indefinite compared with a, (see Art. 6,)

$$A\,B\,D' = a \times A\,B^3 \propto A\,B^3.$$

Q. e. d.

SCHOLIUM TO SECTION I.

62. What Newton asserts in the Scholium, and his commentators Le Seur and Jacquier endeavour (*unsuccessfully*) to elucidate, with regard to the different orders of the angles of contact or curvatures, may be briefly explained, thus.

Let $D\,B \propto A\,D^m$. Then the diameter of curvature, which equals $\frac{A\,D^2}{D\,B}$ (see No. 22 and 24), $\propto A\,D^{2-m}$. Similarly if $D\,B \propto A\,D^n$, the diameter of curvature $\propto A\,D^{2-n}$. Hence D and D′ represents these diameters, we have

$$\frac{D}{D'} = \frac{a \times A\,D^{2-m}}{a' \times A\,D^{2-n}} = \frac{a}{a'} A\,D^{n-m} \text{ (a and a' being finite)}$$

and if $n = 2$ or D′ be *finite*, then D will be *finite*, *infinitesimal*, or *infinite*, according as $m = 2$, or is any number, (whole, fractional, or even transcendental) less than 2, or any number greater than 2. Again, if $m = n$ then D compared with D′ is finite, since $D : D' :: a : a'$. If m be less than n in any finite degree, then $n - m$ is positive, and D is always infinitely less than D′. If m be greater than n, then

$$\frac{D}{D'} = \frac{a}{a'} \times \frac{1}{A\,D^{m-n}}$$

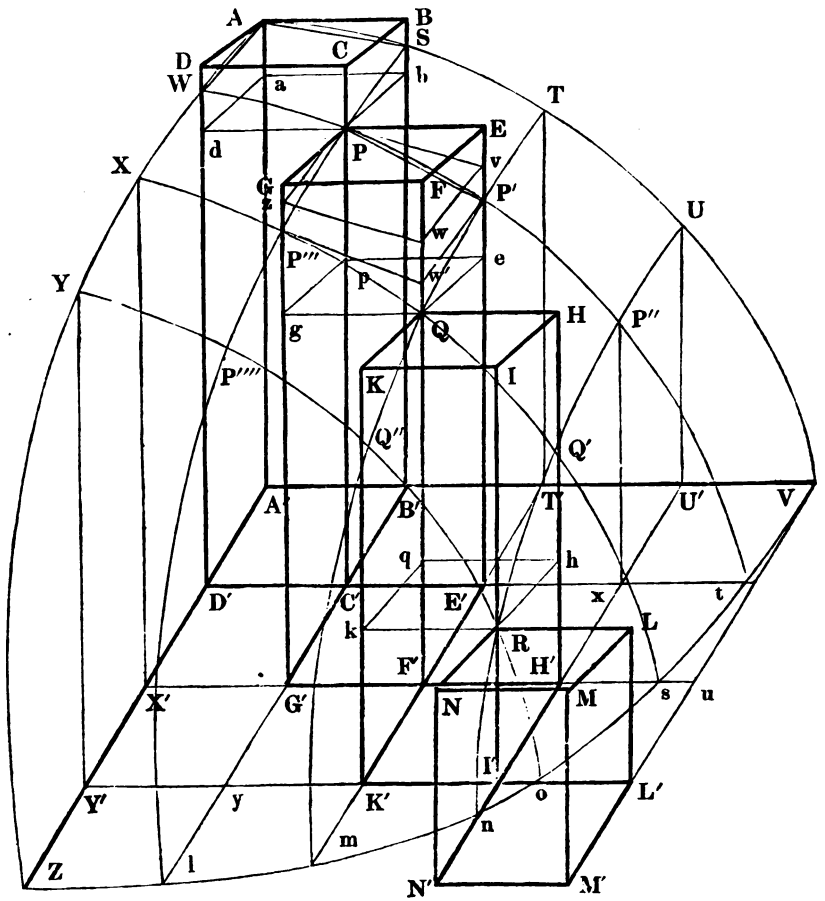
and $m - n$ being positive, D is always infinite compared with D′.

Hence then, there is no limit to the orders of diameters of curvature, with regard to infinite and infinitesimal, and consequently not to the curvatures.

63. In this Scholium Newton says, that "Those things which have been demonstrated of curve lines and the surfaces which they comprehend are easily applied to the curve surfaces and contents of solids." Let us attempt this application, or rather to show,

1st, *That if any number of parallelopipeds of equal bases be inscribed in any solid, and the same number having the same bases be also circumscribed*

about it; then the number of these parallelopipeds being increased and their magnitude diminished IN INFINITUM, the ultimate ratios which the aggregates of the inscribed and circumscribed parallelopipeds have to one another and to the solid, are ratios of equality.



Let A S T U V Z Y X W A be any portion of a solid cut off by three planes A A′ V, A A′ Z and Z A′ V, passing through the same point A′, and perpendicular to one another. Also let the intersections of these planes with one another be A A′, A′ V, A′ Z, and with the surface of the solid be A U V, A Y Z and Z l V. Moreover let A′ V, A′ Z be each divided into any number of equal parts in the points B′, T′, U′; D′, X′, Y′, and through them let planes, parallel to A A′ Z and A A′ V respectively, be supposed to pass, whose intersections with the planes A A′ V, A A′ Z

shall be S B′, T T′, U U′; W D′, X X′, Y Y′, and with the plane A′ Z V, l B′, m T′, n U′; t D′, s X′, o Y′, respectively. Again, let the intersections of these planes with the curve surface be S P l, T Q m, U R n; W P t, X Q s, Y R o respectively. Also suppose their several mutual intersections to be P C′, P′ E′, P″ x, P‴ G′, Q F′, Q′ H′, Q″ K′, &c.; those of these planes taken in pairs and of the plane A′ Z V, being the points C′, E′, x, G′, F′, H′, K′, I′, &c. and those of these pairs of planes and of the curve surface, the points P, P′, P″, P‴, Q, Q′, Q″, R, &c.

Now the planes, passing through B′, T′, U′, being all parallel to A A′ Z, are parallel to one another and perpendicular to A A′ V. Also because the planes passing through D′, X′, Y′ are parallel to A A′ V, they are parallel to one another, and perpendicular to A A′ Z. Hence (Euc. B. XI.) S B′, T T′, U U′, W D′, X X′, Y Y′, as also P C′, P′ E′, P″ x, P‴ G′, Q F′, Q′ H′, Q″ K′, &c. &c. are parallel to A A′ and to one another. It is also evident, for the same reasons, that B′ l, T′ m, U′ n, are parallel to A′ Z and to one another, as also are D′ t, X′ s, Y′ o to A′ V and to one another. Hence also it follows that A′ B′ C′ D′, B′ C′ E′ T′, &c. are rectangles, which rectangles, having their sides equal, are themselves equal.

Again, from the points A, P, Q, R in the curve surface, draw A B, A D; P E, P G; Q H, Q K; R L, R N parallel to A′ B′, A′ D′; C′ E′, C′ G′; F′ H′, T′ K′, I′ o, I′ n and meeting B′ S, D′ W; E′ P′, G′ P‴; H′ Q′, K′ Q″ produced in the points B, D; E, G; H, K, respectively. Then complete the rectangles A C, P F, Q I which, being equal and parallel to A′ C′, C′ F′, F′ I′, will evidently, when C′ P, F′ Q, I′ R are produced to C, F, I, complete the rectangular parallelopipeds A C′, P F′, Q I′. Moreover, supposing F′ I′ the last rectangle wholly within the curve Z V produce K′ I′, H′ I′ and make I′ L′, I′ N′ equal K′ I′, H′ I′, and complete the rectangle I′ M′. Also complete the parallelopiped R M′.

Again, produce E P, G P, H Q, K Q; L R, N R to the points d, b; g, e; k, h, and complete the rectangles P a, Q p, R q thereby dividing the parallelopipeds A C′, P F′, Q I′, each into two others, viz. A P, a C′; P Q, p F′; Q R, q I′.

Now the difference between the sum of the inscribed parallelopipeds a C′, p F′, q I′, and that of the circumscribed ones A C′, P F′, Q I′, R M′, is evidently the sum of the parallelopipeds A P, P Q, Q R, R M′; that is, since their bases are equal and the altitudes I′ R′, R I, Q F, P C are together equal to A A′, this difference is equal to the parallelopiped A C′. In the same manner if a series of inscribed and circumscribed

rectangular parallelopipeds, having the bases B′ E′, E′ H′, H′ L′, be constructed, the difference between their aggregates will equal the parallelopiped whose base is B′ E′ and altitude S B′, and so on with every series that can be constructed on bases succeeding each other *diagonally*. Hence then the difference between the sums of all the parallelopipeds that can be inscribed in the curve surface A Z V and circumscribed about it, is the sum of the parallelopipeds whose bases are each equal to A′ C′ and altitudes are A A′, S B′, T T′, U U′, W D′, X X′, Y Y′. Let now the number of the parts A′ B′, B′ T′, T′ U′, U′ V, and of the parts A D′, D′ X′, X′ Y′, Y′ Z be increased, and their magnitude diminished *in infinitum*, and it is evident the aforesaid sum of the parallelopipeds, which are comprised between the planes A A′ Z, S B′ l and between the planes A A′ V, W D′ t, will also be diminished without limit; that is, the difference between the inscribed and circumscribed whole solid is ultimately less than any that can be assigned, and these solids are *ultimately* equal, and a fortiori is the intermediate curve-surfaced solid equal to either of them (see LEMMA I and Art. 6.) Q. e. d.

Hitherto only such *portions* of solids as are bounded by three planes perpendicular to one another, and passing through the same point, have been considered. But since a *complete curve-surfaced* solid will consist of four such portions, it is evident that what has been demonstrated of any one portion must hold with regard to the whole. Moreover, if the solid should not be curve-surfaced throughout, but have one, two, or three plane faces, there will be no difficulty in modifying the above to suit any particular case.

2dly, *If in two curve-surfaced solids there be inscribed two series of parallelopipeds, each of the same number; and ultimately these parallelopipeds have to each other a given ratio, the solids themselves have to one another that same ratio.*

This follows at once from the above and the composition of ratios.

3dly, *All the corresponding edges or sides, rectilinear or curvilinear, of similar solids are proportionals; also the corresponding surfaces, plane or curved, are in the duplicate ratio of the sides; and the volumes or contents are in the triplicate ratio of the sides.*

When the solids have plane surfaces only, the above is shown to be true by Euclid.

When, however, the solids are curve-surfaced, wholly or in part, we must define them *to be similar when any plane-surfaced solid whatever being inscribed in any one of them, similar ones may also be inscribed in the*

others. Hence it is evident that the corresponding plane surfaces are similar, and consequently, by Lemma V, the corresponding edges are proportional, and the corresponding plane surfaces are in the duplicate ratio of these edges or sides. Moreover, if the same number of similar parallelopipeds be inscribed in the solids, and that number be indefinitely increased, it follows from 63. 1 and the composition of ratios, that the curved surfaces are proportional to the corresponding plane surfaces, and therefore in the duplicate ratio of the corresponding edges; and also that the contents are proportional to the corresponding inscribed parallelopipeds, or (by Euclid) in the triplicate ratio of the edges.

These three cases will enable the student of himself to pursue the analogy as far as he may wish. We shall "leave him to his own devices," after cautioning him against supposing that a curved-surface, at any point of it, has a certain fixed degree of curvature or deflection from the tangent-plane, and therefore that there is a sphere, touching the tangent-plane at that point, whose diameter shall be the limit of the diameters of all the spheres that can be made to touch the tangent-plane or curved-surface —analogously to A I in Lemma XI. Every curvilinear section of a curved-surface, made by a plane passing through a given point, has at that point a different curvature, the curved-surface being taken in the general sense; and it is a problem of Maxima and Minima *To determine those sections which present the greatest and least degrees of curvature.*

The other points of this Scholium require no particular remarks. If the student be desirous of knowing in what consists the distinction between the obsolete methods of Exhaustions, Indivisibles, &c. and that of prime and ultimate ratios, let him go to the original sources—to the works of Archimedes, Cavalerius, &c.

64. Before we close our comments upon this very important part of the *Principia*, we may be excused, perhaps, if we enter into the detail of the Principle delivered in Art. 6, which has already afforded us so much illustration of the text, and, as we shall see hereafter, so many valuable results. We have thence obtained a number of the ordinary rules for deducing indefinite forms from given definite functions of one variable; and it will be confessed, by competent and candid judges, that these applications of the principle strongly confirm it. Enough has indeed been already developed of the principle, to prove it clearly divested of all the metaphysical obscurities and inconsistencies, which render the methods of Fluxions, Differential Calculus, &c. &c. so objectionable as to their logic, and which have given rise to so many theories, all tending to establish

the same rules. It is incredible that the great men, who successively introduced their several theories, should have been satisfied with the reasonings by which they attempted to establish them. So many conflicting opinions, as to the principles of the science, go only to show that all were founded in error. Although it is generally difficult, and often impossible, for even the most sharp-sighted of men, to discern truth through the clouds of error in which she is usually enveloped, yet, when she does break through, it is with such distinct beauty and simplicity that she is instantly recognized by all. In the murkiness around her there are indeed false lights innumerable, and each passing meteor is in turn, by many observers, mistaken for the real presence; but these instantly vanish when exposed to the refulgent brightness of truth herself. Thus we have seen the various systems of the world, as devised by Ptolemy, Tycho Brahe, and Descartes, give way, by the unanimous consent of philosophers, to the demonstrative one of Newton. It is true, the principle of gravitation was received at first with caution, from its non-accordance with astronomical observations; but the moment the cause of this discrepancy, viz. the erroneous admeasurement of an arc of the meridian, was removed, it was hailed universally as truth, and will doubtless be coeval with time itself. The Theories relative to quantities indefinitely variable, present an argument from which may be drawn conclusions directly opposite to the above. Newton himself, dissatisfied with his Fluxions, produces PRIME AND ULTIMATE RATIOS, and again, dissatisfied with these, he introduces the idea of Moments in the second volume of the Principia. He is every where constrained to apologize for his obscurities, first in his Fluxions for the use of time and velocities, and then again in the Scholium, at the end of Sect. I of the Principia, (and in this instance we have shown how little it avails him) for reasoning upon *nothings*. After Newton comes Leibnitz, his great though dishonest rival, (we may so designate him, such being evidently the sentiments of Newton himself), who, bent upon obliterating all traces of his spoil, melts it down into another form, but yet falls into greater errors, as to the true nature of the thing, than the discoverer himself. From his Infinitesimals, considered as absolute *nothings* of the different orders, nothing can be logically deduced, unless by *Him* (we speak with reverence) who made all things from *nothing*. Such *fiats* we mortals cannot issue with the same effect, nor do we therefore admit in science, finite and tangible consequences deduced from the arithmetic of absolute nothings, be they ever so many. Then we have a number of theories promulgated by D'Alembert, Euler, Simpson, Marquise L'Hopital, &c. &c.

all more or less modifications of the others—all struggling to establish and illustrate what the great inventor, with all his almost supernatural genius, failed to accomplish. All these diversities in the views of philosophers make, as it has been already observed, a strong argument that truth had not then unveiled herself to any of them. Newton strove most of any to have a full view, but he caught only a glimpse, as we may perceive by his remaining dissatisfied with it. Hence then it appears, to us at least, that the true metaphysics of the doctrine of quantities indefinitely variable, remain to this day undiscovered. But it may be asked, after this sweeping conclusion, how comes it that the results and rules thence obtained all agree in form, and in their application to physics produce consequences exactly in conformity with experience and observation? The answer is easy. These forms and results are accurately true, although illogically deduced, from a mere *compensation of errors*. This has been clearly shown in the general expression for the subtangent (Art. 29), and all the methods, not even Lagrange's Calcul des Fonctions excepted, are liable to the paralogism. Innumerable other instances might be adduced, but this one we deem amply sufficient to warrant the above assertion.

After these preliminary observations upon the state of darkness and error, which prevails to this day over the scientific horizon, it may perhaps be expected of us to shine forth to dispel the fog. But we arrogate to ourselves no such extraordinary powers. All we pretend to is *self-satisfaction* as to the removal of the difficulties of the science. Having engaged to write a Commentary upon the Principia, we naturally sought to be satisfied as to the correctness of the method of Prime and Ultimate Ratios. The more we endeavoured to remove objections, the more they continually presented themselves; so that after spending many months in the fruitless attempt, we had nearly abandoned the work altogether; when suddenly, in examining the method of *Indeterminate Coefficients* in Dr. Wood's Algebra, it occurred that the aggregates of the coefficients of the like powers of the indefinite variable, must be separately equal to zero, not because the variable might be assumed equal to zero, (which it never is, although it is capable of indefinite diminution,) but because of the different powers being essentially different from, and forming no part of one another.

From this a train of reflections followed, relative to the treatment of homogeneous *definite* quantities in other branches of Algebra. It was soon perceptible that any equation put $= 0$, consisting of an aggregate of

different quantities incapable of amalgamation by the opposition of *plus* and *minus*, must give each of these quantities equal to zero. Reverting to indefinites, it then appeared that their whole theory might be developed on the same principles, and making trial as in Art. 6, and the subsequent parts of the preceding commentary, we have satisfied ourselves most fully of having thus hit upon a method of clearing up all the difficulties of what we shall henceforth, contrary to the intention expressed in Art. 7, entitle

THE CALCULUS

OF

INDEFINITE DIFFERENCES.

65. A *constant quantity* is such, that from its very nature it cannot be made less or greater.

Constants, as such quantities may briefly be called, are denoted generally by the first letters of the alphabet,

a, b, c, d, &c.

A definite quantity is a GIVEN VALUE of a quantity essentially variable.

Definite quantities are denoted by the last letters of the alphabet, as

z, y, x, w, &c.

An INDEFINITE quantity is a quantity essentially variable through all degrees of diminution or of augmentation short of absolute NOTHINGNESS or INFINITUDE.

Thus the ordinate of a curve, considered generally, is an *indefinite*, being capable of every degree of diminution. But if any particular value, as that which to a given abscissa, for instance, be fixed upon, this value is *definite*. All abstract numbers, as 1, 2, 3, &c. and quantities absolutely fixed, are *constants*.

66. The difference between two definite values of the same quantity (y) is a *definite* quantity, and may be represented by

$$\Delta y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

adopting the notation of the Calculus of Finite (or *definite*) Differences.

In the same manner the difference between two definite values of Δy is a definite quantity, and is denoted by

$$\Delta (\Delta y)$$

or more simply by

$$\Delta^2 y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

and so on to

$$\Delta^n y .$$

67. The difference between a *Definite* value and the *Indefinite* value of any quantity y is *Indefinite*, and we call it the *Indefinite Difference* of y, and denote it, agreeably to the received algorithm, by

$$d y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

In the same manner

$$d (d y)$$

or

$$d^2 y$$

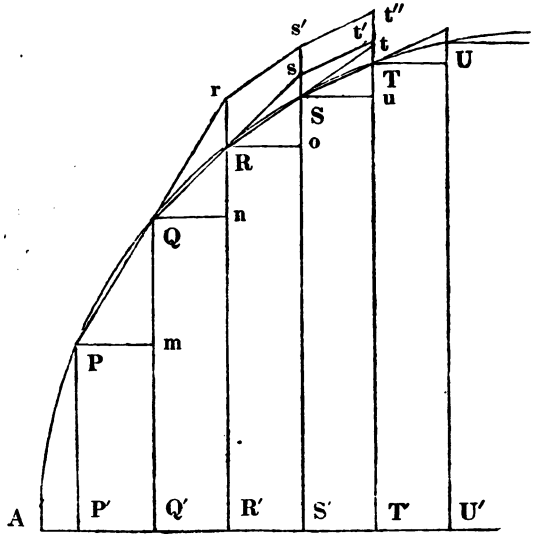
the *Indefinite Difference* of the *Indefinite Difference* of y, or the second indefinite difference of y.

Proceeding thus we arrive at

$$d^n y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

which means the nth indefinite difference of y.

68. *Definite* and *Indefinite* Differences admit of being also represented by lines, as follows:



Let P P′ = y be any fixed or definite ordinate of the curve A U, and taking P′ Q′ = Q′ R′ = R′ S′ = &c. let ordinates be erected meeting the curve in Q, R, S, T, &c. Join P Q, Q R, R S, &c. and produce them to meet the ordinates produced in r, s, t, &c. Also draw r s′, s t′,

&c. parallel to R S, S T, &c. and draw s t″, &c. parallel to s t′, &c.; and finally draw P m, Q n, R o, &c. perpendicular to the ordinates.

Now supposing not only P P′ but also Q Q′, R R′, &c. fixed or definite; then

$$\begin{aligned} Q\,m &= Q\,Q' - P\,P' = \Delta\,P\,P' = \Delta\,y \\ R\,r &= n\,r - n\,R = Q\,m - R\,n = \Delta\,Q\,m \\ &= \Delta\,(\Delta\,P\,P') = \Delta^2\,P\,P' = \Delta^2\,y \\ s\,s' &= S\,s - S\,s' = S\,s - R\,r = \Delta\,R\,r \\ &= \Delta^3\,y \\ t\,t'' &= t\,t' - t'\,t'' = t\,t' - s\,s' = \Delta\,s\,s' \\ &= \Delta\,(\Delta^3\,y) = \Delta^4\,y \end{aligned}$$

and so on to any extent.

But if the equal parts P′ Q′, Q′ R′, &c. be arbitrary or indefinite, then Q m, R r, s s′, t t″, &c. become so, and they represent the several *Indefinite Differences* of y, viz.

$$d\,y,\ d^2\,y,\ d^3\,y,\ d^4\,y, \text{ \&c.}$$

69. The reader will henceforth know the distinction between *Definite* and *Indefinite Differences*. We now proceed to establish, of *Indefinite Differences*, the

FUNDAMENTAL PRINCIPLE.

It is evidently a truth perfectly axiomatic, that *No aggregate of* INDEFINITE *quantities can be a definite quantity, or aggregate of definite quantities, unless these aggregates are equal to zero.*

It may be said that (a — x) + (a + x) = 2 a, in which (x) is indefinite, and (a) constant or definite, is an instance to the contrary; but then the reply is, a — x and a + x are not *indefinites* in the sense of Art. 65.

70. *Hence if in any equation*

$$A + B\,x + C\,x^2 + D\,x^3 + \text{\&c.} = 0$$

A, B, C, &c. be *definite quantities* and x an *indefinite quantity*; then we have

$$A = 0,\ B = 0,\ C = 0, \text{ \&c.}$$

For $B\,x + C\,x^2 + D\,x^3 + \text{\&c.}$ cannot equal — A unless A = 0. But by transposing A to the other side of the equation, it does = — A. Therefore A = 0 and consequently

$$B\,x + C\,x^2 + D\,x^3 + \text{\&c.} = 0$$

or

$$x\,(B + C\,x + D\,x^2 + \text{\&c.}) = 0$$

But x being *indefinite* cannot be equal to 0; ∴

$$B + Cx + Dx^2 + \&c. = 0$$

Hence, as before, it may be shown that $B = 0$, and therefore

$$x(C + Dx + \&c.) = 0$$

Hence $C = 0$, and so on throughout.

71. Again, *if in the equation*

$$A+Bx+B'y+Cx^2+C'xy+C''y^2+Dx^3+D'x^2y+D''xy^2+D'''y^3+\&c.$$

A, B, B′, C, C′, C″, D, &c. *be definite quantities*, and x, y *INDEFINITES; then*

$$\left.\begin{aligned} A &= 0 \\ Bx + B'y &= 0 \\ Cx^2 + C'xy + C''y^2 &= 0 \\ \&c. &= 0 \end{aligned}\right\} \textit{when } y \textit{ is a function of } x.$$

For, let $y = zx$, then substituting

$$A + x(B + B'z) + x^2(C + C'z + C''z^2) + x^3(D + D'z + D''z^2 + D'''z^3) + \&c. = 0$$

Hence by 70,

$$A = 0,\ B + B'z = 0,\ C + C'z + C''z^2 = 0, \&c.$$

and substituting $\frac{y}{x}$ for z and reducing we get

$$A = 0,\ Bx + B'y = 0, \&c.$$

In the same manner, if we have an equation involving three or more indefinites, it may be shown that the aggregates of the homogeneous terms must each equal zero.

This general principle, which is that of *Indeterminate Coefficients* legitimately established and generalized, (the ordinary proofs divide $Bx + Cx^2 + \&c. = 0$ by x, which gives $B + Cx + Dx^2 + \&c. = \frac{0}{x}$ and not 0; x is then put $= 0$, and thence *truly* results $B = \frac{0}{0}$, which instead of being 0, may be any quantity whatever, as we know from algebra; whereas in 70, by considering the *nature* of x, and the absurdity of making it $= 0$ we avoid the paralogism) conducts us by a near route to the *Indefinite Differences of functions of one or* MORE *variables.*

72. *To find the Indefinite Difference of any function of* x.

Let $u = fx$ denote the function.

Then du and dx being the indefinite differences of the function and of x itself, we have

$$u + du = f(x + dx)$$

Assume

$$f(x + dx) = A + B\,dx + C\,dx^2 + \&c.$$

A, B, &c. being independent of d x or definite quantities involving x and constants; then

$$u + d u = A + B\ d x + C\ d x^2 + \&c.$$

and by 71, we have

$$u = A,\ d u = B . d x$$

Hence then this general rule,

The INDEFINITE DIFFERENCE of any function of x, f x, *is the second term in the developement of* f (x + d x) *according to the increasing powers of* d x.

Ex. Let $u = x^n$. Then it may easily be shown independently of the Binomial Theorem that

$$(x + d x)^n = x^n + n . x^{n-1} d x + P d x^2$$
$$\therefore d (x^n) = n . x^{n-1} d x$$

The student may deduce the results also of Art. 9, 10, &c. from this general rule.

73. *To find the indefinite difference of the product of two variables.*

Let $u = x y$. Then

$$u + d u = (x + d x) . (y + d y) = x y + x\ d y + y\ d x + d x\ d y$$
$$\therefore d u = x\ d y + y\ d x + d x\ d y$$

and by 71, or directly from the homogeneity of the quantities, we have

$$d u = x\ d y + y\ d x \quad . . . . . \quad (a)$$

Hence

$$d (x y z) = x\ d (y z) + y z\ d x$$
$$= x z d y + x y\ d z + y z\ d x \quad . . . \quad (b)$$

and so on for any number of variables.

Again, *required* $d . \frac{x}{y}$.

Let $\frac{x}{y} = u$. Then

$$x = y u, \text{ and } d x = u\ d y + y\ d u$$
$$\therefore d \frac{x}{y} = d u = \frac{d x}{y} - \frac{u}{y} d y$$
$$= \frac{y\ d x - x\ d y}{y^2} \quad . . . . . \quad (c)$$

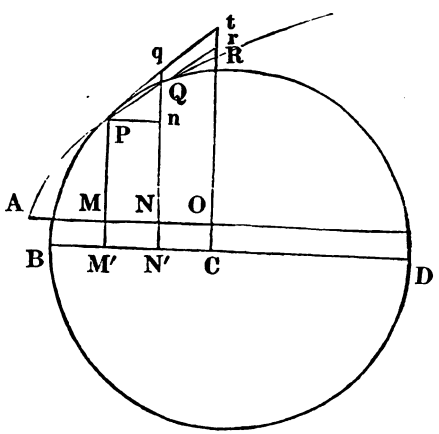
Hence, and from rules already delivered, may be found the *Indefinite Differences* of any functions whatever of two or more variables. We refer the student to Peacock's Examples of the Differential Calculus for practice.

The result (a) may be deduced geometrically from the fig. in Art. 21. The sum of the indefinite rectangles A b, b A′ makes the Indefinite Difference.

We might, in this place, investigate the second, third, &c. Indefinite Differences, and give rules for the *maxima* and *minima* of functions of two or more variables, and extend the Theorems of Maclaurin and Taylor to such cases. Much might also be said upon various other applications, but the complete discussion of the science we reserve for an express Treatise on the subject. We shall hasten to deduce such results as we shall obviously want in the course of our subsequent remarks; beginning with the research of a general expression for the *radius of curvature* of a given curve, or for the radius of that circle whose deflection from the tangent is the same as that of the curve at the point of contact.

74. *Required the radius of curvature for any point of a given curve.*

Let A P Q R be the given curve, referred to the axis A O by the ordinate and abscissa P M, A M or y and x. P M being fixed let Q N, O R be any other ordinates taken at equal indefinite intervals M N, N O. Join P Q and produce it to meet O R in r; and let P t be the tangent at P drawn by Art. 29, meeting Q N, O R in q and t respectively. Again draw a circle (as in construction of Lemma XI, or otherwise) passing through P and Q and touching the tangent P t, and therefore touching the curve; and let B D be its diameter parallel to A O.



Now

$$Q\,n = d\,y,\quad P\,n = d\,x,\quad P\,q = P\,Q \text{ (Lemma VII)} = \sqrt{(d\,x^2 + d\,y^2)} \text{ or } d\,s, \text{ if } s = \text{arc } A\,P.$$

Moreover let

$$P\,M' = y';$$

then it readily appears (see Art. 27) that $d\,s = \frac{R\,d\,x}{y}$, R being the radius of the circle.

Again

$$P\,q^2 = Q\,q \times (Q\,q + 2\,Q\,N')$$
$$= Q\,q\,(Q\,q + 2\,d\,y + 2\,y')$$

or

$$(ds)^2 = Qq\left(Qq + 2dy + \frac{2R\,dx}{ds}\right)$$

But since

$$Rt : Qq :: Pr^2 : PQ^2 :: 4 : 1 \text{ (LEMMA XI)}$$

and

$$Qq : tr :: 1 : 2$$

$$\therefore Rt = 2tr, \text{ or } Rr = tr = 2Qq$$

$$\therefore Qq = \frac{tr}{2} = \frac{d^2y}{2} \text{ (by Art. 68.)}$$

Consequently

$$ds^2 = \frac{d^2y}{2}\left(\frac{d^2y}{2} + 2dy + \frac{2R\,dx}{ds}\right)$$

$$= \frac{(d^2y)^2}{4} + dy\ d^2y + \frac{R\,dx\,d^2y}{ds}$$

and equating *Homogeneous Indefinites*

$$ds^2 = \frac{R\,dx\,d^2y}{ds}$$

$$\therefore R = \frac{ds^3}{dx\,d^2y} = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,d^2y}$$

$$= \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \ldots\ldots\ldots\ldots \text{(d)}$$

the general expression for the radius of curvature.

Ex. 1. In the parabola $y^2 = ax$.

$$\therefore \frac{dy}{dx} = \frac{a}{2y}$$

and since when the curve is concave to the axis d^2y is negative,

$$-\frac{d^2y}{dx^2} = -\frac{a}{2y^2}\cdot\frac{dy}{dx} = -\frac{a^2}{4y^3} =$$

$$\therefore R = \left(1 + \frac{a^2}{4y^2}\right)^{\frac{3}{2}} \times \frac{4y^3}{a^2}$$

$$= (4y^2 + a^2)^{\frac{3}{2}} \times \frac{1}{2a^2}$$

Hence at the vertex $R = \frac{a}{2}$, and at the extremity of the latus rectum,

$$R = \frac{2^{\frac{3}{2}}}{2}\cdot a = a\sqrt{2}.$$

Ex. 2. If p be the parameter or the double ordinate passing through the focus and 2 a the axis-major of any conic section, its equation is

$$y^2 = p\,x \pm \frac{p}{2a}x^2$$

Hence

$$2\,y\,d\,y = p\,d\,x \pm \frac{p}{a}\,x\,d\,x$$

and

$$2\,d\,y^2 + 2\,y\,d^2\,y = \pm\frac{p}{a}\,d\,x^2$$

$$\therefore \frac{d\,y}{d\,x} = \frac{p\left(1 \pm \frac{x}{a}\right)}{2\,y}$$

and

$$-\frac{d^2\,y}{d\,x^2} = \frac{p^2\left(1 \pm \frac{x}{a}\right)^2 \mp \frac{2\,p}{a}\,y^2}{4\,y^3}$$

$$\therefore R = \frac{\left\{4\,y^2 + p^2\left(1 \pm \frac{x}{a}\right)^2\right\}^{\frac{3}{2}}}{2\left\{p^2\left(1 \pm \frac{x}{a}\right)^2 \mp \frac{2p}{a}\,y^2\right\}}$$

which reduces to

$$R = \frac{\left\{p^2 + \frac{2\,p}{a}(2\,a \pm p)\,x + \frac{p}{a^2}(p \pm 2\,a)\,x^2\right\}^{\frac{3}{2}}}{2\,p^2}$$

Ex. 3. In the cycloid it is easy to show that

$$\frac{d\,y}{d\,x} = \sqrt{\frac{2\,r - y}{y}}$$

r being the radius of the generating circle, and x, y referred to the base or path of the circle.

$$\therefore \frac{d^2\,y}{d\,x^2} = -\frac{r}{y^2}$$

$$\therefore R = 2\sqrt{2\,r\,y} = 2 \text{ the normal.}$$

Hence it is an easy problem *to find the equation to the locus of the centres of curvature for the several points of a given curve.*

If y and x be the coordinates of the given curve, and Y and X those of the required locus, all referred to the same origin and axis, then the student will easily prove that

$$X = x - \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{d^2y}{dx^2}\right)}\left(1 + \frac{dy^2}{dx^2}\right)$$

and

$$Y = y + \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}.$$

which will give the equation required, by substituting by means of the equation to the given curve.

In the cycloid for instance

$$X = x + \surd(2ry - y^2)$$
$$Y = -y$$

whence it easily appears that the locus required is the same cycloid, only differing in position from the given one.

75. *Required to express the radius of curvature in terms of the polar co-ordinates of a curve, viz. in terms of the radius vector* ϱ *and traced-angle* θ.

$$\left.\begin{aligned} x &= \varrho \cos.\ \theta \\ \text{and} \qquad y &= \varrho \sin.\ \theta \end{aligned}\right\}$$

$\therefore$ taking the indefinite differences, and substituting in equation (d) of Art. 74, we get

$$R = \frac{\left(\varrho^2 + \frac{d\varrho^2}{d\theta^2}\right)^{\frac{3}{2}}}{2\frac{d\varrho^2}{d\theta^2} - \varrho\frac{d^2\varrho}{d\theta^2} + \varrho^2}$$

which by means of the equation to the curve will give the radius of curvature required.

Ex. 1. *In the logarithmic spiral*

$$\varrho = a^{\theta};$$
$$\therefore \frac{d\varrho}{d\theta} = 1a \times a^{\theta} \text{ (Art. 17.)}$$
$$\therefore -\frac{d^2\varrho}{d\theta^2} = -(1a)^2 \times a^{\theta} = -(1a)^2 \cdot \varrho$$
$$\therefore R = \frac{(\varrho^2 + (1a)^2\varrho^2)^{\frac{3}{2}}}{2(1a)^2\varrho^2 - (1a)^2\varrho^2 + \varrho^2} = \frac{\varrho^3(1 + (1a)^2)^{\frac{3}{2}}}{\varrho^2(1 + (1a)^2)}$$
$$= \varrho\{1 + (1a)^2\}^{\frac{1}{2}}$$

Ex. 2. *In the spiral of Archimedes*

$$\varrho = a\,\theta$$

and

$$R = \frac{(\varrho^2 + a^2)^{\frac{3}{2}}}{2\,a^2 + \varrho^2}.$$

Ex. 3. *In the hyperbolic spiral*

$$\varrho = \frac{a}{\theta}$$

$$\therefore R = \frac{\varrho\,(a^2 + \varrho^2)^{\frac{3}{2}}}{a^3}.$$

Ex. 4. *In the Lituus*

$$\varrho^2 = \frac{a^2}{\theta}$$

$$\therefore R = \frac{\varrho}{2a^2} \cdot \frac{(4\,a^4 + \varrho^4)^{\frac{3}{2}}}{4\,a^4 - \varrho^4}.$$

Ex. 5. *In the Epicycloid*

$$\varrho = (r + r')^2 - 2\,r\,(r + r')\cos.\,\theta$$

r and r′ being the radius of the wheel and globe respectively.

Here

$$R = \frac{(r + r')\,(3\,r^2 - 2\,r\,r' - r'^2 + 2\,\varrho)^{\frac{3}{2}}}{2\,(3\,r^2 - 2\,r\,r' - r'^2) + 3\,\varrho}.$$

Having already given those results of the Calculus of Indefinite Differences which are most useful, we proceed to the reverse of the calculus, which consists in the investigation of the Indefinites themselves from their indefinite differences. In the direct method we seek the Indefinite Difference of a given function. In the inverse method we have given the *Indefinite Difference* to find the function whose Indefinite Difference it is. This inverse method we call

THE INTEGRAL CALCULUS

OR

INDEFINITE DIFFERENCES.

76. The integral of d x is evidently x + C, since the indefinite difference of x + C is d x.

77. *Required the integral of* a d x?

By Art. 9, we have

$$d\,(a\,x) = a\,d\,x.$$

Hence reversely the integral of a d x is a x. This is only one of the innumerable integrals which there are of a d x. We have not only d (a x) = a d x but also

$$d(ax + C) = a\,dx$$

in which C is any constant whatever.

$$\therefore ax + C = \int a\,dx = a\int dx \quad . \quad . \quad . \quad \text{(a)} \quad \text{(see 76)}$$

generally, $\int$ being the characteristic of an integral.

78. *Required the integral of*

$$ax^p\,dx.$$

By Art. 12

$$d(ax^n + C) = n\,ax^{n-1}\,dx$$

$$\therefore ax^n + C = \int n\,ax^{n-1}\,dx$$

$$= n \times \int ax^{n-1}\,dx \quad (77)$$

$$\therefore \int ax^{n-1}\,dx = \frac{ax^n}{n} + \frac{C}{n}.$$

But since C is any constant whatever $\frac{C}{n}$ may be written C.

$$\therefore \int ax^{n-1}\,dx = \frac{ax^n}{n} + C$$

Hence it is plain that

$$\int ax^p\,dx = \frac{ax^{p+1}}{p+1} + C$$

Or *To find the integral of the product of a constant the* pth *power of the variable and the Indefinite Difference of that variable, let the index of the power be increased by* 1, *suppress the Indefinite Difference, multiply by the constant, divide by the increased index, and add an arbitrary constant.*

79. Hence

$$\int (ax^p\,dx + bx^q\,dx + \&c.) =$$

$$\frac{ax^{p+1}}{p+1} + \frac{bx^{q+1}}{q+1} + \&c. + C$$

80. Hence also

$$\int ax^{-n}\,dx = -\frac{a}{(n-1)\,x^{n-1}} + C.$$

81. *Required the integral of*

$$ax^{m-1}\,dx\,(b + ex^m)^p.$$

Let

$$u = b + ex^m$$

$$\therefore du = mex^{m-1}\,dx$$

$$\therefore ax^{m-1}\,dx = \frac{a}{me}.\,du$$

$$\therefore \int ax^{m-1}dx\,(b + ex^m)^p = \int \frac{a}{me}\,u^p\,du$$

$$= \frac{a}{m\,e\,.\,(p+1)} \cdot u^{p+1} + C \quad (78)$$

$$= \frac{a}{m\,e\,(p+1)} \cdot (b \times e\,x^m)^{p+1} + C.$$

82. *Required the integral of* $\frac{dx}{x}$.

By 80 it would seem that

$$\int \frac{dx}{x} = -\frac{1}{0} + C$$

and if when

$$\int \frac{dx}{x} = 0,\ C = c,\ w - \int \frac{dx}{x} = \frac{1}{0} - \frac{1}{0} = \frac{0}{0}.$$

But by Art. 17 a. we know that

$$d\,.\,l\,x = \frac{dx}{x}$$

Therefore

$$\int \frac{dx}{x} = l\,x + C.$$

Here it may be convenient to make the arbitrary constant of the form l C

Therefore

$$\int \frac{dx}{x} = l\,x + l\,C = l\,C\,x$$

Hence the integral of a fraction whose numerator is the Indefinite Difference of the denominator, is the hyperbolic logarithm of the denominator PLUS *an arbitrary constant.*

83. Hence

$$\int \frac{a\,x^{m-1}\,dx}{b\,x^m + e} = \frac{a}{b\,m} \int \frac{m\,x^{m-1}\,dx}{x^m + \frac{e}{b}}$$

$$= \frac{a}{b\,m} \cdot l\,.\,\left(x^m + \frac{e}{b}\right) + C$$

$$= \frac{a}{b\,m} \cdot l\,.\,C\left(x^m + \frac{e}{b}\right),$$

and so on for more complicated forms.

84. *Required the integral of* $a^x\,dx$.

By Art. 17

$$d\,.\,a^x = l\,a\,.\,a^x\,dx$$

$$\therefore \int a^x\,dx = \frac{1}{l\,a} \cdot \int d\,a^x$$

$$= \frac{1}{l\,a} \cdot a^x + C$$

85. If y, x, t, s denote the sine, cosine, tangent, and secant of an angle θ; then we have, Art. 26, 27.

$$d\theta = \frac{dy}{\sqrt{(1-y^2)}} = \frac{-dx}{\sqrt{(1-x^2)}} = \frac{dt}{1+t^2} = \frac{ds}{s\sqrt{2s-s^2}}$$

$$\therefore \int \frac{dy}{\sqrt{(1-y^2)}} = \theta + C = \sin.^{-1} y + C$$

$$\int \frac{-dx}{\sqrt{(1-x^2)}} = \theta + C = \cos.^{-1} x + C$$

$$\int \frac{dt}{1+t^2} = \theta + C = \tan.^{-1} t + C$$

$$\int \frac{ds}{s\sqrt{2s-s^2}} = \theta + C = \sec.^{-1} s + C$$

$\sin.^{-1} y$, $\cos.^{-1} x$, &c. being symbols for the arc whose sine is y, cosine is x, &c. respectively.

86. Hence, more generally,

$$\int \frac{du}{\sqrt{(a-bu^2)}} = \frac{1}{\sqrt{b}} \int \frac{\sqrt{\frac{b}{a}}\, du}{\sqrt{\left(1-\frac{b}{a}u^2\right)}}$$

$$= \frac{1}{\sqrt{b}} \cdot \sin.^{-1} u\sqrt{\frac{b}{a}} + C \quad . \quad . \quad (a)$$

or $= \frac{1}{\sqrt{b}} \times$ angle whose sine is $u\sqrt{\frac{b}{a}}$ to rad. $1 + C$.

Also

$$\int \frac{-du}{\sqrt{(a-bu^2)}} = \frac{1}{\sqrt{b}} \cdot \cos.^{-1} u\sqrt{\frac{b}{a}} + C \quad . \quad . \quad (b)$$

Again

$$\int \frac{du}{a+bu^2} = \frac{1}{\sqrt{ab}} \cdot \int \frac{\sqrt{\frac{b}{a}}\, du}{1+\frac{b}{a}u^2}$$

$$= \frac{1}{\sqrt{ab}} \tan.^{-1} u\sqrt{\frac{b}{a}} + C \quad . \quad . \quad (c)$$

and

$$\int \frac{du}{u\sqrt{(bu^2-a)}} = \frac{1}{\sqrt{a}} \int \frac{\sqrt{\frac{b}{a}}\, du}{u\sqrt{\frac{b}{a}} \times \sqrt{\left(\frac{b}{a}u^2-1\right)}}$$

$$= \frac{1}{\sqrt{a}} \cdot \sec.^{-1} u\sqrt{\frac{b}{a}} + C \quad . \quad . \quad (d)$$

Moreover, if u be the versed sine of an angle θ, then the sine $= \sqrt{(2u - u^2)}$ and

$$d u = d(1 - \cos.\theta) = d\theta . \sin.\theta \text{ (Art. 27.)}$$
$$= d\theta . \sqrt{(2u - u^2)}$$
$$\therefore d\theta = \frac{d u}{\sqrt{(2u - u^2)}}$$

Hence

$$\int \frac{d u}{\sqrt{(2u - u^2)}} = \theta + C$$
$$= \text{vers.}^{-1} u + C$$

and generally

$$\int \frac{d u}{\sqrt{(au - bu^2)}} = \int \frac{\frac{2b}{a} d u}{\sqrt{b}\sqrt{\left(2 . \frac{2b}{a} u - \left(\frac{2b}{a}\right)^2 u^2\right)}}$$
$$= \frac{1}{\sqrt{b}} \text{vers.}^{-1} \frac{2b}{a} u + C \quad . \quad . \quad . \quad \text{(e)}$$

87. *Required the integrals of*

$$\frac{d x}{a + bx}, \quad \frac{d x}{a - bx}, \quad \frac{d x}{a - bx^2}.$$

$$\int \frac{d x}{a + b x} = \frac{1}{b} . \int \frac{d . (a + bx)}{a + bx}$$
$$= \frac{1}{b} . \text{l.} (a + bx) + C \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

and

$$\int \frac{d x}{a - b x} = -\frac{1}{b} \int \frac{d (a - bx)}{a - bx}$$
$$= -\frac{1}{b} . \text{l.} (a - bx) + C \quad . \quad . \quad . \quad \text{(g)}$$

see Art. 17 a.

Hence,

$$\int d x \left\{ \frac{1}{a + bx} + \frac{1}{a - bx} \right\} = \int \frac{2a d x}{a^2 - b^2 x^2}$$
$$= \frac{1}{b} . \text{l.} (a + bx) - \frac{1}{b} . \text{l.} (a - bx) + C$$
$$= \frac{1}{b} . \text{l.} \frac{a + bx}{a - bx} + C.$$

Hence we easily get by analogy

$$\int \frac{dx}{a - bx^2} = \frac{1}{\sqrt{ab}} . l. \frac{\sqrt{a} + \sqrt{b} . x}{\sqrt{a - bx^2}} + C$$
$$= \frac{1}{2\sqrt{ab}} . l. \frac{\sqrt{a} + \sqrt{b} . x}{\sqrt{a} - \sqrt{b} . x} + C \Big\} \ldots (h)$$

88. *Required the integral of*

$$\frac{dx}{ax^2 + bx + c}.$$

In the first place

$$ax^2 + bx + c = a\left\{ x + \frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a} \right\} \times$$
$$\left\{ x + \frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a} \right\} = a\left\{ \left(x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{2a} \right\}$$

Hence, putting

$$x + \frac{b}{2a} = u$$

we have

$$dx = du$$

and

$$\frac{dx}{ax^2 + bx + c} = \frac{du}{a\left(u^2 - \frac{b^2 - 4ac}{2a}\right)}$$

which presents the following cases.

Case 1. *Let* a *be negative and* c *be positive; then*

$$\frac{dx}{-ax^2 + bx + c} = \frac{du}{-a\left(\frac{b^2 + 4ac}{2a} + u^2\right)}$$

$$\therefore \int \frac{dx}{-ax^2 + bx + c} = \frac{\sqrt{2}}{-\sqrt{a}\sqrt{(b^2 + 4ac)}} \tan.^{-1} u \sqrt{\frac{2a}{b^2 + 4ac}} + C$$

$$(\text{see Art. } 86) = -\sqrt{\frac{2}{a(b^2 + 4ac)}} . \tan.^{-1}\left(x + \frac{b}{2a}\right) \sqrt{\frac{2a}{b^2 + 4ac}} + C \ldots (i)$$

Case 2. *Let* c *be negative and* a *positive; then*

$$\int \frac{dx}{ax^2 + bx - c} = \int \frac{du}{a\left(u^2 - \frac{b^2 + 4ac}{2a}\right)}$$
$$= -\frac{1}{a} \int \frac{du}{\frac{b^2 + 4ac}{2a} - u^2}$$
$$= -\sqrt{\frac{1}{2a(b^2 + 4ac)}} . l. \frac{\sqrt{\frac{b^2 + 4ac}{2a}} + x + \frac{b}{2a}}{\sqrt{\frac{b^2 + 4ac}{2a}} - x - \frac{b}{2a}} + C \ldots (k)$$

see Art. 87.

Case 3. *Let* b^2 *be* $> 4ac$ *and* a, c *be both positive; then*

$$\int \frac{dx}{ax^2+bx+c} = \int \frac{du}{a\left(u^2 - \frac{b^2-4ac}{2a}\right)}$$

$$= \frac{1}{-a}\int \frac{du}{\frac{b^2-4ac}{2a} - u^2}$$

$$= -\sqrt{\frac{1}{2a(b^2-4ac)}}.l.\frac{\sqrt{\frac{b^2-4ac}{2a}}+x+\frac{b}{2a}}{\sqrt{\frac{b^2-4ac}{2a}}-x-\frac{b}{2a}}+C..(l)$$

Case 4. *Let* b^2 *be* $< 4ac$ *and* a, c *be both positive;*
Then

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{du}{\frac{4ac-b^2}{2a}+u^2}$$

$$= \sqrt{\frac{2}{a(4ac-b^2)}}.\tan.^{-1}\left(x\pm\frac{b}{2a}\right)\sqrt{\frac{2a}{4ac-b^2}}+C..(m)$$

Case 5. *If* b^2 *be* $> 4ac$ *and* a, c *both negative;*
Then

$$\int \frac{dx}{-ax^2+bx-c} = \frac{1}{-a}\int \frac{du}{\frac{b^2-4ac}{2a}+u^2}$$

$$= -\sqrt{\frac{2}{a(b^2-4ac)}}\tan.^{-1}\left(x+\frac{b}{2a}\right)\sqrt{\frac{2a}{b^2-4ac}}+C..(n)$$

Case 6. *If* b^2 *be* $< 4ac$ *and* a *and* c *both negative;*
Then

$$\int \frac{dx}{-ax^2+bx-c} = \frac{1}{a}\int \frac{du}{\frac{4ac-b^2}{2a}-u^2}$$

$$= \sqrt{\frac{1}{2a(4ac-b^2)}}.l.\frac{\sqrt{\frac{4ac-b^2}{2a}}+x+\frac{b}{2a}}{\sqrt{\frac{4ac-b^2}{2a}}-x-\frac{b}{2a}}+C....(o)$$

89. *Required the integral of any rational function whatever of one variable, multiplied by the indefinite difference of that variable.*

Every rational function of x is comprised under the general form

$$\frac{Ax^m + Bx^{m-1} + Cx^{m-2} + \&c.\ Kx + L}{ax^n + bx^{n-1} + cx^{n-2} + \&c.\ kx + l}$$

in which A, B, C, &c. a, b, c, &c. and m, n are any constants whatever.

If

$$n = 0,$$

then we have (Art. 77)

$$\int (A x^m + B x^{m-1} + \&c.) \frac{dx}{a} = \left(\frac{A x^{m+1}}{m+1} + \frac{B x^m}{m} + \frac{C x^{m-1}}{m-1} + \&c.\right)\frac{1}{a} + \text{constant.}$$

Again, if m be $>$ n the above can always be reduced by actual division to the form

$$A' x^{m-n} + B' x^{m-n-1} + \&c. + \frac{A'' x^{n-1} + B'' x^{n-2} + \&c.}{a x^n + b x^{n-1} + \&c.}$$

and if the whole be multiplied by d x its integral will consist of two parts, one of which is found to be (by 77)

$$\frac{A'}{m-n+1} \cdot x^{m-n+1} + \frac{B' \cdot x^{m-n}}{m-n} + \&c.$$

and the other

$$\int \frac{A'' x^{n-1} + B'' x^{n-2} + \&c.}{a x^n + b x^{n-1} + \&c.} dx.$$

Hence then it is necessary to consider only functions of the general form

$$\frac{x^{n-1} + A x^{n-2} + B x^{n-3} + \&c.}{x^n + a x^{n-1} + b x^{n-2} + \&c.} = \frac{U}{V}$$

in order to integrate an indefinite difference, whose definite part is any rational function whatever.

Case 1. *Let the denominator* V *consist of* n *unequal real factors*, $x - \alpha$, $x - \beta$, &c. *according to the theory of algebraic equations.* Assume

$$\frac{U}{V} = \frac{P}{x-\alpha} + \frac{Q}{x-\beta} + \frac{R}{x-\gamma} + \&c.$$

and reducing to a common denominator we shall have

$$\begin{aligned} U = {} & P \cdot \overline{x-\beta} \cdot \overline{x-\gamma} \ldots \text{ to } (n-1) \text{ terms} \\ & + Q \cdot \overline{x-\alpha} \cdot \overline{x-\gamma} \cdot \ldots \\ & + R \cdot \overline{x-\alpha} \cdot \overline{x-\beta} \cdot \ldots \\ & = (P + Q + R + \&c.) x^{n-1} \\ & - \{P \cdot (\underset{1}{S} - \alpha) + Q \cdot (\underset{1}{S} - \beta) + \&c.\} x^{n-2} \\ & + \{P \cdot (\underset{1.2}{S} - \alpha \cdot \overline{\underset{1}{S} - \alpha}) + Q \cdot (\underset{1.2}{S} - \beta \cdot \overline{\underset{1}{S} - \beta}) + \&c.\} x^{n-3} \\ & - \&c. \end{aligned}$$

where $\underset{1}{S}$, $\underset{1.2}{S}$ &c. denote the sum of α, β, γ &c. the sum of the products of every two of them and so on.

But by the theory of equations

$$\begin{aligned} S_1 &= -a \\ S_{1.2} &= b \\ \&c. &= \&c. \end{aligned}$$

$$\begin{aligned} \therefore U = {} & (P + Q + R + \&c.)\, x^{n-1} \\ & + \{a(P + Q + R + \&c.) + P\alpha + Q\beta + R\gamma + \&c.\} \times x^{n-2} \\ & + \{b(P + Q + R + \&c.) + a(P\alpha + Q\beta + \&c.) + \\ & (P\alpha^2 + Q\beta^2 + R\gamma^2 + \&c.)\}\, x^{n-3} + \&c. \end{aligned}$$

Hence equating like quantities (6)

$$\begin{aligned} P + Q + R + \&c. &= 1 \\ a + P\alpha + Q\beta + R\gamma + \&c. &= A \\ b + a(A - a) + P\alpha^2 + Q\beta^2 + R\gamma^2 + \&c. &= B \\ \&c. &= \&c. \end{aligned}$$

giving n independent equations to determine P, Q, R, &c.

Ex. 1. Let $\frac{U}{V} = \frac{x^2 + 6x + 3}{x^3 + 6x^2 + 11x + 6}$

Here

$$\left.\begin{aligned} P + Q + R &= 1 \\ 6 + P + 2Q + 3R &= 6 \\ 11 + P + 4Q + 9R &= 3 \end{aligned}\right\} \text{whence}$$

$$P = -1,\ Q = 5 \text{ and } R = -3$$

Hence

$$\begin{aligned} \int \frac{U\,dx}{V} &= \int \frac{-dx}{x+1} + \int \frac{5\,dx}{x+2} - \int \frac{3\,dx}{x+3} \\ &= C - \text{l.}(x+1) + 5\,\text{l.}(x+2) - 3\,\text{l.}(x+3). \end{aligned}$$

P, Q, R, &c. may be more easily found as follows:

Since

$$\begin{aligned} x^{n-1} + A x^{n-2} \&c. = {} & P(x - \beta).(x - \gamma).\ \&c. \\ & + Q(x - \alpha).(x - \gamma).\ \&c. \\ & + R(x - \alpha).(x - \beta).\ \&c. \\ & + \&c. \end{aligned}$$

let $x = \alpha, \beta, \gamma$, &c. successively; we shall then have

$$\left.\begin{aligned} \alpha^{n-1} + A\alpha^{n-2} + \&c. &= P.(\alpha - \beta).(\alpha - \gamma)\,\&c. \\ \beta^{n-1} + A\beta^{n-2} + \&c. &= Q.(\beta - \alpha).(\beta - \gamma)\,\&c. \\ \gamma^{n-1} + A\gamma^{n-2} + \&c. &= R.(\gamma - \alpha).(\gamma - \beta)\,\&c. \\ \&c. &= \&c. \end{aligned}\right\} \ldots (A)$$

In the above example we have

$$\alpha = -1,\ \beta = -2,\ \gamma = -3 \text{ and } n = 3$$

$$A = 6 \text{ and } B = 3.$$

$$\therefore P = \frac{1 - 6 + 3}{1.\ 2.} = -1$$

$$Q = \frac{4 - 6.2 + 3}{-1.\ 1} = 5$$

$$R = \frac{9 - 6.3 + 3}{-2 \times -1} = -3$$

as before.

Hence then the factors of V being supposed all unequal, either of the above methods will give the coefficients P, Q, R, &c. and therefore enable us to analyze the general expression $\frac{U}{V}$ into the *partial* fractions as expressed by

$$\frac{U}{V} = \frac{P}{x - \alpha} + \frac{Q}{x - \beta} + \&c.$$

and we then have

$$\int \frac{U\,dx}{V} = P \cdot l\,(x - \alpha) + Q\,l.\,(x - \beta) + \&c. + C.$$

Ex. 2. $$\int \frac{a^3 + b x^2}{a^2 x - x^3} = \int \frac{a\,dx}{x} + \frac{a + b}{2}\int \frac{dx}{a - x} - \frac{a + b}{2}\int \frac{dx}{a + x}$$

$$= a\,l\,x - \frac{a + b}{2}\,l\,(a - x) - \frac{a + b}{2}\,l\,.\,(a + x) + C$$

$$= a\,l\,x - (a + b)\,l\,\sqrt{a^2 - x^2} + C$$

by the nature of logarithms.

Ex. 3. $$\int \frac{3x - 5}{x^2 - 6x + 8}\,dx = -\tfrac{1}{2}\int \frac{dx}{x - 2} + \tfrac{7}{2}\int \frac{dx}{x - 4} + C$$

$$= \tfrac{7}{2}\,.\,l\,(x - 4) - \tfrac{1}{2}\,l\,.\,(x - 2) + C.$$

Ex. 4. $$\int \frac{x\,dx}{x^2 + 4ax - b^2} = \int \frac{P\,dx}{x + \alpha} + \int \frac{Q\,dx}{x + \beta} = P\,l\,(x + \alpha)$$

$$+ Q\,l.\,(x + \beta) + C$$

where

$$\alpha = 2a + \sqrt{(4a^2 + b^2)},\ \beta = 2a - \sqrt{(4a^2 + b^2)}$$

and

$$P = \frac{\alpha}{\alpha - \beta} = \frac{2a + \sqrt{(4a^2 + b^2)}}{2\sqrt{(4a^2 + b^2)}}$$

$$Q = \frac{-\beta}{\alpha - \beta} = \frac{\sqrt{(4a^2 + b^2)} - 2a}{2\sqrt{(4a^2 + b^2)}}$$

Case 2. *Let all the factors of* V *be real and equal, or suppose* $\alpha = \beta = \gamma = \&c.$

Then

$$\frac{U}{V} = \frac{x^{n-1} + A x^{n-2} + \&c.}{(x - \alpha)^n}.$$

and since

$$\alpha - \beta = 0,\ \alpha - \gamma = 0 \text{ \&c.}$$

the forms marked (A) will not give us P, Q, R, &c. In this case we must assume

$$\frac{U}{V} = \frac{P}{(x-\alpha)^n} + \frac{Q}{(x-\alpha)^{n-1}} + \frac{R}{(x-\alpha)^{n-2}} + \text{\&c.}$$

to $\overline{n-1}$ terms, and reducing to a common denominator, we get

$$U = P + Q.(x-\alpha) + R(x-\alpha)^2 + \text{\&c.}$$

now let $x = \alpha$, and we have

$$\alpha^{n-1} + A\alpha^{n-2} + \text{\&c.} = P.$$

Also

$$\frac{dU}{dx} = Q + 2R.(x-\alpha) + 3S.(x-\alpha)^2 + \text{\&c.}$$

$$\frac{d^2U}{dx^2} = 2R + 3.2.S.(x-\alpha) + 4.3.T(x-\alpha)^2 + \text{\&c.}$$

$$\frac{d^3U}{dx^3} = 2.3.S + 4.3.2T(x-\alpha) + \text{\&c.}$$

$$\text{\&c.} = \text{\&c.}$$

and if in each of these x be put $= \alpha$, we have by Maclaurin's theorem the values of Q, R, S, &c.

Ex. 1. Let $\frac{U}{V} = \frac{x^2 - 3x + 2}{(x-4)^3}$.

Then

$$U = x^2 - 3x + 2$$

$$\frac{dU}{dx} = 2x - 3$$

$$\frac{d^2U}{dx^2} = 2$$

$$\therefore P = 6$$

$$Q = 8 - 3 = 5$$

$$R = \tfrac{1}{2}.2 = 1$$

$$\therefore \int \frac{U\,dx}{V} = \int \frac{6\,dx}{(x-4)^3} + \int \frac{5\,dx}{(x-4)^2} + \int \frac{dx}{x-4}$$

$$= C - 3\frac{1}{(x-4)^2} - \frac{5}{x-4} + l(x-4)$$

$$= C + \frac{17 - 5x}{(x-4)^2} + l(x-4).$$

Ex. 2. Let $\frac{U}{V} = \frac{x^5 + x^3}{(x-3)^6}$.

Here

$$U = x^5 + x^3$$

$$\frac{dU}{dx} = 5x^4 + 3x^2$$

$$\frac{d^2U}{dx^2} = 20x^3 + 6x$$

$$\frac{d^3U}{dx^3} = 60x^2 + 6$$

$$\frac{d^4U}{dx^4} = 120x$$

$$\frac{d^5U}{dx^5} = 120$$

$$\therefore P = 3^5 + 3^3 = 27 \times 10 = 270$$

$$Q = 27 \times 16 = 432$$

$$R = \frac{20 \times 27 + 6 \times 3}{2} = 279$$

$$S = \frac{9 \times 60 + 6}{2 \times 3} = 91$$

$$T = \frac{360}{2.3.4.} = 15$$

$$W = \frac{120}{2.3.4.5} = 1.$$

Hence

$$\int \frac{x^5 + x^3}{(x-3)^6} dx = 270 \int \frac{dx}{(x-3)^6} + 432 \int \frac{dx}{(x-3)^5} + 279 \int \frac{dx}{(x-3)^4}$$

$$+ 91 \int \frac{dx}{(x-3)^3} + 15 \int \frac{dx}{(x-3)^2} + \int \frac{dx}{x-3} + C = C - 54 \cdot \frac{1}{(x-3)^5}$$

$$- 108 \cdot \frac{1}{(x-3)^4} - 93 \cdot \frac{1}{(x-3)^3} - \frac{91}{2} \cdot \frac{1}{(x-3)^2} - 15 \cdot \frac{1}{x-3} + 1 \cdot (x-3)$$

which admits of farther reduction.

Ex. 3. Let $\frac{x^2 + x}{(x-1)^5} = \frac{U}{V}$.

Here

$$U = x^2 + x$$

$$\frac{dU}{dx} = 2x + 1$$

and

$$\frac{d^2U}{dx^2} = 2.$$

Hence

$$P = 2$$
$$Q = 3$$
$$R = \tfrac{2}{2} = 1$$

$$\therefore \int \frac{x^2 + x}{(x-1)^5}\, dx = 2\int \frac{dx}{(x-1)^5} + 3\int \frac{dx}{(x-1)^4} + \int \frac{dx}{(x-1)^3}$$
$$= C - \tfrac{1}{2} \cdot \frac{1}{(x-1)^4} - \frac{1}{(x-1)^3} - \tfrac{1}{2} \cdot \frac{1}{(x-1)^2}$$
$$= C - \frac{x^2}{2(x-1)^4}$$

It appears from this example, and indeed is otherwise evident, that the number of partial fractions into which it is necessary to split the function exceeds the dimension of x *in* U, *by unity.*

This is the first time, unless we mistake, that *Maclaurin's Theorem* has been used to analyze rational fractions into partial rational fractions. It produces them with less labour than any other method that has fallen under our notice.

Case 3. *Let the factors of the denominator* V *be all imaginary and unequal.*

We know then if in V, which is real, there is an imaginary factor of the form $x + h + k\sqrt{-1}$, then there is also another of the form $x + h - k\sqrt{-1}$. Hence V must be of an even number of dimensions, and must consist of quadratic real factors of the form arising from

$$(x + h + k\sqrt{-1})(x + h - k\sqrt{-1})$$

or of the form

$$(x + h)^2 + k^2.$$

Hence, assuming

$$\frac{U}{V} = \frac{P + Qx}{(x+\alpha)^2 + \beta^2} + \frac{P' + Q'x}{(x+\alpha')^2 + \beta'^2} + \&c.$$

and reducing to a common denominator, we have

$$U = (P + Qx)\{(x+\alpha')^2 + \beta'^2\}\{(x+a'')^2 + \beta''^2\} \times \&c.$$
$$+ (P' + Q'x)\{(x+\alpha)^2 + \beta^2\}\{(x+\alpha'')^2 + \beta''^2\} \times \&c.$$
$$+ (P'' + Q''x)\{(x+\alpha)^2 + \beta^2\}\{(x+\alpha')^2 + \beta'^2\} \times \&c.$$
$$+ \&c.$$

Now for x substitute successively

$$\alpha + \beta\sqrt{-1},\ \alpha' + \beta'\sqrt{-1},\ \alpha'' + \beta''\sqrt{-1},\ \&c.$$

then U will become for each partly real and partly imaginary, and we have as many equations containing respectively P, Q; P', Q'; P'', Q'', &c. as there are pairs of these coefficients; whence by equating *homogeneous* quantities, viz. real and imaginary ones, we shall obtain P, Q; P', Q', &c.

Ex. 1. *Required the integral of*

$$\frac{x^3\,dx}{x^4+3x^2+2}.$$

Here the quadratic factors of V are x^2+1, x^2+2

$$\therefore \alpha = 0,\ \alpha' = 0,\ \beta = 1,\ \text{and } \beta' = \sqrt{2}.$$

Consequently

$$x^3 = (P+Qx)(x^2+2) + (P'+Q'x)(x^2+1)$$

Let $x = \sqrt{-1}$. Then

$$-\sqrt{-1} = (P+Q\sqrt{-1}).(-1+2) = P+Q\sqrt{-1}$$

$$\therefore P = 0,\ Q = -1$$

Again, let $x = \sqrt{2}.\sqrt{-1}$, and we have

$$-2^{\frac{3}{2}}\sqrt{-1} = (P'+Q'\sqrt{2}.\sqrt{-1})(-2+1) = -P'-Q'\sqrt{2}.\sqrt{-1}$$

$$\therefore P' = 0,\ \text{and } Q' = 2$$

Hence

$$\int\frac{x^3\,dx}{x^4+3x^2+2} = \int\frac{-x\,dx}{x^2+1} + \int\frac{2x\,dx}{x^2+2} = C - \tfrac{1}{2}l(x^2+1) + l.(x^2+2)$$

Ex. 2. *Required the integral of*

$$\frac{dx}{1+x^{2n}}.$$

To find the quadratic factors of

$$1+x^{2n}$$

we assume

$$x^{2n}+1 = 0,$$

and then we have

$$x^{2n} = -1 = \cos.(2p+1)\pi + \sqrt{-1}\sin.(2p+1)\pi$$

π being 180° of the circle whose diameter is 1, and p any integer whatever.

Hence by Demoivre's Theorem

$$x = \cos.\frac{2p+1}{2n}\pi + \sqrt{-1}.\sin.\frac{2p+1}{2n}\pi$$

But since imaginary roots of an equation enter it by pairs of the form $A \pm \sqrt{-1}.B$, we have also

$$x = \cos.\frac{2p+1}{2n}\pi - \sqrt{-1}.\sin.\frac{2p+1}{2n}\pi$$

and

$$\therefore \left(x - \cos.\frac{2p+1}{2n}\pi - \sqrt{-1}.\sin.\frac{2p+1}{2n}\pi\right) \times \left(x - \cos.\frac{2p+1}{2n}\pi + \sqrt{-1}\ \sin.\frac{2p+1}{2n}\pi\right) = x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1$$

which is the general quadratic factor of $x^{2n}+1$. Hence putting $p = 0, 1, 2 \ldots\ldots n-1$ successively,

$$x^{2n}+1 = \left(x^2 - 2x\cos.\frac{\pi}{2n} + 1\right).\left(x^2 - 2x\cos.\frac{3\pi}{2n} + 1\right) \times \left(x^2 - 2x\cos.\frac{5\pi}{2n} + 1\right) \times \ldots\ldots \left(x^2 - 2x\cos.\frac{2n-1}{2n} + 1\right).$$

Hence to get the values of P and Q corresponding to the general factor, assume

$$\frac{1}{1+x^{2n}} = \frac{P + Qx}{x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1} + \frac{N}{M}.$$

Then

$$1 = (P + Qx).M + N\left(x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1\right).$$

But

$$M = \frac{1+x^{2n}}{x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1}$$

and becomes of the form $\frac{0}{0}$ when for x we put $\cos.\frac{2p+1}{2n}\pi + \sqrt{-1}\times \sin.\frac{2p+1}{2n}\pi$; its value however may thus be found

Let $\cos.\frac{2p+1}{2n}\pi + \sqrt{-1}\sin.\frac{2p+1}{2n}\pi = r$

then

$$\cos.\frac{2p+1}{2n}\pi - \sqrt{-1}.\sin.\frac{2p+1}{2n}\pi = \frac{1}{r}$$

and

$$M = \frac{1+x^{2n}}{(x - r.\left(x - \frac{1}{r}\right)}$$

Again let $x - r = y$; then

$$M = \frac{1 + y^{2n} + 2n\ y^{2n-1}r + \&c. \ldots. 2n\,y r^{2n-1} + r^{2n}}{y\left(x - \frac{1}{r}\right)}$$

But

$$r^{2n} = \cos.\overline{2p+1}.\pi + \sqrt{-1}\sin.\overline{2p+1}.\pi = -1$$

$$\therefore M = \frac{y^{2n-1} + 2n\, y^{2n-2}.r + \ldots . 2n\, r^{2n-1}}{x - \frac{1}{r}}.$$

Hence when for x we put r, y = 0, and

$$M = \frac{2n\, r^{2n-1}}{r - \frac{1}{r}}$$

and from the above equation we have

$$1 = (P + Q\,r)\frac{2n\, r^{2n-1}}{r - \frac{1}{r}}.$$

or

$$2\sqrt{-1}\sin.\frac{2p+1}{2n}\pi = 2n\,P.\cos.\frac{\overline{2p+1}.\overline{2n-1}}{2n}\pi + 2n\,P\sqrt{-1}\times$$

$$\sin.\frac{\overline{2p+1}.\overline{2n-1}}{2n}\pi - 2n\,Q \text{ (since } r^{2n} = -1)$$

$\therefore$ equating *homogeneous quantities* we get

$$\sin.\frac{2p+1}{2n}\pi = nP.\sin.\frac{\overline{2p+1}.\overline{2n-1}}{2n}\pi$$

and

$$P.\cos.\frac{\overline{2p+1}.\overline{2n-1}}{2n}\pi = Q.$$

But

$$\frac{\overline{2p+1}.\overline{2n-1}.\pi}{2n} = \overline{2p+1}.\pi - \frac{2p+1}{2n}\pi$$

Hence the above equations become

$$\therefore \sin.\frac{2p+1}{2n}\pi = n\,P\sin.\frac{2p+1}{2n}$$

$$-P\cos.\frac{2p+1}{2n}\pi = Q$$

$$\therefore P = \frac{1}{n}, \text{ and } Q = -\frac{1}{n}.\cos.\frac{2p+1}{2n}\pi.$$

Hence the general partial integral of

$$\frac{dx}{1+x^{2n}} \text{ is}$$

$$\frac{1}{n}\int \frac{\left(1 - x\cos.\frac{2p+1}{2n}\pi\right)dx}{x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1} =$$

$$-\frac{\cos.\frac{2p+1}{2n}\pi}{2n}\cdot\int \frac{2x\,dx - 2\cos.\frac{2p+1}{2n}\pi\,.\,dx}{x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1} +$$

$$\frac{\sin.^2\frac{2p+1}{2n}\pi}{n}\int \frac{dx}{x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1} =$$

$$C - \frac{\cos.\frac{2p+1}{2n}\pi}{2n}\,.\,l\left(x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1\right)$$

$$+\frac{\sin.\frac{2p+1}{2n}\pi}{n} \times \tan.^{-1}\left(\frac{x - \cos.\frac{2p+1}{2n}\pi}{\sin.\frac{2p+1}{2n}\pi}\right)$$

see Art. 88. Case 4.

Hence then the integral of $\frac{dx}{1+x^n}$, which is the aggregate of the results obtained from the above general form by substituting for $p = 0, 1, 2 \ldots n - 1$, may readily be ascertained.

As a particular instance let $\int\frac{dx}{1+x^6}$ *be required.*

Here

$$n = 3$$

and the general term is

$$-\frac{\cos.\frac{2p+1}{6}\pi}{6}\,.\,l.\left(x^2 - 2x\cos.\frac{2p+1}{6}\pi + 1\right)$$

$$+\frac{\sin.\frac{2p+1}{6}\pi}{3}\,.\,\tan.^{-1}\frac{x - \cos.\frac{2p+1}{6}\pi}{\sin.\frac{2p+1}{6}\pi}$$

Let $p = 0, 1, 2$, collect the terms, and reduce them; and it will appear that

$$\int\frac{dx}{1+x^6} = \frac{1}{6}\left\{\frac{\sqrt{3}}{2}\,.\,l.\,\frac{x^2 + x\sqrt{3} + 1}{x^2 - x\sqrt{3} + 1} + \tan.^{-1}\frac{3x(1-x^2)}{x^4 - 4x^2 + 1}\right\} + C.$$

By proceeding according to the above method it will be found, that the general partial fractions to be integrated in the integrals of

$$\frac{dx}{x^n - 1} \text{ and } \frac{x^r dx}{x^n - 1}$$

are respectively

$$\frac{2}{n} \cdot \frac{\cos. \frac{2p\pi}{n} x - 1}{x^2 - 2x \cos. \frac{2p\pi}{n} + 1} \cdot dx$$

and

$$\frac{2}{n} \times \frac{\cos. \frac{(r+1) \cdot 2p\pi}{n} x - \cos. \frac{2rp\pi}{n}}{x^2 - 2 \cos. \frac{2p\pi}{n} x + 1} dx.$$

and when these partial integrals are obtained, the entire ones will be found by putting $p = 0, 1 \ldots . \frac{n}{2}$ or $\frac{n-1}{2}$ according as p is even or odd.

Ex. 3. *Required the integral of*

$$\frac{x^r dx}{x^{2n} - 2ax^n + 1}$$

where a *is* < 1.

First let us find the quadratic factors of $x^{2n} - 2ax^n + 1$. For that purpose put

$$x^{2n} - 2ax^n = -1$$

Then

$$x^n = a \pm \sqrt{a^2 - 1}$$
$$= a \pm \sqrt{-1} \cdot \sqrt{1 - a^2}$$

since a is < 1.

Now put $a = \cos. \delta$; then

$$x^n = \cos. \delta \pm \sqrt{-1} \sin. \delta$$
$$= \cos. (2p\pi + \delta) \pm \sqrt{-1} \sin. (2p\pi + \delta)$$
$$\therefore x = \cos. \frac{2p\pi + \delta}{n} \pm \sqrt{-1} \sin. \frac{2p\pi + \delta}{n}$$

and the general quadratic factor of

$$x^{2n} - 2ax^n + 1$$

is

$$x^2 - 2x \cos. \frac{2p\pi + \delta}{n} + 1$$

where p may be any number from 0, 1, &c. to $n - 1$.

Hence to find the general partial integral of the given indefinite difference, we assume

$$\frac{x^r}{x^{2n} - 2ax^n + 1} = \frac{P + Qx}{x^2 - 2 \cos. \frac{2p\pi + \delta}{n} + 1} + \frac{N}{M}$$

and proceeding as in the last example, we get

$$Q = \text{sin.}\ \frac{(r-1+1)\ (2p\varpi+\delta)}{n} \times \frac{1}{n \text{ sin. } \delta}$$

and

$$P = \text{sin.}\ \frac{(n-r)\,.\,(2p\varpi+\delta)}{n} \times \frac{1}{n \text{ sin. } \delta}$$

whence the remainder of the process is easy.

Case 4. *Let the factors of the denominator be all imaginary and equal in pairs.*

In this Case, we have the form

$$\frac{U}{V} = \frac{U}{\{(x+\alpha)^2+\beta^2\}^n}$$

and assuming as in Case 2.

$$\frac{U}{V} = \frac{P+Qx}{(x+\alpha)^2+\beta^2)^n} + \frac{P'+Q'x}{(x+\alpha)^2+\beta^2)^{n-1}} + \&c.$$
$$+ \frac{K+Lx}{(x+\alpha)^2+\beta^2)^2} + \frac{K'+L'x}{x+\alpha)^2+\beta^2}$$

and reducing to a common denominator,

$$U = P + Qx + (P' + Q'x)\,(x+\alpha)^2 + \beta^2) + \&c.$$

and substituting for x one of its imaginary values, and equating homogeneous terms, in the result we get P and Q. Deriving from hence the values of $\frac{dU}{dx}$, $\frac{d^2U}{dx^2}$, &c. and in each of these values substituting for x one of the quantities which makes $x+\alpha)^2+\beta^2 = 0$, and equating homogeneous terms we shall successively obtain

$$P', Q'; P'', Q'', \&c.$$

This method, however, not being very commodious in practice, for the present case, we shall recommend either the actual developement of the above expression according to the powers of x, and the comparison of the coefficients of the like powers (by art. 6), or the following method.

Having determined P and Q as above, make

$$U' = \frac{U-(P+Qx)}{x+\alpha)^2+\beta^2}$$
$$U'' = \frac{U'-(P'+Q'x)}{x+\alpha)^2+\beta^2}$$
$$U''' = \frac{U''-(P''+Q''x)}{(x+\alpha)^2+\beta^2}$$
$$\&c. = \&c.$$

Then since U', U'', U''', &c. have the same form as U, or have an

integer form, if we put for x that value which makes $(x + \alpha)^2 + \beta^2 = 0$, and afterwards in the several results, equate homogeneous quantities we shall obtain the several coefficients.

$$P', Q'; P'', Q'', \&c.$$

Case 5. *If the denominator* V *consist of one set of Factors simple and unequal of the form*

$$x - \alpha\, x - \alpha', \&c.;$$

of several sets of equal simple Factors, as

$$(x - e)^p, (x - e')^q, \&c.$$

and of equal and unequal sets of quadratic factors of the forms

$$x^2 + ax + b, x^2 + a'x + b', \&c.$$

$$(x^2 + lx + r)^\mu, (x^2 + l'x + r')^\nu, \&c.$$

then the general assumption for obtaining the partial fractions must be

$$\frac{U}{V} = \frac{M}{x - \alpha} + \frac{M'}{x - \alpha'} + \&c.$$

$$+ \frac{E}{(x - e)^p} + \frac{F}{(x - e)^{p-1}} + \&c. \frac{E'}{(x - e')^q} + \frac{F'}{(x - e')^{q-1}} + \&c.$$

$$+ \frac{P + Qx}{x^2 + ax + b} + \frac{P' + Q'x}{x^2 + a'x + b'} + \&c.$$

$$+ \frac{R + Sx}{(x^2 + lx + r)^\mu} + \frac{R' + S'x}{(x^2 + lx + r)^{\mu-1}} + \&c. \frac{G + Hx}{(x^2 + l'x + r')^\nu} + \frac{G' + H'x}{(x^2 + l'x + r')^{\nu-1}} + \&c.$$

and the several coefficients may be found by applying the foregoing rules for each corresponding set. They may also be had at once by reducing to a common denominator both sides of the equation, and arranging the numerators according to the powers of x, and then equating homogeneous quantities.

We have thus shown that every rational fraction, whose denominator can be decomposed into simple or quadratic factors, may be itself analyzed into as many partial fractions as there are factors, and hence it is clear that the integral of the general function

$$\frac{Ax^m + Bx^{m-1} + \&c.\ Kx + L}{ax^n + bx^{n-1} + \&c.\ kx + l}\, dx$$

may, under these restrictions, always be obtained. It is always reducible, in short, to one or other or a combination of the forms

$$\int x^m\, dx, \int \frac{dx}{x + l}, \int \frac{dx}{x^2 + l}.$$

Having disposed of rational forms we next consider irrational ones. Already (see Art. 86, &c.)

$$\int \frac{\pm dx}{\sqrt{(a - bx^2)}}, \int \frac{dx}{x\sqrt{(bx^2 - a)}}, \int \frac{dx}{\sqrt{(ax - bx^2)}}$$

have been found in terms of circular arcs. We now proceed to treat of Irrationals generally; and the most natural and obvious way of so doing is to investigate such forms as admit of being *rationalized.*

90. *Required the integral of*

$$dx \times F\left\{x,\ x^{\frac{1}{m}},\ x^{\frac{1}{n}},\ x^{\frac{1}{p}},\ x^{\frac{1}{q}},\ \&c.\right\}$$

where F *denotes any rational function of the quantities between the brackets.*

Let

$$x = u^{mnpq}, \ \&c.$$

Then

$$x^{\frac{1}{m}} = u^{npqr\ldots.}$$

$$x^{\frac{1}{n}} = u^{mpqr\ldots.}$$

$$x^{\frac{1}{p}} = u^{mnqr\ldots.}$$

$$\&c. = \&c.$$

and

$$dx = mnpq\ldots. \times u^{mnpq\ldots.-1} \times du$$

and substituting for these quantities in the above expression, it becomes rational, and consequently integrable by the preceding article.

Ex. $\dfrac{x^3 + 2ax^{\frac{2}{3}} + x^{\frac{1}{3}}}{b + cx^{\frac{1}{4}}}\, dx$

Here

$$x = u^{60}$$

$$x^3 = u^{180}$$

$$x^{\frac{2}{3}} = u^{94}$$

$$x^{\frac{1}{3}} = u^{20}$$

$$x^{\frac{1}{4}} = u^{15}$$

and

$$dx = 6u^{59}du.$$

Hence the expression is transformed to

$$60u^{79}du\ \frac{u^{160} + 2au^4 + 1}{b + cu^{15}}$$

whose integral may be found by Art. 89, Case 3, Ex. 2.

91. *Required the integral of*

$$dx \times F\left\{x,\ (a + bx)^{\frac{1}{n}},\ (a + bx)^{\frac{1}{m}},\ \&c.\right\}$$

where F, as before, means any rational function.

Put $a + bx = u^{nmp\ldots.}$ then substitute, and we get

$$\frac{nmp\ldots.}{b}.\ u^{nmp\ldots.-1}du \times F\left(\frac{u^{nm\ldots.} - a}{b},\ u^{mp\ldots.},\ u^{np\ldots.},\ \&c.\right)$$

which is rational.

Examples to this general result are

$$\frac{x^4 dx}{cx^5 + (a+bx)^{\frac{3}{4}}} \text{ and } \frac{x^2 dx (a+bx)^{\frac{2}{3}}}{x + c(a+bx)^{\frac{2}{7}}},$$

which are easily resolved.

92. *Required the integral of*

$$dx\, F\left\{x, \left(\frac{a+bx}{f+gx}\right)^{\frac{m}{n}}, \left(\frac{a+bx}{f+gx}\right)^{\frac{p}{q}}, \text{ &c.}\right\}.$$

Assume

$$\frac{a+bx}{f+gx} = u^{nqs\ldots}$$

and then by substituting, the expression becomes rational and integrable.

93. *Required the integral of*

$$dx\, F\{x, \sqrt{(a+bx+cx^2)}\}$$

Case 1. When c is positive, let

$$a + bx + cx^2 = c(x+u)^2.$$

Then

$$x = \frac{a - cu^2}{2cu - b} \text{ and } dx = -\frac{2c(cu^2 - bu + a)\,du}{(2cu-b)^2}$$

$$\sqrt{(a+bx+cx^2)} = \frac{cu^2 - bu + a}{2cu - b}.\sqrt{c}$$

and substituting, the expression becomes *rational.*

Case 2. When c is negative, if r, r′ be the roots of the equation

$$a + bx - cx^2 = 0$$

Then assume

$$\sqrt{c(x-r)(r'-x)} = (x-r)cu$$

and we have

$$x = \frac{cru^2 + r'}{cu^2 + 1},\ dx = \frac{(r-r')\,2cu\,du}{(cu^2+1)^2}$$

$$\sqrt{(a+bx-cx^2)} = \frac{(r'-r)cy}{cy^2+1}$$

and by substitution, the expression becomes rational.

94. *Required the integral of*

$$dx\, F\left\{x, (a+bx)^{\frac{1}{2}}, (a'+b'x)^{\frac{1}{2}}\right\}.$$

Make

$$a + bx = (a' + b'x)u^2;$$

Then

$$x = \frac{a - a'u^2}{b'u^2 - b},\ dx = \frac{(a'b - b'a)\,2u\,du}{(b'u^2-b)^2}$$

$$\sqrt{(a+bx)} = \frac{u\sqrt{(ab'-a'b)}}{\sqrt{(b'u^2-b)}},\ \sqrt{(a'+b'x)} = \frac{\sqrt{(ab'-a'b)}}{\sqrt{(b'u^2-b)}}.$$

Hence, substituting, the above expression becomes of the form

$$d u F' \{u, \surd (b' u^2 - b)\}$$

F′ denoting a rational function different from that represented by F. But this form may be rationalized by 93; whence the expression becomes integrable.

95. *Required the integral of*

$$x^{m-1} d x (a + b x^n)^{\frac{p}{q}}.$$

This form may be rationalized when either $\frac{m}{n}$, or $\frac{m}{n} + \frac{p}{q}$ is an integer.

Case 1. Let $a + b x^n = u^q$; then $(a + b x^n)^{\frac{p}{q}} = u^p$, $x^n = \frac{u^q - a}{b}$, $x^m = \left(\frac{u^q - a}{b}\right)^{\frac{m}{n}}$, $x^{m-1} d x = \frac{q u^{q-1} d u}{n b} \left(\frac{u^q - a}{b}\right)^{\frac{m-n}{n}}$.

Hence the expression becomes

$$\frac{q}{n b} \cdot u^{p+q-1} d u \left(\frac{u^q - a}{b}\right)^{\frac{m-n}{n}}$$

which is rational and integrable when $\frac{m}{n}$ is an integer.

Case 2. Let $a + b x^n = x^n u^q$; then substituting as before, we get the transformed expression

$$\frac{q a^{\frac{m}{n} + \frac{p}{q}}}{n} \cdot \frac{u^{p+q-1} d u}{(u^q - b)^{\frac{m}{n} + \frac{p}{q} + 1}}$$

which is rational and integrable when $\frac{m}{n} + \frac{p}{q}$ is an integer.

Examples are

$$\frac{x^2 d x}{(a^2 + x^2)^{\frac{1}{2}}}, \quad \frac{x^{\pm 2m} d x}{(a^2 + x^2)^{\frac{1}{2}}},$$

$$x^{-2m} d x (a^2 + x^2)^{\frac{2m+1}{2}}, \quad \frac{x^6 d x}{(a^3 + x^3)^{\frac{1}{3}}}.$$

96. *Required the integral of*

$$x^{m-1} d x (a + b x^n)^{\frac{p}{q}} \times F(x^n).$$

This expression becomes rational in the same cases, and by the same substitutions, as that of 95. To this form belongs

$$x^{m+n-1} d x (a + b x^n)^{\frac{p}{q}}$$

and the more general one

$$\frac{P}{Q} x^{m-1} d x \times (a + b x)^{\frac{p}{q}}$$

where

$$P = A + Bx^n + Cx^{2n} + \&c.$$

and

$$Q = A' + B'x^n + C'x^{2n} + \&c.$$

97. *Required the integral of*

$$x^{m-1}dx \times F\left\{x^m, x^n, (a + bx^n)^{\frac{1}{q}}\right\}$$

Make $a + bx^n = u^q$; then

$$x^{m-1}dx = \frac{q u^{q-1} du}{nb} \cdot \left(\frac{u^q - a}{b}\right)^{\frac{m}{n}-1} du$$

and in the cases where $\frac{m}{n}$ is an integer, the whole expression becomes rational and integrable.

98. *Required the integral of*

$$\frac{X dx}{X' + X'' + \sqrt{(a + bx + cx^2)}}$$

where X, X′, X″ *denote any rational functions of* x.

Multiply and divide by

$$X' + X'' - \sqrt{(a + bx + x^2)}$$

and the result is, after reduction,

$$\frac{X X' dx}{X'^2 - X''^2(a + bx + cx^2)} - \frac{X X'' dx \sqrt{(a + bx + cx^2)}}{X'^2 - X''^2(a + bx + cx^2)}$$

consisting of a rational and an irrational part. The irrational part, in many cases, may also be rationalized, and thus the whole made integrable.

99. *Required the integral of*

$$x^m dx F\{x^n, \sqrt{(a + bx^n + cx^{2n})}\}$$

Let $x^n = u$; then the expression may be transformed into

$$\frac{1}{n} u^{\frac{m+1}{n}-1} du F\{u, \sqrt{(a + bu + cu^2)}\}$$

which may be rationalized by Art. 93, when $\frac{m+1}{n}$ is an integer.

100. *Required the integral of*

$$x^m dx F\{x^n, \sqrt{(a + b^2x^{2n})}, bx^n + \sqrt{(a + b^2x^{2n})}\}.$$

Let

$$bx^n + \sqrt{(a + b^2x^{2n})} = u;$$

then

$$x^m dx = \frac{1}{n(2b)^{\frac{m+1}{n}}} \cdot \frac{u^2 + a}{u} \cdot \left(\frac{u^2 - a}{u}\right)^{\frac{m+1}{n}-1} du$$

and the whole expression evidently becomes rational when $\frac{m+1}{n}$ is an integer.

Many other general expressions may be rationalized, and much might

be said further upon the subject; but the foregoing cases will exhibit the general method of such reductions. If the reader be not satisfied let him consult a paper in the Philosophical Transactions for 1816, by E. Ffrench Bromhead, Esq. which is decidedly the best production upon the Integrals of Irrational Functions, which has ever appeared.

Perfect as is the theory of Rational Functions, yet the like has not been attained with regard to Irrational Functions. The above and similar artifices will lead to the integration of a vast number of forms, and to that of many which really occur in the resolution of philosophical and other problems; but a method universally applicable has not yet been discovered, and probably never will be.

Hitherto the integrals of algebraic forms have been investigated. We now proceed to Transcendental Functions.

101. *Required the integral of*

$$a^x dx.$$

By Art. 17,

$$d.a^x = l.a \times a^x dx$$

$$\therefore \int a^x dx = \frac{1}{la} \int da^x$$

$$= \frac{1}{la}.a^x + C \quad \ldots\ldots \quad (a)$$

Hence

$$\int a^{mx} dx = \frac{1}{mla} a^{mx} + C \quad \ldots\ldots \quad (b)$$

102. *Required the integral of*

$$X a^x dx$$

where X *is an algebraic function of* x.

By the form (see 73)

$$d(uv) = u dv + v du$$

we have

$$\int u dv = uv - \int v du.$$

Hence

$$\int X a^x dx = X.\frac{a^x}{la} - \int \frac{a^x}{la} dX$$

$$\int \frac{dX}{dx}.\frac{a^x dx}{la} = \frac{dX}{dx}.\frac{a^x}{(la)^2} - \int \frac{a^x}{(la)^2} \frac{d^2X}{dx}$$

$$\int \frac{d^2X}{dx^2}.\frac{a^x dx}{(la)^2} = \frac{d^2X}{dx^2}.\frac{a^x}{(la)^3} - \int \frac{a^x}{(la)^2} \frac{d^3X}{dx^2}$$

$$\&c. = \&c.$$

the law of continuation being manifest.

Hence, by substitution,

$$\int X a^x dx = X\frac{a^x}{la} - \frac{dX}{dx}\cdot\frac{a^x}{(la)^2} + \frac{d^2X}{dx^2}\cdot\frac{a^x}{(la)^3} - \&c.$$

which will terminate when X is of the form

$$A + Bx + Cx^2 + \&c.$$

$$\text{Ex. } \int x^3 a^x dx = \frac{a^x x^3}{la} - \frac{3a^x x^2}{(la)^2} + \frac{3.2a^x x}{(la)^3} - \frac{3.2a^x}{(la)^4} + C.$$

OTHERWISE

$$\int a^x X dx = a^x \int X dx - \int la . a^x dx \int X dx$$
$$= a^x X' - la \int a^x X' dx$$

putting

$$X' = \int X dx.$$

Hence

$$\int a^x X' dx = a^x X'' - la \int a^x X'' dx$$
$$\&c. = \&c.$$

and substituting, we get

$$\int a^x X dx = a^x X' - la . a^x X'' + (la)^2 a^x X''' - \&c.$$

X', X'', X''', &c. being equal to $\int X dx$, $\int X' dx$, $\int X'' dx$, &c. respectively.

$$\text{Ex. } \int a^x \frac{dx}{x} = a^x lx + \frac{x la}{1} + \frac{x^2 (la)^2}{1.2.2} + \frac{x^3 (la)^3}{1.2.3.3} + \&c. + C.$$

which does not terminate.

By this last example we see how an Indefinite Difference may be integrated in an infinite series. If in that example x be supposed less than 1, the terms of the integral become less and less or the series is convergent. Hence then by taking a few of the first terms we get an approximate value of the integral, which in the absence of an exact one, will frequently suffice in practice.

The general formula for obtaining the integral in an infinite or finite series, corresponding to that of Taylor in the Calculus of Indefinite Differences, is the following one, ascribed to John Bernoulli, and usually termed

JOHN BERNOULLI'S THEOREM.

$$\int X dx = Xx - \int x dX$$
$$\int \frac{dX}{dx}\cdot x dx = \frac{dX}{dx}\cdot\frac{x^2}{2} - \int \frac{x^2 dx}{2}\cdot\frac{d^2X}{dx^2}$$
$$\int \frac{d^2X}{dx^2}\cdot\frac{x^2 dx}{2} = \frac{d^2X}{dx^2}\cdot\frac{x^3}{2.3} - \int \frac{x^3 dx}{2.3}\cdot\frac{d^3X}{dx^3}$$
$$\&c. = \&c.$$

Hence

$$\int X\,dx = Xx - \frac{dX}{dx}\cdot\frac{x^2}{2} + \frac{d^2X}{dx^2}\cdot\frac{x^3}{2.3} - \&c. + C$$

the theorem in question.

Ex. 1. $\int x^m\,dx = x^{m+1} - \frac{m}{2}x^{m+1} + \frac{m.m-1}{2.3}x^{m+1} - \&c. + C$

$$= x^{m+1}\times\left(1 - \frac{m}{2} + \frac{m.m-1}{2.3} + \&c.\right) + C$$

$$= \frac{x^{m+1}}{m+1}\left(m+1 - \frac{\overline{m+1}.m}{2} + \frac{\overline{m+1}.m.m-1}{2.3} - \&c.\right) + C$$

But since

$$(1-1)^{m+1} = 1 - \overline{m+1} + \frac{\overline{m+1}.m}{2} - \&c. = 0$$

$$\therefore \int x^m\,dx = \frac{x^{m+1}}{m+1} + C$$

as in Art. 78.

102. *Required the integral of*

$$X\,dx\,(lx)^n$$

where X *is any Algebraic Function of* x, l x *the Hyperbolic logarithm of* x, *and* n *a positive integer.*

By the formula

$$\int u\,dv = uv - \int v\,du$$

we have

$$\int X\,dx\,(lx)^n = (lx)^n\int X\,dx - n\int (lx)^{n-1}\frac{dx}{x}\int X\,dx$$

$$= (lx)^n X' - n\int (lx)^{n-1}\frac{dx}{x}X'$$

$$\int \frac{X'}{x}\,dx\,(lx)^{n-1} = (lx)^{n-1}X'' - (n-1)\int (lx)^{n-2}\frac{dx}{x}X''$$

$$\&c. = \&c.$$

where X′, X″, X‴, &c. are put for $\int X\,dx$, $\int\frac{X'}{x}\,dx$, $\int\frac{X''}{x}\,dx$, &c. respectively.

Hence

$$\int X\,dx\,(lx)^n = X'(lx)^n - nX''(lx)^{n-1} + n.(n-1)X'''(lx)^{n-2} - \&c. + C.$$

Ex. 1. $\int x^m\,dx\,(lx)^n = \frac{x^{m+1}}{m+1}\left\{(lx)^n - \frac{n}{m+1}(lx)^{n-1}\,\&c.\right\}$

193. *Required the integral of*

$$\frac{dx}{x}.U$$

where U *is any function of* l x.

Let $u = l x.$

Then

$$d u = \frac{d x}{x};$$

and substituting, the expression becomes algebraic, and therefore integrable in many cases.

104. *Required the integral of*

$$X d x (l x)^n$$

where n *is negative.*

Integrating *by Parts*, as it is termed, or by the formula

$$\int u d v = u v - \int v d u$$

we get, since

$$\int \frac{X d x}{(l x)^n} = \int X x . \frac{d x}{x} (l x)^{-n},$$

$$\int \frac{X d x}{(l x)^n} = -\frac{X x}{(n-1)(l x)^{n-1}} + \frac{1}{n-1} . \int \frac{d x}{(l x)^{n-1}} . \frac{d (X x)}{d x}$$

and pursuing the method, and writing

$$X' = \frac{d (X x)}{d x}$$

$$X'' = \frac{d (X' x)}{d x}$$

&c. = &c.

we have

$$\int \frac{X dx}{(lx)^n} = -\frac{X x}{(n-1)(lx)^{n-1}} - \frac{X' x}{n-1 . n-2 . (lx)^{n-2}} - \&c. - \int \frac{X^{(n)'} d x}{(n-1) \ldots 2 . 1 (lx)}$$

or

$$-\frac{X x}{(n-1)(l x)^{n-1}} - \&c. - \int \frac{X^{(m+1)'} d x}{(n-1) . (n-2) \ldots . (n-m) (l x)^{n-m}}$$

according as n is or is not an integer, m being in the latter case the greatest integer in n.

$$\text{Ex.} \int \frac{x^m d x}{(l x)^n} = -\frac{x^{m+1}}{n-1} \left\{ \frac{1}{(l x)^{n-1}} + \frac{m+1}{(n-2)(l x)^{n-2}} + \&c. \right\}$$
$$- \frac{(m+1)^{n-1}}{(n-1)(n-2) \ldots . 1} \int \frac{x^m d x}{l x}$$

when m is an integer.

105. *Required the integrals of*

$$d \theta . \cos. \theta, \quad d \theta . \sin. \theta, \quad d \theta . \tan. \theta, \quad d \theta . \sec. \theta, \quad \frac{d \theta}{\cos. \theta}, \quad \frac{d \theta}{\sin. \theta}, \quad \frac{d \theta}{\tan. \theta}.$$

By Art. 26, &c.

$$d \sin. \theta = d \theta . \cos. \theta, \text{ and } d \cos. \theta = - d \theta \sin. \theta$$

$$\therefore \int d \theta \cos. \theta = \sin. \theta + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and

$$\int d \theta \sin. \theta = C - \cos. \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Again let $\tan.\theta = t$; then

$$d\theta = \frac{dt}{1+t^2}$$

and

$$\int d\theta \tan.\theta = \int \frac{t\,dt}{1+t^2} = \tfrac{1}{2} l(1+t^2) + C$$
$$= C - l.\cos.\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

since

$$1 + t^2 = \sec.^2\theta = \frac{1}{\cos.^2\theta}.$$

Again

$$d\theta \sec.\theta = \frac{d\theta}{\cos.\theta} = \frac{d\theta \cos.\theta}{1 - \sin.^2\theta}$$
$$= \frac{d(\sin.\theta)}{1-\sin.^2\theta}$$
$$= \tfrac{1}{2}\cdot\frac{d(\sin.\theta)}{1-\sin.\theta} + \tfrac{1}{2}\cdot\frac{d\sin.\theta}{1+\sin.\theta}$$
$$\therefore \int d\theta \sec.\theta = \tfrac{1}{2} l.(1+\sin.\theta) - \tfrac{1}{2} l(1-\sin.\theta) + C$$
$$= l.\tan.\left(45^\circ + \frac{\theta}{2}\right) + C \quad . \quad . \quad . \quad (d)$$

which is the same as $\int \frac{d\theta}{\cos.\theta}$.

Again

$$\int \frac{d\theta}{\sin.\theta} = \int d\theta \,\mathrm{cosec.}\,\theta$$
$$= \int d\theta \sec.\left(\frac{\pi}{2} - \theta\right) = -\int d.\left(\frac{\pi}{2} - \theta\right) \sec.\left(\frac{\pi}{2} - \theta\right)$$
$$= -l.\tan.\left(45^\circ + \frac{\pi}{4} - \frac{\theta}{2}\right) + C$$
$$= l.\left(\tan.\frac{\theta}{2}\right) + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Again

$$\int \frac{d\theta}{\tan.\theta} = \int d\theta \cot.\theta = \int d\theta.\tan.\left(\frac{\pi}{2} - \theta\right)$$
$$= l\cos.\left(\frac{\pi}{2} - \theta\right) + C \text{ (by c)}$$
$$= l.\sin.\theta + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

106. *Required the integral of*

$$\sin.^m\theta \cos.^n\theta . d\theta.$$

m *and* n *being positive or negative integers.*

Let $\sin.\theta = u$; then $d\theta \cos.\theta = du$ and the above expression becomes

$$u^m du (1 - u^2)^{\frac{n-1}{2}}$$

which is integrable when either $\frac{m+1}{2}$ or $\frac{m+1}{2} + \frac{n-1}{2} = \frac{m+n}{2}$ is an integer (see 95.) If n be odd, the radical disappears; if n be even and m even also, then $\frac{m+n}{2}$ = an integer; if n be even and m odd, then $\frac{m+1}{2}$ is an integer. Whence

$$u^m du (1 - u^2)^{\frac{n-1}{2}}$$

is integrable by 95.

OTHERWISE,

Integrating by Parts, we have

$$\int d\theta \sin.^m\theta \cos.^n\theta = -\frac{\sin.^{m-1}\theta}{n+1}\cos.^{n+1}\theta + \frac{m-1}{m+1}\int \cos.^{n+2}\theta . \sin.^{m-2}\theta \times d\theta$$

$$= -\frac{\sin.^{m-1}\theta}{m+n}\cos.^{n+1}\theta + \frac{m-1}{m+n}\int dx \sin.^{m-2}\theta \cos.^n\theta$$

and continuing the process m is diminished by 2 each time.

In the same way we find

$$\int d\theta \sin.^m\theta \cos.^n\theta = \frac{\sin.^{m+1}\theta \cos.^{n-1}\theta}{m+n} + \frac{n-1}{m+n}\int dx \sin.^m\theta \cos.^{n-2}\theta$$

and so on.

107. *Required the integrals of*

$$du = d\theta \sin.(a\theta + b) \cos.(a'\theta + b')$$
$$dv = d\theta \sin.(a\theta + b) \sin.(a'\theta + b')$$

and

$$dw = d\theta \cos.(a\theta + b) \cos.(a'\theta + b')$$

By the known forms of Trigonometry we have

$$du = d\theta \{\sin.(\overline{a+a'}.\theta + b + b') + \sin.(\overline{a-a'}.\theta + b - b')\}$$
$$dv = d\theta \{\cos.(\overline{a+a'}.\theta + b + b') - \cos.(\overline{a-a'}.\theta + b - b')\}$$
$$dw = d\theta \{\cos.(\overline{a+a'}.\theta + b + b') + \cos.(\overline{a-a'}.\theta + b - b')\}$$

Hence by 105 we have

$$u = C - \tfrac{1}{2}\left\{\frac{\cos.(\overline{a+a'}.\theta + b + b')}{a+a'} + \frac{\cos.(\overline{a-a'}.\theta + b - b')}{a-a'}\right\}$$

$$v = C + \tfrac{1}{2}\left\{\frac{\sin.(\overline{a+a'}.\theta + b + b')}{a+a'} - \frac{\sin.(\overline{a-a'}.\theta + b - b')}{a-a'}\right\}$$

$$w = C + \tfrac{1}{2}\left\{\frac{\sin.(\overline{a+a'}.\theta + b + b')}{a+a'} + \frac{\sin.(\overline{a-a'}.\theta + b - b')}{a-a'}\right\}$$

These integrals are very useful.

108. *Required the integrals of*

$$\theta^n \, d\theta \sin.\theta, \text{ and } \theta^n \, d\theta \cos.\theta.$$

Integrating by Parts we get

$$\int \theta^n \times d\theta \sin.\theta = C - \theta^n \cos.\theta + n\theta^{n-1} \sin.\theta + n.(n-1)\theta^{n-2} \cos.\theta - \&c.$$

and

$$\int \theta^n \times d\theta \cos.\theta = C + \theta^n \sin.\theta + n\theta^{n-1} \cos.\theta - n.(n-1)\theta^{n-2} \sin.\theta + \&c.$$

109. *Required the integrals of*

$$X \, dx \sin.^{-1} x$$
$$X \, dx \tan.^{-1} x$$
$$X \, dx \sec.^{-1} x$$
$$\&c.$$

Integrating by Parts we have

$$\int X \, dx \sin.^{-1} x = \sin.^{-1} x \int X \, dx - \int \frac{dx \int X \, dx}{\sqrt{(1-x^2)}}$$

$$\int X \, dx \tan.^{-1} x = \tan.^{-1} x \int X \, dx - \int \frac{dx \int X \, dx}{1+x^2}$$

$$\int X \, dx \sec.^{-1} x = \sec.^{-1} x \int X \, dx - \int \frac{dx \int X \, dx}{x\sqrt{(x^2-1)}}$$

$$\&c. = \&c.$$

see Art. 86.

110. *Required the integral of*

$$du = \frac{(f + g\cos.\theta)\, d\theta}{(a + b\cos.\theta)^n}.$$

Integrating by Parts and reducing, we have

$$u = \frac{(ag - bf)\sin.\theta}{(n-1)(a^2-b^2)(a+b\cos.\theta)^{n-1}} + \frac{1}{(n-1)(a^2-b^2)} \times \int \frac{(n-1)(af-bg) + (n-2)(ag-bf)\cos.\theta}{(a+b\cos.\theta)^{n-1}} \, d\theta,$$

which repeated, will finally produce, when n is an integer, the integral required.

$$\text{Ex. } \int \frac{d\theta}{a+b\cos.\theta} = \frac{2}{\sqrt{(a^2-b^2)}} \cdot \tan.^{-1} \frac{(a-b)\tan.\frac{\theta}{2}}{\sqrt{(a^2-b^2)}} + C$$

or

$$\frac{1}{\sqrt{(b^2-a^2)}} \cdot l. \frac{b + a\cos.\theta + \sin.\theta\sqrt{(b^2-a^2)}}{a+b\cos.\theta} + C.$$

Notwithstanding the numerous forms which are integrable by the preceding methods, there are innumerable others which have hitherto resisted all the ingenuity that has been employed to resolve them. If any such appear in the resolution of problems, they must be expanded into con-

verging series, by some such method as that already delivered in Art. 101; or with greater certainty of attaining the requisite degree of convergency, by the following

METHOD OF APPROXIMATION.

111. *Required to integrate between* $x = b$, $x = a$, *any given Indefinite Difference, in a convergent series.*

Let $f(x)$ denote the exact integral of $\int X\, dx$; then by Taylor's Theorem

$$f.(x + h) - fx = Xh + \frac{dX}{dx}\frac{h^2}{1.2} + \&c.$$

and making

$$h = b - a$$

$$f(x + b - a) - fx = X.(b - a) + \frac{dX}{dx}.\frac{(b-a)^2}{1.2} + \&c.$$

Again, make

$$x = a$$

then

$$\frac{dX}{dx},\quad \frac{d^2X}{dx^2},\ \&c.$$

become constants

$$A,\ A',\ \&c.$$

and we obtain

$$f(b) - f(a) = A(b - a) + \frac{A'}{2}.(b - a)^2 + \frac{A''}{2.3}(b - a)^3$$

which, when $b - a$ is small compared with unity, is sufficiently convergent for all practical purposes.

If $b - a$ be not small, assume

$$b - a = p.\beta$$

p being the number of equal parts β, into which the interval $b - a$ is supposed to be divided, in order to make β small compared with unity. Then taking the integral between the several limits

$$a,\ a + \beta$$
$$a,\ a + 2\beta$$
$$\&c.$$
$$a,\ a + p\beta$$

we get

$$f.(a+\beta) - f(a) = A\beta + \frac{A'}{2}.\beta^2 + \frac{A''}{2.3}.\beta^3 + \&c.$$

$$f(a+2\beta) - f(a+\beta) = B\beta + \frac{B'}{2}.\beta^2 + \frac{B''}{2.3}\beta^3 + \&c.$$

$$\&c. = \&c.$$

$$f(a+p\beta) - f(a+\overline{p-1}.\beta) = P\beta + \frac{P'}{2}\beta^2 + \frac{P''}{2.3}\beta^3 + \&c.$$

A, A', &c. B, B', &c. ..... P, P', &c.

being the values of

$$X, \frac{dX}{dx}, \&c.$$

when for x we put

$$a,\ a+\beta,\ a+2\beta,\ \&c.$$

Hence

$$\begin{aligned} f(b) - f(a) = {} & (A + B + \ldots . P)\,\beta \\ & + (A' + B' + \ldots . P')\,\frac{\beta^2}{1.2} \\ & + (A'' + B'' + \ldots . P'')\,\frac{\beta^3}{1.2.3} \\ & + \&c. \end{aligned}$$

the integral required, the convergency of the series being of any degree that may be demanded.

If β be taken very small, then

$$f(b) - f(a) = (A + B + \ldots . P)\,\beta \text{ nearly.}$$

Ex. *Required the approximate value of*

$$\int X^{m-1}\,dx \times (1 - x^n)^{\frac{p}{q}}$$

between the limits of $x = 0$ and $x = 1$, *when neither* $\frac{m}{n}$, *nor* $\frac{m}{n} + \frac{p}{q}$ *is an integer.*

Here

$$X = x^{m-1}(1 - x^n)^{\frac{p}{q}}$$

and

$$\frac{dX}{dx} = \left(m + n\frac{p}{q} - 1\right)x^{m-2}(1 - x^n)^{\frac{p}{q}} - \frac{np}{q}x^{m-2}(1 - x^n)^{\frac{p}{q}-1}$$

$b - a = 1 - 0 = 1.$

Assume $1 = 10 \times \beta$, and we have for limits

$$0, \frac{1}{10};\ 0, \frac{2}{10};\ \&c.$$

Hence m being > 1,

$$A = 0$$

$$B = \frac{1}{10^{m-1}}\left(1 - \frac{1}{10^n}\right)^{\frac{p}{q}}$$

$$C = \left(\frac{2}{10}\right)^{m-1}\left\{1 - \left(\frac{2}{10}\right)^n\right\}^{\frac{p}{q}}$$

$$D = \left(\frac{3}{10}\right)^{m-1}\left\{1 - \left(\frac{3}{10}\right)^n\right\}^{\frac{p}{q}}$$

$$\&c. = \&c.$$

$$P = \left(\frac{9}{10}\right)^{m-1}\left\{1 - \left(\frac{9}{10}\right)^n\right\}^{\frac{p}{q}}.$$

Hence, between the limits $x = 1$ and $x = 0$

$$\int X\,dx = \frac{1}{10^{m+n\frac{p}{q}}} \times \left\{(10^n - 1)^{\frac{p}{q}} + (10^n - 2^n)^{\frac{p}{q}}\right.$$

$$\left. + (10^n - 3^n)^{\frac{p}{q}} + \&c. + (10^n - 9^n)^{\frac{p}{q}}\right\} \text{nearly.}$$

We shall meet with more particular instances in the course of our comments upon the text.

Hitherto the use of the Integral Calculus of Indefinite Differences has not been very apparent. We have contented ourselves so far with making as rapid a sketch as possible of the leading principles on which the Inverse Method depends; but we now come to its

APPLICATIONS.

112. *Required to find the area of any curve, comprised between two given values of its ordinate.*

Let E c C (fig. to LEMMA II of the text) be a given or definite area comprised between 0 and C c, or 0 and y. Then C c being fixed or Definite, let B b be considered Indefinite, or let L b = d y. Hence the Indefinite Difference of the area E c C is the Indefinite area

B C c b.

Hence if E C = x, and S denote the area E c C; then

$$dS = BCcb = CL + Lcb$$

$$= y\,dx + Lcb.$$

But L c b is *heterogeneous* (see Art. 60) compared with C L or y d x.

$$\therefore dS = y\,dx$$

Hence

$$S = \int y \, dx,$$

the area required.

Ex. 1. *Required the area of the common parabola.*

Here

$$y^2 = a x.$$

$$\therefore dx = \frac{2 y \, dy}{a},$$

and

$$S = \int \frac{2 y^2 \, dy}{a} = \frac{2}{3 a} y^3 + C;$$

and between the limits of $y = r$ and $y = r'$ becomes

$$S = \frac{2}{3 a} (r^3 - r'^3)$$

If m and m′ be the corresponding values of x, we have

$$S = \frac{2}{3} (r m - r' m')$$

$$= \frac{2}{3} \text{ of the circumscribing rectangle.}$$

Let $r' = 0$, then

$$S = \frac{2}{3} r m \text{ (see Art. 21.)}$$

Ex. 2. Take the general Parabola whose equation is

$$y^m = a x^n.$$

Here it will be found in like manner that

$$S = \frac{m x y}{m + n} + C$$

$$= \frac{m}{m + n} . \alpha \beta$$

between the limits of $n = y = 0$, and $x = \alpha$, $y = \beta$.

Hence all PARABOLAS *may be squared, as it is termed; or a square may be found whose area shall be equal to that of any Parabola.*

Ex. 3. *Required the area of an* HYPERBOLA *comprised by its asymptote, and one infinite branch.*

If x, y be parallel to the asymptotes, and originate in the center

$$x y = a b$$

is the equation to the curve.

Hence

$$dx = -\frac{a b \, dy}{y^2}$$

and

$$S = \int -\frac{a\,b\,d\,y}{y} = C - a\,b\,l\,y.$$

Let at the vertex $y = \beta$, and $x = 0$; then the area is 0 and

$$C = a\,b\,.\,l\,\beta.$$

Hence

$$S = a\,b\,.\,l\,.\,\frac{\beta}{y}.$$

113. *If the curve be referred to a fixed center by the radius-vector* ϱ *and traced-angle* θ; *then*

$$d\,S = \frac{\varrho^2\,d\,\theta}{2}.$$

For $d\,S$ = the Indefinite Area contained by ϱ, and $\varrho + d\,\varrho = (\varrho + d\,\varrho)\frac{\varrho\,\sin.\,d\,\theta}{2}$

$= \frac{\varrho^2\,d\,\theta}{2} + \frac{\varrho\,d\,\varrho\,d\,\theta}{2}$ (Art. 26) and equating *homogeneous* quantities we have

$$d\,S = \frac{\varrho^2\,d\,\theta}{2}.$$

Ex. 1. In the Spiral of Archimedes

$$\varrho = a\,\theta$$

$$\therefore S = \frac{a^2}{2}\int \theta^2\,d\,\theta = \frac{a^2}{2.\,3}\,.\,\theta^3 + C.$$

Ex. 2. In the Trisectrix

$$\varrho = 2\cos.\,\theta \pm 1$$

$$\therefore d\,S = \tfrac{1}{2}\int (2\cos.\,\theta \pm 1)^2\,d\,\theta$$

which may easily be integrated.'

Hence then the area of every curve could be found, if all integrations were possible. By such as are possible, and the general method of approximation (Art. 111) the quadrature of a curve may be effected either exactly or to any required degree of accuracy. In Section VII and many other parts of the Principia our author integrates Functions by means of curves; that is, he reduces them to areas, and takes it for granted that such areas can be investigated.

114. *To find the length of any curve comprised within given values of the ordinate; or To* RECTIFY *any curve.*

Let s be the length required. Then $d\,s$ = its Indefinite Chord, by Art. 25 and LEMMA VII.

$$\therefore d\,s = \sqrt{(d\,x^2 + d\,y^2)}$$

and

$$s = \int \sqrt{(d\,x^2 + d\,y^2)} \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Ex. 1. *In the general parabola*

$$y^m = a x^n.$$

Hence

$$d x^2 = \frac{m^2}{n^2 a^{\frac{2}{n}}} y^{\frac{2m}{n}-2} . d y^2$$

and

$$d s = d y . \sqrt{\left(1 + \frac{m^2}{n^2 a^{\frac{2}{n}}} y^{\frac{2m}{n}-2}\right)}$$

which is integrable by Art. 95 when either

$$\frac{1}{\frac{2m}{n} - 2} \text{ or } \frac{1}{\frac{2m}{n} - 2} + \frac{1}{2}$$

that is, when either

$$\frac{1}{2} \cdot \frac{n}{m - n} \text{ or } \frac{1}{2} \cdot \frac{m}{m - n}$$

is an integer; that is when either m or n is even.

The common parabola is Rectifiable, because then $m = 2$. In this case

$$d s = d y \sqrt{(1 + \frac{4}{a^2} y^2)} \quad . \quad . \quad . \quad . \quad . \quad (r)$$

Hence assuming according to Case 2 of Art. 95,

$$1 + \frac{4}{a^2} y^2 = y^2 u^2$$

we get the Rational Form

$$d s = \frac{u^2 d u}{\left(u^2 - \frac{4}{a^2}\right)^2}$$

Hence by Art. 89, Case 2,

$$s = \frac{u}{2\left(\frac{4}{a^2} - u^2\right)} - \frac{a}{8} l . \frac{\frac{2}{a} + \sqrt{u}}{\frac{2}{a} - \sqrt{u}} + C$$

$$= \frac{u}{2\left(\frac{4}{a^2} - u^2\right)} - \frac{a}{4} l . \frac{\frac{2}{a} + \sqrt{u}}{\sqrt{\frac{4}{a^2} - u}} + C.$$

But $u = \sqrt{\frac{1 + \frac{4}{a^2} y^2}{y^2}}$. Hence by substituting and making the necessary reductions

$$s = \frac{y\sqrt{\left(y^2+\frac{a^2}{4}\right)}}{a} + a\,l.\,\frac{y+\sqrt{\left(y^2+\frac{a^2}{4}\right)}}{\frac{a}{2}} + C.$$

Let $y = 0$; then $s = 0$ and we get $C = 0$
and $\therefore$ between the Limits of $y = 0$ and $y = \beta$

$$s = \frac{\beta.\sqrt{\left(\beta^2+\frac{a^2}{4}\right)}}{a} + a\,l.\,\frac{\beta+\sqrt{\left(\beta^2+\frac{a^2}{4}\right)}}{\frac{a}{2}}.$$

In the Second Cubical Parabola

$$y^3 = a\,x^2$$

and

$$d\,s = d\,y\sqrt{\left(1+\frac{9\,y}{4\,a}\right)}$$

which gives at once (Art. 91)

$$s = \frac{8}{27}a\left(1+\frac{9\,y}{4\,a}\right)^{\frac{3}{2}} + C.$$

Ex. 2. *In the circle* (Art. 26)

$$d\,s = \frac{d\,y}{\sqrt{(1-y^2)}}$$

which admits of Integration in a series only. Expanding $(1-y^2)^{-\frac{1}{2}}$ by the Binomial Theorem, we have

$$(1-y^2)^{-\frac{1}{2}} = 1 + \frac{y^2}{2} + \frac{1.3}{2.4}\,y^4 + \&c.$$

Hence,

$$d\,s = d\,y + \frac{y^2\,d\,y}{2} + \frac{1.3}{2.4}\,y^4\,d\,y + \&c.$$

and

$$s = y + \frac{y^3}{2.3} + \frac{3}{2.4.5}\cdot y^5 + \&c. + C$$

and between the limits of $y = 0$ and $y = \frac{1}{2}$ or for an arc of 30° we have

$$s = \frac{1}{2} + \frac{1}{2.3.2^3} + \frac{3}{2.4.5.2^5} + \&c.$$

$$= \frac{1}{2} + \frac{1}{3.2^4} + \frac{3}{5.2^8} + \frac{5}{7.2^{11}} + \frac{5.7}{9.2^{16}} + \&c.$$

$$= \left\{\begin{matrix} .5 \\ .0208333333 \\ .0023437500 \\ .0003487720 \\ .0000593390 \\ \&c. \end{matrix}\right\} = .5235851943 \text{ nearly.}$$

Hence 180° of the circle whose *radius* is 1 or the whole circumference π of the circle whose *diameter* is 1 is

$$\pi = .5235851943 \ldots \times 6 \text{ nearly}$$
$$= 3.1415111658$$

which is true to the fourth decimal place; or the defect is less than $\frac{1}{10000}$. By taking more terms any required approximation to the value of π may be obtained.

Ex. 3. *In the Ellipse*

$$s = \int dx \cdot \sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}}$$

where x is the abscissa referred to the center, a the semi-axis major and a e the eccentricity (see Solutions to Cambridge Problems, Vol. II. p. 144.)

115. *If the curve be referred to polar coordinates, ϱ and θ*; then

$$s = \int \sqrt{(\varrho^2 d\theta^2 + d\varrho^2)} \quad \ldots\ldots\ldots \quad (b)$$

For

$$y = \varrho \sin. \theta$$
$$x = m + \varrho \cos. \theta$$

and if dx^2, dy^2 be thence found and substituted in the expression (114. a) the result will be as above.

Ex. 1. *In the Spiral of Archimedes*

$$\varrho = a\theta$$
$$\therefore ds = d\varrho \sqrt{\left(\frac{\varrho^2}{a^2} + 1\right)}$$
$$\therefore s = \frac{\varrho \sqrt{(\varrho^2 + a^2)}}{2a} + 2a \text{ l.} \frac{\varrho + \sqrt{(\varrho^2 + a^2)}}{a} + C$$

see the value for s in the common parabola, Art. 114.

Ex. 2. In the logarithmic Spiral

$$\varrho = e^{\theta}$$

or

$$\theta = \text{l.}\varrho$$

and we find

$$s = \sqrt{2} \int d\varrho = \varrho \sqrt{2} + C.$$

116. *Required the Volume or solid Content of any solid formed by the revolution of a curve round its axis.*

Let V be the volume between the values 0 and y of the ordinate of the generating curve. Then d V = a cylinder whose base is πy^2 and altitude d x + a quantity *Indefinite* or *heterogeneous* compared with either d V or the cylinder.

But the cylinder $= \pi y^2 dx$. Hence equating *homogeneous* terms, we have

$$dV = \pi y^2 dx$$

and

$$V = \pi \int y^2 dx \quad \ldots\ldots\ldots \quad (c)$$

Ex. 1. *In the sphere* (rad. $= r$)

$$y^2 = r^2 - x^2$$

$$\therefore V = \pi \int r^2 dx - \pi \int x^2 dx$$

$$= \pi \left(r^2 x - \frac{x^3}{3} \right) + C;$$

and between the limits $x = 0$ and r

$$V = \frac{2}{3} r^3 \pi$$

which gives the Hemisphere.

Hence for the whole sphere

$$V = \frac{4}{3} r^3 \pi .$$

Ex. 2. *In the Paraboloid.*

$$y^2 = a x$$

$$\therefore V = \pi \int a x \, dx$$

$$= \frac{\pi a}{2} x^2 + C;$$

and between the limits $x = 0$ and α

$$V = \frac{\pi a}{2} . \alpha^2 .$$

Ex. 3. *In the Ellipsoid.*

$$y^2 = \frac{b^2}{a^2} . (a^2 - x^2)$$

$$\therefore V = \frac{\pi b^2}{a^2} . \int (a^2 dx - x^2 dx)$$

$$= \frac{\pi b^2}{a^2} \left(a^2 x - \frac{x^3}{3} \right) + C;$$

and between the limits $x = 0$ and a

$$V = \frac{2 \pi b^2}{3 a^2} . a^3 = \frac{2 \pi}{3} . a b^2 .$$

Hence for the whole Ellipsoid

$$V = \frac{4}{3} \pi a b^2 .$$

The formula (c) may be transformed to

$$V = \pi y S - \pi \int S \, dy \quad \ldots\ldots\ldots \quad (d)$$

where $S = \int y\, dx$ or the area of the generating curve, which is a singular expression, $\int S\, dy$ being also an area.

In philosophical inquiries solids of revolution are the only ones almost that we meet with. Thus the Sun, Planets and Secondaries are Ellipsoids of different eccentricities, or approximately such. Hence then in preparation for such inquiry it would not be of great use to investigate the Volumes of Bodies in general.

If x, y, z, denote the rectangular coordinates, or the perpendiculars let fall from any point of a *curved surface* upon three planes passing through a point given in position at right angles to one another, then it may easily be shown by the principles upon which we have all along proceeded, that

$$\left.\begin{array}{r} dV = dy\int z\, dx \\ \text{or} \\ = dz\int y\, dx \\ \text{or} \\ = dx\int z\, dy \end{array}\right\} \quad \ldots\ldots\ldots \text{(e)}$$

according as we take the base of d V in the planes to which z, y, or x is respectively perpendicular

For let the Volume V be cut off by a plane passing through the point in the surface and parallel to any of the coordinate planes; then the area of the plane section thus made will be

$$\left.\begin{array}{r} \int z\, dx \\ \text{or} \\ \int y\, dx \\ \text{or} \\ \int z\, dy \end{array}\right\} \text{see Art. 112.}$$

Then another section, parallel to $\int z\, dx$, or $\int y\, dx$, or $\int z\, dy$ and at the Indefinite distance d y, or d z, or d x from the former being made, the Indefinite Difference of the Volume will be the portion comprised by these two sections; and the only thing then to be proved is that this portion is $= dy\int z\, dx$ or $dz\int y\, dx$, or $dx\int z\, dy$. But this is easily to be proved by Lemma VII.

This, which is an easier and more comprehensible method of deducing d V than the one usually given by means of Taylor's Theorem, we have merely sketched; it being incompatible with our limits to enter into detail. To conclude we may remark that in Integrating both $\int z\, dx$, and $\int dy \int z\, dx$ must be taken within the prescribed limits, first considering *y Definite* and then *x*.

117. *To find the curved surface of a Solid of Revolution.*

Let the curved surface taken as far as the value y of the ordinate referred to the axis of revolution be σ, and s the length of the generating curve to that point; then $d\sigma$ = the surface of a cylinder the radius of whose base is y and circumference $2\pi y$, and altitude ds, by LEMMA VII. and like considerations. Hence

$$d\sigma = 2\pi y\, ds$$

and

$$\sigma = 2\pi \int y\, ds \quad \ldots\ldots\ldots\ldots \quad (a)$$

or

$$= 2\pi y s - 2\pi \int s\, dy \quad \ldots\ldots\ldots \quad (b)$$

which latter form may be used when s is known in terms of y; this will not often be the case however.

Ex. *In the common Paraboloid.*

$$y^2 = ax$$

and

$$\sigma = \frac{4\pi}{a} \int y\, dy \surd (y^2 + a^2)$$

$$= \frac{4\pi}{3a} (y^2 + a^2)^{\frac{3}{2}} + C.$$

Let $y = 0$ and β, then σ between these limits is expressed by

$$\sigma = \frac{4\pi}{3a} (\beta^2 + a^2)^{\frac{3}{2}}$$

If the surface of any solid whatever were required, by considerations similar to those by which (116. e) is established, we shall have

$$d\sigma = \surd (dy^2 + dz^2) \int \surd (dx^2 + dz^2) \quad \ldots\ldots \quad (c)$$

and substituting for dz in $\surd (dx^2 + dz^2)$ its value deduced from $z = f.(x, y)$ on the supposition that y is Definite; and in $\surd (dy^2 + dz^2)$ its value supposing x Definite. Integrate first $\surd (dx^2 + dz^2)$ between the prescribed limits supposing y Definite and then Integrate $\surd (dy^2 + dz^2) \int \surd (dx^2 + dz^2)$ between its limits making x Definite. This last result will be the surface required.

We must now close our Introduction as it relates to the Integration of Functions of one *Independent* variable.

It remains for us to give a brief notice of the artifices by which Functions of two Independent Variables may be Integrated.

118. *Required the Integral of*

$$X\, dx + Y\, dy = 0,$$

where X *is any function of* x, *and* Y *a function of* y *the same or different.*

When each of the terms can be Integrated separately by the preceding methods for functions of *one* variable, the above form may be Integrated, and we have

$$\int X\,dx + \int Y\,dy = C.$$

This is so plain as to need no illustration from examples. We shall, nowever, give some to show how Integrals apparently Transcendental may in particular cases, be rendered algebraic.

Ex. 1. $\frac{dx}{x} + \frac{dy}{y} = 0.$

$$\therefore l\,x + l\,y = C = l.\,C'$$

$$\therefore l\,(x\,y) = l.\,C'$$

and

$$\therefore x\,y = C' \text{ or } = C.$$

Ex. 2. $\frac{dx}{\sqrt{(1-x^2)}} + \frac{dy}{\sqrt{(1-y^2)}} = 0.$

Here

$$\sin.^{-1} x + \sin.^{-1} y = C = \sin.^{-1} C$$

$$\therefore C = \sin.\{\sin.^{-1} x = \sin.^{-1} y\}$$

$$= \sin.(\sin.^{-1} x).\cos.(\sin.^{-1} y) + \cos.(\sin.^{-1} x)\sin.(\sin.^{-1} y)$$

$$= x.\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}$$

which is algebraic.

Generally if the Integral be of the form

$$f^{-1}(x) + f.^{-1}(y) = C$$

Then assume

$$C = f.^{-1}(C)$$

and take the inverse function of $f^{-1}(C)$ and we have

$$C = f\{f^{-1}(x) + f^{-1}(y)\}$$

which when expanded will be algebraic.

119. *Required the Integral of*

$$Y\,dx + X\,dy = 0.$$

Dividing by X Y we get

$$\frac{dx}{X} + \frac{dy}{Y} = 0$$

which is Integrable by art. 118.

120. *Required the Integral of*

$$P\,dx + Q\,dy = 0;$$

where P *and* Q *are each such functions of* x *and* y *that the sum of the exponents of* x *and* y *in every term of the equation is the same.*

Let $x = u y$. Then if m be the constant sum of the exponents, P and Q will be of the forms

$$U \times y^m - U' y^m$$

U and U′ being functions of u.

Hence, since $d x = u d y + y d u$, we have

$$U . (u d y + y d u) + U' d y = 0$$

and

$$(U u + U') d y + U y d u = 0$$

$$\therefore \frac{d y}{y} + \frac{U d u}{U u + U'} = 0 \quad \ldots\ldots \text{(a)}$$

which is Integrable by art. 118.

Ex. 1. $(a x + b y) d y + (f x + g y) d x = 0.$

Here

$$P = f x + g y, Q = a x + b y$$
$$U = f u + g, U' = a u + b$$
$$\therefore \frac{d y}{y} + \frac{(f u + g) d u}{f u^2 + (g + a) u + b} = 0$$

which being rational is Integrable by art. (88, 89)

Ex. 2. $x d y - y d x = d x \sqrt{(x^2 + y^2)}$

Here

$$Q = x, P = -y - \sqrt{(x^2 + y^2)}$$
$$U' = u, U = -1 - \sqrt{(1 + u^2)}$$
$$\therefore \frac{d y}{y} + \frac{1 + \sqrt{(1 + u^2)}}{u \sqrt{(1 + u^2)}} d u = 0$$

or

$$\frac{d y}{y} + \frac{d u}{u} + \frac{d u}{u \sqrt{(1 + u^2)}} = 0$$

which is Integrable by art. (82, 85.)

These Forms are called *Homogeneous*.

121. *To Integrate*

$$(a x + b y + c) d y + (m x + n y + p) d x = 0.$$

By assuming

$$\left.\begin{aligned} a x + b y + c &= u \\ \text{and} \\ m x + n y + p &= v \end{aligned}\right\}$$

we get

$$d y = \frac{m d u - a d v}{m b - n a}, \text{ and } d x = \frac{b d v - n d u}{m b - n a}$$

and therefore

$$(m u - n v) d u + (b v - a u) d u = 0$$

which being *Homogeneous* is Integrable by Art. 120.

We now come to that class of Integrals which is of the greatest use in Natural Philosophy—to

LINEAR EQUATIONS.

122. *Required to Integrate*

$$dy + yX\,dx = X'\,dx,$$

where X, X′ are functions of X.

Let

$$y = uv.$$

Then

$$u\,dv + v\,du + X\,uv\,dx = X'\,dx$$

Hence assuming

$$dv + v\,X\,dx = 0 \quad \ldots\ldots\ldots \quad (a)$$

we have also

$$v\,du = X'\,dx \quad \ldots\ldots\ldots \quad (b)$$

Hence

$$\frac{dv}{v} + X\,dx = 0$$

$$\therefore lv + \int X\,dx = C$$

or

$$v = e^{C - \int X dx}$$
$$= e^{C} \times e^{-\int X dx}$$
$$= C \times e^{-\int X dx}.$$

Substituting for v in (b) we therefore get

$$du = \frac{1}{C} \cdot e^{\int X dx} X'\,dx$$

which may be Integrated in many cases by Art. 118.

Ex. $dy + y\,dx = a\,x^3\,dx$.

Here

$$X = 1,\ X' = a\,x^3$$
$$\int X\,dx = x$$

and

$$\int X'\,dx\,e^{\int X dx} = a \int x^3 e^x\,dx$$
$$= a\,e^x\,(x^3 - 3\,x^2 + 6\,x - 6)$$

see Art. (102)

Hence

$$y = C\,e^{-x} + a\,(x^3 - 3\,x^2 + 6\,x - 6)$$

122. *Required to Integrate the* LINEAR *Equation of the second order*

$$\frac{d^2y}{dx^2} + X\frac{dy}{dx} + X'y = 0$$

where X, X′ *are functions of* x.

Let $y = e^{\int u\,dx}$; then $\frac{dy}{dx} = u\,e^{\int u\,dx}$

$$\frac{d^2y}{dx^2} = e^{\int u\,dx}\left(\frac{du}{dx} + u^2\right)$$

and ∴ by substitution,

$$\frac{du}{dx} + u^2 + Xu + X' = 0$$

which is an equation of the first order and in certain cases may be Integrable by some one of the preceding methods. When for instance X and X′ are constants and a, b roots of the equation

$$u^2 + Xu + X' = 0$$

then it will be found that

$$y = C\,e^{ax} + C'\,e^{bx}.$$

123. *Required the Integral of*

$$\frac{d^2y}{dx^2} + X\frac{dy}{dx} + X'y = X''$$

where X″ *is a new function of* x.

Let $y = tz$; then Differencing, and substituting, we may assume the result

$$\frac{d^2z}{dx^2} + X\frac{dz}{dx} + X'z = 0 \quad \ldots\ldots\ldots \text{(a)}$$

and

$$\therefore d\left(\frac{dt}{dx}\right) + \left(\frac{dt}{dx}\right)\left(X + \frac{z}{Z}\cdot\frac{dz}{dx}\right)dx = \frac{X''}{z} \quad \ldots \text{(b)}$$

Hence (by 122) deriving z from (a) and substituting in (b) we have a *Linear Equation* of the first order in terms of $\left(\frac{dt}{dx}\right)$; whence $\left(\frac{dt}{dx}\right)$ may be found; and we shall thus finally obtain

$$y = tz = z\int\left(\frac{dx}{z\,e^{2\int X\,dx}}\int X''e^{\int X\,dx}z\,dx\right)\ldots$$

Ex. $\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{1}{x} - \frac{1}{x^2}\cdot y = \frac{a}{x^2-1}$.

Here

$$X = \frac{1}{x},\quad X' = -\frac{1}{x^2},\quad X'' = \frac{a}{x^2-1}.$$

Equat. (a) becomes

$$\frac{d^2 z}{dx^2} + \frac{dz}{dx} \cdot \frac{1}{x} = \frac{z}{x^2}$$

whence

$$du + \left(u^2 + \frac{u}{x} - \frac{1}{x^2}\right) dx = 0$$

wherein $z = e^{\int u\,dx}$; which becomes *homogeneous* when for u we put v^{-1}. Next the variables are separated by putting (see 120)

$$x = vs$$

and we have

$$\frac{dv}{v} + \frac{s^2 + s - 1}{s(s^2 - 1)} ds$$

and

$$\therefore v = \frac{1}{s}\sqrt{\frac{s+1}{s-1}}.$$

Hence

$$u = \frac{x^2 + 1}{x(x^2 - 1)}, \int u\,dx = l.\frac{x^2 - 1}{x}$$

and

$$z = e^{\int u dx} = \frac{x^2 - 1}{x}.$$

Again

$$e^{\int X dx} = e^{lx} = x$$

and

$$\int X'' e^{\int X dx} z\,dx = \int a\,dx = ax + C$$

and

$$y = \frac{x^2 - 1}{x} \int \frac{(ax + C)\,x\,dx}{(x^2 - 1)^2};$$

which being Rational may be farther integrated, and it is found that finally

$$y = -\frac{ax + C}{2x} + \frac{x^2 - 1}{4x} a\,l.\left(C'.\frac{x-1}{x+1}\right).$$

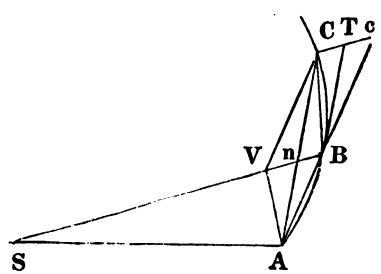
Here we shall terminate our long digression. We have exposed both the Direct and Inverse Calculus sufficiently to make it easy for the reader to comprehend the uses we may hereafter make of them, which was the main object we had in view. Without the Integral Calculus, in some shape or other, it is impossible to prosecute researches in the *higher* branches of philosophy with any chance of success; and we accordingly see Newton, partial as he seems to have been of Geometrical Synthesis, frequently have recourse to its assistance. His Commentators, especially

the Jesuits Le Seur and Jacquier, and Madame Chastellet (or rather Clairaut), have availed themselves on all occasions of its powers. The reader may anticipate, from the trouble we have given ourselves in establishing its rules and formulæ, that we also shall not be very scrupulous in that respect. Our design is, however, not perhaps exactly as he may suspect. As far as the Geometrical Methods will suffice for the comments we may have to offer, so far shall we use them. But if by the use of the Algorithmic Formulæ any additional truths can be elicited, or any illustrations given to the text, we shall adopt them without hesitation.

SECTION II. PROP. I.

124. This Proposition is a generalization of the Law discovered by Kepler from the observations of Tycho Brahe upon the motions of the planets and the satellites.

"*When the body has arrived at* B," says Newton, "*let a centripetal force act at once with a strong impulse, &c.*"] But were the force acting *incessantly* the body will arrive in the next instant at the same point C.



For supposing the centripetal force incessant, the path of the body will evidently be a curve such as A B C. Again, if the body move in the chord A B, and A B, B C be chords described in equal times, the deflection from A B, produced by an impulsive force acting only at B and communicating a velocity which would have been generated by the incessant force in the time through A B, is C c. But if the force had been incessant instead of impulsive, the body would have been moving in the tangent B T at B, and in this case the deflection at the end of the time through B C would have been half the space described with the whole velocity generated through B C (Wood's Mech.) But

$$C\,T = \tfrac{1}{2}\,C\,c$$

$\therefore$ the body would still be at C.

AN ANALYTICAL PROOF.

Let F denote the central force tending constantly to S (see Newton's figure), which take as the origin of the rectangular coordinates (x, y) which determine the place the body is in at the end of the time t. Also let ϱ be the distance of the body at that time from S, and θ the angular distance of ϱ from the axis of x. Then F being resolved parallel to the axis of x, y, its components are

$$\mathrm{F}.\frac{\mathrm{x}}{\varrho} \text{ and } \mathrm{F}.\frac{\mathrm{y}}{\varrho}$$

and (see Art. 46) we $\therefore$ have

$$\frac{\mathrm{d}^2\mathrm{x}}{\mathrm{d}\,\mathrm{t}^2} = -\,\mathrm{F}.\frac{\mathrm{x}}{\varrho}, \quad \frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}\,\mathrm{t}^2} = -\,\mathrm{F}.\frac{\mathrm{y}}{\varrho}.$$

Hence

$$\frac{\mathrm{y}\,\mathrm{d}^2\mathrm{x}}{\mathrm{d}\,\mathrm{t}^2} = -\,\mathrm{F}.\frac{\mathrm{x}\,\mathrm{y}}{\varrho} = \frac{\mathrm{x}\,\mathrm{d}^2\mathrm{y}}{\mathrm{d}\,\mathrm{t}^2}$$

$$\therefore \frac{\mathrm{y}\,\mathrm{d}^2\mathrm{x} - \mathrm{x}\,\mathrm{d}^2\mathrm{y}}{\mathrm{d}\,\mathrm{t}} = 0.$$

But

$$\begin{aligned}\mathrm{y}\,\mathrm{d}^2\mathrm{x} - \mathrm{x}\,\mathrm{d}^2\mathrm{y} &= \mathrm{d}\,\mathrm{y}\,\mathrm{d}\,\mathrm{x} + \mathrm{y}\,\mathrm{d}^2\mathrm{x} - \mathrm{d}\,\mathrm{x}\,\mathrm{d}\,\mathrm{y} - \mathrm{x}\,\mathrm{d}^2\mathrm{y} \\ &= \mathrm{d}.(\mathrm{y}\,\mathrm{d}\,\mathrm{x} - \mathrm{x}\,\mathrm{d}\,\mathrm{y})\end{aligned}$$

$\therefore$ integrating

$$\frac{\mathrm{y}\,\mathrm{d}\,\mathrm{x} - \mathrm{x}\,\mathrm{d}\,\mathrm{y}}{\mathrm{d}\,\mathrm{t}} = \text{constant} = \mathrm{c}.$$

Again,

$$\begin{aligned}\mathrm{x} &= \varrho\cos.\theta,\ \mathrm{y} = \varrho\sin\theta,\ \mathrm{x}^2 + \mathrm{y}^2 = \varrho^2 \\ \therefore \mathrm{d}\,\mathrm{x} &= -\,\varrho\,\mathrm{d}\,\theta\sin.\theta + \mathrm{d}\,\varrho\cos.\theta \\ \mathrm{d}\,\mathrm{y} &= \quad \varrho\,\mathrm{d}\,\theta\cos.\theta + \mathrm{d}\,\varrho\sin.\theta;\end{aligned}$$

whence by substitution we get

$$\mathrm{y}\,\mathrm{d}\,\mathrm{x} - \mathrm{x}\,\mathrm{d}\,\mathrm{y} = \varrho^2\,\mathrm{d}\,\theta$$

$$\therefore \frac{\varrho^2\,\mathrm{d}\,\theta}{\mathrm{d}\,\mathrm{t}} = \mathrm{c}$$

But (see Art. 113)

$$\frac{\varrho^2\,\mathrm{d}\,\theta}{2} = \mathrm{d}.(\text{Area of the curve}) = \mathrm{d}.\,\mathrm{A}$$

$$\therefore \mathrm{d}\,\mathrm{t} = \frac{\varrho^2\,\mathrm{d}\,\theta}{\mathrm{c}} = \frac{2}{\mathrm{c}}.\,\mathrm{d}\,\mathrm{A}.$$

Now since the time and area commence together in the integration there is no constant to be added.

$$\therefore t = \frac{2}{c} \times A \propto A.$$

Q. e. d.

125. COR. 1. PROP. II. By the comment upon LEMMA X, it appears that generally

$$v = \frac{d\,s}{d\,t}$$

and here, since the times of describing A B, B C, &c. are the same by hypothesis, d t is given. Consequently

$$v \propto d\,s$$

that is the velocities at the points A, B, C, &c. are as the elemental spaces described A B, B C, C D, &c. respectively. But since the area of a $\triangle$ generally = semi-base × perpendicular, we have, in symbols,

$$d.A = p \times d\,s$$

$$\therefore v \propto d\,s \propto \frac{d.A}{p};$$

and since the $\triangle$ A B S, B C S, C D S, &c. are all equal, d A is constant, and we finally get

$$v \propto \frac{1}{p} \text{ or } = \frac{c}{p}$$

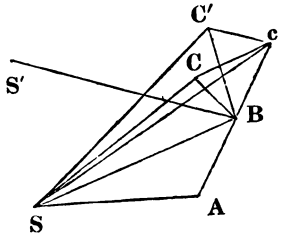
the constant being determinable, as will be shown presently, from the nature of the curve described and the *absolute* attracting force of S.

126. COR. 2. The parallelogram C A being constructed, C V is equal and parallel to A B. But A B = B c by construction and they are in the same line. Therefore C V is equal and parallel to B c. Hence B V is parallel to C c. But S B is also parallel to C c by construction, and B V, B S have one point in common, viz. B. They therefore coincide. That is B V, when produced passes through S.

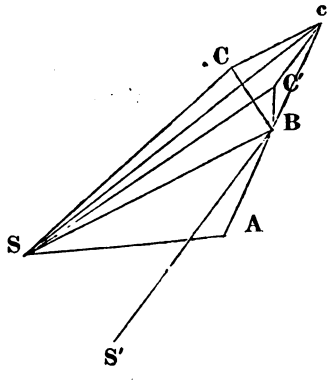
127. COR. 3. The body when at B is acted on by two forces; one in the direction B c, the momentum which is measured by the product of its mass and velocity, and the other the attracting *single* impulse in the direction B S. These acting for an instant produce by composition the momentum in the direction B C measurable by the actual velocity × mass. Now these component and compound momentums being each proportional to the product of the mass and the initial velocity of the body in the directions B c, B V, and B C respectively, will be also proportional to their initial velocities simply, and therefore by (125) to B V, B c, B C.

Hence B V measures the force which attracts the body towards S when the body is at B—and so on for every other position of the body.

128. Cor. 1. Prop. II. In the annexed figure B c = A B, C c is parallel to S B, and C′ c is parallel to S′ B. Now $\triangle$ S C B = S c B = S A B, and if the body by an impulse of S have deflected from its rectilinear course so as to be in C, by the proposition the direction in which the centripetal force acts is that of C c or S B. But if, the body having arrived at C′, the $\triangle$ S B C′ be $>$ S A B (the times of description are equal by hypothesis) and $\therefore >$ S B C, the vertex C′ falls without the $\triangle$ S B C, and the direction of the force along c C′ or B S′, has clearly declined from the course B S in consequentia.



The other case is readily understood from this other diagram.



129. *To prove that a body cannot describe areas proportional to the times round two centers.*

If possible let

$$\triangle S' A B = \triangle S' B C$$

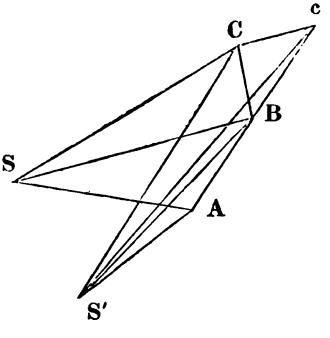
and

$$S A B = S B C.$$

Then

$$\triangle S' B C (= S' A B) = S' B c$$

and C c is parallel to S′ B. But it is also parallel to S B by construction. Therefore S B and S′ B coincide, which is contrary to hypothesis.



130. Prop. III. The demonstration of this proposition, although strictly rigorous, is rather puzzling to those who read it for the first time. At least so I have found it in instruction. It will perhaps be clearer when stated symbolically thus:

Let the central body be called T and the revolving one L. Also let the whole force on L be F, its centripetal force be f, and the force ac-

celerating T be f'. Then supposing a force equal to f' to be applied to L and T in a direction opposite to that of f, by COR. 6. of the Laws, the force f will cause the body L to revolve as before, and we have remaining

$$f = F - f'$$

or

$$F = f + f'.$$

Q. e. d.

ILLUSTRATION.

Suppose on the deck of a vessel in motion, you whirl a body round in a vertical or other plane by means of a string, it is evident the centrifugal force or tension of the string or the power of the hand which counteracts that centrifugal force—i. e. the centripetal force will not be altered by the force which impels the vessel. Now the motion of the vessel gives an equal one to the hand and body and in the same direction; therefore the force on the body = force on the hand + centripetal power of the hand.

131. PROP. IV. Since the motion of the body in a circle is uniform by supposition, the arcs described are proportional to the times. Hence

$$t \propto \text{arc described} \propto \frac{\text{arc} \times \text{radius}}{2}$$

$$\propto \text{area of the sector.}$$

Consequently by PROP. II. the force tends to the center of the circle.

Again the motion being equable and the body always at the same distance from the center of attraction, the centripetal force (F) will clearly be every where the same in the same circle (see COR. 3. PROP. I.) But the absolute value of the force is thus obtained.

Let the arc A B (fig. in the Glasgow edit.) be described in the time T. Then by the centripetal force F, (which supposing A B indefinitely small, may be considered constant,) the sagitta D B (S) will be described in that time, and (Wood's Mechanics) comparing this force with gravity as the unit of force put = 1, we have

$$S = \frac{g}{2} F T^2$$

g being = $32\frac{1}{6}$ feet.

But by similar triangles A B D, A B G

$$S = B D = \frac{(\text{chord A B})^2}{A G} = \frac{(\text{arc A B})^2}{2 R} \text{ ult}^y.$$

(Lemma VII.)

$$\therefore F = \frac{2S}{g T^2} = \frac{(\text{arc A B})^2}{g R T^2}.$$

If T be given

$$F \propto \frac{(\text{arc A B})^2}{R}.$$

If T = arc second

$$F = \frac{(\text{arc A B})^2}{g R}.$$

132. Cor. 1. Since the motion is uniform, the velocity is

$$v = \frac{\text{arc}}{T}$$

$$\therefore F = \frac{v^2}{g R} \propto \frac{v^2}{R}.$$

133. Cor. 2. The Periodic Time is

$$P = \frac{\text{circumference}}{\text{velocity}} = \frac{2\pi R}{v}$$

$$\therefore F = \frac{4\pi^2 R^2}{g R P^2} = \frac{4\pi^2 R}{g P^2} \propto \frac{R}{P^2}.$$

134. Cor. 3, 4, 5, 6, 7. Generally let

$$P = k \times R^n,$$

k being a constant.

Then

$$v = \frac{2\pi R}{P} = \frac{2\pi}{k R^{n-1}} \propto \frac{1}{R^{n-1}}$$

and

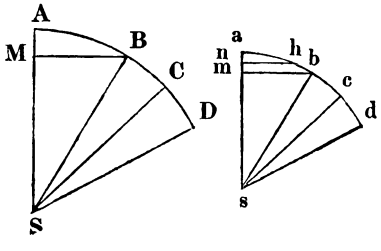
$$F = \frac{4\pi^2 R}{g k^2 R^{2n}} = \frac{4\pi^2}{g k^2 R^{2n-1}} \propto \frac{1}{R^{2n-1}}.$$

Conversely. If $F \propto \frac{1}{R^{2n-1}}$; P will $\propto R^n$.

For (133)

$$P \propto \sqrt{\frac{R}{F}} \propto \sqrt{R^{2n}} \propto R^n.$$

135. Cor. 8. A B, a b are similar arcs, and A B, a h contemporaneously described and indefinitely small.



Now ultimately

$$a n : a m :: a h^2 : a b^2$$

and

$$a m : A M :: a b : A B$$

(Lemma V)

$$\therefore a n : A M :: a h^2 : a b . A B$$

or

$$f : F :: \frac{a\,h^2}{a\,b} : \frac{A\,B^2}{A\,B} :: \frac{a\,h^2}{a\,s} : \frac{A\,B^2}{A\,S} \text{ (LEMMA V)}$$
$$:: \frac{v^2}{a\,s} : \frac{V^2}{A\,S}$$

or

$$F \propto \frac{V^2}{A\,S}.$$

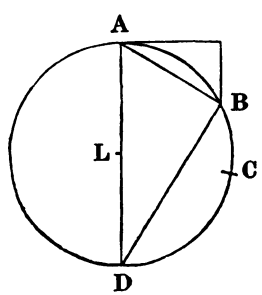
And if the whole similar curves A D, a d be divided into an equal number of indefinitely small equal areas A B S, B C S, &c.; a b s, b c s, &c. these will be similar, and, by composition of ratios, (P and p being the whole times)

$$P : p :: \text{time through A B} : \text{time through a b}$$
$$:: \frac{A\,B}{V} : \frac{a\,b}{v} :: \frac{A\,S}{V} : \frac{a\,s}{v}$$
$$\therefore P \propto \frac{A\,S}{V}.$$

Hence

$$F \propto \frac{V^2}{A\,S} \propto \frac{A\,S}{P^2}.$$

136. COR. 9. Let A C be uniformly described, and with the force considered constant, suppose the body would fall to L in the same time in which it would revolve to C. Then A B being indefinitely small, the force down R B may be considered constant, and we have (131)



$$A\,C^2 : A\,B^2 :: T^2_{AB} : T^2_{AC}$$
$$:: T^2_{AL} : T^2_{RB}$$
$$:: A\,L : R\,B \text{ (131)}$$
$$:: A\,L : \frac{A\,B^2}{A\,D}$$

Hence

$$A\,B^2 = A\,L \times A\,D.$$

PROP. VI. Sagitta $\propto$ F when time is given. Also sag. $\propto$ (arc)^2 by LEMMA XI, $\propto t^2$ when F is given

$\therefore$ when neither force nor time is given

$$\text{sag.} \propto F \times t^2;$$
$$\therefore F \propto \frac{\text{sag.}}{t^2}$$

OTHERWISE.

By Lemma X, Cor. 4,

$$F \propto \frac{\text{space ipso motus initio}}{t^2}$$

$$\propto \frac{\text{sag.}}{t^2}.$$

To generalize this expression, let $\frac{g}{2}$ be the space described in 1″ at the surface of the Earth by Gravity. Also let the unit of force be Gravity. Then

$$F : 1 :: \frac{\text{sag.}}{t^2} : \frac{g}{2 \times 1''^2}$$

$$\therefore F = \frac{2 \text{ sag.}}{g\, t^2} = \frac{2}{g} \times \frac{s}{t^2} \quad \ldots\ldots\ldots \text{(a)}$$

by hypothesis.

137. Cor. 1. $F \propto \frac{Q\,R}{t^2} \propto \frac{Q\,R}{(\text{area } S\,P\,Q)^2}$

$$\propto \frac{Q\,R}{S\,P^2 \times Q\,T^2}.$$

To generalize this, let a be the area described in 1″. Then the area described in $t'' = a \times t = \frac{S\,P \times Q\,T}{2}$.

$$\therefore t = \frac{S\,P \times Q\,T}{2\,a};$$

and substituting in (a) we get

$$F = \frac{8\,a^2}{g} \times \frac{Q\,R}{S\,P^2 \times Q\,T^2} \quad \ldots\ldots \text{(b)}$$

Again, if the Trajectories turn into themselves, there must be

$$a : 1'' :: A \text{ (whole Area)} : T \text{ (Period. Time)}$$

$$\therefore a = \frac{A}{T}.$$

Hence by (b) we have

$$F = \frac{8\,A^2}{g\,T^2} \times \frac{Q\,R}{S\,P^2 \times Q\,T^2} \quad \ldots\ldots\ldots \text{(c)}$$

which, in practice, is the most convenient expression.

138. Cor. 2. $F = \frac{8\,A^2}{g\,T^2} \times \frac{Q\,R}{S\,Y^2 \times Q\,P^2} \quad \ldots\ldots\ldots \text{(d)}$

139. Cor. 3. $F = \frac{8\,A^2}{g\,T^2} \times \frac{1}{S\,Y^2 \times P\,V} \quad \ldots\ldots\ldots \text{(e)}$

Hence is got a differential expression for the force. Since

$$P V = \frac{2 p \, d \varrho}{d p}$$

$$\therefore F = \frac{8 A^2}{g T^2} \times \frac{1}{\frac{2 p^2 p \, d \varrho}{d p}}$$

$$= \frac{4 A^2}{g T^2} \times \frac{d p}{p^3 d \varrho} \quad \ldots\ldots\ldots \quad (f)$$

Another is the following in terms of the reciprocal of the Radius Vector ϱ and the traced-angle θ.

Because

$$p = \frac{\varrho^2 d \theta}{\sqrt{(d \varrho^2 + \varrho^2 d \theta^2)}},$$

$$\therefore \frac{1}{p^2} = \frac{d \varrho^2 + \varrho^2 d \theta^2}{\varrho^4 d \theta^2}$$

$$= \frac{d \varrho^2}{\varrho^4 d \theta^2} + \frac{1}{\varrho^2}.$$

Let $$\frac{1}{\varrho} = u.$$

Then

$$d \varrho = - \frac{d u}{u^2}$$

also

$$\frac{1}{p^2} = \frac{d u^2}{d \theta^2} + u^2$$

$$\therefore - \frac{2 d p}{p^3} = \frac{2 d u d^2 u}{d \theta^2} + 2 u d u$$

$$\therefore \frac{d p}{p^3 d \varrho} = \frac{d^2 u}{d \theta^2} u^2 + u^3$$

and substituting in f we have

$$F = \frac{4 A^2}{g T^2} \times \left(\frac{d^2 u}{d \theta^2} u^2 + u^3 \right) \quad \ldots\ldots \quad (g)$$

140. COR. 4. $$F \propto \frac{1}{S Y^2} \times \frac{1}{P V} \propto V^2 \times \frac{1}{P V}$$

$$\propto \frac{V^2}{P V}.$$

This is generalized thus. Since

$$V = \frac{\text{space}}{\text{Time}} = \frac{P Q}{t}$$

and

$$a \times t \left(= \frac{A}{T} \times t\right) = \text{area described}$$

$$= \frac{P\,Q \times S\,Y}{2}$$

$$\therefore V = \frac{P\,Q}{t} = \frac{2\,A}{T} \times \frac{1}{S\,Y}.$$

Hence

$$\frac{1}{S\,Y^2} = \frac{T^2}{4\,A^2} \times V^2$$

and by Cor. 3.

$$F = \frac{2}{g} \times \frac{V^2}{P\,V} \quad \ldots\ldots\ldots\ldots \quad (h)$$

From this formula we get

$$V^2 = \frac{g}{2} \times F \times P\,V$$

$$= 2\,g\,F \times \frac{P\,V}{4}.$$

But by Mechanics, if s denote the space moved through by a body urged by a constant force F

$$V^2 = 2\,g\,F \times s$$

$$\therefore s = \frac{P\,V}{4} \quad \ldots\ldots\ldots\ldots \quad (i)$$

that is, *the space through which a body must fall when acted on by the force continued constant to acquire the velocity it has at any point of the Trajectory, is $\frac{1}{4}$ of the chord of curvature at that point.*

Also

$$V^2 = 2\,g\,F \times \frac{p\,d\,\varrho}{2\,d\,p} = g\,F \times \frac{p\,d\,\varrho}{d\,p} \quad \ldots\ldots \quad (k)$$

The next four propositions are merely examples to the preceding formulæ.

141. Prop. VII.

$$R\,P^2 \left(= Q\,R \times R\,L\right) : Q\,T^2 :: A\,V^2 : P\,V^2$$

$$\therefore \frac{Q\,R \times R\,L \times P\,V^2}{A\,V^2} = Q\,T^2$$

and multiplying both sides by $\frac{S\,P^2}{Q\,R}$ and putting P V for R L, we have

$$\frac{S\,P^2 \times P\,V^3}{A\,V^2} = \frac{S\,P^2 \times Q\,T^2}{Q\,R}$$

$$\therefore F \propto \frac{A\,V^2}{S\,P^2 \times P\,V^3} \propto \frac{1}{S\,P^2 \times P\,V^3}.$$

Also by (137 c.)

$$F = \frac{8\,A^2}{g\,T^2} \times \frac{A\,V^2}{S\,P^2 \times P\,V^3} = \frac{32\,\pi\,r^4}{g\,T^2} \times \frac{1}{S\,P^2 \times P\,V^3}.$$

OTHERWISE.

From similar triangles we get

$$AV : PV :: SP : SY$$

$$\therefore SY = \frac{SP \times PV}{AV}.$$

$$\therefore SY^2 \times PV = \frac{SP^2 \times PV^2}{AV^2} \times PV$$

$$= \frac{SP^2 \times PV^3}{AV^2}$$

$$\therefore F \propto \frac{1}{SY^2 \times PV} \propto \frac{1}{SP^2 \times PV^3}$$

as before.

OTHERWISE.

$$p = \frac{r^2 - a^2 + \varrho^2}{2r}$$

is the equation to the circle; whence

$$\frac{dp}{d\varrho} = \frac{\varrho}{r}$$

$$\therefore F = \frac{4A^2}{gT^2} \times \frac{dp}{p^3 d\varrho} = \frac{4A^2}{gT^2} \times \frac{\varrho}{rp^3}$$

$$= \frac{4\pi r}{gT^2} \times \varrho \times \frac{8r^3}{(r^2 - a^2 + \varrho^2)^3}$$

$$= \frac{32\pi r^4}{gT^2} \times \frac{\varrho}{(r^2 - a^2 + \varrho^2)^3}.$$

OTHERWISE.

The polar equation to the circle is

$$\varrho = \frac{2a \cos.\theta}{1 + \cos.^2\theta}$$

$$\therefore u \left(= \frac{1}{\varrho}\right) = \frac{1}{2a \cos.\theta} + \frac{\cos.\theta}{2a}$$

$$\therefore \frac{du}{d\theta} = \frac{1}{2a}\left(\frac{\sin.\theta}{\cos.^2\theta} - \sin.\theta\right)$$

$$= \frac{1}{2a} \times \frac{\sin.^3\theta}{\cos.^2\theta}$$

$$\therefore \frac{d^2u}{d\theta^2} = \frac{1}{2a}\left(\frac{3\sin.^2\theta}{\cos.\theta} + \frac{2\sin.^4\theta}{\cos.^3\theta}\right)$$

$$= \frac{1}{2a} \times \frac{\sin.^2\theta}{\cos.^3\theta} \times (3 - \sin.^2\theta).$$

Hence

$$\frac{d^2u}{d\theta^2} + u = \frac{\sin.^2\theta}{2a\cos.^3\theta}\cdot(3 - \sin^2\theta) + \frac{1}{2a\cos.\theta} + \frac{\cos.\theta}{2a}$$

$$= \frac{1}{2a\cos.^3\theta} \times (3\sin^2\theta - \sin.^4\theta + \cos.^2\theta + \cos.^4\theta)$$

$$= \frac{1}{2a\cos.^3\theta} \times (2\sin.^2\theta - \sin.^4\theta + 1 + 1 - 2\sin.^2\theta + \sin.^4\theta)$$

$$= \frac{1}{a\cos.^3\theta}.$$

which by (139) gives

$$F = \frac{4A^2}{gT^2} \times \frac{u^2}{a\cos.^3\theta}$$

$$= \frac{4A^2}{gT^2} \times \frac{(1 + \cos.^2\theta)^2}{4a^3\cos.^5\theta}$$

$$= \frac{A^2}{ga^3T^2} \times \frac{(1 + \cos.^2\theta)^2}{\cos.^5\theta}.$$

142. Cor. 1. $F \propto \frac{1}{SP^2 \times PV^3}$.

But in this case

$$SP = PV.$$

$$\therefore F \propto \frac{1}{SP^5}, \text{ or } = \frac{32\pi r^4}{gT^2} \times \frac{1}{SP^5}.$$

Cor. 2. $\underset{S}{F} : \underset{R}{F} :: RP^2 \times PT^3 : SP^2 \times PV^3$

$$:: SP \times RP^2 : \frac{SP^3 \times PV^3}{PT^3}$$

$$:: SP \times RP^2 : SG^3,$$

by similar triangles.

This is true when the periodic times are the same. When they are different we have

$$\underset{S}{F} : \underset{R}{F} :: SP \times RP^2 \frac{\underset{S}{T}}{\underset{R}{T}} \times SG^3,$$

where the notation explains itself.

143. Prop. VIII.

$$CP^2 : PM^2 :: PR^2 : QT^2$$

and

$$PR^2 = QR \times (RN + QN) = QR \times 2PM$$

$$\therefore CP^2 : PM^2 :: QR \times 2PM : QT^2$$

$$\therefore \frac{QT^2}{QR} = \frac{2PM^3}{CP^2}$$

and

$$\frac{QT^2 \times SP^2}{QR} = \frac{2PM^3 \times SP^2}{CP^2}$$

$$\therefore F \propto \frac{CP^2}{2PM^3 \times SP^2} \propto \frac{1}{PM^3}.$$

Also by 137,

$$F = \frac{4a^2}{g} \times \frac{CP^2}{SP^2 \times PM^3}.$$

But

$$\therefore \quad a = \frac{SP \times \text{velocity}}{2} = \frac{SP \times V}{2}$$

$$\therefore F = \frac{V^2}{g} \times \frac{CP^2}{PM^3}.$$

OTHERWISE.

By PROP. VII,

$$F \propto \frac{1}{SP^2 \times PV^3}.$$

But S P is infinite and P V = 2 P M.

$$\therefore F \propto \frac{1}{PM^3}.$$

OTHERWISE.

The equation to the circle from any point without it is

$$p = \frac{c^2 - r^2 - \varrho^2}{2r}$$

where c is the distance of the point from the center, and r the radius.

$$\therefore \frac{dp}{d\varrho} = -\frac{\varrho}{r}$$

Moreover in this case

$$\varrho = c + PM = c + y$$

$$\therefore p = \frac{c^2 - r^2 - c^2 - 2cy - y^2}{2r}$$

$$= -\frac{cy}{r}$$

$$\therefore \frac{dp}{p^3 d\varrho} = \frac{c+y}{r} \times \frac{r^3}{c^3 y^3}$$

$$= \frac{r^2}{c^2 y^3}.$$

Hence (139)

$$F = \frac{4\,a^2\,r^2}{c^2\,g} \times \frac{1}{y^3} = \frac{V^2\,r^2}{g} \times \frac{1}{y^3}.$$

SCHOLIUM.

144. Generally we have

$$P\,R^2 : Q\,T^2 :: P\,C^2 : P\,M^2$$

$$\therefore \frac{P\,R^2}{Q\,R} : \frac{Q\,T^2}{Q\,R} :: P\,C^2 : P\,M^2$$

But

$$\frac{P\,R^2}{Q\,R} = P\,V,$$

and

$$P\,C : P\,M :: 2\,R\ (R = \text{rad. of curvature}) : P\,V$$

$$\therefore \frac{Q\,T^2}{Q\,R} = P\,V \times \frac{P\,M^2}{P\,C^2} = \frac{2\,R \times P\,M}{P\,C}$$

$$= \frac{2\,R \times P\,M^3}{P\,C^3}.$$

But

$$R = \frac{A\,C^2}{B\,C^4} \times P\,C^3$$

$$\therefore \frac{Q\,T^2}{Q\,R} = \frac{2\,A\,C^2}{B\,C^4} \times P\,M^3$$

and

$$F \propto \frac{1}{P\,M^3}.$$

From the expression (g. 139) we get

$$F = \frac{4\,a^2}{g} \times \frac{d^2\,u}{d\,\theta^2} \times u^2.$$

But

$$a \times t = \frac{\varrho^2\,d\,\theta}{2} = a \times \frac{d\,x}{V}$$

$$\therefore \frac{4\,a^2}{d\,\theta^2} = \frac{V^2\,\varrho^4}{d\,x^2}.$$

Also

$$u = \frac{1}{\varrho}.$$

$$\therefore d\,u = -\frac{d\,\varrho}{\varrho^2}$$

and

$$d^2 u = -\frac{d^2 \varrho}{\varrho^2} + \frac{2 d \varrho^2}{\varrho^3}$$
$$= -\frac{d^2 \varrho}{\varrho^2} \text{ (see 69)}$$

Hence

$$F = \frac{V^2 \varrho^4}{g d x^2} \times -\frac{d^2 \varrho}{\varrho^2} \times \frac{1}{\varrho^2}$$
$$= -\frac{V^2}{g} \times \frac{d^2 \varrho}{d x^2}$$
$$= -\frac{V^2}{g} \times \frac{d^2 y}{d x^2} \quad \ldots\ldots\ldots \text{(l)}.$$

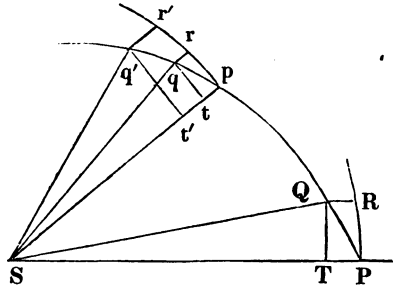
This is moreover to be obtained at once from (see 48)

$$F = \frac{1}{g} \times \frac{d^2 y}{d t^2}.$$

For

$$d t = \frac{d s}{V}$$
$$\therefore F = \frac{V^2}{g} \times -\frac{d^2 y}{d x^2}.$$

145. PROP. IX. Another demonstration is the following:



Let $\angle$ P S Q = $\angle$ p S q. Then from the nature of the spiral the angles at P, Q, p, q being all equal, the triangles S P Q, S p q are similar. Also we have the triangles R P Q, r p q similar, as likewise Q P T, q p t.

Hence

$$\frac{Q T^2}{Q R} : \frac{q t^2}{q r} :: S P : S p$$

and by LEMMA IX.

$$q' r : q r :: p r'^2 : p r^2 :: q' t'^2 : q t^2$$

$$\therefore \frac{q' t'^2}{q' r'} = \frac{q t^2}{q r}.$$

Hence

$$\frac{q' t'^2}{q' r'} : \frac{Q T^2}{Q R} :: Sp : SP$$

$$\therefore \frac{Q T^2}{Q R} \propto SP$$

and

$$\frac{Q T^2 \times S P^2}{Q R} \propto S P^3$$

$$\therefore F \propto \frac{1}{S P^3}.$$

OTHERWISE.

The equation to the logarithmic spiral is

$$p = \frac{b}{a} \times \varrho$$

$$\therefore \frac{dp}{d\varrho} = \frac{b}{a}$$

and by (f. 139) we have

$$F = \frac{4a^2}{g} \times \frac{dp}{p^3 d\varrho} = \frac{4a^2}{g} \times \frac{b}{a} \times \frac{a^3}{b^3 \varrho^3}$$

$$= \frac{4a^2 . a^2}{g . b^2} \times \frac{1}{\varrho^3}.$$

Using the polar equation, viz.

$$\theta = \frac{b}{\sqrt{(a^2 - b^2)}} \times \log. \frac{\varrho}{a}$$

the force may also be found by the formula (g).

146. PROP. X.

$$\left.\begin{array}{r} P v \times v G : Q v^2 :: P C^2 : C D^2 \\ Q v^2 : Q T^2 :: P C^2 : P F^2 \end{array}\right\}$$

$$\therefore P v \times v G : Q T^2 :: P C^4 : C D^2 \times P F^2$$

$$\therefore v G : \frac{Q T^2}{P v} :: P C^2 : \frac{C D^2 \times P F^2}{P C^2}$$

But

$$P V = Q R, \text{ and } C D \times P F = \text{(by Conics) } B C \times C A$$

also

$$\text{ult. } v G = 2 P C.$$

$$\therefore 2 P C : \frac{Q T^2}{Q R} :: P C^2 : \frac{B C^2 \times C A^2}{P C^2}$$

$$\therefore F \propto \frac{Q\,R}{Q\,T^2 \times C\,P^2} \propto \frac{P\,C}{2\,B\,C^2 \times C\,A^2} \propto P\,C.$$

Also by expression (c. 137) we get

$$F = \frac{8\,A^2}{g\,T^2} \times \frac{P\,C}{2\,B\,C^2 \times C\,A^2}$$

But

$$\cdot\ A = \pi \times B\,C \times C\,A$$

$$\therefore F = \frac{4\,\pi^2}{g\,T^2} \times P\,C.$$

The additional figure represents an Hyberbola. The same reasoning shows that the force, being in the center and *repulsive*, also in this curve, $\propto C\,P$.

ALITER.

Take

$$T\,u = T\,v$$

and

$$u\,V : v\,G :: D\,C^2 : P\,C^2$$

Then since

$$Q\,v^2 : P\,v \times v\,G :: D\,C^2 : P\,C^2$$

$$\therefore u\,V : v\,G :: Q\,v^2 : P\,v \times v\,G$$

$$\therefore Q\,v^2 = P\,v \times u\,V$$

$$\therefore Q\,v^2 + u\,P \times P\,v = P\,v \times (u\,V + u\,P)$$

$$= P\,v \times V\,P.$$

But

$$Q\,v^2 = Q\,T^2 + T\,v^2 = Q\,T^2 + T\,u^2$$

$$= P\,Q^2 - P\,T^2 + T\,u^2$$

$$= P\,Q^2 - (P\,T^2 - T\,u^2)$$

$$= P\,Q^2 - P\,u \times P\,v$$

$$(\text{chord } P\,Q)^2 = P\,v \times V\,P.$$

Now suppose a circle touching P R in P and passing through Q to cut P G in some point V′. Then if Q V′ be joined we have

$$\angle P\,Q\,v = \angle Q\,P\,R = \angle Q\,V'\,P$$

and in the $\triangle Q\,P\,v$, Q V′ P the $\angle Q\,P\,V'$ is common. They are therefore similar, and we have

$$P\,v : P\,Q :: P\,Q : P\,V'$$

$$\therefore P\,Q^2 = P\,v \times V'\,P = P\,v \times V\,P$$

$$\therefore V'\,P = V\,P$$

or the circle in question passes through V;

$\therefore$ P V is the chord of curvature passing through C.

Again, since

$$u V = v G \times \frac{D C^2}{P C^2} = C \times v G$$

or

$$P V - P u = C (P G - P v)$$

and

$$P V, P G$$

being homogeneous

$$P V = \frac{2 D C^2}{P C^2} P C = \frac{2 C D^2}{P C}.$$

$\therefore$ (Cor. 3, Prop. VI.)

$$F \propto \frac{P C}{2 P F^2 \times C D^2}.$$

But since by Conics the parallelogram described about an Ellipse is equal to the rectangle under its principal axes, it is constant. $\therefore P F \times C D$ is.

and

$$F \propto P C.$$

OTHERWISE.

By (f. 139) we have

$$F = \frac{4 A^2}{g T^2} \times \frac{d p}{p^3 d \varrho}.$$

But in the ellipse referred to its center

$$p^2 = \frac{a^2 b^2}{a^2 + b^2 - \varrho^2}$$

$$\therefore \frac{1}{p^2} = \frac{a^2 + b^2 - \varrho^2}{a^2 b^2}$$

and differentiating, and dividing by -2, there results

$$\frac{d p}{p^3 d \varrho} = \frac{\varrho}{a^2 b^2}$$

which gives

$$F = \frac{4 A^2}{g T^2} \times \frac{\varrho}{a^2 b^2} = \frac{4 \pi^2}{g T^2} \times \varrho.$$

In like manner may the force be found from the polar equation to the ellipse, viz.

$$\varrho^2 = \frac{b^2}{1 - e^2 \cos.^2 \theta},$$

by means of substituting in equat. (g. 139.)

147. COR. 1. For a geometrical proof of this converse, see the Jesuits' notes, or Thorpe's Commentary. An analytical one is the following.

Let the body at the distance R from the center be projected with the velocity V in a direction whose distance from the center of attraction is P. Also let

$$F = \mu \varrho$$

μ being the force at the distance 1. Then (by f)

$$F = \frac{4 A^2}{g T^2} \times \frac{d p}{p^3 d \varrho} = \mu \varrho$$

which gives by integration, and reduction

$$\frac{1}{p^2} = \frac{\mu g T^2}{4 A^2} \times R^2 + \frac{1}{P^2} - \frac{\mu g T^2}{4 A^2} \times \varrho^2$$

R and P being corresponding values of ϱ and p.

But in the ellipse referred to its center we have

$$\frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} - \frac{\varrho^2}{a^2 b^2}$$

which shows that the orbit is also an ellipse with the force tending to its center, and equating homogeneous quantities, we get

$$\left.\begin{aligned} \frac{a^2 + b^2}{a^2 b^2} &= \frac{\mu g T^2}{4 A^2} \times R^2 + \frac{1}{P^2} \\ \text{and} \\ \frac{1}{a^2 b^2} &= \frac{\mu g T^2}{4 A^2} \end{aligned}\right\}$$

But

$$A = \pi a b$$

$$\therefore T = \frac{2 \pi}{\sqrt{\mu g}} \quad \ldots\ldots\ldots\ldots \quad (1)$$

which gives the value of the periodic time, and also shows it to be constant. (See Cor. 2 to this Proposition.)

Having discovered that the orbit is an ellipse with the force tending to tne center, from the data, we can find the actual orbit by determining its semiaxes a and b.

By 140, we have

$$V = \frac{2 A}{T} \times \frac{1}{P}$$

$$\therefore \frac{a^2 + b^2}{a^2 b^2} = \mu g \times \frac{R^2}{V^2 P^2} + \frac{1}{P^2}$$

and

$$\frac{1}{a^2 b^2} = \mu g \times \frac{1}{V^2 P^2}$$

$$\therefore a^2 + b^2 = R^2 + \frac{V^2}{\mu g}$$

and

$$2ab = \frac{2VP}{\sqrt{\mu g}}$$

$$\therefore a + b = \sqrt{\left(R^2 + \frac{V^2}{\mu g} + \frac{2VP}{\sqrt{\mu g}}\right)}$$

and

$$a - b = \sqrt{\left(R^2 + \frac{V^2}{\mu g} - \frac{2VP}{\sqrt{\mu g}}\right)}$$

which, by addition and subtraction, give a and b.

OTHERWISE.

By formula (g. 139,) we have

$$F = \frac{4A^2}{gT^2} u^2 \left(\frac{d^2u}{d\theta^2} + u\right) = \mu\varrho = \frac{\mu}{u}$$

$$\therefore \frac{d^2u}{d\theta^2} + u - \frac{g\mu T^2}{4A^2} \times \frac{1}{u^3} = 0$$

and multiplying by 2 d u, integrating and putting $\frac{g\mu T^2}{4A^2} = M$, we have

$$\frac{du^2}{d\theta^2} + u^2 + \frac{M}{u^2} + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To determine C, we have

$$\frac{du^2}{d\theta^2} = \frac{1}{\varrho^4} \cdot \frac{d\varrho^2}{d\theta^2}$$

and in all curves it is easily found that

$$\frac{d\varrho}{d\theta} = \frac{\varrho}{p}\sqrt{(\varrho^2 - p^2)}$$

$$\therefore \frac{du^2}{d\theta^2} = \frac{\varrho^2 - p^2}{\varrho^2 p^2} = \frac{1}{p^2} - \frac{1}{\varrho^2}.$$

Hence, when $\varrho = R$, and $p = P$,

$$\frac{1}{P^2} + MR^2 + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives the constant C.

Again from (2) we get

$$d\theta = \frac{u\,du}{\sqrt{(-M - Cu^2 - u^4)}}$$

which being integrated (see Hersch's Tables, p. 160.—Englished edit. published by Baynes & Son, Paternoster Row) and the constants properly determined will finally give ϱ in terms of θ; whence from the equation to the ellipse will be recognised the orbit and its dimensions.

148. COR. 2. This Cor. has already been demonstrated—see (1).

Newton's Proof may thus be rendered a little easier.

By Cor. 3 and 8 of Prop. 1V, in *similar ellipses*

$$T \text{ is constant.}$$

Again for Ellipses having the *same axis-major*, we have

$$T \left(\propto \frac{A}{a} \right) \propto \frac{\pi a b}{a} \propto \frac{b}{a}.$$

But since the forces are the same at the principal vertexes, the sagittæ are equal, and ultimately the arcs, which measure the velocities, are equal to the ordinates, and these are as the axes-minores. Hence, α (which $= \frac{v \times SY}{2}$) $\propto b$.

$$\therefore T \propto \frac{b}{b} \propto 1 \text{ or is constant.}$$

Again, generally if A and B be any two ellipses whatever, and C a third one similar to A, and having the same axis-major as B; then, by what has just been shown,

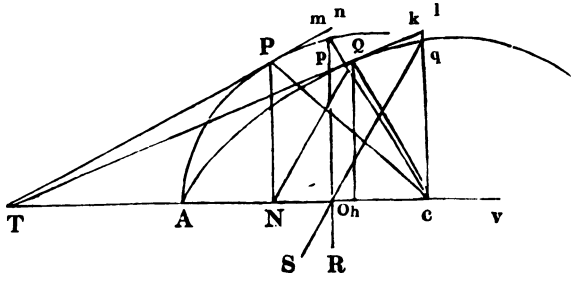
$$T \text{ in } B = T \text{ in } C$$

and

$$T \text{ in } C = T \text{ in } A$$

$$\therefore T \text{ in } B = T \text{ in } A.$$

149. SCHOL. See the Jesuits' Notes. Also take this proof of, "If one curve be related to another on the same axis by having its ordinates in a given ratio, and inclined at a given angle, the forces by which bodies are made to describe these curves in the *same time* about the same center in the axis are, in corresponding points, as the distances from the center."



The construction being intelligible from the figure, we have

$$P N : Q N :: p O : q O$$

$$\therefore P N : p O :: Q N \; q O$$

$$:: N T : O T \text{ ultimately.}$$

$\therefore$ Tangents meet in T,

the triangles C P T, C Q T are in the ratio of P N : Q h or of parallelograms P N O p, Q N O q ultimately, i. e. in the given ratio, and

$$\begin{aligned} C\,p\,P : C\,P\,T &:: p\,P : P\,T \text{ ultimately.} \\ &:: N\,O : N\,T \\ &:: q\,Q : Q\,T \\ &:: CQq : C\,Q\,T \end{aligned}$$

$\therefore$ C p P : C q Q in a given ratio.

$\therefore$ bodies describing equal areas in equal times, are in corresponding points at the same times.

$\therefore$ P p, Q q are described in the same time, and m p and k q are as the forces.

Draw C R, C S parallel to P T, Q T; then

$$p\,O : q\,O :: P\,N : Q\,N :: n\,O : l\,O$$

$$\therefore n\,O : p\,O :: l\,O : q\,O$$

and

$$n\,p : n\,O :: l\,q : l\,Q$$

but

$$n\,O : n\,R :: l\,Q : l\,S$$

(since $n\,O : O\,R :: T\,O : O\,C :: l\,O : O\,S$)

$$\therefore n\,p : n\,R :: l\,q : l\,S$$

$$\therefore n\,p : p\,R :: l\,q : q\,S$$

and

$$n\,p : p\,R :: m\,p : p\,C$$

$$l\,q : q\,S :: k\,q : q\,C$$

$$\therefore m\,p : p\,C :: k\,q : q\,C$$

or

$$Fat\,p : Fat\,q :: p\,C : q\,C.$$

Q. e. d.

SECTION III.

150. Prop. XI. This proposition we shall simplify by arranging the proportions one under another as follows:

$$L \times Q\,R\,(=P\,x) : L \times P\,v :: P\,E : P\,C$$

$$:: A\,C : P\,C$$

But

$$\begin{aligned} L \times P\,v &: G\,v \times P\,v :: L : G\,v \\ G\,v \times P\,v &: Q\,v^2 :: P\,C^2 : C\,D^2 \\ Q\,v^2 &: Q\,x^2 :: 1 : 1 \\ Q\,x^2 &: Q\,T^2 :: P\,E^2 : P\,F^2 \\ &:: C\,A^2 : P\,F^2 \\ &:: C\,D^2 : C\,B^2 \end{aligned}$$

$\therefore L \times QR : QT^2 :: AC \times L \times PC^2 \times CD^2 : PC \times Gv \times CD^2 \times CB^2$

and

$$\frac{QR}{QT^2} = \frac{AC \times PC}{Gv \times CB^2} = \frac{AC \times PC}{2PC \times CB^2} = \frac{AC}{2CB^2}$$

$$\therefore F \propto \frac{QR}{QT^2 \times SP^2} \left(= \frac{AC}{2CB^2 \times SP^2}\right) \propto \frac{1}{SP^2}.$$

Q. e. d.

Hence, by expression (c) Art. 137, we have

$$F = \frac{8A^2}{gT^2} \times \frac{AC}{2CB^2 \times SP^2}$$

$$= \frac{8\pi^2 a^2 b^2}{gT^2} \times \frac{a}{2b^2 \times \varrho^2}$$

$$= \frac{4\pi^2 a^3}{gT^2} \times \frac{1}{\varrho^2} \quad \ldots\ldots\ldots\ldots \text{(a)}$$

where the elements a and T are determinable by observation.

OTHERWISE.

A general expression for the force (g. 139) is

$$F = \frac{4A^2}{gT^2} \times u^2 \left(\frac{d^2u}{d\theta^2} + u\right)$$

But the equation to the Ellipse gives

$$u = \frac{1}{\varrho} = \frac{1 + e \cos.\theta}{a(1 - e^2)}$$

where a is the semi-axis major and a e the eccentricity.

$$\therefore \frac{du}{d\theta} = -\frac{e \sin.\theta}{a(1 - e^2)}$$

and

$$\frac{d^2u}{d\theta^2} = -\frac{e \cos.\theta}{a(1 - e^2)}$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{1}{a(1 - e^2)}$$

and

$$F = \frac{4A^2}{gT^2} \times \frac{u^2}{a(1 - e^2)}.$$

But

$$A^2 = \pi^2 a^2 b^2 = \pi^2 a^2 (a^2 - a^2 e^2)$$

$$\therefore F = \frac{4\pi^2 a^3}{gT^2} \times u^2$$

the same as before.

OTHERWISE.

Another expression is (k. 140)

$$F = \frac{4\,A^2}{g\,T^2} \times \frac{d\,p}{p^3\,d\,\varrho}.$$

Another equation to the Ellipse is also

$$\frac{1}{p^2} = \frac{2\,a - \varrho}{b^2\,\varrho} = \frac{2\,a}{b^2\,\varrho} - \frac{1}{b^2}$$

$$\therefore \frac{d\,p}{p^3\,d\,\varrho} = \frac{a}{b^2\,\varrho^2}$$

$$\therefore F = \frac{4\,A^2}{g\,T^2} \times \frac{a}{b^2\,\varrho^2}$$

$$= \frac{4\,\pi^2\,a^2\,b^2}{g\,T^2} \times \frac{a}{b^2\,\varrho^2} = \frac{4\,\pi^2\,a^3}{g\,T^2} \times \frac{1}{\varrho^2}.$$

151. Prop. XII. The same order of the proportions, which are also lettered in the same manner, as in the case of the ellipse is preserved here.

Moreover the equations to the Hyperbola are

$$\varrho = \frac{a\,(e^2 - 1)}{1 + e\,\cos.\,\theta}$$

and

$$p^2 = \frac{b^2\,\varrho}{\varrho + 2\,a}$$

which will give the same values of F as before excepting that it becomes *negative* and thereby indicates the force to be *repulsive*.

152. Prop. XIII. By Conics

$$4\,S\,P\,.\,P\,v = Q\,v^2 = Q\,x^2 \text{ ultimately.}$$

But

$$P\,v = P\,x = Q\,R.$$

$$\therefore 4\,S\,P\,.\,Q\,R : Q\,x^2 :: 1 : 1$$

and

$$Q\,x^2 : Q\,T^2 :: S\,P^2 : S\,N^2$$

$$:: S\,P : S\,A$$

$$\therefore 4\,S\,P\,.\,Q\,R : Q\,T^2 :: S\,P : S\,A$$

$$\therefore \frac{Q\,R}{Q\,T^2} = \frac{1}{4\,S\,A} = \frac{1}{L}$$

L being the latus rectum.

$$\therefore F \propto \frac{Q\,R}{Q\,T^2 \times S\,P^2} \propto \frac{1}{S\,P^2}.$$

or

$$F = \frac{8\,a^2}{g} \times \frac{Q\,R}{S\,P^2 \times Q\,T^2} \text{ (b. 137)}$$

$$= \frac{8\,a^2}{g\,L} \times \frac{1}{S\,P^2} \text{ or } \frac{2\,P^2\,V^2}{g\,L} \times \frac{1}{S\,P^2}$$

a being the area described by the radius-vector in a second, or P the perpendicular upon the tangent and V the corresponding velocity.

OTHERWISE.

In the parabola we have

$$u = \frac{1}{\varrho} = \frac{2}{L}(1 + \cos.\,\theta) = \frac{2}{L} + \frac{2}{L}\cos.\,\theta$$

and

$$\frac{1}{p^2} = \frac{4}{L} \times \frac{1}{\varrho}$$

which give

$$\frac{d^2 u}{d\theta^2} + u = \frac{2}{L}$$

and

$$\frac{d\,p}{p^3\,d\,\varrho} = \frac{2}{L} \times \frac{1}{\varrho^2}$$

and these give, when substituted in

$$F = \frac{P^2\,V^2}{g} \cdot u^2 \left(\frac{d^2 u}{d\theta^2} + u\right)$$

or

$$= \frac{P^2\,V^2}{g} \cdot \frac{d\,p}{p^3\,d\,\varrho}$$

the same result, viz.

$$F = \frac{2\,P^2\,V^2}{g\,L} \times \frac{1}{\varrho^2} \quad \ldots\ldots\ldots \text{(b)}$$

Newton observes that the two latter propositions may easily be deduced from PROP. XI.

In that we have found (Art. 150)

$$F = \frac{4\,A^2}{g\,T^2} \times \frac{a}{b^2\,\varrho^2}$$

$$= \frac{P^2\,V^2}{g} \times \frac{a}{b^2\,\varrho^2}.$$

Now when the section becomes an Hyperbola the force must be repulsive the trajectory being convex towards the force, and the expression remains the same.

Again by the property of the ellipse

$$b^2 : \frac{L^2}{4} : a \times a : \frac{L}{4}\left(2a - \frac{L}{4}\right)$$

which gives

$$\frac{a}{b^2} = \frac{2}{L} - \frac{1}{4a}$$

and if c be the eccentricity

$$b^2 = a^2 - c^2 = (a + c) \times (a - c)$$

$$\therefore \frac{a}{(a + c) \times (a - c)} = \frac{2}{L} - \frac{1}{4a}.$$

Now when the ellipse becomes a parabola a and c are infinite, a — c is *finite*, and a + c is of the same order of *infinites* as a. Consequently $\frac{a}{b^2}$ is *finite*, and equating like quantities, we have

$$\frac{a}{b^2} = \frac{2}{L},$$

which being substituted above gives

$$F = \frac{2P^2V^2}{gL} \times \frac{1}{\varrho^2}$$

the same as before.

Again, let the Ellipse merge into a circle; then b = a and

$$F = \frac{P^2V^2}{g} \times \frac{a}{b^2\varrho^2}$$

$$= \frac{aV^2}{g} \times \frac{1}{\varrho^2}$$

$$= \frac{V^2}{g \times a} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(c)}$$

153. Prop. XIII. Cor. 1. *For the focus, point of contact, and position of the tangent being given, a conic section can be described having at that point a given curvature.*]

For a geometrical construction see Jesuits' note, No. 268.

The elements of the Conic Section may also be thus found.

The expression for R in Art. 75 may easily be transformed to

$$R = \frac{\frac{\varrho^6}{p^3}}{\frac{\varrho^4}{p^2} + \frac{d\varrho^2}{d\theta^2} - \varrho \cdot \frac{d^2\varrho}{d\theta^2}}$$

for

$$p = \frac{\varrho^2 d\theta}{ds} = \frac{\varrho^2}{\sqrt{\left(\varrho^2 + \frac{d\varrho^2}{d\theta^2}\right)}}.$$

Now the general equation to conic sections being

$$\varrho = \frac{b^2}{a} \times \frac{1}{1 + e \cos. \theta}$$

the denominator of the value of R is easily found to be

$$\frac{a}{b^2 \varrho^3}$$

which gives

$$R = \frac{b^2}{a} \times \frac{\varrho^3}{p^3}.$$

Hence

$$\frac{b^2}{a} = \frac{p^3}{\varrho^3} \times R$$

is known.

Again, by the equation to conic sections we have

$$p^2 = \frac{b^2 \varrho}{2a \mp \varrho}$$

which, by aid of the above, gives

$$a = \frac{\pm \varrho^3}{2\varrho^2 - pR}.$$

And

$$b^2 = \frac{\pm p^3 R}{2\varrho^2 - pR}.$$

Whence the construction is easy.

154. *The Curvature is given from the Centripetal Force and Velocity being given.*]

If the circle of curvature be described passing through P, Q, V, and O (P V being the chord of curvature passing through the center of force, and P O the diameter of curvature); then from the similar triangles P Q R, P V Q, we get

$$QR = \frac{PQ^2}{PV}.$$

Also from the triangles P Q T and P S Y (S Y being the perpendicular upon the tangent) we have

$$PQ = \frac{SP \times QT}{SY}$$

and from P S Y, P V O,

$$PV = \frac{2R \times SY}{SP}$$

whence by substitution, &c.

$$\frac{Q\ R}{Q\,T^2 \times S\,P^2} = \frac{S\,P}{2\,R \times S\,Y^2}$$

$$\therefore F = \frac{2\,P^2\,V^2}{g} \times \frac{Q\,R}{Q\,T^2 \times S\,P^2} = \frac{V^2 \times S\,P}{R \times S\,Y}$$

which gives

$$R = \frac{S\,P}{g.\,S\,Y} \times \frac{V^2}{F}.$$

Hence, S P, S Y and g being given quantities, R is also given if V and F are.

155. *Two orbits which touch one another and have the same centripetal force and velocity cannot be described.*]

This is clear from the "Principle of sufficient Reason." For it is a truth axiomatic that any number of *causes* acting simultaneously under given circumstances, viz. the absolute force, law of force, velocity, direction, and distance, can produce but *one effect*. In the present case that one effect is the motion of the body in some one of the Conic Sections.

OTHERWISE.

Let the given law of force be denoted generally by $f\,\varrho$, where $f\,\varrho$ means any function; then (139)

$$F = \frac{P^2\,V^2}{g} \times \frac{d\,p}{p^3\,d\,\varrho}$$

and since P and V are given

$$F' = \frac{P^2\,V^2}{g} \times \frac{d\,p'}{p'^3\,d\,\varrho'}.$$

But if A be the value of F at the given distance (r) from the center to the point of contact; then

$$F : A :: f\,\varrho : f\,r$$

and

$$F' : A :: f\,\varrho' : f\,r$$

$$\therefore F = \frac{A}{f\,r} \times f\,\varrho$$

and

$$F' = \frac{A}{f\,r} \times f\,\varrho'$$

Hence

$$\frac{P^2 V^2}{g} \cdot \frac{dp}{p^3 d\varrho} = \frac{A}{fr} \times f\varrho$$

and

$$\frac{P^2 V^2}{g} \cdot \frac{dp'}{p'^3 d\varrho'} = \frac{A}{fr} \times f\varrho'$$

and integrating, we have

$$\frac{P^2 V^2 fr}{2gA} \times \left(\frac{1}{P^2} - \frac{1}{p^2}\right) = \int d\varrho f\varrho$$

and

$$\frac{P^2 V^2 fr}{2gA} \cdot \times \left(\frac{1}{P^2} - \frac{1}{p'^2}\right) = \int d\varrho' f\varrho'$$

Now $\int d\varrho f\varrho$ and $\int d\varrho' f\varrho'$ are evidently the same functions of ϱ and ϱ', which therefore assume

$$\varphi\varrho \text{ and } \varphi\varrho';$$

and adding the constant by referring to the point of contact of the two orbits, and putting

$$\frac{P^2 V^2 fr}{2gA} = M,$$

we get

$$M \times \left(\frac{1}{P^2} - \frac{1}{p^2}\right) = \varphi\varrho - \varphi r$$

$$M \times \left(\frac{1}{P^2} - \frac{1}{p'^2}\right) = \varphi\varrho' - \varphi r.$$

$$\left.\begin{aligned} \therefore \frac{1}{p^2} &= \frac{\varphi r}{M} + \frac{1}{P^2} - \frac{\varphi\varrho}{M} \\ \frac{1}{p'^2} &= \frac{\varphi r}{M} + \frac{1}{P^2} - \frac{\varphi\varrho'}{M} \end{aligned}\right\} \quad \ldots\ldots \text{(d)}$$

in which equations the constants being the same, and those with which ϱ and ϱ' are also involved, the curves which are thence descriptible are *identical.* Q. e. d.

These explanations are sufficient to clear up the converse proposition contained in this corollary.

156. It may be demonstrated generally and at once as follows:

By the question

$$f\varrho = \frac{1}{\varrho^2};$$

then

$$f r = \frac{1}{r^2},$$

and

$$\varphi \varrho = \int \frac{d \varrho}{\varrho^2} = -\frac{1}{\varrho}$$

and substituting in (d) we have

$$\frac{1}{p^2} = -\frac{1}{r M} + \frac{1}{P^2} + \frac{1}{M \varrho}.$$

But the general equation to Conic Sections is

$$\frac{1}{p^2} = \frac{2a}{b^2 \varrho} \mp \frac{1}{b^2}.$$

Whence the orbit is a Conic Section whose axes are determinable from

$$\frac{2a}{b^2} = \frac{1}{M} = \frac{2 g A r^2}{P^2 V^2}$$

and

$$\mp \frac{1}{b^2} = -\frac{1}{r M} + \frac{1}{P^2}$$

$$= \frac{1}{P^2} - \frac{2 g A r}{P^2 V^2};$$

and the section is an Ellipse, Parabola or Hyperbola according as

$$V^2 \text{ is } >, \text{ or } = \text{ or } < 2 g A r.$$

Before this subject is quitted it may not be amiss by these forms also to demonstrate the converse of Prop. X, or Cor. 1, Prop. X.

Here

$$f \varrho = \varrho$$
$$f r = r$$
$$\varphi \varrho = \int \varrho \, d \varrho = \frac{\varrho^2}{2}$$
$$\varphi r = \frac{r^2}{2}.$$

Whence

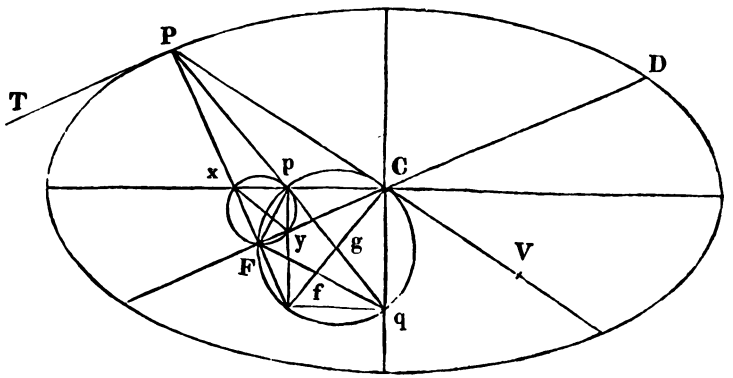
$$\frac{1}{p^2} = \frac{r^2}{2 M} + \frac{1}{P^2} - \frac{\varrho^2}{2 M}.$$

But in the Conic Sections referred to the center, we have

$$\frac{1}{p^2} = \frac{1}{a^2} \pm \frac{1}{b^2} \mp \frac{\varrho^2}{a^2 b^2}$$

which shows the orbit to be an Ellipse or Hyperbola and its axes may be found as before.

In the case of the Ellipse take the following geometrical solution and construction



C, the center of force and distance C P are given. The body is projected at P with the given velocity V. Hence P V is given, (for $V^2 = \frac{g}{2} F . PV$.) Also the position of the tangent is given, ∴ position of D C is given, and $PV = \frac{2 CD^2}{PC}$. Hence C D is given in magnitude. Draw P F perpendicular to C D. Produce and take P f = C D. Join C f and bisect in g. Join P g, and take g C, g f, g p, g q, all equal. Draw C p, C q. These are the positions of the major and minor axis. Also ½ major axis = P q, ½ minor axis = P p.

For from g describe a circle through C, f, p, q, and since C F f is a right ∠, it will pass through F.

$\therefore Pp . Pq = PF . Pf = PF . CD$

Also

$PC^2 + Pf^2 = Pg^2 + gC^2 + Pg^2 + gf^2$, (since base of △ bisected in g)

or

$$\begin{aligned} PC^2 + CD^2 &= Pg^2 + gq^2 + Pg^2 + gp^2 \\ &= Pq^2 - 2Pg . gq + Pp^2 + 2Pg . gp \\ &= Pq^2 + Pp^2 \end{aligned}$$

$\therefore Pp . Pq = PF . CD$
$Pp^2 + Pq^2 = PC^2 + CD^2$ } But a and b are determined by the same equations. $\therefore Pq = a, Pp = b$.

Also since p and F are right angles, the circle on x y will pass through p and F, and ∠ P p x = C p q = C F q = x F p, because ∠ x F C = p F q.

∴ ∠ P p x = ∠ in alternate segment. ∴ P p is tangent.

$Pp^2 = PF . Px$ ∴ $PF . Px = b^2$.

But if in the Ellipse C x be the major axis, $PF . Px = b^2$.

$\therefore$ C x is the major axis, and $\therefore$ C q is the minor axis.

$\therefore$ the Ellipse is constructed.

Prop. XIII, Cor. 2. See Jesuits' note. The case of the body's descent in a straight line to the center is here omitted by Newton, because it is possible in most laws of force, and is moreover reserved for a full discussion in Section VII.

The value of the force is however easily obtained from 140.

157. Prop. XIV. $$L = \frac{Q T^2}{Q R} \propto \frac{Q T^2}{F}$$
$$\propto Q T^2 \times S P^2 \text{ by hypoth.}$$

OTHERWISE.

By Art. 150,
$$F = \frac{4 A^2}{g T^2} \times \frac{a}{b^2 \varrho^2} = \frac{8 A^2}{L g T^2} \times \frac{1}{\varrho^2}$$
for the circle, ellipse, and hyperbola, and by 152.
$$F = \frac{2 P^2 V^2}{L g} \times \frac{1}{\varrho^2}$$
for the parabola.

Now if μ be the value of F at distance 1, we have
$$F = \frac{\mu}{\varrho^2}.$$

Whence in the former case
$$\frac{8 A^2}{g L T^2} = \mu, \text{ or } = \frac{2 P^2 \times V^2}{g L} \quad \ldots\ldots\ldots \text{(a)}$$
and in the latter
$$\frac{2 P^2 \times V^2}{g L} = \mu \quad \ldots\ldots\ldots\ldots\ldots \text{(b)}$$

But
$$\frac{S P^2 \times Q T^2}{4} : 1^2 : A^2 : T^2$$
$$\therefore \frac{A^2}{T^2} = \frac{S P^2 \times Q T^2}{4} = \frac{P^2 \times V^2}{4}$$
$$\therefore S P^2 \times Q T^2 = \frac{\mu g}{2} L \quad \ldots\ldots\ldots\ldots\ldots \text{(c)}$$

158. Prop. XIV. Cor. I. By the form (a) we have
$$A\,(= \pi a b) = \sqrt{\frac{\mu g}{8}} \times \sqrt{L} \times T.$$
$$\propto T \sqrt{L}.$$

159. PROP. XV. From the preceding Art.

$$T = \sqrt{\frac{8}{\mu g}} \times \frac{\pi a b}{\sqrt{L}}.$$

But in the ellipse

$$L = \frac{2 b^2}{a}$$

$$\therefore T = \frac{2\pi}{\sqrt{\mu g}} \times a^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots \quad (e)$$

160. PROP. XVI. For explanations of the text see Jesuits' notes.

OTHERWISE.

By Art. 157 we get

$$V = \sqrt{\frac{g\mu}{2}} \times \frac{\sqrt{L}}{P} \quad \ldots\ldots\ldots\ldots \quad (f)$$

for the circle, ellipse, hyperbola, and parabola.

But in the circle, $L = 2P$.

$$\therefore V = \sqrt{g\mu} \times \frac{1}{\sqrt{P}} = \sqrt{g\mu} \times \frac{1}{\sqrt{r}} \quad \ldots \quad (g)$$

r being its radius.

In the ellipse and hyperbola

$$L = \frac{2 b^2}{a}$$

$$\therefore V = \sqrt{g\mu} \times \frac{b}{\sqrt{a}} \times \frac{1}{P} \quad \ldots\ldots\ldots \quad (h)$$

161. PROP. XVI, COR. I. By 157,

$$L = \frac{2}{g\mu} \times P^2 \times V^2.$$

162. COR. 2. $V = \sqrt{\frac{g\mu}{2}} \times \frac{\sqrt{L}}{D}$,

D being the max. or min. distance.

163. COR. 3. By Art. 160, and the preceding one,

$$V : V' :: \sqrt{\frac{g\mu}{2}} \times \frac{\sqrt{L}}{D} : \sqrt{g\mu} \times \frac{1}{\sqrt{D}}$$

$$:: \sqrt{L} : \sqrt{2D}.$$

164. COR. 4. By Art. 160,

$$v : v' :: \sqrt{\frac{g\mu}{2}} \times \frac{\sqrt{L}}{P} : \sqrt{g\mu} \times \frac{1}{\sqrt{r}}$$

But

$$L = \frac{2b^2}{a},\ P = b,\ \text{and } r = a$$

$$\therefore v : v' :: \frac{b}{b\sqrt{a}} : \frac{1}{\sqrt{a}} :: 1 : 1.$$

165. Cor. 6. By the equations to the parabola, ellipse, and hyperbola, viz.

$$p^2 = \frac{L}{4} \times \varrho,\ p^2 = \frac{b^2}{2a - \varrho},\ \text{and } p^2 = \frac{b^2}{2a + \varrho}$$

the Cor. is manifest.

166. Cor. 7. By Art. 160 we have

$$v^2 : v'^2 :: \frac{1}{2} \cdot \frac{L}{P^2} : \frac{1}{r}$$

which by aid of the above equations to the curves proves the Cor.

OTHERWISE.

By Art. 140 generally for all curves

$$v^2 = g F \times \frac{P V}{2}.$$

But generally

$$P V = \frac{2 p d \varrho}{d p}$$

and in the circle

$$P V = 2 \varrho \text{ (rad. } = \varrho)$$

$$\therefore v^2 : v'^2 :: \frac{p}{\varrho} : \frac{d p}{d \varrho}.$$

An analogy which will give the comparison between v and v′ for any curve whose equation is given.

167. Cor. 9. By Cor. 8,

$$v : v' :: \frac{L}{2} : p$$

and

$$v' : v'' :: \sqrt{\varrho} : \sqrt{\frac{L}{2}}$$

∴ ex equo

$$v : v'' :: \sqrt{\frac{\varrho L}{2}} : p.$$

168. PROP. XVII. The "*absolute quantity of the force*" must be known, viz. the value of μ, or else the *actual* value of V in the assumed orbit will not be determinable; i. e.

$$L : L' :: P^2 V^2 : P'^2 V'^2$$

will not give L'.

It must be observed that it has already been shown (Cor. 1, Prop. XIII) that the orbit is a conic section.

See Jesuits' notes, and also Art. 153 of this Commentary.

169. PROP. XVII, COR. 3. The two motions being compounded, the position of the tangent to the new orbit will thence be given and therefore the perpendicular upon it from the center. Also the new velocity. Whence, as in Prop. XVII, the new orbit may be constructed.

OTHERWISE.

Let the velocity be augmented by the impulse m times.

Now, if μ be the force at the distance 1, and P and V the perpendicular and velocity at distance (R) of projection, by 156 the general equation to the new orbit is such that its semi-axes are

$$a = \frac{R}{2 - m^2}, \text{ or } = \frac{R}{m^2 - 2}$$

and

$$b^2 = \frac{m^2 P}{2 - m^2}, \text{ or } \frac{m^2 P}{m^2 - 2}$$

according as the orbit is an ellipse or hyperbola. Moreover it also thence appears that when $m^2 = 2$, the orbit is a parabola, and that the equations corresponding to these cases are

$$p^2 = \frac{m^2 P}{2 - m^2} \times \frac{\varrho}{2 \times \frac{R}{2 - m^2} - \varrho}$$

or

$$= \frac{m^2 P}{m^2 - 2} \times \frac{\varrho}{2 \times \frac{R}{m^2 - 2} + \varrho}$$

or

$$= P \times \varrho.$$

DEDUCTIONS AND ADDITIONS

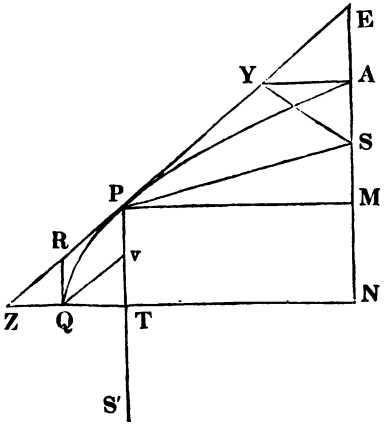
TO

SECTIONS II AND III.

170. In the parabola the force acting in lines parallel to the axis, required F.

$4\,S\,P\,.\,Q\,R : Q\,T^2 :: Q\,v^2 : Q\,T^2 :: Y\,E^2 : Y\,A^2 :: S\,E : S\,A :: S\,P : S\,A$

$\therefore \frac{Q\,R}{Q\,T^2} = \frac{1}{4\,S\,A}$, and $S'\,P$ is constant, $\therefore$ F is constant.



Let u be the velocity resolved parallel to P M then since the force acts perpendicular to P M, u at any point must be same as at A, $\therefore$ if P Q be the velocity in the curve, $Q\,T = u$ = constant quantity, and $a = \frac{S'\,P\,.\,Q\,T}{2}$

$\frac{S'\,P\,.\,u}{2}$.

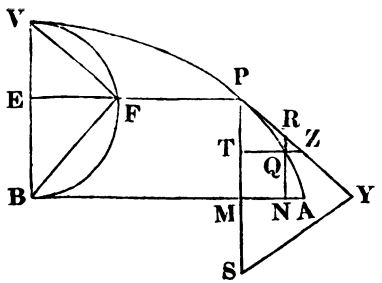
$$\therefore F = \frac{8\,a^2\,.\,Q\,R}{g\,S'\,P^2\,.\,Q\,T^2} = \frac{2\,u^2}{g\,L} \text{ (see 157)}$$

which avoids the consideration of S P being infinite; and

$$u^2 = 2\,g\,F\,.\,\frac{L}{4}$$

$\therefore$ body must fall through $\frac{L}{4}$ to acquire the velocity at vertex, which agrees with Mechanics. $\left(\text{At any point } V = u\sqrt{\frac{S\,P}{S\,A}}.\right)$

171. In the cycloid required the force when acting parallel to the axis.



$R P^2 : Q T^2 :: Z P^2 : Z T^2 :: V F^2 : E F^2 :: V B : B E$
and since the chord of curvature (C. c) $= 4 P M$, $R P^2 = 4 P M . R Q$,

$$\therefore 4 P M . R Q : Q T^2 :: V B : (B E =) P M$$

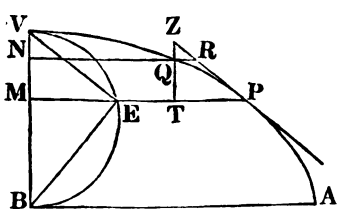
$$\therefore \frac{Q R}{Q T^2} = \frac{V B}{4 P M^2},$$

$$\therefore F \propto \frac{1}{P M^2} \text{ (since S P constant)}$$

$$F = \frac{8 a^2 . Q R}{g . S P^2 . Q T^2} = \frac{u^2 . V B}{2 g . P M^2}, \text{ if } u = \text{velocity parallel to A B.}$$

$$\left(\text{At any point } v = u . \sqrt{\frac{B V}{P M}}.\right)$$

172. In the cycloid the force is parallel to the base



$R P^2 : Q T^2 :: Z P^2 : Z T^2 :: V E^2 : V M^2 :: V B : V M$
and since $C . c = 4 E M$

$$R P^2 = 4 E M . R Q,$$

$$\therefore 4 E M . R Q : Q T^2 :: V B : V M,$$

$$\therefore \frac{Q R}{Q T^2} = \frac{V B}{4 E M . V M} \propto \frac{1}{E M . V M}.$$

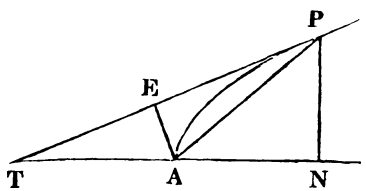
If $V M = y$, $F = \dfrac{u^2 r}{g y \sqrt{2 r y - y^2}} \left(r = \dfrac{V B}{2}\right)$

$u =$ velocity parallel to V B.

$$\left(F = \frac{8 a^2 Q R}{g . S P^2 Q T^2} = \frac{2 u^2 . Q R}{g . Q T^2} = \frac{u^2 . V B}{2 g . E M . V M}.\right)$$

$$\left(\text{At any point } v = u . \sqrt{\frac{B V}{V M}}.\right)$$

173. *Find* F *in a parabola tending to the vertex.*



$$T P : P N : : T A : A E$$

or

$$\sqrt{4 x^2 + y^2} : y : : x : \frac{y x}{\sqrt{4 x^2 + y^2}} = p, \text{ (A E)},$$

$$\therefore \frac{1}{p^2} = \frac{4 x^2 + a x}{a x^3} = \frac{4 x + a}{a x^2}$$

$$\therefore -\frac{2 d p}{p^3} = \frac{4 d x . a x^2 - 2 a x d x (4 x + a)}{a^2 x^4}$$

$$= -\frac{4 x^2 + 2 a x}{a x^4} . d x = -\frac{2}{a} . \frac{2 x + a}{x^3} . d x,$$

$$\therefore \frac{d p}{p^3} = \frac{2 x + a}{a x^3} . d x.$$

Also

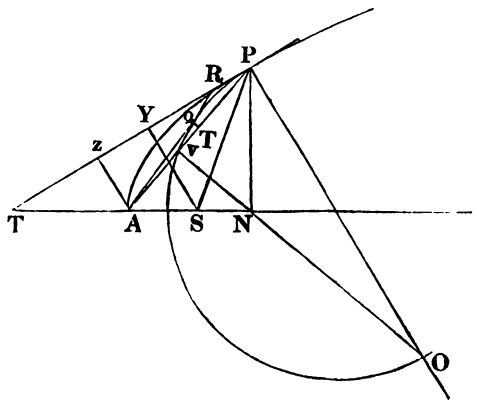
$$\varrho = \sqrt{x^2 + y^2},$$

$$\therefore d \varrho = \frac{x d x + y d y}{\sqrt{x^2 + y^2}} = \frac{x d x + \frac{a d x}{2}}{\sqrt{x^2 + a x}}$$

$$\therefore \frac{d p}{p^3 d \varrho} = \frac{2 x + a}{a x^3} . \frac{2 \sqrt{x^2 + a x}}{2 x + a} = \frac{2 \sqrt{x^2 + a x}}{a x^3}$$

$$\therefore F \propto \frac{A P}{A N^3}.$$

174. *Geometrically.* Let P Q O be the circle of curvature,



$$\text{P v (C. c through the vertex of the parabola)} = \frac{\text{P O} \cdot \text{A z}}{\text{A P}}$$

$$\therefore \frac{\text{P Q}^2}{\text{Q R}} = \frac{\text{P O} \cdot \text{A z}}{\text{A P}}$$

but

$$\frac{\text{P Q}^2}{\text{Q T}^2} = \frac{\text{A P}^2}{\text{A z}^2}$$

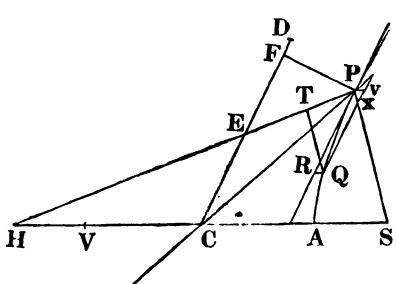
$$\therefore \frac{\text{Q R}}{\text{Q T}^2} = \frac{\text{A P}^3}{\text{P O} \cdot \text{A z}^3}$$

$$\therefore \text{F} = \frac{8\,a^2 \cdot \text{Q R}}{g \cdot \text{A P}^2 \cdot \text{Q T}^2} = \frac{8\,a^2 \cdot \text{A P}}{g \cdot \text{P O} \cdot \text{A z}^3}$$

but

$$\text{P O} \cdot \text{A z}^3 = 2\,\text{A S} \cdot \frac{\text{S P}^3}{\text{S Y}^3} \cdot \frac{\text{S Y}^3 \text{A T}^2}{\text{S P}^3} = 2\,\text{A S} \cdot \text{A N}^3$$

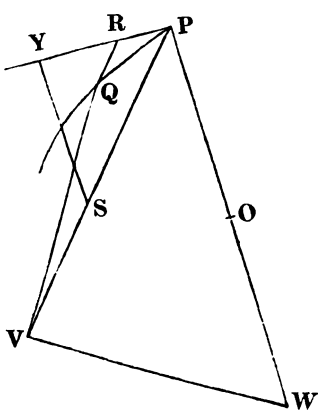
$$\therefore \text{F} = \frac{4\,a^2 \cdot \text{A P}}{g \cdot \text{A S} \cdot \text{A N}^3}.$$



175. If the centripetal be changed into a repelling force, and the body revolve in the opposite hyperbola, $\text{F} \propto \frac{1}{\text{H P}^2}$.

The body is projected in direction P R; R Q is the deflection from the Tangent due to repelling force H P, find the force.

$$L.Px : L.Pv :: Px : Pv :: PE : PC :: AC : PC$$
$$L.Pv : Pv.vG :: L : 2PC$$
$$Pv.vG : Qv^2 :: PC^2 : CD^2$$
$$Qv^2 : Qx^2 :: 1 : 1$$
$$Qx^2 : QT^2 :: PE^2 : PF^2 :: AC^2 : PF^2 :: CD^2 : BC^2$$
$$\therefore L.Px : QT^2 :: AC.L.PC^2.CD^2 : 2PC^2.CD^2.BC^2$$
$$:: L : \frac{2BC^2}{AC} :: 1 : 1$$
$$\therefore L = \frac{QT^2}{QR}$$
$$\therefore F = \frac{8a^2.QR}{g.HP^2.QT^2} = \frac{8a^2}{g.L.HP^2} \propto \frac{1}{HP^2}.$$



176. In any Conic Section the chord of curvature $= \frac{L.SP^2}{SY^2}$

for

$$PV = \frac{QP^2}{QR} = \frac{QT^2.SP^2}{QR.SY^2} = \frac{L.SP^2}{SY^2}.$$

177. Radius of curvature $= \frac{L}{2}\frac{SP^3}{SY^3}$;

for

$$PW = \frac{PV.SP}{SY} = \frac{L.SP^3}{SY^3}.$$

178. Hence in any curve $F = \frac{8a^2}{g.SY^2.PV}$

$$= \frac{8a^2}{g.SY^2.\frac{2R.SY}{SP}} = \frac{4a^2.SP}{g.SY^3.R}.$$ see Art. 74.

179. Hence in Conic Sections

$$F = \frac{8a^2}{g \cdot \overline{SY^2} \cdot PV} = \frac{8a^2}{g \cdot SY^2 \cdot \frac{L \cdot SP^2}{SY^2}} = \frac{8a^2}{g \cdot L \cdot SP^2} \propto \frac{1}{SP^2}.$$

180. If the chord of curvature be proved $= \frac{L \cdot SP^2}{SY^2}$ independently of he proof that $\frac{QT^2}{QR} = L$, this general proof of the variation of force in tonic sections might supersede Newton's; otherwise not.

181. *A body attached to a string, whose length* = b, *is whirled round so as to describe a circle whose center is the fixed extremity of the string parallel to the horizon in* T″; *required the ratio of the tension to the weight.*

Gravity = 1, ∴ v of the revolving body = $\sqrt{g F b}$, if b be the length of the string;

$$\therefore F (= \text{centripetal force} = \text{tension}) = \frac{V^2}{g b} \quad (131)$$

and

$$T = \frac{\text{circumference}}{V} = \frac{2\pi b}{\sqrt{g F b}} = 2\pi \frac{\sqrt{b}}{\sqrt{g F}}$$

$$\therefore F = \frac{\pi^2 4 b}{g T^2}$$

$$\therefore F : \text{Gravity} :: \frac{4\pi^2 b}{g T^2} : 1, \text{ or Tension} : \text{weight} :: 4\pi^2 b : g T^2.$$

If Tension = 3 weight; required T.

$$4\pi^2 b : g T^2 :: 3 : 1,$$

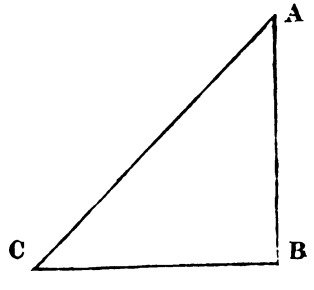
$$\therefore T^2 = \frac{4\pi^2 b}{3 g}.$$

If T be given, and the tension = 3 weight, required the length of the string.

$$T^2 = \frac{4\pi^2 b}{3 g},$$

$$\therefore b = \frac{3 T^2 g}{4\pi^2}.$$

182. *If a body suspended by a string from any point describe a circle, the string describes a cone; required the time of one revolution or of one oscillation.*



Let A C = l, B C = b,

The body is kept at rest by 3 forces, gravity in the direction of A B, tension in the direction C A, and the centripetal force in the direction C B.

As before, centripetal force $= \frac{4\pi^2 b}{g T^2}$,

and centripetal force : gravity : b : $\sqrt{l^2 - b^2}$, (from $\triangle$) : : $\frac{4\pi^2 b}{g T^2} : l$

$$\therefore T^2 = \frac{4\pi^2 \sqrt{l^2 - b^2}}{g}$$

$$\therefore T = 2\pi \sqrt{\frac{\sqrt{l^2 - b^2}}{g}} = \text{a constant quantity if } \sqrt{l^2 - b^2}$$

be given.

$\therefore$ the time of oscillation is the same for all conical pendulums having a common altitude.

183. v *in the Ellipse at the perihelion* : v *in the circle* e. d. : : n : 1, *find the major axis, excentricity, and compare its* T *with that in the circle, and find the limits of* n.

Let S A = c,

v in the Ellipse : that in the circle e. d. : : $\sqrt{H P} : \sqrt{A C}$

: : $\sqrt{H A} : \sqrt{A C}$ in this case

: : n : 1 by supposition,

$$\therefore 2 A C - A S = n^2 A C,$$

$$\therefore A C = \frac{c}{2 - n^2}.$$

$$\text{Excentricity} = A C - A S = \frac{c}{2 - n^2} - c = \frac{c n^2 - c}{2 - n^2}.$$

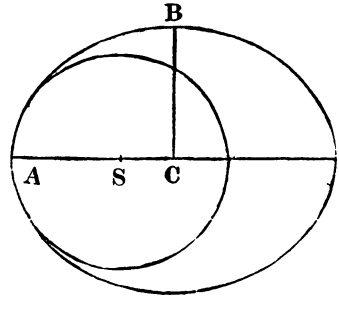
$$T : T \text{ in the circle} : : A C^{\frac{5}{2}} : A S^{\frac{5}{2}} : : \frac{c^{\frac{5}{2}}}{(2 - n^2)^{\frac{5}{2}}} : c^{\frac{5}{2}} : : 1 : (2 - n^2)^{\frac{5}{2}}$$

Also n must be $< \sqrt{2}$,

for if $n = \sqrt{2}$, the orbit is a parabola

if $n > \sqrt{2}$, the orbit is an hyperbola.

184. *Suppose* $\frac{1}{4}$ *of the quantity of matter of* $\oplus$ *to be taken away. How much would* T *of* $\mathbb{D}$ *be increased, and what the excentricity of her new orbit? the* $\mathbb{D}$*'s present orbit being considered circular.*



At any point A her direction is perpendicular to S A,

$\therefore$ if the force be altered at any point A, her v in the new orbit will

$= $ her v in the circle, since $v = \frac{2\alpha}{S\bar{Y}}$, and S Y $=$ S A, and α is the same at A.

Let A S $= c$, P V at A $= L$, and $F = \frac{2V^2}{g\ P\ V} \propto \frac{1}{P\ V}$ in this case,

$$\therefore 3 : 4 :: 2c\ (= L \text{ in the circle}) : \frac{2b^2}{a}\ (= L \text{ in the ellipse})$$

$$\therefore 4c = \frac{3b^2}{a} = \frac{3(a^2 - \overline{a-c}^2)}{a} = \frac{3(2ac - c^2)}{a} = 6c - \frac{3c^2}{a}$$

$$\therefore \frac{3c^2}{a} = 2c,$$

$$\therefore a = \frac{3c}{2}.$$

And T in the circle : T′ in the ellipse $:: \frac{c^{\frac{3}{2}}}{\sqrt{4}} : \frac{\left(\frac{3c}{2}\right)^{\frac{3}{2}}}{\sqrt{3}}$

$$:: \frac{\sqrt{3}}{\sqrt{4}} : \left(\frac{3}{2}\right)^{\frac{3}{2}} :: \frac{1}{\sqrt{2}} : \frac{3}{2}$$

$$:: \sqrt{2} : 3.$$

And the excentricity $= a - c = \frac{3}{2}c - c = \frac{c}{2}$.

185. *What quantity must be destroyed that* ☽*'s* T *may be doubled, and what the excentricity of her new orbit?*

Let F of $\oplus$: f (new force) :: n : 1

$$\therefore v = \sqrt{\frac{g}{2} F . P V}, \text{ and v is given,}$$

$$\therefore F \propto \frac{1}{P\ V},$$

$$\therefore n : 1 :: \frac{2b^2}{a} : 2c :: \frac{a^2 - \overline{a-c}^2}{a} : c :: \frac{2ac - c^2}{a} : c :: 2a - c : a^2$$

$$\therefore n\,a = 2a - c,$$

$$\therefore a = \frac{c}{2-n}.$$

Also T in the circle : T in the ellipse :: 1 : 2

$$:: \frac{c^{\frac{3}{2}}}{\sqrt{n}} : \frac{c^{\frac{3}{2}}}{(2-n)^{\frac{3}{2}}}$$

$$:: (2-n)^{\frac{3}{2}} : n^{\frac{1}{2}}$$

$$\therefore 1 : 4 :: (2-n)^3 : n \therefore n = 4(2-n)^3, \text{ whence n.}$$

And the excentricity

$$= a - c = \frac{c}{2-n} - c = \frac{c-(2c-nc)}{2-n} = \frac{c(n-1)}{2-n}$$

186. *What quantity must be destroyed that* ☽*'s orbit may become a parabola?*

$$L = 4c,$$
$$\therefore F : f :: 4c : 2c :: 2 : 1,$$
$$\therefore \tfrac{1}{2} \text{ the force must be destroyed.}$$

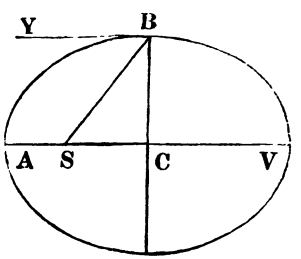
187. $F \propto \frac{1}{D^2}$, *a body is projected at given* D, v = v *in the circle,* $\angle$ *with* S B = 45°, *find axis major, excentricity, and* T.

Since v = v in the circle, ∴ the body is projected from B,

and $\angle$ S B Y = 45°;

∴ $\angle$ S B C, or B S C = 45°,

$$\therefore S C = S B.\ \cos.\ 45° = \frac{S B}{\sqrt{2}}.$$

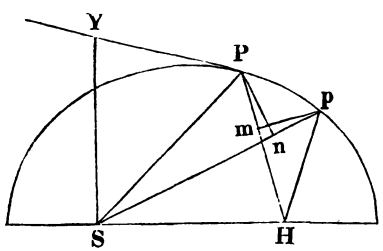


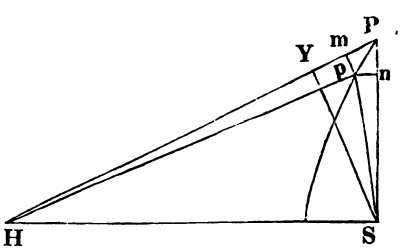
But

$$S B = D = \frac{\text{axis major}}{2},$$

∴ axis major and excentricity are found.

And T may be found from Art. 159.





188. *Prove that the angular* v *round* H : *that round* S :: S P : H P.

This is called Seth Ward's Hypothesis.

In the ellipse. Let P m, p n, be perpendicular to S p, H P,

∴ p m = Increment of S P = Decrement of H P = P n

∴ triangles P m p, P n p, are equal,

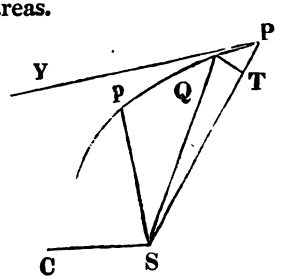
$$\therefore P m = p n, \text{ and angular v} \propto \frac{1}{\text{distance}}$$

189. Similarly in the hyperbola.

$$\text{Angular v of S P : angular v of S Y :: P V : 2 S P :: } \frac{2CD^2}{AC} : 2SP$$
$$:: \frac{CD^2}{SP} : AC$$
$$:: HP : AC.$$

190. *Compare the times of falling to the center of the logarithmic spiral from different points.*

The times are as the areas.



$$\text{d . area} = \frac{\varrho^2 \, \text{d}\,\theta}{2}, \; (\theta = \angle \text{C S P}), \text{ for d . area} = \frac{\text{Q T . S P}}{2}.$$

$$\text{Also } \frac{\text{Q T}}{\text{T P}} = \frac{\varrho \,.\, \text{d}\,\theta}{\text{d}\,\varrho} = \text{tan. } \angle \text{ Y P T} = \text{tan. } \alpha, \; (\alpha \text{ being constant}) = \text{a}$$

$$\therefore \text{d}\,\theta = \frac{\text{a} \,.\, \text{d}\,\varrho}{\varrho},$$

$$\therefore \frac{\varrho^2 \, \text{d}\,\theta}{2} = \frac{\text{a} \,.\, \varrho \,.\, \text{d}\,\varrho}{2},$$

$$\therefore \text{area} = \frac{\text{a} \,.\, \varrho^2}{4} \propto \varrho^2, \text{ (for when } \varrho = 0, \text{ area} = 0, \therefore \text{Cor.} = 0)$$

∴ if P, p, be points given,

T from P to center : t from p to center : : S P^2 : S p^2.

191. *Compare* v *in a logarithmic spiral with that in a circle,* e. d.

$$\text{F} = \frac{2 \, \text{V}^2}{\text{g} \,.\, \text{P V}} \; (140)$$

∴ if F be given, V $\propto \sqrt{\text{P V}}$,

∴ v in spiral : v in the circle : : $\sqrt{\text{P V in spiral}} : \sqrt{2 \, \text{S P}}$: : 1 : 1.

192. *Compare* T *in a logarithmic spiral with that in a circle,* e. d.

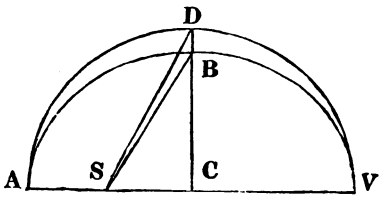
$$\text{T in spiral} = \frac{\text{whole area}}{\text{area in 1}''} = \frac{\text{a} \, \varrho^2}{\frac{4 \,.\, \text{v} \,.\, \text{S Y}}{2}} = \frac{\text{a} \, \varrho^2}{2 \, \text{v} \,.\, \varrho \,.\, \text{sin. } \alpha}$$

$$\text{T}' \text{ in circle} = \frac{\text{whole area}}{\text{area in 1}''} = \frac{\pi \, \varrho^2}{\frac{\text{v} \,.\, \text{S Y}}{2}} = \frac{2 \, \pi \, \varrho^2}{\text{v} \,.\, \varrho} = \frac{2 \, \pi \, \varrho}{\text{v}}$$

$$\therefore \text{T} : \text{T}' : : \frac{\text{a} \, \varrho^2}{2 \, \text{v} \,.\, \varrho \,.\, \text{sin. } \alpha} : \frac{2 \, \pi \, \varrho}{\text{v}} : : \frac{\text{a}}{2 \, \text{sin. } \alpha} : 2 \, \pi : : \text{a} : 4 \, \pi \,.\, \text{sin. } \alpha.$$

$$: : \text{tan. } \alpha : 4 \, \pi \,.\, \text{sin. } \alpha : : \; 1 : 4 \, \pi \, \text{cos. } \alpha.$$

192. *In the Ellipse compare the time from the mean distance to the Aphelion, with the time from the mean distance to the Perihelion. Also given the Excentricity, to find the difference of the times, and conversely.*



A D V is $\frac{\text{Circle}}{2}$ described on A V.

T of passing through Aphelion : t through Perihelion

$$:: S\,B\,V : S\,B\,A$$
$$:: S\,D\,V : S\,D\,A$$
$$:: C\,D\,V + \frac{D\,C \cdot S\,C}{2} : C\,D\,V - \frac{D\,C \cdot S\,C}{2}.$$

Let Q = quadrant C D V,

$$\therefore T : t :: Q + \frac{a \cdot a\,e}{2} : Q - \frac{a \cdot a\,e}{2}$$
$$\therefore (T + t =)\ P : T - t :: 2\,Q : a \cdot a\,e$$
$$\therefore T - t = \frac{P \cdot a \cdot a\,e}{2\,Q}$$

whence T — t, or, if T — t be given, a e may be found.

193. *If the perihelion distance of a comet in a parabola* = 64, ⊕*'s mean distance* = 100, *compare its velocity at the extremity of* L *with* ⊕*'s velocity at mean distance.*

Since ⊕ moves in an ellipse, v at the mean distance = that in the circle e . d . and v in the parabola at the extremity of L

$$: v \text{ in the circle rad. } 2\,S\,A :: \sqrt{2} : 1$$

v in the circle rad. 2 S A

$$: v \text{ in the circle rad. } A\,C :: \sqrt{A\,C} : \sqrt{S\,A}$$

∴ v in the parabola at L

$$: v \text{ in the ellipse at } B :: \sqrt{2 \cdot A\,C} : \sqrt{S\,A \cdot 2}$$
$$:: 10\sqrt{2} : 8\sqrt{2}$$
$$:: 5 : 4$$

194. *What is the difference between* L *of a parabola and ellipse, having the same* $<^{st}$ *distance* = 1, *and axis major of the ellipse* = 300? *Compare the* v *at the extremity of* L *and* $<^{st}$ *distances.*

In the parabola L = 4 A S = 4.

In the ellipse $L' = \frac{2BC^2}{AC} = \frac{2}{300}(AC^2 - \overline{AC - SA^2})$

$$= \frac{1}{150}(2AC \cdot AS - AS^2) = \frac{600-1}{150}.$$

$$\therefore L - L' = 4 - \left(4 - \frac{1}{150}\right) = \frac{1}{150}.$$

v in the parabola at A : v in the circle rad. S A : : $\sqrt{2}$: 1

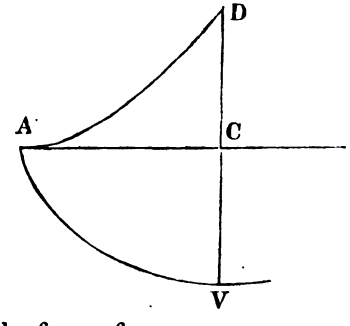
v in the circle rad. S A : v in the ellipse e. d. : : $\sqrt{AC}$: $\sqrt{HP}$

: : $\sqrt{AC}$: $\sqrt{2AC - SA}$: : $\sqrt{150}$: $\sqrt{299}$

$\therefore$ v in the parabola at A : v in the ellipse e. d. : : $\sqrt{300}$: $\sqrt{299}$.

Similarly compare v^s. at the extremity of Lat. R.

195. *Suppose a body to oscillate in a whole cycloidal arc, compare the tension of the string at the lowest point with the weight of the body.*



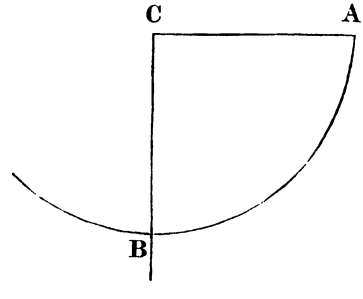
The tension of the string arises from two causes, the weight of the body, and the centrifugal force. At V we may consider the body revolving in the circular arc rad. D V, $\therefore$ the centrifugal = centripetal force. Now the velocity at V = that down C V by the force of grav.

= that with which the body revolves in the circle rad. 2 C V.

$\therefore$ grav. : centrifugal force : : 1 : 1,

$\therefore$ tension : grav. : : 2 : 1

196. *Suppose the body to oscillate through the quadrant* A B, *compare the tension at* B *with the weight.*



At B the string will be in the direction of gravity; $\therefore$ the whole weight will stretch the string; $\therefore$ the tension will = centrifugal force + weight. Now the centrifugal force = centripetal force with which the body would revolve in the circle e. d.

And v in the circle = $\sqrt{2g \cdot F} \cdot \frac{R}{2}$

$$\therefore F = \frac{v^2}{g\,R} = \frac{v^2}{g\,C\,B} \text{ in this case,}$$

also v' at B from grav. $= \sqrt{2\,g\,.\,C\,B}$, grav. $= 1$.

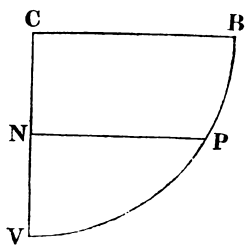
$$\therefore \text{grav.} = 1 = \frac{v'^2}{2\,g\,C\,B}$$

$$\therefore F : \text{grav.} :: \frac{v^2}{2gCB} : \frac{v'^2}{g\,C\,B} :: 2 : 1,$$

since $v = v'$.

$\therefore$ tension : grav. : : 3 : 1.

197. *A body vibrates in a circular arc from the center* C ; *through what arc must it vibrate so that at the lowest point the tension of the string* $= 2 \times$ *weight?*



v from grav. $=$ v d . N V, (if P be the point required) v' of revolution in the circle $=$ v d. $\frac{C\,V}{2}$.

$$\therefore \text{centrifugal force : grav. : : } v : v' :: \sqrt{\frac{C\,V}{2}} : \sqrt{N\,V}$$

$$\therefore \text{centrifugal force + grav. (= tension) : grav. : : } \sqrt{\frac{C\,V}{2}} + \sqrt{N\,V} : \sqrt{N\,V}$$

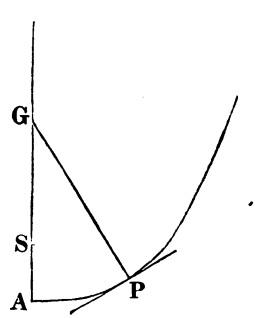
: : 2 : 1 by supposition.

$$\therefore \sqrt{\frac{C\,V}{2}} + \sqrt{N\,V} = 2\sqrt{N\,V}$$

$$\therefore \sqrt{\frac{C\,V}{2}} = \sqrt{N\,V},$$

$$\therefore N\,V = \frac{C\,V}{2}.$$

198. *There is a hollow vessel in form of an inverted paraboloid down which a body descends, the pressure at lowest point* $=$ n . *weight, find from what point it must descend.*



At any point P, the body is in the same situation as if suspended from G, P G being normal, and revolving in the circle whose rad. G P. Now P G $= \sqrt{4\,A\,S\,.\,S\,P}$, $\therefore$ at A, P G $=$

$\sqrt{4\,A\,S^2} = 2\,A\,S$. Also v^2 at A with which the body revolves $= 2g\,.\,F.\frac{2\,A\,S}{2}$.

$$\therefore \text{centrifugal force} = \frac{v^2}{2\,g\,A\,S}$$

$$\text{and grav.} = \frac{v^2}{2\,g\,h}, \text{ if } h = \text{height fallen from.}$$

But the whole pressure arises from grav. + centrifugal force, and $= n$. grav.

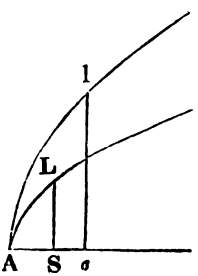
$\therefore$ centrifugal force + grav. : grav. : : n : 1

or

$$\frac{1}{A\,S} + \frac{1}{h} : \frac{1}{h} :: n : 1,$$

$$\therefore \frac{1}{A\,S} : \frac{1}{h} :: n - 1 : 1,$$

$$\therefore h = \overline{n-1}\,.\,A\,S.$$



199. *Compare the time* (T′) *in which a body describes* 90° *of anomaly in a parabola with* T *in the circle rad.* = S A.

Time through A L : 1 : : area A S L : a in 1″

$$\therefore T = \frac{\frac{2}{3}\,A\,S.\,S\,L}{a} = \frac{4\,A\,S^2}{3\,a}$$

T in the circle rad. S A : 1 : : whole circle : a′ in 1″

$$\therefore T = \frac{\pi\,A\,S^2}{a'}$$

$$\therefore T' : T :: \frac{4}{3\,a} : \frac{\pi}{a'}$$

and

$$a : a' :: \sqrt{L} : \sqrt{2\,A\,S} :: \sqrt{4\,A\,S} : \sqrt{2\,A\,S} :: \sqrt{2} : 1$$

$$\therefore T' : T :: \frac{4}{3\sqrt{2}} : \pi :: 2\sqrt{2} : 3\pi.$$

Compare the time of describing 90° *in the parabola* A L *with that in the parabola* A l, (fig. same.)

t : T in the circle rad. S A : : 4 : $3\sqrt{2}\,.\,\pi$

T in the circle S A : T′ in the circle rad. σ A : : S $A^{\frac{5}{2}}$: $\sigma\,A^{\frac{5}{2}}$ (since $T^2 \propto R^3$)

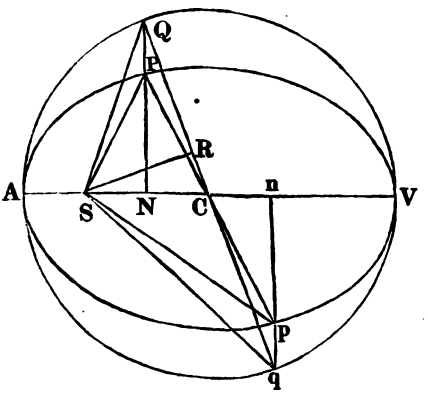
T′ : t′ through l A : : $3\sqrt{2}\,.\,\pi$: 4

$\therefore$ t through S A : t′ through σ A : : S $A^{\frac{5}{2}}$: $\sigma\,A^{\frac{5}{2}}$.

See Sect. VI. Prop. XXX.

200. *Draw the diameter* P p *such that the time through* P V p : *time through* p A P : : n : 1, *force* $\propto \frac{1}{S P^2}$.

Describe the circle on A V.



Let t = time through P V p, and T the periodic time

$$\therefore t = \frac{n}{n+1} = \frac{PVpS}{\text{ellipse}} = \frac{QVqS}{\text{circle}} = \frac{\text{circle} + \triangle QqS}{\text{circle}}$$

$$= \frac{\frac{\text{circle}}{2} + \frac{SR.2CQ}{2}}{\text{circle}}$$

$$= \frac{\frac{\pi a^2}{2} + a e . \sin. u . a}{\pi a^2}, \quad (u = \text{excentric anomaly})$$

$$= \frac{\frac{\pi}{2} + e . \sin. u}{\pi}$$

$$\therefore n\pi = \overline{n+1} . \left(\frac{\pi}{2} + e \sin. u\right)$$

$$= n\frac{\pi}{2} + \frac{\pi}{2} + \overline{n+1} . e \sin. u$$

$$\therefore \sin. u = \frac{n-1}{n+1} \cdot \frac{\pi}{2e},$$

which determines u, &c.

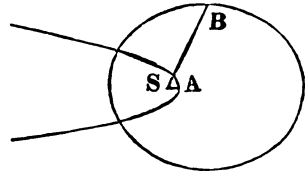
201. *The Moon revolves round the Earth in* 30 *days, the mean distance from the Earth* = 240,000 *miles. Jupiter's Moon revolves in* $\frac{1}{2}$ *day, the mean distance from Jupiter* = 240,000 *miles. Compare the absolute forces of Jupiter and the Earth.*

$T \propto \frac{A^{\frac{3}{2}}}{\sqrt{\mu}}$, A being the major axis of the ellipse,

$\therefore$ If A be given, $\mu \propto \frac{1}{T^2}$;

$$\therefore \frac{\text{Mass of Jupiter}}{\text{Mass of the Earth}} = \frac{T^2 \text{ of the Earth's Moon}}{T'^2 \text{ of Jupiter's Moon}} = \frac{30^2}{\frac{1}{4^2}} = \frac{14{,}400}{1}.$$

202. *A Comet at perihelion is* 400 *times as near to the Sun as the Earth at its mean distance. Compare their velocities at those points.*



$$\frac{\text{Velocity}^2 \text{ of the Comet}}{\text{Velocity}^2 \text{ of the Earth}} = \frac{F \cdot 4\,A\,S}{F' \cdot 2\,B\,S} = \frac{F}{F'} \cdot \frac{4}{2 \cdot 400} = \frac{F}{F'} \cdot \frac{1}{200}$$

$$= \frac{400^2}{1^2} \cdot \frac{1}{200} = 800$$

$$\therefore \frac{V}{v} = \frac{\sqrt{2 \cdot 20}}{1} = \frac{30}{1} \text{ nearly.}$$

203. *Compare the Masses of the Sun and Earth, having the mean distance of the Earth from the Sun* = 400, *the distance of the Moon from the Earth, and Earth's* P^d. = 13. *the Moon's* P^d.

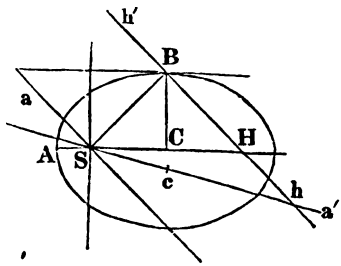
$$T^2 \propto \frac{a^3}{\mu},$$

$$\therefore \mu \propto \frac{a}{T^2},$$

$$\therefore \frac{\text{Mass of the Sun}}{\text{Mass of the Earth}} = \frac{400^3}{1^3} \cdot \frac{1^2}{13^2} = \frac{64{,}000{,}000}{169} = 400{,}000 \text{ nearly.}$$

204. *If the force* $\propto \frac{1}{x^2} - \frac{a}{x^3}$, *where* x *is the distance from the center of force, it will be centripetal whilst* $\frac{1}{x^2} > \frac{a}{x^3}$, *or* $x > a$; *there will be a point of contrary flexure in the orbit when* $\frac{1}{x^2} = \frac{a}{x^3}$, *or* $x = a$, *and afterwards when* $x < a$, *the force will be repulsive, and the curve change its direction.*

205. *The body revolving in an ellipse, at* B *the force becomes* n *times as great. Find the new orbit, and under what values of* n *it will be a parabola, ellipse, or hyperbola.*



S being one focus since the force $\propto \frac{1}{\text{distance}^2}$, the other focus must lie in B H produced both ways, since S B, H B, make equal angles with the tangent. $V^2 = \frac{g}{2}$ F. P V $= \frac{g}{2}$ F. 2 A C in the original ellipse, or $= \frac{g}{2}$ n F. P V in the new orbit.

$$\therefore 2\,A\,C = n\,.\,P\,V = n\,.\,\frac{2\,S\,B\,.\,h\,B}{\frac{S\,B + h\,B}{2}},$$

$$\therefore (S\,B + h\,B)\,A\,C = 2\,n\,.\,S\,B\,.\,h\,B,$$

$$\therefore A\,C^2 + h\,B\,.\,A\,C = 2\,n\,A\,C\,.\,h\,B,$$

$$\therefore h\,B = \frac{A\,C}{2\,n - 1}.$$

If $2\,n - 1 = 0$, or $n = \frac{1}{2}$, the orbit is a parabola; if $n > \frac{1}{2}$, the orbit is an ellipse; if $n < \frac{1}{2}$, the orbit is an hyperbola.

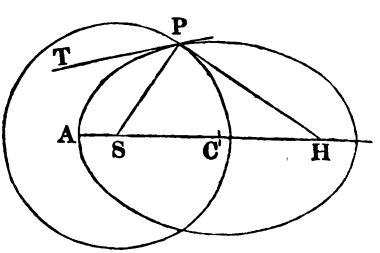
Let S C in the original ellipse be given $=$ B C,

$\therefore$ S B H $=$ right angle, and S B or A C $=$ B h . cot. B S h whence the direction of a a′, the new major axis; also

$$a\,a' = S\,B + B\,h, \text{ and } S\,c = \frac{S\,h}{2} = \frac{\sqrt{Bh^2 - SB^2}}{2}.$$

If the orbit in the parabola a a′ be parallel to B h, and L . R $= 2\,S\,B$, since S B h $=$ right angle.

206. *Suppose a Comet in its orbit to impel the Earth from a circular orbit in a direction making an acute angle with the Earth's distance from the Sun, the velocity after impact being to the velocity before* :: $\sqrt{3} : \sqrt{2}$. *Find the alteration in the length of the year.*



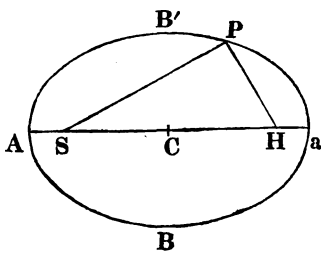
Since $\sqrt{3} : \sqrt{2} <$ ratio than $\sqrt{2} : 1$, $\therefore$ the new orbit will be an ellipse.

$$\frac{V^2}{v^2} = \frac{3}{2} = \frac{PV}{2SP} = \frac{2SP.HP}{AC.2SP} = \frac{HP}{AC}$$
$$= \frac{2AC - SP}{AC}$$

$$\therefore 3AC = 4AC - 2SP$$
$$\therefore 2SP = AC$$

$$\therefore \frac{T \text{ in ellipse}}{T' \text{ in circle}} = \frac{2^{\frac{3}{2}} SP^{\frac{3}{2}}}{SP^{\frac{3}{2}}} = \frac{8}{3} \text{ nearly.}$$

207. *A body revolves in an ellipse, at any given point the force becomes diminished by* $\frac{1}{n}$ *th part. Find the new orbit.*



$$v^2 \propto F.PV$$

$$\therefore \text{in this case } PV \propto \frac{1}{F},$$

$$\therefore \frac{PV \text{ in ellipse}}{pv \text{ in new orbit}} = \frac{1 - \frac{1}{n}}{1} = \frac{n-1}{n}$$

But

$$\frac{v^2 \text{ in conic section}}{v^2 \text{ in circle e. d.}} = \frac{pv}{2SP} = \frac{\frac{n}{n-1}.PV}{2SP} \text{ at P}$$
$$= \frac{n}{n-1}.\frac{HP}{AC}$$

$$\therefore \text{if } \frac{n}{n-1}.HP = AC, \text{ the new orbit is a Circle}$$
$$\text{——————} = 2AC, \text{ ——————} \text{ Parabola}$$
$$\text{——————} < 2AC, \text{ ——————} \text{ Ellipse}$$
$$\text{——————} > 2AC, \text{ ——————} \text{ Hyperbola}$$

If $\frac{n}{n-1} = 2$, or $n = 2$, then when the orbit is a circle or an ellipse, P must be between a, B; when the orbit is a parabola, P must be at B; when the orbit is an hyperbola, P must be between B, A.

208. *If the curvature and inclination of the tangent to the radius be the same at two points in the curve, the forces at those points are inversely as the radii².*

$$F = \frac{8\,a^2}{g \cdot SY^2 \cdot PV} = \frac{8\,a^2}{g \cdot SY \cdot SP \cdot R} = \frac{8\,a^2}{g \cdot \sin.\theta\, SP^2 \cdot R} \propto \frac{1}{SP^2}.$$

This applies to the extremities of major axis in an ellipse (or circle) in the center of force in the axis.

209. *Required the angular velocity of* ϱ.

By 46, θ being the traced-angle,

$$\omega = \frac{d\,\theta}{d\,t}.$$

But by Prop. I. or Art. 124,

$$d\,t : T :: d\,A : A$$

$$:: \frac{\varrho^2 d\,\theta}{2} : A \quad (113)$$

$$\therefore \omega = \frac{d\,\theta}{d\,t} = \frac{2\,A}{T} \times \frac{1}{\varrho^2}$$

or

$$= \frac{P \times V}{\varrho^2} \quad \ldots\ldots\ldots \quad (a)$$

210. *Required the Centrifugal Force* (φ) *in any orbit.*

When the revolving body is at any distance ϱ from the center of force, the Centrifugal Force, which arises from its inertia or tendency to persevere in the direction of the tangent (most authors erroneously attribute this force to the angular motion, see Vince's Flux. p. 283) is clearly the same as it would be were the body with the same Centripetal Force revolving in a circle whose radius is ϱ. Moreover, since in a circle the body is always at the same distance from the center, the Centrifugal Force must always be equal to the Centripetal Force.

But in the circle

$$Q\,T^2 = Q\,R \times 2\,S\,P$$

and $\therefore$ by 137 we have

$$F = \frac{8\,A^2}{g\,T^2} \times \frac{1}{2\,S\,P^3} = \frac{4\,A^2}{g\,T^2} \times \frac{1}{\varrho^3}$$

or

$$= \frac{P^2\,V^2}{g} \times \frac{1}{\varrho^3}$$

P and V belonging to the orbit.

Hence then

$$\varphi = \frac{P^2 V^2}{g} \times \frac{1}{\rho^3} \quad \ldots\ldots\ldots\ldots \quad (a)$$

Hence also and by 209,

$$\varphi = \frac{\omega^2 \times \rho}{g}. \quad \ldots\ldots\ldots\ldots \quad (b)$$

And 139,

$$F : \varphi :: \frac{d\,p}{p^3} : \frac{d\,\rho}{\rho^3} \quad \ldots\ldots\ldots\ldots \quad (c)$$

211. *Required the angular velocity of the perpendicular upon the tangent.*

If two consecutive points in the curve be taken; tangents, perpendiculars and the circle of curvature be described as in Art. 74, it will readily appear that the incremental angle ($d\,\psi$) described by p = that described by the radius of curvature. It will also be seen that

$$d\,\theta : d\,\psi \left(:: \frac{Q\,T}{\rho} : \frac{P\,Q}{R} \right) :: \frac{p}{\rho} : \frac{\rho}{R}$$

But from similar triangles

$$P\,V : 2\,R :: p : \rho.$$

$$\therefore\ d\,\theta : d\,\psi :: P\,V : 2\,\rho$$

P V being the chord of curvature.

Hence

$$\varpi \left(= \frac{d\,\psi}{d\,t} \right) = \frac{d\,\theta}{d\,t} \times \frac{2\,\rho}{P\,V} \text{ (see 209)}$$

$$= \omega \times \frac{2\,\rho}{P\,V} \quad \ldots\ldots\ldots\ldots \quad (d)$$

or

$$= \frac{2\,P \times V}{\rho \times P\,V} \quad \ldots\ldots\ldots\ldots \quad (e)$$

or

$$= \frac{P \times V}{p\,\rho} \times \frac{d\,p}{d\,\rho} \quad \ldots\ldots\ldots\ldots \quad (f)$$

Ex. 1. In the circle $P\,V = 2\,\rho$; whence

$$\varpi = \frac{P \times V}{\rho^2} = \omega.$$

Ex. 2. In the other Conic Sections, we have

$$p^2 = \frac{b^2\,\rho}{2\,a \mp \rho}$$

which gives by taking the logarithms

$$2\,l\,p = l\,b^2 + l\,\varrho - l\,(2\,a \mp \varrho)$$

and (17 a.)

$$\frac{2\,d\,p}{p} \doteq \frac{d\,\varrho}{\varrho} \pm \frac{d\,\varrho}{2\,a \mp \varrho} = \frac{2\,a\,d\,\varrho}{\varrho\,(2\,a \mp \varrho)}$$

whence

$$\varpi = \frac{a\,P \times V}{\varrho^2\,(2\,a \mp \varrho)}.$$

212. *Required the Paracentric Velocity in an orbit.*

It readily appears from the fig. that

$$d\,s : d\,\varrho :: \varrho : \sqrt{\varrho^2 - p^2}.$$

∴ If u denote the velocity towards the center, we have

$$u\left(= \frac{d\,\varrho}{d\,t}\right) = \frac{d\,s}{d\,t} \times \frac{\sqrt{\varrho^2 - p^2}}{\varrho}$$

$$= \frac{P \times V}{p} \times \frac{\sqrt{\varrho^2 - p^2}}{\varrho} \quad (125)$$

$$= P\,V \times \sqrt{\left(\frac{1}{p^2} - \frac{1}{\varrho^2}\right)} \quad . \quad . \quad . \quad . \quad (g)$$

or

$$= \frac{2\,A}{T} \times \sqrt{\left(\frac{1}{p^2} - \frac{1}{\varrho^2}\right)} \quad . \quad . \quad . \quad . \quad (h)$$

Also since

$$\frac{1}{p^2} = \frac{\varrho^2\,d\,\theta^2 + d\,\varrho^2}{\varrho^4\,d\,\theta^2}$$

$$u = P\,V \times \frac{d\,\varrho}{\varrho^2\,d\,\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (k)$$

213. *To find where in an orbit the Paracentric Velocity is a maximum.*

From the equation to the curve substitute in the expression (212. g) for p^2, then put $d\,u = 0$, and the resulting value of ϱ will give the position required.

Thus in the ellipse

$$p^2 = \frac{b^2\,\varrho}{2\,a - \varrho}$$

and

$$u^2 = P^2\,V^2 \times \left(\frac{2\,a - \varrho}{b^2\,\varrho} - \frac{1}{\varrho^2}\right) = \text{max.}$$

$$\therefore \frac{2\,a}{b^2\,\varrho} - \frac{1}{\varrho^2} - \frac{1}{b^2} = \text{max.}$$

$$\therefore -\frac{2\,a\,d\,\varrho}{b^2\,\varrho^2} + \frac{2\,d\,\varrho}{\varrho^3} = 0$$

and

$$\varrho = \frac{b^2}{a} = \frac{\text{Latus-Rectum}}{2}$$

or the point required is the extremity of the Latus-Rectum.

OTHERWISE.

Generally, It neither increases nor decreases when $F = \varphi$. Hence when $u = \max.$ (see 210)

$$\frac{d\,p}{p^3} = \frac{d\,\varrho}{\varrho^3}.$$

which is also got from putting

$$d\,(u^2) = 0$$

in the expression 212. h.

214. *To find where the angular velocity increases fastest.*

By Art. 209 and 125,

$$\frac{d\,\omega}{d\,t} = 2\,P\,V \times \frac{d\,\varrho}{\varrho^3} \times \frac{P\,V}{\varrho^2\,d\,\theta} = \frac{2\,P^2\,V^2}{\varrho^4} \times \frac{d\,\varrho}{\varrho\,d\,\theta},$$

$$\therefore \frac{d\,\varrho}{\varrho^5\,d\,\theta} = \max. \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(a)}$$

But from similar triangles

$$p : \sqrt{(\varrho^2 - p^2)} :: Q\,T : P\,T :: \varrho\,d\,\theta : d\,\varrho$$

$$\therefore \frac{d\,\omega}{d\,t} = \frac{2\,P^2\,V^2}{p\,\varrho^4} \times \sqrt{(\varrho^2 - p^2)} = \max.$$

$$\therefore \frac{\varrho^2 - p^2}{p^2\,\varrho^8} = \frac{1}{p^2\,\varrho^6} - \frac{1}{\varrho^8} = \max. \quad \ldots\ldots\ldots \text{(b)}$$

either of which equations, by aid of that to the curve, will give the point required.

Ex. In the ellipse

$$p^2 = \frac{b^2\,\varrho}{2\,a - \varrho}$$

$$\therefore \frac{2\,a - \varrho}{b^2\,\varrho^7} - \frac{1}{\varrho^8} = \max. = m$$

and $\frac{d\,m}{d\,\varrho} = 0$ gives

$$\varrho^2 - \frac{7}{3}\,a\,\varrho = -\frac{4}{3}\,b^2$$

which gives

$$\varrho = \frac{7}{6}\,a \pm \frac{1}{6}\,\surd\,(49\,a^2 - 48\,b^2)$$

for the maxima or minima positions.

If the equation

$$\varrho = \frac{b^2}{a} \times \frac{1}{1 + e\,\text{cos.}\,\theta}$$

and the first form be used, we have

$$\frac{d\,\varrho}{d\,\theta} = \frac{a\,e}{b^2} \times \varrho^2\,\text{sin.}\,\theta$$

and

$$\frac{\text{sin.}\,\theta}{\varrho^3} = \text{max.} = m.$$

Whence and from $d\,m = 0$, we get finally

$$\text{cos.}\,\theta = -\frac{1}{8\,e} \pm \surd\left(\frac{1}{64\,e^2} + \frac{3}{4}\right).$$

215. *To find where the Linear Velocity increases fastest.*

Here

$$\frac{d\,v}{d\,t} = \text{max.}$$

But (125)

$$v = \frac{P \times V}{p}$$

and

$$d\,t = \frac{\varrho^2\,d\,\theta}{P \times V} = \frac{\varrho}{P \times V} \times \frac{p\,d\,\varrho}{\surd\,\varrho^2 - p^2}$$

$$\therefore \frac{d\,v}{d\,t} = \frac{P^2 V^2\,\surd\,(\varrho^2 - p^2)}{\varrho} \times \frac{d\,p}{p^3\,d\,\varrho}$$

$$= g\,F \times \frac{\surd\,(\varrho^2 - p^2)}{\varrho}.$$

$$\left.\begin{aligned} &\therefore \frac{\surd\,(\varrho^2 - p^2)}{\varrho} \times \frac{d\,p}{p^3\,d\,\varrho} \\ \text{or}\quad &F \times \frac{\surd\,(\varrho^2 - p^2)}{\varrho} \end{aligned}\right\} = \text{max.} = m.$$

and

$$d\,m = 0$$

will give the point required.

Thus in the ellipse

$$F \propto \frac{1}{\varrho^2}$$

$$\therefore \frac{\varrho^2 - p^2}{\varrho^6} = \frac{1}{\varrho^4} - \frac{b^2}{2a\varrho^5 - \varrho^6} = \text{max.}$$

$$\therefore \frac{dm}{d\varrho} = -\frac{4}{\varrho^5} + \frac{10ab^2\varrho^4 - 6b^2\varrho^5}{(2a-\varrho)^2\varrho^{10}} = 0$$

which gives

$$\varrho^3 + 4a\varrho^2 - \frac{8a^2 + 3b^2}{2}\varrho + \frac{5}{2}ab^2 = 0,$$

whence the maxima and minima positions.

In the case of the parabola, a is indefinitely great and the equation becomes

$$4a^2\varrho - \frac{5}{2}ab^2 = 0$$

$$\therefore \varrho = \frac{5}{8} \times \frac{b^2}{a} = \frac{5}{16} \times \text{Latus-Rectum.}$$

Many other problems respecting velocities, &c. might be here added. But instead of dwelling longer upon such matters, which are rather curious than useful, and at best only calculated to exercise the student, I shall refer him to my Solutions of the Cambridge Problems, where he will find a great number of them as well as of problems of great and essential importance.

SECTION IV.

216. PROP. XVIII. If the two points P, p, be given, then circles whose centers are P, p, and radii $AB \mp SP$, $AB \mp Sp$, might be described intersecting in H.

If the positions of two tangents T R, t r be given, then perpendiculars S T, S t must be let fall and doubled, and from V and v with radii each $= AB$, circles must be described intersecting in H.

Having thus in either of the three cases determined the other focus H, the *ellipse* may be described *mechanically*, by taking a thread $= AB$ in length, fixing its ends in S and H, and running the pen all round so as to stretch the string.

This proposition may thus be demonstrated analytically.

1st. Let the focus S, the tangent T R, and the point P be given in position; and the axis-major be given in length, viz. 2 a. Then the perpendicular S T (= p), and the radius-vector S P (= ϱ) are known.

But the equation to Conic Sections is

$$p^2 = \frac{b^2 \varrho}{2a \mp \varrho}$$

whence b is found.

Also the distance (2 c) between the foci is got by making $p = \varrho$, thence finding ϱ and therefore $c = a \mp \varrho$.

This gives the other focus; and the two foci being known, and the axis-major, the curve is easily constructed.

217. 2d. Let two tangents T R, t r, and the focus S be given in position.

Then making S the origin of coordinates, the equations to the trajectory are

$$p^2 = \frac{b^2 \varrho}{2a \mp \varrho}, \text{ and } \varrho = \frac{b^2}{a} \cdot \frac{1}{1 + e \cos.(\theta - \alpha)} \quad . \quad . \quad . \quad (a)$$

α being the inclination of the axis-major to that of the abscissæ.

Now calling the angles which the tangents make with the axis of the abscissæ T and T′, by 31 we have

$$\text{tan. T} = \frac{dy}{dx}.$$

But

$$x = \varrho \cos.\theta, \; y = \varrho \sin.\theta$$

whence

$$\text{tan. T} = \frac{d\varrho \sin.\theta + \varrho\, d\theta \cos.\theta}{d\varrho \cos.\theta - \varrho\, d\theta \sin.\theta}$$

$$= \frac{\frac{\varrho\, d}{\varrho\, d\theta} \text{tan.}\theta + 1}{\frac{d\varrho}{\varrho\, d\theta} - \text{tan.}\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Also from equations (a) we easily get

$$\frac{d\varrho}{\varrho\, d\theta} = \frac{a\,e}{b^2} \cdot \varrho \sin.(\theta - \alpha) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\cos.(\theta - \alpha) = \frac{b^2 - a\varrho}{a\,e\,\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{sin.}(\theta - \alpha) = \frac{b}{a\,e\,\varrho} \times \sqrt{(2a\varrho - \varrho^2 - b^2)} \quad . \quad . \quad (3)$$

and

$$\varrho = \frac{2a p^2}{p^2 \mp b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and putting

$$R = \sqrt{(2a\varrho - \varrho^2 - b^2)} \quad . \quad . \quad . \quad . \quad (5)$$

we have

$$\frac{R}{b^2 - a\varrho} = \tan.(\theta - \alpha) = \frac{\tan.\theta - \tan.\alpha}{1 + \tan.\alpha \,.\, \tan.\theta} \quad . \quad (6)$$

which gives tan. θ in terms of a, b, ϱ, and tan. α.

Hence by successive substitutions by means of these several expressions tan. T may be found in terms of a, b, p, tan. α, all of which are given except b and tan. α. Let, therefore,

$$\tan. T = f(a, p, b, \tan.\alpha).$$

In like manner we also get

$$\tan. T' = f(a, p', b, \tan.\alpha)$$

p' belonging to the tangent whose inclination to the axis is T.

From these two equations b and tan. α may be found, which give $c = \sqrt{a^2 - b^2}$ and α, or the distance between the foci and the position of the axis-major; which being known the Trajectory is easily constructed.

218. 3d. Let the focus and two points in the curve be given in position, &c.

Then the corresponding radii ϱ, ϱ', and traced angles θ, θ', in the equations

$$\varrho = \frac{\pm a(1 - e^2)}{1 + e\cos.(\theta - \alpha)}$$

$$\varrho' = \frac{\pm a(1 - e^2)}{1 + e\cos.(\theta' - \alpha)}$$

are given; and by the formula

$$\cos.(\theta - \alpha) = \cos.\theta \,.\, \cos.\alpha + \sin.\theta \sin.\alpha$$

2 a e and α or the distance between the foci and the position of the axis-major may hence be found.

This is much less concise than Newton's geometrical method. But it may still be useful to students to know both of them.

219. PROP. XIX. To make this clearer we will state the three cases separately.

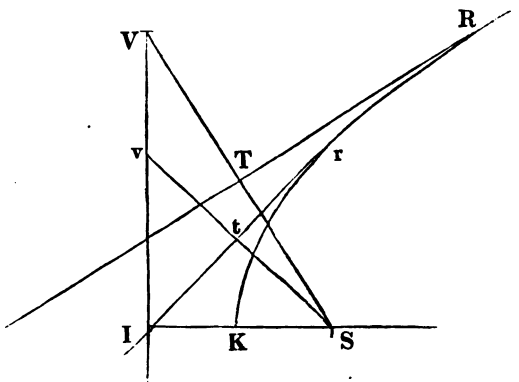
Case 1. Let a point P and tangent T R be given.

Then the figure in the text being taken, we double the perpendicular S T, describe the circle F G, and draw F I touching the circle in F and passing through V. But this last step is thus effected. Join V P, suppose it to cut the circle in M (not shown in the fig.), and take

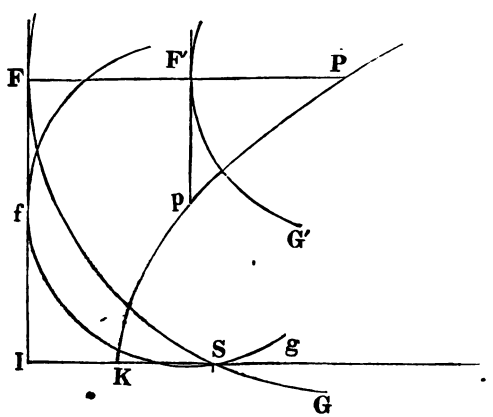
$$V F^2 = V M \times (V P + P M).$$

The rest is easy.

Case. 2. Let two tangents be given. Then V and v being determined the locus of them is the directrix. Whence the rest is plain.



Case 3. Let two points (P, p) be given. Describe from P and p the circles F G, f g intersecting in the focus S. Then draw F f a common tangent to them, &c.



But this is done by describing from P with a radius = S P — S p, a circle F' G', by drawing from p the tangent p F' as in the other case (or by describing a semicircle upon P p, so as to intersect F' G' in F') by producing P F' to F, and drawing F f parallel to F' p.

See my Solutions of the Cambridge Problems, vol. I. Geometry, where tangencies are fully treated.

These three cases may easily be deduced analytically from the general solution above; or in the same way may more simply be done at once, from the equations

$$p^2 = \frac{L}{4}\,\varrho,\ \varrho = \frac{L}{2} \times \frac{1}{1 + \cos.\,(\theta - \alpha)}.$$

220. PROP. XX. Case 1. *Given in species*] means the same as "similar" in the 5th LEMMA.

Since the Trajectory is given in species, &c.] From p. 36 it seems that the ratio of the axes 2 a, 2 b is given in similar ellipses, and thence the same is easily shown of hyperbolas. Hence, since

$$c^2 = a^2 \mp b^2$$

2 c being the distance between the foci, if $\frac{b}{a} = m$, a given quantity, we have

$$\frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a} = \sqrt{(1 \mp m^2)} = e,$$

which is also given.

With the centers B, C, &c.]

The common tangent L K is drawn as in 219.

Cases 2. 3. See Jesuits' Notes.

OTHERWISE.

221. Case 1. Let the two points B, C and the focus S be given. Then

$$\left.\begin{aligned} \varrho &= \frac{\pm a\,(1 - e^2)}{1 + e\cos.\,(\theta - \alpha)} \\ \varrho' &= \frac{\pm a\,(1 - e^2)}{1 + e\cos.\,(\theta' - \alpha)} \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

α being the inclination of the axis of abscissæ to the axis major.

But since the trajectory is given in species

$$e = \frac{c}{a} \text{ is known,}$$

and in equations (1), ϱ, θ; ϱ', θ', are given.

Hence, therefore, by the form

$$\cos.\,(\theta - \alpha) = \cos.\,\theta\,.\,\cos.\,\alpha + \sin.\,\theta\,\sin.\,\alpha,$$

a and α, or the semi-axis-major and its position are found;

also $c = a\,e$ is known;

which gives the construction.

Case 2. By proceeding as in 220, in which expressions (e) will be known, both a, a e, and α may be found.

Case 3. In this case

$$p^2 = \frac{b^2 \varrho}{2a \mp \varrho} = \frac{a^2 \times (1 - e^2)\varrho}{2a \mp \varrho}$$

will give a. Hence $c = a\,e$ is known and

$$\varrho = \frac{\pm a(1 - e^2)}{1 + e \cos.(\theta - \alpha)}$$

gives α.

Case 4. Since the trajectory to be described must be similar to a given one whose a′ and c′ are given,

$$\therefore e = \frac{c}{a} = \frac{c'}{a'}$$

is known (217).

Also ϱ and θ belonging to the given point are known.

Hence we have

$$\varrho = \frac{\pm a.(1 - e^2)}{1 + e \cos.(\theta - \alpha)}$$

And by means of the condition of touching the given line, another equation involving a, α may be found (see 217) which with the former will give a and α.

222. Scholium to Prop. XXI.

Given three points in the Trajectory and the focus to construct it.

ANOTHER SOLUTION.

Let the coordinates to the three points be ϱ, θ; ϱ', θ'; ϱ'', θ'', and α the angle between the major axis and that of the abscissæ. Then

$$\left.\begin{aligned} \varrho &= \frac{\pm a.(1 - e^2)}{1 + e \cos.(\theta - \alpha)} \\ \varrho' &= \frac{\pm a(1 - e^2)}{1 + e \cos.(\theta' - \alpha)} \\ \varrho'' &= \frac{\pm a(1 - e^2)}{1 + e \cos.(\theta'' - \alpha)} \end{aligned}\right\} \quad (A)$$

and eliminating $\pm a(1 - e^2)$ we get

$$\left.\begin{aligned} \varrho - \varrho &= e.\cos.(\theta' - \alpha) - e\cos.(\theta - \alpha) \\ \varrho - \varrho' &= e.\cos.(\theta'' - \alpha) - e\cos.(\theta - \alpha) \end{aligned}\right\} \quad (B)$$

from which eliminating e, there results

$$\frac{\varrho - \varrho'}{\varrho' \cdot \cos.(\theta' - \alpha) - \varrho \cos.(\theta - \alpha)} = \frac{\varrho - \varrho''}{\varrho'' \cos.(\theta'' - \alpha) - \varrho \cos.(\theta - \alpha)}$$

Hence by the formula

$$\cos.(P - Q) = \cos.P \cdot \cos.Q + \sin.P \cdot \sin.Q$$

$$\tan.\alpha = \frac{(\varrho - \varrho')\varrho'' \cos.\theta'' - (\varrho - \varrho'')\varrho' \cos.\theta' + \varrho(\varrho' - \varrho'')\cos.\theta}{(\varrho - \varrho')\varrho'' \sin.\theta'' - (\varrho - \varrho'')\varrho' \sin.\theta' + \varrho(\varrho' - \varrho'')\sin.\theta}$$

which gives α.

Hence by means of equations (B) e will be known; and then by substitution in eq. (A), a is known.

SECTION V.

The preliminary LEMMAS of this section are rendered sufficiently intelligible by the Commentary of the Jesuits P.P. Le Seur, &c.

Moreover we shall be brief in our comments upon it (as we have been upon the former section) for the reason that at Cambridge, the focus of mathematical learning, the students scarcely even touch upon these subjects, but pass at once from the third to the sixth section.

223. PROP. XXII.

This proposition may be analytically resolved as follows:

The general equation to a conic section is that of two dimensions (see Wood's Alg. Part IV.) viz.

$$y^2 + Axy + Bx^2 + Cy + Dx + E = 0$$

in which if A, B, C, D, E were given the curve could be constructed.

Now since five points are given by the question, let their coordinates be

$$\alpha, \beta;\ \underset{1}{\alpha}, \underset{1}{\beta};\ \underset{2}{\alpha}, \underset{2}{\beta};\ \underset{3}{\alpha}, \underset{3}{\beta};\ \underset{4}{\alpha}, \underset{4}{\beta}.$$

These being substituted for x, y, in the above equation will give us five simple equations, involving the five unknown quantities A, B, C, D, E, which may therefore be easily determined; and then the trajectory is easily constructed by the ordinary rules (see Wood's Alg. Lacroix's Diff. Cal. &c.)

224. PROP. XXIII. The analytical determination of the trajectory from these conditions is also easy.

Let

$$\alpha, \beta;\ \underset{1}{\alpha}, \underset{1}{\beta};\ \underset{2}{\alpha}, \underset{2}{\beta};\ \underset{3}{\alpha}, \underset{3}{\beta}$$

be the coordinates of the given point. Also let the tangent given in position be determinable from the equation

$$y' = mx' + n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

in which m, n are given.

Then first substituting the above given values of the coordinates in

$$y^2 + Axy + Bx^2 + Cy + Dx + E = 0 \quad . \quad . \quad . \quad (b)$$

we get *four* simple equations involving the five unknown quantities A, B, C, D, E; and secondly since the inclination of the curve to the axis of abscissæ is the same at the point of contact as that of the tangent,

$$\frac{dy}{dx} = \frac{dy'}{dx'}$$

$$y = y'$$

$$x = x'$$

$$\therefore \frac{Ay + 2Bx + D}{2y + Ax + C} = -m$$

and substituting in this and the general equation for y its value

$$y' = mx + n$$

we have

$$\frac{A(mx + n) + 2Bx + D}{2(mx + n) + Ax + C} = -m$$

and

$$(mx + n)^2 + Ax(mx + n) + Bx^2 + C(mx + n) + Dx + E = 0,$$

from the former of which

$$x = -\frac{nA + mC + D}{2(m^2 + mA + B)}$$

and from the latter

$$x = -\frac{1}{2(m^2 + mA + B)} \times (nA + mC + D + 2mn$$

$$\mp \sqrt{\{(nA + mC + D + 2mn)^2 - (n^2 + nC + E)(m^2 + mA + B)\}}$$

and equating these and reducing the result we get

$$4m^2n^2 = (nA + mC + D + 2mn)^2 - (n^2 + nC + E)(m^2 + mA + B)$$

and this again reduces to

$$n^2A^2 + m^2C^2 + D^2 + mnAC + 2nAD$$
$$+ 2mCD - nBC - mAE - BE + 3mn^2A$$
$$+ 3nm^2C + 4mnD - n^2B - m^2E - n^2m^2 = 0$$

which is a fifth equation involving A, B, C, D, E.

From these five equations let the five unknown quantities be determined, and then construct eq. (b) by the customary methods.

225. PROP. XXIV.

OTHERWISE.

Let

$$\alpha, \beta;\ \alpha', \beta';\ \alpha'', \beta''$$

be the coordinates of the three given points, and

$$y' = m x' + n$$
$$y'' = m' x'' + n'$$

the equations to the two tangents. Then substituting in the general equation for Conic Sections these pairs of values of x, y, we get three *simple* equations involving the unknown coefficients A, B, C, D, E; and from the conditions of contact, viz.

$$\left.\begin{array}{c} \frac{dy}{dx} = \frac{dy'}{dx'} = m \\ y = y' \\ x = x' \end{array}\right\} \qquad \left.\begin{array}{c} \frac{dy}{dx} = \frac{dy''}{dx''} = m' \\ y = y'' \\ x = x'' \end{array}\right\}$$

We also have two other equations (see 224) involving the same five unknowns, whence by the usual methods they may be found, and then the trajectory constructed.

226. PROP. XXV.

Proceeding as in the last two articles, we shall get two simple equations and three quadratics involving A, B, C, D, E, from whence to find them and construct the trajectory.

227. PROP. XXVI.

In this case we shall have one simple equation and four quadratics to find A, B, C, D, E, with, and wherewith to describe the orbit.

228. PROP. XXVII.

In the last case of the five tangents we shall have five quadratics. wherewith to determine the coefficients of the general equation, and to construct.

SECTION VI.

229. Prop. XXX.

OTHERWISE.

After a body has moved t″ *from the vertex of the parabola, let it be required to find its position.*

If A be the area described in that time by the radius vector, and P, V the perpendicular or the tangent and velocity at any point, by 124 and 125 we have

$$A = \frac{c}{2} \times t = \frac{PV}{2} \times t$$

and by 157,

$$PV = \sqrt{\frac{g\mu L}{2}}$$

L being the latus-rectum.

$$\therefore A = \frac{\sqrt{g\mu L}}{2\sqrt{2}} \times t.$$

But

$$ASP = AOP - SOP = \tfrac{2}{3} AO \times PO - \tfrac{1}{2} SO \times PO$$
$$= \tfrac{2}{3} x y - \tfrac{1}{2} \cdot (x - r) y$$

where $r = AS$, &c. (see 21) and

$$y^2 = 4rx$$

$$\therefore A = \sqrt{\frac{g\mu r}{2}} \times t = \frac{1}{6} x y + \frac{r}{2} \cdot y = \frac{y^3}{24r} + \frac{r}{2} y$$

$$\therefore y^3 + 12r^2 y = 12rt\sqrt{g\mu r}$$

by the resolution of which y may be found and therefore the position of P.

OTHERWISE.

230. By 46 and 125,

$$dt = \frac{ds}{v} = \frac{p\,ds}{c}.$$

Also

$$ds = \frac{\varrho\, d\varrho}{c\sqrt{(\varrho^2 - p^2)}}$$

$$\therefore dt = \frac{p\varrho\, d\varrho}{c\sqrt{(\varrho^2 - p^2)}} \quad \ldots\ldots\ldots \quad (a)$$

which is an expression of general use in determining the time in terms of the radius vector, &c.

In the parabola

$$p^2 = r\varrho,$$

whence

$$dt = \frac{\sqrt{r}}{c} \times \frac{\varrho\, d\varrho}{\sqrt{(\varrho - r)}}$$

and integrating *by parts*

$$t = \frac{2\sqrt{r}}{c}\varrho\sqrt{(\varrho - r)} - \frac{2\sqrt{r}}{c}\int d\varrho\sqrt{(\varrho - r)}$$

$$= \frac{2\sqrt{r}}{c}\left\{\varrho\sqrt{(\varrho - r)} - \frac{2}{3}(\varrho - r)^{\frac{3}{2}}\right\}$$

$$= \frac{2\sqrt{r}}{3c}\sqrt{(\varrho - r)} \times (\varrho + 2r)$$

But

$$c = PV = \sqrt{2g\mu r} \quad (229)$$

$$\therefore t = \frac{\sqrt{2}}{3\sqrt{g\mu}} \times (\varrho + 2r)(\varrho - r)^{\frac{1}{2}} \quad \ldots \quad (b)$$

which gives

$$\varrho^3 + 3r\varrho^2 = 4r^3 + \frac{9}{2}g\mu t,$$

whence we have ϱ and the point required.

By the last Article the value of M in Newton's Assumption is easily obtained, and is

$$M = \frac{A}{4r} = \frac{t}{4} \times \sqrt{\frac{g\mu}{2r}}.$$

231. COR. 1. This readily appears upon drawing S Q the semi-latus-rectum and by drawing through its point of bisection a perpendicular to G H.

232. COR. 2. This proportion can easily be obtained as in the note of the Jesuits, by taking the ratio of the increments of G H and of the curve at the vertex; or the absolute value of the velocity of H is directly got thus.

$$v = \frac{d.GH}{dt} = \frac{3\,dM}{dt} = \frac{3}{4}\sqrt{\frac{g\mu}{2r}}.$$

Also the velocity in the curve is given by (see 140)

$$v'^2 = 2gF \times \frac{PV}{4} = \frac{2g\mu}{\varrho}$$

and at the vertex $\varrho = r$,

$$\therefore v' = \sqrt{\frac{2g\mu}{r}}$$

$$\therefore v : v' :: 3 : 8.$$

233. Cor. 3. Either A P, or S P being bisected, &c. will determine the point H and therefore

$$t = \frac{4}{3}\sqrt{\frac{2r}{g\mu}} \times GH.$$

234. Lemma XXVIII. That an oval cannot be squared is differently demonstrated by several authors. See Vince's Fluxions, p. 356; also Waring.

235. Prop. XXXI. This is rendered somewhat easier by the following arrangement of the proportions:

If G is taken so that

$$OG : OA :: OA : OS$$

or

$$OG = \frac{OA^2}{OS}$$

and

$$GK : 2\pi OG :: t : T$$

or

$$GK = \frac{2\pi \times OA^2}{OS} \times \frac{t}{T} \quad \ldots\ldots\ldots \text{(a)}$$

Then, &c. &c. For

$$ASP = ASQ \times \frac{b}{a}$$

$$= \frac{b}{a} \times (OQA - OQS)$$

$$= \frac{b}{2a}(OQ \times AQ - OQ \times SR)$$

$$= \frac{b}{2}(AQ - SR).$$

But

$$SR : \text{sin. } AQ :: SO : OA$$
$$:: OA : OG :: AQ : FG$$

$$\therefore SR = \frac{AQ \text{ sin. } AQ}{FG}$$

and

$$AQ - SR = \frac{AQ}{FG} \times (FG - \text{sin. } AQ)$$

$$= \frac{OS}{OA} \times (FG - \text{sin. } AQ)$$

$$\therefore ASP = \frac{OS \times b}{2a} \times (FG - \text{sin. } AQ)$$

$$= \frac{OS \times b}{2a} \times GK \quad \ldots\ldots\ldots \quad (b)$$

(see the Jesuits' note q.) which is identical with (a), since

$$\frac{t}{T} = \frac{ASP}{\text{Ellipse}}$$

$$= \frac{ASP}{\pi a b}.$$

OTHERWISE.

236. By 230 we have

$$dt = \frac{p \varrho d\varrho}{c \sqrt{(\varrho^2 - p^2)}} = \frac{\varrho d\varrho}{c \sqrt{\left(\frac{\varrho^2}{p^2} - 1\right)}}$$

But in the ellipse

$$p^2 = \frac{b^2 \varrho}{2a - \varrho}$$

$$\therefore dt = \frac{b \varrho d\varrho}{c \sqrt{(2a\varrho - b^2 - \varrho^2)}}$$

and putting

$$\varrho - a = u$$

it becomes

$$dt = \frac{b \cdot (a + u)\, du}{c \sqrt{(a^2 e^2 - u^2)}}$$

2 a e being the excentricity.

Hence

$$t = \frac{b\,a}{c} \int \frac{du}{\sqrt{(a^2 e^2 - u^2)}} + \frac{b}{c} \int \frac{u\,du}{\sqrt{(a^2 e^2 - u^2)}}$$

$$= \frac{b\,a}{c} \text{sin.}^{-1} \frac{u}{a\,e} - \frac{b}{c} \sqrt{(a^2 e^2 - u^2)} + C.$$

Let $t = 0$, when $u = a\,e$; then

$$C = \frac{b\,a}{c} \times \frac{\pi}{2}$$

and we get

$$t = \frac{b\,a}{c} \times \left(\frac{\pi}{2} + \text{sin.}^{-1} \frac{u}{a\,e}\right) - \frac{b}{c} \cdot \sqrt{(a^2 e^2 - u^2)}$$

which is the known form of the equation to the Trochoid, t being the abscissa, &c.

Hence by approximation or by construction u and therefore ϱ may be found, which will give the place of the body in the trajectory.

It need hardly be observed that (157)

$$c = P \times V = \sqrt{\frac{g\mu}{2}}\, L = b \sqrt{\frac{g\mu}{2}}.$$

OTHERWISE.

237. $$dt = \frac{\varrho^2 d\theta}{c};$$

but in the ellipse

$$\varrho = \frac{b^2}{a} \times \frac{1}{1 + e \cos.\theta}$$

$$\therefore dt = \frac{b^4}{a^2 c} \times \frac{d\theta}{(1 + e\cos.\theta)^2}$$

and (see Hirsch's Tables, or art. 110)

$$t = \frac{a^2(1-e^2)}{c} \times \left\{ \frac{1}{\sqrt{(1-e^2)}} \cos.^{-1} \frac{e + \cos.\theta}{1 + e\cos.\theta} - \frac{e \sin.\theta}{1 + e\cos.\theta} \right\}$$

which also indicates the Trochoid.

To simplify this expression let

$$\cos.^{-1} \frac{e + \cos.\theta}{1 + e\cos.\theta} = u$$

then

$$\frac{e + \cos.\theta}{1 + e\cos.\theta} = \cos.u$$

and

$$\cos.\theta = \frac{e - \cos.u}{e\cos.u - 1}$$

Hence

$$\sin.\theta = \frac{\sqrt{(1-e^2)}}{1 - e\cos.u}$$

and

$$\frac{e\sin.\theta}{1 + e\cos.\theta} = \frac{e\sin.u}{\sqrt{(1-e^2)}}$$

$$\therefore t = \frac{a^2\sqrt{(1-e^2)}}{c} \times \{u - e\sin.u\}$$

But (157)

$$c = PV = b \sqrt{\frac{g\mu}{a}} = \sqrt{(1-e^2)}\,\sqrt{g\mu a}$$

$$\therefore t = \frac{a^{\frac{3}{2}}}{\sqrt{g\mu}} \times (u - e \sin. u)$$

Let $\frac{a^{\frac{3}{2}}}{\sqrt{g\mu}} = \frac{1}{n}$.

Then

$$n t = u - e \sin. u \quad \ldots\ldots\ldots\ldots (1)$$

Again, θ may be better expressed in terms of u, thus

$$\tan.^2 \frac{\theta}{2} = \frac{1 - \cos. \theta}{1 + \cos. \theta} = \frac{1 + e}{1 - e} \times \frac{1 - \cos. u}{1 + \cos. u} = \frac{1 + e}{1 - e} \tan.^2 \frac{u}{2}$$

$$\therefore \tan. \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \times \tan. \frac{u}{2} \quad \ldots\ldots\ldots\ldots (2)$$

Moreover ϱ is expressible in terms of u, for

$$\varrho = \frac{a(1 - e^2)}{1 + e \cos. \theta} = a(1 - e \cos. u) \quad \ldots\ldots (3)$$

In these three equations, n t is called the *Mean Anomaly*; u the *Excentric Anomaly*, (because it = the angle at the center of the ellipse subtended by the ordinate of the circle described upon the axis-major corresponding to that of the ellipse); and θ the *True Anomaly*.

238. SCHOLIUM.

Newton says that " the approximation is founded on the Theorem that *The area* A P S α A Q — S F, S F *being the perpendicular let fall from* S *upon* O Q."]

First we have

$$A P R = A Q R \times \frac{b}{a}$$

$$S P R = S Q R \times \frac{b}{a}$$

$$\therefore A S P = A S Q \times \frac{b}{a}$$

But

$$\begin{aligned} A S Q &= A O Q - S O Q \\ &= \tfrac{1}{2} A Q \times A O - \tfrac{1}{2} S F \times O Q \\ &= \tfrac{1}{2} A O \times (A Q - S F). \end{aligned}$$

$$\therefore A S P = \frac{b}{2} \times (A Q - S F)$$

$$= \frac{b}{2} \times (a u' - a e \sin. u') \quad \ldots\ldots (1)$$

u' being the $\angle$ A O Q.

(Hence is suggested this easy determination of eq. 1. 237.

For

$$t = T \times \frac{A\,Sp}{\text{Ellipse}} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu g}} \times \frac{\frac{b}{2}(a\,u - a\,e \sin. u)}{\pi a b}$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{g\mu}} \times (u - e \sin. u).^{*})$$

Again, supposing u' an approximate value of u, let

$$u = u' + \frac{E}{a}$$

Then, by the Theorem, we have

$$\frac{2\,A\,S\,p}{b} = A\,q - S\,O \times \sin. A\,q$$

$$= A\,Q + Q\,q + - S\,O \times \sin. (A\,Q \pm Q\,q)$$

to radius 1.

But A Q being an approximate value of A q, Q q is small compared with A O, and we have

$$\sin. (A\,Q \pm Q\,q) = \sin. A\,Q \cos. Q\,q \pm \cos. A\,Q \sin. Q\,q$$

$$= \sin. A\,Q \pm Q\,q \cos. A\,Q \text{ nearly.}$$

$$\therefore Q\,q = \left(\frac{2\,A\,Sp}{b} - A\,Q + S\,O \sin. A\,Q\right) \times \frac{\frac{1}{SO}}{\frac{1}{SO} \mp \cos. A\,Q} \text{ nearly}$$

which points out the use of these assumptions

$$N' = \frac{2\,A\,S\,p}{b} = \frac{2\,t}{b\,T} \times \text{area of the Ellipse}$$

$$B' = S\,O \quad = \frac{2\pi a\,t}{T}$$

and

$$D' = S\,O.\ \sin. A\,Q = B' \sin. A\,Q$$

$$L' = \frac{1}{SO}$$

Then

$$Q\,q = (N' - A\,Q + D') \times \frac{L'}{L' \mp \cos. A\,Q}$$

in which it is easily seen B′, N′, D′, L′ are identical with B, N, D, L.

Hence

$$E = Q\,q = (N - A\,Q + D) \frac{L}{L \mp \cos. A\,Q}.$$

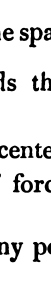
Having augmented or diminished the assumed arc A Q by E, then repeat the process, and thus find successively

G, I, &c.

For a developement of the other mode of approximation in this Scholium, see the Jesuits' note 386. Also see Woodhouse's Plane Astronomy for other methods.

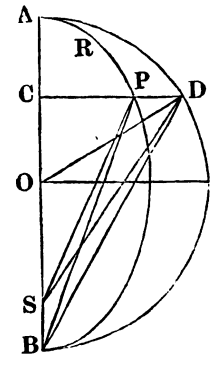
SECTION VII.

239. PROP. XXXII. $F \propto \frac{1}{\text{distance}^2}$. Determine the spaces which a body descending from A in a *straight line* towards the center of force describes in a given time.



If the body did not fall in a straight line to the center, it would describe some conic section round the center of force, as focus (which would be $\left\{\begin{matrix}\text{ellipse}\\ \text{parabola}\\ \text{hyperbola}\end{matrix}\right\}$ if the velocity at any point were to the velocity in the circle, the same distance and force, in R°. $\left\{\begin{matrix}<\\ =\\ >\end{matrix}\right\}$ $\sqrt{2}$: 1.)

(I) Let the Conic Section be an Ellipse A R P B.



Describe a circle on Major Axis A B, draw C P D through the place of the body perpendicular to A B.

The time of describing A P $\propto$ area A S P $\propto$ area A S D, whatever may be the excentricity of the ellipse.

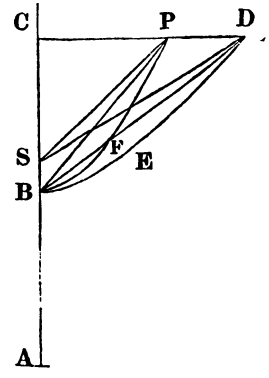
Let the Axis Minor of the ellipse be diminished sine limite and the ellipse becomes a straight line ultimately, A B being constant, and since A S . S B = (Minor Axis)^2 = 0, and A S finite $\therefore$ S B = 0, or B ultimately comes to S, and time d . A C $\propto$ area A D B. $\therefore$ if A D B be taken proportional to time, C is found by the ordinate D C.

(T . A C $\propto$ area A D B $\propto$ A D O + O D B $\propto$ arc A D + C D $\therefore$ take θ + sin. θ proportional to time, and D and C are determined.)

(Hence

$$\frac{\text{the time down A O}}{\text{T. O B}} = \frac{\frac{\pi}{2}+1}{\pi-\frac{\pi}{2}+1} = \frac{\frac{\pi}{2}+1}{\frac{\pi}{2}-1} = \frac{\frac{11}{7}+1}{\frac{11}{7}-1} = \frac{18}{4} = \frac{9}{2} \text{ nearly)}$$

N. B. The time in this case is the time from the beginning of the fall, or the time from A.



(II) Let the conic section be the hyperbola B F P. Describe a rectangular hyperbola on Major Axis A B.

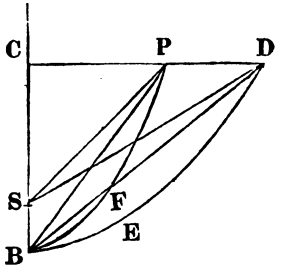
T $\propto$ area S B F P $\propto$ area S B E D.

Let the Minor Axis be diminished sine limite, and the hyperbola becomes a straight line, and T $\propto$ area B D E.

N. The time in this case is the time from the end of motion or time to S.

Let the conic section be the parabola B F P. Describe any fixed parabola B E D.

T $\propto$ area S B F P $\propto$ area S B E D.



Let L. R. of B F P be diminished sine limite the parabola becomes a straight line, and T $\propto$ area B D E.

N. The time in this case is the time from the end of motion, or time to S.

Objection to Newton's method. If a straight line be considered as an evanescent conic section, when the body comes to perihelion i. e. to the center it ought to return to aphelion i. e. to the original point, whereas it will go through the center to the distance below the center = the original point.

240. We shall find by Prop. XXXIX, that the distance from a center from which the body must fall, acted on by α^{ble} force, to acquire the velocity such as to make it describe an ellipse = A B (finite distance), for the hyperbola = — A B, for the parabola = α.

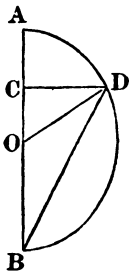
241. Case 1. $v\,dv = -g\mu\frac{dx}{x^2}$, f = force distance 1,

$$\therefore v^2 = g\mu.\left(\frac{a-x}{ax}\right)$$

if a be the original point

$$dt = -\frac{dx}{v} = -\frac{\sqrt{a}}{\sqrt{2g\mu}}dx.\frac{x}{\sqrt{ax-x^2}}$$

$$= \sqrt{\frac{a}{2g\mu}} \cdot \left\{ \frac{\left(\frac{a}{2} - x\right) dx}{\sqrt{ax - x^2}} \quad \frac{-\frac{a}{2} dx}{\sqrt{ax - x^2}} \right\}$$



$$\therefore t = \sqrt{\frac{a}{2g\mu}} \cdot \left\{ \sqrt{ax - x^2} - \text{vers.}^{-1} \frac{x}{\text{rad.} = \frac{a}{2}} \right\}$$

+ C, when $t = 0$, $x = a$,

$$\therefore t = \sqrt{\frac{a}{2g\mu}} \cdot \left\{ \sqrt{ax - x^2} + \left(\frac{\text{circumference}}{2} - \text{vers.}^{-1} \frac{x}{\text{rad.} = \frac{a}{2}} \right) \right\}$$

$$= \sqrt{\frac{a}{2g\mu}} (CD + AD)$$

if the circle be described on BA = a,

$$= \sqrt{\frac{a}{2g\mu}} \cdot \frac{4}{a} \cdot \left(\frac{CD.OB}{2} + \frac{AD.OD}{2} \right) = \frac{2}{\sqrt{2g\mu}} \cdot BAD.$$

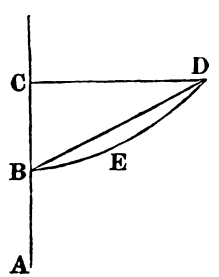
Case 2. $v^2 = 2g\mu \cdot \frac{a + x}{ax}$, if — a be an original point,

$$\therefore t = \sqrt{\frac{a}{2g\mu}} \cdot \int \frac{x\,dx}{\sqrt{ax + x^2}},$$

for t in this case is the time to the center, not the time from the original point,

$$\therefore -dt = -\frac{dx}{v}, \text{ or } dt = \frac{dx}{v}.$$

Now if with the Major Axis AB = a, we describe the rectangular hyperbola,

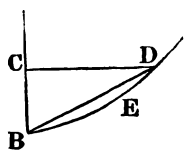


we have

$$d.BED = d.BEDC - d.ABDC = y\,dx - \frac{d.xy}{2} = \frac{y\,dx - x\,dy}{2}$$

$$= \frac{\sqrt{ax+x^2}}{2}.dx - x.\left(\frac{a}{2}+x\right)dx = \frac{a\,x\,dx}{4\sqrt{ax+x^2}} = dt.\frac{\sqrt{a\,g\,\mu}}{2\sqrt{2}}$$

$$\therefore t \text{ from } B = \sqrt{\frac{2}{2\,g\,\mu}}.\ BED, \text{ for they begin or end together at B.}$$



Case 3. $v^2 = 2\,g\,\mu\,\frac{1}{x}$, if a be α,

$$\therefore dt = \frac{dx}{v} = \frac{\sqrt{x}\,dx}{\sqrt{2\,g\,\mu}}, \text{ t being time to B,}$$

$$\therefore t = \frac{1}{\sqrt{2\,g\,\mu}}\frac{2}{3}.x^{\frac{3}{2}} + C, \text{ when } t = 0,\ x = 0,\ \therefore C = 0.$$

Describe a parabola on the line of fall, vertex B, L. R. = any fixed distance a,

$$\therefore t = \frac{1}{\sqrt{2\,g\,\mu}}.\frac{1}{3}.\sqrt{x}.x = \frac{2\sqrt{2}}{\sqrt{a\,g\,\mu}}.\frac{1}{6}.\sqrt{ax}.x = \frac{2\sqrt{2}}{\sqrt{a\,g\,\mu}}.BED.$$

Hence in general, in Newton Prop. XXXII, $t = \frac{2\sqrt{2}}{\sqrt{a\,g\,\mu}}$. curvilinear area, a being L. R. of the figure described.

In the evanescent conic sections, L. R. $= \frac{2\,(\text{Ax. Min.})^2}{\text{Ax. Maj.}}$, $\therefore$ if Ax. Min. be indefinitely small, L. R. will be indefinitely small with respect to the Ax. Min. The chord of curvature at the finite distance from A to B is ultimately finite, for $PV = \frac{2\,SP.AP}{\frac{AB}{2}}$; but at A or B, P V = L, = infinitesimal of the second order. Hence S B is also ultimately of the second order, for at B, $SB = L.\frac{AB}{2AS}$.

Prop. XXXIII. Force $\propto \frac{1}{(\text{distance})^2}$.

$\frac{\text{V at C}}{\text{v in the circle distance S C}} = \frac{\sqrt{AC}}{\sqrt{\frac{SA}{2}}}$ in the ellipse and hyperbola.

$\left(\frac{V}{v} = \frac{\sqrt{HP}}{\sqrt{\frac{\text{Maj. Ax.}}{2}}} = \frac{\sqrt{AC}}{\sqrt{\frac{SA}{2}}}\right.$ when the conic section becomes a straight line$\left.\right)$

NEWTON'S METHOD.

$$\frac{V^2}{v^2} = \frac{\frac{L}{SY^2}}{\frac{2SP}{SP^2}} = \frac{L}{2} \cdot \frac{SP}{SY^2}$$

$$\frac{AC.CB}{CP^2} = \frac{AO^2}{\left(\frac{\text{Min. Ax.}}{2}\right)^2} = \frac{2AO}{2\frac{\left(\frac{\text{Min. Ax.}}{2}\right)^2}{AO}} = \frac{2AO}{L}$$

$$\therefore \frac{L}{2} = \frac{AO.CP^2}{AC.CB}$$

$$\therefore \frac{V^2}{v^2} = \frac{AO.CP^2.SP}{AC.CB.SY^2},$$

but

$$\frac{CO}{BO} = \frac{BO}{TO},$$

$$\therefore \frac{CO}{BO} = \frac{CB}{BT}, \begin{matrix}\text{comp. in the ellipse} \\ \text{div. in the hyperbola,}\end{matrix}$$

$$\therefore \frac{AC}{BO} = \frac{CT}{BT}, \begin{matrix}\text{div. in the ellipse} \\ \text{comp. in the hyperbola}\end{matrix} = \frac{CP}{BQ},$$

$$\therefore \frac{AC^2}{AO^2} = \frac{CP^2}{BQ^2},$$

$$\therefore \frac{BQ^2.AC}{AO} = \frac{AO.CP^2}{AC},$$

$$\therefore \frac{V^2}{v^2} = \frac{BQ^2.AC.SP}{AO.BC.SY^2},$$

but ultimately

$$BQ = SY, SP = BC,$$

$$\therefore \text{ultimately } \frac{V^2 \text{ in a straight line}}{v^2 \text{ in the circle}} = \frac{AC}{AO},$$

$$\therefore \frac{V}{v} = \sqrt{\frac{AC}{AO}}.$$

COR. 1. It appeared in the proof that $\frac{AC}{AO} = \frac{CT}{BT}$,

$$\therefore \text{ ultimately } \frac{A\,C}{A\,O} = \frac{C\,T}{S\,T}.$$

(This will be used to prove next Prop.)

Cor. 2. Let C come to O, then A C = A O and V = v,

∴ the velocity in the circle = the velocity acquired by falling externally through distance = rad. towards the center of the force $\propto \frac{1}{\text{distance}^2}$.

242. *Find actual Velocity at* C.

$$\frac{V^2 \text{ at } C}{v^2 \text{ in the circle distance } B\,C} = \frac{A\,C}{\frac{B\,A}{2}},$$

$$\therefore V^2 = \frac{2\,A\,C}{B\,A}.v^2 = \frac{2\,A\,C}{B\,A}.\frac{g\,\mu}{B\,C^2}.B\,C,$$

if μ = the force at distance 1,

$$\therefore V^2 = 2\,g\,\mu\,\frac{A\,C}{B\,A\,.\,B\,C},$$

$$\therefore V = \sqrt{2\,g\,\mu}.\frac{\sqrt{a-x}}{\sqrt{a\,x}}, \text{ if } B\,A = a,\ B\,C = x.$$

If a is given, $V \propto \frac{\sqrt{\text{space described}}}{\sqrt{\text{space to be described}}}$.

In descents from different points,

$$V \propto \frac{\sqrt{\text{space described}}}{\sqrt{\text{space to be described} \times \text{initial height}}}.$$

In descents from different points to different centers,

$$V \propto \frac{\sqrt{\text{space described} \times \text{absolute force}}}{\sqrt{\text{space to be described} \times \text{initial height}}}$$

243. Otherwise. $v\,dv = -\frac{g\,\mu}{x^2}\,dx$,

$$\therefore v^2 = 2\,g\,\mu.\frac{a-x}{a\,x}, \text{ when a is positive, as in the ellipse}$$

$$= 2\,g\,\mu.\frac{a+x}{a\,x}, \text{ when a is negative as in the hyperbola}$$

$$= 2\,g\,\mu.\frac{1}{x}, \text{ when a is } \alpha, \text{ as in parabola}$$

(when x = 0, v is infinite)

V^2 in the circle radius x (in the ellipse and hyperbola)

$$= \frac{g\,\mu}{x^2}.x = \frac{g\,\mu}{x}$$

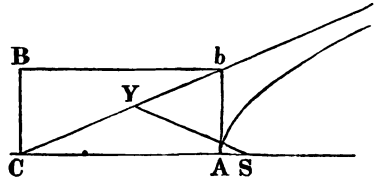
$$\therefore \frac{v^2}{V^2} = \frac{2\,\overline{a-x}}{a} \text{ in the ellipse, } = \frac{a-x}{\left(\frac{a}{2}\right)}.$$

$\frac{v^2}{V^2} = \frac{2\overline{a+x}}{a}$ in the hyperbola, $= \frac{a+x}{\left(\frac{a}{2}\right)}$

V^2 in the circle radius $= \frac{x}{2}$ (in the parabola) $= \frac{g\mu}{\frac{x^2}{4}} \cdot \frac{x}{2} = \frac{2g\mu}{x}$

$\therefore \frac{v^2}{V^2} = \frac{1}{1}$ in the parabola.

244. In the hyperbola not evanescent $\frac{\text{Velocity at the infinite distance}}{\text{velocity at A}} = \frac{S\,A}{S\,Y}$ finite R°., but when the hyperbola vanishes, S Y ultimately = Min. Ax. for $\frac{S\,Y}{A\,b} = \frac{S\,C}{b\,C}$, and ultimately S C =



A C, and b C = A C, ∴ ultimately S Y = A b = C B, ∴ ultimately $\frac{S\,Y}{S\,A} = \frac{\text{infinitesimal of the first order}}{\text{of the 2d order}}$

$= \frac{\text{velocity at A}}{\text{velocity at } \propto \text{ distance}}$

245. PROP. XXXIV. $\frac{\text{Velocity at C}}{\text{velocity in the circle, distance } \frac{S\,C}{2}} = \frac{1}{1}$, for the parabola.

For the velocity in the parabola at P = velocity in the circle $\frac{S\,P}{2}$ whatever be L. R. of the parabola.

246. PROP. XXXV. Force $\propto \frac{1}{(\text{distance})^2}$.

The same things being assumed, the area swept out by the indefinite radius S D in fig. D E S = area of a circular sector (rad. = $\frac{L.R}{2}$ of fig.) uniformly described about the center S in the same time. Whilst the falling body describes C c indefinitely small, let K k be the arc described by the body uniformly revolving in the circle.

Case 1. If D E S be an ellipse or rectangular hyperbola, $\frac{L.R}{2} = \frac{S\,A}{2}$,

$$\frac{C\,c}{D\,d} = \frac{C\,T}{D\,T}$$

$$\frac{C\,D}{S\,Y} = \frac{D\,T}{T\,S}$$

$$\therefore \frac{C c \cdot C D}{D d \cdot S Y} = \frac{C T}{T S} = \frac{A C}{A O} \text{ ultimately.}$$

(Cor. Prop. XXXIII.)

But

$$\frac{\text{velocity at C}}{\text{v in the circle rad. S C}} = \frac{\sqrt{A C}}{\sqrt{A O}}$$

$$\frac{\text{v in the circle rad. S C}}{\text{v in the circle rad. S K}} = \sqrt{\frac{S K}{S C}} = \sqrt{\frac{A O}{S C}}$$

$$\therefore \left(\frac{\text{velocity at C}}{\text{v in the circle rad. S K}}\right) = \frac{C c}{K k} = \sqrt{\frac{A C}{S C}} = \frac{A C}{C D}$$

$$\therefore C c \cdot C D = K k \cdot A C$$

$$\therefore \frac{K k \cdot A C}{D d \cdot S Y} = \frac{A C}{A O},$$

$$\therefore A O \cdot K k = D d \cdot S Y,$$

$\therefore$ the area S K k = the area S D d,

$\therefore$ the nascent areas traced out by S D and S K are equal

$\therefore$ the sums of these areas are equal.

Case 2. If D E S be a parabola $S K = \frac{L \cdot R}{2}$.

As above

$$\frac{C c \cdot C D}{D d \cdot S Y} = \frac{C T}{T S} = \frac{2}{1}$$

also

$$\frac{C c}{K k} = \frac{\text{velocity at C}}{\text{velocity in the circle } \frac{L \cdot R}{2}} = \frac{\text{velocity in the circle } \frac{S C}{2}}{\text{velocity in the circle } \frac{L \cdot R}{2}}$$

$$= \frac{\sqrt{S K}}{\sqrt{\frac{S C}{2}}} = \frac{S K}{\frac{C D}{2}}$$

$$\therefore C c \cdot C D = 2 \cdot K k \cdot S K$$

$$\therefore K k \cdot S K = D d \cdot S Y.$$

247. Prop. XXXVI. Force $\propto \frac{1}{(\text{distance})^2}$.

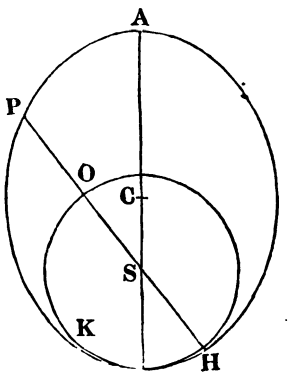
To determine the times of descent of a body falling from the given (and $\therefore$ finite) altitude A S

On A S describe a circle and an equal circle round the center S.

From any point of descent C erect the ordinate C D, join S D. Make the sector O S K = the area A D S (O K = A D + D C) the body will fall from A to C in the time of describing O K about the center S

uniformly, the force $\propto \frac{1}{\text{distance}^2}$. Also S K being given, the period in the circle may be found, ($P = \sqrt{\frac{4\mu}{g}} \cdot \pi \cdot S K^{\frac{3}{2}}$), and the time through $O K = P \cdot \frac{O K}{\text{circumference}}$. ∴ the time through O K is known. ∴ the time through A C is known.

248. *Find the time in which a Planet would fall from any point in its orbit to the Sun.*



Time of fall = time of describing $\frac{\text{circle}}{2}$ O K H, $S O = \frac{S P}{2}$,

$$\frac{\text{period in the circle O K H}}{\text{period in the ellipse}} = \frac{\text{period in the circle rad. S O}}{\text{period in the circle rad. A C}} = \frac{S O^{\frac{3}{2}}}{A C^{\frac{3}{2}}}$$

∴ the time of fall $= \frac{1}{2} \cdot P \cdot \left(\frac{S O}{A C}\right)^{\frac{3}{2}}$, P = period of the planet. If the orbit be considered a circle

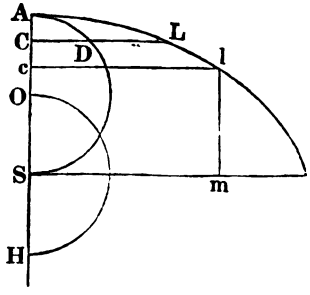
$$\left(\frac{S O}{A C}\right)^{\frac{3}{2}} = \left(\frac{1}{2}\right)^{\frac{3}{2}} = \frac{1}{\sqrt{8}}$$

and the time of fall

$$= \frac{P}{4\sqrt{2}} = P \cdot \frac{\sqrt{2}}{8} = P \cdot \frac{4}{3\sqrt{8}} \text{ nearly.}$$

$$= \frac{P}{6} \text{ nearly.}$$

249. The time down A C $\propto$ (arc = A D + C D), $\propto$ C L, if the cycloid be described on A S. Hence, having given the place of a body at a given time, we can determine the place at another given time.



Cut off $S\,m = C\,L \cdot \frac{\text{time d. A c}}{\text{time d. A C}}$.

Draw the ordinate m l; l c will determine c the place of the body.

250. Prop. XXXVII. *To determine the times of ascent and descent of a body projected upwards or downwards from a given point,* $F \propto \frac{1}{\text{distance}^2}$.

Let the body move off from the point G with a given velocity. Let $\frac{V^2 \text{ at G}}{v^2 \text{ in the circle e. d.}} = \frac{m^2}{1}$, (V and v known, $\therefore$ m known).

To determine the point A, take

$$\frac{GA}{\frac{SA}{2}} = \frac{m^2}{1}$$

$$\therefore \frac{GA}{GA + GS} = \frac{m^2}{2}$$

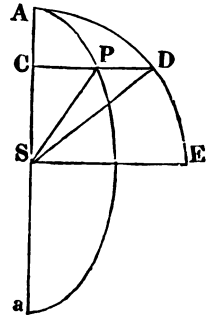
$$\therefore \frac{GA}{GS} = \frac{m^2}{2 - m^2}$$

$\therefore$ if $m^2 = 2$, G A is + and ∞, $\therefore$ the parabola
if $m^2 < 2$, G A is + and fin. $\therefore$ the circle
if $m^2 > 2$, G A is — and fin. $\therefore$ the rectangular hyperbola
} must be described on the axis S A.

With the center S and rad. $= \frac{L}{2}$ of the conic section, describe the circle k K H, and erecting the ordinates G I, C D, c d, from any places of the body, the body will describe G C, G c, in times of describing the areas S K k, S K k′, which are respectively = S I D, S I d.

251. Prop. XXXVIII. Force $\propto$ distance.

Let a body fall from A to any point C, by a force tending to S, and $\propto$ s. as the distance. Time $\propto$ arc A D, and V acquired $\propto$ C D. Conceive a body to fall in an evanescent ellipse about S as the center.



$\therefore$ the time down A P or A C

$\propto$ A S P $\propto$ A S D $\propto$ A D. $\frac{A\,S}{2}$

$\propto$ A D for the same descent, i. e. when A is given.

The velocity at any point P

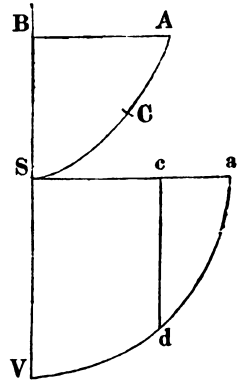
$$\propto \sqrt{F.PV}$$

$$\propto \sqrt{SP.\frac{2AC.Ca}{SP}} \text{ ultimately.}$$

$$\propto CD.$$

COR. 1. T. from A to S = ¼ period in an evanescent ellipse.
= ¼ period in the circle A D E.
= T. through A E.

COR. 2. T. from different altitudes to S ∝ time of describing different quadrants about S as the center ∝ 1.



N. In the common cycloid A C S it is proved in Mechanics that if S c a = S C A and the circle be described on 2 . S c a, and if a c = A C, the space fallen through, then the time through A C ∝ arc a d, and V acquired ∝ c d, which is analogous to Newton's Prop.

Newton's Prop. might be proved in the same way that the properties of the cycloid are proved.

OTHERWISE.

252. $v\,dv = -g\mu x\,.\,dx,$

$\therefore v^2 = 2g\mu(a^2 - x^2)$, if a = the height fallen from

$$\therefore v = \sqrt{2g\mu}.\sqrt{a^2-x^2} = \sqrt{2g\mu}.\,CD.$$

$$dt = -\frac{dx}{v} = -\frac{dx}{\sqrt{2g\mu}}.\frac{1}{\sqrt{a^2-x^2}}$$

$$\therefore t = +\frac{\text{arc}}{a\sqrt{2g\mu}}.\begin{pmatrix}\text{cos.} = x\\ \text{rad.} = a\end{pmatrix} + C,\ C = 0,$$

$$= \frac{1}{a\sqrt{2g\mu}}.\,AD.$$

$\therefore$ velocity ∝ *sine* of the arc whose *versed sine* = space, and the *arc* ∝ time, (rad. = original distance.)

253. *The velocity is velocity from a finite altitude.*

If the velocity had been that from infinity, it would have been infinite

and constant. $\therefore d t = -\frac{d x}{v}$, and $t = \frac{x}{a . \sqrt{g \mu}} + C$, when $t = 0$,

$= \sqrt{g \mu} . a$, $a = \alpha$.

$x = a$, $\therefore c$ is finite, $\therefore t = C = \frac{1}{\sqrt{g \mu}}$.

Similarly if the velocity had been $>$ velocity from infinity, it would have been infinite.

254. Prop. XXXIX. *Force* α (*distance*)^n, *or any function of distance.* Assuming any α^n. of the centripetal force, and also that quadratures of all curves can be determined (i. e. that all fluents can be taken); Required the velocity of a body, when ascending or descending perpendicularly, at different points, and the time in which a body will arrive at any point.

(The proof of the Prop. is inverse. Newton assumes the area A B F D to α V^2 and A D to α space described, whence he shows that the force α D F the ordinate. Conversely, he concludes, if F α D F, A B F D α V^2.)

$$v^2 \alpha \int v \, d v \alpha \int F . d s.$$

Let D E be a small given increment of space, and I a corresponding increment of velocity. By hypothesis

$$\frac{A B F D}{A B G E} = \frac{V^2}{v'^2} = \frac{V^2}{V^2 + 2 V . I + I^2}$$

$$\therefore \frac{A B F D}{D F G E} = \frac{V^2}{2 V . I + I^2} = \frac{V^2}{2 V . I} \text{ ultimately.}$$

But

$A B F D \alpha V^2$ $\therefore D F G E \alpha 2 V . I$

$\therefore D E . D F$ ultimately. α 2. V. I

$$\therefore D F \alpha \frac{2 V . I}{D E} \alpha \frac{I . V}{D E}.$$

But in motions where the forces are constant if I be the velocity generated in T, $F \alpha \frac{I}{T}$, $\left(F \alpha \frac{d v}{d t}\right)$ and if S be the space described with uniform velocity V in T, $\frac{V}{S} = \frac{1}{T}$, $\left(d t = \frac{d s}{v}\right)$. Also when the force is α^{ble}, the same holds for nascent spaces. $\therefore F \alpha \frac{I . V}{S}$, and D E represents S. $\therefore$ D F represents F.

Let D L $\propto \frac{1}{\sqrt{A B F D}} \propto \frac{1}{V}$, $\therefore$ D L M E ultimately = D L . D E $\propto \frac{D E}{V} \propto$ time through D E ultimately.

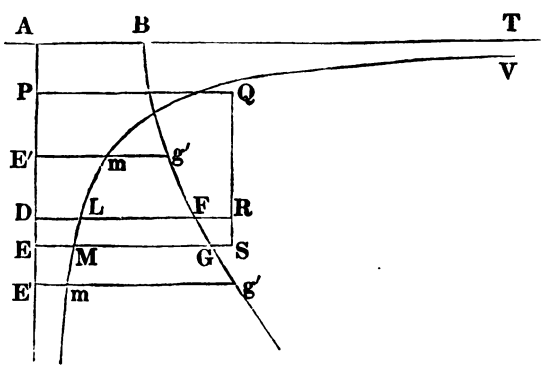
$\therefore$ Increment of the area A T V M E $\propto$ increment of the time down A D.

$\therefore$ A T V M E $\propto$ T.

$$\int d t \propto \int \frac{d s}{v}$$

(Since A B F D vanishes at A, $\therefore$ A T is an asymptote to the time curve. And since E M becomes indefinitely small when A B F D is infinite, $\therefore$ A E is also an asymptote.)

255. COR. 1. Let a body fall from P, and be acted on by a constant force given. If the velocity at D = the velocity of a body falling by the action of $\propto^{ble}$ force, then A, the point of fall, will be found by making A B F D = P Q R D.



For

$$\frac{A B F D}{D F G E} = \frac{V}{2 I} \text{ by Prop.}$$

$$\frac{D F G E}{D R S E} = \frac{D F}{D R} = \frac{I}{i}$$

if i be the increment of the velocity generated through D E by a constant force.

$$\frac{D R S E}{P Q R D} = \frac{V^2 (V + i)^2}{V^2} = \frac{2 i}{V} \text{ ultimately.}$$

$$\therefore \frac{A B F D}{P Q R D} = \frac{1}{I}.$$

256. COR. 2. If a body be projected up or down in a straight line from the center of force with a given velocity, and the law of force given; Find the velocity at any other point E′. Take E′ g′ for the force at E′.

velocity at E′ = velocity at D. $\frac{\sqrt{PQRD \pm DFg'E'}}{\sqrt{PQRD}}$, + if projected down, — if projected up.

$$\left(\text{For } \frac{\sqrt{PQRD \pm DFg'E'}}{\sqrt{PQRD}} = \frac{\sqrt{ABg'E'}}{\sqrt{ABFD}}\right).$$

257. Cor. 3. Find the time through D E.

Take E′ m inversely proportional to $\sqrt{PQRD \pm DFg'E'}$ (or to the velocity at E′).

$$\frac{T.PD}{T.PE} = \frac{\sqrt{PD}}{\sqrt{PE}} = \frac{\sqrt{PD}}{\sqrt{(PD+DE)}} = \frac{\sqrt{PD}}{\sqrt{PD} + \frac{DE}{2\sqrt{PD}}} \text{ (D E small)}$$

$$= \frac{PD}{PD + \frac{DE}{2}}$$

$$\therefore \frac{T.PD}{T.DE} = \frac{2PD}{DE} = \frac{2PD.DL}{DLME}$$

also

$$\frac{T.DE \text{ by } \alpha^{ble} \text{ force}}{T.DE' \text{ by do.}} = \frac{DLME}{DLmE'},$$

but T. D E by a constant force = T. D E by α^{ble} force since the velocities at D are equal $\left(dt = \frac{ds}{v}\right)$

$$\therefore \frac{T.PD}{T.DE'} = \frac{2PD.DL}{DLmE'}.$$

258. It is taken for granted in Prop. XXXIX, that $F \propto \frac{dv}{dt}$ (46), and that $v = \frac{ds}{dt}$, whence it follows that if $c.F = \frac{dv}{dt}$, $dv = c.F.dt$, and $v\,dv = c\,F.ds$.

$$\therefore v^2 = 2c\int F\,ds$$

Newton represents $\int F\,ds$ by the area A B F D, whose ordinate D F always = F.

$$dt = \frac{ds}{v} = \frac{ds}{\sqrt{2c.\int F\,ds}},$$

$$\therefore t = \int \frac{ds}{\sqrt{2c\int F\,ds}}.$$

Newton represents $\int \frac{ds}{\sqrt{\int F\,ds}}$ by the area A B T U M E, whose ordinate D L always $= \frac{1}{V}$

$$\left(= \frac{1}{\sqrt{2g.ABFD}}\right).$$

In COR. 1. If F′ be a constant force $V^2 = 2gF'.PD$, by Mechanics

but $$V^2 = 2c.\int F\,ds$$

And F′. P D or P Q R D is proved $= \int F\,ds$ or A B F D,

$$\therefore c = g$$

and

$$v^2 = 2g.\int F\,ds.$$

In COR. 2. $$\frac{\text{velocity at E}'}{\text{velocity at D}} = \frac{\sqrt{\int F\,ds \text{ when } s = AE'}}{\sqrt{\int F\,ds \text{ when } s = AD}} = \frac{\sqrt{ABg'E'}}{\sqrt{ABFD}}.$$

In COR. 3. t = time through D E′ $= \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2g\int F\,ds}} =$ D L m E′,

$$T = \text{time through P D} = \frac{2PD}{V \text{ at D}} = \frac{2PD}{\sqrt{2g.PQRD}} = 2PD.DL$$

$$\therefore \frac{T}{t} = \frac{2PD.DL}{DLmE'}.$$

259. The force $\propto x^n$.

$$\therefore v\,dv = -g\mu x^n\,dx, \quad \mu = \text{the force distance 1.}$$

$$\therefore v^2 = \frac{2g\mu}{n+1}(a^{n+1} - x^{n+1})$$

if a be the original height.

Let n be positive.

V from a finite distance to the center is finite
V from ∞ to a finite distance is infinite.

Let n be negative but less than 1.

V from a finite distance to the center is finite
V from ∞ to a finite distance is infinite.

Let $n = -1$ the above Integral fails, because x disappears, which cannot be.

$$v\,dv = -g\mu\frac{dx}{x}$$

$$\therefore v^2 = 2g\mu . l\frac{a}{x}$$

$\therefore$ V from a finite distance to the center is infinite
V from ∞ to a finite distance is infinite.

But the log. of an infinite quantity is ∞^{ly} less than the quantity itself $\frac{lx}{x}$ when x is infinite $= \frac{0}{0}$. Diff. and it becomes $\frac{\frac{dx}{x}}{dx} = \frac{1}{x} = \frac{1}{\infty}$.

Let n be negative and greater than 1.

V from a finite distance to the center is infinite
V from ∞ to a finite distance is finite.

260. If the force be constant, the velocity-curve is a straight line parallel to the line of fall, as Q R in Prop. XXXIX.

DEDUCTIONS.

261. To find under what laws of force the velocity from ∞ to a finite distance will be infinite or finite, and from a finite distance to the center will be finite or infinite.

If (1) $F \propto x^2$, $V \propto \sqrt{a^3 - x^3}$
(2) —— x —— $\sqrt{a^2 - x^2}$
(3) —— 1 —— $\sqrt{a - x}$
(4) —— $\frac{1}{x}$ —— $\sqrt{.l\frac{a}{x}}$
(5) —— $\frac{1}{x^2}$ —— $\sqrt{\frac{a - x}{ax}}$
(6) —— $\frac{1}{x^3}$ —— $\sqrt{\frac{a^2 - x^2}{a^2x^2}}$
(7) —— $\frac{1}{x^n}$ —— $\sqrt{\frac{a^{n-1} - x^{n-1}}{a^{n-1}x^{n-1}}}$

In the former cases, or in all cases where F $\propto$ some direct power of distance, the velocity acquired in falling from ∞ to a finite distance or to the center will be infinite, and from a finite distance to the center will be finite.

In the 4th case, the velocity from ∞ to a finite, and from a finite distance to the center will be infinite.

In the following cases, when the force $\propto$ as some inverse power of distance, the velocity from ∞ to a finite distance will be finite, for

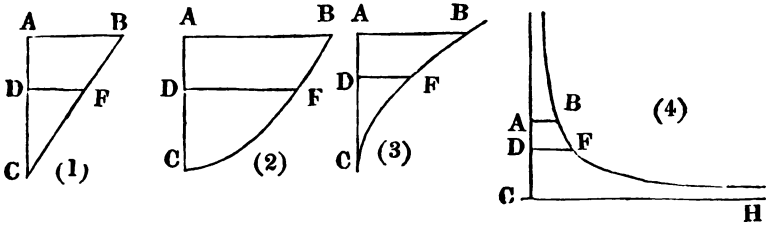
$$\sqrt{\frac{a^{n-1} - x^{n-1}}{a^{n-1}x^{n-1}}} = \sqrt{\frac{1}{x^{n-1}}}$$

when a is infinite. And the velocity from a finite distance to the center will be infinite, for

$$\sqrt{\frac{a^{n-1} - x^{n-1}}{a^{n-1}x^{n-1}}} = \sqrt{\frac{1}{0}}$$

when $x = 0$.

262. *On the Velocity and Time-Curves.*



(1) Let F $\propto$ D, the area which represents V^2 becomes a $\triangle$.

For D F $\propto$ D C.

(2) Let F $\propto \sqrt{\text{D}}$, $\therefore$ D $F^2 \propto$ D C and V-curve is a parabola.

(3) Let F $\propto D^2$, $\therefore$ D F $\propto$ D C^2, and V-curve is a parabola the axis parallel to A B.

(4) Let F $\propto \frac{1}{D}$, $\therefore$ D F $\propto \frac{1}{D\,C}$, $\therefore$ V-curve is an hyperbola referred to the asymptotes A C, C H.

(5) If F $\propto$ D, and be repulsive, $V^2 \propto$ D C. D F $\propto$ D C^2, $\therefore$ V $\propto$ D C, $\therefore$ the ordinate of the time curve $\propto \frac{1}{V} \propto \frac{1}{D\,C}$, $\therefore$ T-curve is an hyperbola between asymptotes.

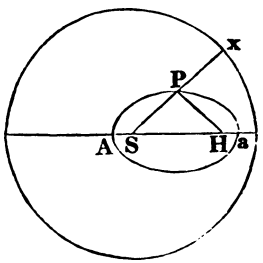
(6) If a body fall from ∞ distance, and F $\propto \frac{1}{D^3}$, V $\propto \frac{1}{D}$, $\therefore$ the ordinate of the time-curve D, $\therefore$ T-curve is a straight line.

(7) If a body fall from ∞, and F $\propto \frac{1}{D^2}$, V $\propto \frac{1}{\sqrt{D}}$, $\therefore$ the ordinate of T-curve $\propto \sqrt{\text{D C}}$, $\therefore$ T-curve is a parabola.

(8) If a body fall from ∞, and F $\propto \frac{1}{D^5}$, V $\propto \frac{1}{D^2}$, $\therefore$ the ordinate of T-curve $\propto$ D C^2, $\therefore$ T-curve is a parabola as in case 3.

EXTERNAL AND INTERNAL FALLS.

263. *Find the external fall in the ellipse, the force in the focus.*



Let x P be the space required to acquire the velocity in the curve at P.

$$\frac{V^2 \text{ down } P\,x}{V^2 \text{ in the circle distance } S\,P} = \frac{P\,x}{\frac{S\,x}{2}}$$

$$\frac{V^2 \text{ in the circle distance } S\,P}{V^2 \text{ in the ellipse at } P} = \frac{A\,a}{2.\,H\,P}$$

$$\therefore \frac{V^2 \text{ down } P\,x}{V^2 \text{ in the ellipse at } P} = \frac{A\,a.\,P\,x}{S\,x.\,H\,P}$$

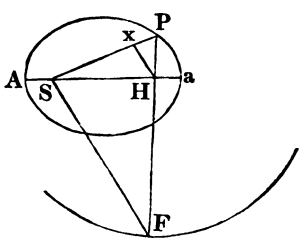
$$\therefore \frac{P\,x}{S\,x} = \frac{H\,P}{A\,a}$$

$$\therefore \frac{P\,x}{S\,P} = \frac{H\,P}{S\,P}$$

$$\therefore P\,x = H\,P$$

$\therefore S\,x = S\,P + P\,x = A\,a$, and the locus of x is the circle on 2 A a, the center S.

264. *Find the internal fall in the ellipse, the force in the focus.*



$$\frac{V^2 \text{ down } P\,x}{V^2 \text{ in the circle } S\,x} = \frac{P\,x}{\frac{S\,P}{2}}$$

$$\frac{V^2 \text{ in the circle } S\,x}{V^2 \text{ in the circle } S\,P} = \frac{S\,P}{S\,x}, \text{ force } \propto \frac{1}{\text{distance}^2}$$

$$\frac{V^2 \text{ in the circle S P}}{V^2 \text{ in the ellipse at P}} = \frac{A\,a}{2\,H\,P}$$

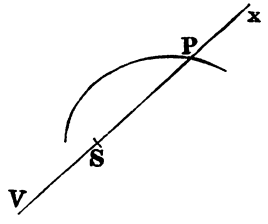
$$\therefore \frac{V^2 \text{ down P x}}{V^2 \text{ in the ellipse at P}} = \frac{P\,x \,.\, A\,a}{S\,x \,.\, H\,P}$$

$$\therefore \frac{P\,x}{S\,x} = \frac{H\,P}{A\,a}$$

$$\therefore \frac{P\,x}{S\,P} = \frac{H\,P}{A\,a + H\,P}$$

Describe a circle from H with the radius A a. Produce P H to the circumference in F. Join F S. Draw H x parallel to F S.

265. *Generally.*



For *external* falls.

$$\frac{V^2 \text{ down P x}}{V^2 \text{ in the circle distance S P}} = \frac{2\,g \,.\, \text{area A B F D Newton's fig.}}{g\,F \,.\, S\,P} \quad F = \text{force at distance S P}$$

$$\frac{V^2 \text{ in the circle S P}}{V^2 \text{ in the curve at P}} = \frac{2\,S\,P}{P\,V}$$

$$\therefore \frac{V^2 \text{ down P x}}{V^2 \text{ in the curve}} = \frac{4 \,.\, A\,B\,F\,D}{F \,.\, P\,V}$$

$$\therefore 4 \,.\, A\,B\,F\,D = F \,.\, P\,V$$

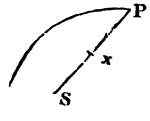
Find the area in general $\left\{\begin{matrix} \text{ordinate} = F \\ \text{abscissa} = \text{space} \end{matrix}\right\}$

In the general expression make the distance from the center = S P, and a the original height, S x will be found.

266. For *internal* falls.

$$\frac{V^2 \text{ down P x}}{V^2 \text{ in the circle S P}} = \frac{2\,g \,.\, A\,B\,F\,D \text{ Newton's fig.}}{2\,g\,F \,.\, S\,P} \quad F = \text{force at P}$$

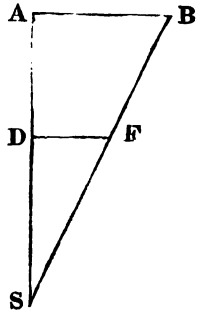
$$\frac{V^2 \text{ in the circle S P}}{V^2 \text{ in the curve at P}} = \frac{2\,S\,P}{P\,V}$$

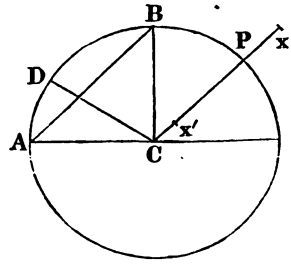


$$\therefore \frac{V^2 \text{ down P x}}{V^2 \text{ in the curve at P}} = \frac{4\,A\,B\,F\,D}{F \,.\, P\,V}$$

$\therefore$ if the velocities aré equal, $4\,A\,B\,F\,D = F \,.\, P\,V$.

267. Ex. For internal and external falls.





In the ellipse the force tending to the center.

In this case, D F $\propto$ D S. Take A B for the force at A. Join B S.

$\therefore$ D F is the force at D, and the area $ABFD = \frac{AD}{2}(AB + DF)$

$= \overline{AS - SD}.\ \overline{AB + DF}$. Let μ equal the absolute force at the distance 1. Let S A = a, S D = x, $\therefore$ A B = a μ

D F = x μ

$$\therefore ABFD = \mu . \frac{\overline{a - x}.\ \overline{a + x}}{2} = \mu . \frac{a^2 - x^2}{2}$$

and

$$4\,ABFD = F . PV,$$

or

$$a^2 - x^2 = CP . \frac{CD^2}{CP} \text{ in the ellipse,}$$

or

$$a^2 - x^2 = CD^2.$$

For the *external* fall, make x = C P, then a = Cx, and $Cx^2 - CP^2 = CD^2$,

or
$$Cx^2 = CP^2 + CD^2$$
$$= AC^2 + BC^2$$
$$= AB^2$$
$$\therefore Cx = AB.$$

For the *internal* fall, make a = C P, then x = C x′, and

$$CP^2 - Cx'^2 = CD^2,$$

or
$$Cx'^2 = CP^2 - CD^2,$$
$$\therefore Cx' = \sqrt{CP^2 - CD^2}.$$

268. Similarly, in all cases where the velocity in the curve is quadrable, without the Integral Calculus we may find internal and external falls. But generally the process must be by that method.

Thus in the above Ex.

$$v\,dv = -g\mu x.\,dx$$

$$\therefore v^2 = g\mu(a^2 - x^2)$$

$$\therefore A\,B\,F\,D = \frac{V^2}{2g} = \mu\frac{a^2 - x^2}{2}, \text{ as above, \&c.}$$

269. And in general,

$$v^2 = \frac{2g\mu}{n+1}(a^{n+1} - x^{n+1}), \text{ if the force } \propto x^n.$$

Also

$$v^2 = \frac{g}{2}F.PV = \frac{g}{2}\mu\varrho^n.\frac{2p\,d\varrho}{dp}$$

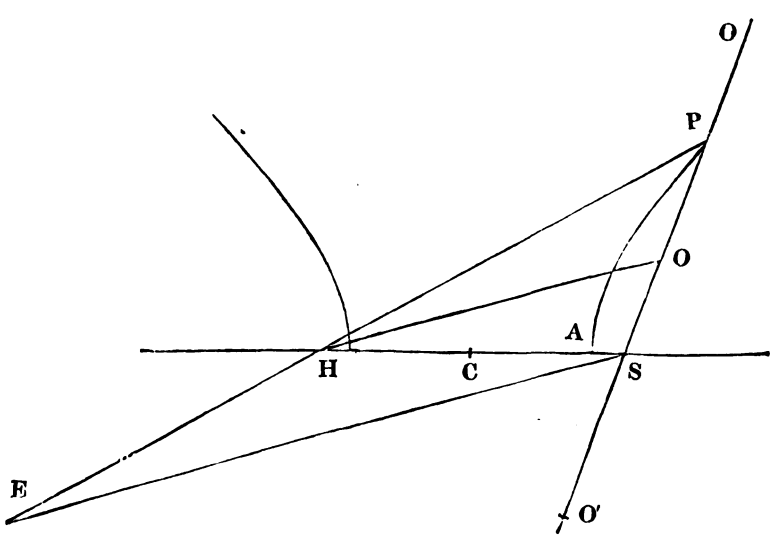
$$\therefore \frac{2g\mu}{n+1}(a^{n+1} - x^{n+1}) = g\mu\varrho^n.\frac{p\,d\varrho}{dp}$$

$$\therefore \frac{2}{n+1}(a^{n+1} - x^{n+1}) = \varrho^n.p.\frac{d\varrho}{dp}$$

And to find the external fall, make $x = \varrho$, and from the equation find a, the distance required.

And to find the internal fall make $a = r$, and from the equation find x, the distance required.

270. *Find the external fall in the hyperbola, the force* $\propto \frac{1}{D^2}$ *from the focus.*



V^2 down O P : V^2 in the circle rad. S P : : O P : $\frac{S\,O}{2}$

V^2 in the circle S P : V^2 in the hyperbola at P : : A C : H P

$\therefore$ V^2 down O P : V^2 in the hyperbola : : A C. O P : $\frac{SO.HP}{2}$

$$\therefore 2AC.OP = SO.HP$$
$$\therefore 2AC.SO - 2AC.SP = SO.HP$$
$$\therefore SO = -\frac{2AC.SP}{HP-2AC} = -2AC$$

To find what this denotes, find the actual velocity in the hyperbola.

Let the force $= \beta$, at a distance $= r$, $\therefore$ the force at the distance $= \frac{\beta . r^2}{x^2}$.

Also

$$\frac{V^2 \text{ in the circle S P}}{2g} = \frac{\beta . r^2}{x^2} \cdot \frac{x}{2} = \frac{\beta x^2}{2x}$$
$$\therefore \frac{V^2 \text{ in the hyperbola}}{2g} = \frac{(2a + x)\beta r^2}{a . 2x}$$
$$= \frac{\beta r^2}{x} + \frac{\beta r^2}{2a}$$

But $\frac{V^2}{2g}$ when the body has been projected from $\infty = \frac{\beta r^2}{x} + \frac{V^2}{2g}$ of projection from ∞, $\therefore$ $\frac{V^2}{2g}$ of projection from $\infty = \frac{\beta . r^2}{2a} = \frac{V^2}{2g}$ down 2 a, F being constant and $= \frac{\beta r^2}{4a^2}$, or $= V^2$ from ∞ to O', when S O' $= 2$ A C.

$\therefore$ V in the hyperbola is such as would be acquired by the body ascending from the distance x to ∞ by the action of force considered as repulsive, and then being projected from ∞ back to O', S O being $= 2$ A C.

In the opposite hyperbola the velocity is found in the same way, the force repulsive, p externally $= \frac{2HC.SP}{2AC-HP}$.

271. *Internal fall.*

V^2 down P O : V^2 in the circle rad. S O : : P O : $\frac{SP}{2}$

V^2 in the circle S O : V^2 in the circle S P : : S P : S O

V^2 in the circle S P : V^2 in the hyperbola at P : : A C : H P

$\therefore$ V^2 down P O : V^2 in the hyperbola : : A C. P O : $\frac{SO.HP}{2}$

$$\therefore 2AC.PO = SO.HP$$

or

$$2AC(SP - SO) = SO.HP$$
$$\therefore SO = \frac{2AC.SP}{2AC + HP},$$

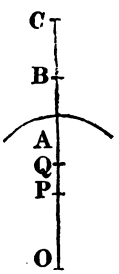
and

$$P O = S P - S O = \frac{S P . H P}{2 A C + H P}.$$

Hence make H E = 2 A C, join S E, and draw H O parallel to S E.

Hence the external and internal falls are found, by making V acquired down a certain space p with α^{ble} force equal that down $\frac{1}{2}$. P V by a constant force, P V being known from the curve.

272. *Find how far the body must fall externally to the circumference to acquire* V *in the circle,* F α *distance towards the center of the circle.*



Let O C = p, O B = x, O A = a, C being the point required from which a body falls.

Let the force at A = 1, $\therefore$ the force at B = $\frac{x}{a}$

$v\,dv = -g . F . dx$, (for the velocity increases as x decreases)

$$= -g \frac{x}{a} . dx$$

$$\therefore v^2 = \frac{g}{a} . x^2 + C$$

and when v = 0, x = p,

$$\therefore C = \frac{g}{a} . p^2$$

$$\therefore v^2 = \frac{g}{a} (p^2 - x^2)$$

and when x = a,

$$v^2 \text{ at } A = \frac{g}{a} (p^2 - a^2).$$

But

$$v^2 \text{ at } A = 2 g . \frac{a}{2}$$

the force at A being constant, and

$$\frac{a}{2} = \frac{P V}{4},$$

$$= g a$$

$$\therefore p^2 - a^2 = a^2, \therefore p^2 = 2 a^2, \therefore p = \sqrt{2} . a.$$

273. *Find how far the body must fall internally from the circumference to acquire* V *in the circle,* F α *distance towards the center of the circle.*

Let P be the point to which the body must fall, O A = a, O P = p, O Q = x, F at A = 1, $\therefore$ the force at Q = $\frac{x}{a}$.

$$\therefore v\,dv = -g \cdot \frac{x}{a} \cdot dx$$

$$\therefore v^2 = -\frac{g}{a} \cdot x^2 + C,$$

and when $v = 0$, $x = a$,

$$\therefore C = \frac{g}{a} \cdot a^2$$

$$\therefore v^2 = \frac{g}{a}(a^2 - x^2)$$

and when $x = p$,

$$v^2 = \frac{g}{a}(a^2 - p^2) \text{ from } \propto^{\text{ble}} \text{ force}$$

and

$$v^2 = g \cdot a, \text{ from the constant force 1 at A.}$$

$\therefore a^2 - p^2 = a^2$, $\therefore p = 0$, $\therefore$ the body falls from the circumference to the center.

274. Similarly, when $F \propto \frac{1}{\text{distance}}$.

$$\text{O C, or p externally} = a\sqrt{e}, \text{ (e = base of hyp. log.)}$$

and

$$\text{O P, or p internally} = \frac{a}{\sqrt{e}}.$$

275. When $F \propto \frac{1}{\text{distance}^2}$.

$$\text{p externally} = 2a$$

$$\text{p internally} = \frac{2a}{3}.$$

276. When $F \propto \frac{1}{\text{distance}^3}$.

$$\text{p externally} = \infty.$$

$$\text{p internally} = \frac{a}{\sqrt{2}}.$$

277. When $F \propto \frac{1}{\text{distance}^{n+1}}$.

$$\text{p externally} = a\sqrt[n]{2 - n}$$

$$\text{p internally} = a\sqrt[n]{\frac{2}{2 + n}}$$

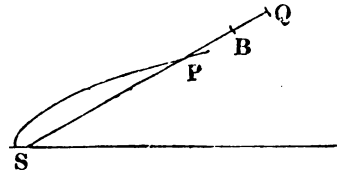
If the force be repulsive, the velocity increases as the distance increases,

$$\therefore v\,dv = g\,F \cdot dx$$

278. *Find how far a body must fall externally to any point* P *in the parabola, to acquire* v *in the curve.* $F \propto \frac{1}{D^2}$, *towards the focus.*

P V = 4 S P = c, S Q = p, S B = x, S P = a, force at P = 1,

$$\therefore F \text{ at } B = \frac{a^2}{x^2},$$



$$\therefore v\,dv = -g\frac{a^2}{x^2}.dx$$

$$\therefore \frac{v^2}{2} = \frac{g a^2}{x} + C,$$

when v = 0, x = p

$$\therefore C = \frac{g a^2}{p},$$

$$\therefore v^2 = 2 g a^2 \left(\frac{1}{x} - \frac{1}{p}\right) = 2 g a^2 \left(\frac{1}{a} - \frac{1}{p}\right) \text{ at P,}$$

but

$$v^2 = 2 g . \frac{c}{4} = 2 g a,$$

$$\therefore \frac{1}{a} - \frac{1}{p} = \frac{1}{a},$$

$$\therefore \frac{1}{p} = 0, \therefore p = \infty.$$

279. Similarly, internally, $p = \frac{a}{2}$.

280. In the ellipse, $F \propto \frac{1}{D^2}$ towards a focus

p externally = $\overline{PH + PS}$. (∴ describe a circle with the center S, rad. = 2 A C)

p internally = $\frac{PH . PS}{2AC + PH}$.

(Hence V at P = V in the circle e. d.)

281. In the hyperbola, $F \propto \frac{1}{D^2}$ towards focus

p externally = − 2 A C (Hence V at P = V in the circle e. d.)

p internally = $\frac{PH . PS}{2AC + PH}$. (Hence V at P = V in the circle e. d., p. 190)

282. In the ellipse F ∝ D from the center

p externally = $\sqrt{AC^2 + BC^2}$, (= A B)} (Hence construction)

or (= $\sqrt{CD^2 + CP^2}$)

(Hence also V at P = V in the circle radius C P, when C D = C P)

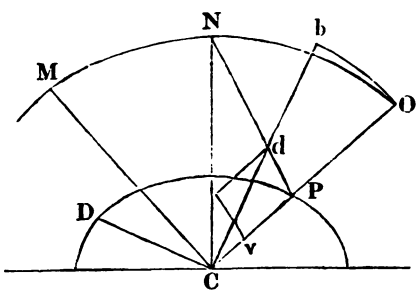
p internally = $\sqrt{CP^2 - CD^2}$.

(Hence if C P = C D, p = 0, and V at P = V in the circle e. d., as was deduced before)

(If C P < C D, p impossible, ∴ the body cannot fall from any distance to C and thus acquire the V in the curve)

283. In the ellipse, F ∝ D from the center.

External fall.



The velocity-curve is a straight line, (since D F ∝ C D, also since F = 0, when C P = 0, this straight line comes to C, as C d b, V ∝ $\sqrt{\text{C O b}}$ ∝ C O, O being the point fallen from, to acquire V at P.

∴ V from O to C : V from P to C : : O C : P C

Also since v d v = — g F . d x, and if the force at the distance 1 = 1, the force at x = x. ∴ v d v = — g x d x, and integrating and correcting, $v^2 = g(p^2 - x^2)$, where p = the distance fallen from.

∴ v ∝ $\sqrt{p^2 - x^2}$, and if a circle be described, with center C, rad. C O ∝ P N (the right sine of the arc whose versed P O is the space fallen through).

∴ V from O to P : V from O to C : : P N : (C M =) O C

and

V from P to C : V in the circle rad. C P : : 1 : 1

(for if P v = ½ P C, v d = C d P) and

V in the circle C P : V in the ellipse : : C P : C D.

Compounding the 4 ratios,

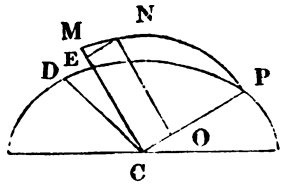
V down O P : V in the ellipse : : P N : C D

∴ Take P N = C D, and

V down O P = V in the ellipse,

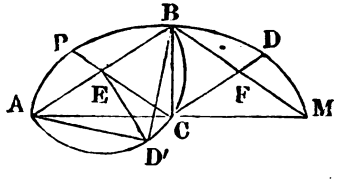
∴ C O = C N = $\sqrt{\text{C P}^2 + \text{C D}^2}$.

Internal fall.



V in the ellipse : V in the circle rad. C P : : C D : C P
V in the circle : V down C P : : 1 : 1
V down C P : V down P O : : (C M =) C P : O N
∴ V in the ellipse : V down P O : : C D : O N
∴ Take O N = C D, and V in the curve = V down P O, and C O $= \sqrt{C P^2 - C D^2}$.

284. *Find the point in the ellipse, the force in the center, where* V = *the velocity in the circle,* e. d.



In this case C P = C D, whence the construction.

Join A B, describe $\frac{\text{circle}}{2}$ on it, bisect the circumference in D', join B D', A D'. From C with A D' cut the ellipse in P.

$2 A D'^2 (= 2 P C^2) = A B^2 = A C^2 + B C^2 (= C P^2 + C D^2)$
$\therefore 2 C P^2 = C P^2 + C D^2$
$\therefore C P^2 = C D^2$. (C P will pass through E.)

A simpler construction is to bisect A B in E, B M in F, then C P is the diameter to the ordinate A B, and from the triangles C E B, C F B, C F is parallel to A B, ∴ C D' is a conjugate to C P and = C P.

285. In the hyperbola, force repulsive, ∝ D, from the center
- p externally (to which body must rise from P,) $= \sqrt{C D^2 + C P^2}$
- p internally (to which body must rise from the center) $= \sqrt{C P^2 - C D^2}$

(Hence if the hyperbola be rectangular p internally = 0, or the body must rise through C P.)

286. *In any curve,* $F \propto \frac{1}{D^{n+1}}$, *find* p *externally.*

$$p = a \cdot \left(\frac{a}{a - \frac{n c}{4}} \right)^{\frac{1}{n}},$$

where $a = S P$, $c = P V$.

287. If the curve be a logarithmic spiral, $c = 2a$,

$$\therefore p = a \left(\frac{a}{a - \frac{n a}{2}} \right)^{\frac{1}{n}}$$

$$= a \left(\frac{1}{1 - \frac{n}{2}} \right)^{\frac{1}{n}}$$

also $F \propto \frac{1}{D^3}$, $\therefore n = 2$ $\Big\}$ $\therefore p = a \left(\frac{1}{1-1} \right)^{\frac{1}{n}} = \infty$.

288. *In any curve,* $F \propto \frac{1}{D^{n+1}}$, *find* p *internally.*

$$p = a \left(\frac{a}{a + \frac{n c}{4}} \right)^{\frac{1}{2}} \cdot \left(p^n = \frac{4 a^{n+1}}{4 a + n c} \right)$$

289. If the curve be a logarithmic spiral, $c = 2a$, $n = 2$,

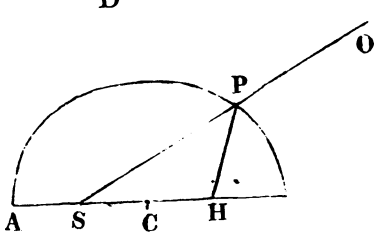
$$\therefore p = a \left(\frac{a}{a+a} \right)^{\frac{1}{n}} = \frac{a}{\sqrt{2}}.$$

290. If the curve be a circle, F in the circumference, $c = a$, and $n = 4$,

$$\therefore p \text{ externally} = a \left(\frac{a}{a - a} \right)^{\frac{1}{4}} = \infty$$

$$\text{and p internally} = a \left(\frac{a}{a + a} \right)^{\frac{1}{4}} = \frac{a}{\sqrt[4]{2}}.$$

291. *In the ellipse,* $F \propto \frac{1}{D^2}$ *from focus. External fall.*



V^2 down O P : V^2 in the circle radius S P : : O P : $\frac{S O}{2}$, Sect. VII.

V^2 in the circle S P : V^2 in the ellipse at P : : A C : H P,

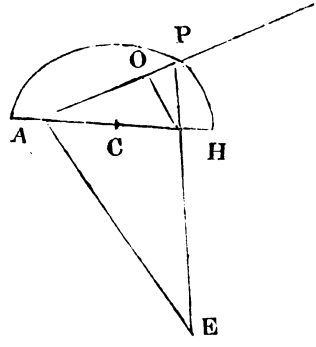
$$\therefore V^2 \text{ down } O\,P : V^2 \text{ in the ellipse} :: A\,C \,.\, O\,P : \frac{S\,O \,.\, H\,P}{2}$$

$$\therefore 2\,A\,C \,.\, O\,P = S\,O \,.\, H\,P$$

$$\therefore S\,O = \frac{2\,A\,C \,.\, O\,P}{H\,P} = \frac{2\,A\,C \,.\, S\,O - 2\,A\,C \,.\, S\,P}{H\,P}$$

$$\therefore S\,O = \frac{2\,A\,C \,.\, S\,P}{2\,A\,C - H\,P} = 2\,A\,C.$$

Internal fall.



V^2 down $P\,O : V^2$ in the circle radius $S\,O :: P\,O : \frac{S\,P}{2}$,

V^2 in the circle $S\,O : V^2$ in the circle $S\,P :: S\,P : S\,O$

V^2 in the circle $S\,P : V^2$ in the ellipse at $P :: A\,C : H\,P$

$$\therefore V^2 \text{ down } P\,O : V^2 \text{ in the ellipse} :: P\,O \,.\, A\,C : \frac{S\,O \,.\, H\,P}{2}$$

$$\therefore 2\,P\,O \,.\, A\,C = S\,O \,.\, H\,P$$

$$\therefore 2\,S\,P \,.\, A\,C - 2\,S\,O \,.\, A\,C = S\,O \,.\, H\,P$$

$$\therefore S\,O = \frac{2\,A\,C \,.\, S\,P}{2\,A\,C + H\,P}$$

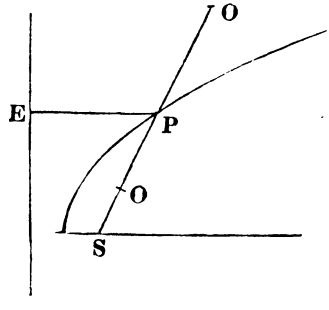
Hence, make $H\,E = 2\,A\,C$, join $S\,E$, and draw $H\,O$ parallel to $E\,S$.

292. *External fall in the parabola,*

$F \propto \frac{1}{D^2}$ from focus.

V^2 d. $O\,P : V^2$ in the circle radius $S\,P$ $:: O\,P : \frac{S\,O}{2}$, Sect. VII.

V^2 in the circle $S\,P : V^2$ in the parabola at $P :: 1 : 2$,



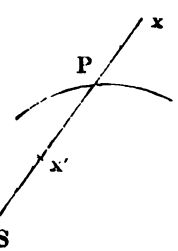
$$\therefore V^2 \text{ down } O\,P : V^2 \text{ in the parabola} :: O\,P : S\,O$$
$$\therefore O\,P = S\,O, \therefore S\,O = \alpha$$

Internal fall.

$$V^2 \text{ down } O\,P : V^2 \text{ in the circle } S\,O :: O\,P : \frac{S\,P}{2}$$
$$V^2 \text{ in the circle } S\,O : V^2 \text{ in the circle } S\,P :: S\,P : S\,O$$
$$V^2 \text{ in the circle } S\,P : V^2 \text{ in the parabola at } P :: 1 : 2$$
$$\therefore V^2 \text{ down } O\,P : V^2 \text{ in the parabola} :: O\,P : S\,O,$$
$$\therefore O\,P = S\,O,$$
$$\therefore S\,O = \frac{S\,P}{2}.$$

$V = V$ down $\frac{P\,V}{4} = V$ down $S\,P = V$. down $E\,P = V$ of a body describing the parabola by a constant vertical force = force at P.

293. *Find the external fall so that the velocity² acquired = n′. velocity in the curve,* $F \propto x^n$.



$$v\,dv = -g\mu . x^n . dx, \quad (\mu = \text{force distance 1}),$$
$$\therefore v^2 = \frac{2g\mu}{n+1} . (a^{n+1} - x^{n+1})\ a = \text{original height},$$
$$V^2 \text{ in the curve} = g\mu . \frac{p\,d\varrho}{d\,p} = \frac{g}{2}\mu . c, \text{ if } c = \frac{2p\,d\varrho}{d\,p},$$
$$\therefore n' . \frac{g}{2}\mu . c = \frac{2g\mu}{n+1} . (a^{n+1} - x^{n+1}), \text{ or } n' . c = \frac{4}{n+1} . (a^{n+1} - x^{n+1})$$

Make $x = S\,P = \varrho$, and from the equation we get a, which = S x.

For the internal fall, make $a = S\,P = \varrho$, and from the equation we get x, which = S x′.

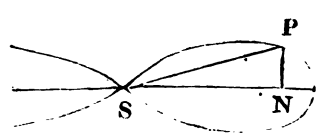
294. *Find the external fall in a* LEMNISCATA.

$$(x^2 + y^2)^2 = a^2 (x^2 - y^2)$$

is a rectangular equation whence we must get a polar one

Let $\angle N\,S\,P = \theta$,

$$\therefore y = \varrho . \sin.\theta,\ x = \varrho . \cos.\theta,\ \varrho^2 = (x^2 + y^2)$$
$$\therefore \varrho^4 = a^2 . (\varrho^2 (\cos.^2\theta - \sin.^2\theta)) = a^2 \varrho^2 . \cos.2\theta,$$
$$\therefore \varrho^2 = a^2 . \cos.2\theta$$
$$\therefore 2\theta = \angle \left(\cos. = \frac{\varrho^2}{a^2}\right),$$
$$\therefore 2\,d\theta = -\frac{2\varrho\,d\varrho}{a^2} = \frac{-2\varrho\,d\varrho}{\sqrt{a^4 - \varrho^4}}$$



$$\sqrt{1 - \frac{\varrho^4}{a^4}},$$

$$\therefore d\theta^2 = \frac{\varrho^2 d\varrho^2}{a^4 - \varrho^4},$$

$$\therefore \frac{d\varrho^2}{d\theta^2} = \frac{a^4 - \varrho^4}{\varrho^2}$$

but in general

$$\varrho . d\theta = \frac{d\varrho . p}{\sqrt{\varrho^2 - p^2}},$$

$$\therefore \varrho^4 d\theta^2 - \varrho^2 d\theta^2 p^2 = d\varrho^2 . p^2,$$

$$\therefore p^2 = \frac{\varrho^4 d\theta^2}{\varrho^2 d\theta^2 + d\varrho^2} = \frac{\varrho^4}{\varrho^2 + \frac{d\varrho^2}{d\theta^2}}$$

$$= \text{in this case } \frac{\varrho^4}{\varrho^2 + \frac{a^4 - \varrho^4}{\varrho^2}}$$

$$\therefore p^2 = \frac{\varrho^6}{a^4},$$

$$\therefore \frac{1}{p^2} = \frac{a^4}{\varrho^6},$$

$$\therefore -\frac{2 dp}{p^3 d\varrho} = -\frac{6 . a^4}{\varrho^7},$$

$$\therefore \text{force to } S \propto \frac{1}{\varrho^7}$$

$$v dv = -\frac{g\mu .}{x^7} . dx,$$

$$\therefore v^2 = \frac{2 g \mu}{6}\left(\frac{1}{x^6} - \frac{1}{a^6}\right)$$

Also

$$PV = \frac{2 p d\varrho}{dp} = 2p . \frac{a^2}{3\varrho^2} = \frac{2 . \varrho^3}{a^2} . \frac{a^2}{3\varrho^2} = \frac{2\varrho}{3},$$

$$\therefore v^2 = \frac{g\mu}{\varrho^7} . \frac{2\varrho}{3} = \frac{2 g \mu .}{6} . \frac{1}{\varrho^6},$$

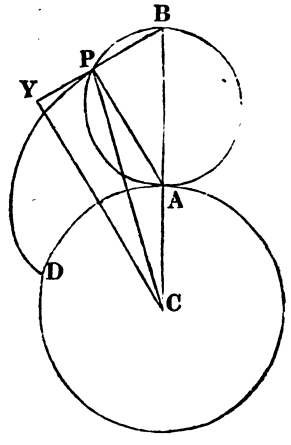
Make x in the formula above $= \varrho$,

$$\therefore \frac{1}{\varrho^6} - \frac{1}{a^6} = \frac{1}{\varrho^6},$$

$$\therefore \frac{1}{a^6} = 0, \therefore a \text{ is infinite.}$$

295. *Find the force and external fall in an* EPICYCLOID

$$CY^2 = CP^2 - YP^2 = CP^2 - CA^2 \cdot \frac{YB^2}{CB^2}$$



Let

$$CY = p,\ CP = \varrho,\ CB = c,\ CA = b,$$

$$\therefore p^2 = \varrho^2 - b^2 \cdot \frac{c^2 - p^2}{c^2}$$

$$\therefore c^2 p^2 = c^2 \varrho^2 - b^2 c^2 + b^2 p^2$$

$$\therefore p^2 = \frac{c^2 (\varrho^2 - b^2)}{c^2 - b^2}$$

$$\therefore \frac{1}{p^2} = \frac{c^2 - b^2}{c^2 (\varrho^2 - b^2)},$$

$$\therefore -\frac{2\,d\,p}{p^3} = \frac{c^2 - b^2}{c^2}\left(\frac{-2\,d\,\varrho \cdot \varrho}{(\varrho^2 - b^2)^2}\right)$$

$$\therefore \text{force} \propto \frac{\varrho}{(\varrho^2 - b^2)^2} \propto \frac{\varrho}{p^4}$$

(as in the Involute of the circle which is an Epicycloid, when the radius of the rota becomes infinite.)

Having got $\propto^n$ of force, we can easily get the external (or internal) fall.

296. *Find in what cases we can integrate for the Velocity and Time.*

Case 1. Let force $\propto x^n$,

$$\therefore v\,d\,v = g\mu \cdot x^n\,d\,x,$$

$$\therefore v^2 = \frac{2g\mu}{n+1}(a^{n+1} - x^{n+1})$$

$$\therefore t = \int \frac{-d\,x}{v} = \sqrt{\frac{n+1}{2g\mu}} \cdot \int \frac{-d\,x}{\sqrt{(a^{n+1} - x^{n+1})}}$$

Now in general we can integrate $x^m\,d\,x \cdot (a + b\,x^n)^{\frac{p}{q}}$, when

$$\frac{m+1}{n} \text{ is whole or } \frac{m+1}{n} + \frac{p}{q} \text{ whole.}$$

$\therefore$ in this case, we can integrate, when $\frac{1}{n+1}$, or $\frac{1}{n+1} - \frac{1}{2}$, is whole.

Let

$$\frac{1}{n+1} = p \text{ any whole number}$$

$$\therefore n + 1 = \frac{1}{p},$$

$$\therefore n = \frac{1-p}{p}, \text{ (p being positive), } (\alpha)$$

Let

$$\frac{1}{n+1} - \frac{1}{2} = p,$$

$$\therefore \frac{1}{n+1} = p + \frac{1}{2} = \frac{2p+1}{2},$$

$$\therefore n + 1 = \frac{2}{2p+1},$$

$$\therefore n = \frac{1-2p}{1+2p}. \quad (\beta)$$

$\therefore$ these formulæ admit only 0 and 1 for integer positive values of n, and no positive fractional values. $\therefore$ we can integrate when $F \propto x$, or $F \propto 1$.

297. Case 2. Let force $\propto \frac{1}{x^n}$,

$$\therefore v\,dv = -g\mu\frac{dx}{x^n},$$

$$\therefore v^2 = \frac{2g\mu}{n-1}\cdot\left(\frac{a^{n-1} - x^{n-1}}{a^{n-1}x^{n-1}}\right)$$

$$\therefore t = \int\frac{-dx}{v} = \sqrt{\frac{n-1.a^{n-1}}{2g\mu}}\cdot\int\frac{-dx.x^{\frac{n-1}{2}}}{\sqrt{a^{n-1}-x^{n-1}}}$$

in which case we can integrate, when $\frac{\frac{n-1}{2}+1}{n-1}$, or $\frac{\frac{n-1}{2}+1-\frac{1}{2}}{n-1}$, whole,

i. e. if $\frac{1}{2} + \frac{1}{n-1}$, or $\frac{1}{n-1}$, be whole.

Let $\frac{1}{n-1} = p$, any whole positive No.,

$$\therefore n - 1 = \frac{1}{p}, \therefore n = \frac{p+1}{p}, \quad (\alpha')$$

Let $\frac{1}{2} + \frac{1}{n-1} = p$,

$$\therefore \frac{1}{n-1} = \frac{2p-1}{2},$$

$$\therefore n - 1 = \frac{2}{2p-1},$$

$$\therefore n = \frac{2p+1}{2p-1}. \quad (\beta')$$

$\therefore$ these formulæ admit any values of n, in which the numerator exceeds the denominator by 1, or in which the numerator and denominator are any two successive odd numbers, the numerator being the greater.

$\therefore$ we can integrate, when $F \propto \frac{1}{x^2}, \frac{1}{x^{\frac{3}{2}}}, \frac{1}{x^{\frac{4}{3}}}, \frac{1}{x^{\frac{5}{4}}}$, &c.

or

$\frac{1}{x^3}, \frac{1}{x^{\frac{5}{3}}}, \frac{1}{x^{\frac{7}{5}}}, \frac{1}{x^{\frac{9}{7}}}$, &c.

298. Case 3. The formulæ (α') (β'), in which p is positive, cannot become negative. But the formulæ (α) and (β) may. From which we can integrate, when $F \propto \frac{1}{x^{\frac{1}{2}}}, \frac{1}{x^{\frac{2}{3}}}, \frac{1}{x^{\frac{3}{4}}}, \frac{1}{x^{\frac{4}{5}}}$, &c.

or when $F \propto \frac{1}{x^{\frac{1}{3}}}, \frac{1}{x^{\frac{3}{5}}}, \frac{1}{x^{\frac{5}{7}}}, \frac{1}{x^{\frac{7}{9}}}$, &c.

299. *When the force* $\propto x^n$, *find* $\propto^n$. *of times from different altitudes to the center of force. Find the same, force* $\propto^g \frac{1}{x^n}$.

$$F \propto x^n, \therefore v\,dv = -g\mu x^n\,dx,$$

$$\therefore v^2 = \frac{2g\mu}{n+1} \cdot (a^{n+1} - x^{n+1})$$

$$\therefore dt = -\frac{dx}{v} \propto \frac{-dx}{\sqrt{a^{n+1} - x^{n+1}}} \text{ which is of } -\frac{n+1}{2} \text{ dimensions,}$$

$$\therefore t \text{ will be of } -\frac{n-1}{2} \text{ dimensions.}$$

and when $x = 0$, t will $\propto \frac{1}{a^{\frac{n-1}{2}}}$.

$$F \propto \frac{1}{x^n}, \therefore t \propto \frac{1}{a^{-\frac{n-1}{2}}} \propto a^{\frac{n+1}{2}}.$$

$$t \propto \int \frac{-dx}{\sqrt{a^{n+1} - x^{n+1}}} \propto \frac{1}{a^{\frac{n+1}{2}}} \int \frac{-dx}{\sqrt{1 - \left(\frac{x}{a}\right)^{n+1}}}$$

$$\propto \frac{1}{a^{\frac{n+1}{2}}} \int -dx \left\{1 - \left(\frac{x}{a}\right)^{n+1}\right\}^{-\frac{1}{2}}$$

$$\propto \frac{1}{a^{\frac{n+1}{2}}} \int -dx. \left\{1 + \frac{1}{2} \cdot \left(\frac{x}{a}\right)^{n+1} + \frac{1.3}{2.4} \cdot \left(\frac{x}{a}\right)^{2.\overline{n+1}} + \&c.\right\}$$

$$\propto \frac{1}{a^{\frac{n+1}{2}}} \left(C - \left\{x + \frac{1}{2} \cdot \frac{x^{n+2}}{n+2\,a^{n-1}} + \frac{1.3}{2.4} \cdot \frac{x^{2n+3}}{2n+3 \,.\, a^{2n-2}} + \&c.\right\}\right)$$

when $t = 0$, $x = a$,

$$\therefore C = \left\{a + \frac{1}{2} \cdot \frac{a}{n+2} + \frac{1.3}{2.4} \cdot \frac{a}{2n+3} + \&c.\right\}$$

$$= a\left\{1 + \frac{1}{2} \cdot \frac{1}{n+2} + \frac{1.3}{2.4} \cdot \frac{1}{2n+3} + \&c.\right\}$$

$$\therefore t \propto \frac{1}{a^{\frac{n+1}{2}}} \cdot \left(a \cdot \left\{1 + \frac{1}{2} \cdot \frac{1}{n+2} + \&c.\right\} - \left\{x + \frac{1}{2} \cdot \frac{x^{n+2}}{n+2} \cdot \frac{1}{a^{n-1}} + \&c.\right\}\right)$$

$$\therefore \text{when } x = 0,\ t \propto \frac{a}{a^{\frac{n+1}{2}}} \propto \frac{1}{a^{\frac{n-1}{2}}}$$

when n is negative $t \propto \frac{1}{a^{\frac{-n-1}{2}}} \propto a^{\frac{n+1}{2}}$.

COR. If n be positive and greater than 1, the greater the altitude, the less the time to the center.

300. *A body is projected up* P A *with the velocity* V *from the given point* A, *force in* S $\propto x^n$, *find the height to which the body will rise.*

P

$$v\,dv = -g\mu x^n\,dx,$$

for the velocity decreases as x increases, A

$$\therefore v^2 = \frac{2g\mu}{n+1}\cdot x^{n+1} + C$$

when $v = V$, $x = a$,

$$\therefore C = V^2 + \frac{2g\mu}{n+1}\cdot a^{n+1}$$ S

$$\therefore \frac{2g\mu}{n+1}\cdot(x^{n+1} - a^{n+1}) = V^2 - v^2$$

Let $v = 0$,

$$\therefore \frac{2g\mu}{n+1}\cdot(x^{n+1} - a^{n+1}) = V^2$$

$$\therefore x^{n+1} - a^{n+1} = \frac{V^2.\overline{n+1}}{2g\mu}$$

$$\therefore x^{n+1} = \frac{V^2.\overline{n+1} + 2g\mu.\overline{a^{n+1}}}{2g\mu}$$

$$\therefore x = \left(\frac{V^2.\overline{n+1} + 2g\mu.\overline{a^{n+1}}}{2g\mu}\right)^{\frac{1}{n+1}}.$$

COR. Let $n = -2$, and $V =$ the velocity down $\frac{a}{2}$, force at A constant, $=$ velocity in the circle distance S A.

$$\therefore x = \left(\frac{-V^2 + \frac{2g\mu}{a}\cdot}{2g\mu}\right)^{-1} = \frac{2g\mu}{\frac{2g\mu}{a} - V^2}$$

$$= \frac{2g\mu}{\frac{2g\mu}{a} - \frac{g\mu}{a^2}.a} = \frac{2}{\frac{2}{a} - \frac{1}{a}} = 2a.$$

SECTION VIII.

301. Prop. XLI. Resolving the centripetal force I N or D E (F) into the tangential one I T (F′) and the perpendicular one T N, we have (46)

$$I N : I T :: F : F' :: \frac{d v}{d t} : \frac{d v'}{d t'}$$

$$\therefore d v : d v' :: d t \times I N : d t' \times I T.$$

But since (46)

$$d t = \frac{d s}{v}, \; d t' = \frac{d s'}{v'}$$

and by hypothesis

$$v = v'$$

$$\therefore d t : d t' :: d s : d s' :: I N : I K$$

$$\therefore d v : d v' :: I N^2 : I K \times I T$$

$$:: 1 \quad : 1$$

or

$$d v = d v',$$

&c. &c.

OTHERWISE.

302. By 46, we have generally

$$v \, d v = g \, F \, d s$$

s being the direction of the force F. Hence if s′ be the straight line and s the trajectory, &c. we have

$$v \, d v = g \, F \, d s$$

$$v' \, d v' = g \, F' \, d s'$$

$$\therefore v^2 - V^2 = 2 g \int F \, d s$$

$$v'^2 - V'^2 = 2 g \int F' \, d s'$$

V and V′ being the given values of v and v′ at given distances by which the integrals are corrected.

Now since the central body is the same at the same distance the central force must be the same in both curve and line. Therefore, resolving F

when at the distance s into the tangential and perpendicular forces, we have

$$F' = F \times \frac{I\,T}{I\,N} = F \times \frac{I\,N}{I\,K}$$

$$= F \times \frac{d\,s}{d\,s'}$$

$$\therefore F'\,d\,s' = F\,d\,s$$

and

$$v'^2 - V'^2 = 2\,g \int F\,d\,s = v^2 - V^2$$

which shows that if the velocities be the same at any two equal distances, they are equal at all equal distances — i. e. if

$$V = V'$$

then

$$v = v'.$$

303. COR. 2. By Prop. XXXIX,

$$v^2 \propto A\,B\,G\,E.$$

But in the curve

$$y \propto F \propto A^{n-1}$$

$$\therefore y\,d\,x \propto A^{n-1}\,d\,A$$

Therefore (112)

$$A\,B\,G\,E = \int y\,d\,x \propto -\frac{A^n}{n} + C$$

$$\propto \frac{P^n - A^n}{n}$$

Hence

$$v^2 \propto P^n - A^n.$$

OTHERWISE.

304. Generally (46)

$$v\,d\,v = -g\,F\,d\,s$$

and if

$$F = \mu\,s^{n-1}$$

then

$$v^2 = \frac{2\,g\,\mu}{n}(C - s^n)$$

But when $v = 0$, let $s = P$; then

$$0 = \frac{2\,g\,\mu}{n}(C - P^n)$$

and

$$C = P^n.$$

$$\therefore v^2 = \frac{2g\mu}{n}(P^n - s^n)$$

in which s is any quantity whatever and may therefore be the radius vector of the Trajectory A; that is

$$v^2 = \frac{2g\mu}{n}(P^n - A^n) \text{ or } = \frac{2g\mu}{n}(D^n - \varrho^n)$$

in more convenient notation.

N. B. From this formula may be found the spaces through which a body must fall *externally* to acquire the velocity in the curve (286, &c.)

305. Prop. XLI. *Given the centripetal force to construct the Trajectory, and to find the time of describing any portion of it.*

By Prop. XXXIX,

$$v = \sqrt{2g}.\sqrt{ABFD} = \frac{ds}{dt}(46) = \frac{IK}{dt}$$

But

$$dt = ICK \times \frac{\text{Time}}{\text{Area}} = IC \times KN \times \frac{\text{Time}}{2\,\text{Area}}$$

$$= \frac{A \times KN}{P \times V} \text{ (P being the perpendicular upon the}$$

tangent when the velocity is V. See 125, &c.)

Moreover, if V be the velocity at V, by Prop. XXXIX,

$$V = \sqrt{2g}.\sqrt{ABLV}.$$

Whence

$$\sqrt{ABFD} = \frac{P\sqrt{ABLV}}{A} \times \frac{IK}{KN}$$

$\therefore$ putting

$$Z = \frac{P\sqrt{ABLV}}{A}\left(\text{or } = \frac{Q}{A} = \frac{P \times V}{\sqrt{2gA}}\right) \quad . \quad . \quad (1)$$

we have

$$ABFD : Z^2 :: IK^2 : KN^2$$

$$\therefore ABFD - Z^2 ; Z^2 :: IK^2 - KN^2 : KN^2$$

and

$$\sqrt{ABFD - Z^2} : Z = \frac{Q}{A} :: IN : KN$$

$$\therefore A \times KN = \frac{Q \times IN}{\sqrt{(ABFD - Z^2)}} \quad . \quad . \quad . \quad . \quad (2)$$

Also since similar triangles are to one another in the duplicate ratio of their homologous sides

$$YX \times XC = A \times KN \times \frac{CX^2}{A^2}$$

$$= \frac{Q \times CX^2 \times IN}{A^2 \sqrt{(ABFD - Z^2)}} \quad . \quad . \quad . \quad (3)$$

and putting

$$y = Db = \frac{Q}{2\sqrt{(ABFD - Z^2)}}$$

and

$$y' = Dc = \frac{Q \times CX^2}{2A^2\sqrt{(ABFD - Z^2)}}.$$

Then

$$\left.\begin{aligned} \text{Area } VCI &= \int y\,dx = VDba \\ \text{Area } VCX &= \int y'\,dx = VDca \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

Now (124)

$$t = \frac{2VCI}{P \times V} = \frac{2VDba}{P \times V}$$

or

$$= \frac{2VDba}{\sqrt{2g}.P \times \sqrt{ABLV}} \quad . \quad . \quad (5)$$

the time of describing V I.

Also, if $\angle VCI = \theta$, we have

$$VDca = VCX = \frac{XV \times CV}{2} = \frac{\theta \times CV^2}{2}$$

$$\theta = \frac{2VDca}{P^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which gives the Trajectory.

306. *To express equations* (5) *and* (6) *in terms of* ϱ *and* θ, ($\varrho = A$).

First

$$ABFD = \frac{v^2}{2g}$$

$$ABLV = \frac{V^2}{2g}$$

and

$$Q = \frac{P \times V}{\sqrt{2g}}$$

$$\therefore Z^2 = \frac{Q^2}{\varrho^2} = \frac{P^2 \times V^2}{2g\varrho^2}$$

$$\therefore ABFD - Z^2 = \frac{v^2}{2g} - \frac{P^2 \times V^2}{2g\varrho^2}$$

$$= \frac{1}{2g\varrho^2} \times (\varrho^2 v^2 - P^2 V^2)$$

Hence

$$y = \frac{P \times V\varrho}{2\sqrt{(\varrho^2 v^2 - P^2 V^2)}}$$

and

$$y' = \frac{P^3 \times V}{2\varrho\sqrt{(\varrho^2 v^2 - P^2 V^2)}}$$

$$\therefore VDba = \frac{PV}{2}\int \frac{\varrho d\varrho}{\sqrt{(\varrho^2 v^2 - P^2 V^2)}}$$

and

$$VDca = \frac{P^3 V}{2} \times \int \frac{d\varrho}{\varrho\sqrt{(\varrho^2 v^2 - P^2 V^2)}}$$

$$\therefore t = \int \frac{\varrho d\varrho}{\sqrt{(\varrho^2 v^2 - P^2 V^2)}}$$

But by Prop. XL.

$$v^2 = 2\int g F d\varrho$$

the integral being taken from $v = 0$, or from $\varrho = D$, D being the same as P in 304.

$$\therefore t = \int \frac{\int \varrho d\varrho}{\sqrt{(2\varrho^2 \int g F d\varrho - P^2 V^2)}}, \text{ or } = \int \frac{\int \varrho d\varrho}{\sqrt{(\varrho^2 v^2 - P^2 V^2)}} \quad . . \quad (7)$$

$$\theta = \int \frac{P \times V d\varrho}{\varrho\sqrt{(-2\varrho^2 \int g F d\varrho - P^2 V^2)}}, \text{ or} = \int \frac{PV d\varrho}{\varrho\sqrt{(\varrho^2 v^2 - P^2 V^2)}} \quad . \quad (8)$$

307. *To find* t *and* θ *in terms of* ϱ *and* p.

Since (125)

$$v^2 = \frac{P^2 V^2}{p^2} = -2\int g F d\varrho$$

$$\therefore t = \int \frac{\int p\varrho d\varrho}{\sqrt{(P^2 V^2 \varrho^2 - P^2 V^2 p^2)}}$$

$$= \frac{1}{PV} \times \int \frac{\varrho d\varrho}{\sqrt{\left(\frac{\varrho^2}{p^2} - 1\right)}} \quad . . . . . \quad (9)$$

and

$$\theta = \int \frac{d\varrho}{\varrho\sqrt{\left(\frac{\varrho^2}{p^2} - 1\right)}} \quad . . . . . \quad (10)$$

But previous to using these forms we must find the equation to the trajectory, thus (139)

$$\frac{P^2 V^2}{g} \times \frac{dp}{p^3 d\varrho} = F = f(\varrho)$$

f denoting the law of force.

$$\therefore \frac{P^2 V^2}{2g} \times \left(\frac{1}{P^2} - \frac{1}{p^2}\right) = \int d\varrho f\varrho$$

or

$$p^2 = \frac{P^2 V^2}{V^2 - 2g\int d\varrho\, f\varrho} \quad \ldots\ldots \quad (11)$$

308. To these different methods the following are examples:

1st. Let $F \propto \varrho = \mu\varrho$. Then (see 304)

$$\therefore v^2 = g\mu(D^2 - \varrho^2)$$

and if P and V belong to an apse or when $P = \varrho$;

$$V^2 = g\mu(D^2 - P^2)$$

$$\therefore t = \frac{1}{\sqrt{g\mu}} \times \int \frac{\varrho\, d\varrho}{\sqrt{\{\varrho^2(D^2-\varrho^2) - P^2(D^2-P^2)\}}}.$$

Let $\varrho^2 - \frac{D^2}{2} = u$. Then we easily get

$$2t\sqrt{g\mu} = \int \frac{du}{\sqrt{\left\{\left(P^2 - \frac{D^2}{2}\right)^2 - u^2\right\}}}$$

$$= \text{sin.}^{-1} \frac{u}{P^2 - \frac{D^2}{2}} + C$$

and making $t = 0$ at an apse or when $\varrho = P$, we find

$$C = -\text{sin.}^{-1} \frac{P^2 - \frac{D^2}{2}}{P^2 - \frac{D^2}{2}} = -\text{sin.}^{-1} 1$$

$$= -\frac{\pi}{2}.$$

$$\therefore t = \frac{1}{2\sqrt{g\mu}} \times \left\{\text{sin.}^{-1} \frac{\varrho^2 - \frac{D^2}{2}}{P^2 - \frac{D^2}{2}} - \frac{\pi}{2}\right\} \quad \ldots \quad (1)$$

Also

$$\frac{\theta}{PV} = \int \frac{dt}{\varrho^2} = \frac{1}{2\sqrt{g\mu}} \int \frac{du}{\left(u + \frac{D^2}{2}\right)\sqrt{\left\{\left(P^2 - \frac{D^2}{2}\right)^2 - u^2\right\}}}$$

and assuming

$$P^2 - \frac{D^2}{2} - u = v^2 \times \left(P^2 - \frac{D^2}{2} + u\right)$$

we get

$$\frac{\theta}{PV} = \frac{1}{2\sqrt{g\mu}\, P.\sqrt{D^2 - P^2}} \times \left\{\text{sin.}^{-1} \frac{\left(P^2 - \frac{D^2}{2}\right)^2 + \frac{D^2}{2}\left(\varrho^2 - \frac{D^2}{2}\right)}{\varrho^2\left(P^2 - \frac{D^2}{2}\right)} + C\right\}$$

and making $\theta = 0$, when $\varrho = P$ we find

$$C = -\sin.^{-1} 1 = -\frac{\pi}{2}.$$

Also

$$V = \sqrt{g\mu}.\ \sqrt{(D^2 - P^2)}$$

$$\therefore 2\theta + \frac{\pi}{2} = \sin.^{-1} \frac{\left(P^2 - \frac{D^2}{2}\right)^2 + \frac{D^2}{2}\left(\varrho^2 - \frac{D^2}{2}\right)}{\varrho^2\left(P^2 - \frac{D^2}{2}\right)}$$

$$\therefore \frac{\left(P^2 - \frac{D^2}{2}\right)^2 + \frac{D^2}{2}\left(\varrho^2 - \frac{D^2}{2}\right)}{\varrho^2\left(P^2 - \frac{D^2}{2}\right)} = \sin.\left(2\theta + \frac{\pi}{2}\right)$$

$$= \cos. 2\theta = 2\cos.^2\theta - 1$$

which gives

$$\varrho^2 = \frac{P^2(D^2 - P^2)}{P^2 - (2P^2 - D^2)\cos.^2\theta} \quad \ldots\ldots\ldots \quad (2)$$

Now the equation to the ellipse, ϱ and θ being referred to its center, is

$$\varrho^2 = \frac{b^2}{1 - e^2\cos.^2\theta}.$$

Therefore the trajectory is an ellipse the center of force being in its center, and we have its semiaxes from

$$b^2 = D^2 - P^2$$

$$e^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = \frac{2P^2 - D^2}{P^2}$$

viz.

$$\left.\begin{aligned} b &= \sqrt{(D^2 - P^2)} \\ \text{and} \quad a &= P \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (3)$$

which latter value was already assumed.

To find the Periodic time.

From (3) it appears that when

$$t = \frac{T}{4}, \text{ or } \theta = \frac{\pi}{2}, \varrho = b = \sqrt{(D^2 - P^2)}$$

and substituting in (1) we have

$$\frac{T}{4} = \frac{1}{2\sqrt{g\mu}} \times \left\{\sin.^{-1} \frac{\frac{D^2}{2} - P^2}{P^2 - \frac{D^2}{2}} - \frac{\pi}{2}\right\}$$

$$= \frac{1}{2\sqrt{g\mu}} \times \left\{\sin.^{-1}(-1) - \frac{\pi}{2}\right\}$$

But

$$\sin.^{-1}(-1) = \frac{3\pi}{2}.$$

$$\therefore \frac{T}{4} = \frac{\pi}{2\sqrt{g\mu}}$$

and

$$T = \frac{2\pi}{\sqrt{g\mu}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

which has already been found otherwise (see 147).

To apply (9) and (10) of 307 to this example we must first integrate (11) where $f\varrho = \mu\varrho$; that is since

$$\int \mu\varrho\, d\varrho = \frac{\mu}{2}\varrho^2 + C$$

$$= \frac{\mu}{2}\varrho^2 - \frac{\mu}{2}P^2$$

we have

$$p^2 = \frac{P^2V^2}{V^2 - \mu g\varrho^2 + \mu g P^2}.$$

But

$$V^2 = g\mu(D^2 - P^2)$$

$$\therefore p^2 = \frac{P^2.(D^2 - P^2)}{D^2 - \varrho^2} \quad \ldots\ldots\ldots \quad (5)$$

which also indicates an ellipse referred to its center, the equation being generally

$$p^2 = \frac{a^2b^2}{a^2 + b^2 - \varrho^2}.$$

Hence

$$\frac{\varrho^2}{p^2} - 1 = \frac{\varrho^2(D^2 - \varrho^2) - P^2(D^2 - P^2)}{P^2(D^2 - P^2)}$$

$$\therefore t = \frac{1}{\sqrt{g\mu}} \int \frac{\varrho\, d\varrho}{\sqrt{\{\varrho^2(D^2 - \varrho^2) - P^2(D^2 - P^2)\}}}$$

the same as before.

With regard to θ, the axes of the ellipse being known from (5) we have the polar equation, viz.

$$\varrho^2 = \frac{b^2}{1 - e^2\cos.^2\theta}.$$

309. Ex. 2. Let $F = \frac{\mu}{\varrho^2}$. Then (304)

$$v^2 = \frac{2g\mu}{-1} \times (D^{-1} - \varrho^{-1})$$

$$= 2 g \mu \times \frac{D - \varrho}{D \varrho}$$

and

$$V^2 = 2 g \mu \times \frac{D - P}{D P}$$

P and V belonging to an apse.

$$\therefore t = \int \frac{\sqrt{D}}{\sqrt{2 g \mu}} \times \frac{\varrho \, d \varrho}{\sqrt{(D \varrho - \varrho^2 - D P + P^2)}}$$

which, adding and subtracting $\frac{D^2}{4}$, transforms to

$$t = \frac{\sqrt{D}}{\sqrt{2 g \mu}} \times \int \frac{\varrho \, d \varrho}{\sqrt{\left\{ \left(P - \frac{D}{2} \right)^2 - \left(\varrho - \frac{D}{2} \right)^2 \right\}}}$$

and making $\varrho - \frac{D}{2} = u$

$$t = \frac{\sqrt{D}}{\sqrt{2 g \mu}} \times \int \frac{\left(u + \frac{D}{2} \right) d u}{\sqrt{\left\{ \left(P - \frac{D}{2} \right)^2 - u^2 \right\}}}$$

$$= \sqrt{\frac{D}{2 g \mu}} \times \left\{ C - \sqrt{\left\{ \left(P - \frac{D}{2} \right)^2 - u^2 \right\}} + \frac{D}{2} \sin.^{-1} \frac{u}{P - \frac{D}{2}} \right\}$$

(see 86).

Let $t = 0$, when $\varrho = P$. Then

$$C = - \frac{D}{2} \sin.^{-1} 1 = - \frac{D}{2} \times \frac{\pi}{2} .$$

$$\therefore t = \sqrt{\frac{D}{2 g \mu}} \times \left\{ \frac{D}{2} \left(\sin.^{-1} \frac{\varrho - \frac{D}{2}}{P - \frac{D}{2}} - \frac{\pi}{2} \right) - \sqrt{\left\{ \left(P - \frac{D}{2} \right)^2 - \left(\varrho - \frac{D}{2} \right)^2 \right\}} \right\} \quad (1)$$

Also

$$\frac{\theta}{PV} = \int \frac{d t}{\varrho^2} = \sqrt{\frac{D}{2 g \mu}} \times \int \frac{d u}{\left(u + \frac{D}{2} \right) \sqrt{\left\{ \left(P - \frac{D}{2} \right)^2 - u^2 \right\}}} .$$

But assuming

$$P - \frac{D}{2} - u = v^2 \times \left(P - \frac{D}{2} + u \right)$$

the above becomes rationalized, and we readily find

$$\int \frac{du}{\left(u+\frac{D}{2}\right)\sqrt{\left\{\left(P-\frac{D}{2}\right)^2-u^2\right\}}} =$$

$$\frac{1}{\sqrt{P.(D-P)}} \times \left\{ \tan.^{-1} \frac{\left(P-\frac{D}{2}\right)^2+\frac{D}{2}u}{\sqrt{P.(D-P)}\times\sqrt{\left\{\left(P-\frac{D}{2}\right)^2-u^2\right\}}} + C \right\}$$

and making $\theta = 0$, when $\varrho = P$, or when $u = P - \frac{D}{2}$, we get

$$C = -\tan.^{-1}\frac{1}{0} = -\frac{\pi}{2}.$$

Hence, since moreover

$$V = \sqrt{2g\mu}.\sqrt{\frac{D-P}{DP}}$$

$$\theta + \frac{\pi}{2} = \tan.^{-1} \frac{\left(P-\frac{D}{2}\right)^2+\frac{D}{2}\left(\varrho-\frac{D}{2}\right)}{\sqrt{P.(D-P)}\times\sqrt{\left\{\left(P-\frac{D}{2}\right)^2-\left(\varrho-\frac{D}{2}\right)^2\right\}}}$$

or

$$= \sin.^{-} \frac{\left(P-\frac{D}{2}\right)^2-\frac{D}{2}\left(\varrho-\frac{D}{2}\right)}{\left(P-\frac{D}{2}\right)\varrho}$$

$$\therefore \frac{\left(P-\frac{D}{2}\right)^2+\frac{D}{2}\cdot\left(\varrho-\frac{D}{2}\right)}{\left(P-\frac{D}{2}\right)\varrho} = \sin.\left(\theta+\frac{\pi}{2}\right) = \cos.\theta$$

$$\therefore \varrho = \frac{P^2-PD}{-\frac{D}{2}+\left(P-\frac{D}{2}\right)\cos.\theta}$$

$$= \frac{2P.(D-P)}{D} \times \frac{1}{1+\left(1-\frac{2P}{D}\right)\cos.\theta} \quad \ldots\ldots (2)$$

But the equation to the ellipse referred to its focus is

$$\varrho = \frac{b^2}{a} \times \frac{1}{1+e\cos.\theta}$$

$$\therefore \frac{b^2}{a} = \frac{2P(D-P)}{D}$$

and

$$e^2 = \frac{a^2-b^2}{a^2} = 1-\frac{b^2}{a^2} = \left(1-\frac{2P}{D}\right)^2$$

$$\therefore \frac{b^2}{a^2} = \frac{4P}{D} - \frac{4P^2}{D^2} = \frac{4P}{D^2} \times (D - P)$$

$$= \frac{b^2}{a} \times \frac{2}{D}$$

$$\therefore a = \frac{D}{2}$$

and

$$b = \sqrt{P \times (D - P)}$$

. . . . . (3)

To find the Periodic Time; let $\theta = \pi$. Then $\varrho = 2a - P = D - P$, and equation (1) gives

$$\frac{T}{2} = \sqrt{\frac{D}{2g\mu}} \times \frac{D}{2} \times \left(\sin.^{-1} - 1 - \frac{\pi}{2}\right)$$

$$= \frac{a^{\frac{5}{2}}}{\sqrt{g\mu}} \times \left(\frac{3\pi}{2} - \frac{\pi}{2}\right)$$

$$\therefore T = \frac{2\pi a^{\frac{5}{2}}}{\sqrt{g\mu}},$$

see 159.

OTHERWISE.

First find the Trajectory by formula (11. 307); then substitute for $\frac{\varrho^2}{P^2}$ in 9 and 10, &c.

310. *Required the Time and Trajectory when* $F = \frac{\mu}{\varrho^3}$.

By 304,

$$v^2 = -g\mu \times (D^{-2} - \varrho^{-2})$$

$$= \frac{g\mu}{D^2} \times \frac{D^2 - \varrho^2}{\varrho^2}$$

$\therefore$ if V and P belong to an apse we have

$$V^2 = \frac{g\mu}{D^2} \times \frac{D^2 - P^2}{P^2}$$

$$\therefore \varrho^2 v - P^2 V^2 = \frac{g\mu}{D^2} \times \{D^2 - \varrho^2 - D^2 + P^2\}$$

$$= \frac{g\mu}{D^2}(P^2 - \varrho^2)$$

$$\therefore t = \frac{D}{\sqrt{g\mu}} \int \frac{\varrho\, d\varrho}{\sqrt{P^2 - \varrho^2}}$$

$$= \frac{D}{\sqrt{g\mu}} \times (C \pm \sqrt{P^2 - \varrho^2})$$

and taking $t = 0$ at an apse or when $\varrho = P$, $C = 0$,

$$t = \frac{D}{\sqrt{g\mu}} \times \sqrt{P^2 - \varrho^2} \quad \ldots\ldots\ldots \quad (1)$$

also

$$\frac{\theta}{PV} = \int \frac{dt}{\varrho^2} = \frac{D}{\sqrt{g\mu}} \times \int \frac{d\varrho}{\varrho \sqrt{(P^2 - \varrho^2)}}$$

But

$$\int \frac{d\varrho}{\varrho \sqrt{(P^2 - \varrho^2)}} = \frac{1}{\pm P} \times \left\{ l. \frac{\sqrt{(P^2 - \varrho^2)} \mp P}{\varrho} + C \right\}$$

and

$$V = \frac{\sqrt{g\mu}}{PD} \times \sqrt{(D^2 - P^2)}.$$

$$\therefore \frac{\pm P\theta}{\sqrt{(D^2 - P^2)}} = l. \frac{\sqrt{(P^2 - \varrho^2)} \mp P}{\varrho} + C$$

and making $\theta = 0$ at the apse or where $\varrho = P$,

$$C = -l. \frac{P}{P} = 0$$

$$\therefore \pm \frac{P\theta}{\sqrt{D^2 - P^2}} = l. \frac{\sqrt{(P^2 - \varrho^2)} \mp P}{\varrho}$$

$$\therefore \pm \frac{P\theta}{e\sqrt{(D^2 - P^2)}} = \frac{\sqrt{(P^2 - \varrho^2)} \mp P}{\varrho}$$

which gives

$$\varrho = \frac{2P e^{\frac{P\theta}{\sqrt{(D^2 - P^2)}}}}{1 + e^{\frac{2P\theta}{\sqrt{(D^2 - P^2)}}}} \quad \ldots\ldots \quad (2)$$

311. *Required the Trajectory and circumstances of motion when*

$$F = \frac{\mu}{\varrho^n}$$

or for any inverse law of the distance.

The readiest method is this; By (11) 307, if r, and P be the values of ϱ and p for the given velocity V (P is no longer an apsidal distance)

$$\frac{P^2 V^2}{p^2} = V^2 + \frac{2\mu}{(n-1) r^{n-1}} \times \frac{r^{n-1} - \varrho^{n-1}}{\varrho^{n-1}} \quad \ldots\ldots \quad (1)$$

the equation to the Trajectory.

Also since

$$v\, dv = -g F\, d\varrho$$

$$\therefore v^2 = \frac{2g\mu}{(n-1)\varrho^{n-1}} \text{ (from } \infty \text{ to } \varrho)$$

Hence

$$V^2 = \frac{2g\mu}{(n-1)r^{n-1}} \text{ (from } \infty \text{ to } \varrho)$$

and if we put

$$\dot{V}^2 = \frac{2mg\mu}{(n-1)r^{n-1}}$$

in which m may be $>$ = or $<$ 1 we easily get

$$\left.\begin{aligned} p &= \sqrt{\frac{m}{m-1}} \times \frac{P\varrho^{\frac{n-1}{2}}}{\sqrt{\left(\varrho^{n-1} + \frac{r^{n-1}}{m-1}\right)}} \;\ldots m > 1 \\ p &= \frac{P}{r^{\frac{n-1}{2}}} \times \varrho^{\frac{n-1}{2}} \;\ldots\ldots\ldots\; m = 1 \\ p &= \frac{\sqrt{\frac{m}{m-1}} \times P\varrho^{\frac{n-1}{2}}}{\sqrt{\left(\frac{r^{n-1}}{1-m} - \varrho^{n-1}\right)}} \;\ldots\; m < 1 \end{aligned}\right\} \quad \ldots \quad (2)$$

To find θ on this hypothesis.

We have (307)

$$d\theta = \frac{rp\,d\varrho}{\varrho\sqrt{(\varrho^2 - p^2)}}$$

which gives by substitution

$$d\theta = \pm r\sqrt{\frac{m}{m-1}}\,P \times \frac{\varrho^{\frac{n-3}{2}}\,d\varrho}{\varrho\sqrt{\left(\varrho^{n-1} - \frac{m}{m-1}P^2\varrho^{\frac{n-3}{2}} + \frac{r^{n-1}}{m-1}\right)}} \;\ldots m > 1$$

$$d\theta = \pm \frac{r\varrho^{\frac{n-5}{2}}\,d\varrho}{\sqrt{\left(\frac{r^{n-1}}{P^2} - \varrho^{n-3}\right)}} \;\ldots\ldots\ldots\ldots\; m = 1$$

$$d\theta = \pm r\sqrt{\frac{m}{1-m}} \times P \times \frac{\varrho^{\frac{n-3}{2}}\,d\varrho}{\varrho\sqrt{\left(\frac{r^{n-1}}{1-m} - \frac{m}{1-m}P^2\varrho^{n-3} - \varrho^{n-1}\right)}}$$

the positive or negative sign being used according as the body ascends or descends.

Ex. If n = 2, we get

$$p = \sqrt{\frac{m}{m-1}}\,P \times \frac{\varrho}{\sqrt{\left(\varrho^2 + \frac{r}{m-1}\varrho\right)}} \;\ldots\ldots\; m > 1$$

$$p = \frac{P}{r^{\frac{1}{2}}} \cdot \varrho^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad m = 1$$

$$p = \sqrt{\frac{m}{1-m}}\, P \cdot \frac{\varrho}{\sqrt{\left(\frac{r\varrho}{1-m} - \varrho^2\right)}} \quad \ldots\ldots \quad m < 1$$

the equations to the ellipse, parabola and hyperbola respectively.

Also we have correspondingly

$$d\theta = \pm \frac{r\sqrt{\frac{m}{m-1}}\, P\, d\varrho}{\varrho\sqrt{\left(\varrho^2 + \frac{r}{m-1}\varrho - \frac{m}{m-1} \cdot P^2\right)}}$$

$$d\theta = \pm r\, P \cdot \frac{d\varrho}{\varrho\sqrt{(r\varrho - P^2)}}$$

$$d\theta = \pm r\sqrt{\frac{m}{1-m}} \cdot \frac{d\varrho}{\varrho\sqrt{\left(-\frac{m}{1-m}P^2 + \frac{r}{1-m}\varrho - \varrho^2\right)}}$$

which are easily integrated.

Ex. 2. Let n = 3. Then we get

$$p = \sqrt{\frac{m}{m-1}} \times P \times \frac{\varrho}{\sqrt{\left(\varrho^2 + \frac{r^2}{m-1}\right)}} \quad \ldots \quad m > 1$$

$$p = \frac{P}{r}\varrho \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad m = 1$$

$$p = \sqrt{\frac{m}{m-1}} \times P \times \frac{\varrho}{\sqrt{\left(\frac{r^2}{1-m} - \varrho^2\right)}} \quad \ldots \quad m < 1$$

$$d\theta = \pm \sqrt{\frac{m}{m-1}} \cdot Pr \times \frac{d\varrho}{\varrho\sqrt{\left(\varrho^2 - \frac{mP^2 - r^2}{m-1}\right)}} \cdot m > 1$$

$$d\theta = \pm \frac{Pr}{\sqrt{(r^2 - P^2)}} \cdot \frac{d\varrho}{\varrho} \quad \ldots\ldots\ldots \quad m = 1$$

$$d\theta = \pm \sqrt{\frac{m}{1-m}} \times r\, P \times \frac{d\varrho}{\varrho\sqrt{\left(\frac{r^2 - mP^2}{1-m} - \varrho^2\right)}} \cdot m < 1$$

312. In the first of these values of θ, $m P^2$ may be $>$ $=$ or $< r^2$.

(1). Let $m P^2 > r^2$. Then (see 86)

$$\theta - \alpha = \pm \sqrt{\frac{m}{mP^2 - r^2}} \times P\, r \left(\sec.^{-1}\varrho\sqrt{\frac{m-1}{mP^2 - r^2}} - \sec.^{-1} r\sqrt{\frac{m-1}{mP^2 - r^2}}\right)$$

and at an apse or when $r = P$

$$\theta = \pm \sqrt{\frac{m}{m-1}} \times P \times \sec.^{-1}\frac{\varrho}{P} \quad \ldots \quad (a)$$

for

$$\sqrt{\frac{m-1}{mP^2-r^2}} = \frac{1}{P} \text{ or } = \frac{1}{r}.$$

(2) Let $mP^2 = r^2$. Then we have

$$p = \frac{\frac{r}{\sqrt{(m-1)}}\varrho}{\sqrt{\left(\varrho^2 + \frac{r^2}{m-1}\right)}} \quad \ldots\ldots\ldots \text{(b)}$$

$$\theta - \alpha = \pm \frac{r^2}{\sqrt{(m-1)}} \times \int \frac{d\varrho}{\varrho^2}$$

$$= \pm \frac{r^2}{\sqrt{m-1}} \times \left(\frac{1}{r} - \frac{1}{\varrho}\right)$$

$$= \pm \frac{r}{\sqrt{m-1}} \times \frac{\varrho - r}{\varrho} \quad \ldots\ldots \text{(c)}$$

which indicates the *Reciprocal* or *Hyperbolic Spiral.*

(3) Let mP^2 be $< r^2$. Then

$$p = \frac{\sqrt{\frac{m}{1-m}}P\varrho}{\sqrt{\left(\frac{r^2}{1-m} + \varrho^2\right)}} \quad \ldots\ldots\ldots \text{(d)}$$

$$\theta = \pm r P \sqrt{\frac{m}{m-1}} \int \frac{d\varrho}{\varrho \sqrt{\left(\varrho^2 + \frac{r^2 - mP^2}{m-1}\right)}}$$

$$= \pm rP\sqrt{\frac{m}{r^2-mP^2}} \times \frac{\text{l.}\dfrac{\sqrt{\left(\varrho^2+\frac{r^2-mP^2}{m-1}\right)}-\sqrt{\frac{r^2-mP^2}{m-1}}}{\varrho}}{} + C$$

$$= \pm rP\sqrt{\frac{m}{r^2-mP^2}} \times \text{l.}\frac{r}{\varrho}\frac{\sqrt{(m-1).\varrho^2+r^2-mP^2)}-\sqrt{(r^2-mP^2)}}{\sqrt{m.(r^2-P^2)}-\sqrt{(r^2-mP^2)}} \quad \ldots \text{(e)}$$

at an apse $r = P$; and then

$$\theta = \pm P \sqrt{\frac{m}{1-m}} \times \text{l.}\frac{\sqrt{(r^2-\varrho^2)}-r}{\varrho} \quad \ldots \text{(f)}$$

Thus the first of the values of θ has been split into three, and integrating the other two we also get

$$\theta - \alpha = \pm \frac{Pr}{\sqrt{(r^2-P^2)}} \times (\text{l}\,\varrho - \text{l}\,r)$$

$$= \pm \frac{Pr}{\sqrt{(r^2-P^2)}} \times \text{l.}\frac{\varrho}{r}$$

$$\theta - \alpha = \pm r P \sqrt{\frac{m}{1-m}} \int \frac{d\varrho}{\varrho \sqrt{\left(\frac{r^2-mP^2}{1-m} - \varrho^2\right)}}$$

$$=\pm rP\sqrt{\frac{m}{r^2-mP^2}}\times l.\frac{\sqrt{\left(\frac{r^2-mP^2}{1-m}-\varrho^2\right)}\sqrt{\frac{r^2-mP^2}{1-m}}}{\varrho}+C$$

$$=\pm rP\sqrt{\frac{m}{r^2-mP^2}}\times l.\frac{r}{\varrho}\frac{\sqrt{(r^2-mP^2-1-m.\varrho^2)}-\sqrt{(r^2-mP^2)}}{\sqrt{(m.r^2-\varrho^2)}-\sqrt{(r^2-mP^2)}}$$

and if θ is measured from an apse or $r = P$ it reduces to

$$\theta = \pm P\sqrt{\frac{m}{1-m}}l.\frac{r+\sqrt{\varrho^2-r^2}}{\varrho}.$$

313. Hence recapitulating we have these pairs of equations, viz.

$$(1)\, p=\frac{P\sqrt{\frac{m}{m-1}}\varrho}{\sqrt{\left(\varrho^2+\frac{r^2}{m-1}\right)}}$$

$$\theta-\alpha=\pm Pr\sqrt{\frac{m}{mP^2-r^2}}\times\left(\sec.^{-1}\varrho\sqrt{\frac{m-1}{mP^2-r^2}}-\sec.^{-1}\sqrt{\frac{m-1}{mP^2-r^2}}\right)$$

or

$$\theta=\pm P\sqrt{\frac{m}{m-1}}\times\sec.^{-1}\frac{\varrho}{P}.$$

To construct the Trajectory,

put $\theta = 0$, then

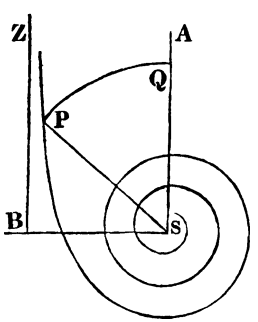
$\varrho = P = S A$;

let $\varrho = \infty$, then

$$p=\pm P\sqrt{\frac{m}{m-1}}$$

and

$$\theta=\pm\frac{P\pi}{2}\sqrt{\frac{m}{m-1}};$$



and taking A S B, A S B′ for these values of θ, and S B, S B′ for those of p and drawing B Z, B′ Z′ at right angles we have two asymptotes; S C being found by putting $\theta = \pi$. Thus and by the rules in (35, 36, 37, 38.) the curve may be traced in all its ramifications.

2. $$p=\frac{r}{\sqrt{(m-1)}}\times\frac{\varrho}{\sqrt{\left(\varrho^2+\frac{r^2}{m-1}\right)}}$$

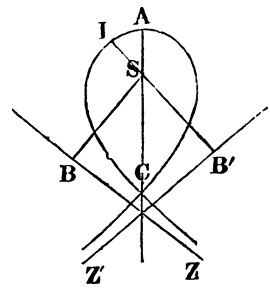
and

$$\theta-\alpha=\pm\frac{r}{\sqrt{(m-1)}}\times\frac{\varrho-r}{\varrho}$$

This equation becomes more simple when we make θ originate from $\varrho = \infty$; for then it is

$$\theta = \frac{r^2}{\sqrt{(m-1)}} \times \frac{1}{\varrho}$$

and following the above hinted method the curve, viz. the *Reciprocal Spiral*, may easily be described as in the annexed diagram.



3. $$p = P\sqrt{\frac{m}{1-m}} \times \frac{\varrho}{\sqrt{\left(\frac{r^2}{1-m} + \varrho^2\right)}}$$

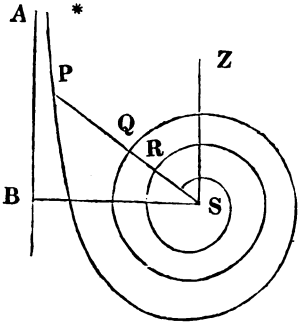
and

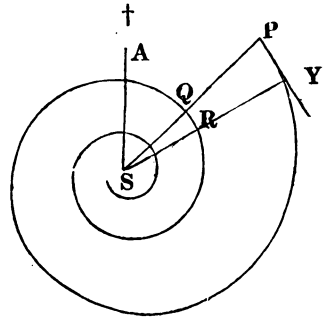
$$\theta - \alpha = \pm r\,P\sqrt{\frac{m}{r^2 - m\,P^2}} \times 1.\frac{r}{\varrho}.\frac{\sqrt{(m-1)\varrho^2 + r^2 - mP^2)} - \sqrt{(r^2 - mP^2}}{\sqrt{m\,(r^2 - P^2)} - \sqrt{(r^2 - m\,P^2)}}$$

and when θ is measured from an apse or when $P' = r$

$$\theta = \pm P.\sqrt{\frac{m}{1-m}}.1\,\frac{\sqrt{(r^2 + \varrho^2)} - r}{\varrho}$$

Whence may easily be traced this figure.*





4. $$p = \frac{P}{r}\varrho$$

and

$$\theta - \alpha = \pm \frac{P\,r}{\sqrt{(r^2 - P^2)}} 1.\frac{\varrho}{r}.$$

From which may be described the *Logarithmic Spiral.*†

5. $$p = P\sqrt{\frac{m}{m-1}} \times \frac{\varrho}{\sqrt{\left(\frac{r^2}{1-m} - \varrho^2\right)}}$$

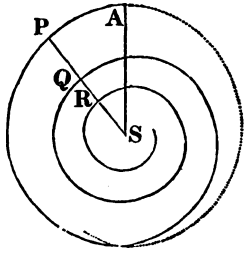
$$\theta - \alpha = \pm r\,P\sqrt{\frac{m}{r^2 - m\,P^2}} \times 1.\frac{r}{\varrho}.\frac{\sqrt{(r^2 - m\,P^2 - 1-m.\varrho^2)} - \sqrt{(r^2 - m\,P^2)}}{\sqrt{(m.r^2 - \varrho^2)} - \sqrt{(r^2 - m\,P^2)}}$$

or

$$\theta = \pm r \sqrt{\frac{m}{1-m}} \, l. \frac{r - \sqrt{(\varrho^2 - r^2)}}{\varrho}$$

when $P = r$.

Whence this spiral.



These several spirals are called *Cotes' Spirals*, because he was the first to construct them as Trajectories.

314. If $n = 4$. Then the Trajectory, &c. are had by the following equations, viz.

$$p = \frac{P \sqrt{\frac{m}{m-1}} \cdot \varrho^{\frac{3}{2}}}{\sqrt{\left(\varrho^3 + \frac{r^3}{m-1}\right)}}$$

$$d\theta = r P \sqrt{\frac{m}{m-1}} \times \frac{d\varrho}{\varrho^{\frac{1}{2}} \sqrt{\left(\varrho^3 - \frac{m}{m-1} P^2 \varrho + \frac{r^3}{m-1}\right)}}$$

315. If $n = 5$. Then

$$p = P \sqrt{m} \frac{\varrho^2}{\sqrt{(m-1 \cdot \varrho^2 + r^4)}}$$

$$d\theta = r P \sqrt{\frac{m}{m-1}} \times \frac{d\varrho}{\sqrt{\left(\varrho^4 - \frac{m}{m-1} P^2 \varrho^2 + \frac{r^4}{m-1}\right)}}$$

which as well as the former expression is not integrable by the usual methods.

When

$$\varrho^4 - \frac{m}{m-1} P^2 \varrho^2 + \frac{r^4}{m-1}$$

is a perfect square, or when

$$\frac{r^4}{m-1} = \frac{m^2 P^4}{4 (m-1)^2}$$

then we have

$$d\theta = \pm r P \sqrt{\frac{m}{m-1}} \times \frac{d\varrho}{\varrho^2 - \frac{m P^2}{2 (m-1)}}$$

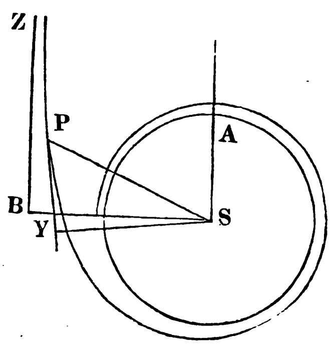
Therefore (87)

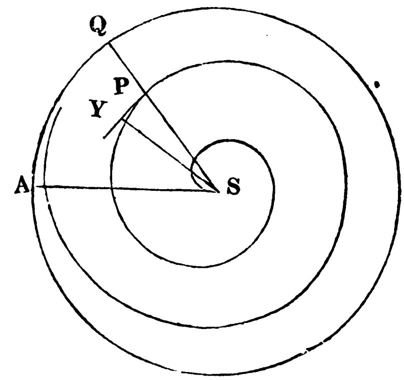
$$\theta - \alpha = r P \sqrt{\frac{m}{m-1}} \times \sqrt{\frac{2(m-1)}{m P^2}} \times l. \frac{\sqrt{\frac{m P^2}{2 (m-1)}} - \varrho}{\sqrt{\left(\varrho^2 - \frac{m P^2}{2(m-1)}\right)}} + C$$

$$\theta - \alpha = r\sqrt{2} \times l . \frac{P\sqrt{m} - \varrho\sqrt{2(m-1)}}{\sqrt{(2.\overline{m-1}.\varrho^2 - m P^2)}}$$

$$\alpha - \theta \text{ or } = r\sqrt{2}\, l . \frac{\varrho\sqrt{2(m-1)} + P\sqrt{m}}{\sqrt{(m P^2 - 2.\overline{m-1}.\varrho^2)}}$$

and these being constructed will be as subjoined.





316. Cor. 1. OTHERWISE.

To find the apses of an orbit where $F = \frac{\mu}{\varrho^n}$.

Let

$$p = \varrho.$$

Then

$$\left.\begin{array}{ll} \varrho^{n-1} - \frac{m}{m-1} P^2 \varrho^{n-3} + \frac{r^{n-1}}{m-1} = 0 & m > 1 \\ \varrho = \frac{r^{\frac{n-1}{n-3}}}{P^{\frac{2}{n-3}}} \ldots\ldots\ldots\ldots & m = 1 \\ \text{and} & \\ \varrho^{n-1} + \frac{m}{1-m} P^2 \varrho^{n-3} - \frac{r^{n-1}}{1-m} \ldots & m < 1 \end{array}\right\} \ldots A$$

which being resolved all the possible values of ϱ will be discovered in each case, and thence by substituting in θ, we get the position as well as the number of apses.

Ex. 1. Let n = 2. Then

$$\varrho^2 + \frac{r}{m-1} . \varrho - \frac{m P^2}{m-1} = 0$$

$$\varrho = \frac{P^2}{r} = \frac{\frac{L}{4} r}{r} = \frac{L}{4}$$

$$\varrho^2 - \frac{r}{1-m} \varrho + \frac{m P^2}{1-m} = 0$$

which give

$$\varrho = -\frac{r}{2(m-1)} \pm \sqrt{\frac{r^2 + 4mP^2.(m-1)}{4(m-1)^2}}$$

$$\varrho = \frac{L}{4}$$

and

$$\varrho = \frac{r}{2(1-m)} \pm \sqrt{\frac{r^2 - 4mP^2.(1-m)}{4.(1-m)^2}}.$$

Whence in the ellipse and hyperbola there are two apses (force in the focus); in the former lying on different sides of the focus; in the latter both lying on the same side of the focus, as is seen by substituting the values of ϱ in those of θ. Also there is but one in the parabola.

Ex. 2. Let $n = 3$. Then eq. (A) become

(1) $$\varrho^2 = \frac{mP^2 + r^2}{m-1}$$

which indicate two apses in the same straight line, and on different sides of the center, whose position will be given by hence finding θ;

(2) $$\varrho = \frac{r^{\frac{2}{0}}}{P^{\frac{2}{0}}} = \infty$$

because r is $> P$,

whence there is no apse.

(3) $$\varrho^2 = \frac{r^2 - mP^2}{1-m}$$

which gives two apses, r^2 being $> mP^2$ because m is < 1 and $P < r$; and their position is found from θ.

317. COR. 2. This is done also by the equation

$$p = \varrho.\ \sin.\ \varphi, \text{ or } \sin.\ \varphi = \frac{p}{\varrho}$$

φ being the $\angle$ required.

Ex. When $n = 3$, and $m = 1$, we have (4. 313)

$$p = \frac{P}{r}\varrho$$

$$\therefore \sin.\ \varphi = \frac{P}{V}$$

$\therefore \varphi$ is constant, a known property of the logarithmic spiral.

318. *To find when the body reaches the center of force.*

Put in the equations to the Trajectory involving p, ϱ; or ϱ, θ

$$\varrho = 0.$$

Ex. 1. When $n = 3$, in all the five cases it is found that

$$p = 0$$

and

$$\theta = -\infty.$$

Ex. 2. When n = 5 in the particular case of 315, the former value of θ becomes impossible, being the logarithm of a negative quantity, and the latter is $= -\infty$.

319. *To find when the Trajectory has an asymptotic circle.*

If at an apse for $\theta = \infty$ the velocity be the same as that in a circle at the same distance (166), or if when

$$\theta = \infty$$

and

$$p = \varrho$$

we also have

$$\frac{p}{\varrho} = \frac{d\,p}{d\,\varrho}$$

then it is clear there is an asymptotic circle.

Examples are in hypothesis of 315.

320. *To find the number of revolutions from an apse to $\varrho = \infty$.*

Let θ' be the value of $\theta - \alpha$ when $\varrho = p$ or at an apse, and θ'' when $\varrho = \infty$. Then

$$v = \frac{\theta' \sim \theta''}{2\pi} = \text{the number of revolutions required.}$$

Ex. By 313, we have

$$\theta' \sim \theta'' = P\sqrt{\frac{m}{m-1}}\,\text{sec.}^{-1}\frac{\infty}{P}$$

$$= \sqrt{\frac{m}{m-1}} \cdot \frac{\pi}{2}$$

$$\therefore v = \tfrac{1}{4}\sqrt{\frac{m}{m-1}}.$$

321. Cor. 3. First let V R S be an hyperbola whose equation, x being measured from C, is

$$y^2 = \frac{b^2}{a^2}.(x^2 - a^2).$$

Then

$$V\,C\,R = \frac{y \times x}{2} - \int y\,dx$$

But

$$\int y\,dx = \frac{b}{a}\int dx\,\sqrt{(x^2 - a^2)}$$

$$= \frac{b}{a}\,x\,\sqrt{x^2 - a^2} - \frac{b}{a}\int \frac{x^2\,dx}{\sqrt{(x^2 - a^2)}}$$

$$= \frac{b}{a} x \sqrt{(x^2 - a^2)} - \frac{b}{a} \int dx \sqrt{(x^2 - a^2)} - \frac{b}{a} \int \frac{a^2 \, dx}{\sqrt{(x^2 - a^2)}}$$

$$\therefore 2 \int y \, dx = \frac{b}{a} x \sqrt{(x^2 - a^2)} - a b \, l. \frac{x + \sqrt{(x^2 - a^2)}}{a}$$

and

$$V C R = \frac{a b}{2} l. \frac{x + \sqrt{(x^2 - a^2)}}{a} \quad . \quad . \quad . \quad . \quad (1)$$

Again

$$\varrho = C P = C T = x - \text{subtangent}$$

$$= x - \frac{y \, dx}{dy} \; (29)$$

$$= x - \frac{x^2 - a^2}{x} = \frac{a^2}{x}$$

and substituting for x in (1) we have

$$V C R = \frac{a b}{2} . l. \frac{\frac{a^2}{\varrho} + \sqrt{\left(\frac{a^4}{\varrho^2} - a^2\right)}}{a}$$

$$\therefore \theta = V C P \propto V C R = \frac{a b}{2} N l. \frac{a + \sqrt{(a^2 - \varrho^2)}}{a \varrho} \quad . \quad . \quad . \quad . \quad (2)$$

N being a constant quantity.

322. Hence conversely

$$\frac{2}{C^{a b N}} \theta = \frac{1}{\varrho} - \sqrt{\left(\frac{1}{\varrho^2} - \frac{1}{a^2}\right)}$$

$$= u - \sqrt{\left(u^2 - \frac{1}{a^2}\right)}$$

and differentiating (17) we get

$$\frac{d u^2}{d \theta^2} = \frac{4}{a^2 b^2 N^2} \times \left(u^2 - \frac{1}{a^2}\right) \quad . \quad . \quad . \quad . \quad (1)$$

and again differentiating (d θ being constant)

$$\frac{d^2 u}{d \theta^2} = \frac{4}{a^2 b^2 N^2} \times U$$

Hence (139)

$$F = \frac{P^2 V^2}{g} \cdot \left(\frac{4}{a^2 b^2 . N^2} + 1\right) \frac{1}{\varrho^3} \propto \frac{1}{\varrho^3}$$

322. By the text it would appear that the body must proceed from V in a direction perpendicular to C V — i. e. that V is an apse.

From (1) 322, we easily get

$$\frac{d \varrho^2}{d \theta^2} = \frac{4}{a^4 b^2 N^2} (a^2 \varrho^2 - \varrho^4)$$

and since generally

$$p^2 = \frac{\varrho^4 \, d\theta^2}{\varrho^2 \, d\theta^2 + d\varrho^2}$$

$$\frac{d\varrho^2}{d\theta^2} = \frac{\varrho^2}{p^2} \cdot (\varrho^2 - p^2)$$

$$\therefore \frac{4}{a^4 b^2 N^2} \times p^2 \times (a^2 - \varrho^2) = \varrho^2 - p^2$$

$$\therefore p^2 = \frac{\varrho^2}{1 + \frac{4}{a^4 b^2 N^2} \times (a^2 - \varrho^2)} \quad . \quad . \quad . \quad . \quad (1)$$

which is another equation to the trajectory involving the perpendicular upon the tangent.

Now at an apse

$$p = \varrho$$

and substituting in equation (1) we get easily

$$\varrho = a$$

which shows V to be an apse.

OTHERWISE.

Put $d\varrho = 0$, for ϱ is then = max. or min.

324. *With a proper velocity.*]

The velocity with which the body must be projected from V is found from

$$v \, dv = -g F \, d\varrho.$$

325. *Descend to the center*]. When

$$\varrho = 0, \; p = 0 \text{ (1. 323) and } \theta = \infty \text{ (2. 321).}$$

326. Secondly, let V R S be an ellipse, whose equation referred to the center C is

$$y^2 = \frac{b^2}{a^2} \cdot (a^2 - x^2)$$

Then

$$V C R = \frac{y x}{2} + \int - y \, dx$$

$$= \frac{y x}{2} - \frac{b}{a} \int dx \sqrt{(a^2 - x^2)}$$

and as above, integrating by parts,

$$\int dx \sqrt{(a^2 - x^2)} = \frac{x \sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2} \int \frac{dx}{\sqrt{(a^2 - x^2)}}$$

$$= \frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2}\left(\sin.^{-1}\frac{x}{a} - \frac{\pi}{2}\right)$$

$$\therefore \text{V C R} = \frac{a\,b}{2}\left(\frac{\pi}{2} - \sin.^{-1}\frac{x}{a}\right).$$

Also

$$\varrho = x + \text{N T} = x + \frac{-y\,dx}{dy}$$

$$= x + \frac{a^2 - x^2}{x} = \frac{a^2}{x}$$

and

$$\theta = \text{N . V C R} = \frac{a\,b\,N}{2}\cdot\left(\frac{\pi}{2} - \sin.^{-1}\frac{a}{\varrho}\right) \ldots (1)$$

$$\therefore \sin.^{-1}\frac{a}{\varrho} = \frac{\pi}{2} - \frac{2\theta}{a\,b\,N}$$

$$\therefore \frac{a}{\varrho} = \sin.\left(\frac{\pi}{2} - \frac{2\theta}{a\,b\,N}\right) = \cos.\frac{2\theta}{a\,b\,N}$$

and

$$\varrho = a\sec.\frac{2\theta}{a\,b\,N} \quad \ldots\ldots\ldots\ldots \quad (2)$$

Conversely by the expression for F in 139, we have

$$F \propto \frac{1}{\varrho^3}$$

327. *To find when the body is at an apse, either proceed as in* 323, *or put*

$$d\varrho = 0.$$

By (27) $d.\sec. x = \dfrac{dx.\sin. x}{\cos.^2 x}$

$$\therefore \frac{\sin.\theta}{\cos.^2\theta} = 0$$

or

$$\theta = 0$$

that is the point V is an apse.

328. The proper velocity of projection is easily found as indicated in 324.

329. *Will ascend perpetually and go off to infinity.*]

From (2) 327, we learn that when

$$\frac{2\theta}{a\,b\,N} = \frac{\pi}{2}$$

ϱ is ∞;

also that ϱ can never $= 0$.

330. When the force is changed from centripetal to centrifugal, the sign of its expression (139) must be changed.

331. Prop. XLII. The preceding comments together with the Jesuits' notes will render this proposition easily intelligible.

The expression (139)

$$F = \frac{P^2 V^2}{g} \times \frac{d p}{p^3 d \varrho}$$

or rather (307)

$$p^2 = \frac{P^2 V^2}{V^2 - 2 g \int d \varrho f \varrho}$$

in which P and V are given, will lead to a more direct and convenient resolution of the problem.

It must, however, be remarked, that the difference between the first part of Prop. XLI. and this, is that the *force itself is given in the former and only the law of force* in the latter. That is, if for instance $F = \mu \varrho^{n-1}$, in the former μ is given, in the latter not. But since V is given in the latter, we have μ from 304.

SECTION IX.

332. Prop. XLIII. *To make a body move in an orbit revolving about the center of force, in the same way as in the same orbit quiescent*] that is, To adjust the angular velocity of the orbit, and centripetal force so that the body may be at any time at the same point in the revolving orbit as in the orbit at rest, and tend to the same center.

That it may tend to the same center (see Prop. II), the area of the new orbit in a fixed plane (V C p) must $\propto$ time $\propto$ area in the given orbit (V C P); and since these areas begin together their increments must also be proportional, that is (see fig. next Prop.)

$$CP \times KR \propto Cp \times kr$$

But

$$KR = CK \times \angle KCP$$
$$kr = Ck \times \angle kCp$$

and $CP = Cp$, and $CK = Ck$

$$\therefore \angle KCP \propto kCp$$

and the angles V C P, V C p begin together

$$\therefore \angle VCP \propto \angle VCp.$$

Hence in order that the centripetal force in the new orbit may tend to C, it is *necessary* that

$$\angle \mathrm{V\,C\,p} \propto \angle \mathrm{V\,C\,P}.$$

Again, taking always

$$\mathrm{C\,P} = \mathrm{C\,p}$$

and

$$\mathrm{V\,C\,p} : \mathrm{V\,C\,P} :: \mathrm{G} : \mathrm{F}$$

G : F being an invariable ratio, the equation to the locus of p or the orbit in fixed space can be determined; and thence (by 137, 139, or by Cor. 1, 2, 3 of Prop. VI) may be found the centripetal force in that locus.

333. *To find the orbit in fixed space or the locus of* p.

Let the equation to the given orbit V C P be

$$\varrho = \mathrm{f}(\theta)$$

where $\varrho = \mathrm{C\,P}$, and $\theta = \mathrm{V\,C\,P}$, and f means any function; then that of the locus is

$$\varrho = \mathrm{f}\left(\frac{\mathrm{G}}{\mathrm{F}}\theta\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

which will give the orbit required.

OTHERWISE.

Let p′ be the perpendicular upon the tangent in the given orbit, and p that in the locus; then it is easily got by drawing the incremental figures and similar triangles (see fig. Prop. XLIV) that

$$\mathrm{K\,R} : \mathrm{k\,r} :: \mathrm{F} : \mathrm{G}$$
$$\mathrm{k\,r} : \mathrm{p\,r} :: \mathrm{p} : \sqrt{(\varrho^2 - \mathrm{p}^2)}$$
$$\mathrm{p\,r} : \mathrm{PR} :: 1 : 1$$
$$\mathrm{P\,R} : \mathrm{KR} :: \sqrt{(\varrho^2 - \mathrm{p}'^2)} : \mathrm{p}'$$

whence

$$1 : 1 :: \mathrm{F}.\mathrm{p}\sqrt{(\varrho^2 - \mathrm{p}'^2)} : \mathrm{G}\,\mathrm{p}'\sqrt{(\varrho^2 - \mathrm{p}^2)}$$

and

$$\therefore \mathrm{p}^2 = \frac{\mathrm{G}^2\mathrm{p}'^2\varrho^2}{\mathrm{F}^2\varrho^2 + (\mathrm{G}^2 - \mathrm{F}^2)\mathrm{p}'^2} \quad \ldots\ldots \quad (2)$$

334. Ex. 1. Let the given Trajectory be the ellipse with the force in its focus; then

$$\mathrm{p}'^2 = \frac{\mathrm{b}^2\varrho}{2\mathrm{a} - \varrho}, \text{ and } \varrho = \frac{\mathrm{a}.(1 - \mathrm{e}^2)}{1 + \mathrm{e}\cos.\theta}$$

and therefore

$$\mathrm{p}^2 = \frac{\mathrm{b}^2\mathrm{G}^2(2\mathrm{a} - \varrho)\varrho^2}{\mathrm{b}^2(\mathrm{G}^2 - \mathrm{F}^2) + \mathrm{F}^2(2\mathrm{a}\varrho - \varrho^2)}$$

and

$$\varrho = \frac{a.(1-e^2)}{1+e\cos.\left(\frac{F}{G}\theta\right)}.$$

Hence since the force is (139)

$$\frac{P^2V^2}{g}u^2\left(\frac{d^2u}{d\theta^2}+u\right)$$

and here we have

$$a(1-e^2)u = 1+e\cos.\frac{F}{G}\theta$$

$$\therefore \frac{du^2}{d\theta^2} = \frac{F^2}{a^2G^2(1-c^2)^2}\times\{e^2-(1-a(1-e^2)u)^2\}$$

$$= Q+\frac{2F^2}{aG^2(1-e^2)}u-\frac{F^2}{G^2}u^2$$

and again differentiating, &c. we have

$$\frac{d^2u}{d\theta^2}+u = \frac{F^2}{G^2a(1-e^2)}+\frac{G^2-F^2}{G^2}\times u.$$

But if 2 R = latus-rectum we have

$$R = \frac{L}{2} = \frac{b^2}{a} = a(1-e^2)$$

$\therefore$ the force in the new orbit is

$$\frac{P^2V^2}{gRG^2}\times\left\{\frac{F^2}{\varrho^2}+\frac{RG^2-RF^2}{\varrho^3}\right\}$$

335. Ex. 2. Generally let the equations to the given trajectory be

$$\left.\begin{aligned}\varrho &= f(\theta')\\ p' &= \varphi(\varrho)\end{aligned}\right\}$$ and

Then since

$$\theta = \frac{G}{F}\theta'$$

$$\therefore d\theta^2 = \frac{G^2}{F^2}d\theta'^2$$

$$\therefore \frac{d^2u}{d\theta^2}+u = \frac{F^2}{G^2}\frac{d^2u}{d\theta'^2}+u$$

$$= \frac{F^2}{G^2}\times\left(\frac{d^2u}{d\theta^2}+u\right)+u-\frac{F^2}{G^2}u$$

and if the centripetal forces in the given trajectory and locus be named X, X', by 139 we have

$$\frac{gX'}{P'^2V'^2} = \frac{F^2}{G^2}\times\frac{gX}{P'^2V'^2}+\frac{G^2-F^2}{G^2}\times\frac{1}{\varrho^3}$$

$$\therefore X' = \frac{P^2 V^2}{G^2}\left(\frac{F^2 X}{P'^2 V'^2} + \frac{G^2 - F^2}{g} \times \frac{1}{\varrho^3}\right) \quad . \quad . \quad (1)$$

Also from (2. 333) we have

$$\frac{1}{p^2} = \frac{F^2}{G^2} \times \frac{1}{p'^2} + \frac{G^2 - F^2}{G^2} \times \frac{1}{\varrho^2}$$

$$\therefore \frac{dp}{p^3 d\varrho} = \frac{F^2}{G^2} \times \frac{dp'}{p'^3 d\varrho} + \frac{G^2 - F^2}{G^2} \times \frac{1}{\varrho^3}$$

$\therefore$ by 139

$$\frac{g X'}{P^2 V^2} = \frac{F^2 g X}{P'^2 V'^2} + \frac{G^2 - F^2}{G^2} \times \frac{1}{\varrho^3}$$

the same as before.

This second general example includes the first, as well as Prop. XLIV, &c. of the text.

336. *Another determination of the force tending to* C *and which shall make the body describe the locus of* p.

First, as before, we must show that in order to make the force X tend to C, the ratio $\angle$ V C P : $\angle$ V C p must be constant or $= F : G$.

Next, since they begin together the corresponding angular velocities ω, ω' of C P, C p are in that same ratio; i. e.

$$\omega : \omega' :: F : G.$$

Now in order to exactly counteract the centrifugal force which arises from the angular motion of the orbit, we must add the same quantity to the centripetal force. Hence if φ, φ' denote the centrifugal forces in the given orbit and the locus, we have

$$X' = X + \varphi' - \varphi$$

X being the force in the given orbit.

But (210)

$$\varphi = \frac{P^2 V^2}{g} \times \frac{1}{\varrho^3}$$

and

$$\propto \omega^2$$

when ϱ is given.

$$\therefore \varphi' = \varphi \times \frac{\omega'^2}{\omega^2} = \varphi \times \frac{G^2}{F^2} = \frac{P^2 V^2}{g} \times \frac{G^2}{F^2} \times \frac{1}{\varrho^3}$$

$$\therefore \varphi' - \varphi = \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3}$$

$$X' = X + \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$= \frac{P^2 V^2}{g} \times \left(u^2 . \frac{d^2 u}{d \theta^2} + u + \frac{G^2 - F^2}{F^2 \varrho^3} \right) \quad . \quad . \quad . \quad (2)$$

or

$$= \frac{P^2 V^2}{g} \times \left(\frac{d p}{p^3 d \varrho} + \frac{G^2 - F^2}{F^2 \varrho^3} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

337. Prop. XLIV. Take u p, u k similar and equal to V P and V K; also

$$m r : k r :: \angle V C p : V C P.$$

Then since always C P = p c, we have

$$p r = P R.$$

Resolve the motions P K, p k into P R, R K and p r, r k. Then

$$R K (= r k) : r m :: \angle V C P : \angle V C p$$

and therefore when the centripetal forces P R, p r are equal, the body would be at m. But if

$$p C n : p C k :: V C p : V C P$$

and

$$C n = C k$$

the body will really be in n.

Hence the difference of the forces is

$$m n = \frac{m k \times m s}{m t} = \frac{(m r - k r) . (m r + k r)}{m t}.$$

But since the triangles p C k, p C n are given,

$$K r \propto m r \propto \frac{1}{C p}$$

$$\therefore m n \propto \frac{1}{C p^2} \times \frac{1}{m t}.$$

Again since

$$\begin{aligned} p C k : p C n &:: P C K : p C n :: V C P : V C p \\ &:: k r \quad : m r \quad \text{by construction} \\ &:: p C k : p C m \text{ ultimately} \end{aligned}$$

$$\therefore p C n = p C m$$

and m n ultimately passes through the center. Consequently

$$m t = 2 C p \text{ ultimately}$$

and

$$m n \propto \frac{1}{C p^3}$$

OTHERWISE.

338. By 336,

$$X' - X = \varphi' - \varphi$$
$$= \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3}$$
$$\propto \frac{1}{\varrho^3}.$$

339. *To trace the variations of sign of* m n.

If the orbit move in consequentia, that is in the same direction as C P, the new centrifugal force would throw the body farther from the center, that is

C m is $>$ C n or C k .

or m n is positive.

Again, when the orbit is projected in antecedentia with a velocity $<$ than twice that of C P, the velocity of C p is less than that of C P. Therefore

C m is $<$ C n

or m n is negative.

Again, when the orbit is projected in antecedentia with a velocity $=$ twice that of C P, the angular velocity of the orbit just counteracts the velocity of C P, and

$$m n = 0.$$

And finally, when the orbit is projected in antecedentia with a velocity $>$ 2 vel. of C P, the velocity of C p is $>$ vel. of C P or C m is $>$ C n, or

m n is positive.

OTHERWISE.

By 338,

$$m n \propto \varphi' - \varphi$$
$$\propto \omega'^2 - \omega^2$$

But

$$\omega' = \omega \pm W$$

W being the angular velocity of the orbit.

$$\therefore m n \propto \pm 2 \omega W + W^2$$
$$\propto \pm 2 \omega + W$$

$+$ or $-$ being used according as W is in consequentia or antecedentia.

Hence m n is positive or negative according as W is positive, and negative and $> 2\omega$; or negative and $< 2\omega$. That is, &c. &c.

Also when W is negative and $= 2\omega$, $m = 0$. Therefore, &c.

340. Cor. 1. Let D be the difference of the forces in the orbit and in the locus, and f the force in the circle K R, we have

$$\begin{aligned} D : f &:: m n : z r \\ &:: \frac{m k \times m s}{m t} : \frac{r k^2}{2 k c} \\ &:: \frac{(m r + r k)(m r - r k)}{2 k c} : \frac{r k^2}{2 k c} \\ &:: m r^2 - r k^2 : r k^2 \\ &:: G^2 - F^2 : F^2. \end{aligned}$$

341. Cor. 2. *In the ellipse with the force in the focus, we have*

$$X' \propto \frac{F^2}{A^2} + \frac{R G^2 - R F^2}{A^3}.$$

For (C V being put = T)

$$\text{Force at V in Ellipse : Do. in circle} :: \frac{v^2}{\text{chord } P V} : \frac{v'^2}{P' V'}$$

$$:: \frac{1}{2R} : \frac{1}{2T}$$

$$:: T : R$$

$$\text{Also F in Circle : m n at V} :: F^2 : G^2 - F^2$$

$$\text{m n at V : m n at p} :: \frac{1}{T^3} : \frac{1}{A^3}$$

$$\therefore \text{F at V in ellipse : m n at p} :: \frac{T F^2}{T^3} : \frac{R G^2 - R F^2}{A^3}.$$

Hence

$$X = \frac{F^2}{A^2}$$

we have

$$F \text{ in ellipse at V} = \frac{F^2}{T^2}$$

and

$$m n = \frac{R G^2 - R F^2}{A^3}$$

and

$$X' = X + m n \propto \frac{F^2}{A^2} + \frac{R G^2 - R F^2}{A^3}$$

see 334.

OTHERWISE.

342. By 336,

$$X' = X + \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3}$$

But

$$X = \frac{\mu}{\varrho^2}$$

and

$$\frac{P^2 V^2}{g} = \frac{L}{2}\mu = R\mu \text{ (157)}$$

$$\therefore X' = \frac{\mu}{F^2} \times \left\{ \frac{F^2}{\varrho^2} + \frac{G^2 - F^2}{\varrho^3} \right\}$$

343. COR. 3. *In the ellipse with the force in the center.*

$$X' \propto \frac{F^2 A}{T^3} + \frac{R G^2 - R F^2}{A^3}.$$

For here $X \propto A$ and the force generally $\propto \frac{v^2}{PV}$ (140)

$$\therefore \begin{cases} \text{Force in ellipse at V} : \text{Force in circle at V} :: T : R \\ \text{F in circle} : \text{m n at V} :: F^2 : G^2 - F^2 \\ \text{m n at V} : \text{m n at p} :: \frac{1}{T^3} : \frac{1}{A^3} \end{cases}$$

$$\therefore \text{F in ellipse at V} : \text{m n at p} :: \frac{F^2}{T^3} \cdot T : \frac{R G^2 - R F^2}{A^3}$$

$\therefore$ assuming F in ellipse at $P = \frac{F^2 A}{T^3}$, we have

$$\text{F in ellipse at V} = \frac{F^2}{T^3} \times T$$

and

$$\therefore m n = \frac{R G^2 - R F^2}{A^3}$$

$$\therefore X' \propto X + m n \propto, \text{ \&c.}$$

OTHERWISE.

344. $$X = \mu\varrho, \text{ and } \frac{P^2 V^2}{g} = \frac{4 (\text{Area of Ellipse})^2}{g (\text{Period})^2}$$

$$= \frac{4\pi^2 a^2 b^2}{g(\text{Period})^2} = \mu a^2 b^2 \text{ (147)}$$

Therefore by 336

$$X' = \mu\varrho + \mu a^2 b^2 \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3}$$
$$= \frac{\mu a^3}{F^2} \times \left\{ \frac{F^2 \varrho}{a^3} + \frac{b^2 \times (G^2 - F^2)}{a \varrho^3} \right\}$$
$$= \frac{\mu a^3}{F^2} \times \left\{ \frac{F^2 \varrho}{a^3} + \frac{R G^2 - R F^2}{\varrho^3} \right\}$$

345. Cor. 4. *Generally let* X *be the force at* P, V $\frac{F^2}{T^2}$ *at* V, R *the radius of curvature in* V, C V = T, &c. *then*

$$X' \propto X + \frac{V R G^2 - V R F^2}{A^3}$$

For

$$\begin{cases} F \text{ in orbit at V} & : F' \text{ in circle at V} & :: T : R \\ F' & : m\,n \text{ at V} & :: F^2 : G^2 - F^2 \\ m\,n \text{ at V} & : m\,n & :: A^3 : T^3 \end{cases}$$
$$\therefore F \text{ in orbit at V} : m\,n :: \frac{V F^2}{T^2} : V R . \frac{G^2 - F^2}{A^3}$$

$\therefore$ since by the assumption

$$F \text{ in orbit at V} = \frac{V F^2}{T^2}$$
$$\therefore m\,n = \frac{V R (G^2 - F^2)}{A^3}$$

and

$$X' \propto X + \frac{V R (G^2 - F^2)}{A^3}.$$

OTHERWISE.

This may better be done after 336, where it must be observed V is not the same as the indeterminate quantity V in this corollary.

346. Cor. 5. The equation to the new orbit is (333)

$$p^2 = \frac{G^2 p'^2 \varrho^2}{F^2 \varrho^2 + (G^2 - F^2) p'^2}$$

p′ belonging to the given orbit.

Ex. 1. *Let the given orbit be a common parabola.*

Then

$$p'^2 = r \varrho$$
$$\therefore p^2 = \frac{G^2 r \varrho^2}{F^2 \varrho + (G^2 - F^2) r}$$

and the new force is obtained from 336.

Ex. 2. *Let the given orbit be any one of Cotes' Spirals, whose general equation is*

$$p'^2 = \frac{b^2 \varrho^2}{a^2 \mp \varrho^2}.$$

Then the equation of 333 becomes

$$p^2 = \frac{\frac{G^2}{F^2} b^2 \varrho^2}{\frac{G^2}{F^2} b^2 + a^2 - b^2 \mp \varrho^2}$$

which being of the same form as the former shows the locus to be similar in each case to the given spiral.

This is also evident from the law of force being in each case the same (see 336) viz.

$$X' = \frac{\mu}{\varrho^3} + \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{G^2} \times \frac{1}{\varrho^3}$$
$$\propto \frac{1}{\varrho^3}.$$

Ex. 3. If the given orbit be a circle, the new one is also.

Ex. 4. *Let the given trajectory be a straight line.*

Here p' is constant. Therefore

$$p^2 = \frac{G^2}{F^2} \frac{p'^2 \times \varrho^2}{\varrho^2 + \frac{G^2 - F^2}{F^2} p'^2}$$

the equation to the elliptic spiral, &c. &c.

Ex. 5. *Let the given orbit be a circle with the force in its circumference.*

Here

$$p'^2 = \frac{\varrho^2 (4 r^2 - \varrho^2)}{4 r^2}$$

and we have from 333

$$p^2 = \frac{G^2 \varrho^4}{4 r^2 F^2 + (G^2 - F^2) \varrho^2}.$$

Ex. 6. *Let the given orbit be an ellipse with force in the focus.*

Here

$$p'^2 = \frac{b^2 \varrho}{2a - \varrho}$$

and this gives

$$p^2 = \frac{G^2 b^2 \varrho^2}{F^2 \varrho (2a - \varrho) + b^2 (G^2 - F^2)}.$$

347. *To find the points of contrary flexure*, in the locus put

$$d\,p = 0;$$

and this gives in the case of the ellipse

$$\varrho = \frac{b^2}{a} \cdot \frac{F^2 - G^2}{F^2}$$

OTHERWISE.

In passing from convex to concave towards the center, the force in the locus must have changed signs. That is, at the point of contrary flexure, the force equals nothing or in this same case

$$F^2 A + R G^2 - R F^2 = 0$$

$$\therefore A = \frac{R}{F^2} \times (F^2 - G^2)$$

$$= \frac{b^2}{a} \cdot \frac{F^2 - G^2}{F^2}.$$

And generally by (336) we have in the case of a contrary flexure

$$X' = X + \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3} = 0$$

which will give all the points of that nature in the locus.

348. *To find the points where the locus and given Trajectory intersect one another.*

It is clear that at such points

$$\varrho = \varrho', \text{ and } \theta' = 2\,W\,\pi \pm \theta$$

W being any integer whatever. But

$$\theta' = \frac{G}{F}\,\theta = m\,\theta$$

$$\therefore \theta = \frac{2\,W\,\pi}{m \pm 1}.$$

This is independent of either the Trajectory or Locus.

349. *To find the number of such intersections during an entire revolution* of C P.

Since θ cannot be $> 2\,\pi$

W is $< m + 1$ and also $< m - 1$

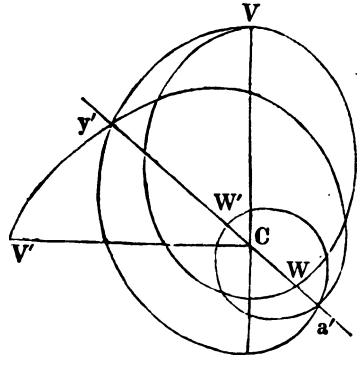
$\therefore 2\,W$ is $< 2\,m$.

Or the number required is the greatest integer in $2\,m$ or $\frac{2\,G}{F}$.

This is also independent of either Trajectory or Locus.

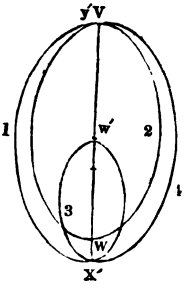
350. *To find the number and position of the double points of the Locus,* i. e. *of those points where it cuts or touches itself.*

Having obtained the equation to the Locus find its singular points whether double, triple, &c. by the usual methods; or more simply, consider the double points which are owing to apses and pairs of equal values of C P, one on one side of C V and the other on the other, thus:



The given Trajectory V W being symmetrical on either side of V W, let W′ be the point in the locus corresponding to W. Join C W′ and produce it indefinitely both ways. Then it is clear that W′ is an apse; also that the angle subtended by V v′ x′ W′ is $= \frac{G}{F} \times \pi = w\pi + \angle$ V C y′, w being the greatest whole number in $\frac{G}{F}$ (this supposes the motion to be in consequentia). Hence it appears that wherever the Locus cuts the line C W′ there is a double point or an apse, and also that there are w + 1 such points.

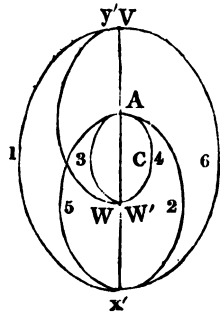
Ex. 1. Let $\frac{G}{F} = 2$; i. e. *let the orbit move in consequentia with a velocity = the velocity of* C P. Then $\angle$ V C y′ = 0, y′ coincides with V, and the double points are y′ V, x′ and W′.



The course of the Locus is indicated by the order of the figures 1, 2, 3, 4.

Ex. 2. Let $\frac{G}{F} = 3$.

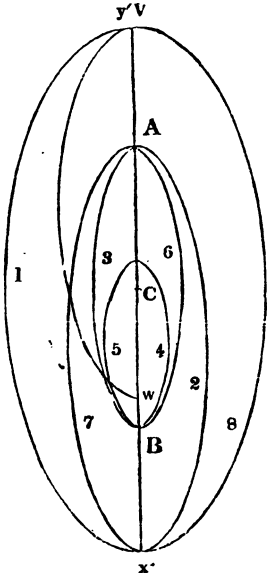
Then the Locus resembles this figure, 1, 2, 3, 4, 5, 6. showing the course of the curve in which V, x′, A, W′ are double points and also apses.



Ex. 3. Let $\frac{G}{F} = 4$.

Then this figure sufficiently traces the Locus. Its five double points, viz. V, x′, A, B, W′ are also apses.

Higher *integer* values of $\frac{G}{F}$ will give the Locus



still more complicated. If $\frac{G}{F}$ be not integer, the figure will be as in the first of this article, the double points lying out of the line C V. Moreover if $\frac{G}{F}$ be less than 1, or if the orbit move in antecedentia this method must be somewhat varied, but not greatly. These and other curiosities hence deducible, we leave to the student.

351. *To investigate the motion of* (p) *when the ellipse, the force being in the focus, moves in antecedentia with a velocity = velocity of* C P *in consequentia.*

Since in this case

$$G = 0$$

$\therefore$ (333) also

$$p = 0$$

or the Locus is the straight line C V.

Also (342)

$$X' = \frac{\mu}{F^2}\left(\frac{F^2}{\varrho^2} - \frac{R\,F^2}{\varrho^3}\right)$$

$$= \mu \times \frac{\varrho - R}{\varrho^2}.$$

Hence

$$v\,dv \propto X'\,d\varrho \propto \frac{-\,d\varrho}{\varrho^2} + \frac{R\,d\varrho}{\varrho^3}$$

$$\therefore v^2 \propto \frac{2\varrho - R}{\varrho^2} - 1 \propto \frac{2\varrho - 1 - e^2 - \varrho^2}{\varrho^2}$$

(where $\frac{\text{axis major}}{2} = 1$;) and the body stops when

$$2\varrho - 1 + e^2 - A^2 = 0,$$

or when

$$\varrho = 1 \pm e.$$

Hence then the body moves in a straight line C V, the force increasing to $\frac{3}{4}$ of the latus-rectum from the center, when it = max. Then it decreases until the distance = $\frac{L}{2}$ or R. Here the centrifugal force prevails, but the velocity being then = max. the body goes forward till the

distance = the least distance when v = 0, and afterwards it is repelled and so on in infinitum.

352. *To find when the velocity in the Locus = max. or min.*

Since in either case

$$d.v^2 = 2v\,dv = 0$$

and

$$v\,dv = X'\,d\varrho$$

$$\therefore X' = 0$$

$\therefore$ (336)

$$X + \frac{P^2V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3} = 0$$

Ex. In the ellipse with the force in the focus, we have (342)

$$X' = \frac{\mu}{F^2}\left(\frac{F^2}{\varrho^2} + \frac{R\,G^2 - R\,F^2}{\varrho^3}\right)$$

$$\therefore \frac{F^2}{\varrho^2} + \frac{R\,G^2 - R\,F^2}{\varrho^3} = 0$$

$$\therefore \varrho = R \times \frac{F^2 - G^2}{F^2}$$

$$= \frac{b^2}{a} \times \frac{F^2 - G^2}{F^2}.$$

If G = 0, v = max. when $\varrho = \frac{b^2}{a} - \frac{L}{2}$, or when P is at the extremity of the latus-rectum.

If F = 2 G, v = max. when $\varrho = R.\frac{4G^2 - G^2}{4G^2} = \frac{3}{4}R = \frac{3}{8}$ lat. rectum.

353. *To find when the force X′ in the Locus = max. or min.*

Put d X′ = 0, which gives (see 336)

$$d\,X = \frac{3\,P^2V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^4}$$

Ex. In the ellipse

$$X = \frac{\mu}{\varrho^2}$$

and (157)

$$\frac{P^2V^2}{g} = \mu R$$

$$\therefore \frac{-2F^2 d\varrho}{\varrho^3} - \frac{3RG^2 d\varrho - 3RF^2 d\varrho}{\varrho^4} = 0$$

which gives

$$\varrho = \frac{3R}{2} \times \frac{F^2 - G^2}{F^2}.$$

Hence when

$$G = 0$$

$$X = \text{max. when } \varrho = \frac{3\,R}{2}.$$

When $\varrho = R$, and $G = 0$. Then

$$X = \frac{F^2}{R^2} - \frac{R\,F^2}{R^3} = 0.$$

When $F = 2\,G$, or the ellipse moves in consequentia with $\frac{1}{2}$ the velocity of C p; then

$$X = \text{max. when}$$

$$\varrho = \frac{3\,R}{2} \cdot \frac{4\,G^2 - G^2}{4\,G^2} = \frac{9}{8}\,R.$$

354. Cor. 6. Since the given trajectory is a straight line and the *center of force* C *not in it*, this force cannot act at all upon the body, or (336)

$$X = 0.$$

Hence in this case

$$X' = \frac{P^2\,V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3}$$

where $P = C\,V$ and V the given uniform velocity along V P.

In this case the Locus is found as in 346.

355. If the given Trajectory is a circle, it is clear that the Locus of p is likewise a circle, the radius-vector being in both cases invariable.

356. Prop. XLV. *The orbits (round the same center of force) acquire the same form, if the centripetal forces by which they are described at equal altitudes be rendered proportional.*]

Let f and f' be two forces, then if at all equal altitudes

$$f \propto f'$$

the orbits are of the same form.

For (46)

$$f \propto \frac{d^2\varrho}{dt^2} \propto \frac{1}{dt^2} \propto \frac{1}{SP^2 \times QT^2}$$

$$\propto \frac{1}{QT^2} \propto \frac{1}{SP^2 \times d\theta^2}$$

$$\propto \frac{1}{d\theta^2}.$$

$$\therefore \frac{1}{d\theta^2} \propto \frac{1}{d\theta'^2}$$

and

$$d\theta \propto d\theta'.$$

But they begin together and therefore

$$\theta \propto \theta'$$

and

$$\varrho = \varrho'.$$

Hence it is clear the orbits have the same form, and hence is also suggested the necessity for making the angles θ, θ' proportional.

Hence then X′, and X being given, we can find $\frac{G}{F}$ such as shall render the Trajectory traced by p, very nearly a circle. This is done approximately by considering the given fixed orbit nearly a circle, and equating as in 336.

357. Ex. 1. *To find the angle between the apsides when* X′ *is constant.*

In this case (342)

$$X' \propto 1 \propto \frac{\varrho^3}{\varrho^3} \propto \frac{F^2\varrho + R G^2 - R F^2}{\varrho^3}.$$

Now making $\varrho = T - x$, where x is indefinitely diminishable, and equating, we have

$$\begin{aligned}(T-x)^3 &= F^2 T - F^2 x + R G^2 - R F^2 \\ &= T^3 - 3 T^2 x + 3 T x^2 - x^3\end{aligned}$$

and equating homologous terms (6)

$$T^3 = F^2 T + R G^2 - R F^2 = F^2 \times (T - R) + R G^2$$

and

$$F^2 = 3 T^2$$

$$\begin{aligned}\therefore \frac{G^2}{F^2} &= \frac{T^3}{R F^2} - \frac{T-R}{R} \\ &= \frac{T^3}{3 R T^2} - \frac{T-R}{R} \\ &= \frac{T}{3R} - \frac{T-R}{R} = \frac{3R - 2T}{3R} \\ &= \frac{1}{3} \text{nearly}\end{aligned}$$

since R is = T nearly.

Hence when $F = 180^\circ = \pi$

$$\gamma = G = \frac{\pi}{\sqrt{3}} \quad \ldots\ldots\ldots\ldots \quad (1)$$

the angle between the apsides of the Locus in which the force is constant.

358. Ex. 2. Let $X' \propto \varrho^{n-3}$. Then as before

$$(T-x)^n = F^2(T-x) + R G^2 - R F^2$$

and expanding and equating homologous terms

$$T^n = F^2 T + R G^2 - R F^2$$

and

$$F^2 = n\,T^{n-1}.$$

But since T nearly = R

$$T^{n-1} = G^2$$

$$\therefore \frac{G^2}{F^2} = \frac{1}{n}$$

and when $F = \pi$

$$\gamma = G = \frac{\pi}{\sqrt{n}}.$$

Thus when $n - 3 = 1$, we have

$$\gamma = \frac{\pi}{\sqrt{4}} = \frac{\pi}{2} = 90^\circ.$$

When $n - 3 = -1$, $n = 2$, and $\gamma = \frac{\pi}{\sqrt{2}} = 127^\circ.\ 16'.\ 45''$.

When $n - 3 = -\frac{11}{4}$, $n = \frac{1}{4}$, and $\gamma = 2\pi = 360^\circ$.

359. Let $X' \propto \frac{b\varrho^m \pm c\varrho^n}{\varrho^3}$. Then

$$b.(T-x)^m \pm c\,(T-x)^n = F^2.(T-x) + R.(G^2 - F^2)$$

and expanding and equating homologous terms we get

$$b\,T^m \pm c\,T^n = F^2\,(T - R) + R\,G^2$$

and

$$b\,m\,T^{m-1} \pm c\,n\,T^{n-1} = F^2.$$

But R being nearly = T, we have

$$b\,T^{m-1} \pm c\,T^{n-1} = G^2$$

$$\therefore \frac{G^2}{F^2} = \frac{b\,T^{m-1} \pm c\,T^{n-1}}{b\,m\,T^{m-1} \pm c\,n\,T^{n-1}} = \frac{b\,T^m \pm c\,T^n}{m\,b\,T^m \pm n\,c\,T^n}$$

which is more simply expressed by putting $T = 1$. Then we have

$$\frac{G^2}{F^2} = \frac{b \pm c}{m\,b \pm n\,c}$$

and when $F = \pi$

$$\gamma = G = \pi \sqrt{\frac{b \pm c}{m\,b \pm n\,c}}.$$

360. Cor. 1. *Given the ∠ between the apsides to find the force.*

Let $n : m :: 360^\circ : 2\gamma$

$$:: 180^\circ = \pi : \gamma$$

$$\therefore \gamma = \frac{m}{n}\,\pi$$

But if $X' \propto \varrho^{p-3}$

$$\gamma = \frac{\pi}{\sqrt{p}}.$$

$$\therefore p = \frac{n^2}{m^2}$$

$$\therefore X' \propto \varrho^{\frac{n^2}{m^2}-3}$$

Ex. 1. If $n : m :: 1 : 1$,

$$X' \propto \frac{1}{\varrho^2}$$

as in the ellipse about the focus.

2. If $n : m :: 363 : 360$

$$X' \propto \varrho^{\left(\frac{120}{121}\right)^2-3}$$

3. If $n : m :: 1 : 2$

$$X' \propto \frac{1}{\varrho^{\frac{11}{4}}}.$$

And so on.

$$\text{Again if } X' \propto \frac{1}{\varrho^3}$$

$$\gamma = \frac{\pi}{\sqrt{0}} = \infty$$

and the body having reached one apse can never reach another.

$$\text{If } X' \propto \frac{1}{\varrho^{3+q}}$$

$$\gamma = \frac{\pi}{\sqrt{-q}}$$

∴ the body never reaches another apse, and since the centrifugal force $\propto \frac{1}{\varrho^3}$, if the body depart from an apse and centrifugal force be $>$ centripetal force, then centrifugal is always $>$ centripetal force and the body will continue to ascend in infinitum.

Again if at an apse the centrifugal be $<$ the centripetal force, the centrifugal is always $<$ centripetal force and the body will descend to the center.

The same is true if $X' \propto \frac{1}{\varrho^3}$ and in all these cases, if

$$\text{centrifugal} = \text{centripetal}$$

the body describes a circle.

361. COR. 2. First let us compare the force $\frac{1}{A^2} - c\,A$, belonging to the moon's orbit, with

$$\frac{F^2}{A^2} + \frac{R\,G^2 - R\,F^2}{A^3}.$$

Since the moon's apse proceeds, (n m) is positive.

$\therefore -c\,A$ does not correspond to n m and $\therefore \frac{1}{A^2}$ does not correspond to $\frac{F^2}{A^2}$.

Now

$$\frac{1}{A^2} - c\,A \propto \frac{A - c\,A^4}{A^3} \propto \frac{b\,A^m - c\,A^n}{A^3}$$

$$\therefore X' \propto A^{\frac{1-4c}{1-c}-3} \propto A^{\frac{F^2}{G^2}-3}$$

$$\therefore \frac{1-4c}{1-2} = \frac{F^2}{G^2}$$

$$\therefore \frac{F^2}{A^2} + \frac{R\,G^2 - R\,F^2}{A^3} = \frac{1-4c}{A^3} + \frac{3\,c\,R}{A^3}$$

$$\therefore \frac{F^2}{A^2} = \frac{1-4c}{A^2} \text{ and not } \frac{1}{A^2}$$

and

$$m\,n = \frac{3\,c\,R}{A^2}.$$

Hence also

$$\gamma = \pi \sqrt{\frac{1-c}{1-4c}}. \text{ \&c. \&c. \&c.}$$

362. *To determine the angle between the apsides generally.*

Let

$$X \propto \frac{f(A)}{A^3}$$

f (A) meaning any function whatever of A. Then for Trajectories which are nearly circular, put

$$\frac{f(A)}{A^3} = \frac{F^2\,A + R.(G^2 - F^2)}{A^3}$$

$$\therefore f.\,A = F^2\,A + R\,(G^2 - F^2)$$

or

$$f.\,(T - x) = F^2\,(T - x) + R\,(G^2 - F^2)$$

But expanding f (T — x) by Maclaurin's Theorem (32)

$$u = f(T - x) = U - U'x + U''\frac{x^2}{2} - \text{\&c.}$$

U, U' &c. being the values of $u, \frac{du}{dx}, \frac{d^2u}{dx^2}$ &c.

when x = 0, and therefore independent of x. Hence comparing homologous terms (6) we have

$$U = F^2\,T + R\,(G^2 - F^2)$$

$$U' = F^2$$

Also since $R = T$ nearly

$$U = T G^2$$

$$\frac{G^2}{F^2} = \frac{U}{T . U'} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Hence when $F = \pi$, the angle between the apsides is

$$\left.\begin{aligned} \gamma = G &= \pi \sqrt{\frac{U}{T . U'}} \\ \text{or} \\ &= \pi \sqrt{\frac{U}{U'}} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

making $T = 1$.

Ex. 1. Let $f(A) = b A^m + c A^n = u$

Then

$$\frac{d u}{d x} = m b A^{m-1} + n c A^{n-1}.$$

Hence since $A = T$ when $x = 0$

$$U = f T = b T^m + c T^n$$

$$U' = m b T^{m-1} + n c T^{n-1}$$

$$\therefore \frac{G^2}{F^2} = \frac{b T^m + c T^n}{m b T^m + n c T^n}$$

or

$$\frac{G^2}{F^2} = \frac{b + c}{m b + n c}$$

and

$$\gamma = \pi \sqrt{\frac{b + c}{m b + n c}}$$

as in 359.

Ex. 2. Let $f.(A) = b A^m + c A^n + e A^r + \&c.$

$$\therefore \frac{d u}{d x} = m b A^{m-1} + n c A^{n-1} + r e A^{r-1} + \&c.$$

$$\therefore U = b T^m + c T^n + e T^r + \&c.$$

and

$$T \times U' = m b T^m + n c T^n + r e T^r + \&c.$$

$$\therefore \frac{G^2}{F^2} = \frac{b T^m + c T^n + e T^r + \&c.}{m b T^m + n c T^n + r e T^r + \&c.}$$

or

$$= \frac{b + c + e + f + \&c.}{m b + n c + r e + s f + \&c.}$$

when $T = 1$.

Also

$$\gamma = \pi \sqrt{\frac{b + c + e \ldots\ldots}{m b + n c + r e + \ldots\ldots}}$$

Ex. 3. Let $\frac{f(A)}{A^3} = a^A = u$.

Here (17)

$$\frac{du}{dx} = A^2 a^A \times (3 + A \, l \, a)$$

Hence

$$U = T^2 a^T \times (3 + T \, l \, a)$$
$$T \times U' = T^3 a^T (3 + T \, l \, a)$$
$$\frac{G^2}{F^2} = \frac{1}{T \times (3 + T \, l \, a)}$$

and when $T = 1$

$$\frac{G^2}{F^2} = \frac{1}{3 + l \, a}$$
$$\therefore \gamma = \pi \sqrt{\frac{1}{3 + l \, a}}$$

Hence if $a = e$ the hyperbolic base, since $l \, e = 1$, we have

$$\gamma = \frac{\pi}{2}.$$

Ex. 4. Let $f(A) = e^A = u$.

Then

$$\frac{du}{dx} = e^A$$
$$\therefore U = e^T$$

and

$$T \cdot U' = T e^T$$
$$\therefore \frac{G^2}{F^2} = \frac{1}{T}$$
$$\therefore \gamma = \pi.$$

Ex. 5. Let $\frac{f(A)}{A^3} = \sin. A$.

$$u = f(A) = A^3 \sin. A$$
$$\therefore U = T^3 \sin. T$$

and

$$\frac{du}{dx} = 3 A^2 \sin. A + A^3 \cos. A$$
$$\therefore T U' = 3 T^3 \sin. T + T^4 \cos. T$$
$$\therefore \frac{G^2}{F^2} = \frac{\sin. T}{3 \sin. T + T \cos. T}$$
$$\therefore \gamma = \pi \sqrt{\frac{\sin. T}{3 \sin. T + T \cos. T}}.$$

If $T = \frac{\pi}{4}$. Then

$$\gamma = \pi \sqrt{\frac{1}{3 + \frac{\pi}{4}}}.$$

363. *To prove that*

$$\frac{b A^m + c A^n}{A^3} = \frac{1}{b+c} \cdot A^{\frac{mb+nc}{b+c} - 3}.$$

$$\begin{aligned} b A^m + c A^n &= b.(1-x)^m + c.(1-x)^n \\ &= b + c - (mb + nc)x + \&c. \\ &= \frac{1}{b+c}\left(1 - \frac{mb+nc}{b+c}x + \&c.\right). \\ &= \frac{1}{b+c} \times (1-x)^{\frac{mb+nc}{b+c}} \\ &= \frac{1}{b+c} \cdot A^{\frac{mb+nc}{b+c}}. \end{aligned}$$

364. *To find the apsides when the excentricity is infinitely great.*

Make

$2q : \sqrt{(n+1)} ::$ velocity in the curve : velocity in the circle of the same distance a.

Then (306) it easily appears that when $F \propto \varrho^n$

$$d\theta = \frac{q a^{\frac{n+3}{2}} d\varrho}{\varrho \sqrt{(a^{n+1} - \varrho^{n+1})\varrho^2 - q^2 a^{n+1}(a^2 - \varrho^2)}}$$

and

$$\frac{d\varrho}{d\theta} = 0$$

gives the equation to the apsides, viz.

$$(a^{n+1} - \varrho^{n+1})\varrho^2 - q^2 a^{n+1}(a^2 - \varrho^2) = 0$$

whose roots are

a (and $-$ a when n is odd) and a positive and negative quantity (and when n is odd another negative quantity).

Now when $q = 0$

$$(a^{n+1} - \varrho^{n+1})\varrho^2 = 0$$

two of whose roots are 0, 0, and the roots above-mentioned consequently arise from q, which will be very small when q is.

Again since

$$1 - \frac{\varrho^{n+1}}{a^{n+1}} - \frac{a^2 q^2}{\varrho^2} + q^2 = 0$$

when q and ϱ are both very small

$$1 - \frac{a^2 q^2}{\varrho^2} = 0$$

and

$$\varrho = \pm a q.$$

$\therefore$ the lower apsidal distance is a q.

A nearer approximation is

$$\varrho = \pm \frac{a q}{\sqrt{(1 + q^2)}}.$$

Hence

$$d\theta = \frac{q a^{\frac{n+3}{2}} d\varrho}{\varrho \sqrt{(\varrho^2 - a^2 q^2 + \beta) \times Q}}$$

where β contains q^4 &c. &c., and this must be integrated from $\varrho = b$ to $\varrho = a$ ($b = a q$).

But since in the variation of ϱ from b to c, Q may be considered constant, we get

$$\theta = \text{sec.}^{-1}. \frac{\varrho}{a q} + C = \text{sec.}^{-1}. \frac{c}{a q}.$$

and

$$\gamma = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \text{ \&c. ultimately}$$

the apsidal distances required.

Next let

$$F \propto \frac{1}{\varrho^n} \text{ and } = \frac{f a^n}{\varrho^n}.$$

Then again, make

v : v in a circle of the same distance : : q $\sqrt{}$ 2 : $\sqrt{}$ (n — 1)

and we get (306)

$$'d\theta = \frac{q a d\varrho}{\varrho \sqrt{a^{n-1} \varrho^{3-n} - (1 - q^2) \varrho^2 - a^2 q^2}}$$

and for the apsidal distances

$$1 - \frac{\varrho^{n-1}}{a^{n-1}} + \frac{q^2 \varrho^{n-1}}{a^{n-1}} - \frac{q^2 a^{3-n}}{\varrho^{3-n}} = 0$$

which gives ($n > 1$ and < 3)

$$\varrho = a q^{\frac{2}{3-n}}$$

Hence

$$\theta = \int \frac{a q d\varrho}{\varrho \sqrt{(\varrho^{3-n} - q^2 a^{3-n} + \beta) \times Q}}$$

$$= \frac{1}{\sqrt{Q}} \cdot \int \frac{a q d\varrho}{\varrho \sqrt{(\varrho^{3-n} - q^2 a^{3-n}}}$$

and

$$\gamma = \frac{2}{3-n} \sec.^{-1} \frac{c^{\frac{3-n}{2}}}{q a^{\frac{3-n}{2}}} = \frac{\pi}{3-n} = \frac{3\pi}{3-n}, \text{ \&c.}$$

Hence, the orbit being indefinitely excentric, when

$$F \propto \varrho \quad . \quad . \quad . \quad . \quad \text{we have} \quad . \quad . \quad . \quad . \quad \gamma = \frac{\pi}{2}$$

for

$$F \propto \frac{1}{\text{any number} < 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \gamma = \frac{\pi}{2}$$

$$F \propto \frac{1}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \gamma = \frac{\pi}{2}$$

$$F \propto \frac{1}{\varrho \text{ number between 1 and 2}} \quad . \quad . \quad . \quad . \quad \gamma > \frac{\pi}{2} < \pi$$

$$F \propto \frac{1}{\varrho^{>2<3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \gamma > \pi.$$

But by the principles of this 9th Section when the excentricity is indefinitely small, and $F \propto \varrho^n$

$$\gamma = \frac{\pi}{\sqrt{(n+3)}}$$

(see 358), and when

$$F \propto \frac{1}{\varrho^n}$$

$$\gamma = \frac{\pi}{\sqrt{(3-n)}}$$

Wherefore when n is > 1
γ increases as the excentricity from

$$\frac{\pi}{\sqrt{(3+n)}} \text{ to } \frac{\pi}{2}.$$

When $F \propto \varrho$

$\gamma = \frac{\pi}{2}$ is the same for all excentricities.

When $F \propto \varrho^{\pm \frac{p}{q}}$
γ decreases as the excentricity increases from

$$\frac{\pi}{\sqrt{(3-n)}} \text{ to } \frac{\pi}{2}$$

which is also true for $F \propto \frac{1}{\varrho}$.

When $F \propto \frac{1}{\rho^{>1<2}}$

γ decreases as the excentricity increases from

$$\frac{\pi}{\sqrt{(3-n)}} \text{ to } \frac{\pi}{3-n}.$$

When $F \propto \frac{1}{\rho^2}$

$$\gamma = \pi.$$

When $F \propto \frac{1}{\rho^{>2<3}}$

γ increases with the excentricity from

$$\frac{\pi}{\sqrt{(3-n)}} \text{ to } \frac{\pi}{3-n}.$$

If the above concise view of the method of finding the apsides in this particular case, the opposite of the one in the text, should prove obscure; the student is referred to the original paper from which it is drawn, viz. a very able one in the Cambridge Philosophical Transactions, Vol. I, Part I, p. 179, by Mr. Whewell.

365. We shall terminate our remarks upon this Section by a brief discussion of the general apsidal equations, or rather a recapitulation of results—the details being developed in Leybourne's Mathematical Repository, —by Mr. Dawson of Sedburgh.

It will have been seen that the equation of the apsides is of the form

$$x^n - A x^m - B = 0 \quad \ldots\ldots\ldots \quad (1)$$

the equation of Limits to which is (see Wood's Algeb.)

$$n x^{n-1} - m A x^{m-1} = 0 \quad \ldots\ldots\ldots \quad (2)$$

and gives

$$x = \left(\frac{m}{n} A\right)^{\frac{1}{n-m}}.$$

If n and m are even and A positive, x has two values, and the number of real roots cannot exceed 4 in that case.

Multiply (1) by n and (2) by x and then we have

$$(m-n) A x^m - n B = 0$$

which gives

$$x = \left(\frac{n}{m-n}\right)^{\frac{1}{m}} \left(\frac{B}{A}\right)^{\frac{1}{m}}$$

and this will give two other limits if A, B be positive and m even; and if (1) have two real roots they must each = x.

If m, n be even and B, A positive, there will be two pairs of equal roots. Make them so and we get

$$\frac{(m-n)^{n-m}}{n^{n-m}} A^n - \left(\frac{n}{m}\right)^m B^{n-m} = 0$$

which will give the number of real roots.

(1). If n be even and B positive there are two real roots.

(2). If n be even, m odd, and B negative and (M), the coefficient to A^n, negative, there are two; otherwise none.

(3). If n, m, be even, A, B, negative, there are no real roots.

(4). If m, n be even, B negative, and A positive, and (M) positive there are four real roots; otherwise none.

(5). If m, n be odd, and (M) positive there will be three or one real.

(6). If m be even, n odd, and A, B have the same sign, there will be but one.

(7). If m be even, n odd, and A, B have different signs, and M's sign differs from B's, there will be three or only one.

(8). If

$$x^n + A n^m - B = 0$$

then

$$\left(\frac{m-n}{n}\right)^{n-m} A^n.$$

is positive, and it must be $>$ B, and the whole must be positive.

If

$$x^n - A x^m + B = 0$$

the result is negative.

SECTION X.

366. PROP. XLVI. The shortest line that can be drawn to a plane from a given point is the perpendicular let fall upon it. For since Q C S = right $\angle$, any line Q S which subtends it must be $>$ than either of the others in the same triangle, or S C is $<$ than any other S C.

A familiar application of this proposition is this:

367. *Let* S Q *be a sling with a body* Q *at the end of it, and by the hand* S *let it be whirled so as to describe a right cone whose altitude is* S C, *and base the circle whose radius is* Q C; *required the time of a revolution.*

Let S C = h, S Q = l, Q C = r = $\sqrt{l^2 - h^2}$.

Then if F denote the resolved part of the tension S Q in the direction Q C, or that part which would cause the body to describe the circle P Q, and gravity be denoted by 1, we have

$$F : 1 :: r : h$$

$$\therefore F = \frac{r}{h}.$$

But by 134, or Prop. IV,

$$F \times P^2 = \frac{4\pi^2 \times r}{g} = \frac{r}{h} \times P^2$$

$$\therefore P = 2\pi \sqrt{\frac{h}{g}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

the time required.

If the time of revolution (P) be *observed*, then h may be hence obtained.

If a body were to revolve round a circle in a paraboloidal surface, whose axis is vertical, then the reaction of the surface in the direction of the normal will correspond to the tension of the string, and the subnormal, which is constant, will represent h. Consequently the times of all such revolutions is constant for every such circle.

368. Prop. XLVII. When the excentricity of the ellipse is indefinitely diminished it becomes a straight line in the limit, &c. &c. &c.

369. Scholium. In these cases it is sufficient to consider the motion in the generating curves.]

Since the surface is supposed perfectly smooth, whilst the body moves through the generating curve, the surface, always in contact with the body, may revolve about the axis of the curve with any velocity whatever, without deranging in the least the motion of the body; and thus by adjusting the angular velocity of the surface, the body may be made to trace any proposed path on the surface.

If the surface were not perfectly smooth the friction would give the body a tangential velocity, and thence a centrifugal force, which would cause a departure from both the curve and surface, unless opposed by their material; and even then in consequence of the resolved pressure a rise or fall in the surface.

Hence it is clear that the time of describing any portion of a path in a surface of revolution, is equal to the time of describing the corresponding portion of the generating curve.

Thus when the force is in the center of a sphere, and whilst this force causes the body to describe a fixed great-circle, the sphere itself revolves with a uniform angular velocity, the path described by the body on the surface of the sphere will be the Spiral of Pappus.

370. PROP. XLVIII and XLIX. *In the Epicycloid and Hypocycloid,*

$$s : 2\,\text{vers.}\,\frac{s'}{2} :: 2(R \pm r) : R$$

where s *is any arc of the curve,* s′ *the corresponding one of the wheel, and* R *the radius of the globe and* r *that of the wheel, the* + *sign being used for the former and* — *in the Hypocycloid.* (See Jesuits' notes.)

OTHERWISE.

If p be the perpendicular let fall from C upon the tangent V P, we have from similar triangles in the Epicycloid and Hypocycloid

$$PY : CB :: VY : VC$$

or

$$\varrho^2 - p^2 : R^2 :: (R \pm 2r)^2 - p^2 : (R \pm 2r)^2$$

which gives

$$p^2 = (R \pm 2r)^2 \times \frac{\varrho^2 - R^2}{(R \pm 2r)^2 - R^2} \quad \ldots\ldots \quad (1)$$

Now from the incremental figure of a curve we have generally

$$\frac{ds}{d\varrho} = \frac{\varrho}{\sqrt{(\varrho^2 - p^2)}} \quad \ldots\ldots\ldots \quad (2)$$

But

$$\varrho^2 - p^2 = \frac{R^2}{(R \pm 2r)^2 - R^2} \times \{(R \pm 2r)^2 - \varrho^2\}$$

$$\therefore ds = \frac{2\sqrt{r^2 \pm Rr}}{R} \times \frac{\varrho\, d\varrho}{\sqrt{(R \pm 2r)^2 - \varrho^2}}$$

and integrating from

$$s = 0, \text{ when } \varrho = R$$

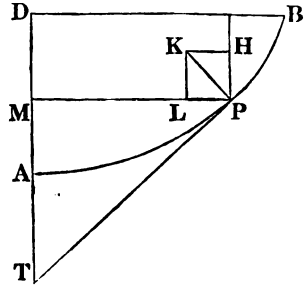
we get

$$s = \frac{2\sqrt{r^2 \pm Rr}}{R} \times \{\sqrt{(R \pm 2r)^2 - R^2} - \sqrt{(R \pm 2r)^2 - \varrho^2}\}$$

which is easily transformed to the proportion enunciated.

The subsequent propositions of this section shall now be headed by a succinct view of the analytical method of treating the same subject.

371. Generally, *A body being constrained to move along a given curve by known forces, required its velocity.*



Let the body P move along the curve P A, referred to the coordinates x, y originating in A; and let the forces be resolved into others which shall act parallel to x, y and call the respective aggregates X, Y. Besides these we have to consider the reaction (R) of the

curve along the normal P K, which being resolved into the same directions gives (d s, being the element of the curve)

$$R\frac{dx}{ds}, \text{ and } R\frac{dy}{ds}.$$

Hence the whole forces along x and y are (see 46)

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= X + R\frac{dy}{ds}\\ \frac{d^2y}{dt^2} &= Y - R\frac{dx}{ds}\end{aligned}\right\}$$

Again, eliminating R, we get

$$\frac{2\,dx\,d^2x + 2\,dy\,d^2y}{dt^2} = 2X\,dx + 2Y\,dy$$

and

$$\frac{dx^2 + dy^2}{dt^2} = 2\int(X\,dx + Y\,dy)$$

But

$$v^2 = \frac{ds^2}{dt^2} = \frac{dx^2 + dy^2}{dt^2} \quad (46)$$

$$\therefore v^2 = 2\int(X\,dx + Y\,dy) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence it appears that *The velocity is independent of the reaction of the curve.*

372. If the force be constant and in parallel lines, such as gravity, and x be vertical; then

$$X = -g$$

and

$$Y = 0$$

and we have

$$\begin{aligned}v^2 &= 2\int -g\,dx\\ &= 2g(c - x)\\ &= 2g(h - x)\end{aligned}$$

h being the value of x, when $v = 0$; and the height from which it begins to fall.

373. *To determine the motion in a common cycloid, when the force is gravity.*

The equation to the curve A P is

$$dy = dx\sqrt{\frac{2r - x}{x}}$$

r being the radius of the generating circle.

$$\therefore ds = dx\sqrt{\frac{2r}{x}}$$

and

$$dt = -\frac{ds}{\sqrt{2g}.\sqrt{(h-x)}} = \sqrt{\frac{r}{g}} \times \frac{dx}{\sqrt{(hx-x^2)}}$$

$$\therefore t = C - \sqrt{\frac{r}{g}}\,\text{vers.}^{-1}\frac{2x}{h} \quad (86)$$

$$= \sqrt{\frac{r}{g}}\left\{\pi - \text{vers.}^{-1}\frac{2x}{h}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

t being = 0, when x = h.

Hence the whole time of descent to the lowest point is

$$\frac{T}{2} = \pi\sqrt{\frac{r}{g}}$$

which also gives the time of an oscillation.

374. *Required the time of an oscillation in a small circular arc.*

Here

$$y = \sqrt{(2rx - x^2)}$$

r being the radius of the circle, and

$$ds = \frac{r\,dx}{\sqrt{(2rx-x^2)}}$$

$$\therefore dt = \frac{ds}{\sqrt{2g}\sqrt{(h-x)}}$$

$$= -\frac{r}{\sqrt{2g}} \times \frac{dx}{\sqrt{\{(h-x)(2rx-x^2)\}}}$$

$$= -\frac{r}{\sqrt{2g}} \times \frac{dx}{\sqrt{\{(hx-x^2)(2r-x)\}}}$$

to integrate which, put

$$\theta = \text{sin.}^{-1}\sqrt{\frac{x}{h}},$$

$$\therefore d\theta = \frac{dx}{2\sqrt{(hx-x^2)}}$$

and since

$$\sqrt{\frac{x}{h}} = \text{sin.}\,\theta$$

$$x = h\,\text{sin.}^2\theta,\ 2r - x = 2r - h\,\text{sin.}^2\theta$$

$$= 2r(1 - \delta^2\,\text{sin.}^2\theta),\ \delta^2 \text{ being put} = \frac{h}{2r}$$

$$\therefore dt = -\sqrt{\frac{r}{g}} \times \frac{d\theta}{\sqrt{(1-\delta^2\,\text{sin.}^2\theta)}}.$$

Now since the circular arc is small, h is small; and therefore δ is so. And by expanding the denominator we get

$$\int \frac{d\theta}{\sqrt{(1-\delta^2\,\text{sin.}^2\theta)}} = \int d\theta\left\{1 + \frac{\delta^2}{2}\,\text{sin.}^2\theta + \frac{1.3}{2.4}\delta^4\,\text{sin.}^4\theta + \&c.\right\}$$

and integrating by parts or by the formula

$$\int d\theta \,.\, \sin.^{m}\theta = -\frac{1}{m}\cos.\theta \sin.^{m-1}\theta + \frac{m-1}{m}\int d\theta \sin.^{m-2}\theta$$

and taking it from

$$\theta = 0 \text{ to } \theta = \frac{\pi}{2}$$

we get

$$\int' d\theta \sin.^{m}\theta = \frac{m-1}{m}\int' d\theta \sin.^{m-2}\theta$$

the accented $\int'$ denoting the Definite Integration from $\theta = 0$, to $\theta = \frac{\pi}{2}$.

In like manner

$$\int' d\theta \sin.^{m-2}\theta = \frac{m-3}{m-2}\int' d\theta \sin.^{m-4}\theta$$

and so on to

$$\int' d\theta \sin.^{2}\theta = \tfrac{1}{2}\int' d\theta = \tfrac{1}{2}\cdot\frac{\pi}{2}.$$

Hence

$$\int' d\theta \sin.^{m}\theta = \frac{(m-1)(m-3)\ldots\ldots 1}{m(m-2)\ldots\ldots\ldots 2}\times\frac{\pi}{2}$$

and

$$\int \frac{d\theta}{\sqrt{(1-\delta^{2}\sin.^{2}\theta}} \text{ from } \left.\begin{matrix}\theta = 0\\ \theta = \frac{\pi}{2}\end{matrix}\right\}$$

is the same as

$$\int -\frac{d\theta}{\sqrt{(1-\delta^{2}\sin.^{2}\theta)}} \text{ from } \left.\begin{matrix}\theta = \frac{\pi}{2}\\ \theta = 0\end{matrix}\right\}$$

whence then

$$t = \frac{\pi}{2}\sqrt{\frac{r}{g}}\left\{1 + \left(\frac{\delta}{2}\right)^{2} + \left(\frac{1.3\,\delta^{2}}{2.4}\right)^{2} + \left(\frac{1.3.5.\delta^{2}}{2.4.6}\right)^{3} + \&c.\right\} \quad . \quad (2)$$

and taking the first term only as an approximate value

$$t = \frac{\pi}{2}\sqrt{\frac{r}{g}} \quad \ldots\ldots\ldots\ldots \quad (3)$$

which equals the time down a cycloidal arc whose radius is $\frac{r}{4}$.

If we take two terms we have

$$t = \frac{\pi}{2}\sqrt{\frac{r}{g}}\left(1 + \frac{\delta^{2}}{4}\right)$$

$$= \frac{\pi}{2}\sqrt{\frac{r}{g}}\left(1 + \frac{h}{8r}\right) \quad \ldots\ldots\ldots \quad (4)$$

375. To determine the velocity and time in a Hypocycloid, the force tending to the center of the globe and $\propto \rho$.

By (370)

the equation to the Hypocycloid is

$$p^2 = \frac{R^2 - \rho^2}{R^2 - (2r - R)^2} \times (2r - R)^2$$
$$= \frac{R^2 - \rho^2}{R^2 - D^2} \times D^2$$

by hypothesis.

Now calling the force tending to the center F, we have

$$X = -F \times \frac{x}{\rho}, \; Y = -F \times \frac{y}{\rho}$$
$$\therefore \int (X dx + Y dy) = -\int F \frac{x\,dx + y\,dy}{r}$$
$$= -\int F\,d\rho$$
$$\therefore v^2 = C - 2\int F\,d\rho \quad \ldots\ldots\ldots\ldots \quad (1)$$

But by the supposition

$$F = \mu\rho$$
$$\therefore v^2 = \mu(h^2 - \rho^2) \quad \ldots\ldots\ldots\ldots \quad (2)$$

Hence

$$dt = \frac{ds}{v} = -\frac{\sqrt{R^2 - D^2}}{R\sqrt{\mu}} \times \frac{\rho\,d\rho}{\sqrt{\{(\rho^2 - D^2)(h^2 - \rho^2)\}}}.$$

To integrate it, put

$$\rho^2 - D^2 = u^2$$
$$\therefore \frac{\rho\,d\rho}{\sqrt{(\rho^2 - D^2)}} = du$$
$$h^2 - \rho^2 = h^2 - D^2 - u^2$$

and

$$dt = -\frac{\sqrt{R^2 - D^2}}{R\sqrt{\mu}} \cdot \frac{du}{\sqrt{(h^2 - D^2 - u^2)}}$$
$$\therefore t = \frac{\sqrt{(R^2 - D^2)}}{R\sqrt{\mu}} \cos.^{-1} \sqrt{\frac{\rho^2 - D^2}{h^2 - D^2}}.$$

Hence making $\rho = D$, we have

$$\frac{\text{Oscill.}}{2} = \frac{\pi}{2}\sqrt{\frac{R^2 - D^2}{R^2\mu}} \quad \ldots\ldots\ldots \quad (3)$$

376. Since h does not enter the above expression the descents are Isochronous.

We also have it in another form, viz.

$$\frac{T}{2} = \pi\sqrt{\left(\frac{r}{R\mu} - \frac{r^2}{R^2\mu}\right)}.$$

If $R\mu = g$ or force of gravity and R be large compared with b,

$$\frac{T}{2} = \pi\sqrt{\frac{r}{g}}$$

the same as in the common cycloid.

377. *Required to find the value of the reaction* R, *when a body is constrained to move along a given curve.*

As before (46)

$$\frac{d^2x}{dt^2} = X + R\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = Y - R\frac{dx}{ds}.$$

Hence

$$\frac{dy\,d^2x - dx\,d^2y}{dt^2} = X\,dy - Y\,dx + R\,ds$$

$$\therefore R = -\frac{X\,dy - Y\,dx}{ds} + \frac{dy\,d^2x - dx\,d^2y}{dt^2\,ds}.$$

But if r be the radius of curvature, we have (74)

$$r = \frac{ds^3}{dy\,d^2x - dx\,d^2y}.$$

Hence

$$R = \frac{Y\,dx - X\,dy}{ds} + \frac{ds^2}{r\,dt^2} \quad \ldots \quad (1)$$

Another expression is

$$\left.\begin{aligned} R &= \frac{Y\,dx - X\,dy}{ds} + \frac{v^2}{r} \\ \text{or} \qquad &= \frac{Y\,dx - X\,dy}{ds} + \varphi \end{aligned}\right\} \quad \ldots \quad (2)$$

φ being the centrifugal force.

If the body be acted on by gravity only

$$\left.\begin{aligned} R &= \frac{g\,dy}{ds} + \frac{ds^2}{r\,dt^2} \\ \text{or} \qquad &= \frac{g\,dy}{ds} + \frac{v^2}{r} \\ \text{or} \qquad &= \frac{g\,dy}{ds} + \varphi \end{aligned}\right\} \quad \ldots \quad (3)$$

If the body be moved by a constant force in the origin of x, y, we have

$$Y\,dx - X\,dy = F\frac{x\,dy - y\,dx}{\varrho}$$

$$= F\,\varrho\,d\theta.$$

for

$$x\,dy - y\,dx = \varrho^2\,d\theta$$

$$\therefore R = \frac{F\varrho\,d\theta}{ds} + \frac{ds^2}{r\,dt^2}$$

or

$$= \frac{F\varrho\,d\theta}{ds} + \frac{v^2}{r}$$

or

$$= \frac{F\varrho\,d\theta}{ds} + \varphi \qquad \ldots\ldots (4)$$

378. *To find the tension of the string in the oscillation of a common cycloid.*

Here

$$R = g\frac{dy}{ds} + \frac{ds^2}{r\,dt^2}$$

but

$$dy = dx\sqrt{\frac{2a - x}{x}}$$

$$ds = dx\sqrt{\frac{2a}{x}}$$

$$\frac{dy}{ds} = \sqrt{\frac{2a - x}{2a}}$$

and

$$r = 2\sqrt{2a}\sqrt{(2a - x)}$$

$$\frac{ds^2}{dt^2} = 2g(h - x)$$

$$\therefore R = g\sqrt{\frac{2a - x}{2a}} + \frac{g(h - x)}{\sqrt{2a}\sqrt{(2a - x)}}$$

$$= g\cdot\frac{2a + h - 2x}{\sqrt{(4a^2 - 2ax)}}.$$

When $x = h$

$$R = g\cdot\frac{2a - h}{\sqrt{(4a^2 - 2ah)}} = g\frac{\sqrt{(2a - h)}}{\sqrt{(2a)}}.$$

When $x = 0$

$$R = g\cdot\frac{2a + h}{2a} = g\left(1 + \frac{h}{2a}\right).$$

When moreover $h = 2a$, the pressure at A the lowest point is $= 2g$.

379. *To find the tension when the body oscillates in a circular arc by gravity.*

Here

$$dy = \frac{(c - x)\,dx}{\sqrt{(2cx - x^2)}}$$

$$ds = \frac{c\,dx}{\sqrt{(2cx - x^2)}}$$

$$\frac{dy}{dx} = \frac{c - x}{c}$$

$$r = c$$

$$\frac{ds^2}{dt^2} = 2g(h - x)$$

$$R = g \cdot \frac{c - x}{c} + \frac{2g(h - x)}{c}$$

$$= g \cdot \frac{c + 2h - 3x}{c}.$$

When $x = 0$

$$R = g \cdot \frac{c + 2h}{c}$$

$$= 3g \text{ or } h = c.$$

If it fall through the whole semicircle from the highest point

$$h = 2c,$$

and

$$R = 5g$$

or the tension at the lowest point is five times the weight.

When this tension $= 0$,

$$c + 2h - 3x = 0, \text{ or } x = \frac{c + 2h}{3}.$$

A body moving along a curve whose plane is vertical will quit it when

$$R = 0$$

that is when

$$x = \frac{c + 2h}{3}$$

and then proceed to describe a parabola.

380. *To find the motion of a body upon a surface of revolution, when acted on by forces in a plane passing through the axis.*

Referring the surface to three rectangular axes x, y, z, one of which (z) is the axis of revolution, another is also situated in the plane of forces, and the third perpendicular to the other two.

Let the forces which act in the plane be resolved into two, one parallel to the axis of revolution Z, and the other F, into the direction of the radius-vector, projected upon the plane perpendicular to this axis. Then,

calling this projected radius ϱ, and resolving the reaction R (which also takes place in the same plane as the forces) into the same directions, these components are

$$R\frac{dz}{ds}$$

$$R\frac{d\varrho}{ds}$$

supposing $ds = \sqrt{(dz^2 + d\varrho^2)}$ and the whole force in the direction of ϱ' is

$$F + R\frac{dz}{ds}$$

and resolving this again parallel to x and y, we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\left(F + R\frac{dz}{ds}\right)\frac{x}{\varrho} \\ \frac{d^2y}{dt^2} &= -\left(F + R\frac{dz}{ds}\right)\frac{y}{\varrho} \\ \text{and} \\ \frac{d^2z}{dt^2} &= -Z + R\frac{d\varrho}{ds} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

Hence we get

$$\frac{x\,d^2y - y\,d^2x}{dt^2} = 0 = d.\frac{x\,dy - y\,dx}{dt} \quad \ldots\ldots\ldots \quad (2)$$

and

$$\frac{dx\,d^2x + dy\,d^2y + dz\,d^2z}{dt^2} = -F.\frac{x\,dx + y\,dy}{\varrho}$$

$$-Z\,dz - R.\frac{dz}{ds}\left\{\frac{x\,dx + y\,dy}{\varrho} - \frac{d\varrho\,dz}{ds}\right\} \quad \ldots \quad (3)$$

Which, since

$$\frac{x\,dx + y\,dy}{\varrho} = d\varrho$$

$$d\left(\frac{dx^2 + dy^2 + dx^2}{dt^2}\right) = -2F\,d\varrho - 2Z\,dz.$$

Again

$$\frac{dz^2}{dt^2} = \frac{dz^2}{d\varrho^2}\cdot\frac{d\varrho^2}{dt^2},$$

and from the nature of the section of the surface made by a plane passing through the axis and body, $\frac{dz}{d\varrho}$ is known in terms of ϱ. Let therefore

$$\frac{dz}{d\varrho} = p$$

and we have

$$\frac{d z^2}{d t^2} = p^2 \frac{d \varrho^2}{d t^2}.$$

Also let the angle corresponding to ϱ be θ, then

$$x\,dy - y\,dx = \varrho^2\,d\theta$$

and

$$dx^2 + dy^2 = d\varrho^2 + \varrho^2\,d\theta^2,$$

and substituting the equations (2) and (3) become

$$d.\frac{\varrho^2\,d\theta}{dt} = 0$$

$$d\left\{\frac{d\varrho^2}{dt^2} + \frac{\varrho^2\,d\theta^2}{dt^2} + p^2\frac{d\varrho^2}{dt^2}\right\} = -2F\,d\varrho - 2Z\,dz.$$

Integrating the first we have

$$\varrho^2\,d\theta = h\,dt$$

h being the arbitrary constant.

or

$$dt = \frac{\varrho^2\,d\theta^2}{h} \quad \ldots\ldots\ldots\ldots \quad (4)$$

The second can be integrated when

$$-2F\,d\varrho - 2Z\,dz$$

is integrable. Now if for F, Z, z we substitute their values in terms of ϱ, the expression will become a function of ϱ and its integral will be also a function of ϱ. Let therefore

$$\int(F\,d\varrho + Z\,dz) = Q$$

and we get

$$\frac{d\varrho^2}{dt^2} + \frac{\varrho^2\,d\theta^2}{dt^2} + p^2.\frac{d\varrho^2}{dt^2} = C - 2Q \quad \ldots\ldots\ldots \quad (5)$$

which gives, putting for dt its value

$$(1 + p^2)\frac{h^2\,d\varrho^2}{\varrho^4\,d\theta^2} + \frac{h^2}{\varrho^2} = C - 2Q$$

$$\therefore d\theta = \frac{\sqrt{(1 + p^2)}\,h\,d\varrho}{\sqrt{\{(c - 2Q)\varrho^2 - h^2\}}} \quad \ldots\ldots\ldots \quad (6)$$

Hence also

$$dt = \frac{\sqrt{(1 + p^2)}.\varrho\,d\varrho}{\sqrt{\{(c - 2Q)\varrho^2 - h^2\}}} \quad \ldots\ldots\ldots \quad (7)$$

If the force be always parallel to the axis, we have

$$F = 0$$

and if also Z be a constant force, or if

$$Z = g$$

we then have

$$Q = \int Z\,dz = g\,z \quad \ldots\ldots\ldots \quad (8)$$

Z being to be expressed in terms of ϱ.

381. *To find under what circumstances a body will describe a circle on a surface of revolution.*

For this purpose it must always move in a plane perpendicular to the axis of revolution; ϱ, z will be constant; also (Prop. IV)

$$\varrho \cos. \theta = x$$

$$\therefore \frac{d^2 x}{d t^2} = - \frac{\varrho \cos. \theta \, d \theta^2}{d t^2}$$

Also

$$v = \frac{\varrho \, d \theta}{d t}$$

$$\therefore \frac{d^2 x}{d t^2} = - \frac{v^2 \cos. \theta}{\varrho}.$$

Hence as in the last art.

$$\left.\begin{aligned} \frac{v^2}{\varrho} &= F + R \frac{d z}{d s} \\ 0 &= - Z + R \frac{d \varrho}{d s} \end{aligned}\right\}$$

$$\therefore \frac{v^2}{\varrho} = F + Z \frac{d z}{d \varrho} \quad \ldots\ldots\ldots \quad (1)$$

If the force be gravity acting vertically along z, we have

$$Z = g$$

$$\frac{v^2}{\varrho} = g \frac{d z}{d \varrho}.$$

Hence may be found the time of revolution of a Conical Pendulum. (See also 367.)

382. To determine the motion of a body moving so as not to describe a circle, when acted on by gravity.

Here

$$Q = g z$$

and

$$C - 2 Q = 2 g . (k - z)$$

k being an arbitrary quantity.

Also

$$\varrho^2 = 2 r z - z^2$$

z being measured from the surface.

$$\therefore \varrho \, d \varrho = (r - z) \, d z$$

and

$$1 + p^2 = 1 + \frac{\varrho^2}{(r - z)^2} = \frac{r^2}{(r - z)^2}.$$

Hence (380)

$$d t = \frac{\sqrt{(1 + p^2)}\, \varrho\, d \varrho}{\sqrt{\{2 g (k - z) . (2 r z - z^2) - h^2\}}}.$$

In order that

$$\frac{d z}{d t} = 0$$

the denominator of the above must be put $= 0$; i. e.

$$2 g (k - z) (2 r z - z^2) - h^2 = 0$$

or

$$z^3 - (k + 2 r) z^2 + 2 k r z - \frac{h^2}{2 g} = 0$$

which has two possible roots; because as the body moves, it will reach one highest and one lowest point, and therefore two places when

$$\frac{d z}{d t} = 0.$$

Hence the equation has also a third root. Suppose these roots to be

$$\alpha, \beta, \gamma$$

where α is the greatest value of z, and β the least, which occur during the body's motion.

Hence

$$d t = \frac{r\, d z}{\sqrt{(2 g)}\ \sqrt{\{(\alpha - z) . (z - \beta)\ (\gamma - z)}}}.$$

To integrate which let

$$\theta = \text{sin.}^{-1} \sqrt{\frac{z - \beta}{\alpha - \beta}}.$$

Then

$$d \theta = \frac{d z}{2 \sqrt{\{(z - \beta) . (\alpha - \beta)\}} \sqrt{\left\{1 - \frac{z - \beta}{\alpha - \beta}\right\}}}$$

$$= \frac{d z}{2 \sqrt{\{(\alpha - z)\ (z - \beta)\}}}$$

Also

$$\text{sin.}^2 \theta = \frac{z - \beta}{\alpha - \beta}$$

$$\therefore z = \beta + (\alpha - \beta) \text{ sin.}^2 \theta$$

and

$$\gamma - z = \gamma - \{\beta + (\alpha - \beta) \text{ sin.}^2 \theta\}$$

$$= (\gamma - \beta) \{1 - \delta^2 \text{ sin.}^2 \theta\},$$

if

$$\delta = \sqrt{\frac{\alpha - \beta}{\gamma - \beta}}$$

$$\therefore dt = \frac{2r\,d\theta}{\sqrt{2g.(\gamma-\beta).\sqrt{\{1-\delta^2\sin.^2\theta\}}}}$$

which is to be integrated from $z=\beta$, to $z=\alpha$; that is from

$$\theta = 0 \text{ to } \theta = \frac{\pi}{2}$$

this expanded in the same way as in 374 gives

$$t = \frac{2r}{\sqrt{2g(\gamma-\beta)}} \times \left\{1+\left(\frac{\delta}{2}\right)^2+\left(\frac{1.3\,\delta^2}{2.4}\right)^2+\&c.\right\}_{2}^{\pi}$$

which is the time of a whole oscillation from the least to the greatest distance.

Also

$$d\theta = \frac{h\,dt}{\varrho^2} = \frac{h\,dt}{2rz-z^2}$$

and θ is hence known in terms of z.

383. *A body acted on by gravity moves on a surface of revolution whose axis is vertical: when its path is nearly circular, it is required to find the angle between the apsides of the path projected in the plane of* x, y.

In this case

$$\int Z\,dz = gz = Q$$

and if at an apse

$$\varrho = a,\ z = k$$

we have

$$(C-2gk)a^2-h^2 = 0$$

$$\therefore C = \frac{h^2}{a^2}+2gk.$$

Hence (380)

$$d\theta = \frac{\sqrt{(1+p^2)}\,h\,d\varrho}{\varrho\sqrt{\left\{2g(k-z)\varrho^2-\frac{h^2(a^2-\varrho^2)}{a^2}\right\}}}$$

$$= \frac{\sqrt{(1+p^2)}\,\frac{h\,d\varrho}{\varrho^2}}{\sqrt{\left\{2g(k-z)-h^2\left(\frac{1}{\varrho^2}-\frac{1}{a^2}\right)\right\}}}$$

Let $\frac{1}{\varrho} = \frac{1}{a}+\omega$

$$\therefore -\frac{d\varrho}{\varrho^2} = d\omega$$

$$\therefore d\theta = -\frac{\sqrt{(1+p^2)}\,h\,d\omega}{\sqrt{\left\{2g(k-z)-h^2\left(\frac{1}{\varrho^2}-\frac{1}{a^2}\right)\right\}}}$$

$$\therefore \frac{d\omega^2}{d\theta^2} = \frac{2g(k-z) - h^2\left(\frac{1}{\varrho^2} - \frac{(1}{a^2}\right)}{h^2(1+p^2)}.$$

It is requisite to express the right-hand side of this equation in terms of ω

Now since at an apse we have

$$\omega = 0,\ z = k, \text{ and } \varrho = a$$

we have generally

$$z = k + \frac{dz}{d\omega}\omega + \frac{d^2z}{d\omega^2}\cdot\frac{\omega^2}{1.2} + \&c.$$

the values of the differential coefficients being taken for

$$\omega = 0 \text{ (see 32)}$$

And

$$dz = p\,d\varrho = -p\varrho^2\,d\omega$$
$$d^2z = -2p\varrho\,d\varrho\,d\omega - \varrho^2\,d\omega\,dp$$

or, making

$$dp = q\,d\varrho$$
$$d^2z = -(2p+q\varrho)\varrho\,d\varrho\,d\omega = (2p+q\varrho)\varrho^3 d\omega^2.$$

And if $p_{,}$ and $q_{,}$ be the values which p and q assume when $\omega = 0$, $\varrho = a$, we have for that case,

$$\frac{d^2z}{d\omega^2} = (2p_{,} + q_{,}a)\,a^3$$

$$Z = k - p_{,}a^2\omega + (2p + q_{,}a)\,a^3\cdot\frac{\omega^2}{2} - \&c.$$

Also

$$\frac{1}{\varrho^2} = \left(\frac{1}{a} + \omega\right)^2 = \frac{1}{a^2} + \frac{2\omega}{a} + \omega^2.$$

Hence

$$2g(k-z) - h^2\left(\frac{1}{\varrho^2} - \frac{1}{a^2}\right)$$

becomes

$$2g\left(p_{,}a^2\omega - (2p_{,} + q_{,}a)\,a^3\cdot\frac{\omega^2}{2} + \&c.\right) - h^2\left(\frac{2\omega}{a} + \omega^2\right).$$

But when a body moves in a circle of radius $= a$, we have

$$h^2 = g\varrho^3 p = g a^3 p_{,}$$

in this case. And when the body moves *nearly* in a circle, h^2 will have nearly this value. If we put

$$h^2 = (1+\delta)\,g a^3 p_{,}$$

we shall finally have to put

$$\delta = 0$$

in order to get the ultimate angle when the orbit becomes indefinitely near a circle. Hence we may put

$$h^2 = g a^3 p_,$$

and

$$2 g (k - z) - h^2 \left(\frac{1}{\varrho^3} - \frac{1}{a^2}\right)$$

becomes

$$- \{3 g a^3 p_, + g a^4 q_,\} \omega^2 + \&c.$$

in which the higher powers of ω may be neglected in comparison of ω^2;

$$\therefore \frac{d \omega^2}{d \theta^2} = - \frac{g a^3 (3 p_, + q_, a) \omega^2}{h^2 (1 + p^2)} = \frac{- (3 p_, + q_, a) \omega^2}{p_, (1 + p^2)}$$

$$= - \frac{(3 p_, + q_, a) \omega^2}{p_, (1 + p_,^2)}$$

again omitting powers above ω^2: for $p = p_, + A \omega + \&c.$

Differentiate and divide by $2 d \omega$, and we have

$$\frac{d^2 \omega}{d \theta^2} = - \frac{3 p_, + q_, a}{p_, (1 + p_,^2)} \cdot \omega = - N \omega$$

suppose; of which the integral is taken so that

$$\theta = 0, \text{ when } \omega = 0$$

is

$$\omega = C \sin. \theta \surd N.$$

And ω passes from 0 to its greatest value, and consequently ϱ passes from the value a, to another maximum or minimum, while the arc $\theta \surd N$ passes from 0 to π. Hence, for the angle A between the apsides we have

$$A \surd N = \pi \text{ or } A = \frac{\pi}{\surd N}$$

where

$$N = \frac{3 p_, + q_, a}{p_, (1 + p_,^2)}.$$

384. *Let the surface be a sphere and let the path described be nearly a circle; to find the horizontal angle between the apsides.*

Supposing the origin to be at the lowest point of the surface, we have

$$z = r - \surd (r^2 - \varrho^2)$$

$$p = \frac{d z}{d \varrho} = \frac{\varrho}{\surd (r^2 - \varrho^2)}$$

$$q = \frac{d p}{d \varrho} = \frac{r^2}{(r^2 - \varrho^2)^{\frac{3}{2}}}$$

$$\therefore p_, = \frac{a}{\surd (r^2 - a^2)}$$

$$q_{,} = \frac{r^2}{(r^2 - a^2)^{\frac{3}{2}}}$$

$$1 + p_{,}^2 = \frac{r^2}{r^2 - a^2}$$

$$\therefore N = \frac{4r^2 - 3a^2}{r^2}.$$

Hence the angle between the apsides is

$$A = \frac{\pi r}{\sqrt{(4r^2 - 3a^2)}}.$$

The motion of a point on a spherical surface is manifestly the same as the motion of a simple pendulum or heavy body, suspended by an inextensible string from a fixed point; the body being considered as a point and the string without weight. If the pendulum begin to move in a vertical plane, it will go on oscillating in the same plane in the manner already considered. But if the pendulum have any lateral motion it will go on revolving about the lowest point, and generally alternately approaching to it, and receding from it. By a proper adjustment of the velocity and direction it may describe a circle (134); and if the velocity when it is moving parallel to the horizon be *nearly* equal to the velocity in a circle, it will describe a curve little differing from a circle. In this case we can find the angle between the greatest and least distances, by the formula just deduced.

Since

$$A = \frac{\pi r}{\sqrt{(4r^2 - 3a^2)}}$$

if $a = 0$, $A = \frac{\pi}{2}$, the apsides are 90° from each other, which also appears from observing that when the amplitude of the pendulum's revolution is very small, the force is nearly as the distance; and the body describes ellipses nearly; of which the lowest point is the center.

If $a = r$,

$$A = \pi = 180^\circ$$

this is when the pendulum string is horizontal; and requires an infinite velocity.

If $a = \frac{r}{2}$; so that the string is inclined 30° to the vertical;

$$A = \frac{2\pi}{\sqrt{13}} = 99^\circ\ 50'.$$

If $a^2 = \frac{r^2}{2}$; so that the string is inclined 45° to the vertical;

$$A = \pi \sqrt{\frac{2}{5}} = 113°.56'.$$

If $a^2 = \frac{3r^2}{4}$; so that the string is inclined 60° to the vertical;

$$A = \frac{2\pi}{\sqrt{7}} = 136° \text{ nearly.}$$

385. *Let the surface be an inverted cone, with its axis vertical: to find the horizontal angle between the apsides when the orbit is nearly a circle.*

Let r be the radius of the circle and γ the angle which the slant side makes with the horizon. Then

$$z = \varrho \tan. \gamma$$

$$p = \tan. \gamma$$

$$q = 0$$

$$\therefore N = \frac{3 \tan. \gamma}{\tan. \gamma. \sec.^2 \gamma} = 3 \cos.^2 \gamma$$

and

$$A = \frac{\pi}{\cos. \gamma \sqrt{3}}.$$

$$\text{If } \gamma = 60°$$

$$A = \frac{2\pi}{3} = 120°.$$

386. *Let the surface be an inverted paraboloid whose parameter is* c.

$$\varrho^2 = c z$$

$$\therefore p = \frac{dz}{d\varrho} = \frac{2\varrho}{c}$$

$$q = \frac{2}{c}$$

$$\therefore N = \frac{\frac{6a}{c} + \frac{2a}{c}}{\frac{2a}{c}\left(1 + \frac{4a^2}{c^2}\right)} = \frac{4c^2}{c^2 + 4a^2}.$$

If $a = \frac{c}{2}$, or the body revolve at the extremity of the focal ordinate,

$$N = 2$$

and

$$A = \frac{\pi}{\sqrt{2}}.$$

387. *When a body moves on a conical surface, acted on by a force tending to the vertex; its motion in the surface will be the same, as if the surface were unwrapped, and made plane, the force remaining at the vertex.*

Measuring the radius-vector (ϱ) from the vertex, let the force be F, and the angle which the slant side makes with the base $= \gamma$; then

$$z = \varrho \tan. \gamma$$
$$p = \tan. \gamma$$
$$1 + p^2 = \sec.^2 \gamma$$

also

$$Q = f(F\, d\varrho + Z\, dz) = f F'\, d\varrho'.$$

Hence (380)

$$d\theta = \frac{\sec. \gamma\, h\, d\varrho}{\varrho \sqrt{\{(C - 2f F'\, d\varrho')\, \varrho^2 - h^2\}}}$$

or putting

$$h' \cos. \gamma \text{ for } h$$
$$d\theta' \sec. \gamma \text{ for } d\theta$$

and

$$\varrho' \cos. \gamma \text{ for } \varrho$$

we have

$$d\theta' = \frac{h'\, d\varrho'}{\varrho' \sqrt{\{(C - 2f F'\, d\varrho')\, \varrho'^2 - h'^2\}}}.$$

Now $d\theta'$ is the differential of the angle described *along the conical surface*, and it appears that the relation between θ' and ϱ' will be the same as in a plane, where a body is acted upon by a central force F. For in that case we have

$$d\left\{\frac{h'^2 d\varrho'^2}{\varrho'^4 d\theta'^2} + \frac{h'^2}{\varrho'^2}\right\} = -2 F' d\varrho'$$

and integrating

$$\frac{h'^2 d\varrho'^2}{\varrho'^4 d\theta'^2} + \frac{h'^2}{\varrho'^2} = C - 2f F' d\varrho'$$

which agrees with the equation just found.

388. *When a body moves on a surface of revolution, to find the reaction* R.

Take the three original equations (380) and multiply them by $x\, dz$, $y\, dz$, $\varrho\, d\varrho$; and the two first become

$$\frac{x\, d^2 x\, dz}{dt^2} = -\frac{F' x^2 dz}{\varrho} - R \frac{dz^2}{ds} \cdot \frac{x^2}{\varrho}$$
$$\frac{y\, d^2 y\, dz}{dt^2} = -\frac{F' y^2 dz}{\varrho} - R \frac{dz^2}{ds} \cdot \frac{y^2}{\varrho}$$

add these, observing that

$$x^2 + y^2 = \varrho^2$$

and we have

$$\frac{(x\,d^2x + y\,d^2y)\,dz}{dt^2} = -F'\varrho\,dz - R\varrho\frac{dz^2}{ds}.$$

Also the third is

$$\frac{\varrho\,d\varrho\,d^2z}{dt^2} = -Z\varrho\,d\varrho + R\varrho\frac{d\varrho^2}{ds}.$$

Subtract this, observing that $dz^2 + d\varrho^2 = ds^2$, and we have

$$\frac{(x\,d^2x + y\,d^2y)\,dz - \varrho\,d\varrho\,d^2z}{dt^2} =$$

$$\varrho\,(Z\,d\varrho - F'\,dz) - R\varrho\,ds.$$

But

$$x^2 + y^2 = \varrho^2$$

$$x\,dx + y\,dy = \varrho\,d\varrho$$

$$x\,d^2x + y\,d^2y + dx^2 + dy^2 = \varrho\,d^2\varrho + d\varrho^2.$$

Hence

$$\frac{(d\varrho^2 - dx^2 - dy^2)\,dz}{dt^2} + \frac{\varrho\,dz\,d^2\varrho - \varrho\,d\varrho\,d^2z}{dt^2} =$$

$$\varrho\,(Z\,d\varrho - F'\,dz) - R\varrho\,ds$$

and

$$d\varrho^2 = ds^2 - dz^2.$$

Hence

$$R = \frac{Z\,d\varrho - F'\,dz}{ds} + \frac{d\varrho\,d^2z - dz\,d^2\varrho}{dt^2\,ds}$$

$$+ \frac{(dx^2 + dy^2 + dz^2 - ds^2)\,dz}{\varrho\,dt^2\,ds}.$$

Now if r be the radius of curvature, we have (74)

$$r = \frac{ds^3}{d\varrho\,d^2z - dz\,d^2\varrho}$$

and

$$dx^2 + dy^2 + dz^2 = d\sigma^2$$

σ being the arc described.

Hence

$$R = \frac{Z\,d\varrho - F'\,dz}{ds} + \frac{ds^2}{r\,dt^2}$$

$$+ \frac{d\sigma^2 - ds^2}{\varrho\,dt^2}\cdot\frac{dz}{ds} \quad \ldots\ldots\ldots \quad (1)$$

Here it is manifest that

$$\frac{ds^2}{dt^2}$$

is the square of the velocity resolved into the generating curve, and that

$$\frac{d\sigma^2 - ds^2}{dt^2}$$

is the square of the velocity resolved perpendicular to ϱ. The two last terms which involve these quantities, form that part of the resistance which is due to the centrifugal force; the first term is that which arises from the resolved part of the forces.

From this expression we know the value of R; for we have, as before

$$\frac{d\sigma^2}{dt^2} = C - 2\int(F'd\varrho + Z dz).$$

Also

$$\frac{d\sigma^2 - ds^2}{dt^2} = \frac{\varrho^2 d\theta^2}{dt^2} = \frac{h^2}{\varrho^2}.$$

Hence

$$\frac{ds^2}{dt^2} = C - 2\int(F'd\varrho + Z dz)$$
$$= \frac{h^2}{\varrho^2}.$$

389. *To find the tension of a pendulum moving in a spherical surface.*

$$C - 2\int(F'd\varrho + Z dz) = 2g(k - z)$$
$$\varrho = \sqrt{(2rz - z^2)}$$
$$\frac{d\varrho}{dz} = \frac{r - z}{\sqrt{(2rz - z^2)}}$$
$$\frac{ds}{d\varrho} = \frac{r}{r - z}$$
$$\frac{ds}{dz} = \frac{r}{\sqrt{(2rz - z^2)}} = \frac{r}{\varrho}.$$

Hence

$$R = \frac{g(r - z)}{r} + \frac{2g(k - z) - \frac{h^2}{\varrho^2}}{r} + \frac{h^2}{\varrho^3}\cdot\frac{\varrho}{r}$$
$$= \frac{g(r + 2k - 3z)}{r}$$

and hence it is the same as that of the pendulum oscillating in a vertical plane with the same velocity at the same distances.

390. *To find the Velocity, Reaction, and Motion of a body upon any surface whatever.*

Let R be the reaction of the surface, which is in the direction of a normal to it at each point. Also let ε, ε', ε'' be the angles which this normal

makes with the axes of x, y, z respectively; we shall then have, considering the resolved parts of R among the forces which act on the point

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= X + R\cos.\,\varepsilon\\ \frac{d^2y}{dt^2} &= Y + R.\cos.\,\varepsilon'\\ \frac{d^2z}{dt^2} &= Z + R.\cos.\,\varepsilon''\end{aligned}\right\}$$

Now the nature of the surface is expressed by an equation between x, y, z: and if we suppose that we have deduced from this equation

$$dz = p\,dx + q\,dy$$

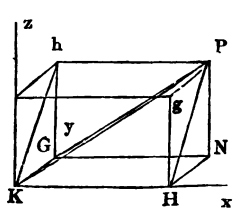
$$\text{where } p = \frac{dz}{dx} \text{ and } q = \frac{dz}{dy},$$

p and q being taken on the supposition of y and x being constants respectively; we have for the equations to the normal of the points whose coordinates are

$$\left.\begin{aligned}x, y, z\\ x' - x + p(z' - z) &= 0\\ y' - y + q(z' - z) &= 0\end{aligned}\right\}$$

x', y', z' being coordinates to any point in the normal (see Lacroix, No. 143.)

Hence it appears that if P K be the normal, P G, P H its projections on planes parallel to x z, y z respectively.



The equation of P G is

$$x' - x + p.(z' - z) = 0,$$

and hence

$$GN + p\,PN = 0$$

and

$$GN = -p.PN.$$

Similarly the equation of P H is

$$y' - y + q(z' - z) = 0$$

whence

$$HN + q.PN = 0$$

$$HN = -q.PN.$$

And hence,

$$\cos.\,\varepsilon = \cos.\,KPh = \frac{Ph}{PK} = \frac{GN}{\sqrt{(PN^2 + NG^2 + HN^2)}}$$

$$= -\frac{p}{\sqrt{(1+p^2+q^2)}}$$

$$\cos.\,\varepsilon' = \cos.\,K\,P\,g = \frac{Pg}{PK}$$

$$= -\frac{HN}{\sqrt{(PN^2+NG^2+HN^2)}}$$

$$= -\frac{q}{\sqrt{(1+p^2+q^2)}}.$$

Whence, since

$$\cos.^2\varepsilon + \cos.^2\varepsilon' + \cos.^2\varepsilon'' = 1$$

$$\cos.^2\varepsilon'' = \sqrt{(1-\cos.^2\varepsilon-\cos.^2\varepsilon')}$$

$$= \frac{1}{\sqrt{(1+p^2+q^2)}}.$$

Substituting these values; multiplying by d x, d y, d z respectively, in the three equations; and observing that

$$dz - p\,dx - q\,dy = 0$$

we have

$$\frac{dx\,d^2x + dy\,d^2y + dz\,d^2z}{dt^2} = X\,dx + Y\,dy + Z\,dz$$

and integrating

$$\frac{dx^2+dy^2+dz^2}{dt^2} = 2f(X\,dx + Y\,dy + Z\,dz)$$

and if this can be integrated, we have the velocity.

If we take the three original equations, and multiply them respectively by $-p$, $-q$, and 1, and then add, we obtain

$$-p\frac{d^2x}{dt^2} - q\cdot\frac{d^2y}{dt^2} + \frac{d^2z}{dt^2} =$$

$$-pX - qY + Z + R\sqrt{(1+p^2+q^2)}.$$

But

$$dz = p\,dx + q\,dy.$$

Hence

$$\frac{d^2z}{dt^2} = p\frac{d^2x}{dt^2} + q\frac{d^2y}{dt^2} + \frac{dp\,dx + dq\,dy}{dt^2}.$$

Substituting this on the first side of the above equation, and taking the value of R, we find

$$R = \frac{pX+qY-Z}{\sqrt{(1+p^2+q^2)}} + \frac{dp\,dx + dq\,dy}{dt^2\sqrt{(1+p^2+q^2)}}$$

If in the three original equations we eliminate R, we find two second differential equations, involving the known forces

X, Y, Z

and p, q, which are also known when the surface is known, combining with these the equation to the surface, by which z is known in terms of x, y, we have equations from which we can find the relation between the time and the three coordinates.

391. *To find the path which a body will describe upon a given surface, when acted upon by no force.*

In this case we must make

$$X, Y, Z \text{ each } = 0.$$

Then, if we multiply the three equations of the last art. respectively by

$$-(q\,dz + dy),\ p\,dz + dx,\ q\,dx - p\,dy$$

and add them, we find,

$$-(q\,dz + dy)\,d^2x + (p\,dz + dx)\,d^2y + (q\,dx - p\,dy)\,d^2z$$
$$= R\,dt^2 \left\{ \begin{array}{l} -(q\,dz + dy)\cos.\,\epsilon \\ +(p\,dz + dx)\cos.\,\epsilon' \\ +(q\,dx - p\,dy)\cos.\,\epsilon'' \end{array} \right\}$$

or putting for cos. ϵ, cos. ϵ', cos. ϵ'' their values

$$= \frac{R\,dt^2}{\sqrt{(1+p^2+q^2)}} \{p(q\,dz+dy) - q(p\,dz+dx) + q\,dx - p\,dy\} = 0.$$

Hence, for the curve described in this case, we have

$$(p\,dz+dx)\,d^2y = (p\,dy - q\,dx)\,d^2z + (q\,dz+dy)\,d^2x.$$

This equation expresses a relation between x, y, z, without any regard to the time. Hence, we may suppose x the independent variable, and $d^2x = 0$; whence we have

$$(p\,dz + dx)\,d^2y = (p\,dy - q\,dx)\,d^2z.$$

This equation, combined with

$$dz = p\,dx + q\,dy,$$

gives the curve described, where the body is left to itself, and moves along the surface.

The curve thus described is the shortest line which can be drawn from one of its points to another, upon the surface.

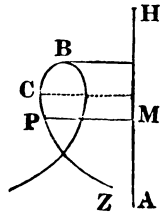
The velocity is constant as appears from the equation

$$v^2 = 2\int(X\,dx + Y\,dy + Z\,dz).$$

By methods somewhat similar we might determine the motion of a point upon a given curve of double curvature, or such as lies not in one plane when acted upon by given forces.

392. *To find the curve of equal pressure, or that on which a body descending by the force of gravity, presses equally at all points.*

Let A M be the vertical abscissa = x, M P the horizontal ordinate = y; the arc of the curve s, the time t, and the radius of curvature at P = r, r being positive when the curve is concave to the axis; then R being the reaction at P, we have by what has preceded.



$$R = \frac{g\,dy}{ds} + \frac{ds^2}{r\,dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But if H M be the height due to the velocity at P, A H = h, we have

$$\frac{ds^2}{dt^2} = 2g(h - x).$$

Also, if we suppose d s constant, we have (74)

$$r = -\frac{ds\,dx}{d^2y},$$

and if the constant value of R be k, equation (1) becomes

$$k = \frac{g\,dy}{ds} - \frac{2g(h-x)\,d^2y}{ds\,dx}$$

$$\therefore -\frac{k}{g}\cdot\frac{dx}{2\sqrt{(h-x)}} = \sqrt{(h-x)}\cdot\frac{d^2y}{ds} - \frac{dy}{ds}\cdot\frac{dx}{2\sqrt{(h-x)}}$$

The right-hand side is obviously the differential of

$$\sqrt{(h-x)}\,\frac{dy}{ds};$$

hence, integrating

$$\frac{k}{g}\cdot\sqrt{(h-x)} = \sqrt{(h-x)}\cdot\frac{dy}{ds} + C,$$

$$\frac{dy}{ds} = \frac{k}{g} - \frac{C}{\sqrt{(h-x)}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If C = 0, the curve becomes a straight line inclined to the horizon, which obviously answers the condition. The sine of inclination is $\frac{k}{g}$.

In other cases the curve is found by equation (2), putting

$$\sqrt{(dx^2 + dy^2)} \text{ for } ds$$

and integrating.

If we differentiate equation (2), d s being constant, we have

$$\frac{d^2y}{ds} = -\frac{C\,dx}{2(h-x)^{\frac{3}{2}}}$$

$$r = -\frac{ds\,dx}{d^2y} = \frac{2(h-x)^{\frac{3}{2}}}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

And if C be positive, r is positive, and the curve is concave to the axis.

We have the curve parallel to the axis, as at C, when $\frac{dy}{ds} = 0$, that is, when $\frac{k}{g} = \frac{C}{\sqrt{(h-x)}}$; when

$$x = h - \frac{C^2 g^2}{k^2}.$$

When x increases beyond this, the curve approaches the axis, and $\frac{dy}{dx}$ is negative; it can never become < -1; hence B the limit of x is found by making

$$\frac{k}{g} - \frac{C}{\sqrt{(h-x)}} = -1;$$

or

$$x = h - \frac{C^2 g^2}{(k+g)^2}.$$

If k be $< g$, as the curve descends towards Z, it approximates perpetually to the inclination, the sine of which is $\frac{k}{g}$.

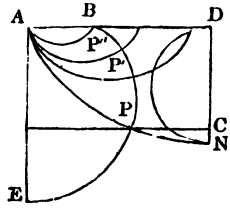
If k be $> g$ there will be a point at which the curve becomes horizontal.

C is known from (2), (3), if we knew the pressure or the radius of curvature at a given point.

If C be negative, the curve is convex to the axis. In this case the part of the pressure arising from centrifugal force diminishes the part arising from gravity, and k must be less than g.

393. *To find the curve which cuts a given assemblage of curves, so as to make them Synchronous, or descriptible by the force of gravity in the same time.*

Let A P, A P′, A P″, &c. be curves of the same kind, referred to a common base A D, and differing only in their parameters, (or the constants in their equations, such as the radius of a circle, the axes of an ellipse, &c.)



Let the vertical A M = x, M P (horizontal) = y; y and x being connected by an equation involving a. The time down A P is

$$\int \frac{dx}{\sqrt{(2gx)}}$$

the integral being taken between

$$x = 0 \text{ and } x = AM;$$

and this must be the same for all curves, whatever (a) may be.

Hence, we may put

$$\int \frac{ds}{\sqrt{(2gx)}} = k \quad \ldots \ldots \ldots \ldots \quad (1)$$

k being a constant quantity, and in differentiating, we must suppose (a) variable as well as x and s.

Let

$$ds = p\,dx$$

p being a function of x, and a which will be of 0 dimensions, because d x, and d s are quantities of the same dimensions. Hence

$$\int \frac{p\,dx}{\sqrt{(2gx)}} = k$$

and differentiating

$$\frac{p\,dx}{\sqrt{(2gx)}} + q\,da = 0 \quad \ldots \ldots \ldots \quad (2)$$

Now, since p is of 0 dimensions in x, and a, it is easily seen that

$$\int \frac{p\,dx}{\sqrt{(2gx)}}$$

is a function whose dimensions in x and a are $\frac{1}{2}$, because the dimensions of an expression are increased by 1 in integrating. Hence by a known property of homogeneous functions, we have

$$\frac{px}{\sqrt{(2gx)}} + qa = \tfrac{1}{2}k;$$

$$\therefore q = \frac{k}{2a} - \frac{p\sqrt{x}}{a\sqrt{(2g)}}$$

substituting this in equation (2) it becomes

$$\frac{p\,dx}{\sqrt{(2gx)}} + \frac{k\,da}{2a} - \frac{p\,da\sqrt{x}}{a\sqrt{(2g)}} = 0 \quad \ldots \ldots \ldots \quad (3)$$

in which, if we put for (a) its value in x and y, we have an equation to the curve P P′ P″.

If the given time (k) be that of falling down a vertical height (h), we have

$$k = \sqrt{\frac{2h}{g}},$$

and hence, equation (3) becomes

$$p(a\,dx - x\,da) + da\sqrt{(hx)} = 0 \quad \ldots \quad (4)$$

Ex. *Let the curves* A P, A P′, A P″ *be all cycloids of which the bases coincide with* A D.

Let C D be the axis of any one of these cycloids and = 2 a, a being the radius of the generating circle. If C N = x′, we shall have as before

$$-ds = dx'\sqrt{\frac{2a}{x'}}$$

and since

$$x' = 2a - x$$

$$ds = dx \sqrt{\frac{2a}{2a - x}}.$$

Hence

$$p = \sqrt{\frac{2a}{2a - x}},$$

and equation (4) becomes

$$\frac{\sqrt{(2a)}\,(a\,dx - x\,da)}{\sqrt{(2a - x)}} + d\sqrt{(hx)} = 0 \quad . \quad . \quad . \quad . \quad (5)$$

Let $\frac{x}{a} = u$

so that

$$a\,dx - x\,da = a^2\,du$$

$$x = au;$$

and substituting

$$\frac{a^2\,du\sqrt{2}}{\sqrt{(2 - u)}} + da\sqrt{(hau)} = 0$$

$$\therefore \frac{du\sqrt{2}}{\sqrt{(2u - u^2)}} + \frac{da\sqrt{h}}{a^{\frac{3}{2}}} = 0$$

$$\therefore \sqrt{2} \times \text{vers.}^{-1} u - 2\sqrt{\frac{h}{a}} = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When a is infinite, the portion A P of the cycloid becomes a vertical line, and

$$x = h, \quad \therefore u = 0, \quad \therefore C = 0.$$

Hence

$$\frac{x}{a} = \text{vers.} \sqrt{\frac{2h}{a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From this equation (a) should be eliminated by the equation to the cycloid, which is

$$y = a\,\text{vers.}^{-1}\frac{x}{a} - \sqrt{(2ax - x^2)} \quad . \quad . \quad . \quad . \quad (8)$$

and we should have the equation to the curve required.

Substituting in (8) from (7), we have

$$y = \sqrt{(2ah)} - \sqrt{(2ax - x^2)}$$

$$dy = \frac{da\sqrt{h}}{\sqrt{(2a)}} - \frac{x\,da + a\,dx - x\,dx}{\sqrt{(2ax - x^2)}}$$

and eliminating d a by (5)

$$\frac{dy}{dx} = -\frac{2a - x}{\sqrt{(2ax - x^2)}} = -\sqrt{\frac{2a - x}{x}}.$$

But differentiating (8) supposing (a) constant, we have in the cycloid

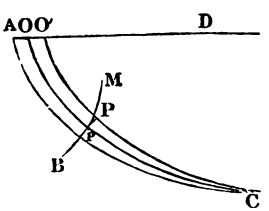
$$dy = \sqrt{\frac{x}{2a-x}}.$$

And hence (31) the curve P P′ P″ cuts the cycloids all at right angles, the subnormal of the former coinciding with the subtangent of the latter, each being

$$y\sqrt{\frac{2a-x}{x}}.$$

The curve P P′ P″ will meet A D in the point B, such that the given time is that of describing the whole cycloid A B. It will meet the vertical line in E, so that the body falls through A E in the given time.

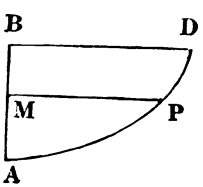
394. If instead of supposing all the cycloids to meet in the point A, we suppose them all to pass through any point C, their bases still being in the same line A D; a curve P P′ drawn so that the times down P C, P′ C, &c. are all equal, will cut all the cycloids at right angles. This may easily be demonstrated.



395. *To find Tautochronous curves or those down which to a given fixed point a body descending all distances shall move in the same time.*

(1) let the force be constant and act in parallel lines.

Let A the lowest point be the fixed point, D that from which the body falls, A B vertical, B D, M P horizontal. A M = x, A P = s, A B = h, and the constant force = g.



Then the velocity at P is

$$v = \sqrt{(2g.\overline{h-x})}$$

and

$$dt = -\frac{ds}{v} = \frac{-ds}{\sqrt{2g}\sqrt{(h-x)}}$$

and the whole time of descent will be found by integrating this from x = h, to x = 0.

Now, since the time is to be the same, from whatever point D the body falls, that is whatever be h, the integral just mentioned, taken between the limits, must be independent of h. That is, if we take the integral so as to vanish when

$$x = 0$$

and then put h for x, h will disappear altogether from the result. This must manifestly arise from its being possible to put the result in a form

involving only $\frac{x}{h}$, as $\frac{x^2}{h^2}$, &c.; that is from its being of 0 dimensions in x and h.

Let

$$ds = p\,dx$$

where p depends only on the curve, and does not involve h. Then, we have

$$t = -\int \frac{p\,dx}{\sqrt{\{2g(h-x)\}}}$$
$$= -\frac{1}{\sqrt{(2g)}}\int\left\{\frac{p\,dx}{h^{\frac{1}{2}}} + \frac{1}{2}\cdot\frac{p\,x\,dx}{h^{\frac{3}{2}}} + \frac{1.3}{2.4}\frac{p\,x^2\,dx}{h^{\frac{5}{2}}} + \&c.\right\}$$

and from what has been said, it is evident, that each of the quantities

$$\int\frac{p\,dx}{h^{\frac{1}{2}}}, \int\frac{p\,x\,dx}{h^{\frac{3}{2}}}, \int\frac{p\,x^n\,dx}{h^{\frac{2n+1}{2}}}$$

must be of the form

$$\frac{c\,x^{\frac{2n+1}{2}}}{h^{\frac{2n+1}{2}}};$$

that is

$$\int p\,x^n\,dx \text{ must} = c\,x^{\frac{2n+1}{2}};$$

hence

$$p\,x^n\,dx = \frac{2n+1}{2}c\,x^{\frac{2n-1}{2}}\,dx;$$

$$p = \frac{2n+1}{2}\cdot\frac{c}{x^{\frac{1}{2}}};$$

or if

$$\frac{2n+1}{2}c = a^{\frac{1}{2}}$$

$$p = \sqrt{\frac{a}{x}}$$

and

$$ds = dx\sqrt{\frac{a}{x}}$$

which is a property of the cycloid.

Without expanding, the thing may thus be proved. If p be a function of m dimensions in x, $\frac{p}{\sqrt{(h-x)}}$ is of $m - \frac{1}{2}$ dimensions; and as the dimensions of an expression are increased by 1 in integrating

$$\int\frac{p\,dx}{\sqrt{(h-x)}}$$

is of m + 1 dimensions in x, and when h is put for x, of m + $\frac{1}{2}$ dimensions in h. But it ought to be independent of h or of 0 dimensions. Hence

$$m + \tfrac{1}{2} = 0$$
$$\therefore p = a^{\frac{1}{2}} x^{-\frac{1}{2}}$$

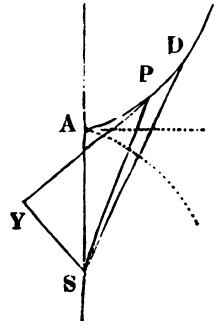
as before.

396. (2) *Let the force tend to a center and vary as any function of the distance. Required the Tautochronous Curve.*

Let S be the center of force, A the point to which the body must descend; D the point from which it descends. Let also

$$SA = e,\ SD = f,\ SP = \varrho,\ AP = s$$

P being any point whatever.



Now we have

$$v^2 = C - 2f F d\varrho$$

or if

$$2P d\varrho = \varphi(\varrho)$$
$$v^2 = \varphi(f) - \varphi(\varrho)$$

the velocity being 0 when $\varrho = f$.

Hence the time of describing D A is

$$t = f - \frac{ds}{\sqrt{(\varphi f - \varphi \varrho)}}$$

taken from $\varrho = f$, to $\varrho = e$. And since the time must be the same whatever is D, the integral so taken must be independent of f.

Let

$$\varphi\varrho - \varphi e = z$$
$$\varphi f - \varphi e = h$$
$$ds = p\,dz$$

p depending on the nature of the curve, and not involving f. Then

$$t = -\int \frac{p\,dz}{\sqrt{(h-z)}},\ \text{from } z = h \text{ to } z = 0$$
$$= \int \frac{p\,dz}{\sqrt{(h-z)}},\ \text{from } z = 0 \text{ to } z = h.$$

And this must be independent of f, and therefore of φf, and of h. Hence, after taking the integral the result must be 0 when z = 0, and independent of h, when h is put for z. Therefore it must be of 0 dimensions in z and h. But if p be of n dimensions in z, or if

$$p = c z^n$$

$\frac{p}{\sqrt{(h-z)}}$ will be of $n - \frac{1}{2}$ dimensions,

and

$$\int \frac{p\,dz}{\sqrt{(h-z)}} \text{ of } n + \tfrac{1}{2} \text{ dimensions.}$$

Hence, $n + \frac{1}{2} = 0$, $n = -\frac{1}{2}$, and

$$p = \sqrt{\frac{C}{z}}$$

Therefore

$$ds = dz\sqrt{\frac{C}{z}} = \varphi'\varrho\,d\varrho\sqrt{\frac{C}{\varphi\varrho - \varphi e'}};$$

whence the curve is known.

If θ be the angle A S O, we have

$$ds^2 = d\varrho^2 + \varrho^2 d\theta^2$$

and

$$d\theta^2 = \frac{ds^2 - d\varrho^2}{\varrho^2}$$

whence may be found a polar equation to the curve.

397. Ex. 1. Let the force vary as the distance, and be attractive. Then

$$F = \mu\varrho,\ \varphi\varrho = \mu\varrho^2;$$
$$z = \varphi\varrho - \varphi e = \mu(\varrho^2 - e^2);$$
$$dz = 2\mu\varrho\,d\varrho$$
$$ds = dz\sqrt{\frac{C}{z}} = 2\mu\varrho\,d\varrho\sqrt{\frac{C}{\mu(\varrho^2 - e^2)}}$$
$$= \varrho\,d\varrho\sqrt{\frac{4c\mu}{\varrho^2 - e^2}}$$

when $\varrho = e$, $\frac{ds}{d\varrho}$ is infinite or the curve is perpendicular to S A at A.

If S Y, perpendicular upon the tangent P Y, be called p, we have

$$\frac{p^2}{\varrho^2} = \frac{ds^2 - d\varrho^2}{ds^2}$$
$$= 1 - \frac{d\varrho^2}{ds^2}$$
$$= 1 - \frac{\varrho^2 - e^2}{4c\mu\varrho^2}$$
$$p^2 = \frac{e^2 - (1 - 4c\mu)\varrho^2}{4c\mu}.$$

If $e = 0$, or the body descend to the center, this gives the logarithmic spiral.

In other cases let

$$1 - 4c\mu = \frac{e^2}{a^2},$$

$$\therefore 4c\mu = \frac{a^2 - e^2}{a^2}$$

and

$$p^2 = \frac{e^2(a^2 - \varrho^2)}{a^2 - e^2}$$

the equation to the Hypocycloid (370)

If $4c\mu = 1$, the curve becomes a straight line, to which S A is perpendicular at A.

If $4c\mu$ be > 1 the curve will be concave to the center and go off to infinity.

398. Ex. 2. Let the force vary inversely as the square of the distance. Then

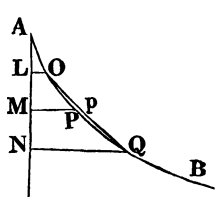
$$F = \frac{\mu}{\varrho^2};$$

and as before we shall find

$$p^2 = \varrho^2 - \frac{\varrho^5(\varrho - e)}{2\mu c e}.$$

399. *A body being acted upon by a force in parallel lines, in its descent from one point to another, to find the Brachystochron, or the curve of quickest descent between them.*

Let A, B be the given points, and A O P Q B the required curve. Since the time down A O P Q B is less than down any other curve, if we take another as A O p Q B, which coincides with the former, except for the arc O P Q, we shall have



Time down A O : T. O P Q + T. Q B, less than
Time down A O + T. O p Q + T. Q B

and if the times down Q B be the same on the two suppositions, we shall have

T. O P Q less than the time down any other arc O p Q.

The times down Q B will be the same in the two cases if the velocity at Q be the same. But we know that the velocity acquired at Q is the same, whether the body descend down

A O P Q, or A O p Q.

Hence it appears that *if the time down* A O P Q B *be a minimum, the time down any portion* O P Q *is also a minimum.*

Let a vertical line of abscissas be taken in the direction of the force; and perpendicular ordinates, O L, P M, Q N be drawn, it being supposed that

$$L M = M N.$$

Then, if L M, M N be taken indefinitely small, we may consider them as representing the differential of x: On this supposition, O P, P Q, will represent the differentials of the curve, and the velocity may be supposed constant in O P, and in P Q. Let

$$A L = x, L O = y, O A = s,$$

and let d x, d y, d s be the differentials of the abscissa, ordinate, and curve at Q, and v the velocity there; and d x′, d y′, d s′, v′ be the corresponding quantities at P. Hence the time of describing O P Q will be (46)

$$\frac{d s}{v} + \frac{d s'}{v'}$$

which is a minimum; and consequently its differential = 0. This differential is that which arises from supposing P to assume any position as p out of the curve O P Q; and as the differentials indicated by d arise from supposing P to vary its position along the curve O P Q, we shall use δ to indicate the differentiation, on hypothesis of passing from one curve to another, or the *variations* of the quantities to which it is prefixed.

We shall also suppose p to be in the line M P, so that d x is not supposed to vary. These considerations being introduced, we may proceed thus,

$$\delta\left\{\frac{d s}{v} + \frac{d s'}{v'}\right\} = 0 \quad \ldots\ldots\ldots \quad (1)$$

And v, v′ are the same whether we take O P Q, or O p Q; for the velocity at p = velocity at P. Hence

$$\delta v = 0, \delta v' = 0$$

and

$$\frac{\delta d s}{v} + \frac{\delta d s'}{v'} = 0.$$

Now

$$d s^2 = d x^2 + d y^2$$
$$\therefore d s \delta d s = d y \delta d y,$$
(for $\delta d x = 0$).

Similarly

$$d s' \delta d s' = d y' \delta d y.$$

Substituting the value of δ d s, δ d s′ which these equations give, we have

$$\frac{dy\,\delta dy}{v\,ds} + \frac{dy'\,\delta dy'}{v'\,ds'} = 0.$$

And since the points O, Q, remain fixed during the variation of P's position, we have

$$dy + dy' = \text{const.}$$
$$\delta dy' = -\delta dy.$$

Substituting, and omitting δ d y,

$$\frac{dy}{v\,ds} - \frac{dy'}{v'\,ds'} = 0.$$

Or, since the two terms belong to the successive points O, P, their difference will be the differential indicated by d; hence,

$$d.\frac{dy}{v\,ds} = 0$$

$$\therefore \frac{dy}{v\,ds} = \text{const.} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Which is the property of the curve; and v being known in terms of x, we may determine its nature.

Let the force be gravity; then

$$v = \sqrt{(2gx)};$$

$$\frac{dy}{ds\sqrt{(2gx)}} = \text{const.}$$

$$\frac{dy}{ds\sqrt{x}} = \frac{1}{\sqrt{a}}$$

a being a constant.

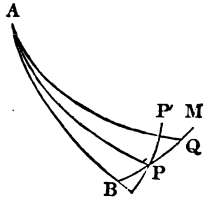
$$\therefore \frac{dy}{ds} = \sqrt{\frac{x}{a}}$$

which is a property of the cycloid, of which the axis is parallel to x, and of which the base passes through the point from which the body falls.

If the body fall from a given point to another given point, setting off with the velocity acquired down a given height; the curve of quickest descent is a cycloid, of which the base coincides with the horizontal line, from which the body acquires its velocity.

400. *If a body be acted on by gravity, the curve of its quickest descent from a given point to a given curve, cuts the latter at right angles.*

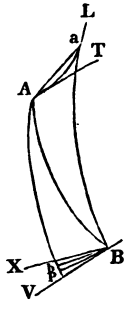
Let A be the given point, and B M the given curve; A B the curve of quickest descent cuts B M at right angles.



It is manifest the curve A B must be a cycloid, for otherwise a cycloid might be drawn from A to B, in which the descent would be shorter. If possible, let A Q be the cycloid of quickest descent, the angle A Q B being acute. Draw another cycloid A P, and let P P′ be the curve which cuts A P, A Q so as to make the arcs A P, A P′ *synchronous*. Then (394) P P′ is perpendicular to A Q, and therefore manifestly P′ is between A and Q, and the time down A P is less than the time down A Q; therefore, this latter is not the curve of quickest descent. Hence, if A Q be not perpendicular to B M, it is not the curve of quickest descent.

The cycloid which is perpendicular to B M may be the cycloid of *longest* descent from A to B M.

401. *If a body be acted on by gravity, and if* A B *be the curve of quickest descent from the curve* A L *to the point* B; A T, *the tangent of* A L *at* A, *is parallel to* B V, *a perpendicular to the curve* A B *at* B.



If B V be not parallel to A T, draw B X parallel to A T, and falling between B V and A. In the curve A L take a point *a* near to A. Let a B be the cycloid of quickest descent from the point a to the point B; and B b being taken equal and parallel to a A, let A b be a cycloid equal and similar to a B. Since A B V is a right angle, the curve B P, which cuts off A P synchronous to A B, has B V for a tangent. Also, ultimately A a coincides with A T, and therefore B b with B X. Hence B is between A and P. Hence, the time down A b is less than the time down A P, and therefore, than that down A B. And hence the time down a B (which is the same as that down A b) is less than that down A B. Hence, if B V be not parallel to A T, A B is not the line of quickest descent from A L to B.

402. *Supposing a body to be acted on by any forces whatever, to determine the Brachystochron.*

Making the same notations and suppositions as before, A L, L O, (see a preceding figure) being any rectangular coordinates; since, as before, the time down O P Q is a minimum, we have

$$\delta\left\{\frac{ds}{v}+\frac{ds'}{v'}\right\}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\delta d s}{v} + \frac{\delta d s'}{v'} - \frac{d s \delta v}{v^2} - \frac{d s' \delta v'}{v'^2} = 0.$$

Now as before we also have

$$\delta d s = \frac{d y \delta d y}{d s}$$

supposing $\delta d x = 0$, and

$$\delta d s' = \frac{d y' . \delta d y'}{d s'} = - \frac{d y' . \delta d y}{d s'}.$$

$$d v = 0$$

for v is the velocity at O and does not vary by altering the curve.

$$v' = v + d v$$

$$d v' = \delta v + \delta d v = \delta d v.$$

Hence

$$\frac{d y \delta d y}{v d s} - \frac{d y' \delta d y}{v' d s'} - \frac{d s' \delta d v}{v'^2} = 0.$$

Also

$$\frac{1}{v'} = \frac{1}{v + d v} = \frac{1}{v} - \frac{d v}{v^2};$$

for $d v^2$, &c. must be omitted. Substituting this in the second term of the above equation, we have

$$\frac{d y . \delta d y}{v d s} - \frac{d y' \delta d y}{v d s'} + \frac{d y' d v \delta d y}{v^2 d s'} - \frac{d s' \delta d v}{v'^2} = 0$$

or

$$- \left(\frac{d y'}{d s'} - \frac{d y}{d s}\right) . \frac{1}{v} + \frac{d y' . d v}{d s' . v^2} - \frac{d s'}{v'^2} . \frac{\delta d v}{\delta d y} = 0$$

Now as before

$$\frac{d y'}{d s'} - \frac{d y}{d s} = d . \frac{d y}{d s}.$$

And in the other terms we may, since O, P, are indefinitely near, put

$$d s, d y, v \text{ for } d s', d y', v':$$

if we do this, and multiply by $-v$, we have

$$d . \frac{d y}{d s} - \frac{d y . d v}{d s . v} + \frac{d s}{v} . \frac{\delta d v}{\delta d y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which will give the nature of the curve.

If the forces which act on the body at O, be equivalent to X in the direction of x, and Y in the direction of y, we have (371)

$$v d v = X d x + Y d y$$

$$\therefore d v = \frac{X d x + Y d y}{v}$$

$$\therefore \delta d v = \frac{Y \delta d y}{v}$$

because $\delta v = 0$, $\delta d x = 0$; also X and Y are functions of A L, and L O, and therefore not affected by δ.

Substituting these values in the equation to the curve, we have

$$d.\frac{dy}{ds} - \frac{dy}{ds}.\frac{X dx + Y dy}{v^2} + \frac{ds}{v}.\frac{Y}{v} = 0$$

or

$$d.\frac{dy}{ds} - \frac{dx}{ds}.\frac{X dy - Y dx}{v^2} = 0$$

which will give the nature of the curve.

If r be the radius of curvature, and d s constant, we have (from 74)

$$r = \frac{ds\,dx}{d^2y}$$

r being positive when the curve is convex to A M;

$$d.\frac{dy}{ds} = \frac{dx}{r}$$

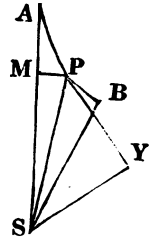
and hence

$$\frac{v^2}{r} = \frac{X dy - Y dx}{ds}.$$

The quantity $\frac{v^2}{r}$ is the centrifugal force (210), and therefore that part of the pressure which arises from it. And $\frac{X dy - Y dx}{ds}$ is the pressure which arises from resolving the forces perpendicular to the axis. Hence, it appears then in the Brachystochron for any given forces, the parts of the pressure which arise from the given forces and from the centrifugal force must be equal.

403. If we suppose the force to tend to a center S, which may be assumed to be in the line A M, and F to be the whole force; also if

$$SA = \alpha,\ SP = \varrho,\ SY = p;$$



then we have

$$\frac{X dy - Y dx}{ds} = \text{force in P S resolved parallel to } YS = F \times \frac{p}{\varrho}$$

and

$$v^2 = C - 2\int g F d\varrho$$

$$\therefore \frac{C - 2g\int F d\varrho}{r} = \frac{Fp}{\varrho}$$

also

$$r = -\frac{\varrho\, d\varrho}{dp}$$

$$\therefore C - 2g\int F\,d\varrho = -\frac{F p\,d\varrho}{dp}$$

$$\therefore \frac{2\,dp}{p} = \frac{-2F\,d\varrho}{C - 2g\int F\,d\varrho}$$

and integrating

$$p^2 = C'\{C - 2g\int F\,d\varrho\}$$

whence the relation of p and ϱ is known.

If the body begin to descend from A

$$C - 2g\int F\,d\varrho = 0$$

when $\varrho = a$.

404. Ex. 1. *Let the force vary directly as the distance.*

Here

$$F = \mu\varrho$$

$$C - 2g\int F\,d\varrho = v^2 = \mu g(a^2 - \varrho^2)$$

$$p^2 = C'\mu(a^2 - \varrho^2)$$

which agrees with the equation to the Hypocycloid (370).

405. Ex. 2. *Let the force vary inversely as the square of the distance;* then

$$F = \frac{\mu}{\varrho^2}$$

$$C - 2g\int F\,d\varrho = \frac{2g\mu}{\varrho} - \frac{2g\mu}{a}$$

$$p^2 = \frac{2\mu C'}{a}\cdot\frac{a-\varrho}{\varrho} = c^2\cdot\frac{a-\varrho}{\varrho}$$

by supposition.

$$\therefore \varrho^2 - p^2 = \frac{\varrho^3 + c^2\varrho - c^2 a}{\varrho},$$

$$d\theta = \frac{p\,d\varrho}{\varrho\sqrt{(\varrho^2 - p^2)}}$$

$$= \frac{c\sqrt{(a-\varrho)}\cdot d\varrho}{\varrho\sqrt{(\varrho^3 + c^2\varrho - c^2 a)}}$$

$$= \frac{c\,d\varrho}{\varrho\sqrt{\left\{\frac{\varrho^3}{a-\varrho} - c^2\right\}}}$$

When $\varrho = a$, $d\theta = 0$; when

$$\varrho^3 + c^2\varrho - c^2 a = 0$$

$d\theta$ is infinite, and the curve is perpendicular to the radius as at B. This equation has only one real root.

If we have $c = \frac{a}{2}$, $SB = \frac{a}{2}$

B being an apse.

$$\text{If } c = \frac{a}{10}, \; S\,B = \frac{a}{5}.$$

$$\text{If } c = \frac{a}{30}, \; S\,B = \frac{a}{10}.$$

$$\text{If } c = \frac{a}{n^3 + n}, \; S\,B = \frac{a}{n^2 + 1}.$$

406. *When a body moves on a given surface, to determine the Brachystochron.*

Let x, y, z be rectangular coordinates, x being vertical; and as before let d s, d s′ be two successive elements of the curve; and let

$$d\,x,\ d\,y,\ d\,z,$$
$$d\,x',\ d\,y',\ d\,z'$$

be the corresponding elements of x, y, z; then since the minimum property will be true of the indefinitely small portion of the curve, we have as before, supposing v, v′ the velocities,

$$\frac{d\,s}{v} + \frac{d\,s'}{v'} = \text{min.}$$

$$\therefore \delta . \left\{ \frac{d\,s}{v} + \frac{d\,s'}{v'} \right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The variations indicated by δ are those which arise, supposing d x, d x′ to be equal and constant, and d y, d z, d y′, d z′ to vary

Now

$$d\,s^2 = d\,x^2 + d\,y^2 + d\,z^2$$
$$\therefore d\,s\,\delta\,d\,s = d\,y\,\delta\,d\,y + d\,z\,\delta\,d\,z.$$

Similarly

$$d\,s'\,\delta\,d\,s' = d\,y'\,\delta\,d\,y' + d\,z'\,\delta\,d\,z.$$

Also, the extremities of the arc

$$d\,s + d\,s'$$

being fixed, we have

$$d\,y + d\,y' = \text{const.}$$
$$\therefore \delta\,d\,y + \delta\,d\,y' = 0$$
$$d\,z + d\,z' = \text{const.}$$
$$\therefore \delta\,d\,z + \delta\,d\,z' = 0.$$

Hence

$$\left.\begin{aligned} \delta\,d\,s &= \frac{d\,y}{d\,s}\,\delta\,d\,y + \frac{d\,z}{d\,s}\,\delta\,d\,z \\ \delta\,d\,s' &= -\frac{d\,y'}{d\,s'}\,\delta\,d\,y - \frac{d\,z'}{d\,s'}\,\delta\,d\,z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And the surface is defined by an equation between x, y, z, which we may call

$$L = 0.$$

Let this differentiated give

$$dz = p\,dx + q\,dy \quad \ldots\ldots\ldots \quad (3)$$

Hence, since dx, p, q are not affected by δ

$$\delta\,dz = q\,\delta\,dy \quad \ldots\ldots\ldots \quad (4)$$

For the sake of simplicity, we will suppose the body to be acted on only by a force in the direction of x, so that v, v′ will depend on x alone, and will not be affected by the variation of dy, dz. Hence, we have by (1)

$$\frac{\delta\,ds}{v} + \frac{\delta\,ds'}{v'} = 0$$

which, by substituting from (2) becomes

$$\left\{\frac{\delta\,y'}{v'\,ds'} - \frac{dy}{v\,ds}\right\}\delta\,dy + \left\{\frac{dz'}{v'\,ds'} - \frac{dz}{v\,ds}\right\}\delta\,dz = 0.$$

Therefore we shall have, as before

$$d.\frac{dy}{v\,ds}\,\delta\,dy + d.\frac{dz}{v\,ds}\,\delta\,dz = 0;$$

and by equation (4), this becomes

$$d.\frac{dy}{v\,ds} + q\,d.\frac{dz}{v\,ds} = 0 \quad \ldots\ldots\ldots \quad (5)$$

whence the equation to the curve is known.

If we suppose the body not to be acted on by any force, v will be constant, and the path described will manifestly be the shortest line which can be drawn on the given surface, and will be determined by

$$d.\frac{dy}{ds} + q.d.\frac{dz}{ds} = 0 \quad \ldots\ldots\ldots \quad (6)$$

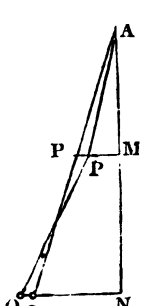
If we suppose ds to be constant, we have

$$d^2y + q\,d^2z = 0$$

which agrees with the equation there deduced for the path, when the body is acted on by no forces.

Hence, it appears that when a body moves along a surface undisturbed, it will describe the shortest line which can be drawn on that surface, between any points of its path.

407. *Let* P *and* Q *be two bodies, of which the first hangs from a fixed point and the second from the first by means of inextensible strings* AP, PQ; *it is required to determine the small oscillations.*



Let

$$AM = x,\ MP = y,$$
$$AN = x',\ NQ = y'$$
$$AP = a,\ PQ = a'$$
$$\text{mass of } P = \mu, \text{ of } Q = \mu'$$
$$\text{tension of } AP = p, \text{ of } PQ = p'.$$

Then resolving the forces p, p′, we have

$$\left.\begin{aligned}\frac{d^2y}{dt^2} &= \frac{p'g}{\mu}\cdot\frac{y'-y}{a'} - \frac{pg}{\mu}\cdot\frac{y}{a}\\ \frac{d^2y'}{dt^2} &= -\frac{p'g}{\mu'}\cdot\frac{y'-y}{a'}\end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

By combining these with the equations in x, x′ and with the two

$$x^2 + y^2 = a^2,$$
$$(x'-x)^2 + (y'-y)^2 = a'^2;$$

we should, by eliminating p, p′ find the motion. But when the oscillations are small, we may approximate in a more simple manner.

Let β, β' be the initial values of y, y′. Then manifestly, p, p′ will depend on the initial position of the bodies, and on their position at the time t: and hence we may suppose

$$p = M + P\beta + Q\beta' + R\,y + S\,y' + \&c.$$

and similarly for p′.

Now, in the equations of motion above, p, p′ are multiplied by y, y′ − y which, since the oscillations are very small are also very small quantities, (viz. of the order β). Hence their products with β will be of the order β^2, and may be neglected, and we may suppose p reduced to its first term M.

M is the tension of A P, when β, β' &c. are all = 0. Hence it is the tension when P, Q, hang at rest from A, and consequently

$$M = \mu + \mu'.$$

Similarly, the first term of p′, which may be put for it is m′. Substituting these values and dividing by g, equations (1) become

$$\left.\begin{aligned}\frac{d^2y}{g\,dt^2} &= -\left(\frac{\mu'}{\mu a'} + \frac{\mu+\mu'}{\mu a}\right)y + \frac{\mu'}{\mu a'}\,y'\\ \frac{d^2y'}{g\,dt^2} &= \frac{y}{a'} - \frac{y'}{a'}\end{aligned}\right\} \quad \ldots\ldots \quad (2$$

Multiply the second of these equations by λ and add it to the first, and we have

$$\frac{d^2y + \lambda\,d^2y'}{g\,dt^2} = -\left(\frac{\mu'}{\mu a'} + \frac{\mu+\mu'}{\mu a} - \frac{\lambda}{a'}\right)y - \left(\frac{\lambda}{a'} - \frac{\mu}{\mu a'}\right)y'$$

and manifestly this can be solved if the second member can be put in the form

$$-k\,.\,(y + \lambda y')$$

that is, if

$$k = \frac{\mu'}{\mu a'} + \frac{\mu+\mu'}{\mu a} - \frac{\lambda}{a'}$$

$$k\lambda = \frac{\lambda}{a'} - \frac{\mu'}{\mu a'}$$

or

$$\left.\begin{aligned} a' k &= \frac{\mu'}{\mu} + \frac{a'}{a} + \frac{\mu' a'}{\mu a} - \lambda \\ -\frac{\mu'}{\mu} &= (a' k - 1)\lambda \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3)$$

Eliminating λ we have

$$(a' k - 1) a' k - \frac{\mu'}{\mu} = (a' k - 1)\left(\frac{\mu'}{\mu} + \frac{a'}{a} + \frac{\mu' a'}{\mu a}\right)$$

Hence

$$(a' k)^2 - \left(1 + \frac{\mu'}{\mu}\right)\left(1 + \frac{a'}{a}\right) a' k = -\frac{a'}{a} - \frac{\mu' a'}{\mu a} \quad \ldots\ldots \quad (4)$$

From this equation we obtain two values of k. Let these be denoted by

$$^1k,\ ^2k$$

and let the corresponding values of λ, be

$$^1\lambda,\ ^2\lambda.$$

Then, we have these equations.

$$\frac{d^2 y + {}^1\lambda\, d^2 y'}{g\, dt^2} = -{}^1k\,(y + {}^1\lambda\, y')$$

$$\frac{d^2 y + {}^2\lambda\, d^2 y'}{g\, dt^2} = -{}^2k\,(y + {}^2\lambda\, y')$$

and it is easily seen that the integrals of these equations are

$$y + {}^1\lambda\, y' = {}^1C \cos.\, t\sqrt{({}^1k\, g)} + {}^1D \sin.\, t\sqrt{({}^1k\, g)}$$

$$y + {}^2\lambda\, y' = {}^2C \cos.\, t\sqrt{({}^2k\, g)} + {}^2D \sin.\, t\sqrt{({}^2k\, g)}$$

^1C, ^1D, ^2C, ^2D being arbitrary constants. But we may suppose

$$^1C = {}^1E \cos.\, {}^1e$$

$$^1D = {}^1E \sin.\, {}^1e$$

$$^2C = {}^2E \cos.\, {}^2e$$

$$D^2 = {}^2E \sin.\, {}^2e$$

By introducing these values we find

$$\left.\begin{aligned} y + {}^1\lambda\, y' &= {}^1E \cos.\, \{t\sqrt{({}^1k\, g)} + {}^1e\} \\ y + {}^2\lambda\, y' &= {}^2E \cos.\, \{t\sqrt{({}^2k\, g)} + {}^2e\} \end{aligned}\right\} \quad \ldots\ldots \quad (5)$$

From these we easily find

$$\left.\begin{aligned} y &= \frac{{}^2\lambda\, {}^1E}{{}^2\lambda - {}^1\lambda} \cos.\, \{t\sqrt{({}^1k\, g)} + {}^1e\} + \frac{{}^1\lambda\, {}^2E}{{}^2\lambda - {}^1\lambda} \cos.\, \{t\sqrt{({}^2k\, g)} + {}^2e\} \\ y' &= \frac{{}^1E}{{}^1\lambda - {}^2\lambda} \cos.\, \{t\sqrt{({}^1k\, g)} + {}^1e\} + \frac{{}^2E}{{}^1\lambda - {}^2\lambda} \cos.\, \{t\sqrt{({}^2k\, g)} + {}^2e\} \end{aligned}\right\} \quad \ldots \quad (6)$$

The arbitrary quantities ^1E, ^1e, &c. depend on the initial position and

velocity of the points. If the velocities of P, Q = 0, when t = 0, we shall have

$$^1E, {}^2e, \text{ each } = 0$$

as appears by taking the Differentials of y, y'.

If either of the two ^1E, ^2E be = 0, we shall have (supposing the latter case and omitting ^1e)

$$y = \frac{{}^2\lambda\, {}^1E}{{}^2\lambda - {}^1\lambda} \cos.\, t \surd ({}^1k\, g)$$

$$y' = \frac{{}^1E}{{}^1\lambda - {}^2\lambda} \cos.\, t \surd ({}^1k\, g).$$

Hence it appears that the oscillations in this case are *symmetrical*: that is, the bodies P, Q come to the vertical line at the same time, have similar and equal motions on the two sides of it, and reach their greatest distances from it at the same time. It is easy to see that in this case, the motion has the same law of time and velocity as in a cycloidal pendulum; and the time of an oscillation, in this case, extends from when t = 0 to when $t \surd ({}^1k\, g) = \pi$. Also if β, β' be the greatest horizontal deviation of P, Q, we shall have

$$y = \beta .\ \cos.\, t \surd ({}^1k\, g)$$

$$y' = \beta' .\ \cos.\, t \surd ({}^1k\, g).$$

In order to find the original relation of β, β', (the oscillations will be symmetrical if the forces which urge P, Q to the vertical be as P M, Q N, as is easily seen. Hence the conditions for symmetrical oscillation might be determined by finding the position of P, Q that this might originally be the relation of the forces) that the oscillations may be of this kind, the original velocities being 0, we must have by equation (5) since $^2E = 0$.

$$\beta + {}^2\lambda\, \beta' = 0.$$

Similarly, if we had

$$\beta + {}^1\lambda\, \beta' = 0$$

we should have $^1E = 0$, and the oscillations would be symmetrical, and would employ a time

$$\frac{\pi}{\surd ({}^2k\, g)}.$$

When neither of these relations obtains, the oscillations may be considered as compounded of two in the following manner: Suppose that we put

$$y = H \cos.\, t \surd ({}^1k\, g) + \dot{K} \cos.\, t \surd ({}^2k\, g) \quad . \quad . \quad . \quad (7)$$

omitting ^1e, ^2e, and altering the constants in equation (6); and suppose that we take

$$M\, p = H .\ \cos.\, t \surd ({}^1k\, g);$$

Then p will oscillate about M according to the law of a cycloidal pendulum (neglecting the vertical motion). Also

$$p\,P \text{ will} = K \,.\, \cos.\, t \sqrt{({}^2k\, g)}.$$

Hence, P oscillates about p according to a similar law, while p oscillates about M. And in the same way, we may have a point q so moved, that Q shall oscillate about q in a time

$$\frac{\pi}{\sqrt{({}^2k\, g)}}$$

while q oscillates about N in a time

$$\frac{\pi}{\sqrt{({}^1k\, g)}}$$

And hence, the motion of the pendulum A P Q is compounded of the motion A p q oscillating symmetrically about a vertical line, and of A P Q oscillating symmetrically about A p q, as if that were a fixed vertical line.

When a pendulum oscillates in this manner it will never return exactly to its original position if $\sqrt{{}^1k}$, $\sqrt{{}^2k}$ are incommensurable.

If $\sqrt{{}^1k}$, $\sqrt{{}^2k}$ are commensurable so that we have

$$m \sqrt{{}^1k} = n \sqrt{{}^2k}$$

m and n being whole numbers, the pendulum will at certain intervals, return to its original position. For let

$$t \sqrt{({}^1k\, g)} = 2\, n\, \pi$$

then

$$t \sqrt{({}^2k\, g)} = 2\, m\, \pi$$

and by (7)

$$\begin{aligned} y &= H \cos.\, 2\, n\, \pi + K \,.\, \cos.\, 2\, m\, \pi \\ &= H + K, \end{aligned}$$

which is the same as when

$$t = 0.$$

And similarly, after an interval such that

$$t \sqrt{({}^1k\, g)} = 4\, n\, \pi,\ 6\, n\, \pi,\ \&c.$$

the pendulum will return to its original position, having described in the intermediate times, similar cycles of oscillations.

408. Ex. *Let* $\mu' = \mu$

$$a' = a$$

to determine the oscillations.

Here equation (4) becomes

$$a^2 k^2 - 4\, a\, k = -2$$

and

$$a\, k = 2 \pm \sqrt{2}.$$

Also, by equation (3)

$$a k = 3 - \lambda$$

$$\therefore {}^1\lambda = 1 + \sqrt{2},\ {}^2\lambda = 1 - \sqrt{2}.$$

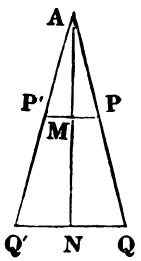
Hence, in order that the oscillations may be symmetrical, we must either have

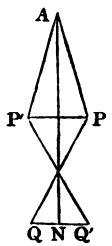
$$\beta + (1 + \sqrt{2})\,\beta' = 0, \text{ whence } \beta' = -(\sqrt{2} - 1)\,\beta$$

or

$$\beta - (\sqrt{2} - 1)\,\beta' = 0, \text{ whence } \beta' = (\sqrt{2} + 1)\,\beta.$$

The two arrangements indicated by these equations are thus represented.





The first corresponds to

$$\beta' = (\sqrt{2} + 1)\,\beta$$

or

$$Q\,N = (\sqrt{2} + 1)\,P\,M.$$

In this case, the pendulum will oscillate into the position A P' Q', similarly situated on the other side of the line; and the time of this complete oscillation will be

$$\frac{\pi}{\sqrt{\left\{\frac{g}{a}(2 - \sqrt{2})\right\}}} = \frac{\pi}{\sqrt{(2 - \sqrt{2})}}\sqrt{\frac{a}{g}}.$$

In the other case, corresponding to

$$\beta' = -(\sqrt{2} - 1)\,\beta$$

Q is on the other side of the vertical line, and

$$Q\,N = (\sqrt{2} - 1)\,P\,M.$$

The pendulum oscillates into the position A P' Q', the point O remaining always in the vertical line; and the time of an oscillation is

$$\frac{\pi}{\sqrt{(2 + \sqrt{2})}}\sqrt{\frac{a}{g}}.$$

The lengths of simple pendulums which would oscillate respectively in these times would be

$$\frac{a}{2 - \sqrt{2}} \text{ and } \frac{a}{2 + \sqrt{2}}$$

or

$$1.707\ a \text{ and } .293\ a.$$

If neither of these arrangements exist originally, let β, β' be the original values of y, y′ when t is 0. Then making $t = 0$ in equation (5), we have

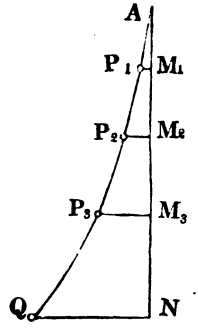
$$^1E = \beta + (\sqrt{2} + 1)\,\beta'$$

and

$$^2E = \beta - (\sqrt{2} - 1)\,\beta'.$$

And these being known, we have the motion by equation (6).

409. *Any number of material points* P_1, P_2, $P_3 \ldots Q$, *hang by means of a string without weight, from a point* A; *it is required to determine their small oscillations in a vertical plane.*



Let A N be a vertical abscissa, and $P_1 M_1$, $P_2 M_2$, &c. horizontal ordinates; so that

$$\begin{aligned} A\,M_1 &= x_1,\ A\,M_2 = x_2, \text{ \&c.} \\ P_1\,M_1 &= y_1,\ P_2\,M_2 = y_2, \text{ \&c.} \\ A\,P_1 &= a_1,\ P_1\,P_2 = a_2, \text{ \&c.} \\ \text{tension of } A\,P_1 &= p_1, \text{ of } P_1P_2 = p_2, \text{ \&c.} \\ \text{mass of } P_1 &= \mu_1, \text{ of } P_2 = \mu_2, \text{ \&c.} \end{aligned}$$

Hence, we have three equations, by resolving the forces parallel to the horizon.

$$\left.\begin{aligned} \frac{d^2 y_1}{dt^2} &= -\frac{p_1 g}{\mu_1}\cdot\frac{y_1}{a_1} + \frac{p_2 g}{\mu_1}\cdot\frac{y_2 - y_1}{a_2} \\ \frac{d^2 y_2}{dt^2} &= -\frac{p_2 g}{\mu_2}\cdot\frac{y_2 - y_1}{a_2} + \frac{p_3 g}{\mu_2}\cdot\frac{y_3 - y_2}{a_3} \\ \frac{d^2 y_3}{dt^2} &= -\frac{p_3 g}{\mu_3}\cdot\frac{y_3 - y_2}{a_3} + \frac{p_4 g}{\mu_3}\cdot\frac{y_4 - y_3}{a_4} \\ &\text{- - - - - - - - - - - - -} \\ \frac{d^2 y_n}{dt^2} &= \frac{p_n g}{\mu_n}\cdot\frac{y_n - y_{n-1}}{a_n} \end{aligned}\right\} \quad \ldots\ (1)$$

And as in the last, it will appear that p_1, p_2, &c. may, for these small oscillations, be considered constant, and the same as in the state of rest. Hence if

$$\mu_1 + \mu_2 + \ldots \mu_n = M,$$

then

$$p_1 = M,\ p_2 = M - \mu_1,\ p_3 = M - \mu_1 - \mu_2, \text{ \&c.}$$

Also, dividing by g, and arranging, the above equations may be put in this form:

$$\left.\begin{aligned}
\frac{d^2 y_1}{g\,d\,t^2} &= -\left(\frac{p_1}{\mu_1 a_1} + \frac{p^2}{\mu_1 a_2}\right) y_1 + \frac{p_2 y_2}{\mu_1 a_2} \\
\frac{d^2 y_2}{g\,d\,t^2} &= \frac{p_2 y_1}{\mu_2 a_2} - \left(\frac{p_2}{\mu_2 a_2} + \frac{p_3}{\mu_2 a_3}\right) y_2 + \frac{p_3 y_3}{\mu_2 a_3} \\
\frac{d^2 y_3}{g\,d\,t^2} &= \frac{p_3 y_2}{\mu_3 a_3} - \left(\frac{p_3}{\mu_3 a_3} + \frac{p_4}{\mu_3 a_4}\right) y_3 + \frac{p_4 y_4}{\mu_3 a_4} \\
&\text{- - - - - - - - - - - -} \\
\frac{d^2 y_n}{g\,d\,t^2} &= \frac{p_n y_{n-1}}{\mu_n a_n} - \frac{p_n y_n}{\mu_n a_n}.
\end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

The first and last of these equations become symmetrical with the rest if we observe that

$$y_0 = 0$$

and

$$p_{n+1} = 0.$$

Now if we multiply these equations respectively by

$$1, \lambda, \lambda', \lambda'', \text{ \&c.}$$

and add them, we have

$$\begin{aligned}
&\frac{d^2 y_1 + \lambda\, d^2 y_2 + \lambda'\, d^2 y_3 + \text{\&c.}}{g\,d\,t^2} = \\
&\qquad \left\{-\frac{p_1}{\mu_1 a_1} - \frac{p_2}{\mu_1 a_2} + \frac{\lambda p_2}{\mu_2 a_2}\right\} y_1 \\
&+ \left\{\frac{p_2}{\mu_1 a_2} - \lambda\left(\frac{p_2}{\mu_2 a_2} + \frac{p_3}{\mu_2 a_3}\right) + \frac{\lambda' p_3}{\mu_3 a_3}\right\} y_2 \\
&+ \left\{\frac{\lambda p_3}{\mu_3 a_3} - \lambda'\left(\frac{p_3}{\mu_3 a_3} + \frac{p_4}{\mu_3 a_4}\right) + \frac{\lambda'' p_4}{\mu_4 a_4}\right\} y_3 \\
&\qquad \text{- - - - - - - - - - - -} \\
&+ \left\{\frac{\lambda'^{(n-3)} p_n}{\mu_{n-1} a_n} - \frac{\lambda'^{(n-2)} p_n}{\mu_n a_n}\right\} y_n
\end{aligned}$$

and this will be integrable, if the right-hand side of the equation be reducible to this form

$$-k\,(y_1 + \lambda\, y_2 + \lambda'\, y_3 + \text{\&c.}).$$

That is, if

$$\left.\begin{aligned}
k &= \frac{p_1}{\mu_1 a_1} + \frac{p_2}{\mu_1 a_2} - \frac{\lambda p_2}{\mu_2 a_2} \\
k\lambda &= -\frac{p_2}{\mu_1 a_2} + \lambda\left(\frac{p_2}{\mu_2 a_2} + \frac{p_3}{\mu_2 a_3}\right) - \frac{\lambda' p_3}{\mu_3 a} \\
k\lambda' &= -\frac{\lambda p_3}{\mu_2 a_3} + \lambda'\left(\frac{p_3}{\mu_3 a_3} + \frac{p_4}{\mu_3 a_4}\right) + \frac{\lambda'' p_4}{\mu_4 a_4} \\
&\text{- - - - - - - - - - - -} \\
k\lambda'^{(n-2)} &= -\frac{\lambda'^{(n-3)} p_n}{\mu_{n-1} a_n} + \frac{\lambda'^{(n-2)} p_n}{\mu_n a_n}.
\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

If we now eliminate

$$\lambda,\ \lambda',\ \lambda'',\ \&c.$$

from these n equations, it is easily seen that we shall have an equation of n dimensions in k.

Let

$$^1k,\ ^2k,\ ^3k \ldots\ldots\ ^nk$$

be the n values of k; then for each of these there is a value of

$$\lambda',\ \lambda'',\ \lambda'''$$

easily deducible from equations (3), which we may represent by

$$^1\lambda,\ ^1\lambda',\ ^1\lambda'',\ \&c.$$
$$^2\lambda',\ ^2\lambda'',\ ^2\lambda''',\ \&c.$$

Hence we have these equations by taking corresponding values λ and k,

$$\frac{d^2y_2 + {}^1\lambda\, d^2y_2 + {}^1\lambda'\, d^2y_3 + \&c.}{g\, dt^2} = -{}^1k\,(y_1 + {}^1\lambda\, y_2 + {}^1\lambda'\, y_3 + \&c.)$$

$$\frac{d^2y_1 + {}^2\lambda\, d^2y_2 + {}^2\lambda''\, d^2y_3 + \&c.}{g\, dt^2} = -{}^2k\,(y_2 + {}^2\lambda\, y_2 + {}^2\lambda'\, y_3 + \&c.)$$

and so on, making n equations.

Integrating each of these equations we get, as in the last problem

$$\left.\begin{aligned} y_1 + {}^1\lambda\, y_2 + {}^1\lambda'\, y_3 + \&c. &= {}^1E\cos.\{t\surd({}^1k\,g) + {}^1e\} \\ y_1 + {}^2\lambda\, y_2 + {}^2\lambda'\, y_3 + \&c. &= {}^2E\cos.\{t\surd({}^2k\,g) + {}^2e\} \end{aligned}\right\} \quad \ldots\ (5)$$

^1E, ^2E, &c. ^1e, ^2e, &c. being arbitrary constants.

From these n simple equations, we can, without difficulty, obtain the n quantities y_1, y_2, &c. And it is manifest that the results will be of this form

$$\left.\begin{aligned} y_1 &= {}^1H_1\cos.\{t\surd({}^1k\,g) + {}^1e\} + {}^2H_1\cos.\{t\surd({}^2k\,g) + {}^2e\} + \&c. \\ y_2 &= {}^1H_2\cos.\{t\surd({}^1k\,g) + {}^1e\} + {}^2H_2\cos.\{t\surd({}^2k\,g) + {}^2e\} + \&c. \\ \&c. &= \&c. \end{aligned}\right\} \quad \ldots\ (6)$$

where $^1H_1$, $^1H_2$, &c. must be deduced from β_1, β_2, &c. the original values of y_1, y_2, &c.

If the points have no initial velocities (i. e. when $t = 0$) we shall have $^1E = 0$, $^2E = 0$, &c.

We may have symmetrical oscillations in the following manner. If, of the quantities ^1E, ^2E, ^3E, &c. all vanish except one, for instance ^nE; we have

$$\left.\begin{aligned} y_1 + {}^1\lambda\, y_2 + {}^1\lambda'\, y_3 + \&c. &= 0 \\ y_1 + {}^2\lambda\, y_2 + {}^2\lambda'\, y_3 + \&c. &= 0 \\ y_1 + {}^3\lambda\, y_2 + {}^3\lambda'\, y_3 + \&c. &= 0 \\ \text{- - - - - - - - - -} & \\ y_1 + {}^n\lambda\, y_2 + {}^n\lambda'\, y_3 + \&c. &= {}^nE\cos.t\surd({}^nk\,g) \end{aligned}\right\} \quad \ldots\ (7)$$

omitting ^nE.

From the n — 1 of these equations, it appears that y_2, y_3, &c. are in a given ratio to y_1; and hence

$$y_1 + {}^n\lambda\, y_2 + {}^n\lambda'\, y_3 + \&c.$$

is a given multiple of y_1 and $= m\, y_1$ suppose. Hence, we have

$$m\, y_1 = {}^nE \cos.\, \surd ({}^nk\, g);$$

or, omitting the index n, which is now unnecessary,

$$m\, y_1 = E \cos.\, t \surd (k\, g).$$

Also if $y_2 = e_2\, y_1$,

$$m\, y_2 = E\, e_2 \cos.\, t \surd (k\, g)$$

and similarly for y_3 &c.

Hence, it appears that in this case the oscillations are *symmetrical.* All the points come into the vertical line at the same time, and move similarly, and contemporaneously on the two sides of it. The relation among the original ordinates β_1, β_2, β_3, &c. which must subsist in order that the oscillations may be of this kind, is given by the n — 1 equations (7),

$$\begin{aligned}
\beta_1 + {}^1\lambda\, \beta_2 + {}^1\lambda'\, \beta_3 + \&c. &= 0 \\
\beta_1 + {}^2\lambda\, \beta_2 + {}^2\lambda'\, \beta_3 + \&c. &= 0 \\
\beta_1 + {}^3\lambda\, \beta_2 + {}^3\lambda'\, \beta_3 + \&c. &= 0 \\
\&c. &= \&c.
\end{aligned}$$

These give the proportion of β_1 β_2, &c; the arbitrary constant ${}^nE$, in the remaining equation, gives the *actual quantity* of the original displacement.

Also, we may take any one of the quantities ${}^1E$, ${}^2E$, ${}^3E$, &c. for that which does not vanish; and hence obtain, in a different way, such a system of n — 1 equations as has just been described. Hence, there are n different relations among β_1 β_2, &c. or n different modes of arrangement, in which the points may be placed, so as to oscillate symmetrically.

(We might here also find these positions, which give symmetrical oscillations, by requiring the force in each of the ordinates $P_1\, M_1$, $P_2\, M_2$ to be as the distance; in which case the points P_1, P_2, &c. would all come to the vertical at the same time.

If the quantities $\surd {}^1k$, $\surd {}^2k$ have one common measure, there will be a time after which the pendulum will come into its original position. And it will describe similar successive cycles of vibrations. If these quantities be not commensurable, no portion of its motion will be similar to any preceding portion.)

The time of oscillation in each of these arrangements is easily known; the equation

$$m\, y_1 = {}^nE \cos.\, t \surd ({}^nk\, g)$$

shows that an oscillation employs a time

$$t = \frac{\pi}{\sqrt{({}^{n}k\, g)}}.$$

And hence, if all the roots ${}^{1}k$, ${}^{2}k$, ${}^{3}k$, &c. be different, the time is different for each different arrangement.

If the initial arrangement of the points be different from all those thus obtained, the oscillations of the pendulum may always be considered as compounded of n symmetrical oscillations. That is, if an imaginary pendulum oscillate symmetrically about the vertical line in a time

$$\frac{\pi}{\sqrt{({}^{1}k\, g)}};$$

and a second imaginary pendulum oscillate about the place of the first, considered as a fixed line, in the time

$$\frac{\pi}{\sqrt{({}^{2}k\, g)}};$$

and a third about the second, in the same manner, in the time

$$\frac{\pi}{\sqrt{({}^{3}k\, g)}};$$

and so on; the nth pendulum may always be made to coincide perpetually with the real pendulum, by properly adjusting the amplitudes of the imaginary oscillations. This appears by considering the equations (6), viz.

$$y_1 = {}^{1}H_1 \cos.\, t \sqrt{({}^{1}k\, g)} + {}^{2}H_1 \cos.\, t \sqrt{({}^{2}k\, g)} + \&c.$$
$$\&c. = \&c.$$

This principle of the *coexistence of vibrations* is applicable in all cases where the vibrations are indefinitely small. In all such cases each set of symmetrical vibrations takes place, and affects the system as if that were the only motion which it experienced.

A familiar instance of this principle is seen in the manner in which the circular vibrations, produced by dropping stones into still water, spread from their respective centers, and cross without disfiguring each other.

If the oscillations be not all made in one vertical plane, we may take a horizontal ordinate z perpendicular to y. The oscillations in the direction of y will be the same as before, and there will be similar results obtained with respect to the oscillations in the direction of z.

We have supposed that the motion in the direction of x, the vertical axis, may be neglected, which is true when the oscillations are very small.

410. Ex. Let there be three bodies all equal (each $= \mu$), and also their distances a_1, a_2, a_3 all equal (each $= a$).

Here

$$p = 3\mu,\ p_2 = 2\mu,\ p_3 = a$$

and equations (3) become

$$a k = 5 - 2\lambda$$
$$a k \lambda = -2 + 3\lambda - \lambda'$$
$$a k \lambda' = -\lambda + \lambda'.$$

Eliminating k, we have

$$5\lambda - 2\lambda^2 = -2 + 3\lambda - \lambda',$$
$$5\lambda' - 2\lambda\lambda' = -\lambda + \lambda',$$

or

$$\lambda' = 2\lambda^2 - 2\lambda - 2,$$
$$4\lambda' - 2\lambda\lambda' = -\lambda$$
$$\therefore \lambda' = \frac{\lambda}{2\lambda - 4}$$
$$\therefore (2\lambda^2 - 2\lambda - 2)(2\lambda - 4) = \lambda$$

or

$$\lambda^3 - 3\lambda^2 + \frac{3}{4}\lambda + 2 = 0,$$

which may be solved by Trigonometrical Tables. We shall find three values of λ.

Hence, we have a value of λ' corresponding to each value of λ; and then by equations (7)

$$\left.\begin{aligned} \beta_1 + {}^1\lambda\,\beta_2 + {}^1\lambda'\,\beta_3 &= 0 \\ \beta + {}^2\lambda\,\beta_2 + {}^2\lambda'\,\beta_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots \quad (7)$$

whence we find β_2, β_3 in terms of β_1.

We shall thus find

$$\beta_2 = 2.295\,\beta_1$$

or

$$\beta_2 = 1.348\,\beta_1$$

or

$$\beta_2 = -.643\,\beta_1$$

according as we take the different values of λ.

And the times of oscillation in each case will be found by taking the value of

$$a k = 5 - 2\lambda;$$

that value of λ being taken which is not used in equation (7′). For the time of oscillation will be given by making

$$t\sqrt{(k g)} = \pi.$$

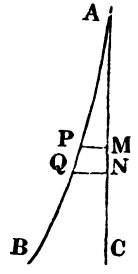
If the values of β_1, β_2, β_3 have not this initial relation, the oscillations

will be compounded in a manner similar to that described in the example for two bodies only.

411. *A flexible chain, of uniform thickness, hangs from a fixed point: to find its initial form, that its small oscillations may be symmetrical.*

Let A M, the vertical abscissa = x; M P the horizontal ordinate = y; A P = s, and the whole length A C = a;

$$\therefore \text{A P} = a - s.$$



And as before, the tension at P, when the oscillations are small, will be the weight of P C, and may be represented by a—s. This tension will act in the direction of a tangent at P, and hence the part of it in the direction P M will be

$$\text{tension} \times \frac{dy}{ds}$$

or

$$(a - s)\frac{dy}{ds}.$$

Now, if we take any portion P Q = h, we shall find the horizontal force at Q in the same manner. For the point Q, supposing d s constant

$$\frac{dy}{ds} \text{ becomes } \frac{dy}{ds} + \frac{d^2y}{ds^2}\cdot\frac{h}{1} + \frac{d^3y}{ds_3} - \frac{h^2}{1.2} \&c.$$

(see 32).

Also, the tension will be a — s + h. Hence the horizontal force in the direction N Q, is

$$(a - s + h)\left(\frac{dy}{ds} + \frac{d^2y}{ds^2}\cdot\frac{h}{1} + \&c.\right)$$

Subtracting from this the force in P M, we have the force on P Q horizontally.

$$= (a - s)\left(\frac{d^2y}{ds^2}\cdot\frac{h}{1} + \frac{d^3y}{ds^3}\cdot\frac{h^2}{1.2} + \&c.\right)$$

$$- h\left(\frac{dy}{ds} + \frac{d^2y}{ds^2}\cdot\frac{h}{1} + \&c.\right)$$

and the mass of P Q being represented by h, the accelerating force $\left(= \frac{\text{pressure} \times g}{\text{mass}}\right)$ is found. But since the different points of P Q move with different velocities, this expression is only applicable when h is indefinitely small. Hence, supposing Q to approach to and coincide with P, we have, when h vanishes

$$\text{accelerating force on P} = (a - s)\frac{d^2y}{ds^2} - \frac{dy}{ds}.$$

But since the oscillations are indefinitely small, x coincides with s and we have

$$\text{accelerating force on P} = (a-x)\frac{d^2y}{dx^2} - \frac{dy}{dx}.$$

Now, in order that the oscillations may be symmetrical, this force must be in the direction P M, and proportional to P M, in which case all the points of A C, will come to the vertical A B at once. Hence, we must have

$$(a-x)\frac{d^2y}{dx^2} - \frac{dy}{dx} = -k\,dy \quad \ldots\ldots \quad (1)$$

k being some constant quantity to be determined.

This equation cannot be integrated in finite terms. To obtain a series let

$$y = A + B.(a-x) + C(a-x)^2 + \&c.$$

$$\therefore \frac{dy}{dx} = -B - 2C(a-x) - 3D(a-x)^2$$

$$\therefore \frac{d^2y}{dx^2} = 1.2.C + 2.3D(a-x) + \&c.$$

Hence

$$0 = (a-x)\frac{d^2y}{dx^2} - \frac{dy}{dx} + ky$$

gives.

$$\begin{aligned} 0 = {} & 1.2.C(a-x) + 2.3D(a-x)^2 + \&c. \\ & + B + 2C(a-x) + 3D(a-x)^2 + \&c. \\ & + kA + kB(a-x) + kC(a-x)^2 + \&c. \end{aligned}$$

Equating coefficients; we have

$$\begin{aligned} B &= -kA, \\ 2^2C &= -kB \\ 3^2D &= -kC \\ \&c. &= \&c. \\ \therefore B &= kA \\ C &= \frac{k^2A}{2^2} \\ D &= \frac{k^3A}{2^2.3^2} \\ \&c. &= \&c. \end{aligned}$$

and

$$v = A\left\{1 - k(a-x) + \frac{k^2}{2^2}(a-x)^2 - \frac{k^3}{2^2.3^2}.(a-x) + \&c.\right\} \quad \ldots \quad (2)$$

Here

A is B C, the value of y when $x = a$. When $x = 0$, $y = 0$;

$$\therefore 1 - k\,a + \frac{k^2 a^2}{2^2} - \frac{k^3 a^3}{2^2 . 3^2} + \&c. = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From this equation (k) may be found. The equation has an infinite number of dimensions, and hence k will have an infinite number of values, which we may call

$$^1k, ^2k, \ldots {}^n k \ldots 1,$$

and these give an infinite number of initial forms, for which the chain may perform symmetrical oscillations.

The time of oscillation for each of these forms will be found thus. At the distance y, the force is $k\,g\,y$: hence by what has preceded, the time to the vertical is

$$\frac{\pi}{2\sqrt{(k\,g)}}$$

and the time of oscillation is

$$\frac{\pi}{\sqrt{(k\,g)}}.$$

(The greatest value of k a is about 1.44 (Euler Com. Acad. Petrop. tom. viii. p. 43). And the time of oscillation for this value is the same as that of a simple pendulum, whose length is $\frac{2}{3}$ a nearly.)

The points where the curve cuts the axis will be found by putting $y = 0$. Hence taking the value $^n k$ of k, we have

$$0 = 1 - {}^n k\,(a - x) + \frac{{}^n k^2 (a - x)^2}{2^2} + \frac{{}^n k^3 (a - x)^2}{2 . 3^2} + \&c.$$

which will manifestly be verified, if

$$^n k\,(a - x) = {}^1k\,a$$

or

$$^n k\,(a - x) = {}^2k\,a$$

or

$$^n k\,(a - x) = {}^3k\,a$$

$$\&c. = \&c.$$

because $^1k\,a$, $^2k\,a$, &c. are roots of equation (3).

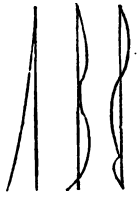
That is if

$$x = a\left(1 - \frac{{}^1k}{{}^n k}\right) \text{ or } = a\left(1 - \frac{{}^2k}{{}^n k}\right) \text{ or } = \&c.$$

Suppose ^1k, ^2k, ^3k, &c. to be the roots in the order of their magnitude ^1k being the least.

Then if for $^n k$, we take ^1k, all these values of x will be negative, and the curve will never cut the vertical axis below A.

If for $^n k$, we take $^2 k$, all the values of x will be negative except the first; therefore, the curve will cut A B in one point. If we take $^3 k$, all the values will be negative except the two first, and the curve cuts A B in two points; and so on.



Hence, the forms for which the oscillations will be symmetrical, are of the kind thus represented.

And there are an infinite number of them, each cutting the axis in a different number of points.

If we represent equation (2) in this manner

$$y = A\ \varphi\ (k, x)$$

it is evident that

$$y = {}^1A\ \varphi\ ({}^1k, x)$$
$$y = {}^2A\ \varphi\ ({}^2k, x)$$
$$\&c. = \&c.$$

will each satisfy equation (1). Hence as before, if we put

$$y = {}^1A\ \varphi\ ({}^1k, x) + {}^2A\ \varphi\ ({}^2k, x) + \&c.$$

and if ^1A, ^2A, &c. can be so assumed that this shall represent a given initial form of the chain, its oscillations shall be compounded of as many coexisting symmetrical ones, as there are terms ^1A, ^2A, &c.

We shall now terminate this long digression upon constrained motion. The reader who wishes for more complete information may consult Whewell's Dynamics, one of the most useful and elegant treatises ever written, the various speculations of Euler in the work above quoted, or rather the comprehensive methods of Lagrange in his Mecanique Analytique.

We now proceed to simplify the text of this Xth Section.

412. PROP. L. First, S R Q is formed by an entire revolution of the generating circle or wheel, whose diameter is O R, upon the globe S O Q.

413. Secondly, by taking

$$CA : CO :: CO : CR$$

we have

$$CA : CO :: CA - CO : CO - CR$$
$$:: AO : OR$$

and therefore if C S be joined and produced to meet the exterior globe in D, we have also

$$AD : SO (:: CA : CO) :: AO : OR.$$

But

$$SO = \text{the semi-circumference of the wheel } OR = \frac{\pi . OR}{2}.$$

$\therefore$ A D $= \frac{\pi \cdot A O}{2} = \frac{1}{2}$ the circumference of the wheel whose diameter is A O. That is S is the vertex of the Hypocycloid A S, and A S is perpendicular in S to C S. But O S is also perpendicular to C S. Therefore A S touches O S in S, &c.

414. *The similar figures* A S, S R.]

By 39 it readily appears that Hypocycloids are similar when

$$R : r :: R' : r'$$

R and r being the radii of the globe and wheel; that is when

$$C A : A O :: C O : O R$$

or when

$$C A : C O :: C O : C R$$

$\therefore$ A S, S R are similar

415. V B, V W *are equal to* O A, O R.]

If B be not in the circumference A D let C V meet it in B′. Then V P being a tangent at P, and since the element of the curve A P is the same as would be generated by the revolution of B′ P around B′ as a center, and $\therefore$ B′ P is perpendicular both to the curve and its tangent P V, therefore P B, P B′ and $\therefore$ B, B′ coincide. That is

$$V B = O A.$$

Also if the wheel O R describes O V whilst A O describes A B, the angular velocity B P in each must be the same, although at first, viz. at O and A, they are at right angles to each other. Hence when they shall have arrived at V and B their distances from C B must be complements of each other. But

$$\angle T V W = B V P = \frac{\pi}{2} - P B V$$

$\therefore$ T V is a chord in the wheel O R, and

$$\therefore V W = O R.$$

See also the Jesuits' note.

OTHERWISE.

416. Construct the curve S P, to which the radius of curvature to every point of S R Q is a tangent; or which is the same, find S A the Locus of the Centers of Curvature to S R Q.

Hence is suggested the following generalization of the Problem, viz.

417. *To make a body oscillate in any given curve.*

Let S R Q (Newton's fig.) the given curve be symmetrical on both sides

of R. Then if x, y be the rectangular coordinates referred to the vertex R, and α, β those of the centers of curvature (P) we have

$$r^2 = P\,T^2 = (y-\beta)^2 + (x-\alpha)^2.$$

Hence, the contact being of the second order (74)

$$x - \alpha + (y-\beta)\frac{dy}{dx} = 0 \quad \ldots\ldots\ldots \quad (1)$$

and

$$1 + \frac{dy^2}{dx^2} + (y-\beta)\frac{d^2y}{dx^2} = 0 \quad \ldots\ldots\ldots \quad (2)$$

These two equations by means of that of the given curve, will give us β in terms of α, or the equation to the Locus of the centers of curvature.

Let S A be the Locus corresponding to S R, and A Q the other half. Then suspending a body from A attached to a string whose length is R A, when this string shall be stretched into any position A P T, it is evident that P being the point where the string quits the locus is a tangent to it, and that T is a point in S R Q.

Ex. 1. *Let* S R Q *be the common parabola.*

Here

$$y^2 = 2ax$$

$$\therefore \frac{dy}{dx} = \frac{a}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2}\cdot\frac{dy}{dx} = -\frac{a^2}{y^3} = -\frac{a}{2xy}$$

$\therefore$ substituting we get

$$x - \alpha + (y-\beta)\cdot\frac{a}{y} = 0$$

and

$$1 + \frac{a}{2x} - (y-\beta)\cdot\frac{a}{2xy} = 0$$

$$\therefore x - \alpha + \frac{a}{y}\left(1 + \frac{a}{2x}\right)\cdot\frac{2xy}{a} = 0 = 3x - \alpha + a$$

or

$$\left.\begin{aligned} \alpha &= 3x + a \\ \text{and} \quad \therefore \beta &= -\frac{2xy}{a} \end{aligned}\right\}$$

But

$$y^2 = 2ax$$

$$\therefore \beta^2 = \frac{4x^2\cdot y^2}{a^2} = \frac{8x^3}{a}$$

$$= \frac{8}{a} \times \frac{(\alpha - a)^3}{27} = \frac{8}{27\,a} \cdot (\alpha - a)^3 \quad . \quad . \quad . \quad (3)$$

Now when $\beta = 0$, $\alpha = a$; which shows that A R the length of the string must equal a. Also making A the origin of abscissas, that is, augmenting α by a, we have

$$\beta^2 = \frac{8}{27\,a} \times \alpha'^3$$

the equation to the semicubical parabola A S, A Q, which may be traced by the ordinary rules (35, &c.); and thereby the body be made to oscillate in the common parabola S Q R.

Ex. 2. *Let* S R Q *be an ellipse.*

Then, referring to its center, instead of the vertex,

$$y^2 = \frac{b^2}{a^2} \cdot (a^2 - x^2)$$

or

$$a^2 y^2 + b^2 x^2 = a^2 b^2$$

$$\therefore a^2 y \frac{dy}{dx} + b^2 x = 0$$

and

$$a^2 y \frac{d^2 y}{dx} + a^2 \frac{dy^2}{dx^2} + b^2 = 0.$$

These give

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

and

$$\frac{d^2 y}{dx^2} = -\frac{b^4}{a^2 y^3}.$$

Hence

$$\alpha = \frac{(a^2 - b^2)\, x^3}{a^4}$$

and

$$\beta = -\frac{(a^2 - b^2)\, y^3}{b^4}.$$

Hence substituting the values of y and x in

$$a^2 y^2 + b^2 x^2 = a^2 b^2$$

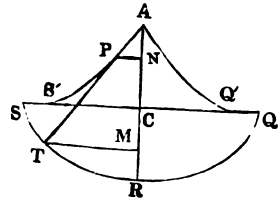
we get

$$\left(\frac{\beta\, b}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{\alpha\, a}{a^2 - b^2}\right)^{\frac{2}{3}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

the equation to the Locus of the centers of curvature.

In the annexed figure let

$$S C = b, \quad C R = a$$
$$C M = x, \quad T M = y.$$



Then

$$P N = \beta, \quad C N = \alpha.$$

And to construct A S′ by points, first put

$$\beta = 0$$

whence by equation (a)

$$\alpha = \pm \frac{a^2 - b^2}{a}$$

the value of A C. Let

$$\alpha = 0$$

then

$$\beta = \pm \frac{a^2 - b^2}{b}$$

the value of S′ C or C Q′.

Hence to make a body oscillate in the semi-ellipse S R Q we must take a pendulum of the length A R, (part = A P S′ flexible, and part = S S′ rigid; because S S′ is horizontal, and no string however stretched can be horizontal—see Whewell's Mechanics,) and suspend it at A. Then A P being in contact with the Locus A S′, P T will also touch A S in P, &c. &c.

Ex. 3. *Let* S R Q *be the common cycloid;*

The equation to the cycloid is

$$\frac{dy}{dx} = \sqrt{\frac{2r - y}{y}} = \sqrt{\left(\frac{2r}{y} - 1\right)}$$
$$\therefore \frac{d^2 y}{dx^2} = -\frac{r}{y^2}$$

whence it is found that

$$\left.\begin{aligned} \alpha &= x + 2\sqrt{(2ry - y^2)} \\ \beta &= -y \end{aligned}\right\}.$$

Hence

$$\frac{d\alpha}{dx} = \frac{2r - y}{y}$$

and

$$\frac{d\beta}{dx} = -\frac{dy}{dx} = -\sqrt{\frac{2r - y}{y}}$$
$$\therefore \frac{d\beta}{d\alpha} = -\sqrt{\frac{y}{2r - y}} = -\sqrt{\frac{-\beta}{2r + \beta}}$$

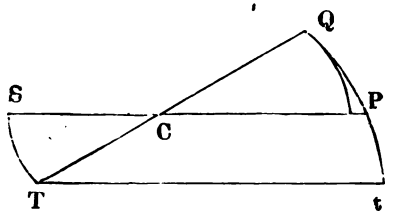
which is also the equation of a cycloid, of which the generating circle is

precisely the same as the former, the only difference consisting in a change of sign of the ordinate, and of the origin of the abscissæ.

The rest of this section is rendered sufficiently intelligible by the Notes of P. P. Le Seur and Jacquier; and by the ample supplementary matter we have inserted.

SECTION XI.

417. Prop. LVII. Two bodies attracting one another, describe round each other and round the center of gravity similar figures.



Since the mutual actions will not affect the center of gravity, the bodies will always lie in a straight line passing through C, and their distances from C will always be in the same proportion.

$$\therefore S C : T C :: P C : Q C$$

and

$$\angle S C T = Q C P.$$

$\therefore$ the figures described round each other are similar.

Also if T t be taken $=$ S P, the figure which P seems to describe round S will be t Q, and

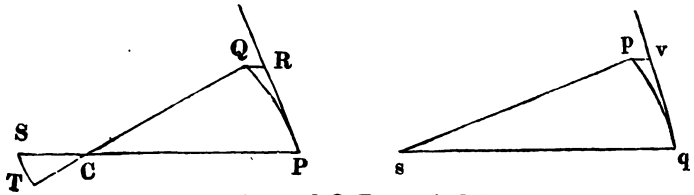
$$T t : T Q :: S P : T Q$$
$$:: C P : C Q$$

and

$$\angle t T Q = P C Q.$$

$\therefore$ the figures t Q, P Q, are similar; and the figure which S seems to describe round P is similar, and equal to the figure which P seems to describe round S.

418. Prop. LVIII. If S remained at rest, a figure might be described by P round S, similar and equal to the figures which P and S seem to describe round each other, and by an equal force.



Curves are supposed similar and Q R, q r indefinitely small. Let P and p be projected in directions P R, p r (making equal angles C P R, s p r) with such velocities that

$$\frac{V}{v} = \frac{\sqrt{S}}{\sqrt{S+P}} = \frac{\sqrt{CP}}{\sqrt{sp}} = \frac{\sqrt{PQ}}{\sqrt{pq}}.$$

Then $\left(\text{since } dt = \frac{ds}{v}\right)$

$$\therefore \frac{T}{t} = \frac{PQ}{pq} \cdot \frac{\sqrt{pq}}{\sqrt{PQ}} = \frac{\sqrt{PQ}}{\sqrt{pq}} = \frac{\sqrt{QR}}{\sqrt{qr}}.$$

But in the beginning of the motion $f = \frac{S}{\frac{gt^2}{2}}$

$$\therefore \frac{F}{f} = \frac{QR}{qr} \cdot \frac{qr}{QR} = \frac{1}{1}.$$

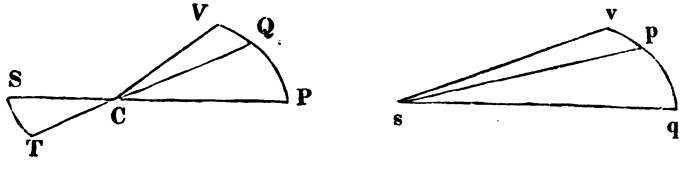
The same thing takes place if the center of gravity and the whole system move uniformly forward in a straight line in fixed space.

419. COR. 1. If $F \propto D$, the bodies will describe round the common center of gravity, and round each other, concentric ellipses, for such would be described by P round S at rest with the same force.

Conversely, if the figures be ellipses concentric, $F \propto D$.

420. COR. 2. If $F \propto \frac{1}{D^2}$ the figures will be conic sections, the foci in the centers of force, and the converse.

421. COR. 3. Equal areas are described round the center of gravity, and round each other, in equal times.



422. COR. 3. Otherwise. Since the curves are similar, the areas, bounded by similar parts of the curves, are similar or proportional.

$\therefore spq : CPQ :: sp^2 : CP^2 :: (S+P)^2 : s^2$ in a given ratio;

and T. through s p q : T. through C P Q : : $\sqrt{S+P}$: $\sqrt{S}$, in a given ratio
and ∴ : : T. through spv : T. through CPV
∴ T. through C P Q : T. through C P V : : T. through s p q : T. through s p v
: : s p q : s p v (by Sect. II.)
: : C P Q : C P V

∴ the areas described round C are proportional to the times, and the areas described round each other in the same times, which are similar to the areas round C, are also proportional to the times.

423. Prop. LIX. The period in the figure described in last Prop. : the period round C : : $\sqrt{S+P}$: $\sqrt{S}$; for the times through similar arcs p q, P Q, are in that proportion.

424. Prop. LX. The major axis of an ellipse which P seems to describe round S in motion $\left(\text{Force} \propto \frac{1}{D^2}\right)$: major axis of an ellipse which would be described by P *in the same time* round S at rest : : $\overline{S+P}$: first of two mean proportionals between $\overline{S+P}$ and S.

Let A = major axis of an ellipse described (or seemed to be described) round S in motion, and which is similar and equal to the ellipse described in Prop. LVIII.

Let x = major axis of an ellipse which would be described round S at rest in the same time.

$$\therefore \frac{\text{period in ellipse round S in motion}}{\text{period in same ellipse round S at rest}} = \frac{\sqrt{S}}{\sqrt{S+P}} \text{ (Prop. LIX)}$$

and by Sect. III,

$$\frac{\text{period in ellipse round S at rest}}{\text{period in required ellipse round S at rest}} = \frac{A^{\frac{3}{2}}}{x^{\frac{3}{2}}}$$

$$\therefore \frac{\text{period in ellipse round S in motion}}{\text{period in required ellipse round S at rest}} = \frac{A^{\frac{3}{2}}\sqrt{S}}{x^{\frac{3}{2}}\sqrt{S+P}}$$

but these periods are to be equal,

$$\therefore A^3 S = x^3 . \overline{S+P}$$

∴ A : x : : $\sqrt[3]{S+P}$: $\sqrt[3]{S}$: : S + P : first of two mean proportionals

(for if a, a r, a r^2, a r^3, be proportionals, $\sqrt[3]{a}$: $\sqrt[3]{a r^3}$: : a : a r.)

425. At what mean distance from the earth would the moon revolve round the earth at rest, in the same time as she now revolves round the earth in motion? This is easily resolved.

426. Prop. LXI. The bodies will move as if acted upon by bodies at the center of gravity with the same force, and the law of force with re-

spect to the distances from the center of gravity will be the same as with respect to the distances from each other.

For the force is always in the line of the center of gravity, and ∴ the bodies will be acted upon as if it came from the center of gravity.

And the distance from the center of gravity is in a given ratio to the distance from each other, ∴ the forces which are the same functions of these distances will be proportional.

427. PROP. LXII. *Problem of two bodies with no initial Velocities.*

$F \propto \frac{1}{D^2}$. Two bodies are let fall towards each other. Determine the motions.

The center of gravity will remain at rest, and the bodies will move as if acted on by bodies placed at the center of gravity, (and exerting the same force at any given distance that the real bodies exert),

∴ the motions may be determined by the 7th Sect.

428. PROP. LXIII. *Problem of two bodies with given initial Velocities.*

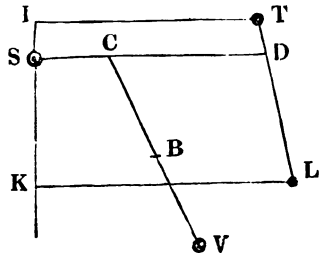
$F \propto \frac{1}{D^2}$. Two bodies are projected in given directions, with given velocities. Determine the motions.

The motion of the center of gravity is known from the velocities and directions of projection. Subtract the velocity of the center of gravity from each of the given velocities, and the remainders will be the velocities with which the bodies will move in respect of each other, and of the center of gravity, as if the center of gravity were at rest. Hence since they are acted upon as if by bodies at the center of gravity, (whose magnitudes are determined by the equality of the forces), the motions may be determined by Prop. XVII, Sect. III, (velocities being supposed to be acquired down the finite distance), if the directions of projection do not tend to the center, or by Prop. XXXVII, Sect. VII, if they tend to or directly from the center. Thus the motions of the bodies with respect to the center of gravity will be determined, and these motions compounded with the uniform motion of the center of gravity will determine the motions of the bodies in absolute space.

429. PROP. LXIV. $F \propto D$, determine the motions of any number of bodies attracting each other.

T and L will describe concentric ellipses round D.

Now add a third body S.



Attraction of S on T may be represented by the distance T S, and on L by L S, (attraction at distance being 1) resolve T S, L S, into T D, D S; L D, D S, whereof the parts T D, L D, being in given ratios to the whole, T L, L T, will only increase the forces with which L and T act on each other, and the bodies L and T will continue to describe ellipses (as far as respects these new forces) but with accelerated velocities, (for in similar parts of similar figures $V^2 \propto F.R$ Prop. IV. Cor. 1 and 8.) The remaining forces D S, and D S, being equal and parallel, will not alter the relative motions of the bodies L and T, $\therefore$ they will continue to describe ellipses round D, which will move towards the line I K, but will be impeded in its approach by making the bodies S and D (D being T + L) describe concentric ellipses round the center of gravity C, being projected with proper velocities, in opposite and parallel directions. Now add a fourth body V, and all the previous motions will continue the same, only accelerated, and C and V will describe ellipses round B, being projected with proper velocities.

And so on, for any number of bodies.

Also the periods in all the ellipses will be the same, for the accelerating force on $T = L.TL + S.TD = (T+L).TD + S.TD = (T+L+S).TD$, i. e. when a third body S is added, T is acted on as if by the sum of the three bodies at the distance T D, and the accelerating force on D towards $C = S.SD = S.CS + S.DC = (T+L).DC + S.DC = (T+L+S).DC$.

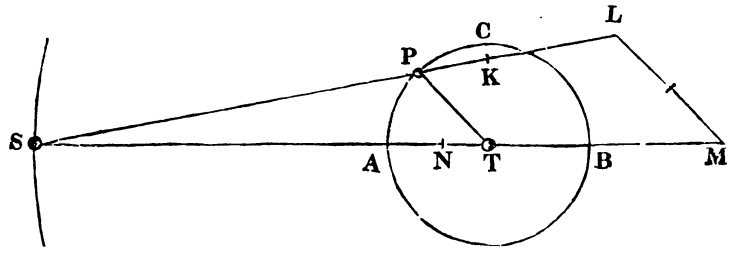
$\therefore$ accelerating force on T towards D : do. on D towards C :: TD : DC

$\therefore$ the absolute accelerating forces on T and D are equal, or T and D move as if they revolved round a common center, the absolute force the same, and varying as the distance from the center, i. e. they describe ellipses, in the same periods.

Similarly when a fourth body V is added, T, L, D, S, C, and V, move as if the four bodies were placed at D, C, B, i. e. as if the absolute forces were the same, and with forces proportional to their respective distances from the centers of gravity, and $\therefore$ in equal periods.

And so on, for any number of bodies.

430. PROP. LXVI. S and P revolve round T, S in the exterior orbit, P in the interior,



$F \propto \frac{1}{D^2}$, find when P will describe round T an orbit nearest to the ellipse, and areas most nearly proportional to the times.

(1st.) Let S, P, revolve round the greatest body T *in the same plane.*

Take K S for the force of S on P at the mean distance S K,

and

$$L S = S K . \frac{S K^2}{S P^2} = \text{force at P},$$

resolve L S into L M, M S,

L M is parallel to P T, and ∴ tends to the center T, ∴ P will continue to describe areas round T proportional to the times, as when acted on only by P T, but since L M does not $\propto \frac{1}{P T^2}$, the sum of L M and P T will not $\propto \frac{1}{P T^2}$, ∴ the form of the elliptic orbit P A B will be disturbed by this force, L M, M S neither tends from P to the center T, nor $\propto \frac{1}{P T^2}$, ∴ from the force M S both the proportionality of areas to times, and the elliptic form of the orbit, will be disturbed, and the elliptic form on two accounts, because M S does not tend to C, and because it does not $\propto \frac{1}{P T^2}$.

∴ the areas will be most proportional to the times, when the force M S is least, and the elliptic form will be most complete, when the forces M S, L M, but particularly L M, are least.

Now let the force of S on T = N S, then this first part of the force M S being common to P and T will not affect their mutual motions, ∴ the

disturbing forces will be least when L M, M N, are least, or L M remaining, when M N is least, i. e. when the forces of S on P and T are nearly equal, or S N nearly = S K.

(2dly) Let S and P revolve round T *in different planes.*

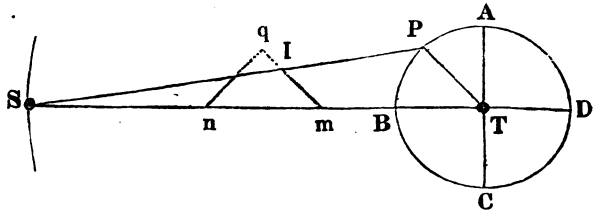
Then L M will act as before.

But M N acting parallel to T S, when S is not in the line of the Nodes, (and M N does not pass through T), will cause a disturbance not only in the longitude as before, but also in the latitude, by deflecting P from the plane of its orbit. And this disturbance will be least, when M N is least, or S N nearly = S K.

431. Cor. 1. If more bodies revolve round the greatest body T, the motion of the inmost body P will be least disturbed when T is attracted by the others equally, according to the distances, as they are attracted by each other.

432. Cor. 2. In the system of T, if the attractions of any two on the third be as $\frac{1}{D^2}$, P will describe areas round T with greater velocity near conjunction and opposition, than near the quadratures.

433. To prove this, the following investigation is necessary.



Take l S to represent the attraction of S on P,

n S ——————————— T,

Then the disturbing forces are l m (parallel to P T) and m n.

Now

$$S\,l = \frac{S}{S\,P^2}\left(\text{force} \propto \frac{1}{D^2}\right),$$

$$= \frac{S}{R^2 - 2\,R\,r\cos.\,A + r^2}, (R = S\,T, r = P\,T)\ A = \angle\, S\,T\,P$$

$$\therefore S\,m = S\,l.\frac{S\,T}{S\,P} = \frac{S.R}{(R^2 - 2\,R\,r\cos.\,A + r^2)}\sqrt{R^2 - 2\,R\,r\cos.\,A + r^2}$$

$$= \frac{S.R}{R^3\left(1 - \frac{2\,r\cos.\,A}{R} + \frac{r^2}{R^2}\right)}\sqrt{1 - \frac{2\,r\cos.\,A}{R} + \frac{r^2}{R^2}}$$

$$= \frac{S}{R^2}\left(1 - \frac{2r}{R}\cos. A + \frac{r^2}{R^2}\right)^{-\frac{3}{2}}$$

$$= \frac{S}{R^2}\left(1 + \frac{3}{2}\left(\frac{2r}{R}\cos. A - \frac{r^2}{R^2}\right) + \frac{3.5}{2.4}.\left(\frac{2r\cos. A}{R} - \frac{r^2}{R^2}\right)^2 \&c.\right)$$

$$= \frac{S}{R^2}\left(1 + \frac{3r}{R}\cos. A - \left(\frac{3}{2} - \frac{3.5}{2.4}\cos.^2 A\right)\frac{r^2}{R^2} \&c.\right)$$

$$= \frac{S}{R^2}\left(1 + \frac{3r.\cos. A}{R}\right),$$

where R is indefinitely great with respect to r.

Also

$$m n = S m - S n = \frac{S}{R^2}\left(1 + \frac{3r\cos. A}{R}\right) - \frac{S}{R^2} = \frac{S.3r\cos.}{R^3},$$

ultimately

and $l m = S l . \frac{r}{R} = \frac{S.r}{R}(R^2 - 2Rr\cos. A + r^2)$

$$= \frac{S.r}{R}.(R^2 - 2Rr\cos. A + r^2)^{-1}$$

$$= \frac{S.r}{R^3} + \frac{S.2r^2}{R^4}, \&c.$$

$$= \frac{S.r}{R^3} \text{ ultimately.}$$

434. Call l m the addititious force

and m n the ablatitious force

and m n = l m 3 cos. A.

Resolve m n into m q, q n.

The part of the ablatitious force which acts in the direction m q = m n . cos. A

$$= \frac{3.S.r.\cos.^2 A}{R^3} = \text{central ablatitious force.}$$

The tangential part = m n . sin. A $= \frac{3\,S.r}{R^3}$. sin. A . cos. A

$$= \frac{3}{2}.\frac{S.r}{R^3}.\text{ sin. } 2A = \text{tangential ablatitious force}$$

∴ the whole force in the direction P T = l m − m q $= \frac{S.r}{R^3} - \frac{3.S.r.\cos.^2 A}{R^3}$

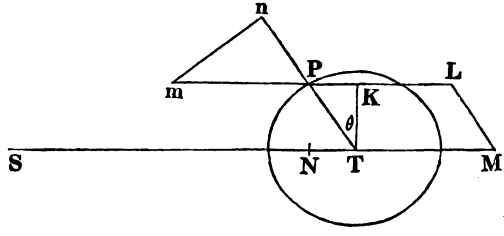
$$= \frac{S.r}{R^3}(1 - 3\cos.^2 A) \text{ and the}$$

whole force in the direction of the Tangent = q n $= \frac{3}{2}.\frac{S.r}{R^3}.\text{ sin. } 2A.$

435. Hence COR. 2. is manifest, for of the four forces acting on P, the

three first, namely, attraction of T, addititious force, and central ablatitious force, do not disturb the equable description of areas, but the fourth or tangential ablatitious force does, and this is + from A to B, —from B to C, + from C to D, — from D to A. ∴ the velocity is accelerated from A to B, and retarded from B to C, ∴ it is greatest at B. Similarly it is a maximum at D. And it is a minimum at A and C. This is Cor. 3.

436. *To otherwise calculate the central and tangential ablititious forces.*



On account of the great distance of S, S M, P L may be considered parallel, and

$$\therefore P T = L M, \text{ and } S P = S K = S T.$$

∴ the ablatitious force $= 3 P T. \sin. \theta = 3 P K.$

Take $P m = 3 P K$, and resolve it into $P n$, $n m$.

$P n = P m . \sin. \theta = 3 P T. \sin.^2 \theta =$ central ablatitious force

$$= 3 P T. \frac{1 - \cos. 2\theta}{2}$$

$n m = P m . \cos. \theta = 3 P T. \sin. \theta \cos. \theta = \frac{3}{2} . P T. \sin. 2\theta =$ tangential ablatitious force.

The same conclusions may be got in terms of l m from the fig. in Art 433, which would be better.

437. *Find the disturbing force on* P *in the direction* P T.

This = (addititious + central ablatitious) force $= l m + 3 l m . \sin.^2 \theta$

$$= l m - 3 l m \left(\frac{1 - \cos. 2\theta}{2}\right)$$

$$= -l m \left(\frac{1 - 3 \cos. 2\theta}{2}\right).$$

438. *To find the mean disturbing force of* S *during a whole revolution in the direction* P T.

Let P T at the mean distance = m, then $- l m \left(\frac{1 - 3 \cos. 2\theta}{2}\right)$

$= -\frac{1\,m}{2} = -\frac{m}{2}$ since cos. 2θ is destroyed during a whole revolution.

439. The disturbing forces on P are

(1) addititious $= \frac{S \cdot r}{R^3} = A.$

(2) ablatitious $= 3 \cdot A \cdot \sin. \theta$

which is (1) tangential ablatitious force $\frac{3 \cdot A}{2} \cdot \cos. 2\theta$

and (2) central ablatitious force $= 3\,A \cdot \frac{1 - \cos. 2\theta}{2}$

$\therefore$ whole disturbing force in the direction P T $= A - \frac{3\,A}{2} + \frac{3\,A}{2} \cdot \cos. 2\theta$

$$= -\frac{A}{2} + \frac{3\,A}{2} \cdot \cos. 2\theta.$$

But in a whole revolution cos. 2θ will destroy itself, $\therefore$ the whole disturbing force in the direction P T in a complete revolution is ablatitious and $= \frac{1}{2}$ addititious force.

The whole force in the direction P T $= \frac{S \cdot r}{R^3}(1 - 3 \sin.^2\theta)$ (Art. 433)

$$= \frac{S\,r}{R^3}\left(1 - \frac{3}{2}(1 - \cos. 2\theta)\right)$$

multiply this by d.θ, and the integral $= \frac{S \cdot r}{R^3}\left(\theta - \frac{3}{2}\theta + \frac{3}{4} \cdot \sin. 2\theta\right)$

= sum of the disturbing forces; and this when $\theta = \pi$ becomes $-\frac{S \cdot r}{R^3} \cdot \frac{\pi}{2}$.

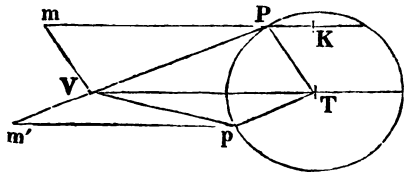
This must be divided by π, and it gives the mean disturbing force acting on P in the direction of radius vector $= -\frac{1}{2}\frac{S \cdot r}{R^3}$.

440. The 2d COR. will appear from Art. 433 and 434.

For the tangential ablatitious force $= \frac{3}{2} \cdot \sin\ 2\theta \cdot \times$ addititious force,

$\therefore$ this force will accelerate the description of the areas from the quadratures to the syzygies and retard it from the syzygies to the quadratures, since in the former case sin. 2θ is $+$, and in the latter $-$.

441. COR. 3 is contained in COR. 2. (Hence the Variation in astronomy.)

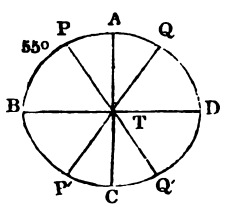


442. P V is equivalent to P T, T V, and accelerates the motion; p V is equivalent to p T, T V, and retards the motion.

443. Cor. 4. Cæt. par., the curve is of greater curvature in the quadratures than in the syzygies.

For since the velocity is greatest in the syzygies, (and the central ablatitious force being the greatest, the remaining force of P to T is the least) the body will be less deflected from a right line, and the orbit will be less curved. The contrary takes place in the quadratures.

444. The whole force from S in the direction $P T = \frac{S \cdot r}{R^3}(1 - 3 \sin.^2\theta)$ (see 433) and the force from T in the direction $P T = \frac{T}{r^2}$.



$\therefore$ the whole force in the direction $P T = \frac{T}{r^2} + \frac{S \cdot r}{R^3}(1 - 3 \sin.^2\theta)$

and at A this becomes $\frac{T}{r^2} + \frac{S \cdot r}{R^3}$

at B . . . . . $\frac{T}{r^2} - \frac{2 \cdot S \cdot r}{R^3}$

at C . . . . . $\frac{T}{r^2} + \frac{S \cdot r}{R^3}$

at D . . . . . $\frac{T}{r^2} - \frac{2\,S \cdot r}{R^3}$

(for though sin. 270 is —, yet its syzygy is +).

Thus it appears that on two accounts the orbit is more curved in the quadratures than in the syzygies, and assumes the form of an ellipse at the major axis A C.

$\therefore$ the body is at a greater distance from the center in the quadratures than in the syzygies, which is Cor. 5.

445. COR. 5. Hence the body P, cæt. par., will recede farther from T in the quadratures than in the syzygies; for since the orbit is less curved in the syzygies than in the quadratures, it is evident that the body must be farther from the center in the quadratures than in the syzygies.

446. COR. 6. The addititious central force is greater than the ablatitious from Q′ to P, and from P′ to Q, but less from P to P′, and from Q to Q′, $\therefore$ on the whole, the central attraction is diminished. But it may be said, that the areas are accelerated towards B and D, and $\therefore$ the time through P P′ may not exceed the time through P′ Q, or the time through Q Q′ exceed that through Q′ P. But in all the corollories, since the errors are very small, when we are seeking the quantity of an error, and have ascertained it without taking into account some other error, there will be an error in our error, but this error in the error will be an error of the second order, and may $\therefore$ be neglected.

The attraction of P to T being diminished in the course of a revolution, the absolute force towards T is diminished, (being diminished by the mean disturbing force $-\frac{1}{2}\frac{S\,r}{R^3}$, 439,) $\therefore$ the period which $\propto \frac{r^{\frac{3}{2}}}{\sqrt{f}}$, is increased, supposing r constant.

But as T approaches S (which it will do from its higher apse to the lower) R is diminished, the disturbing force $\left(\text{which involves } \frac{r}{R}\right)$ will be increased, and the gravity of P to T still more diminished, and $\therefore$ r will be increased; $\therefore$ on both accounts (the diminution of f and increase of r) the period will be increased.

(Thus the period of the moon round the earth is shorter in summer than in winter. Hence the Annual equation in astronomy.)

When T recedes from S, R is increased, and the disturbing force diminished and r diminished. $\therefore$ the period will be diminished (not in comparison with the period round T if there were no body S, but in comparison with what the period was before, from the actual disturbance.)

447. COR. 6. The whole force of P to T in the quadratures $=\frac{T}{r^2}+\frac{S.r}{R^3}$

——————————— the syzygies $=\frac{T}{r^2}-\frac{2\,S\,r}{R^3}$

.. on the whole the attraction of P to T is diminished in a revolution.

For the ablatitious force in the syzygies equals twice the addititious force in the quadratures.

At a certain point the ablatitious force = the addititious; when

$$1 = 3 \sin.^2 \theta$$

or

$$\sin. \theta = \frac{1}{\sqrt{3}}$$

and

$$A = 55°, \&c.$$

$\left(\text{the whole force being then} = \frac{\Gamma}{r^2}.\right)$

Up to this point from the quadratures the addititious force is greater than the ablatitious force, and from this point to one equally distant from the syzygies on the other side, the ablatitious is greater than the addititious; $\therefore$ in a whole revolution P's gravity to T is diminished.

Again since T alternately approaches to and recedes from S, the radius P T is increased when T approaches S, and the period $\propto \frac{r^{\frac{3}{2}}}{\sqrt{\text{absolute force}}}$ and since f is diminished, and $\therefore$ r increased, $\therefore$ the periodic time is increased on both accounts, (for f is diminished by the increase of the disturbing forces which involve $\frac{r}{R}$.) If the distance of S be diminished, the absolute force of S on P will be increased. $\therefore$ the disturbing forces which $\propto \frac{1}{D^2}$ from S are increased, and P's gravity to T diminished, and $\therefore$ the periodic time is increased in a greater ratio than $r^{\frac{3}{2}}$ (because of the diminution of f in the expression $\frac{r^{\frac{3}{2}}}{\sqrt{f}}$) and when the distance of S is increased, the disturbing force will be diminished, (but still the attraction of P to T will be diminished by the disturbance of S) and r will be decreased, $\therefore$ the period will be diminished in a less ratio than $r^{\frac{3}{2}}$.

448. Cor. 7. To find the effect of the disturbing force on the motion of the apsides of P's orbit during one whole revolution.

Whole force in the direction P T $= \frac{T}{r^2} + \frac{S \cdot r}{R^3}(1 - 3 \cos.^2 A)$

$= \frac{T}{r^3} + T \cdot c \cdot r$, (if $T \cdot c = \frac{S}{R^3}(1 - 3 \cos.^2 A) = \frac{T r + T \cdot c \cdot r^4}{r^3}$,

$\therefore$ the $\angle$ between the apsides $= 180 \quad \frac{1 + c}{1 + 4c}$ by the IXth Sect. which is less than 180 when c is positive, i. e. from Q′ to P and from P′ to P,

(fig. (446,)) and greater than 180 when c is negative, i. e. from P to P′ and from Q to Q′,

∴ upon the whole the apsides are progressive, (regressive in the quadratures and progressive in the syzygies);

$$\text{force} = \frac{T}{r^2} - \frac{3\,S\,r}{R^3} = \text{force in conjunction}$$

$$\frac{T}{r'^2} - \frac{3\,.\,S\,r'}{R^3} = \text{force in opposition}$$

Now

$$\frac{R^3\,T - 3\,S\,r^3}{r^2\,R^3} \text{ and } \frac{R^3\,T - 3\,S\,r'^3}{r'^2\,R^3}$$

differ most from $\frac{1}{r^2}$ and $\frac{1}{r'^2}$

when r is least with respect to r′,

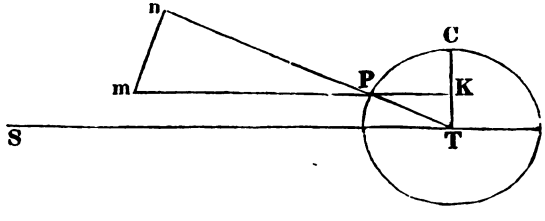
which is the case when the Apsides are in the syzygies.

But

$$\frac{R^3\,T + S\,r^3}{r^2\,R^3} \quad \frac{R^3\,T + S\,r'^3}{r'^2\,R^3}$$

differ least from $\frac{1}{r^2}$ and $\frac{1}{r'^2}$ when r is most nearly equal to r′,

449. COR. 7. Ex. Find the angle from the quadratures, when the apses are stationary.



Draw P m parallel to T S, and = 3 P K, m n perpendicular to T P, resolve P m into P n, n m, whereof n m neither increases nor diminishes the accelerating force of P to T, but P n lessens that force, ∴ when P n = P T, the accelerating force of P is neither increased nor diminished, and the apses are quiescent,

by the triangles P T : P K :: P M = 3 P K : P n = P T

∴ in the required position $3\,P\,K^2 = P\,T^2$

or

$$P\,K = \frac{P\,T}{\sqrt{3}} = P\,T\,.\,\text{sin.}\,\phi,$$

$$\therefore \text{sin. } \theta = \frac{1}{\sqrt{3}},$$

or

$$\theta = 35^{\circ}\ 26'.$$

The addititious force P T — P n is a maximum in quadratures.

$$\text{F or P T} : \text{P K} :: 3\text{ P K} : \text{P n} = \frac{3\text{ P K}^2}{\text{P T}}$$

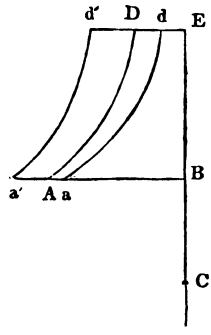
$\therefore$ P T — P n = P T — $\frac{3\text{ P K}^2}{\text{P T}}$, which is a maximum when P K = 0, or the body is in syzygy.

450. Cor. 8. Since the progression or regression of the Apsides depends on the decrement of the force in a greater or less ratio than D^2, from the lower apse to the upper, and on a similar increment from the upper to the lower, (by the IXth Sect.), and is $\therefore$ greatest when the proportion of the force in the upper apse to that in the lower, recedes the most from the inverse square of D, it is manifest that the Apsides progress the fastest from the ablatitious force, when they are in the syzygies, (because the whole forces in conjunction and opposition, i. e. at the upper and lower apses being $\frac{T}{r^2} - \frac{2\text{ S r}}{R^3}$, when the apsides are in the syzygies and when r is greatest at the upper apse, $\frac{T}{r^2}$ being least, and the negative part of the expression $\frac{2\text{ S r}}{R^3}$ being greatest, the whole expression is $\therefore$ least, and when r is least, at the lower apse, $\frac{T}{r^2}$ being greatest, and the negative part least, $\therefore$ the whole expression is greatest, and $\therefore$ the disproportion between the forces at the upper and lower apse is greatest), and that they regress the slowest in that case from the addititious force, (for $\frac{T}{r^2} + \frac{S\text{ r}}{R^3}$, which is the whole force in the quadratures, both before and after conjunction, r being the semi minor axis in each case, differs least from the inverse square); therefore, on the whole the progression in the course of a revolution is greatest when the apsides are in the syzygies.

Similarly the regression is greatest when the apsides are in the quadratures, but still it is not equal to the progression in the course of the revolution.

451. Cor. 8. Let the apsides be in the syzygies, and let the force at the upper apse : that at the lower, : : D E : A B, D A′

being the curve whose ordinate is inversely as the distance^2 from C, $\therefore$ these forces being diminished, the force D E at the upper apse by the greatest quantity $\frac{2\,r\,S}{R^3}$, and the force A B at the lower apse by the least quantity $\frac{2\,r'\,S}{R^3}$; the curve a d which is the new force curve has its ordinates decreasing in a greater ratio than $\frac{1}{D^2}$.



Let the apsides be in the quadratures, then the force E D will be increased by the greatest quantity $\frac{S\,r}{R^3}$, and the force A B by the least quantity $\frac{S\,r'}{R^3}$, $\therefore$ the curve a' d' which is the new force curve will have its ordinates decreasing in a less ratio than $\frac{1}{D^2}$.

451. COR. 9. Suppose the line of apsides to be in quadratures, then while the body moves from a higher to a lower apse, it is acted on by a force which does not increase so fast as $\frac{1}{D^2}$ (for the force $= \frac{R^3\,T + S\,r^3}{r^2\,R^3}$, $\therefore$ the numerator decreases as the denominator increases), $\therefore$ the orbit will be exterior to the elliptic orbit and the excentricity will be decreased. Also as the descent is caused by the force $\frac{S\,r}{R^3}$ (1 — 3 cos.^{2} A), the less this force is with respect to $\frac{T}{r^2}$, the less will the excentricity be diminished. Now while the line of the apsides moves from the line of quadratures, the force $\frac{S\,r}{R^3}$ (1 — 3 cos.^{2} A) is diminished, and when it is inclined at $\angle$ 35° 16′ the disturbing force $= 0$, and $\therefore$ at those four points the excentricity is unaltered. After this, it may be shown in the same manner that the excentricity will be continually increased until the line of apsides coincides with the line of syzygies. Here it is a maximum, since the disturbing force is negative. Afterwards it will decrease as before it increased until the line of apsides again coincides with the quadrature, and then the excentricity $=$ maximum.

(Hence Evection in Astron.)

452. Lemma. To calculate that part of the ablatitious force which is employed in drawing P from the plane of its orbit.

Let A = angular distance from syzygy.

Q = angular distance of nodes from syzygy.

I = inclination of orbit to orbit of S and T.

Then the force required $= \frac{3\,S\,r}{R^3}$. cos. A. sin. Q. sin. I. (not quite accurately.)

When P is in quadratures, this force vanishes, since cos. A = 0.

When nodes are in syzygy, ——— since sin. Q = 0,

——— quadratures, this force (cæt. par.) = maximum, since sin. Q = sin. 90 = rad.

453. Cor. 12. The effects produced by the disturbing forces are all greater when P is in conjunction than when in opposition.

For they involve $\frac{1}{R^3}$, ∴ when R is least, they are greatest.

454. Cor. 13. Let S be supposed so great that the system P and T revolve round S fixed. Then the disturbing forces will be of the same kind as before, when we supposed S to revolve round T at rest.

The only difference will be in the magnitude of these forces, which will be increased in the same ratio as S is increased.

455. Cor. 14. If we suppose the different systems in which S and S T ∝, but P T and P and T remain the same, and the period (p) of P round T remains the same, all the errors $\propto \frac{S}{R^3} \propto \frac{\Delta . d^3}{R^3}$, if Δ = density of S, and d its diameter,

$\propto \delta^3$, if Δ given, and δ = apparent diam.

also

$$\frac{1}{P^2} \propto \frac{S}{R^3} \text{ if P = period of T round S,}$$

$$\therefore \text{ the errors} \propto \frac{1}{P^2}.$$

These are the linear errors, and angular errors ∝ in the same ratio, since P T is given.

456. Cor. 15. If S and T be varied in the same ratio,

Accelerating force of S : that of T :: $\frac{S}{R^2} : \frac{T}{r^2}$ the same ratio as before.

∴ the disturbances remain the same as before.

(The same will hold if R and r be also varied proportionally.)

∴ the linear errors described in P's orbit ∝ P T, (since they involve r), if P T ∝, the rest remaining constant.

also the angular errors of P as seen from T $\propto \frac{\text{linear errors}}{PT} \propto \frac{PT}{PT} \propto 1$, and are $\therefore$ the same in the two systems.

The similar linear errors $\propto$ f. T^2, $\therefore$ P T $\propto$ f. T^2, and f $\propto \frac{PT}{T^2}$, but f $\propto$ accelerating force of T on P $\propto \frac{PT}{p^2}$, (p = period of P round T,)

$$\therefore T \propto p \text{ and } \therefore \propto P$$

$$\left(\text{for } P^2 \propto \frac{S T^3}{S} \propto \frac{P T^3}{T} \propto p^2\right)$$

COR. 14. In the systems

S, T, P, Radii R, r Periods P, p
S′, T, P —— R′, r ——— P′, p.

Linear errors dato t. in 1st. : do. in second :: $\frac{1}{P^2} : \frac{1}{P'^2}$

$\therefore$ angular errors in the period of P —— : —— :: $\frac{1}{P^2} : \frac{1}{P'^2}$.

COR. 15. In the systems

S, T, P, —— R, r ——— P, p
S′, T′, P —— R′, r′——— P′, p′,

so that $\frac{S'}{S} = \frac{T'}{T}$ and $\frac{R'}{R} = \frac{r'}{r}$

$$\therefore \frac{P}{P'} = \frac{p}{p'}.$$

Linear errors in a revolution of P in 1st. : do. in second :: r : r′
angular errors : :: 1 : 1.

COR. 16. In the systems

S, T, P, —— R, r ——— P, p
S, T′, P, —— R, r′ ——— P, p′.

Linear errors in a revolution of P in 1st. : do in second :: r p^2 : r′ p'^2
angular errors in a revolution of P : :: p^2 : p'^2.

To compare the systems

(1) S, T, P —— R, r ——— P, p
(2) S′, T′, P′ —— R′, r′ ——— P′, p′.

Assume the system

(3) S′, T, P —— R′, r ——— P′, p

$\therefore$ by (14) angular errors in P S revolution in (1) : in (3) :: $\frac{1}{P^2} : \frac{1}{P'^2}$

by (16) angular errors in (3) : in (2) :: p^2 : p'^2

therefore errors in (1) : in (2) :: $\frac{p^2}{P^2} : \frac{p'^2}{P'^2}$.

Or assume the system (3) Σ, T, P — ϱ, r — Π, p

$$\text{so that } \frac{\Sigma}{S'} = \frac{T}{T'}, \frac{Q}{R'} = \frac{r}{r'}$$

$\therefore$ the errors in (1) : errors in (3) :: $\frac{1}{P^2} : \frac{1}{\Pi^2} :: \frac{S}{R^3} : \frac{\Sigma}{\varrho^3} :: \frac{S}{\Sigma} : \frac{R^3}{\varrho^3}$

(3) : (2) :: 1 : 1

$$:: \frac{S}{S'}\frac{S'}{\Sigma} : \frac{R^3}{R'^3} \cdot \frac{R'^3}{\varrho^3} :: \frac{S}{S'} \cdot \frac{T'}{T} : \frac{R^3}{R'^3} \cdot \frac{r'^3}{r^3}$$

$$:: \frac{S}{R^3}\frac{r^3}{T} : \frac{S'}{R'^3} \cdot \frac{r'^3}{T} :: \frac{p^2}{P^2} : \frac{p'^2}{P'^2}.$$

457. Cor. 16. In the different systems the mean angular errors of P $\propto \frac{p}{P^2}$ whether we consider the motion of apses or of nodes (or errors in latitude and longitude.)

For first, suppose every thing in the two different systems to be the same except P T, $\therefore$ p will vary. Divide the whole times p, p', into the same number of indefinitely small portions proportional to the wholes. Then if the position of P be given, the disturbing forces all $\propto$ each other $\propto$ P T; and the space $\propto$ f . T^2, $\therefore$ the linear errors generated in any two corresponding portions of time $\propto$ P T . p^2.

$\therefore$ the angular errors generated in these portions, as seen from T, $\propto p^2$.

$\therefore$ Comp°. the periodic angular errors as seen from T $\propto p^2$.

Now by Cor. 14, if in two different systems P T and $\therefore$ p be the same, every thing else varying, the angular errors generated in a given time, as in p, $\propto \frac{1}{P^2}$.

$\therefore$ neutris datis, in different systems the angular errors generated in the time p $\propto \frac{p^2}{P^2}$.

Now

$$p'' : 1'' :: \frac{p^2}{P^2} : \frac{p}{P^2},$$

$\therefore$ the angular errors generated in 1″ (or the mean angular errors) $\propto \frac{p}{P^2}$.

Hence the mean motion of the nodes as seen from T $\propto$ mean motion of the apses, for each $\propto \frac{p}{P^2}$.

458. Cor. 17.

Mean addititious force : mean force of P on T :: $p^2 : P^2$.

For

mean addititious force : force of S on T :: P T : S T,

$$\left(:: \frac{S\,r}{R^3} : \frac{S}{R^2} :: r : R\right)$$

force of S on T : mean force of T on P: : $\frac{S\,T}{p^2} : \frac{P\,T}{p^2}$

$$\left(\text{force} \propto \frac{\text{rad.}}{p^2}\right)$$

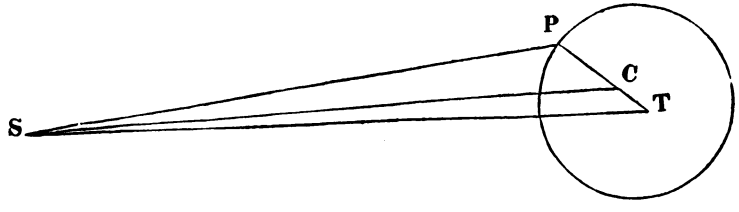
$\therefore$ mean addititious force : mean force of T on P: : $p^2 : P^2$

$\therefore$ ablatitious force : mean force of T on P: : 3 cos. θ . $p^2 : P$.

Similarly, the tangential and central ablatitious and all the forces may be found in terms of the mean force of T on P.

459. PROP. LXVII. Things being as in Prop. LXVI, S describes the areas more nearly proportional to the times, and the orbit more elliptical round the center of gravity of P and T than round T.

For the forces on S are $\frac{P}{PS^2}$ and $\frac{T}{TS^2}$.



$\therefore$ the direction of the compound force lies between S P, S T; and T attracts S more than P.

$\therefore$ it lies nearer T than P, and $\therefore$ nearer C the center of gravity of T and P.

$\therefore$ the areas round C are more proportional to the times, than when round T.

Also as S P increases or decreases, S C increases or decreases, but S T remains the same; $\therefore$ the compound force is more nearly proportional to the inverse square of S C than of S T; $\therefore$ also the orbit round C is more nearly elliptic (having C in the focus) than the orbit round T.

A

SECOND COMMENTARY

ON

SECTION XI.

460. To find the axis major of an ellipse, whose periodic time round S at rest would equal the periodic time of P round S in motion.

Let A equal the axis major of an ellipse described round P at rest equal the axis major of P Q v.

Let x equal the axis major required,

P. T. of P round S in motion : p S at rest : : $\surd$ S : $\surd$ S + P

P. T. of p in the elliptic axis A : P. T. in the elliptic axis x : : $A^{\frac{3}{2}} : x^{\frac{3}{2}}$

$\therefore$ P. T. of P round S in motion : P.T. in the el.ax. x : : $\surd \overline{A^3 S} : \surd \overline{x^3 (S+P)}$.

By hyp. the 1st term equals the 2d,

$$\therefore A^3 S = x^3 . \overline{S + P}$$

$$\therefore A : x : : (S+P)^{\frac{1}{3}} : S^{\frac{1}{3}}.$$

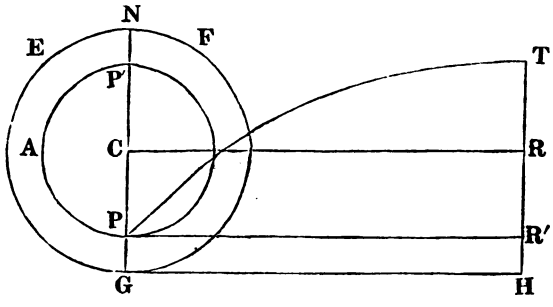
461. PROP. LXIII. Having given the velocity, places, and directions of two bodies attracted to their common center of gravity, the forces varying inversely as the distance^2, to determine the actual motions of bodies in fixed space.

Since the initial motions of the bodies are given, the motions of the center of gravity are given. And the bodies describe the same moveable curve round the center of gravity as if the center were at rest, while the center moves uniformly in a right line.

* Take therefore the motion of the center proportional to the time, i. e. proportional to the area described in moveable orbits.

* Since a body describes some curve in fixed space, it describes areas in proportion to the times in this curve, and since the center moves uniformly forward, the space described by it is in proportion to the time, therefore, &c.

462. Ex. 1. Let the body P describe a circle round C, while the center C moves uniformly forward. Take C G : C P : : v of C : v of P, and with the



center C and rad. C G describe a circle G C N, and suppose it to move round along G H, then P will describe the trochoid P L T, and when P has described the semicircle P A B, P will be at the summit of the trochoid

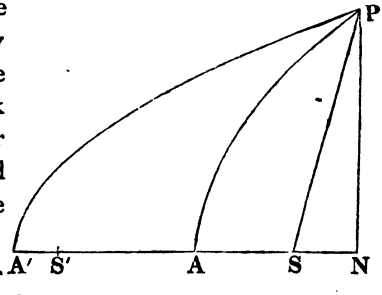
∴ every point of the semicircumference G F N will have touched G H,

∴ G H equals the semicircumference G F N,

∴ v of P : v of C : : P A B semicircumference : C R = G F N semicircle

* : : C P : C G Q. e. d.

463. Ex. 2. Let the moveable curve be a parabola, and let the center of gravity move in the direction of its primitive axis. When the body is at the vertex A′, let S′ be the position of the center of gravity, and while S′ has described uniformly S′ S, let A have described the arc of the parabola A P.



Let A′ N = x, N P = y, be the abscissa and ordinate of the curve A P in fixed space.

Let 4 p equal the parameter of the parabola A P.

$$\therefore A\,N = \frac{y^2}{4p},\quad A'\,S = S'\,S = x - \frac{y^2}{4p} = \frac{4px - y^2}{4p}$$

$$S\,N = A\,N - A\,S = A\,N - p = \frac{y^2}{4p} - p' = \frac{y^2 - 4p^2}{4p}$$

$$\text{Area } A\,S\,P = A\,N\,P - S\,N\,P = \tfrac{2}{3}\,A\,N \times N\,P - \tfrac{1}{2}\,N\,S \times N\,P$$

$$= \tfrac{2}{3}\cdot\frac{y^3}{4p} - \tfrac{1}{2}\,\frac{y^3 - 4p^2y}{4p} = \frac{y^3 + 12p^2y}{24p}.$$

By Prop. S′ S ∞ A S P; therefore they are in some given ratio.

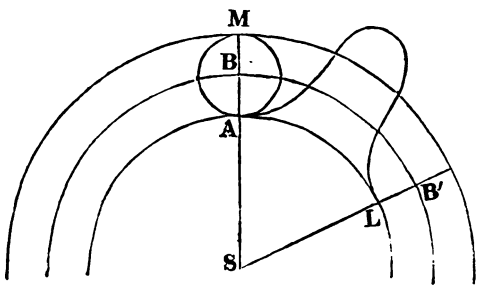
$$\text{Let } A\,S\,P : S'\,S : : a : b : : \frac{y^3 + 12p^2y}{24p} : \frac{4px - y^2}{4p}$$

* If C P = C G the curve in fixed space becomes the common cycloid.
If C P > C G ———— the oblongated trochoid.

$$\therefore y^3 + 12 p^2 y = 4 p a x - a y^2$$
$$\therefore y^3 + a y^2 + 12 p^2 y - 4 p a x = 0.$$

Equation to the curve in fixed space.

464. Ex. 3. * Let B B′ be the orbit of the earth round the sun, M A

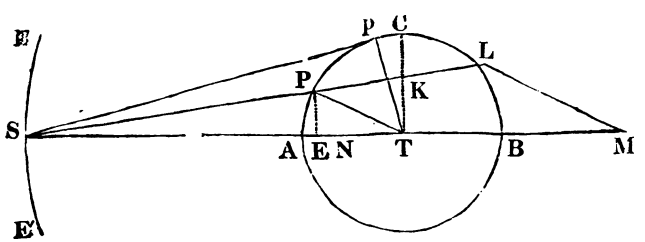


that of the moon round the earth, then the moon will, during a revolution, trace out a contracted or protracted epicycloid according as A L has a greater or less circumference than A M, and the orbit of the moon round the sun will consist of twelve epicycloids, and it will be always concave to the sun. For

$$\text{F of the earth to the sun : F of the moon to the earth} :: \frac{R}{P^2} : \frac{r}{p^2}$$
$$:: \frac{400}{(365)^2} : \frac{1}{(27)^2}$$

in a greater ratio than 2 : 1. But the force of the earth to the sun is nearly equal to the force of the moon to the sun, ∴ the force of the moon to the earth, ∴ the deflection to the sun will always be within the tangential or the curve is always concave towards the sun.

465. Prop. LXVI. If three bodies attract each other with forces



varying inversely as the square of the distance, but the two least revolve

* To determine the nature of the curve described by the moon with respect to the sun.

about the greatest, the innermost of the two will more nearly describe the areas proportional to the time, and a figure more nearly similar to an ellipse, if the greatest body be attracted by the others, than if it were at rest, or than if it were attracted much more or much less than the other bodies.

$$\left(\begin{array}{l} \text{L M} : \text{P T} :: \text{S L} : \text{S P}, \\ \therefore \text{L M} \propto \frac{\text{P T}}{\text{S P}^3}, \\ \therefore \text{L M} = \frac{\text{P T} \times \text{S L}}{\text{S P}} = \frac{\text{S K}^3 \times \text{P T}}{\text{S P}^3}\Big), \\ \therefore \text{S K}^2 : \text{S P}^2 :: \text{S L} : \text{S P}). \end{array}\right.$$

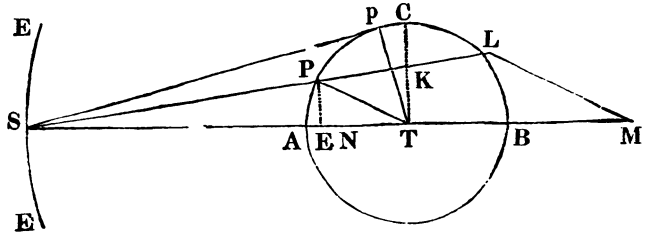
Let P and S revolve in the same plane about the greatest body T, and P describe the orbit P A B, and S, E S E. Take S K the mean distance of P from S, and let S K represent the attraction of P to S at that distance. Take S L : S K :: S R^2 : S P^2, and S L will represent the attraction of S on P at the distance S P. Resolve it into two S M, and L M parallel to P T, and P will be acted upon by three forces P T, L M, S M. The first force P T tends to T', and varies inversely as the distance^2, $\therefore$ P ought by this force to describe an ellipse, whose focus is T. The second, L M, being parallel to P T may be made to coincide with it in this direction, and $\therefore$ the body P will still, being acted upon by a centripetal force to T, describe areas proportional to the time. But since L M does not vary inversely as P T, it will make P describe a curve different from an ellipse, and $\therefore$ the longer L M is compared with P T, the more will the curves differ from an ellipse. The third force S M, being neither in the direction P T, nor varying in the inverse square of the distance, will make the body no longer describe areas in proportion to the times, and the curve differ more from the form of an ellipse. The body P will $\therefore$ describe areas most nearly proportional to the times, when this third force is a minimum, and P A B will approach nearest to the form of an ellipse, when both second and third forces are minima. Now let S N represent the attraction of S on T towards S, and if S N and S M were equal, P and T being equally attracted in parallel directions would have relatively the same situation, and if S N be greater or less then S M, their difference M N is the disturbing force, and the body P will approach most nearly the equable description of areas, and P A B to the form of an ellipse, when M N is either nothing or a minimum.

Case 2. If the bodies P and S revolve about T in different planes, L M being parallel to P S will have the same effect as before, and will not

tend to move P from its plane. But N M acting in a different plane, will tend to draw P out of its plane, besides disturbing the equable description of areas, &c. and as before this disturbing force is a minimum, when M N is a minimum, or when S N = nearly S K.

466. To estimate the magnitude of the disturbing forces on P, when P moves in a circular orbit, and in the same plane with S and T.

Let the angle from the quadratures P C T = θ,



S T = d, P T = r, F at the distance (a) = M,

F on P = $\frac{M a^2}{S P^2}$,

$\therefore$ From P in the direction S P : — P T :: S P : P T,

$\therefore$ F in the direction P T = $\frac{M a^2}{S P^2} \times \frac{P T}{S P}$.

But $S P^2 = d^2 + r^2 - 2 d r \sin. \theta$,

$\therefore$ F in the direction P T = $\frac{M a^2 r}{(d^2 + r^2 - 2 d r \sin. \theta)^{\frac{3}{2}}}$

$= \frac{M a^2 r}{d^3} \left\{ 1 - \frac{3}{2} \frac{r^2 - 2 d r \sin. \theta}{d^2} \right\}$

$= \frac{M a^2 r}{d^3}$ = A nearly, since d being indefinitely great compared with r in the expansion, all the terms may be neglected except two. First $\frac{1}{d}$ vanishes when compared with $\frac{1}{d^3}$, $\therefore$ the addititious force in the direction P T = A. By proportion as before, force in the direction S T

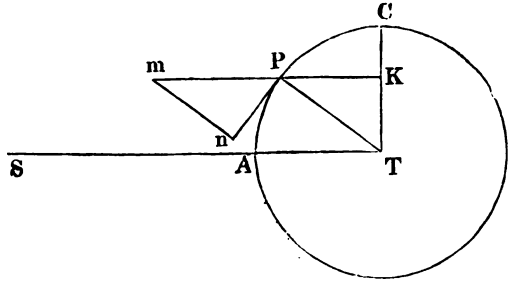
$= \frac{M a^2}{S P^2} \frac{S T}{S P} = d^3 \left(1 + \frac{M a^2 d}{(r^2 - 2 d r \sin. \theta)^{\frac{3}{2}}}\right)$

$= \frac{M a^2}{d^2} \left(1 - \frac{3}{2} \frac{r^2 - 2 d r \sin. \theta}{d^2}\right)$

$- \frac{M a^2}{d^2} - \frac{3 M a^2 r^2}{2 d^4} + \frac{3 M a^2 r \sin. \theta}{d^3}$,

$\therefore$ force in the direction S T $= \frac{M a^2}{d^2} + \frac{3 M a^2 r}{d^3}$ sin. θ nearly, since $\frac{1}{d^4}$ vanishes when compared with $\frac{1}{d^3}$, and the force of S on T $= \frac{M a^2}{d^2}$,

$$\therefore \text{ablatitious } F = \frac{M a^2}{d^2} + \frac{3 M a\ r}{d^3} . \text{ sin. } \theta - \frac{M a^2}{d^2}$$
$$= 3 A . \text{ sin. } \theta.$$



If P T equal the addititious force, then the ablatitious force equals 3 P K, for P K : P T : : sin. θ : (1 = r),

$$\therefore 3 P K = 3 P T . \text{ sin. } \theta = 3 A . \text{ sin. } \theta.$$

To resolve the ablatitious force. Take

$$P m : P n : : P T : T K : : 1 : \text{cos. } \theta,$$
$$\therefore P n = P m \times \text{cos. } \theta = 3 A \times \text{sin. } \theta \text{ cos. } \theta = \frac{3 A}{2} . \text{ sin. } 2\theta$$
$$m n = P m \times P K = 3 A . \text{sin.}^2 \theta = 3 A . \frac{1 - \text{cos. } 2\theta}{2},$$

$\therefore$ the disturbing forces of S on P are

1. The addititious force $= \frac{M a^2 r}{d^3} = A.$

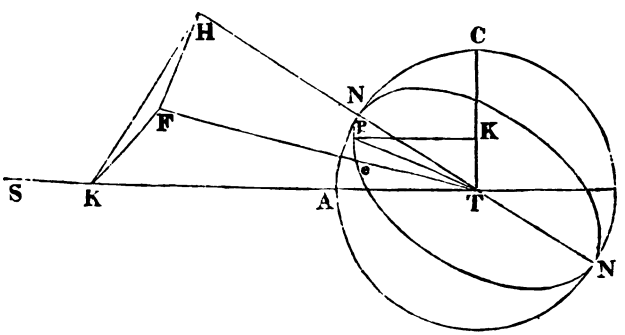
2. The ablatitious force which is resolved into the tangential part $= \frac{3 A}{2} . \text{ sin. } 2\theta$, and that in the direction T P $= 3 A . \frac{1 - 2 \text{ cos. } A}{2}$,

$\therefore$ whole disturbing force in the direction P T $= A - 3 A . \frac{1 - 2 \text{ cos. } A}{2}$
$= A - \frac{3 A}{2} + \frac{3 A}{2} . \text{ cos. } 2\theta = -\frac{A}{2} + \frac{3 A}{2} . \text{ cos. } 2\theta$, and in the whole revolution the positive cosine destroys the negative, therefore the whole disturbing force in a complete revolution is ablatitious, and equal to one half of the mean addititious force.

467. To compare N M and L M.

$$L M : P T : : \left(S L = \frac{S K^3}{S P^2}\right) : S P,$$
$$\therefore L M = \frac{S K^3}{S P^3} \times P T$$

$$MN = \frac{SK^3}{SP^3} \times ST - ST = \frac{SK^3 - SP^3}{SP^3} \times ST$$
$$= \frac{SK^3 - (SK - KP)^3}{SP^3} \times ST$$
$$= \frac{SK^3 - SK^3 + 3SR^2 \times KP}{SP^3} \times ST \text{ nearly}$$
$$= \frac{3SK^2 \times PK}{SP^3} \times ST \text{ nearly} = \frac{3SK^3}{SP^3} \times PK$$
$$= \frac{3SK^3}{SP^3} \times PT \times \sin.\ \theta,$$
$$\therefore MN : LM :: 1 : 3 \sin.\ \theta.$$



468. Next let S and P revolve about T in different planes, and let N P N′ be P's orbit, N N′ the line of the nodes. Take T K in T S = 3 A. sin. θ. Pass a plane through T K and turn it round till it is perpendicular to P's orbit. Let T e be the intersection of it with P's orbit. Produce T E and draw K F perpendicular to it, ∴ K F is perpendicular to the plane of P's orbit, and therefore perpendicular to every line meeting it in that orbit, T in the plane of S's orbit; draw K H perpendicular to N′ N produced; join H F, then F H K equals the inclination of the planes of the two orbits. For K H T, K F T, K F H being all right angles,

$$KT^2 = KH^2 + HT^2$$
$$KF^2 + H^2 = KF^2 + FH^2 + HT^2,$$
$$*\therefore FT^2 = FH^2 + HT^2,$$
$$\therefore FH \text{ is perpendicular to } HT.$$

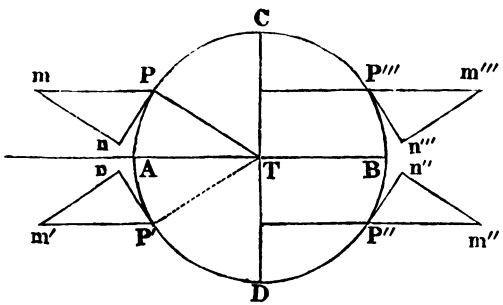
Since P T = A, T K = A × sin. θ

* Let the angle K H T = T, H T K = φ = angular distance of the line of the nodes from S y z.

$$P\,T : T\,K :: 1 : 3 \sin.\theta$$
$$T\,K : K\,H :: 1 : \sin.\varphi$$
$$K\,H : K\,F :: 1 : \sin.\,T,$$
$$\therefore P\,T : K\,F :: 1 : 3 \sin.\theta . \sin.\varphi . \sin.\,T,$$

$\therefore$ ablatitious force perpendicular to P's orbit $= K\,F$
$= 3\,P\,T \times \sin.\theta . \sin.\varphi \times \sin.\,T = 3\,A \times \sin.\theta . \sin.\varphi \times \sin.\,T.$

2d. Hence it appears that there are four forces acting on P.



1. Attraction of P to T $\propto \frac{1}{r^2}$.

2. Addititious F in the direction $P\,T = \frac{M\,a^2\,r}{d^3}$.

3. Ablatitious F in the direction $P\,T = \frac{3\,M\,a^2\,r}{d^3} . \sin.^2\theta$.

4. Tangential part of the ablatitious force $= \frac{3}{2} . \frac{M\,a^2\,r}{d^3} . \sin.^2\theta$.

Of these the three first acting in the direction of the radius-vector do not disturb the equable description of areas, the fourth acting in the direction of a tangent at P does interrupt it.

Since the tangential part of F is formed by the revolution of $P\,M = 3\,A \times \sin.\theta$ at C, $\theta = 0$, therefore $P\,m = 0$, and consequently the tangential $F = 0$; from C to A, P n is in consequentia, and therefore accelerates the body P at A, it again equals 0, and from A to D is in antecedentia, and therefore retards P; from D to B it accelerates; from B to C it retards.

Therefore the velocity of P is greatest at A and B, because these are the points at which the accelerations cease and retardations begin, and the velocity is least at D and C. To find the velocity gained by the action of the tangential force.*

$$d\,Z = F\,d\,x = \tfrac{3}{2}\,A . \sin.\,2\theta\,d\,\theta$$

* F in the direction P T is a maximum at the quadrature, because the ablatitious F in the quadrature is 0, and at every other point it is something.

$$\sin. 2\theta \times 2\theta' = -(\cos. 2\theta)',$$

$$\therefore Z = \frac{v^2}{2g} = \text{Cor.} -\tfrac{3}{4} A. \cos. 2\theta.$$

But when $\theta = 0$, the tangential $F = 0$, and no velocity is produced,

$$\therefore \cos. 2\theta = R = 1,$$

$$\therefore \frac{v^2}{2g} = \frac{3A}{4}(1 - \cos. 2\theta) = \tfrac{3}{4} A. 2 \sin.^2 \theta,$$

$$\therefore v^2 = 3g A. \sin.^2 \theta,$$

$$\therefore v = \sqrt{3g A.} \sin. \theta,$$

$$\therefore v' \propto (\sin. \theta)',$$

$\therefore$ whole f on the moon at the mean distance : f of S on T : : $\frac{1}{p^2} : \frac{d}{P^2}$

and the force of S on T : add. f at the mean distance (m) : : $\frac{q}{d^2} : \frac{q}{d^3}$,

$\therefore$ whole f at the mean distance : m : : $P^2 : p^2$ and $\frac{p^2}{P^2} \times$ whole f &c. $= m$.

Now f on the moon at any distance (r) $= \frac{f}{r^2} - \frac{q\,r}{2\,d^3}$ and at the mean distance (1) $= f - \frac{q}{2\,d^3} = f - \frac{m}{2}$,

$$\therefore m = \frac{p^2 f}{P^2} - \frac{m\,p^2}{2\,P^2},$$

$$\therefore m = \frac{2\,p^2 f}{2\,P^2 + p^2},$$

and therefore nearly $= \frac{p^2 f}{\varphi} - \frac{p^4 f}{2\,P^4}$,

$\therefore$ m r (which equals the addititious force) $= f.\ r. \left\{\frac{p^2}{P^2} - \frac{2\,p^4}{P^4}\right\}$.

469. To compare the ablatitious and addititious forces upon the moon, with the force of gravity upon the earth's surface. (Newton, Vol. III. Prop. XXV.)

add. f : f of S on T :: P T : S T

f of S on T : f of the earth on the moon :: $\frac{S\,T}{P^2} : \frac{P\,V}{p^2} = \frac{P\,T}{p^2}$,

$\therefore$ add. f : f of the earth on the moon :: $p^2 : P^2$

f of the earth on the moon : force of gravity :: 1 : 60^2,

$\therefore$ add. f : force of gravity :: $p^2 : P^2 . 60^2$ . . . (1)

Also ablat. f : additious force :: 3 P K : P T,

$\therefore$ ablat. f : additious force :: $3\,P\,K . p^2 : 60^2 . P\,T . P^2$. (2)

470. Cor. 2. In a system of three bodies S, P, T, force $\propto^s \frac{1}{d^2}$, the

body P will describe greater areas in a given time at the syzygies than at the quadrature.

The tangent ablatitious $f = \frac{3}{2}$. P T. sin. 2θ; therefore this force will accelerate the description of areas from quadratures to syzygies and retard it from syzygies to quadratures, since in the former case sin. 2θ is positive, and in the latter negative.

COR. 3. is contained in Cor. 2.

The first quadrant d. sin. being positive the velocity increases, in the second d. sin. negative the velocity decreases, &c. for the 1st Cor. 2d Cor. &c.

Also v is a maximum when sin. θ is a maximum, i. e. at A and B.

471. COR. 4. The curvature of P's orbit is greater in quadratures than in the syzygy.

The whole F on $P = \frac{M a^2}{r^2} + \frac{M a^2 r}{d^3} - \frac{3 M a^2 r}{2 d^3} (1 - \cos. 2\theta) \times \left(\frac{3 M a^3 r . \text{ sin. } 2\theta}{2 d^3}\right)$.

In quadratures sin. $2\theta = 0$,

$$\therefore F = \frac{M a^2}{r^2} + \frac{M a^2 r}{d^3}.$$

And in syz. $2\theta = 180$,

$$\therefore \text{sin. } 2\theta = 0, \quad \cos. 2\theta = 1$$

$$\therefore \frac{3 M a^2 r}{2 d^3} . (1 - \cos. 2\theta) = \frac{3 M a^2 r}{d^3},$$

$\therefore$ the whole F on P in the syz. $= \frac{M a^2}{r^2} - \frac{2 M a^2 r}{d}$

$\therefore$ F is greater in the quadratures than in the syzygies; and the velocity is greater in the syzygies than in the quadratures.

But the curvature $\propto \frac{1}{P V} \propto \frac{F}{V^2}$, $\therefore$ is greatest in the quadratures and least in the syzygies.

472. COR. 5. Since the curvature of P's orbit is greatest in the quadrature and least in the syzygy, the circular orbit must assume the form of an ellipse whose major axis is C D and minor A B.

$\therefore$ P recedes farther from T in the quadrature than in the syzygy.

473. COR. 6.

The whole F on P in the line $PT = \frac{M a^2}{r^2} + \frac{M a^2 r}{d^3} - \frac{3 M a^2 r}{d^3}$. sin.$^2\theta$

$$= \text{in quad. } \frac{M a^2}{r^2} + \frac{M a^2 r}{d^3}$$

$$\text{and in syz.} = \frac{M a^2}{r^2} - \frac{2 M a^2 r}{d^3}$$

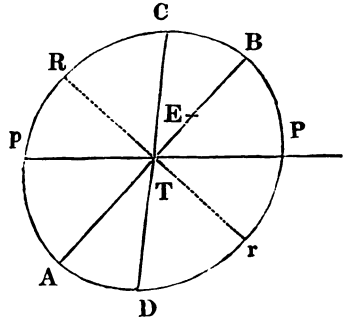
let the ablatitious force on P equal the addititious, and

$$\frac{M a^2 r}{d^3} = \frac{3 M a^2 r}{d^3} . \sin.^2 \theta$$

$$\therefore \sin. \theta = \frac{1}{\sqrt{3}} \sin. 35^\circ . 16.$$

Therefore up to this point from quadrature the ablatitious force is less than the addititious, and from this to one equally distant from the other point of quadrature, the ablatitious is greater than the addititious, therefore in a whole revolution the gravity of P to T is diminutive from what it would be if the orbit were circular or if S did not act, and $P \propto \frac{R^{\frac{3}{2}}}{\sqrt{\text{abl. F}}}$ and since the action of S is alternately increased or diminished, therefore P $\propto$ from what it would be were P T constant, both on account of the variation, and of the absolute force.

474. Cor. 7. * Let P revolve round T in an elliptic orbit, the force on P in the quad. $= \frac{M a^2}{r^2} + \frac{M a^2 r}{d^3} + \frac{b}{r^2} + c r.$



$\therefore G + 180 \sqrt{\frac{b + c}{b + 4 c}}$ and since the number is greater than the denomination G is less than 180. $\therefore$ the apsides are regressive if the same effect is produced as long as the addititious force is greater than the ablatitious, i. e. through 35°. 16′.

The force on P in the syz. $= \frac{M a^2}{r} - \frac{2 M a^2 r}{d^3} = \frac{b}{r^2} - 2 c r$

* Since $P \propto \frac{R^{\frac{3}{2}}}{\sqrt{\text{ablatitious force}}}$ and in winter the sun is nearer the earth than in summer, R is increased in winter, and A is diminished, therefore the lunar months are shorter in winter than in summer.

$$\therefore G = 180 \cdot \sqrt{\frac{b - 2c}{b - 8c}} > 180^\circ$$

$\therefore$ in the syz. the apsides are progressive, and since $\sqrt{\frac{b - m c}{b - n c}}$ will be an improper fraction as long as the ablatitious force is greater than the addititious, and when the disturbing forces are equal, $m c = n c$, therefore $G = 180^\circ$, i. e. the line of apsides is at rest (or it lies in V C produced 9th.) $\therefore$ since they are regressive through $141^\circ. 4'$ and progressive $218^\circ. 56'$ they are on the whole progressive.

To find the effect produced by the tangential ablatitious force, on the velocity of P in its orbit. Assume u = velocity of a body at the mean distance 1, then $\frac{u}{r}$ = velocity at any other distance r nearly, the orbit being nearly circular.

Let v be the true velocity of P at any distance (r), $v\,dv = g F dx$ $\left(\frac{g}{2} = 16\frac{1}{12}\right.$. For the tangent ablatitious $f = \frac{3}{2}$. P T . 2θ, and $x' = r\theta'\Big)$ $= 3$ P T . m r . sin. $2\theta . \theta'$,

$$\therefore v^2 = -3 \text{ P T m r cos. } 2\theta + C,$$

and

$$C = \frac{u^2}{r^2},$$

$$\therefore v^2 = \frac{u^2}{r^2} - \&c.$$

Hence it appears that the velocity is greatest in syzygy and least in quadrature, since in the former case, cos. 2θ is greatest and negative, and in the latter, greatest and positive.

To find the increment of the moon's velocity by the tangential force while she moves from quadrature to syzygy.

$$v^2 = -3 \text{ P T . m . r . cos. } 2\theta + C,$$

but (v) the increment = 0, when $\theta = 0$,

$$\therefore C = 3 \text{ P T . m . r},$$

$$\therefore v^2 = 3 \text{ P T . m . r } (1 - \cos. 2\theta) = 6 \text{ P T. m. r. sin.}^2\theta,$$

and when $\theta = 90^\circ$, or the body is in syzygy $v^2 = 6$ P T m . r.

475. COR. 6. Since the gravity of P to T is twice as much diminished in syzygy as it is increased in quadrature, by the action of the disturbing force S, the gravity of P to T during a whole revolution is diminished. Now the disturbing forces depend on the proportion between P T and T S, and therefore they become less or greater as T S becomes greater

or less. If therefore T approach S, the gravity of P to T will be still more diminished, and therefore P T will be the increment.

Now $P.T \propto \frac{R^{\frac{5}{2}}}{\sqrt{\text{absolute force}}}$; since, therefore, when S T is diminished, R is increased and the absolute force diminished (for the absolute force to T is diminished by the increase of the disturbing force) the P . T is increased. In the same way when S T is increased the P . T is diminished, therefore P . T is increased or diminished according as S T is diminished or increased. Hence per. t of the moon is shorter in winter than in summer.

OTHERWISE.

476. Cor. 7. To find the effect of the disturbing force on the motion of the apsides of P's orbit during a whole revolution.

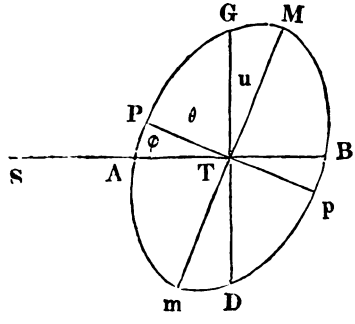
Let f = gravity of P to T at the mean distance (1), then $\frac{f}{r^2}$ = gravity of P at any other distance r.

Now in quadrature the whole force of P to T $= \frac{f}{r^2}$ + add. $f = \frac{f}{r^2} + r = \frac{fr + r^4}{r^3}$ and with this force the distance of the apsides $= 180^\circ \sqrt{\frac{f+1}{f+4}}$ which is less than 180°, therefore the apsides are regressive when the body is in quadrature. Now in syz. the whole force of P to T $= \frac{f}{r^2} - 2r = \frac{fr - 2r^4}{r^3}$, therefore the distance between the apsides $= 180^\circ \sqrt{\frac{f-2}{f-8}}$ which is greater than 180°, therefore the apsides are progressive when the body is in syzygy.

But as the force (2 r) which causes the progression in syzygy is double the force (r) which causes the regression in quadrature, the progressive motion in syzygy is greater than the regressive motion in the quadrature. Hence, upon the whole, the motion of the apsides will be progressive during a whole revolution.

At any other point, the motion of the apsides will be progressive or retrograde, according as the whole central force $-\frac{PT}{2} + \frac{3PT}{2} . \cos. 2\theta$ is negative or positive.

477. COR. 8. To calculate the disturbing force when P's orbit is excentric.



The whole central disturbing force $= -\frac{P\,T}{2} + \frac{3\,P\,T}{2}\cos.\,2\,\theta =$ $-\frac{m\,r}{2} + \frac{3\,m\,r}{2}.\cos.\,2\,\theta$ (m is the mean add. f). Now $r = \frac{1 - e^2}{1 - e\cos.\,u}$ = by div. $1 - e^2 + e.\cos.\,u + e^2.\cos.^2 u$, &c. neglecting terms involving e^3, &c. $= 1 - \frac{e^2}{2} + e.\cos.\,u + \frac{e^2}{2}.\cos.\,2\,u$; therefore the whole central disturbing force $= -\frac{m}{2} + \frac{m\,e^2}{4} - \frac{m.e.\cos.\,u}{2} -$ $\frac{m\,e^2\cos.\,2\,u}{4} + \frac{3}{2}\,m\cos.\,2\,\theta - \frac{3\,m\,e^2}{2}.\cos.\,2\,\theta + \frac{3}{2}\,m\,e.\cos.\,u.\cos.\,2\,\theta$ $+ \frac{3}{4}\,m\,e^2.\cos.\,2\,u.\cos.\,2\,\theta$.

478. COR. 8. It has been shown that the apsides are progressive in syzygy in consequence of the ablatitious force, and that they are regressive in quadrature from the effect of the ablatitious force, and also, that they are upon the whole progressive. It follows, therefore, that the greater the excess of the ablatitious over the addititious force, the more will the apsides be progressive in the course of a revolution. Now in any position m M of the line of the apsides, the excess of the ablatitious in conjunction = 2 A T in opposition = T B, therefore the whole excess = 2 A B. Again, the excess of the addititious above the ablatitious force in quadrature = C D. Therefore the apsides in a whole revolution will be retrograde if 2 A B be less than C D, and progressive if 2 A B be greater than C D. Also their progression will be greater, the greater the excess of 2 A B above C D; but the excess is the greatest when M m is in syzygy, for then A B is greatest and C D the least. Also, when M m is in syzygy the apsides being progressive are moving in the same direction with S, and therefore will remain for some length of time in syzygy. Again, when the apsides are in quadrature A B = P p, and C D = M m,

but if the orbit be nearly circular, 2 A B is greater than C D; therefore the apsides are still in a whole revolution progressive, though not so much as in the former case.

In orbits nearly circular it follows from $G = \frac{F}{\sqrt{r}}$ when $F \propto A^{p-3}$, that if the force vary in a greater ratio than the inverse square, the apsides are progressive. If therefore in the inverse square they are stationary,—if in a less ratio they are regressive. Now from quadrature to 35° a force which $\propto$ the distance is added to one varying inversely as the square, therefore the compound varies in a less ratio than the inverse square, therefore the apsides are regressive up to this point. At this point $F \propto \frac{1}{\text{distance}^2}$, therefore they are stationary. From this to 35° from another □ a quantity varying as the distance is subtracted from one varying inversely as the square, therefore the resulting quantity varies in a greater ratio than the inverse square, therefore the apsides are progressive through 218°.

OTHERWISE.

479. Cor. 8. It has been shown that the apsides are progressive in syzygy in consequence of the ablatitious force, and that they are regressive in the quadratures on account of the addititious force, and they are on the whole progressive, because the ablatitious force is on the whole greater than the addititious. ∴ the greater the excess of the ablatitious force above the addititious the more will be the apsides progressive.

In any position of the line A B in conjunction the excess of the ablatitious force above the addititious is 2 P T, in opposition 2 p t. ∴ the whole excess in the syzygies = 2 P p. In the quadratures at C the ablatitious force vanishes. ∴ the excess of the addititious = additious = C T. ∴ the whole addititious in the quadratures = C D.

Now the apsides will, in the whole revolution, be progressive or regressive, according as 2 P p is greater or less than C D, and then the progression will be greatest in that position of the line of the apses when 2 P p— C D is the greatest, i. e. when A B is in the syzygy, for then 2 P p = 2 A B, the greatest line in the ellipse, and C D = R r = ordinate = least through the focus. ∴ 2 P p — C D is a maximum. Also when A B is in the syzygy, the line of apsides being progressive, will move the same way as S. ∴ it will remain in the syzygy longer, and on this account the apsides will be more progressive. But when the apsides are in the quadratures S P = R r and C D = A B, and the orbit being nearly circular, R r nearly equals A B. ∴ 2 P p — C D is positive, and the

apsides are progressive on the whole, though not so much as in the last case; and the apsides being regressive in the quadratures move in the opposite direction to S, ∴ are sooner out of the quadratures, ∴ the regression in the quadrature is less than the progression in the syzygy.

480. COR. 9. LEMMA. If from a quantity which $\propto \frac{1}{A^2}$ any quantity be subtracted which $\propto$ A the remainder will vary in a higher ratio than the inverse square of A, but if to a quantity varying as $\frac{1}{A^2}$ another be added which $\propto$ A, the sum will vary in a lower ratio than $\frac{1}{A^2}$.

If $\frac{1}{A^2}$ be diminished C A $= \frac{1 - c A^2}{A^2}$. If A increases $1 - c A^2$ decreases, and $\frac{1}{A^2}$ increases. ∴ the quantity decreases, $1 - c A$ increases and $\frac{1}{A^2}$ increases. ∴ increases from both these accounts. ∴ the whole quantity varies in a higher ratio than $\frac{1}{A^2}$.

If C A be added $\frac{1 + c A^2}{A^2}$, as A is increased the numerator increases, and $\frac{1}{A^2}$ decreases. ∴ the quantity does not decrease so fast as $\frac{1}{A^2}$, and if A be diminished $1 + c A^2$ is diminished, and $\frac{1}{A^2}$ increased. ∴ the quantity is not increased as fast as $\frac{1}{A^2}$. ∴ &c. Q. e. d.

OTHERWISE.

481. COR. 9. To find the effect of the disturbing force on the excentricity of P's orbit. If P were acted on by a force $\propto \frac{1}{d^2}$, the excentricity of its orbit would not be altered. But since P is acted on by a force varying partly as $\frac{1}{d^2}$ and partly as the distance, the excentricity will continually vary.

Suppose the line of the apsides to coincide with the quadrature, then while the body moves from the higher to the lower apse, it is acted upon by a force which does not increase so fast as $\frac{1}{d^2}$, for the force at the quadrature $= \frac{f}{r^2} + m r$, and ∴ the body will describe an orbit exterior to the elliptic which would be described by the force $\propto \frac{1}{d^2}$. Hence the body

will be farther from the focus at the lower apse than it would have been had it moved in an elliptic orbit, or the excentricity is diminished. Also as the decrease in excentricity is caused by the force (m r), the less this force is with respect to $\frac{f}{r^2}$, the less will be the diminution of excentricity. Now while the line of apsides moves from the line of quadratures, the force (m r) is diminished, and when it is inclined at an angle of 35° 16′ the disturbing force is nothing, and ∴ at those four points the excentricity remains unaltered. After this it may be shown in the same manner that the excentricity will be continually increased, until the line of apsides coincides with the syzygies. Hence it is a maximum, since the disturbing force in these is negative. Afterwards it will decrease as before it increased, until the line of apsides again coincides with the line of quadrature, and the excentricity is a minimum.

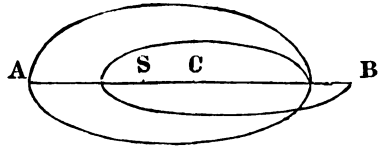
Cor. 14. Let P T = r, S T = d, f = force of T on P at the distance 1, g = force of S on T at the distance, then the ablatitious force $= \frac{3\,g\,r\,\sin.\,\theta}{d^3}$; if ∴ the position of P be given, and d varies, the ablatitious force $\propto \frac{1}{d^3}$. But when the position of P is given, the ablatitious : addititious : : in a given ratio, ∴ addititious force $\propto \frac{1}{d^3}$, or the disturbing force $\propto \frac{1}{d^3}$. Hence if the absolute force of S should ∝ the disturbing force $\propto \frac{\text{absol. f.}}{d^3}$. Let P = the periodical time of T about S, ∴ $\frac{1}{P^2} \propto \frac{\text{absol. f.}}{d^3}$. Let Δ = density, δ = diameter of the sun, then the absolute force $\propto \Delta\,\delta^3$, then the disturbing force $\propto \frac{\Delta\,\delta^3}{d^3} \propto \frac{1}{P^2} \propto \Delta$ (apparent diameter)^3 of the sun. Or since P T is constant, the linear as well as the angular errors ∝ in the same ratio.

483. Cor. 15. If the bodies S and T either remain unchanged, or their absolute forces are changed in any given ratio, and the magnitude of the orbits described by S and P be so changed that they remain similar to what they were before, and their inclination be unaltered, since the accelerating force of P to T : accelerating force of S : : $\frac{\text{absolute force of T}}{P\,T^2}$: $\frac{\text{absolute force of S}}{S\,T^2}$, and the numerators and denominators of the last terms are changed in the same given ratio, the accelerating forces remain in the same ratio as before, and the linear or angular errors ∝ as before,

i. e. as the diameter of the orbits, and the times of those errors $\propto$ P T's of the bodies.

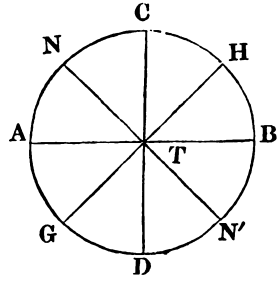
COR. 16. Hence if the forms and inclinations of the orbits remain, and the magnitude of the forces and the distances of the bodies be changed; to find the variation of the errors and the times of the errors. In Cor. 14. it was shown, how that when P T remained constant, the errors $\propto \frac{1}{P^2}$. Now let P T also $\propto$, then since the addititious force in a given position of P $\propto$ P T, and in a given position of P the addititious : ablatitious in a given ratio.

COR. If a body in an ellipse be acted upon by a force which varies in a ratio greater than the inverse square of the distance, it will in descending from the higher apse B to the lower apse A, be drawn nearer to the center. $\therefore$ as S is fixed, the excentricity is increased, and from A to B the excentricity will be increased also, because the force decreases the faster the distance^2 increases.



484. (COR. 10.) Let the plane of P's orbit be inclined to the plane of T's orbit remaining fixed. Then the addititious force being parallel to P T, is in the same plane with it, and $\therefore$ does not alter the inclination of the plane. But the ablatitious force acting from P to S may be resolved into two, one parallel, and one perpendicular to the plane of P's orbit. The force perpendicular to P's orbit $= 3$ A $\times$ sin. $\theta \times$ sin. Q $\times$ sin. T when θ = perpendicular distance of P from the quadratures, Q = angular distance of the line of the nodes from the syzygy, T = first inclination of the planes.

Hence when the line of the nodes is in the syzygy, $\theta = 0$, $\therefore$ sin. $= 0$ $\therefore$ no force acts perpendicular to the plane, and the inclination is not changed. When the line of the nodes is in the quadratures, $\theta = 90°$, $\therefore$ sin. is a maximum, $\therefore$ force perpendicular produces the greatest change in the inclination, and sin. θ being positive from C to D, the force to change the inclination continually acts from C to D pulling the plane down from D to C. Sin. θ is negative, $\therefore$ force which before was posi-



tive pulling down to the plane of S's orbit (or to the plane of the paper) now is negative, and $\therefore$ pulls up to the plane of the paper. But P's orbit is now below the plane of the paper, $\therefore$ force still acts to change the inclination. Now since the force from C to D continually draws P towards the plane of S's orbit, P will arrive at that plane before it gets to D.

If the nodes be in the octants past the quadrature, that is between C and A. Then from N to D, sin. θ being positive, the inclination is diminished, and from D to N′ increased, $\therefore$ inclination is diminished through 270°, and increased through 90°, $\therefore$ in this, as in the former case, it is more diminished than increased. When the nodes are in the octants before the quadratures, i. e. in G H, inclination is decreased from H to C, diminished from C to N, (and at N the body having got to the highest point) increased from N to D, diminished from D′ to N′, and increased from 2 N′ to H, $\therefore$ inclination is increased through 270°, and diminished through 90°, $\therefore$ it is increased upon the whole. Now the inclination of P's orbit is a maximum when the force perpendicular to it is a minimum, i. e. when (by expression) the line of the nodes is in the syzygies. When is the quadratures, and the body is in the syzygies, the least it is increased when the apsides move from the syzygies to the quadratures; it is diminished and again increased as they return to the syzygies.

485. (Cor. 11.) While P moves from the quadrature in C, the nodes being in the quadrature it is drawn towards S, and $\therefore$ comes to the plane of S's orbit at a point nearer S than N or D, i. e. cuts the plane before it arrives at the node. $\therefore$ in this case the line of the nodes is regressive. In the syzygies the nodes rest, and in the points between the syzygies and quadratures, they are sometimes progressive and sometimes regressive, but on the whole regressive; $\therefore$ they are either retrograde or stationary.

486. (Cor. 12.) All the errors mentioned in the preceding corollaries are greater in the syzygies than in any other points, because the disturbing force is greater at the conjunction and opposition.

487. (Cor. 13.) And since in deducing the preceding corollaries, no regard was had to the magnitude of S, the principles are true if S be so great that P and T revolve about it, and since S is increased, the disturbing force is increased; $\therefore$ irregularities will be greater than they were before.

488. (Cor. 14.) L M $= \frac{M a^2 r}{d^3} =$ N N M $= \frac{3 M a^2 r}{d^3}$ sin. θ, $\therefore$ in a given position of P, if P T remain unaltered, the forces N M and L M

$\propto \frac{1}{d^3} \times$ absolute force $\propto \frac{1}{(\text{Per. T})^2}$ of T for (sect. 3 . $P^2 \propto \frac{d^3}{\text{absolute f.}}$) whether the absolute force vary or be constant. Let D = diameter of S, δ = density of S, and attractive force of S $\propto$ magnitude or quantity of matter $\propto D^3 \delta$,

$\therefore$ forces L M and N M $\propto \frac{D^3 \delta}{d^3}$.

But $\frac{D}{d}$ = apparent diameter of S,

$\therefore$ forces $\propto$ (apparent diameter)^3 δ another expression.

489. (COR. 15.) Let another body as P′ revolve round T′ in an orbit similar to the orbit of P round T, while T′ is carried round S′ in an orbit similar to that of T round S, and let the orbit of P′ be equally inclined to that of T′ with the orbit P to that of T. Let A, a, be the absolute forces of S, T, A′, a′, of S′, T′,

accelerating force of P by S : that of P by T :: $\frac{A}{S P^2} : \frac{a}{P T^2}$,

and the orbits being similar

accelerating force of P′ by S′ : that of P′ by T′ :: $\frac{A'}{S P'^2} : \frac{a}{P' T'^2}$,

$\therefore$ if A′ : a′ :: A : a, and the orbits being similar,

S P : P T^2 :: S′ P′ : P′ T′,

accelerating force of P′ by S′ : that of P′ by T′

:: force on P by S : force on P′ by T′,

and the errors due to the disturbing forces in the case of P are as $\frac{A}{S T^3} \times r$, in the case of P′ and S′ are as $\frac{A'}{S' T'^3} \times R$,

$\therefore$ linear errors in the first case : that in the second :: r : R.

Angular errors $\propto \frac{\text{sin. errors}}{R}$,

angular errors in the first case : that in the second :: 1 : 1.

Now Cor. 2. Lem. X. $T^2 \propto \frac{\text{linear errors}}{T C}$

$\propto \frac{\text{angular errors}}{R} \times R$,

$\therefore T^2 \propto$ angular errors,

$\therefore$ angular errors : 360 :: T^2 : P^2,

$\therefore T^2 \propto P^2 \times$ angular errors,

$\therefore T \propto P$ for = angular errors.

490. (Cor. 16.) Suppose the forces of S, P T, S T to vary in any manner, it is required to compare the angular errors that P describes in similar, and similarly situated orbits. Suppose the force of S and T to be constant, $\therefore$ addititious force $\propto$ P T, $\therefore$ if two bodies describe in similar orbits = evanescent arcs. Linear errors $\propto p^2 \times P T$.

$\therefore$ angular errors $\propto p^2$ (p = per. time of P round T, P = that of T round S). But by Cor. 14. if P T be given, the absolute force of A and S T $\propto$.

$$\text{Angular errors} \propto \frac{1}{P^2}$$

$\therefore$ if P T, S T and the absolute force alternately vary,

$$\text{angular errors} \propto \frac{p^2}{P^2},$$

$$\left.\begin{matrix} P = \text{per. time of P round T} \\ p = \text{per. time of T round S} \end{matrix}\right\} \text{force} \propto \frac{M a^2 r}{d^3},$$

$$\text{angular errors} \propto \frac{\text{linear errors}}{\text{radius}}.$$

$$\therefore \text{lin. errors} \propto \text{force } T^2 \propto \frac{M a^2 r}{d^3} \times P^2 \text{ by last Cor.}$$

$$\therefore \text{angular errors} \propto \frac{r P^2}{d^3 \times r} \propto \frac{P^2}{p^2}\Big).$$

$$\frac{}{M a^2}$$

Now the errors d t $\times$ p = whole angular errors $\propto \frac{p}{P^2}$.

$\therefore$ error d t $\propto \frac{p}{P^2}$ thence the mean motion of the apsides $\propto$ mean motion of the nodes, for each $\propto \frac{p}{P^2}$, for each error is formed by forces varying as

proof of the preceding corollaries, both the disturbing forces, and $\therefore$ the errors produced by them in a given time will $\propto$ P T. Let P describe an indefinite small angle about T (in a given position of P), then the linear errors generated in that time $\propto$ force T P time^2, but the time of describing = angles about T $\propto$ whole periodic time (p), $\therefore$ linear errors $\propto$ P T p^2, and as the same is true for every small portion, similar; the linear errors during a whole revolution $\propto$ P T p^2. Angular errors $\propto \frac{\text{linear er.}}{\text{rad.}} \therefore \propto p^2 \therefore$ when S T, P T, and the absolute force vary, the angular errors $\propto \frac{p^2}{P^2} \propto \frac{\text{absolute } p^2}{S T^3} \propto \frac{p^2}{S T^3}$ (when the absolute force is

given.) Now the error in any given time $\times$ p varies the whole errors during a revolution $\propto \frac{p^2}{P^2}$. $\therefore$ the errors in any given time $\propto \frac{p}{P^2}$. Hence the mean motion of the apsides of P's orbit varies the mean motion of the nodes, and each will $\propto \frac{p}{P^3}$ the excentricities and inclination being small and remaining the same.

491. (COR. 17.) To compare the disturbing forces with the force of P to T.

$$\text{F of S on T} : \text{F of P on T} :: \frac{\text{absolute F}}{S\,T^2} : \frac{a}{T\,P^2}$$

$$P^2 \propto \frac{\text{absolute F}}{\text{axis major}^3} \qquad :: \frac{A.\,S\,T}{S\,S^3} : \frac{a\,T\,P}{T\,P^3}$$

$$:: \frac{S\,T}{P^2} : \frac{TP}{p^2} :: \frac{d}{P^2} : \frac{r}{p^2}$$

$$\text{mean add. F} : \text{F of S on T} :: \frac{M\,a^2\,r}{d^3} : \frac{M\,a^2}{d^2} :: r : d$$

$\therefore$ mean add. F : F P on T $:: p^2 : P^2$.

492. To compare the densities of different planets.

Let P and P′ be the periodic times of A and B, r and r′ their distances from the body round which they revolve.

$$\text{F of A to S} : \text{F of B to S} :: \frac{r}{P^2} : \frac{r'}{P'^2}$$

$$\frac{\text{quantity of matter in A}}{\text{distance}^2} : \frac{\text{do. in B}}{\text{distance}^2} :: \frac{D^3 \text{ of A} \times \text{density}}{\text{distance}^2} : \frac{D^3 \text{ of B} \times \text{density}}{\text{distance}^2}$$

$$:: \frac{r}{P^2} : \frac{r'}{P'^2}$$

$$\therefore \frac{D^3 \times d}{r^3} : \frac{D'^3 \times d'}{r'^3} :: \frac{1}{P^2} : \frac{1}{P'^2}$$

$$\therefore d : d' :: \frac{r^3}{D^3 P^2} : \frac{r'^3}{D'^3 P'^2}$$

$$:: \frac{1}{S^3 P^2} : \frac{1}{S'^3 P'^2}$$

where S and S′ represent the apparent diameters of the two planets.

493. In what part of the moon's orbit is her gravity towards the earth unaffected by the action of the sun.

$$F = \frac{M\,a^2}{r} + \frac{M\,a^2\,r}{d^3} - \frac{3\,M\,a^2\,r}{d^3} \cdot \frac{1 - \cos.^2\theta}{2} + \frac{3}{2}\,\frac{M\,a^2\,r}{d^3} \sin.^2\theta$$

and when it is acted upon only by the force of gravity $= \frac{M\,a^2}{r}$ for the other forces then have no effect.

$$\therefore \frac{M a^2 r}{d^3} - \frac{3 M a^2 r}{d^3} \cdot \frac{1 - \cos. 2\theta}{2} + \frac{3}{2} \frac{M a^2 r}{d^3} \sin. \theta = 0$$

$$1 - 3 . \frac{1 - \cos. 2\theta}{2} + \frac{2}{3} \sin. 2\theta = 0$$

$$1 - \frac{3}{2} + \frac{3}{2} \cos. 2\theta + \frac{3}{2} \sin. 2\theta = 0$$

$$1 - \frac{3}{2} + \frac{3}{2} \cdot \frac{1 - \sin.^2 C}{2} + \frac{3}{2} \sin. 2\theta = 0$$

Let $x = \sin. \theta$,

$$(\therefore 1 - \tfrac{3}{2} + \tfrac{3}{2} - \tfrac{3}{2} \sin.^2 \theta + \tfrac{3}{2} \times 2 \sin. \theta \times \cos. \theta = 0)$$

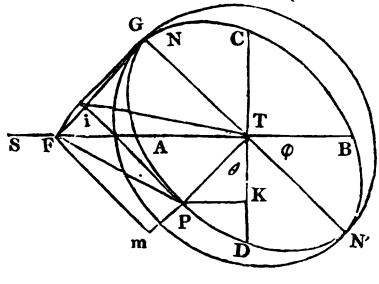
and

$$1 - \frac{3 x^2}{2} + 3 x \sqrt{1 - x^2} = 0.$$

An equation from which x may be found.

494. Lemma. If a body moving towards a plane given in position, be acted upon by a force perpendicular to its motion tending towards that plane, the inclination of the orbit to the plane will be increased. Again, if the body be moving from the plane, and the force acts from the plane, the inclination is also increased. But if the body be moving towards the plane, and the force tends from the plane, or if the body be moving from the plane, and the force tends towards the plane, the inclination of the orbit to the plane is diminished.

495. To calculate that part of the ablatitious tangential force which is employed in drawing P from the plane of its orbit.



Let the dotted line upon the ecliptic N A P N′ be that part of P's orbit which lies above it. Let C D be the intersection of a plane drawn perpendicular to the ecliptic; P K perpendicular to this plane, and there-

fore parallel to the ecliptic. Take T F = 3 P K; join P F and it will represent the disturbing force of the sun. Draw P i a tangent to, and F i perpendicular to the plane of the orbit. Complete the rectangle i m, and P F may be resolved into P m, P i, of which P m is the effective force to alter the inclination. Draw the plane F G i perpendicular to N N'; then F G is perpendicular to N N'. Also F i G is a right angle. Assume P T tabular rad. Then

$$\left.\begin{array}{l} P\,T : T\,F :: R : 3\,g \\ T\,F : F\,G :: R : s \\ F\,G : P\,m :: R : i \end{array}\right\} \therefore P\,T : P\,m :: R^3 : 3\,g.\,s.\,i$$

$$\therefore P\,m = \frac{P\,T.\,3\,g.\,s.\,i}{R^3}$$

g = sin. θ = sin. $\angle$ dist. from quad.

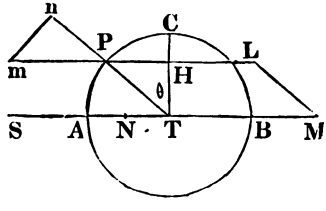
s = sin. φ = sin. $\angle$ dist. of nodes from syz.

i = sin. F T i = sin. F G i = sin. inclination of orbit to ecliptic.

Hence the force to draw P from its orbit $= \frac{P.\,3\,g\,i\,s}{R^3}$ when P is in the quadratures. Since g vanishes this force vanishes. When the nodes are in the syzygies s vanishes, and when in the quadratures this force is a maximum. Since s = rad. cotan. parte.

496. To calculate the quantity of the forces.

Let S T = d, P T = r, the mean distance from T = 1. The force



of T on P at the mean distance = f; the force of S on P at the mean distance = g.

Then the force S T $= \frac{g}{d^2}$, and the force S T : f. P T :: d : r,

$\therefore$ force P T $= \frac{g\,r}{d^3}$, hence the add. f $= \frac{g\,r}{d^3}$; ablat. f $= \frac{3\,g\,r}{d^3}$ sin. θ, the mean add. force at distance 1 $= \frac{g}{d^3}$, the central ablat. $= \frac{3\,g\,r}{d^3}$ sin.^{2} θ, the tangential ablat. f $= \frac{3\,g\,r}{2\,d^3}$. sin. 2 θ.

The whole disturbing force of S on P $= \frac{-gr}{2d^3} + \frac{3gr}{2d^3} \cdot \cos. 2\theta$; the mean disturbing $f = \frac{-gr}{2d^3}$ (since cos. 2θ vanishes) $= -\frac{m}{n}$ by supposition.

Hence we have the whole gravitation of P to T $= \frac{f}{r^2} - \frac{gr}{2d^3} + \frac{3gr}{2d^3} \times$ cos. 2θ, and the mean $= \frac{f}{r^2} - \frac{gr}{2d^2}$ (since cos. 2θ vanishes).

PROBLEM.

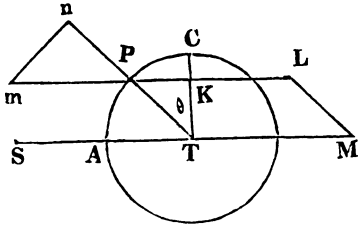
497. Required the whole effect, and also the mean effect of the sun to diminish the lunar gravity; and show that if P and p be the periodic times of the earth and moon, f the earth's attraction at the mean distance of the moon, r the radius-vector of the moon's orbit; the additious force will be nearly represented by the formula $\left\{\frac{p^2}{P^2} - \frac{p^4}{2P^4}\right\} fr$.

$Pn = 3PT.\sin.^2\theta$, and $PT - 3PT.\sin.^2\theta = -\frac{PT}{2} + \frac{3}{2}PT \times$ cos. 2θ = whole diminution of gravity of the moon, and the mean diminution $= -\frac{PT}{2} + \frac{-gr}{2d^3}$ by supposition.

Again,

$$P^2 \propto d^3$$
$$\therefore \frac{ab.f}{d^2} \propto \frac{d}{P^2}. \quad \text{Vid. seq.}$$

498. To find the central and ablatitious tangential forces.



Take $Pm = 3PK = 3PT.\sin.\theta$ = ablatitious force.
Then $Pn = Pm.\sin.\theta = 3PT.\sin.^2\theta$ = central force
$mn = Pm.\cos.\theta = 3PT.\sin.\theta.\cos.\theta$
$= \frac{3}{2}.PT\sin. 2\theta$ = tangential ablatitious force.
To find what is the disturbing force of S on P.

The disturbing force $= P T - 3 P T . \sin.^2 \theta = \left(\frac{-1+3\cos.2\theta}{2}\right) \times P T = -\frac{P T}{2} + \frac{3}{2} P T . \cos. 2\theta.$

To find the mean disturbing force of S during a whole revolution.

Let P T at the mean distance $= m$, then $-\frac{P T}{2} + \frac{3}{2} . P T \cos. 2\theta = -\frac{P T}{2} = \frac{-m}{2}$ since cos. 2θ is destroyed during a whole revolution.

499. To find the disturbing force in syzygy.

$$3 A T - A T = 2 A T = \text{disturbing force in syzygy;}$$

the force in quadrature is wholly effective and equal P T,

$\therefore$ force in quadrature : f in syzygy :: P T : 2 P T :: 1 : 2.

To find that point in P's orbit when the force of P to T is neither increased nor diminished by the force of S to T.

In this point P n = P T or $3 P T \sin.^2 \theta = P T$,

$$\therefore \sin. \theta = \frac{1}{\sqrt{3}}$$

and

$$\theta = 35^\circ\ 16'.$$

To find when the central ablatitious force is a maximum.

$$P n = 3 P T . \sin.^2 \theta = \text{maximum},$$

$$\therefore d . (\sin.^2 \theta) \text{ or } 2 \sin. \theta . \cos. \theta - d \theta = 0,$$

$$\therefore \sin. \theta . \cos. \theta = 0,$$

or

$$\sin. \theta . \sqrt{1 - \sin.^2 \theta} = 0,$$

and

$$\sin. \theta = 1,$$

or the body is in opposition.

Then (Prop. LVIII, LIX,)

$$T^2 : t^2 :: S P : C P :: S + P : S$$

and

$$T^2 : t^2 :: A^3 : x^3$$

$$\therefore A^3 : x^3 :: S + P : S$$

and

$$A : x :: (S + P)^{\frac{1}{3}} : S^{\frac{1}{3}}.$$

500. PROB. Hence to correct for the axis major of the moon's orbit.

Let S be the earth, P the moon, and let per. t of a body moving in a secondary at the earth's surface be found, and also the periodic time of

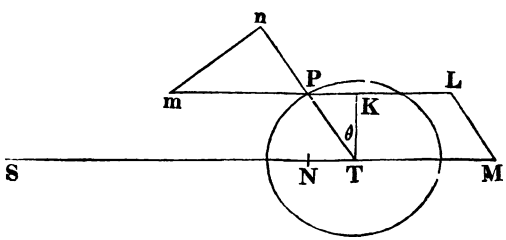
the moon. Then we may find the axis major of the moon's orbit round the earth supposed at rest = x, by supposition. Then the corrected axis or axis major round the earth in motion : $x :: (S + P)^{\frac{1}{3}} : S^{\frac{1}{3}}$

$\therefore$ axis major round the earth in motion $= x \cdot \frac{(S+P)^{\frac{1}{3}}}{S^{\frac{1}{3}}} = y.$

Hence to compare the quantity of matter in the earth and moon,

$$y : x :: \sqrt[3]{S+P} : \sqrt[3]{S}$$
$$\therefore y^3 - x^3 : x^3 :: P : S.$$

501. To define the additititious and ablatitious forces. Let S T repre-



sent the attractive force of T to S. Take

$$S L : S T :: \frac{1}{S P^2} : \frac{1}{S T^2} :: S T^2 : S P^2$$

and S L will represent the attractive force of P to S. Resolve this into S M, and L M; then L M, that part of the force in the direction P T is called the additititious force, and S M — S T = N M is the ablatitious force.

502. To compare these forces.

Since $S L : S T :: S T^2 : S P^2$, $\therefore S L = \frac{S T^3}{S P^2}$ = attractive force of P to S in the direction S P, and $S P : S T :: \frac{S T^3}{S P^2} : \frac{S T^4}{S P^3}$ = attractive force of P to S in the direction T S = $S T^4 (S T - P K)^{-}$ = S T + 3 P K = S M nearly,

$\therefore$ 3 P K = T M = P L = ablatitious force = 3 P T . sin. θ.

Also $S P : P T :: \frac{S T^3}{S P^2} : \frac{S T^3}{S P^3}$.

P T = attractive force of P to S in the direction L M = P T nearly.

Hence the additititious force : ablatitious force : : P T : 3 P T . sin. θ : 1 : 3 sin. θ. Q. e. d.

BOOK III.

1. PROP. I. All secondaries are found to describe areas round the primary proportional to the time, and these periodic times to be to each other in the sesquiplicate ratio of their radii. Therefore the center of force is in the primary, and the force $\propto \frac{1}{D^2}$.

2. PROP. II. In the same way, it may be proved, that the sun is the center of force to the primaries, and that the forces $\propto \frac{1}{\text{dist.}^2}$. Also the Aphelion points are *nearly* at rest, which would not be the case if the force varied in a greater or less ratio than the inverse square of the distance, by principles of the 9th Section, Book 1st.

3. PROP. III. The foregoing applies to the moon. The motion of the moon's apogee is very slow—about 3° 3′ in a revolution, whence the force will $\propto \frac{1}{\text{dist.}^{2\,\frac{4}{243}}}$. It was proved in the 9th Section, that if the ablatitious force of the sun were to the centripetal force of the earth :: 1 : 357.45, that the motion of the moon's apogee would be $\frac{1}{2}$ the real motion.

$\therefore$ the ablatitious force of the sun : centripetal force :: 2 : 357.45
:: 1 : 178 $\frac{29}{40}$.

This being very small may be neglected, the remainder $\propto \frac{1}{D^2}$.

4. COR. The mean force of the earth on the moon : force of attraction
:: 177 $\frac{29}{40}$: 178 $\frac{29}{40}$.

The centripetal force at the distance of the moon : centripetal force at the earth :: 1 : D^2.

5. PROP. IV. By the best observations, the distance of the moon from the earth equals about 60 semidiameters of the earth in syzygies. If the moon or any heavy body at the same distance were deprived of motion in the space of one minute, it would fall through a space = 16 $\frac{1}{12}$ feet. For the

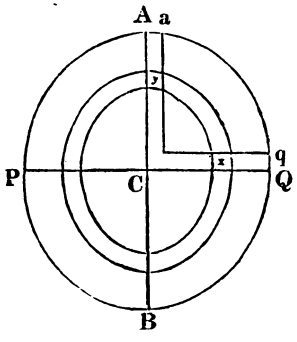
deflexion from the tangent in the same time = 16 $\frac{1}{12}$ feet. Therefore the space fallen through at the surface of the earth in 1″ = 16 $\frac{1}{12}$ feet.

For 60″ : t :: D : 1,

$$\therefore t = \frac{60''}{60} = 1'',$$

thence the moon is retained in its orbit by the force of the earth's gravity like heavy bodies on the earth's surface.

6. Prop. XIX. By the figure of the earth, the force of gravity at



the pole : force of gravity at the equator :: 289 : 288. Suppose A B Q q a spheroid revolving, the lesser diameter P Q, and A C Q q c a a canal filled with water. Then the weight of the arm Q q c C : ditto of A a c C : : 288 : 289. The centrifugal force at the equator, therefore 1 suppose $\frac{1}{289}$ of the weight.

Again, supposing the ratio of the diameters to be 100 : 101. By computation, the attraction to the earth at Q : attraction to a sphere whose radius = Q C : : 126 : 125. And the attraction to a sphere whose radius A C : attraction of a spheroid at A formed by the revolution of an ellipse about its major axis :: 126 : 125.

The attraction to the earth at A is a mean proportional between the attractions to the sphere whose radius = A C, and the oblong spheroid, since the attraction varies as the quantity of matter, and the quantity of matter in the oblate spheroid is a mean to the quantities of matter in the oblong spheroid and the circumscribing sphere.

Hence the attraction to the sphere whose radius = A C : attraction to the earth at A :: 126 : 125 $\frac{1}{2}$.

∴ attraction to the earth at the pole : attraction to the earth at the equator :: 501 : 500.

Now the weights in the canals $\propto$ whole weights $\propto$ magnitudes $\times$ gra-

vity, therefore the weight of the equatorial arm : weight of the polar

$$:: 500 \times 101 : 501 \times 100$$
$$:: 505 : 501.$$

Therefore the centrifugal force at the equator supports $\frac{4}{505}$ to make an equilibrium.

But the centrifugal force of the earth supports $\frac{1}{289}$,

$\therefore \frac{4}{505} : \frac{1}{289} :: \frac{1}{100} : \frac{1}{229}$ = the excess of the equatorial over the polar radius.

Hence the equatorial radius : polar $:: 1 + \frac{1}{229} : 1$

$$:: 230 : 229.$$

Again, since when the times of rotation and density are different the difference of the diameter $\propto \frac{V^2}{\text{dens.}}$, and that the time of the earth's rotation = 23h. 56′.

The time of Jupiter's rotation = 9h. 56′.

The ratio of the squares of the velocity are as 29 : 5, and the density of the earth : density of Jupiter : : 400 : 94.5.

d the difference of Jupiter's diameter is as $\frac{29}{5} \times \frac{400}{94.5} \times \frac{1}{229}$,

$\therefore$ d : Jupiter's least diameter $:: \frac{29}{5} \times \frac{400}{94.5} \times \frac{1}{229} :: 29 \times 80 : 94.5 \times 229$

$$:: 2320 : 21640$$
$$:: 232 : 2164$$
$$:: 1 : 9\tfrac{1}{3}$$

The polar diameter : equatorial diameter $:: 9\frac{1}{3} : 10\frac{1}{3}$

ON THE TIDES.

7. THE PHENOMENA OF THE TIDES.

1. The interval between two succeeding high waters is 12 hours 25 minutes. The diminution varies nearly as the squares of the times from high water.

2. Twenty-four hours 50 minutes may be called the lunar day. The interval between two complete tides, the tide day. The first may be call-

ed the superior, the other inferior, and at the time of new moon, the morning and evening.

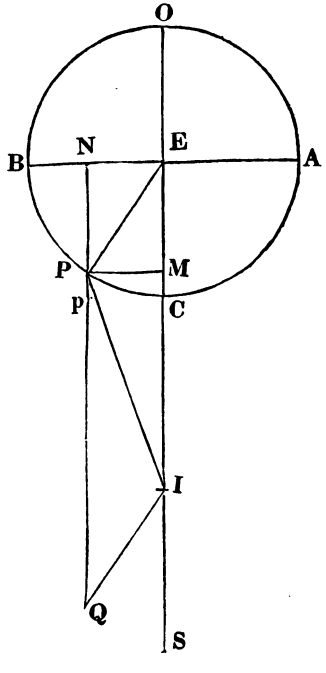
3. The high water is when the moon is in S.W. to us. The highest tide at Brest is a day and a half after full or change. The third full sea after the high water at the full moon is the highest; the third after quadrature is the lowest or neap tide.

4. Also the highest spring tide is when the moon is in perigee, the next spring tide is the lowest, since the moon is nearly in the apogee.

5. In winter the spring tides are greater than in summer, and from the same reasoning the neap tides are lower.

6. In north latitude, when the moon's declination is north, that tide in which the moon is above the horizon is greater than the other of the same day in which the moon is below the horizon. The contrary will take place if either the observer be in south latitude or the moon's declination south.

7. Prop. I. Suppose P to be any particle attracted towards a center E, and let the gravity of E to S be represented by E S. Draw B A perpendicular to E S, which will therefore represent the diameter of the plane of illumination. Draw Q P N perpendicular to B A, P M perpendicular to E C. Then take E I = 3 P N, and join P I, P I will represent the disturbing force of P. P I may be resolved into the two P E, P Q, of which P E is counterbalanced by an equal and opposite force, P Q acts in the direction N P.



Hence if the whole body be supposed to be fluid, the fluid in the canal N P will lose its equilibrium, and therefore cannot remain at rest. Now, the equilibrium may be restored by adding a small portion P p to the canal, or by supposing the water to subside round the circle B A, and to be collected towards O and C, so that the earth may put on the form of a prolate spheroid, whose axis is in the line O C, and poles in O and C, which may be

the case since the forces which are superadded $\propto$ N P, or the distance from B A, so that this mass may acquire such a protuberancy at O and C, that the force at O shall be to the force at B : : E A : E C; and by the above formula

$$\frac{x}{r} = \frac{5\,C}{4\,g} = \frac{E\,C - E\,A}{E\,A}.$$

8. Prop. II. Let W equal the terrestrial gravitation of C; G equal its gravitation to the sun; F the disturbing force of a particle acting at O and C; S and E the quantities of matter in the sun and earth.

$$\therefore F : W :: \frac{3\,S}{C\,S^2 \times C\,G} : \frac{C}{C\,E^3}$$

Since the gravitation to the sun $\propto \frac{1}{\text{dist.}^2}$

$$C\,S^2 : E\,S^2 :: E\,S : C\,G$$
$$\therefore C\,G \times C\,S^2 = E\,S^3.$$
$$\therefore F : W :: \frac{3\,S}{E\,S^3} : \frac{E}{C\,E^3}$$

and

$$E : S :: 1 : 338343$$
$$E\,C : E\,S :: 1 : 23668$$
$$\therefore \frac{3\,S}{E\,S^3} : \frac{E}{C\,E^3} :: 1 : 12773541 :: F : W.$$
$$\therefore 4\,W : 5\,F :: C\,E : E\,C - E\,A.$$

Attraction to the pole : attraction to the equator : : $1 - \frac{4\,d}{5} : 1 - \frac{3\,d}{5}$

Quantity of matter at the pole : do. at equator : : 1 : 1 — d.

Weight of the polar arm : weight of the equatorial arm : : $1 - \frac{4\,d}{5} : 1 - \frac{8\,d}{5}$

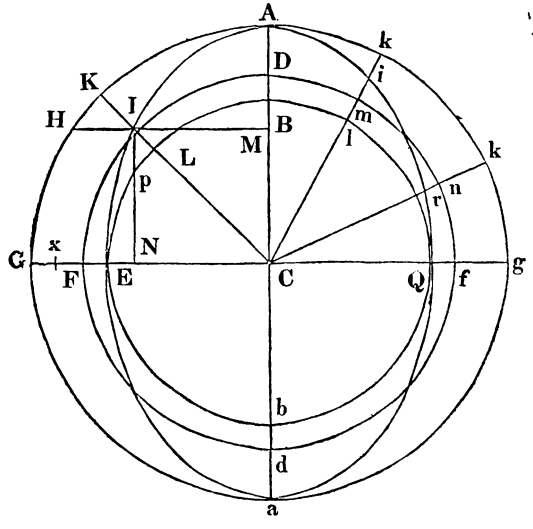
$$:: 1 + \frac{4\,d}{5} : 1$$

$\therefore$ Excess of the polar = attractive force : weight of the equator or mean weight W : : $\frac{4\,d}{5} : 1$

$$\therefore d = \frac{5\,F}{4\,W}.$$

9. Prop. III. Let A E a Q be the spheroid, B E b Q the inscribed

sphere, A G a g the circumscribed sphere, and D F d f the sphere equal (in capacity) to the spheroid.



Then since spheres and spheroids are equal to $\frac{2}{3}$ of their circumscribing cylinder, and that the spheroid = sphere D F d f.

$$C F^2 \times C D = C E^2 \times C A$$
$$C E^2 : C F^2 :: C D : C A,$$

and make

$$C E : C F :: C F : C x$$
$$\therefore C E^2 : C F^2 :: C E : C x$$
$$\therefore C D : C A :: C E : C x$$
$$\therefore C D : C E :: C A : C x$$

but

$$C D = C E \text{ nearly}$$
$$\therefore C A = C x.$$

Also

$$E x = 2 E F \text{ nearly}$$
$$\therefore A D = 2 E F.*$$

* Let $C E = a$, $C F = a + x$,

$$\therefore C x = \frac{a^2 + 2ax + x^2}{a} = \frac{a^2 + 2x}{a}$$
$$= a + 2x \text{ nearly}$$

$\therefore E x = 2x$ nearly.

Prop. IV. By the triangles p I L, C I N,

$$A B : I L :: r^2 : (\cos.)^2 \angle T C A$$
$$\therefore I L = A B \times (\cos.)^2 \angle I C A = S \times (\cos.)^2 x$$

(if S = A B and x = angular distance from the sun's place.)

Again,

$$G E : K I :: r^2 : (\sin.)^2 \angle T C A$$
$$\therefore K I = S \times (\sin.)^2 \angle K.$$

Cor. 1. The elevation of a spheroid above the level of the undisturbed ocean $= l i - l m = S \times (\cos.)^2 x - \frac{S}{3} = S \times \overline{(\cos.)^2 x - \frac{1}{3}}$.

The depression of the same $= S \times (\sin.)^2 x - S = S \times \overline{(\sin.)^2 x - \frac{2}{3}}$.

Cor. 2. The spheroid cuts the sphere equal in capacity to itself in a point where $S \times (\cos.)^2 x = \frac{S}{3} = 0$, or $(\cos.)^2 x = \frac{1}{3}$.

$$\therefore \cos. x = .57734, \text{ \&c.}$$
$$= \cos. 54^\circ. 44'.$$

10. Prop. V. The unequal gravitation of the earth to the moon is $(4000)^3$ times greater than towards the sun.

Let M equal the elevation above the inscribed sphere at the pole of the spheroid, γ equal the angular distance from the pole.

$\therefore$ the elevation above the equally capacious sphere $= M \times \overline{(\cos.)^2 \gamma - \frac{1}{3}}$

the depression $= M \times \overline{(\sin.)^2 \gamma - \frac{2}{3}}$.

Hence the effect of the joint action of the sun and moon is equal to the sum or difference of their separate actions.

$\therefore$ the elevation at any place $= S \times (\cos.)^3 x + M \times (\cos.)^2 \gamma - \frac{1}{3} \overline{S + M}$

the depression $= S \times (\sin.)^2 x + M \times (\sin.)^2 \gamma - \frac{2}{3} \overline{S + M}$.

1. Suppose the sun and moon in the same place in the heavens.

Then the elevation at the pole $= S + M - \frac{1}{3} \overline{S + M} = \frac{2}{3} \overline{S + M}$, and the depression at the equator $= S + M - \frac{2}{3} \overline{S + M} = \frac{1}{3} \overline{S + M}$,

$\therefore$ the elevation above the inscribed sphere $= S + M$.

2. Suppose the moon to be in the quadratures.

The elevation at $S = S - \frac{1}{3} \overline{S + M} = \frac{2}{3} S - \frac{1}{3} M$.

the depression at $M = S - \frac{2}{3} \overline{S + M} = \frac{1}{3} S - \frac{2}{3} M$,

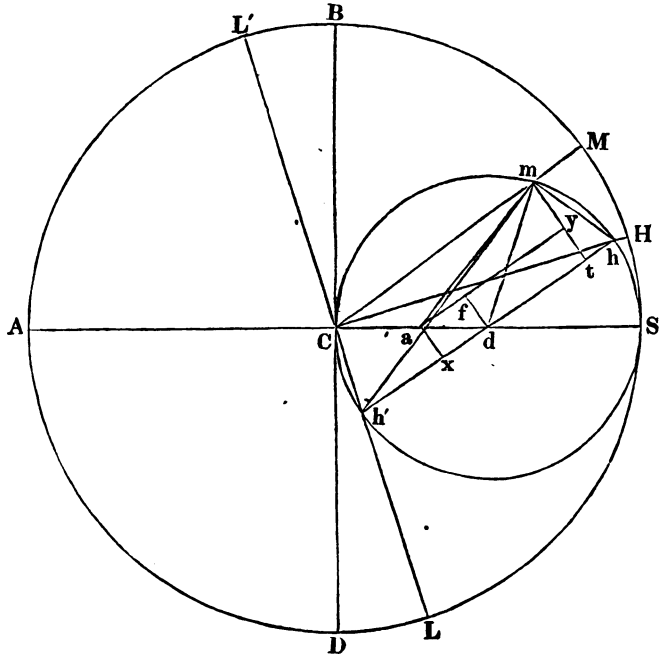
the elevation at S above the inscribed sphere $= S - M$,

the elevation at M (by the same reasoning) $= M - S$.

But (by observation) it is found that it is high water under the moon when it is in the quadratures, also that the depression at S is below the natural level of the ocean; hence M is more than twice S, and although

the high water is never directly under the sun or moon, when the moon is in the quadratures high water is always 6 hours after the high water at full or change.

Suppose the moon to be in neither of the former positions.



Then the place of high water is where the elevation = maximum,

or when S × cos.^{2} x + M × cos.^{2} y = maximum,

and since

$$\cos.^2 x = \tfrac{1}{2} + \tfrac{1}{2} \cos. 2x,$$

and

$$\cos. y = \tfrac{1}{2} + \tfrac{1}{2} \cos. 2y,$$

elevation = maximum, when S × cos. 2 x + M × cos. 2 y = maximum.

Therefore, let A B S D be a great circle of the earth passing through S and M, (those places on its surface which have the sun and moon in the zenith). Join C M, cutting the circle described on C S in (m). Make S d : d a : : force of the moon : force of the sun (which force is supposed

known). Join m a, m d, and let H be any point on the surface of the ocean. Join C H cutting the circle C m S in (h); draw the diameter h d h′, and draw m t, a x perpendicular to h h′, and a y parallel to it.

Then

$$M = Sd,\ S = ad$$

and

$$\angle MCH = y,\ \angle SCH = x,$$
$$\therefore \angle mdh = 2\angle MCH = 2y$$

and

$$\angle adx = \angle SdH = 2x.$$
$$\therefore dt = M \times \cos. 2y,\ dx = S \times \cos. 2x,$$

$\therefore$ elevation = maximum when $tx = ay$ = maximum,

or when $ay = am$, i. e. when h h′ is parallel to a m, hence

CONSTRUCTION.

Make

$$Sd : da :: M : S,$$

and join m a, draw h h′ parallel to a m, and from C draw C h H cutting the surface of the ocean in H, which is the point of high water.

Again, through h′ draw L C h′, meeting the circle in L, L′; these are the points of low water. For let

$$LCS = u,\ LCM = z.$$
$$\cos. \angle adx = \cos. \angle Sdh' = \cos. 2\angle SCh' = \cos. 2u = dx$$

and

$$\cos. 2z = \cos. 2LCM = dt.$$
$$\therefore S \times \cos. 2u + M \times \cos. 2z = \max.$$

Cor. If d f be drawn perpendicular to a m, a m represents the whole difference between high and low water, a f equals the point effected by the sun, m f that by the moon.

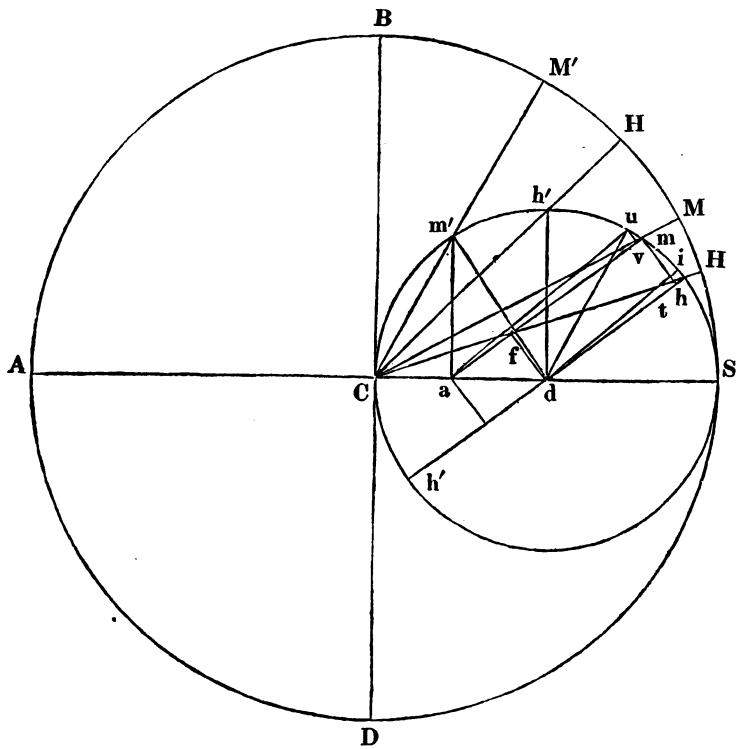
For

$$\sin.^2 u = \cos.^2 x,$$
$$\sin.^2 y = \cos.^2 x.$$
$$\therefore \text{elevation} + \text{depression} = S \times : \overline{\cos.^2 x - \tfrac{1}{3}} + M \times : \overline{\cos.^2 y - \tfrac{1}{3}}$$
$$+ S \times \overline{\cos.^2 x - \tfrac{2}{3}}$$
$$+ M \times : \cos.^2 y - \tfrac{2}{3} = S \times : \overline{2\cos.^2 x - 1} + M \times : \overline{2\cos.^2 y - 1}$$
$$= S \times \cos. 2x + M \times \cos. 2y$$

and

$$dt = M \times \cos. 2y$$
$$dx = S \times \cos. 2x.$$

12. Conclusions deduced from the above (supposing that both the sun and moon are in the equator.)



1. At new and full moon, high water will be at noon and midnight. For in this case C M, a m, C S, d h, C H, all coincide.

2. When the moon is in the quadrature at B, the place of high water is also at B under the moon, when the moon is on the meridian, for C M is perpendicular to C S, (m) coincides with C, (a m) with (a C), d h with d C.

3. While the moon passes from the syzygy to the quadrature the place of high water follows the moon's place, and is to the westward of it, overtakes the moon at the quadratures, and is again overtaken at the next syzygy. Hence in the first and third quadrants high water is after noon or midnight, but before the moon's southing, and in second and fourth *vice versâ*.

4. $\angle$ M C H = max. when S C H = 45°. S d h′ = 90°. and m′ a perpendicular to S C, and $\angle$ a m′ d = max., and a m′ d = m′ d h′ = 2 y′.

Hence in the octants, the motion of the high water = moon's easterly motion; in syzygy it is slower; in quadratures faster. Therefore the tide day in the octants = 24h. 50′ = the lunar day; in syzygy it is less = 24h. 35′; in quadratures = 25h. 25′.

For take any point (u) near (m), draw u a, u d, and d i parallel to a u and with the center (a) and radius a u, describe an arc (u v) which may be considered as a straight line perpendicular to a m; u m and i h are respectively equal to the motions of M and H, and by triangles u m v, d m f.

$$u\,m : i\,h :: m\,a : m\,f.$$

Therefore the synodic motion of the moon's place : synodic motion of high water : : m a : m f.

Cor. 1. At new or full moon, m a coincides with S a, and m f with S d; at the quadratures, m a coincides with C a, and m f with C d; therefore the retardation of the tides at new or full moon : retardation at quadratures : : S a : C a : : M + S : M — S.

Cor. 2. In the octants, m a is perpendicular to S a, therefore m a, m f coincide. Therefore the synodic motion of high water equals the synodic motion of the moon.

Cor. 3. The variation of the tide during a lunation is represented by (m a); at S, m a = S a, at C = C a.

Therefore the spring tide : neap tide : : M + S : M — S.

Cor. 4. The sun contributes to the elevation, till the high water is in the octants, after which (a f) is — v e, therefore the sun diminishes the elevation.

Cor. 5. Let m u be a given arc of the moon's synodic motion, m v is the difference between the tides m a, u a corresponding to it.

Therefore by the triangles m u v, m d f.

$$m\,u : m\,v :: m\,d : d\,f.$$
$$\therefore m\,v \propto d\,f;$$

and since

$$m\,d : d\,f :: r : \text{sin. } d\,m\,f :: r : \text{sin. } m\,d\,h :: r : \text{sin. } 2\,M\,C\,H$$
$$m\,v \propto \text{sin. } 2 \text{ arc } M\,H.$$

13. Prop. VI. In the triangle m d a, m d, d a and ∠ m d a are known when the proportion M : S is known and the moon's elongation.

Let the angle m d a = a,

and make

$$M + S : M - S :: \text{tan. } a : \text{tan. } b$$

then

$$y = \frac{a - b}{2},\ x = \frac{a + b}{2}.$$

For

$$M + S : M - S :: m\,d + d\,a : m\,d - d\,a$$
$$:: \tan.\frac{m\,a\,d + a\,m\,d}{2} : \tan.\frac{m\,a\,d - a\,m\,d}{2}$$
$$:: \tan.\frac{2\,x + 2\,y}{2} : \tan.\frac{2\,x - 2\,y}{2}$$
$$:: \tan.\overline{x + y} : \tan.\overline{x - y}$$
$$:: \tan.\ a : \tan.\ b,$$
$$\therefore x + y : x - y :: a : b,$$
$$\therefore 2\,x = a + b,\ 2\,y = a - b,$$
$$\therefore x = \frac{a + b}{2}$$

and

$$y = \frac{a - b}{2}.$$

14. Prop. VII. To find the proportion between the accelerating forces of the moon and sun. 1st. By comparing the tide day at new and full moon with the tide day at quadratures.

$$35 : 85 :: M : S,$$
$$\therefore M : M :: \frac{35 + 85}{2} : \frac{85 - 85}{2} :: 5 : 2\tfrac{1}{12}.$$

Also, at the time of the greatest separation of high water from the moon in the triangle m′ d a, m d : d a :: r : sin. 2 y :: M : S,

$$\therefore \frac{S}{M} = \text{sin. } 2\,y,$$

at the octants y is found $= 12^\circ\ 30'$,

$$\therefore \frac{S}{M} = \text{sin. } 25^\circ,$$
$$\therefore M : S :: 5 : 2\tfrac{1}{2} \text{ nearly.}$$

Hence taking this as the mean proportion at the mean distances of the moon and sun (if the earth $= 1$) the moon $= \frac{1}{70}$.

Cor. 1. If the disturbing forces were equal there would be no high or low water at quadratures, but there would be an elevation above the inscribed spheroid all round the circle, passing through the sun and moon $= \frac{1}{2}\overline{M + S}$.

Cor. The gravitation of the sun produces an elevation of 24 inches, the gravitation of the moon produces an elevation of 58 inches.

$\therefore$ the spring tide = 82 inches, and the neap tide = $33\frac{3}{4}$ inches.

15. Cor. 3. Though M : S : : 5 : 2, this ratio varies nearly from (6 : 2) to 4 : 2, for supposing the sun and moon's distance each = 1000.

In January, the distance of the sun = 983, perigee distance of the moon = 945.

In July, the distance of the sun = 1017, apogee distance of the moon = 1055.

Disturbing force $\propto \frac{1}{D^3}$; hence

	S	M
apogee	1.901	4.258
mean	2	5
perigee	2.105	5.925.*

The general expression is $M = \frac{5}{2} S \times \frac{\Delta^3}{D^3} \times \frac{d^3}{\delta^3}$.

To find the general expression above.

Disturbing force of different bodies (See Newton, Sect. 11th, p. 66, Cor. 14.) $\propto \frac{1}{D^3}$,

$\therefore$ disturbing force S : disturbing force at mean distance : : $D^3 : \Delta^3$
disturbing force M : disturbing force at mean distance : : $d^3 : \delta^3$,

$$\therefore \frac{M}{S} : \frac{5}{2} :: \frac{d^3}{D^3} : \frac{\delta^3}{\Delta^3},$$

$$\therefore \frac{M}{S} = \frac{5}{2} \times \frac{\Delta^3}{D^3} \times \frac{d^3}{\delta^3},$$

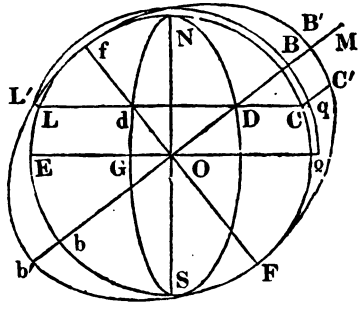
$$\therefore M = \frac{5}{2} \times S \times \frac{\Delta^3}{D^3} \times \frac{d^3}{\delta^3}$$

(or supposing that the absolute force of the sun and moon are the same).

16. Prop. VIII. Let N Q S E be the earth, N S its axis, E Q its equator, O its center; let the moon be in the direction O M having the declination B Q.

* The solar force may be neglected, but the variation of the moon's distance, and proportionally the variation of its action, produces an effect on the times, and a much greater on the heights of the tides.

Let D be any point on the surface of the earth, D C L its parallel of latitude, N D S its meridian; and let B′ F b′ f be the elliptical spheroid of the ocean, having its poles in O M, and its equator F O f.



As the point D is carried along its parallel of latitude, it will pass through all the states of the tide, having high water at C and L, and low water when it comes to (d) the intersection of its parallel of latitude with the equator of the watery spheroid.

Draw the meridian N d G cutting the terrestrial equator in G. Then the arc Q G (converted into lunar hours) will give the duration of the ebb of the superior tide, G E in the same way the flood of the inferior. N. B., the whole tide G Q C′, consisting of the ebb Q G, and the flood G Q is more than four times G O greater than the inferior tide.

Cor. If the spheroid touch the sphere in F and f, C C′ is the height at C, L L′ the height at L, hence if L′ q be a concentric circle C′ q will be the difference of superior and inferior tides.

CONCLUSIONS DRAWN FROM PROP. VIII.

1. If the moon has no declination, the duration of the inferior and superior tides is equal for one day over all the earth.

2. If the moon has declination, the duration of the superior will be longer or shorter than the duration of the inferior according as the moon's declination and the latitude of the place are of the same or different denominations.

3. When the moon's declination equals the colatitude or exceeds it,

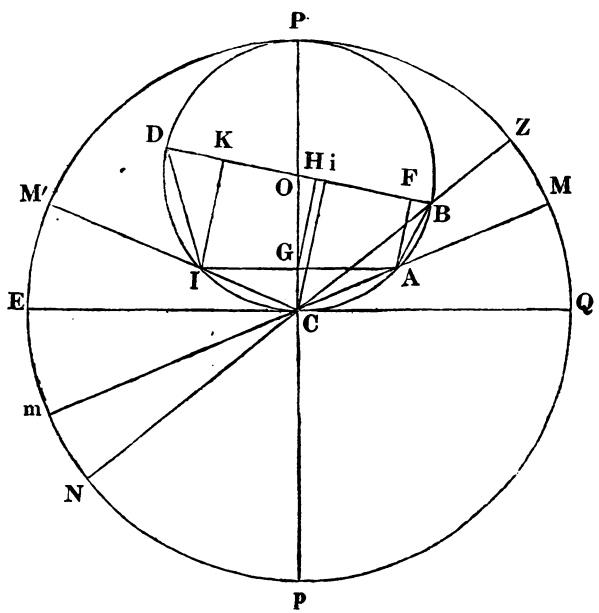
there will only be a superior or inferior tide in the same day, (the parallel of latitude passing through f or between N and f.)

4. The sin. of arc G O = tan. of latitude × tan. declination.

For

$$\text{rad.} : \text{cot. d O G} :: \text{tan. d G} : \text{sin. G O},$$
$$\therefore \text{sin. G O} = \text{cot. d O G} \times \text{tan. G d}$$
$$= \text{tan. declination} \times \text{tan. latitude.}$$

17. Prop. IX. With the center C and radius C Q (representing the



whole elevation of the lunar tide) describe a circle which may represent the terrestrial meridian of any place, whose poles are P, p, and equator E Q. Bisect P C in O, and round O describe a circle P B C D; let M be the place on the earth's surface which has the moon in its zenith, Z the place of the observer. Draw M C m, cutting the small circle in A, and Z C N cutting the small circle in B; draw the diameter B O D and A I parallel to E Q, draw A F, G H, I K perpendicular to B D, and join I D, A B, A D, and through I draw C M′ cutting the meridian in M′. Then after $\frac{1}{2}$ a diurnal revolution the moon will come into the situation M′, and the angle M′ C N (= the nadir distance) = supplement the angle I C B = ∠ I D B.

Also the ∠ A D B = B C A = zenith distance of the moon.

Hence D F, D K $\propto$ cos.^{2} of the zenith and nadir distances to rad. D B.
$\propto$ elevation of the superior and inferior tides.

CONCLUSIONS FROM PROP. IX.

1. The greatest tides are when the moon is in the zenith or nadir of the observer. For in this case (when M approaches to Z) A and I move towards D, B, and F coincides with B; but in this case, the medium tide which is represented by D H (an arithmetic mean to D K, D F) is diminished.

If Z approach to M, D and I separate; and hence, the superior and inferior and the medium tides all increase.

2. If the moon be in the equator, the inferior and superior tides are equal, and equal M $\times$ (cos)^2 latitude. For since A and I coincide with C, and F and K with (i) D i = D B $\times$ (cos.)^2 B D C = M $\times$ (cos.)^2 latitude.

3. If the observer be in the equator, the superior and inferior tides are equal every where, and = M $\times$ (cos.)^2 of the declination of the moon. For B coincides with C, and F and K with G; P G = P C $\times$ cos.^{2} of the moon's declination = M $\times$ (cos.)^2 of the moon's declination.

4. The superior tides are greater or less than the inferior, according as the moon and place of the observer are on the same or different sides of the equator.

5. If the colatitude of the place equal the moon's declination or is less than it, there will be no superior or inferior tide, according as the latitude and the declination have the same or different denominations. For when P Z = M Q, D coincides with I, and if it be less than M Q, D falls between I and C, so that Z will not pass through the equator of the watery spheroid.

6. At the pole there are no diurnal tides, but a rise and subsidence of the water twice in the month, owing to the moon's declining to both sides of the equator.

18. Prop. X. To find the value of the mean tide.

A G = sin. 2 declination (to rad. = O C.)

and

O G = cos. 2 declination (to the same radius).

$$\therefore \text{O H} = \text{cos. 2 declination} \times \text{cos. 2 lat.} \times \frac{\text{M}}{2},$$

$$\therefore \text{D H} = \text{O D} + \text{O H}$$

$$= \text{M} \times \frac{1 + \text{cos. 2 lat.} \times \text{cos. 2 declination}}{2}.$$

Now as the moon's declination never exceeds 30°, the cos. 2 declination is always + v^2, and never greater than $\frac{1}{2}$; if the latitude be less than 45°, the cos. 2 lat. is + v e, after which it becomes — v e.

Hence

1. The mean tide is equally affected by north and south declination of the moon.

2. If the latitude = 45°, the mean tide $\frac{1}{2}$ M.

3. If the lat. be less than 45°, the mean tide decreases as the declination increases.

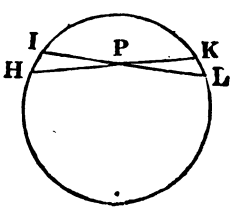
4. If the latitude be greater than 45°, the mean tide decreases as the declination diminishes.

5. If the latitude = 0, the mean tide = $M \times \frac{1 + \text{cos. 2 declination}}{2}$

BOOK I.

SECTION XII.

503. PROP. LXX. To find the attraction on a particle placed *within* a spherical surface, force $\propto \frac{1}{\text{distance}^2}$.



Let P be a particle, and through P draw H P K, I P L making a very small angle, and let them revolve and generate conical surfaces I P H, L P K. Now since the angles at P are equal and the angles at H and L are also equal (for both are on the same segment of the circle), therefore the triangles H I P, P L K, are similar.

$$\therefore HI : KL :: HP : PL$$

Now since the surface of a cone $\propto$ (slant side)^2,

$\therefore$ surface intercepted by revolution of I P H : that of L P K $:: PH^2 : PL^2$

$$:: HI^2 : KL^2$$

and attractions of each particle in I P H : that of L P K $:: \frac{1}{HP^2} : \frac{1}{PL^2}$

$$:: \frac{1}{HI^2} : \frac{1}{KL^2}$$

but the whole attraction of P $\propto$ the number of particles $\times$ attraction of each,

$\therefore$ the whole attraction on P from H I : from K L $:: \frac{HI^2}{HI^2} : \frac{KL^2}{KL^2}$

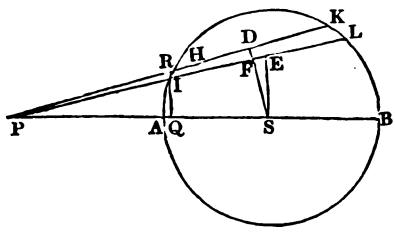
$$:: 1 : 1;$$

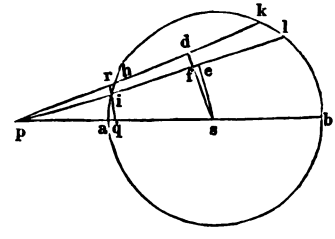
and the same may be proved of any other part of the spherical surface;

$\therefore$ P is at rest.

504. PROP. LXXI. To find the attraction on a particle placed *without* a spherical surface, force $\propto \frac{1}{\text{distance}^2}$.

Let A B, a b, be two equal spherical surfaces, and let P, p be two particles at any distances P S, p s from their centers; draw P H K,





P I L very near each other, and S F D, S E perpendicular upon them, and from (p) draw p h k, p i l, so that h k, i l may equal H K, I L respectively, and s f d, s e, i r perpendiculars upon them may equal S F D, S E, I R respectively; then ultimately P E = P F = p e = p f, and D F = d f. Draw I Q, i q perpendicular upon P S, p s.

Now

$$\left.\begin{array}{l} PI : PF :: IR : DF \\ \text{and} \\ pf : pi :: df : ir \end{array}\right\} \therefore PI \cdot pf : pi \cdot PF :: IR : ir :: IH : ih$$

Again

$$\left.\begin{array}{l} PI : PS :: IQ : SF \\ \text{and} \\ ps : pi :: sf : iq \end{array}\right\} \therefore PI \cdot ps : pi \cdot PS :: IQ : iq$$

$$\therefore PI^2 \cdot pf \cdot ps : (pi)^2 \cdot PF \cdot PS :: IQ \cdot IH : iq \cdot ih$$

:: circumfer. of circle rad. I Q × I H : circumfer. of circle rad i q × i h

:: annulus described by revolution of I Q : that by revolution of i q.

Now

$$\text{attraction on 1st annulus : attraction on 2d} :: \frac{\text{1st annulus}}{\text{distance}^2} : \frac{\text{2d annulus}}{\text{distance}^2}$$

$$:: \frac{PI^2 \cdot pf \cdot ps}{PI^2} : \frac{(pi)^2 \cdot PF \cdot PS}{(pi)^2}$$

$$:: pf \cdot ps : PF \cdot PS.$$

And

attraction on the annulus : attraction in the direction P S :: P I : P Q

:: P S : P F

$$\therefore \text{attraction in direction P S} = pf \cdot ps \cdot \frac{PF}{PS}$$

$$\therefore \text{whole att}^n. \text{ of P to S : whole att}^n. \text{ of (p) to s} :: pf \cdot ps \cdot \frac{PF}{PS} : PF \cdot PS \cdot \frac{pf}{ps}$$

$$:: ps^2 : PS^2 :: \frac{1}{PS^2} : \frac{1}{ps^2}$$

and the same may be proved of all the annuli of which the surfaces are composed, and therefore the attraction of P $\propto \frac{1}{PS^2} \propto \frac{1}{\text{distance}^2}$ from the center.

Cor. The attraction of the particles within the surface on P equals the attraction of the particles without the surface.

For K L : I H :: P L : P I :: L N : I Q.

$\therefore$ annulus described by I H : annulus described by K L

$$:: IQ.IH : KL.LN :: PI^2 : PL^2$$

$\therefore$ attraction on the annulus I H : attraction on the annulus K L

$$:: \frac{PI^2}{PI^2} : \frac{PL^2}{PL^2} :: 1 : 1,$$

and so on for every other annulus, and one set of annuli equals the part within the surface, and the other set equals the part without.

506. Prop. LXXII. To find the attraction on a particle placed *without* a solid sphere, force $\propto^{E.} \frac{1}{\text{distance}^2}$.

Let the sphere be supposed to be made up of spherical surfaces, and the attraction of these surfaces upon P will $\propto \frac{1}{\text{distance}^2}$, and therefore the whole attractions

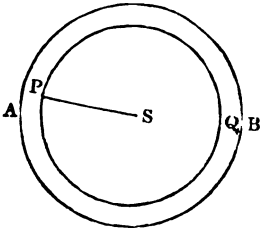
$$\propto \frac{\text{number of surfaces}}{PS^2} \propto \frac{\text{content of sphere}}{PS^2} \propto \frac{\text{diameter}^3}{PS^2}$$

and if P S bear a given ratio to the diameter, then

$$\text{the whole attraction on P} \propto \frac{\text{diameter}^3}{\text{diameter}^2} \propto \text{diameter}.$$

507. Prop. LXXIII. To find the attraction on the particle placed *within*.

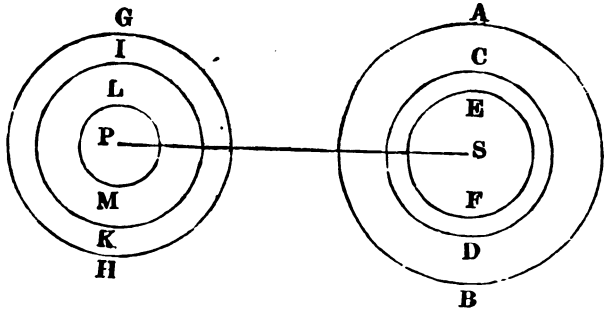
Let P be the particle; with rad. S P describe the interior sphere P Q; then by Prop. LXX. (considering the sphere to be made of spherical surfaces,) the attraction of all the particles contained between the circumferences of the two circles on P will be nothing, inasmuch as they are equal on each side of P, and the attraction of the other part by the last Prop. $\propto \frac{PS^3}{PS^2} \propto PS$.



508. PROP. LXXIV. If the attractions of the particles of a sphere $\propto \frac{1}{\text{distance}^2}$, and two similar spheres attract each other, then the spheres will attract with a force $\propto^s$ as $\frac{1}{\text{distance}^2}$ of their centers.

For the attraction of each particle $\propto \frac{1}{\text{distance}^2}$ from the center of the attracting sphere (A), and therefore with respect to the attracted particle the attracting sphere is the same as if all its particles were concentrated in its center. Hence the attraction of each particle in (A) upon the whole of (B) will $\propto \frac{1}{\text{distance}^2}$ of each particle in B from the center of P, and if all the particles in B were concentrated in the center, the attraction would be the same; and hence the attractions of A and B upon each other will be the same as if each of them were concentrated in its center, and therefore $\propto \frac{1}{\text{distance}^2}$.

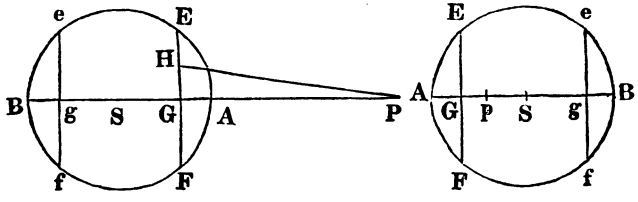
509. PROP. LXXVI. Let the spheres attract each other, and let them not be homogeneous, but let them be homogeneous at corresponding distances from the center, then they attract each other with forces $\propto^s . \frac{1}{\text{distance}^2}$.



Suppose any number of spheres C D and E F, I K and L M, &c. to be concentric with the spheres A B, G H, respectively; and let C D and I K, E F and L M be homogeneous respectively; then each of these spheres will attract each other with forces $\propto^s . \frac{1}{\text{distance}^2}$. Now suppose the original spheres to be made up by the addition and subtraction of similar and homogeneous spheres, each of these spheres attracting each

other with a force $\propto^{s} \frac{1}{\text{distance}^{s}}$; then the sum or differences will attract each other in the same ratio.

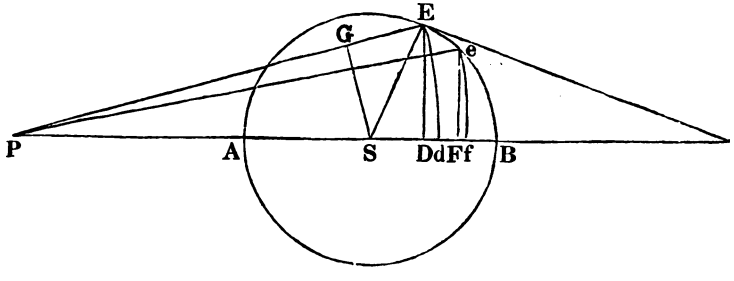
510. Prop. LXXVII. Let the force $\propto$ distance, to find the attraction of a sphere on a particle placed without or within it.



Let P be the particle, S the center, draw two planes E F, e f, equally distant from S; let H be a particle in the plane E F, then the attraction of H on P $\propto$ H P, and therefore the attraction in the direction S P $\propto$ P G, and the attraction of the sum of the particles in E F on P towards S $\propto$ circle E F . P G, and the attraction of the sum of the particles in (e f) on P towards S $\propto$ circle e f . P g, therefore the whole attraction of E F, e f, $\propto$ circle E F (P G + P g) $\propto$ circle E F . 2 P S, therefore the whole attraction of the sphere $\propto$ sphere $\times$ P S.

When P is within the sphere, the attraction of the circle E F on P towards S $\propto$ circle E F . P G, and the attraction of the circle (e f) towards S $\propto$ circle e f . P g, and the differencc of these attractions on the whole attraction to S $\propto$ circle E F (P g — P G) $\propto$ circle E F . 2 P S. Therefore the whole attraction of the sphere on P $\propto$ sphere $\times$ P S.

511. Lemma XXIX. If any arc be described with the center S, rad.



S B, and with the center P, two circles be described very near each other

cutting, first, the circle in E, e, and P S in F, f; and E D, e d, be drawn perpendicular to P S, then ultimately,

$$D d : F f :: P E : P S.$$

For

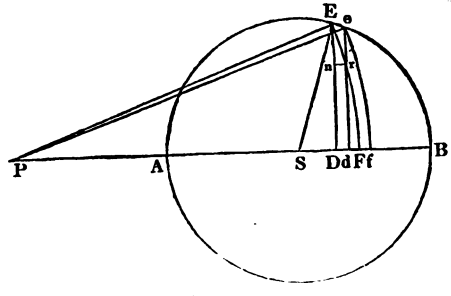
$$D d : E e :: D T : E T :: D E : E S$$

and

$$E e : F f :: E e : e r :: S E : S G$$

$$\therefore D d : F f :: \frac{\quad}{\quad} :: D E : S G :: P E : P S.$$

512. PROP. LXXIX. Let a solid be generated by the revolutions of an evanescent lamina E F f e round the axis P S, then the force with which the solid attracts P $\propto$ D E^2. F f $\times$ force of each particle.



Draw E D, e d perpendiculars upon P S; let e d intersect E F in r; draw r n perpendicular upon E D. Then E r : n r :: P E : E D, $\therefore$ E r . E D = n r . P S = D d . P E, $\therefore$ the annular surface generated by the revolution of E r $\propto$ E r . E D $\propto$ D d . P E, and (P E remaining the same) $\propto$ D d. But the attraction of this annular surface on P $\propto$ D d . P E, and the attraction in the direction P E : the attraction in the direction P S :: P E : P D,

$\therefore$ the attraction in the direction P S $\propto \frac{P D}{P E}$. D d . P E $\propto$ P D . D d

and the whole attraction on P of the surface described by E F $\propto$ sum of the P D . D d.

Let P E = r, D F = x,

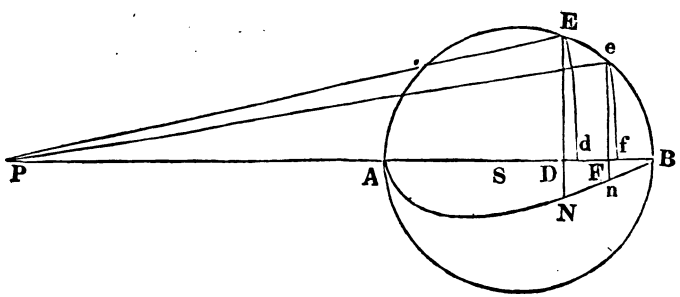
$\therefore$ P D = r — x,

$\therefore$ P D . D d = r d x — x d x,

$\therefore$ sum of P D . D d S = $\int$ r d x — x d x = $\frac{2 r x - x^2}{2}$ = $\frac{D E^2}{2}$ $\propto$ D E^2,

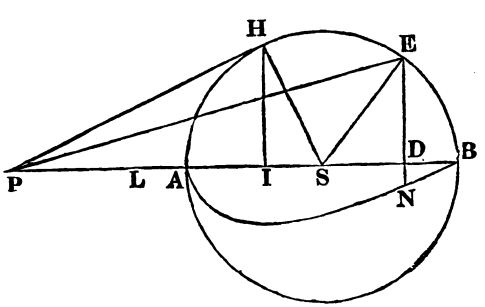
and therefore the attraction of lamina $\propto$ D E^2. F f $\times$ force of each particle.

513. Prop. LXXX. Take D N proportional to $\frac{D E^2 . P S}{P E} \times$ force of each particle at the distance P E, or if $\frac{1}{V}$ represent that force, let D N $\propto \frac{D E^2 . P S}{P E . V}$, then the area traced out by D N will be proportional to the whole attraction of the sphere.



For the attraction of lamina E F f e $\propto$ D E^2. F f $\times$ force of each particle $\propto$ (Lemma XXIII) $\frac{D E^2 . P S}{P E}$. D d $\times$ force of each particle, or $\propto \frac{D E^2 . P S}{P E . V}$ D d, $\therefore$ D N . D d $\propto$ attraction of lamina E F f e, and the sum of these areas or area A N B will represent the whole attraction of the sphere on P.

514. Prop. LXXXI. To find the area A N B.



Draw the tangent P H and H I perpendicular on P S, and bisect P I in L; then

$$PE^2 = PS^2 + SE^2 + 2PS.SD$$

But

$$SE^2 = SH^2 = PS.SI,$$

$$\therefore PE^2 = PS^2 + PS.SI + 2PS.SD$$
$$= PS\{PS + SI + 2SD\}$$
$$= PS\{(PI + IS) + SI + 2SD\}$$
$$= PS\{2LI + 2SI + 2SD\}$$
$$= 2PS\{LI + SI + SD\} = 2PS.LD$$

$$DE^2 = SE^2 - SD^2 = SE^2 - (LD - LS)^2$$
$$= SE^2 - LD^2 - LS^2 + 2LD.LS$$
$$= 2LD.LS - LD^2 - (LS + SE)(LS - SE)$$
$$= 2LD.LS - LD^2 - LB.LA,$$

$$\therefore DN \propto \frac{DE^2.PS}{PE.V} \propto \frac{2LD.LS.PS}{\sqrt{2SD.PS.V}}$$
$$- \frac{LD^2.PS}{\sqrt{2LD.PS.V}} - \frac{LB.LA.PS}{\sqrt{2LD.PS.V}}$$

and hence if V be given, D N may be represented in terms of L D and known quantities.

515. Ex. 1. Let the force $\propto \frac{1}{\text{distance}}$; to find the area A N B.

Since $F \propto \frac{1}{\text{distance}} \propto \frac{1}{V}$, $\therefore V \propto PE$,

$$\therefore DN \propto \frac{2LS.LD.PS}{2LD.PS} - \frac{LD^2.PS}{2LD.PS} - \frac{AL.LB.PS}{2LD.PS}$$
$$\propto LS - \frac{LD}{2} - \frac{AL.LB}{2LD},$$

$$\therefore DN.Dd, \text{ or } d.\text{area} \propto LS.Dd - \frac{LD.Dd}{2} - \frac{AL.LB.Dd}{2LD},$$

$\therefore$ area A N D between the values of L A and L B

$$= LS.(LB - LA) - \frac{LB^2 - LA^2}{4} - \frac{AL.LB}{2}\, l\frac{LB}{LA}.$$

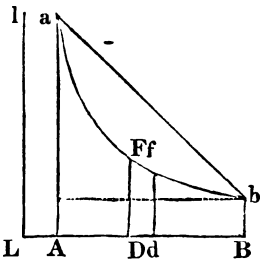
Now

$$LB^2 - LA^2 = (LB + LA).(LB - LA)$$
$$= (\overline{LS + AS} + \overline{LS - AS})\,AB = 2LS.AB,$$

$$\therefore \text{area } AND = LS.AB - \frac{2LS.AB}{4} - \frac{AL.LB}{2}\, l\frac{LB}{LA}$$
$$= \frac{LS.AB}{2} - \frac{AL.LB}{2}\, l\frac{LB}{LA}.$$

516. To construct this area.

To the points L, A, B erect L l, A a, B b, perpendiculars, and let A a = L B, and B b = L A, through the points (a), (b), describe an hyperbola to which L l, L B are asymptotes. Then by property of the hyperbola, A L . A a = L D . D F,



$$\therefore DF = \frac{AL.Aa}{LD} = \frac{AL.LB}{LD},$$

$$\therefore DF.Dd = \frac{AL.LB.Dd}{LD},$$

$$\therefore \text{area } AaFD = \int DF.Dd = AL.LB \int \frac{LD}{},$$

$$\therefore \text{hyperbolic area } AafbB = AL.LB \int \frac{LB}{LA}.$$

$$\text{The area } AaBb = Bb.AB + \frac{AB.an}{2}$$

$$= \frac{Bb.AB}{2} + \frac{an+Bb}{2}.AB = \frac{Aa+Bb}{2}AB$$

$$= \frac{LB+LA}{2}.AB = LS.AB,$$

$$\therefore \text{area } afba = \text{area } AaBb - \text{area } AafbB$$

$$= LS.AB - AL.LB \int \frac{LB}{LA}.$$

517. Ex. 2. Let the force $\propto \frac{1}{\text{distance}^3}$; to find the area A N B.

$$\text{Let } V = \frac{PE^3}{2AS^2},$$

$$\therefore DN = \frac{2LS.LD.PS}{PE.V} - \frac{LD^2.PS}{PE.V} - \frac{AL.LB.PS}{PE.V}$$

but

$$V.PE = \frac{PE^4}{2AS^2} = \frac{4PS^2.LD^2}{2AS^2} = 2PS\frac{PS}{AS^2}.LD^2,$$

$$\therefore DN = \frac{SI.LS}{LD} - \frac{SI}{2} - \frac{AL.LB.SI}{2LD^2} = 2PS.\frac{1}{SI}.LD^2,$$

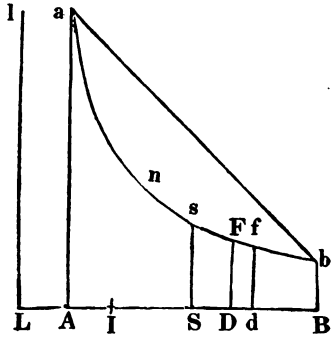
$$\therefore \int DN.x' = SI.LS \int LD - \frac{SI.LD}{2} + \frac{AL.LB.SI}{2LD},$$

$\therefore$ area between the values of L A and L B

$$= SI.LS \int \frac{LB}{LA} - \frac{SI.(LB-LA)}{2} - \left(\frac{LB.SI}{2} - \frac{AL.SI}{2}\right)$$

$$= SI.LS \int \frac{LB}{LA} - SI.AB.$$

To construct this area.



Take S I = S s, and describe a hyperbola passing through a, s, b, to which L l, L B are asymptotes; then as in the former case, the area A a n b B

$$= A L . S B . \int \frac{L B}{L A} = L S . S s \int \frac{L B}{L A} = S I . L S \int \frac{L B}{L A}$$

$$\therefore \text{ the area } A N B = S I . L S \int \frac{L B}{L A} - S I . A B .$$

518. PROP. LXXXII. Let I be a particle within the sphere, and P the same particle without the sphere, and take

$$S P : S A :: S A : S I,$$

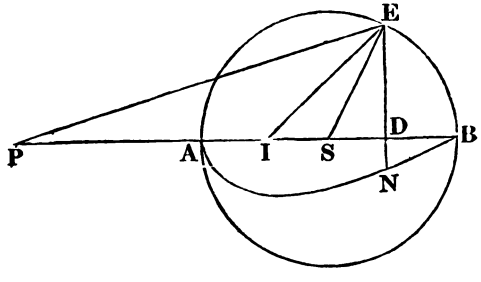
then will the attracting power of the sphere on I : attracting power of the sphere on P

$$:: \sqrt{S I} . \sqrt{\text{force on I}} : \sqrt{S P} . \sqrt{\text{force on P}}.$$

D N force on the point P : D′ N′ force on the point I

$$:: \frac{D E^2 P S}{P E . V} : \frac{D E^2 I S}{I E . V'}$$

$$:: P S . I E . V' : I S . P E . V.$$



Let

$$V : V' :: P E^n : I E^n,$$

then

$$DN : D'N' :: PS.IE.IE^n : IS.PE.PE^n,$$

but

$$PS : SE :: SE : SI,$$

and the angle at S is common,

$\therefore$ triangles P S E, I S E are similar,

$$\therefore PE : IE :: PS : SE :: SE : SI,$$

$$\therefore DN : D'N' :: PS.SE.IE^n : PS.SI.PE^n,$$

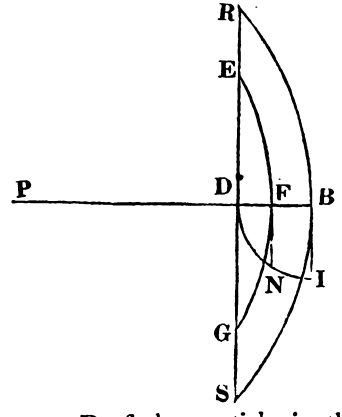
$$:: SE.IE^n : SI.PE^n$$

$$:: \sqrt{SP}.IE^n : \sqrt{SI}.PE^n$$

$$:: \sqrt{SP} : SI^{\frac{n}{2}} \sqrt{SI}.PS^{\frac{n}{2}}.$$

519. Prop. LXXXIII. To find the attraction of a segment of a sphere upon a corpuscle placed within its centre.

Draw the circle F E G with the center P, let R B S be the segment of the sphere, and let the attraction of the spherical lamina E F G upon P be proportional to F N, then the area described by F N $\propto$ whole attraction of the segment to P.



Now the surface of the segment E F G $\propto$ P F D F, and the content of the lamina whose thickness is O $\propto$ P F D F O.

Let F $\propto \frac{1}{\text{distance}^n}$ and the attraction on P of the particle in that spherical lamina, $\propto$ (Prop. LXXIII.) $\frac{DE^2 O}{PF^n}$

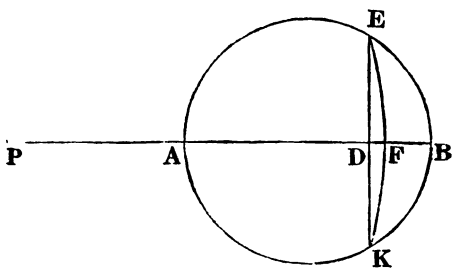
$$\propto \frac{(2PF\,FD - FD^2)\,O}{PF^n}$$

$$\propto \frac{2FDO}{PF^{n-1}} - \frac{FD^2 O}{PF^n},$$

$\therefore$ if F N be taken proportional to $\frac{2FD}{PF^{n-1}} - \frac{FD^2}{PF^n}$, the area traced out by F N will be the whole attraction on P.

520. Prop. LXXXIV. To find the attraction when the body is placed in the axis of the segment, but not in the center of the sphere.

Describe a circle with the radius P E, and the segment cut off by the revolution of this circle E F K round P B, will have P in its center, and



the attraction on P of this part may be found by the preceding Proposition, and of the other part by PROP. LXXXI. and the sum of these attractions will be the whole attraction on P.

SECTION XIII.

521. PROP. LXXXV. If the attraction of a body on a particle placed in contact with it, be much greater than if the particle were removed at any the least distance from contact, the force of the attraction of the particles $\propto$ in a higher ratio than that of $\frac{1}{\text{distance}^2}$.

For if the force $\propto \frac{1}{\text{distance}^2}$, and the particle be placed at any distance from the sphere, then the attraction $\propto \frac{1}{\text{distance}^2}$ from the center of the sphere, and $\therefore$ is not sensibly increased by being placed in contact with the sphere, and it is still less increased when the force $\propto$ in a less ratio than that of $\frac{1}{\text{distance}^2}$, and it is indifferent whether the sphere be homogeneous or not; if it be homogeneous at equal distances, or whether the body be placed within or without the sphere, the attraction still varying in the same ratio, or whether any parts of this orbit remote from the point of contact be taken away, and be supplied by other parts, whether attractive or not, $\therefore$ so far as attraction is concerned, the attracting power of this sphere, and of any other body will not sensibly differ; $\therefore$ if the pheno-

mena stated in the Proposition be observed, the force must vary in a higher ratio than that of $\frac{1}{\text{distance}^2}$.

522. Prop. LXXXVI. If the attraction of the particles $\propto$ in a higher ratio than $\frac{1}{\text{distance}^3}$, or $\propto \frac{1}{\text{distance}^3}$, then the attraction of a body placed in contact with any body, is much greater than if they were separated even by an evanescent distance.

For if the force of each particle of the sphere $\propto$ in a higher ratio than that of $\frac{1}{\text{distance}^3}$, the attraction of the sphere on the particle is indefinitely increased by their being placed in contact, and the same is the case for any meniscus of a sphere; and by the addition and subtraction of attractive particles to a sphere, the body may assume any given figure, and $\therefore$ the increase or decrease of the attraction of this body will not be sensibly different from the attraction of a sphere, if the body be placed in contact with it.

523. Prop. LXXXVII. Let two similar bodies, composed of particles equally attractive, be placed at proportional distances from two particles which are also proportional to the bodies themselves, then the accelerating attractions of corpuscles to the attracting bodies will be proportional to the whole bodies of which they are a part, and in which they are similarly situated.

For if the bodies be supposed to consist of particles which are proportional to the bodies themselves, then the attraction of each particle in one body : the attraction of each particle in the other body, : : the attraction of all the particles in the first body : the attraction of all the particles in the second body, which is the Proposition.

Cor. Let the attracting forces $\propto \frac{1}{\text{distance}^n}$, then the attraction of a particle in a body whose side is A : — B

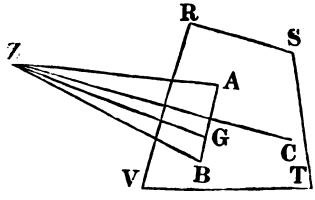
$$\therefore : \frac{A^3}{\text{distance}^n \text{ from } A} : \frac{B^3}{\text{distance}^n \text{ from } B}$$

$$:: \frac{A^3}{A^n} : \frac{B^3}{B^n}$$

$$:: \frac{1}{A^{n-3}} : \frac{1}{B^{n-3}},$$

if the distances $\propto$ as A and B.

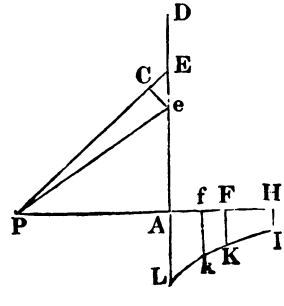
524. PROP. LXXXVIII. If the particles of any body attract with a force $\propto$ distance, then the whole body will be acted upon by a particle without it, in the same manner as if all the particles of which the body is composed, were concentrated in its center of gravity.



Let R S T V be the body, Z the particle without it, let A and B be any two particles of the body, G their center of gravity, then A A G = B B G, and then the forces of Z of these particles $\propto$ A A Z, B B Z, and these forces may be resolved into A A G + A G Z, B B G + B G Z, and A A G being = B B G and acting in opposite directions, they will destroy each other, and $\therefore$ force of Z upon A and B will be proportional to A Z G + B Z G, or to (A + B) Z G, $\therefore$ particles A and B will be equally acted upon by Z, whether they be at A and B, or collected in their center of gravity. And if there be three bodies A, B, C, the same may be proved of the center of gravity of A and B (G) and C, and $\therefore$ of A, B, and C, and so on for all the particles of which the body is composed, or for the body itself.

525. PROP. LXXXIX. The same applies to any number of bodies acting upon a particle, the force of each body being the same as if it were collected in its center of gravity, and the force of the whole system of bodies being the same as if the several centers of gravity were collected in the common center of the whole.

526. PROP. XC. Let a body be placed in a perpendicular to the plane of a given circle drawn from its center; to find the attraction of the circular area upon the body.



With the center A, radius = A D, let a circle be supposed to be described, to whose plane A P is perpendicular. From any point E in this circle draw P E, in P A or it produced take P F = P E, and draw F K perpendicular to P F, and let F K $\propto$ attracting force at E on P. Let I K L be the curve described by the point K, and let I K L meet A D in L, take P H = P D, and draw H I perpendicular

to P H meeting this curve in I, then the attraction on P of the circle $\propto$ A P the area A H I L.

For take E e an evanescent part of A D, and join P e, draw e C perpendicular upon P E, $\therefore$ E e : E C : : P E : A E, $\therefore$ E e . A E = E C $\times$ P E $\propto$ annulus described by A E, and the attraction of that annulus in the direction P A $\propto$ E C . P E . $\frac{P A}{P E}$ $\times$ force of each particle at E $\propto$ E C $\times$ P A $\times$ force of each particle at E, but E C = F f, $\therefore$ F K . F f $\propto$ E C $\times$ the force of each particle at E, $\therefore$ attraction of the annulus in the direction P A $\propto$ P A . F f . F K, and $\therefore$ P A $\times$ sum of the areas F K . F f or P A the area A H I L is proportional to the attraction of the whole part described by the revolution of A E.

527. Cor. 1. Let the force of each particle $\propto \frac{1}{\text{distance}^2}$, at P F = x, let b = force at the distance a,

$$\therefore \text{F K the force at the distance x} = \frac{b a^2}{x^2},$$

$$\therefore \text{F K . F f} = \frac{b a^2 dx}{x^2},$$

$$\therefore \text{attraction} = P A . F K . F f = P A \int \frac{b a^2 dx}{x^2}$$

$$\propto P A - \frac{1}{x} \propto A - \frac{1}{P F},$$

and between the values of P A and P H, the attraction

$$\propto P A \frac{1}{P A} - \frac{1}{P H} \propto 1 - \frac{P A}{P H}.$$

528. Cor. 2. Let the force $\propto \frac{1}{\text{distance}^n}$, then T K = $\frac{b a^n}{x^n}$,

$$\therefore \text{attraction} = P A \int \frac{b a^n}{x^n} dx \propto \frac{P A}{n-1} \times - \frac{1}{x^{n-1}} + \text{Cor.},$$

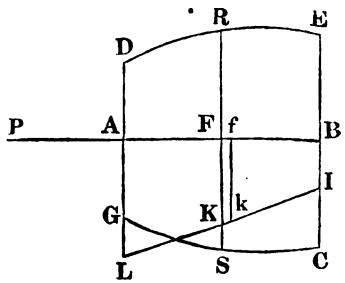
and between the values of P A and P H,

$$\text{attraction} = \frac{P A}{n-1} \left\{ \frac{1}{P A^{n-2}} - \frac{1}{P H^{n-1}} \right\}$$

$$\propto \frac{1}{P A^{n-1}} - \frac{P A}{P H^{n-1}}.$$

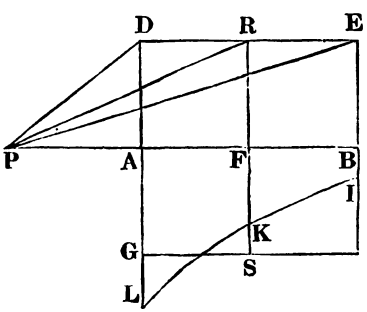
529. Cor. 3. Let the diameter of a circle become infinite, or P H $\propto \infty$, then the attraction $\propto \frac{1}{P A^{n-2}}$.

530. Prop. XCI. To find the attraction on a particle placed in the axis produced of a regular solid.



Let P be a body situated in the axis A B of the curve D E C G, by the revolution of which the solid is generated. Let any circle R F S perpendicular to the axis, cut the solid, and in the semidiameter F S of the solid, take F K proportional to the attraction of the circle on P, then F K . F f $\propto$ attraction of the solid whose base = circle R F S, and depth = F f, let I K L be the curve traced out by F K, $\therefore$ A L K F $\propto$ attraction of the solid.

COR. 1. Let the solid be a cylinder, the force varying as $\frac{1}{\text{distance}^2}$.



Then the attraction of the circle R F S, or F K which is proportional to that attraction $\propto 1 - \frac{P F}{P R}$.

Let P F = x, F R = b,

$$\therefore F K \propto 1 - \frac{x}{\sqrt{x^2 + b^2}},$$

$$\therefore F K . F f \propto d x - \frac{x x'}{\sqrt{x^2 + b^2}},$$

$$\therefore \text{area} \propto - x \sqrt{x^2 + b^2}.$$

Now if P A = x, attraction = 0,

$\therefore$ Cor. = P D — P A,

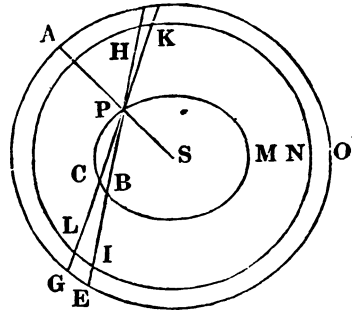
$\therefore$ whole attraction = P B — P E + P D — P A

= A B — P E + P D.

Let A B = ∞ = P E = P D,

$\therefore$ atraction = A B.

531. Cor. 3. Let the body P be placed within a spheroid, let a spheroidical shell be included between the two similar spheroids D O G, K N I, and let the spheroid be described round S which will pass through P, and which is similar to the original spheroid, draw D P E, F P G, very near each other. Now P D = B E, P F = C G, P H = B I, P K = C L.



$\therefore$ F K = L G, and D H = I E,

and the parts of the spheroidical shell which are intercepted between these lines, are of equal thickness, as also the conical frustums intercepted by the revolution of these lines, and

$\therefore$ attraction on P by the part D K : .... G I

$$:: \frac{\text{number of particles in D K}}{\text{P D}^2} : \frac{\dots \text{G}^2}{\text{P G}^2}$$

$$:: \frac{\text{P D}^2}{\text{P D}^2} : \frac{\text{P G}^2}{\text{P G}^2} :: 1 : 1,$$

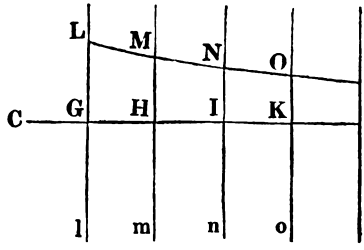
and the same may be proved of every other part of a spheroidical shell, and $\therefore$ body is not at all attracted by it; and the same may be proved of all the other spheroidical shells which are included between the spheroids, A O G, and C P M, and $\therefore$ P is not affected by the parts external to C P M, and $\therefore$ (Prop. LXXII.),

attraction on P : attraction on A :: P S : A S.

532. Prop. XCIII. To find the attraction of a body placed without an infinite solid, the force of each particle varying as $\frac{1}{\text{distance}^n}$, where n is greater than 3.

Let C be the body, and let G L, H M, K O, &c. be the attractions at the several infinite planes of which a solid is composed on the

body C; then the area G L O K equals the whole attraction of a solid on C.



Now if the force $\propto \frac{1}{\text{distance}^n}$.

Then

$$H\,M \propto \frac{1}{C\,H^{n-2}} \text{ (Cor. 3. Prop. XC.)}$$

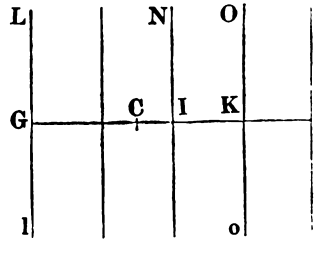
$$\therefore \int H\,M \,.\, d\,x \propto \int \frac{d\,x}{x^{n-2}} \propto -\frac{1}{x^{n-3}} + \text{Cor.}$$

$$\propto \frac{1}{C\,G^{n-3}} - \frac{1}{C\,H^{n-3}};$$

and if $H\,C = \infty$

then the area $G\,L\,O\,K \propto \frac{1}{G\,C^{n-3}}$.

Case 2. Let a body be placed within the solid.

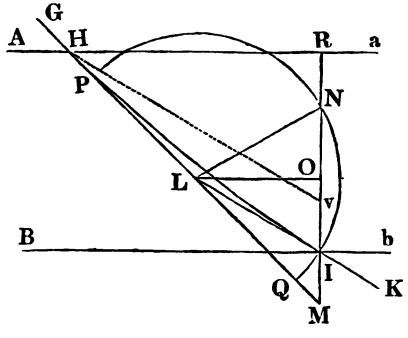


Let C be the place of the body, and take $C\,K = C\,G$; the part of the solid between G and K will have no effect on the body C, and therefore it is attracted to remain as if it were placed without it at the distance C K.

$$\therefore \text{attraction} \propto \frac{1}{C\,K^{n-3}} \propto \frac{1}{C\,G^{n-3}}.$$

SECTION XIV.

534. Prop. XCIV. Let a body move through a similar medium, terminated by parallel plane surfaces, and let the body, in its passage through this medium, be attracted by a force varying according to any law of its distance from the plane of incidence. Then will the sine of inclination be to the sine of refraction in a given ratio.



Let A a, B b be the planes which terminate the medium, and G H be the direction of the body's incidence, and I R that of its emergence.

Case 1. Let the force to the plane A a be constant, then the body will describe a parabola, the force acting parallel to I R, which will be a diameter of the parabola described. H M will be a tangent to the parabola, and if K I be produced I L will also be a tangent to the parabola at I. Let K I produced meet G M in L with the center L, and distance L I describe a circle cutting I R in N, and draw L O perpendicular to I R. Now by a property of the parabola M I = I v,

$$\therefore M L = H L, \therefore M O = O R, \text{ and } \therefore M N = I R.$$

The angle L M I = the angle of incidence, and the angle M I L = supplement of M I K = supplemental angle of emergence.

Now

$$L . M I = M H^2 = 4 M L^2$$

but

$$MN.MI=MI.IR=MQ.MP=\overline{ML+LQ}.\overline{ML-LQ}$$
$$=ML^2-LQ^2$$

$$\therefore MI=\frac{ML^2-LQ^2}{IR}$$

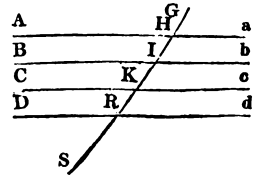
$$\therefore L:IR::4ML^2:ML^2-LQ^2$$

but L and I R are given

$$\therefore 4ML^2 \propto ML^2-LQ^2$$
$$\therefore ML^2 \propto LQ^2 \propto LI^2$$

$\therefore$ M L $\propto$ L I or sin. refraction : sin. inclination in a given ratio.

Case 2. Let the force vary according to any law of distance from A a.



Divide the medium by parallel planes A a, B b, C c, D d, &c. and let the planes be at evanescent distances from each other, and let the force in passing from A a to B b, from B b to C c, from C c to D d, &c. be uniform.

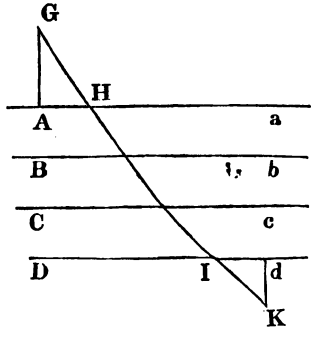
$\therefore$ sin. I at H : sin. R at H : : a : b

sin. R or I at I : sin. R at K : : c : d

sin. R or I at K : sin. R at R : : e : f, and so on.

$\therefore$ sin. I at H : sin. R at R : : a . c . e : b . d . f and in a constant proportion.

535. PROP. XCV. The velocity of a particle before incidence : velocity after emergence : : sin. emergence : sin. incidence.



Take A H = I d, and draw A G, d K perpendicular upon A a, D d, meeting the directions of incidence and emergence in G, K. Let the motion of the body be resolved into the two G A, A H, I d, d k, the ve-

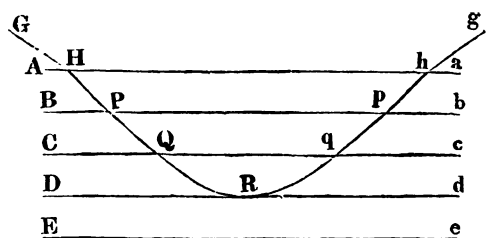
locity perpendicular to A a cannot alter the motion in the direction A a; therefore the body will describe G H, I K in the same time as the spaces A H, I d are described, that is, it will describe G H, I K in equal times before the incidence and after the emergence.

Velocity before incidence : velocity after emergence : : G H : I K

$$:: \frac{\text{A H}}{\text{sin. incidence}} : \frac{\text{I d}}{\text{sin. emergence}}$$

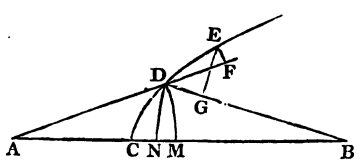
: : sin. emergence : sin. incidence.

536. Prop. XCVI. Let the velocity before incidence be greater than the velocity after emergence, then, by inclining the direction of the incident particle perpetually, the ray will be refracted back again in a similar curve, and the angle of reflection will equal the angle of incidence.



Let the medium be separated by parallel planes A a, B b, C c, D d, E e, &c. and since the velocity before incidence is greater than the velocity after emergence. ∴ sin. of emergence is greater than sin. of incidence. ∴ H P, P Q, Q R, &c. will continually make a less angle with H a, P b, Q c, R d, &c. till at last it coincides with it as at R; and after this it will be reflected back again and describe the curve R q p h g similar to R Q P H G, and the angle of emergence at h will equal the angle of incidence at H.

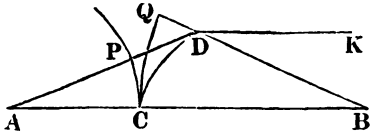
537. Prop. XCVII. Let sin. incidence : sin. refraction in a given ratio, and let the rays diverge from a given point; to find the surface of medium so that they may be refracted to another given point.



Let A be the focus of incident, B of refracted rays, and let C D E be the surface which it is required to determine. Take D E a small arc,

and draw E F, E G perpendiculars upon A D and D B; then D F, D G are the sines of incidence and refraction; or increment of A D : decrement of B D : : sin. incidence : sin. refraction. Take ∴ a point C in the axis through which the curve ought to pass, and let C M : C N : : sin. incidence : sin. refraction, and points where the circles described with radii A M, B N intersect each other will trace out the curve.

538. COR. 1. If A and B be either of them at an infinite distance or at any assigned situation, all the curves, which are the loci of D in different situations of A and B with respect to C, will be traced out by this process.



539. COR. 2. Describe circles with radii A C and C B, meeting A D, B D in P and Q; then P D : D Q : : sin. incidence : sin. refraction, since P D, D Q are the increments of B C and A C.

BOOK II.

SECTION I.

1. PROP. I. Suppose the resistance $\propto$ velocity, and supposing the whole time to be divided into equal portions, the motion lost will $\propto$ velocity, and $\propto$ space described. Therefore by composition, the whole decrement of the velocity $\propto$ space described.

COR. Hence the whole velocity at the beginning of motion : that part which is lost : : the whole space which the velocity can describe : space already described.

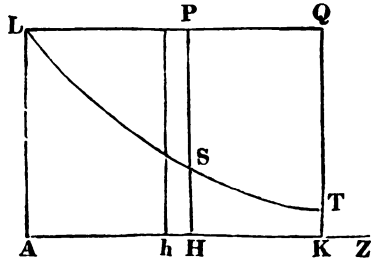
2. PROP. II. Suppose the resistance $\propto$ velocity.

Case 1. Suppose the whole time to be divided into equal portions, and at the beginning of each portion, the force of resistance to make a single impulse which will $\propto$ velocity, and the decrement of the velocity $\propto$ resistance in a given time, $\propto$ velocity. Therefore the velocities at the beginning of the respective portions of time will be in a continued progression. Now suppose the portions of time to be diminished *sine limite*, and then the number increased *ad infinitum*, then the force of resistance will act constantly, and the velocity at the beginning of equal successive portions of time will be in geometric progression.

Case 2. The spaces described will be as the decrements of the velocity $\propto$ velocity.

3. COR. 1. Hence if the time be represented by any line and be divided into equal portions, and ordinates be drawn perpendicular to this line in geometric progression, the ordinates will represent the velocities, and the area of the curve which is the logarithmic curve, will be as the spaces described.

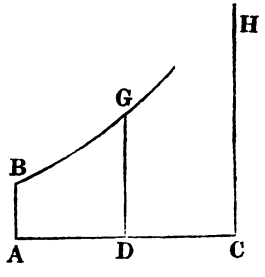
Suppose L S T to be the logarithmic curve to the asymptote A Z. A L, the velocity of the body at the beginning of the motion.



The space described in the time A H with the first velocity continued uniform : space described in the resisting medium, in the same time : : A H P L : area A L S H : : rect. A L × P L : rect. A L × P S*
: : P L : P S (if A L = subtan. of the curve).

Also since H S, K T representing the velocities in the times A H, A K; P S, Q T are the velocities lost, and therefore ∝ spaces described.

4. COR. 1. Suppose the resistance as well as the velocity at the begin-



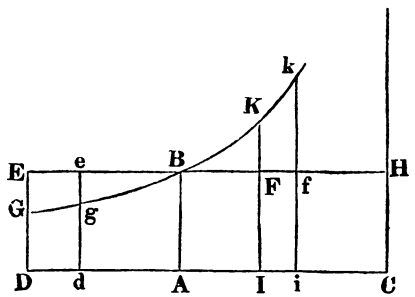
ning of the motion to be represented by the line C A, and after any time by the line C D. The area A B G D will be as the time, and A D as the space described.

For if A B G D increase in arithmetical progression the areas being the hyperbolic logarithms of the abscissas, the abscissa will decrease in geometrical progression, and therefore A D will increase in the same proportion.

5. PROP. III. Let the force of gravity be represented by the rectangle

* Let the subtangent = M. Then the whole area of the curve = M × A L.
∴ the area A L S H = M × A L − M × H S = M × P S = A L × P S.

B A C H, and the force of resistance at the beginning of the motion by the rectangle B A D E on the other side of A B.

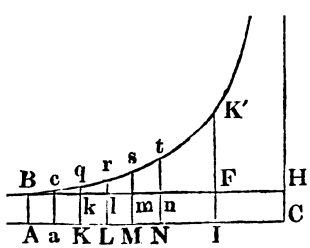


Describe the hyperbola G B K between the asymptotes A C and C H cutting the perpendiculars D E, d e, in G and g.

Then if the body ascend in the time represented by the area D G g d, the body will describe a space proportional to the area E G g e, and the whole space through which it can ascend will be proportional to the area E G B.

If the body descend in the time A B K I, the area described is B F K.

For suppose the whole area of the parallelogram B A C H to be di-



vided into portions, which shall be as the increments of the velocity in equal times, therefore A k, A l, A m, A n, &c. will $\propto$ velocity, and therefore $\propto$ resistances at the beginning of the respective times.

Let A C : A K : : force of gravity : resistance at the beginning of the second portion of time, then the parallelograms B A C H, k K C H, &c. will represent the absolute forces on the body, and will decrease in geometrical progression. Hence if the lines K k, L l, &c. be produced to meet

the curve in q, r, &c. these hyperbolic areas being all equal will represent the times, and also the force of gravity which is constant. But the area B A K q : area B q k : : K q : ½ k q : : A C : ½ A K : : force of gravity : resistance in the middle of the first portion of time.

In the same way, the areas q K L r, r L M s, &c. are to the areas q k l r, r l m s, &c. as the force of gravity to the force of resistance in the middle of the second, third, &c. portions of time. And since the first term is constant and proportional to the third, the second is proportional to the fourth, similarly as to the velocities, and therefore to the spaces described.

∴ by composition B k q, B r l, B s m, &c. will be as the whole spaces described, Q. e. d.

The same may be proved of the ascent of the body in the same way.

6. COR. 1. The greatest velocity which the body can acquire : the velocity acquired in any given time : : force of gravity : force of resistance at the end of the given time.

7. COR. 2. The times are logarithms of the velocities.

8. COR. 4. The space described by the body is the difference of the space representing the time, and the area representing the velocity, which at the beginning of the motion are mutually equal to each other.

Suppose the resistance to α velocity.

$\therefore c^2 : v^2 : : r : \frac{r v^2}{c^2}$ = retarding force corresponding with the velocity (v)

$$\therefore v\,dv = -g \times \frac{r v^2}{c^2} \times dx,$$

$$\therefore dx = \frac{c^2}{g} \times \frac{dv}{v}$$

$$\therefore x = -b \times l\, v + C,$$

$$\therefore x = b \times l\, \frac{c}{v}$$

$$t = \frac{dx}{v} = -\frac{b\,dv}{v^2},$$

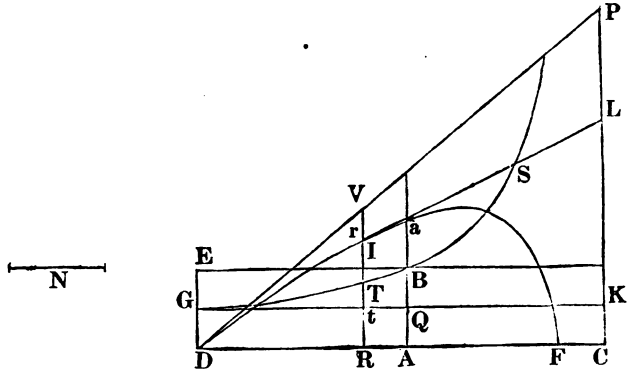
$$\therefore t = -b \times -\frac{1}{v} + \text{Cor.}$$

$$= b \times \frac{1}{v} - \frac{1}{c} \propto \frac{1}{v} - \frac{1}{c},$$

∴ the times being in geometrical progression, the velocities C, d, E, &c. will be in the same inverse geometrical progression.

Also the spaces will be in arithmetical progression.

9. Prop. IV. Let D P be the direction of the projectile, and let it represent the initial velocity; draw C P perpendicular to C D, and



let D A : A C : : resistance : gravity. Also D P : C P : : resistance : gravity, ∴ D A × D P : C P × C A : : R : G. Between D C, C P describe a hyperbola cutting D G and A B perpendicular to D C in G and B, from R draw R V perpendicular cutting D P in V and the hyperbola in T, complete the parallelogram G K C D and make N : Q B : : C D : C P.

Take

$$V r = \frac{G T t}{N} \text{ or } R r = \frac{G T E I}{N},$$

for since

$$N : Q B : : C D : C P : : D R : R V,$$

$$R V = \frac{D R \times Q B}{N},$$

and

$$\frac{G T E I}{N} = \frac{D R \times Q B - G T t}{N} = R r$$

in the time represented by D R T G the body will be at (r), and the greatest altitude = a, and the velocity $\propto$ r L.

For the motion may be resolved into two, ascending and lateral. The lateral motion is represented by D R, and the motion in ascent by R r, which

$$\propto D R \times Q B - G T t,$$

or

$$\frac{D R \times A B - D G . R T}{N},$$

or

$$\frac{DR \times AB - DR \times AQ}{N},$$

$\therefore DR : Rr :: N : AB - AQ$, or QB

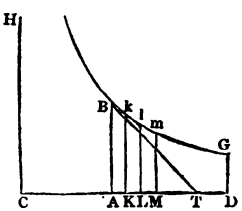
$:: CD : CP$,

$::$ lateral motion . ascending motion at the beginning,

$\therefore$ (r) will be the place of the body required.

SECTION II.

10. PROP. V. Suppose the resistance to vary as the velocity^2.
Then as before, the decrement of velocity $\propto$ resistance $\propto$ velocity^2.



Let the whole time A D be divided into a great number of equal portions, and draw the ordinates A B, K k, L l, M m, &c. to the hyperbola described between the two rectangular asymptotes, C H, C D; then by the property of the hyperbola,

$$AB : Kk :: CK : CA,$$

$$\therefore AB - Kk : Kk :: AK : CA$$

$$:: AB \times AK : AB \times CA.$$

$\therefore AB - Kk \propto AB \times Kk$.

In the same way

$$Kk - Ll \propto Kk^2, \text{ &c.}$$

or

$$AB^2, Kk^2, Ll^2, \text{ &c.}$$

are proportional to their differences.

$\therefore$ velocities will decrease in the same proportion. Also the spaces described are represented by the areas described by the ordinates; hence in

the time A M the space described may be represented by the whole area A M m B.

Now suppose the lines C A, C K, &c. and similarly A K, K L, &c. in geometrical progression, then the ordinates will decrease in the inverse geometrical progression, and the spaces will be *all* equal to each other.

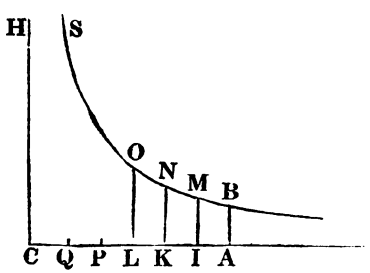
Q. e. d.

11. Cor. 1. The space described in the resisting medium : the space described with the first velocity continued uniform for the time A D : : the hyperbolic area A D G B : rectangle A B × A D.

12. Cor. 3. The first resistance equals the centripetal force which would generate the first velocity in the time A C, for if the tangent B T be drawn to the hyperbola at B, since the hyperbola is rectangular A T = A C, and with the first resistance continued uniform for the time A C the whole velocity A B would be destroyed, which is the time in which the same velocity would be generated by a force equal the first resistance. For the first decrement is A B — K k, and in equal times there would be equal decrements of velocity.

13. Cor. 4. The first resistance : force of gravity : : velocity generated by the force equal the first resistance in the time A C : velocity generated by the force of gravity in the same time.

14. Cor. 5. *Vice versâ*, if this ratio is given, every thing else may be found.



15. Prop. VIII. Let C A represent the force of gravity, A K the resistance, ∴ C K represents the absolute force at any time (if the body descend); A P, a mean proportional to A C and A K, represents the velocity; K L, P Q are contemporaneous increments of the resistance and the velocity.

Then since

$$A P^2 \propto A K, \; K L \propto 2 A P \times P Q \propto A P \times K C,$$

the increment of velocity $\propto$ force when the time is given,

$\therefore$ K L $\times$ K N $\propto$ A P $\times$ K C $\times$ K N,

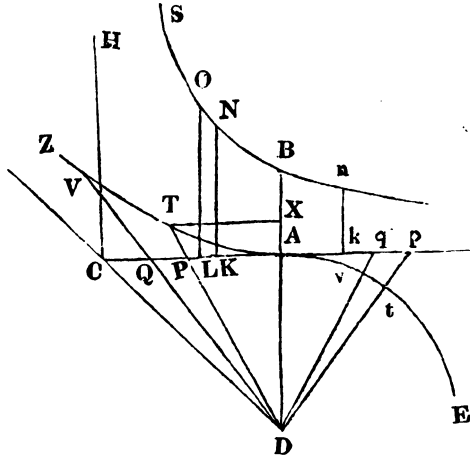
$\therefore$ ultimately K L O N (equal the increment of the hyperbolic area) $\propto$ A P $\propto$ velocity, $\propto$ space described, and the whole hyperbolic area = the sum of all the K L O Ns which are proportional to the velocity, and $\therefore$ space described. $\therefore$ If the whole hyperbolic area be divided into equal portions the absolute force C A, C I, C K, &c. are in geometrical progression. Q. e. d.

16. COR. 1. Hence if the space described be represented by a hyperbolic area, the force of gravity, velocity, and resistance, may be represented by lines which are in continued proportion.

17. COR. 2. The greatest velocity = A C.

18. COR. 3. If the resistance is known for a given velocity, the greatest velocity : given velocity : : $\sqrt{\text{force of gravity}}$: $\sqrt{\text{given resistance.}}$

19. PROP. IX. Let A C represent the greatest velocity, and A D be per-



pendicular and equal to it. With the center D and radius A D describe the quadrant A t E and the hyperbola A V Z. Draw the radii D P, D p. Then

Case 1. If the body ascend; draw D v q near to D p, $\therefore$ since the sector and the triangle are small,

$$\text{D v t} : \text{D p q} : : \text{D t}^2 : \text{D p}^2$$

$$\therefore \text{D v t} \propto \frac{\text{D q p}}{\text{D p}^2}$$

$$\propto \frac{\frac{1}{2} A D \times p q}{A D^2 + A D \times A K} \propto \frac{p q}{C K}$$

$\propto$ increment of the time.

$\therefore$ by composition, the whole sector $\propto$ whole time till the whole $V = 0$.

Case 2. If the body descend; as before

$$D V T : D P Q :: D T^2 : D P^2$$
$$:: D X^2 : D A^2 :: T X^2 : A P^2$$
$$:: D X^2 - T X^2 : D A^2 - A P^2$$
$$:: A D^2 : A D^2 - A D \times A K$$
$$:: A D : C K.$$

By the property of the hyperbola,

$$T X^2 = D X^2 - D A^2$$
$$\therefore D A^2 = D X^2 - T X^2$$
$$\therefore D V T \propto \frac{D P Q}{A D \times C K} \propto \frac{P Q}{C k}$$

$\propto$ increment of the time.

$\therefore$ by composition, the whole time of descent till the body acquire its greatest $V =$ the whole hyperbolic sector D A T.

20. Cor. 1. If $A B = \frac{1}{4} A C$.

The space which the descending body describes in any time : space which it would describe in a non-resisting medium to acquire the greatest velocity : : area A B N K : $\triangle$ A T D, which represents the time. For since A C : A P : : A P : A K

$$K L : \tfrac{1}{2} P Q :: A P : \tfrac{1}{4} A C$$

and

$$K N : A C :: A B : C K$$

$\therefore$ K L O N : D T V : : A P : A C

: : vel. of the body at any time : the greatest vel.

Hence the increments of the areas $\propto$ velocity $\propto$ spaces described.

$\therefore$ by composition the whole A B N K : sector A T D : : space described to acquire any velocity : space described in a non-resisting medium for the same time.

21. Cor. 2. In the same way, if the body ascend, the space described till the velocity = A p : space through which a body would move : : A B n k : A D t.

22. Cor. 3. Also, the velocity of a body falling for the time A T D : velocity which a body would acquire in a non-resisting medium in the same time : : $\triangle$ A D P : sector T D A; for since the force is constant,

the velocity in a non-resisting medium $\propto$ time, and the force in a resisting medium $\propto$ A P $\propto$ $\triangle$ A D P.

23. COR. 4. In the same way, the velocity in the ascent: velocity with which a body should move, to lose its whole motion in the same time : : $\triangle$ A p D : sector A t D : : A p : arc A t.

For let A Y be any other velocity acquired in a non-resisting medium in the same time with A P.

$$\therefore \text{A P : A C : : A P D : this area}$$

and

$$\text{A P : A C : : A P D : A C D.}$$

Therefore the area which represents the time of acquiring the greatest velocity in a non-resisting medium = A C D.

In the same way, let A y be velocity lost in a non-resisting medium in the same time as A p in a resisting medium.

$\therefore$ A p : A y : : $\triangle$ A p D : area which represents the time of losing the velocity A p.

$\therefore$ time of losing the velocity A y = $\triangle$ A p D.

24. COR. 5. Hence the time in which a falling body would acquire the velocity A P : time in which, in a non-resisting medium, it would acquire the greatest velocity : : sector A D T : $\triangle$ C A D.

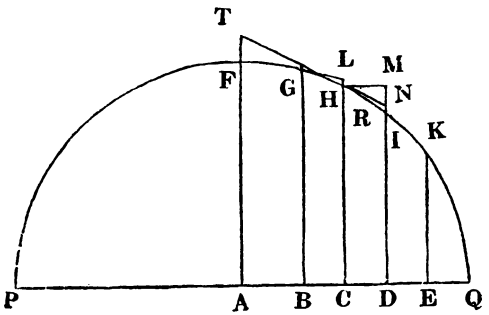
Also the time in which it would lose the velocity A p : time in which, in a non-resisting medium, it would lose the same velocity : : arc A t : tangent A p.

25. COR. 6. Hence the time being given, the space described in ascent or descent may be known, for the greatest velocity which the body can acquire is constant, therefore the time in which a body falling in a non-resisting medium, would acquire that velocity is also known. Then the sector A D T or A D t : $\triangle$ A D C : : given time : time just found; therefore the velocity A P is known or A p.

Then the area A B N K or A B n k : A D T or A D t : : space sought for : space which the body would describe uniformly with its greatest velocity.

26. COR. 7. Hence *vice versâ*, if the space be given, the time will be known.

27. Prop. X. Let P F Q be the curve meeting the plane P Q. Let



G, H, I, K be the points in the curve, draw the ordinates; let B C = C D = D E, &c.

Draw H N, G L tangents at H and G, meeting the ordinates produced in L and N, complete the parallelogram C H M D. Then the times $\propto \sqrt{L H}$ and $\sqrt{N I}$, and the velocities $\propto$ G H and H I, and the times $\propto$; let T and t = times, and the velocities $\propto \frac{G H}{T}$ and $\frac{H I}{t}$, therefore the decrement of the velocity arising from the retardation of resistance and the acceleration of gravity $\propto \frac{G H}{T} - \frac{H T}{t}$, also the accelerating force of gravity would cause a body to describe 2 I N in the same time, therefore the increment of the velocity from G $= \frac{2 N I}{t}$, again the arc is increased by the space $= H I - H N = R I = \frac{M I \times N I}{H I}$, therefore the decrement from the resistance alone $= \frac{G H}{T} - \frac{H I}{t} + \frac{2 M I \times N I}{t \times H I}$, $\therefore$ resistance : gravity : : $\frac{G H \times t}{T} - H I + \frac{2 M I \times N I}{H I} : 2 N I$.

Again, let

A B, C D, C E, &c. be $- o + o, 2o, 3o$, &c.

C H = P

and

$$M I = Q o + R o^2 + S o^3 + \&c.$$

$$\therefore D I = P - Q o + \&c.$$

$$E K = P - 2 Q o - 4 R o^2 - \&c.$$

$$B G = P + Q o + \&c.$$

$$(B G - C H)^2 + B C^2 (= G H^2) = o^2 + Q^2 o^2 + 3 Q R o^3 + \&c.$$

$$\therefore G H^2 = \overline{1 + Q^2} \times o^2 + 3 Q R o^3,$$

$$\therefore G H = \sqrt{1 + Q^2} \times o + \frac{Q R o^2}{\sqrt{1 + Q^2}}$$

and

$$H T = o \sqrt{1 + Q^2} + \frac{Q R o^3}{\sqrt{1 + Q^2}}.$$

Subtract from C H $\frac{1}{2}$ the sum G B and D I, and R o^2 and R o^2 + 3 S o^3 will be the remainder, equal to the sagittæ of the arcs, and which are proportional to L H and N I, and therefore, in the subtracted number of the times,

$$\therefore \frac{t}{T} \propto \sqrt{\frac{R + 3 S o}{R}} \propto \frac{R + \frac{3}{2} S o}{2 R} \propto 1 + \frac{3 S o}{2 R}.$$

$$\therefore \frac{G H \times t}{T} = \overline{o \sqrt{1 + Q^2} + \frac{Q R o^2}{\sqrt{1 + Q^2}}} \times \overline{1 + \frac{3 S o^2}{2 R}}$$

$$= o.\sqrt{1+Q^2} + \frac{Q R o^2}{\sqrt{1+Q^2}} + \frac{3 S o^3 \sqrt{1+Q^2}}{2 R} + \frac{3So}{2 R} \times \frac{Q R o^2}{\sqrt{1+Q^2}}$$

$$H I = o. \sqrt{1 + Q^2} + \frac{Q R o^2}{\sqrt{1 + Q^2}}$$

$$\frac{M I \times N I}{H I} = \frac{R o^2 \times Q o + R o^2 + \&c.}{o. \sqrt{1 + Q^2} + \frac{Q R o^2}{\sqrt{1 + Q^2}}}$$

$$\therefore \text{resistance : gravity} :: \frac{G H \times t}{T} - H I + \frac{2 M I \times N I}{H I} : 2 N I$$

$$:: \frac{3 S o^2 \sqrt{1 + Q^2}}{2 R} : 2 R o^2$$

$$:: 3 S \sqrt{1 + Q^2} : 4 R^2.$$

The velocity is equal to that in the parabola whose diameter = H C, and the lat. rect. $= \frac{H N^2}{T N}$ or $\frac{1 + Q^2}{R}$. The resistance $\propto$ density $\times V^2$, therefore the density $\propto \frac{\text{resistance}}{V^2} \propto \frac{3 S \sqrt{1 + Q^2}}{4 R^2}$ directly $\propto \frac{R}{1 + Q^2}$ directly $\propto \frac{S}{R \sqrt{1 + Q^2}}$.

28. Ex. 1. Let it be a circular arc, C H = e, A Q = n, A C = a, C D = o,

$\therefore D I^2 = n^2 - (a + o)^2 = n^2 - a^2 - 2 a o - o^2 = e^2 - 2 a o - o^2,$

$$\therefore D I = e - \frac{a o}{e} - \frac{n^2 o^2}{2 e^3} - \frac{a n^2 o^3}{2 e^5},$$

and therefore

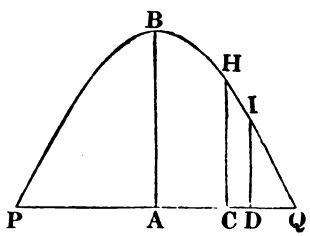
$$P = e,\ Q = \frac{a}{e},\ R = \frac{n^2}{2 e^3},\ S = \frac{a n^2}{2 e^5},$$

$$\therefore \text{density} \propto \frac{S}{R \sqrt{1 + Q^2}} \propto \frac{a n^2}{2 e^5} \times \frac{2 e^3}{n^2} \times \frac{e}{n},$$

$$\propto \frac{a}{n e} \propto \frac{a}{e} \propto \frac{\text{sin.}}{\text{cos.}} \propto \text{tangent.}$$

The resistance : gravity : : $\frac{3 a n^2}{2 e^5} \times \frac{n}{e} : \frac{n^4}{e^5}$: : 3 a : 2 n.

29. Ex. 2. Of the hyperbola.



$$P I \times b = P D^2,$$

$$\therefore \text{put } P C = a,\ C D = o,\ Q P = c,$$

$$\therefore \overline{a + o} \times \overline{c - a - o} = a c - a^2 - 2 a o + c o - o^2$$

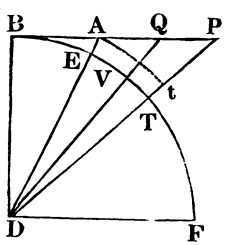
$$\therefore D I = \frac{a c - a^2}{b} - \frac{2 a + c}{b} . o - \frac{o^2}{b},$$

and since there is no fourth term,

$$S = 0,$$

$$\therefore \text{draw } y = 0.$$

30. Prop. XIII. Suppose the resistance to $\propto V + V^2$.



Case 1. Suppose the body to ascend; with the center D and rad. D B,

describe the quadrant B T F; draw B P an indefinite line perpendicular to B D, and parallel to D F. Let A P represent the velocity; join D P, D A, and draw D Q near D P.

$\therefore$ resistance $\propto A P^2 + 2 B A \times A P$, suppose gravity $\propto D A^2$,

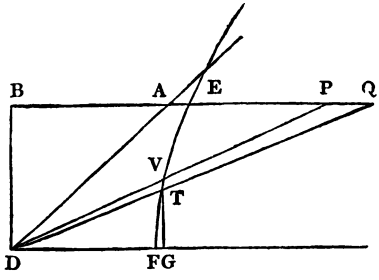
$\therefore$ decrement of V $\propto$ gravity + resistance $\propto A D^2 + A P^2 + 2 B A \times A P$.
$\propto D P^2$

$D P Q (\propto P Q) : D T V :: D P^2 : D T^2$,

$\therefore D T V \propto D T^2 \propto 1$,

therefore the whole sector E T D, is proportional to the time.

Case 2. Suppose the force of gravity proportional to a less quantity than $D A^2$, draw B D perpendicular to B P, and let the force of gravity



$\propto A B^2 - B D^2$. Draw D F parallel to P B and = D B and with the center D — $\frac{1}{2}$ axis-major = $\frac{1}{2}$ axis-minor = D B, describe a hyperbola from the vertex F, cutting A D produced in E, and D P, D Q in T, V.

Now since the body is supposed to ascend.

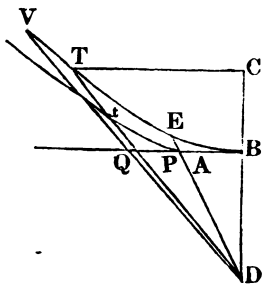
The decrement of the velocity $\propto A P^2 + 2 A B \times A P + A B^2 - B D^2 \propto B P^2 - B D^2$ ($B P^2 = A P^2 + A B^2 + 2 A B \times B P$).

Also, D T V : D P Q : : $D T^2$: $D P^2$ (by similar triangles)
: : $T G^2$: $B D^2$ (T G perpendicular to G)
: : $D F^2$: $P B^2 - D B^2$.

Now D P Q $\propto$ decrement of velocity $\propto P B^2 - D B^2$,

$\therefore$ D T V $\propto D F^2 \propto 1 \propto$ increment of the time, since the time flows uniformly.

Case 3. If the body descend; let gravity $\propto B D^2 - A B^2$.



With center D and vertex B, describe the rectangular hyperbola B T V, cutting the lines D A, D P, D Q produced in E, T, V.

The increment of $V \propto B D^2 - A B^2 - 2 A B \times A P - A P^2$

$$\propto B D^2 - (A B + A P)^2 \propto B D^2 - B P^2$$

$$D T V : D P Q (\propto P Q) :: D T^2 : D P^2$$

$$:: G T^2 : B P^2 :: G D^2 - B D^2 : B P^2$$

$$:: G D^2 : B D^2 :: B D^2 : B D^2 - B P^2,$$

$$\therefore D T V \propto B D^2 \propto 1,$$

$\therefore$ the whole sector E D T $\propto$ time.

31. Cor. With the center C and distance D A describe an arc similar to B T.

Then the velocity A P : the velocity which in the time E D t a body would lose or acquire in a non-resisting medium : : $\triangle$ D A P : sector A D t.

For V in a non-resisting medium $\propto$ time.

32. In the case of the ascent,

Let the force of gravity $\propto 1$. Resistance $\propto 2 a v + v^2$

$$\therefore d v \propto 1 + 2 a v + v^2$$

$$\therefore \frac{d v}{1 + 2 a v + v^2} \propto \text{time}.$$

$\therefore$ by Demoivre's first formula,

$$\text{f. or time} = 0$$

when

$$\text{f.} \frac{d v}{1 + 2 a v + v^2} = \frac{1}{g^2} \times \text{cir. arc. rad.} = g \text{ and}$$

$$\text{tangent} = v + a$$

The whole time $\therefore$ when $v = 0 = \frac{1}{g^2} \times$ cir. arc rad. $= g$ and tangent $= a + C$.

$\therefore$ cor^t. time $= \frac{1}{g^2} \times$ cir. arc rad. $= g$ and tangent $v + a -$ cir. arc rad. $= g$ and tangent a.

$\therefore$ the time of ascent $=$ sector E D T $- g^2 = 1 - a^2$.

33. In the case of descent,

$$d v \propto 1 - 2 a v - v^2$$

let

$$v + a = x$$

$$\therefore d v = d x$$

$$\therefore v^2 + 2 a v + a^2 = x^2$$

$$\therefore 1 + a^2 - x^2 = 1 - 2 a v - v^2$$

$$\therefore f. = \frac{1}{2g} \times \int \frac{g + x}{g - x} + C, \; (g^2 = 1 + a^2)$$

Time $= 0$, $v = 0$,

$$\therefore x = a,$$

$$\therefore o = \frac{1}{2g} \times \int \frac{g + a}{g - a} + C$$

$$\therefore \text{Cor}^t. \text{ time} = \frac{1}{2g} \times \int \frac{g + x}{g - x} - \int \frac{g + a}{g - a}.$$

34. PROP. XIV. Take A C proportional to gravity, and A K to the resistance on contrary sides if the body ascend, and *vice versâ*.

Between the asymptotes describe a hyperbola, &c. &c.

Draw A b perpendicular to C A, and

$$A b : D B :: D B^2 : 4 B A \times A C.$$

The area A b N K increases or decreases in arithmetic progression if the forces be taken in geometric progression.

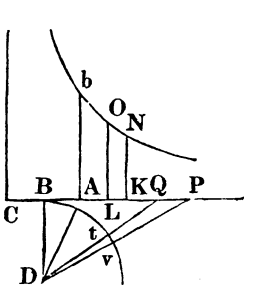
Now

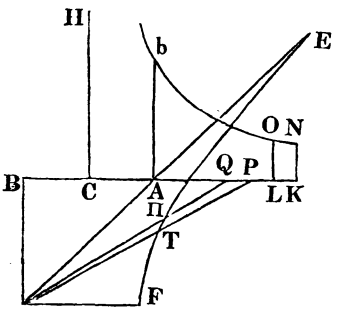
$$A K \propto \text{resistance} \propto 2 B A P + A P^2.$$

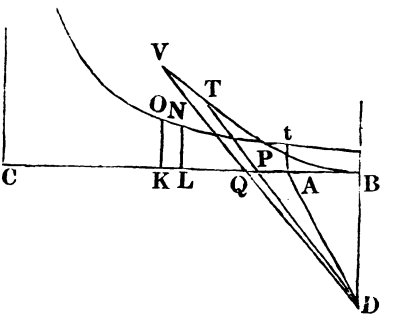
Let

$$A K = \frac{2 B A P + A P^2}{Z}$$

$$\therefore K L = \frac{2 B A \times P Q + 2 A P \times P Q}{Z},$$







$$KL = \frac{2BPQ}{Z}$$

$$\therefore KLON = \frac{2BP \times PQ}{Z} \times LO.$$

Now

$$Ab : LO :: CK : CA$$

$$DB : Ab :: 4BA \times CA : DB^2$$

$$\therefore LO = \frac{BD^3}{4BA \times CK}$$

$$\therefore KLON = \frac{2PB \times PQ \times BD^3}{4BA \times CK \times Z}.$$

Case 1. Suppose the body to ascend.

$$\text{gravity} \propto AB^2 + BD^2 = \frac{AB^2 + BD}{Z}$$

$$AK = \frac{AP^2 + 2BAP}{Z}$$

$$\therefore DP^2 = CK \times Z.$$

$$\therefore DT^2 : DP^2 :: DB^2 : CK \times Z$$

and in the other two cases the same result will obtain.

Make

$$DTV = DB \times m.$$

$$\therefore DB \times m : \tfrac{1}{2} DB \times PQ :: DB^2 : CK \times Z$$

$$\therefore BD^3 \times PQ = 2BD \times m \times CK \times Z.$$

$$\therefore AbNK = \frac{BP \times BD \times m}{AB}$$

$$\therefore AbNK - DTV = \frac{\overline{BP - AB} \times BD \times m}{AB} \propto AP. \propto \text{velocity.}$$

$\therefore$ it will represent the space.

SECTION IV.

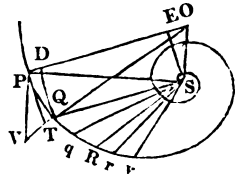
35. PROP. XV. LEMMA. The

$$\angle OPQ = \text{a rectangle} = \angle OQR$$

and

$$\angle SPQ = \angle \text{of the spiral} = \angle SQR$$

$$\therefore \angle OPS = \angle OQS.$$



$\therefore$ the circle which passes through the points P, S, O, also passes through Q. Also when Q coincides with P, this $\frac{\text{circle}}{2}$ touches the spiral.

$$\therefore \angle PSO \quad \angle \text{ in a } \frac{\text{circle}}{2} \text{ whose diameter} = PO.$$

Also

$$T Q : P Q :: P Q : 2 P S.$$
$$\therefore P Q^2 = 2 P S \times T O$$

which also follows from the general property of every curve.

$$P Q^2 = P V \times Q R.$$
$$\therefore Q R = \frac{P Q^2}{2 W}.$$

36. Hence the resistance $\propto$ density $\times$ square of the velocity.

37. Density $\propto \frac{1}{\text{distance}}$, centripetal force $\propto$ density$^2 \propto \frac{1}{\text{distance}^2}$.

Then produce S Q to V so that S V = S P, and let P Q be an arc described in a small time, P R described in twice that time, $\therefore$ the decrements of the arcs from what would be described in a non-resisting medium $\propto T^2$.

$\therefore$ decrement of the arc P Q $= \frac{1}{4}$ decrement of the arc P R

$\therefore$ decrement of the arc P Q $= \frac{1}{2}$ R r (if Q S r = area P S Q).

For let P q, q v be arcs described (in the same time as P Q, Q R) in a non-resisting medium,

$$P S q - P S Q = Q S q = q S v - Q S r$$
$$= r S v - Q S q$$
$$\therefore 2 Q S q = r S v$$

$\therefore$ if S T ultimately = S t be the perpendicular on the tangents

$$S T \times Q q = \tfrac{1}{2} S t \times r v$$
$$\therefore 2 Q q = r v$$

and

$$R v = 4 Q q.$$
$$\therefore 2 Q q = R r.$$

Hence

Resistance : centripetal force : : $\frac{1}{2}$ R r : T Q,

Also

$$T Q \times S P^2 \propto \text{time}^2, \text{ (Newt. Sect. II.)}$$
$$\therefore P Q^2 \times S P \propto \text{time}^2$$
$$\therefore \text{time} \propto P Q \times \sqrt{S P}$$
$$\therefore V \propto \frac{P Q}{P Q \times \sqrt{S P}} \propto \frac{1}{\sqrt{S P}}$$

also

$$V \text{ at } Q \propto \frac{1}{\sqrt{S Q}}$$

$$P\,Q : Q\,R :: \sqrt{S\,Q} : \sqrt{S\,P}$$
$$:: S\,Q : \sqrt{S\,Q \times S\,P}$$
$$P\,Q : Q\,r :: S\,Q : S\,P$$

since the areas are equal, and the angles at P and Q are equal.

$$\therefore P\,Q : R\,r :: S\,Q : S\,P - \sqrt{S\,Q \times S\,P}$$
$$:: S\,Q : \tfrac{1}{2}\,V\,Q$$

For

$$S\,Q = S\,P - V\,Q$$
$$\therefore S\,Q \times S\,P = S\,P^2 - V\,Q \times S\,P$$
$$\therefore \sqrt{S\,Q \times S\,P} = S\,P - \tfrac{1}{2}\,V\,Q - \frac{V\,Q^2}{8\,S\,P} - \&c.$$
$$\therefore \tfrac{1}{2}\,V\,Q \text{ ultimately} = S\,P - \sqrt{S\,P \times S\,Q}$$
$$\text{Resistance} \propto \frac{\text{decrement of V}}{\text{time}^2} \propto \frac{R\,r}{P\,Q^2 \times S\,P}$$
$$\propto \frac{\tfrac{1}{2}\,V\,Q}{P\,Q \times S\,Q \times S\,P}$$
$$\tfrac{1}{2}\,V\,Q : P\,Q :: \tfrac{1}{2}\,O\,S : P\,O$$

and

$$S\,Q = S\,P \propto \frac{\tfrac{1}{2}\,O\,S}{O\,P \times S\,P^2}$$

$\therefore$ density $\times$ square of the velocity $\propto$ resistance $\propto \dfrac{O\,S}{O\,P \times S\,P^2}$

$$\therefore \text{density} \propto \frac{O\,S}{O\,P \times S\,P}$$

and in the logarithmic spiral $\dfrac{O\,S}{O\,P}$ is constant.

$$\therefore \text{density} \propto \frac{1}{S\,P}.$$ Q. e. d.

38. COR. 1. V in spiral = V in the circle in a non-resisting medium at the same distance.

39. COR. 3. Resistance : centripetal force $:: \tfrac{1}{2}\,R\,r : T\,Q$
$$:: \frac{\tfrac{1}{2}\,V\,Q \times P\,Q}{S\,Q} : \frac{\tfrac{1}{2}\,P\,Q^2}{S\,P}$$
$$:: \tfrac{1}{2}\,V\,Q : P\,Q$$
$$:: \tfrac{1}{2}\,O\,S : O\,P.$$

$\therefore$ the ratio of resistance to the centripetal force is known if the spiral be given, and *vice versâ*.

40. COR. 4. If the resistance exceed $\frac{1}{2}$ the centripetal force, the body cannot move in this spiral. For if the resistance equal $\frac{1}{2}$ the centripetal

force, O S = O P, ∴ the body will descend to the center in a straight line P S.

V of descent in a straight line : V in a non-resisting medium of descent in an evanescent parabola :: 1 : $\sqrt{2}$; for V in the spiral = V in the circle at the same distance, V in the parabola = V in the circle at $\frac{1}{2}$ distance.

Hence since time $\propto \frac{1}{V}$,

time of descent in the 1st case : that in 2d :: $\sqrt{2}$: 1.

41. Cor. 5. V in the spiral P Q R = V in the line P S at the same distance. Also

P Q R : P S in a given ratio :: P S : P T :: O P : O S

∴ time of descending P Q R : that of P S : : O P : O S.*

Length of the spiral = T P = sector of the ∠ T P S.

a : b :: b : c :: c : d :: d : e

a + b + c + &c. : b + c + d + &c. : : a : b

∴ a + b + c + &c. : a :: a : $\overline{a - b}$.

42. Cor. 6. If with the center S and any two given radii, two circles be described, the number of revolutions which the body makes between the two circumferences in the different spirals $\propto$ tangent of the angle of the spiral $\propto \frac{P S}{O S}$.

The time of describing the revolution : time down the difference of the radii :: length of the revolution : that difference.

2d $\propto$ 4th,

∴ time $\propto$ length of the revolution $\propto$ secant of the angle of the spiral

$$\propto \frac{O P}{O S}.$$

* p q : p t : : S p : S y

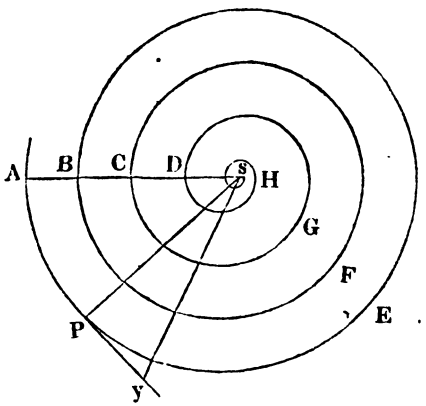
or

$$d w : \frac{p\, d x}{\sqrt{r^2 - p^2}} : : x : p.$$

$$\therefore d w = \frac{x\, d x}{\sqrt{r^2 - p^2}}$$

$$\therefore w = \frac{x^2}{2\sqrt{r^2 - p^2}}.$$

43. Con. 7. Suppose a body to revolve as in the proposition, and to cut



the radius in the points A, B, C, D, the intersections by the nature of the spiral are in continued proportion.

$$\text{Times of revolution} \propto \frac{\text{perimeters described}}{V}$$

$$\text{and velocity} \propto \frac{1}{\sqrt{\text{distance}}}$$

$$\propto A S^{\frac{3}{2}}, B S^{\frac{3}{2}}, C S^{\frac{3}{2}},$$

$\therefore$ the whole time : time of one revolution : : $A S^{\frac{3}{2}} + B S^{\frac{3}{2}} + \&c. : A S^{\frac{3}{2}}$

$$:: A S^{\frac{3}{2}} : A S^{\frac{3}{2}} - B S^{\frac{3}{2}}.$$

44. PROP. XVI. Suppose the centripetal force $\propto \frac{1}{S P^{n+1}}$,

$$\text{time} \propto P Q \times S P^{\frac{n}{2}}$$

$$\text{and velocity} \propto \frac{1}{S P^{\frac{n}{2}}}$$

$$P Q : Q R :: S Q^{\frac{n}{2}} : S P^{\frac{n}{2}}$$

$$Q r : P Q :: S P : S Q$$

$$Q r : Q R :: S Q^{\frac{n}{2}-1} : S P^{\frac{n}{2}-1}$$

$$\therefore Q r : R r :: S Q^{\frac{n}{2}-1} : S Q^{\frac{n}{2}-1} - S P^{\frac{n}{2}-1}$$

$$:: S Q : 1 - \tfrac{1}{2} n . V Q.$$

For

$$S P = S Q + V Q,$$

$$\therefore S P^{\frac{n}{2}-1} = S Q^{\frac{n}{2}-1} + \overline{\frac{n}{2} - 1}.\ V Q \times S Q^{\frac{n}{2}-2} + \&c.$$

$$\therefore S Q^{\frac{n}{2}-1} - S P^{\frac{n}{2}-1} = \overline{1 - \frac{n}{2}} \times V Q \times S Q^{\frac{n}{2}-2}.$$

Then as before it may be proved, if the spiral be given, that the density $\propto \frac{1}{S P^3}$. Q. e. d.

45. Cor. 1.

Resistance : centripetal force : : $\overline{1 - \frac{1}{2}n}$. O S : O P,

for the resistance : centripetal force : : $\frac{1}{2}$ R r : T Q

$$:: \frac{\left(1 - \frac{n}{2}\right) \times V Q \times P Q}{2 S Q} : \frac{P Q^2}{2 S P}$$

$$:: \overline{1 - \frac{n}{2}} \times V Q : P Q$$

$$:: \overline{1 - \frac{n}{2}} \times O S : O P.$$

46. Cor. 2. If $n + 1 = 3$, $1 - \frac{n}{2} = 0$,

$\therefore$ resistance $= 0$.

Cor. 3. If $n + 1$ be greater than 3, the resistance is propelling.

SECTION VI.

47. Prop. XXIV. The distances of any bodies' centers of oscillation from the axis of motion being the same, the quantities of matter $\propto$ weight $\times$ squares of the times of oscillation *in vacuo*.

For the velocity generated $\propto \frac{\text{force} \times \text{time}}{\text{quantities of matter}}$. Force on bodies at equal distances from the lowest points $\propto$ weights, times of describing corresponding parts of the motion $\propto$ whole time of oscillation,

$$\therefore \text{quantities of matter} \propto \frac{\text{force} \times \text{time of oscil.}}{\text{velocities}}$$

$$\propto \text{weights} \times \text{squares of the times},$$

since the velocities generated $\propto \frac{1}{\text{times}}$ for equal spaces.

48. Cor. 1. Hence the times being the same, the quantities of matter $\propto$ weights.

Cor. 2. If the weights be the same, the quantities of matter $\propto$ time^2.

Cor. 3. If the quantities of matter be the same, the weights $\propto \frac{1}{\text{time}^2}$.

49. COR. 4. Generally the accelerating force $\propto \frac{\text{Weights}}{\text{quantities}}$ of matter,

and $L \propto T T^2$,

$\therefore L \propto \frac{W \times T^2}{Q}$,

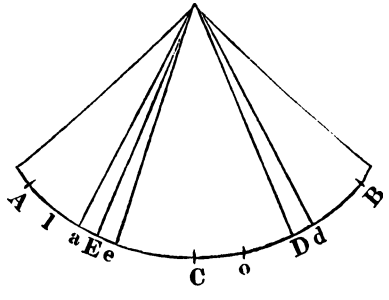
$\therefore Q \propto \frac{W \times T^2}{L}$,

$\therefore$ if W and Q be given $L \propto T^2$.

If T and Q be given $L \propto W$.

50. COR. 5. generally the quantity of matter $\propto \frac{\text{weight} \times \text{time}^2 \text{ of oscillation}}{\text{length}}$.

51 PROP. XXV. Let A B be the arc which a body would describe in a



non-resisting medium in any time. Then the accelerating force at any point D $\propto$ C D; let C D represent it, and since the resistance $\propto$ time, it may be represented by the arc C o.

$\therefore$ the accelerating force in a resisting medium of any body d, $=$ o d.

Take

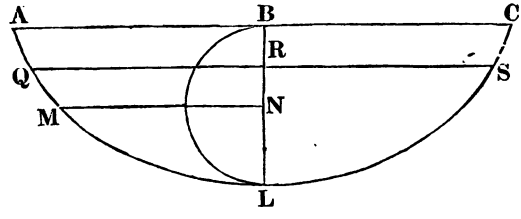
o d : C D :: o B : C B.

Therefore at the beginning of motion, the accelerating force will be in this ratio, $\therefore$ the initial velocities and spaces described will be in the same ratio, $\therefore$ the spaces to be described will also be in the same ratio, and vanish together, $\therefore$ the bodies will arrive at the same time at the points C and o.

In the same way when the bodies ascend, it may be proved that they will arrive at their highest points at the same time. $\therefore$ If A B : a B in the ratio C B : o B, the oscillations in a non-resisting and resisting medium will be isochronous. Q. e. d.

Cor. The greatest velocity in a resisting medium is at the point o.

The expression for the $\frac{1}{2}$ time of an oscillation in vacuo, or time of descent down to the lowest point $\propto$ quadrant whose radius = 1. Now



suppose the body to move in a resisting medium when the resistance : force of gravity : : r : 1.

Then $v\,dv = -g\,F\,dx + g\,r\,dz = -g\,d^2x + g\,r\,dz$. Now by a property of the cycloid, if $\frac{a}{2}$ be the axis, $dx : dz :: x : \frac{z}{2} :: z : a$,

$$\therefore dx = \frac{z\,dz}{a},$$

$$\therefore v\,dv = -\frac{g}{a} \times z\,dz + g\,r\,dz - \frac{v^2}{2}$$

$$= -\frac{g}{a} \times z^2 + g\,r\,z,$$

$$\therefore v^2 = \frac{-g}{a} \times z^2 + 2\,g\,r\,z + C.$$

Now

$$z = d,\ v = o,$$

$$\therefore v^2 = \frac{g}{a} \times \overline{d^2 - z^2} - 2\,g\,r \times \overline{d - z}$$

$$= \frac{g}{a} \times \overline{d^2 - 4\,a\,r\,d + 2\,a\,d\,r\,z - z^2},$$

$$\therefore v = \int \frac{g}{a} \times \sqrt{d^2 - 2\,a\,r\,d + 2\,a\,r\,z - z^2},$$

$$\therefore dt = \frac{-dz}{v} = \int \frac{a}{g} \times \frac{-dz}{\sqrt{d^2 - 2\,a\,r\,d + 2\,a\,r\,z - z^2}}.$$

Assume

$$z - a\,r = y,$$

$$\therefore z^2 - 2\,a\,r\,z + a^2 r^2 = y^2,$$

$$\therefore 2\,a\,r\,z - z^2 = a^2 v^2 - y^2,$$

$$d^2 - 2\,a\,r\,d + 2\,a\,r\,z - z^2 = (d - a\,r)^2 - y^2 = (b^2 - y^2.)$$

and

$$d z = d y$$

$$\therefore d t = \int \frac{a}{g} \times \frac{-dy}{\sqrt{b^2 - y^2}}$$

$$\therefore t = \int \frac{a}{g} \times \text{circular arc, radius} = 1,$$

and

$$\text{cos.} = \frac{z - a r}{d - a r} + C \text{ and } C = o.$$

$\therefore$ the whole time of descent to the lowest point $= \int \frac{a}{g} \times$ circular arc whose cos. $= \frac{-a r}{d - a r}$, $\therefore$ time in vacuo : time in resisting medium :: quadrant : arc whose cos. $= \frac{-a r}{d - a r}$.

COR. 1. Time of descent to the point of greatest acceleration is constant, for in that case $z = a r$,

$$\therefore t = \int \frac{a}{g} \times \text{quadrant, for } d v = 0,$$

$$\therefore v d v = 0,$$

$$\therefore -g z d z + g a r z = 0,$$

$$\therefore z = a r,$$

$$\therefore z : r :: a : 1.$$

COR. 2. To find the excess of arc in descent above that in ascent.

$$v d v = + g T d x + g r d z,$$

$$\therefore v d v = -\frac{g z d z}{a} - g r d z$$

$$\therefore \frac{v^2}{2} = -\frac{m z^2}{a} - g r z + C,$$

$$\therefore v^2 = \frac{g}{a} (d^2 - z^2) - (z - d) \times 2 a r$$

$$= \frac{g}{a} \times (d^2 - 2 a r d) - (2 a r z - z^2)$$

which when the body arrives to the highest point $= 0$,

$$d^2 - 2 a r d - 2 a r z - z^2 = 0,$$

$$\therefore d^2 - 2 a r d = z^2 + 2 a r z,$$

$$\therefore z + a r = d - a r,$$

$$\therefore z = d - 2 a r,$$

$$\therefore d - z = 2 a r.$$

52. Prop. XXVI. Since V $\propto$ arc, and resistance $\propto$ V, resistance $\propto$ arc.

$\therefore$ Accelerating force in the resisting medium $\propto$ arcs.

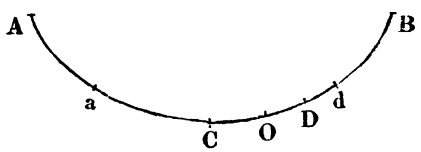
Also the increments or decrements of V $\propto$ accelerating force.

$\therefore$ the V will always $\propto$ arc.

But in the beginning of the motion, the forces which ∞ arcs will generate velocities which are proportional to the arcs to be described. $\therefore$ the velocities will always ∞ arcs to be described.

$\therefore$ the times of oscillation will be constant.

53. Prop. XXVIII. Let C B be the arc described in the descent, C a in the ascent.



$\therefore$ A a = the difference (if A C = C B)

Force of gravity at D : resistance : : C D : C O.

C A = C B

O a = O B

$\therefore$ C A — O a or A a — C O = C B — O B = C O

$\therefore$ C O = $\frac{1}{2}$ A a

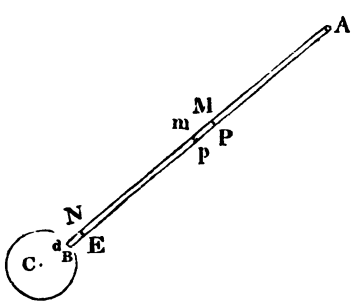
$\therefore$ Force of gravity at D : resistance : : C D : $\frac{1}{2}$ A a

$\therefore$ At the beginning of the motion,

Force of gravity : resistance : : 2 C B : A a

: : 2 length of pendulum : A a.

54. Prob. To find the resistance on a thread of a sensible thickness.



Resistance ∞ $V^2 \times D^2$ of suspended globe.

$\therefore$ resistance on the whole thread : resistance on the globe C : :

$:: 2a^3b^2.(a-b)^2 : a^3r^2c^2 - r^2c^2.(a-2b)^3$, $c = a + r$.

$:: a^3b^2.(a-b)^2 : 3a^2r^2c^2b - bab^2r^2c^2 + 4b^3r^2c^2$,

$:: a^3b.(a-b)^2 : 3a^2r^2c^2 - babr^2c^2 + 4b^2r^2c^2$,

$\therefore$ resistance on the thread : whole resistance

$:: a^3b.(a-b)^2 : r^2c^2.(3a^2 - bab + 4b^2)$.

COR. If the thickness (b) be small when compared with the length (a)

$3a^2 - bab + 4b^2 = 3a^2 - bab + 3b^2$ (nearly) $= 3.(a-b)^2$.

$\therefore$ Resistance on the whole thread : resistance on the globe

$:: a^3b : 3r^2c^2$

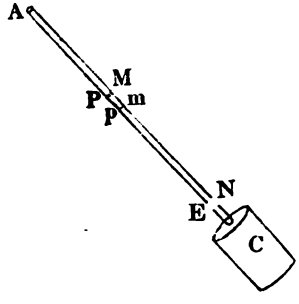
and

Resistance on the thread : whole resistance to the pendulum

$:: a^3b : a^3b + 3r^2c^2$.

Suppose, instead of a globe, a cylinder be suspended whose ax. $= 2r$. Now by differentials

the resistance on the circumference : resistance on the base : : 2 : 3.



By composition the resistance to the cylinder : resistance on the square $= 2r :: 2 : 3$.

Resistance $\propto x^2x'$,

$\therefore$ resistance $\propto x^3$,

$\therefore$ resistance to the whole thread $\propto x^3$.

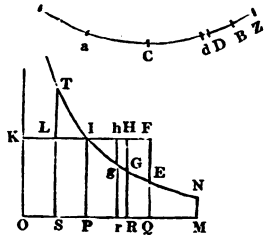
Resistance on A E $\propto (a - 2b)^3$ if $2b = E D$.

$\therefore$ Resistance on the thread : resistance of the globe

$:: 16 . a^3b^2.(a-b)^2 : 3p.\overline{a^3 - (a-2b)^3} \times r^2.(a+1)^2$.

55. PROP. XXIX. B a is the whole arc of oscillation. In the line O Q take four points S, P, Q, R, so that if O K, S T, P I, Q E be erected

perpendiculars to O Q meeting a rectangular hyperbola between the asymptotes O Q, O K in T, I, G, E, and through I, K F be drawn



parallel to O Q, meeting Q E produced in F. The area P I E Q may be : area P I S T : : C B : C a. Also I E F : I L T : : O R : O S.

Draw M N perpendicular to O Q meeting the hyperbola in N, so that P L M N may be proportional to C Z, and P I G R to C D.

Then the resistance : gravity : : $\frac{O R}{O Q} \times T E F - I G H : P I N M.$

Now since the force $\propto$ distance, the arcs and forces are as the hyperbolic areas. $\therefore$ D d is proportional to R r G g.

Now by taking the differentials the increment of $\left(\frac{O R}{O Q} T E F - I G H\right)$ $= G H g h - \frac{R r . I E F}{O Q} : R r \times G R :: H G - \frac{I E F}{O Q} : G R :: O R \times H G - I E Z \times \frac{O R}{O Q} : O P \times P I - (O R \times H G = O R \times H R - O P \times P I = P I H R = P I R G + I G H) : : P I R G + I G H - \frac{O R}{O Q} \times I E F : O P \times P I.$

Now if $Y = \frac{O R}{O Q} \times I E F - I G H$, the increment $Y \propto P I G R - Y.$

Let V = the whole from gravity. $\therefore$ V $-$ R = actual accelerating force. $\therefore$ Increment of the velocity $\propto \overline{V - R} \times$ increment of the time. As the resistance $\propto V^2$ the increment of resistance $\propto$ V $\times$ increment of the velocity, and the velocity $\propto \frac{\text{increment of the space}}{\text{increment of the time}}$. $\therefore$ Increment of resistance $\propto$ V $-$ R if the space be given, $\propto$ P I G R $-$ Z, if Z be the area which represents the resistance R e.

Since the increment Y $\propto$ P I G R $-$ Y, and the increment of Z

$\propto$ P I G R — Z. If Y and Z be equal at the beginning of the motion and begin at the same time by the addition of equal increments, they will still remain equal, and vanish at the same time.

Now both Z and Y begin and end when resistance = 0, i. e. when

$$\frac{O\,R}{O\,Q}.\,I\,E\,F - I\,G\,H = 0$$

or

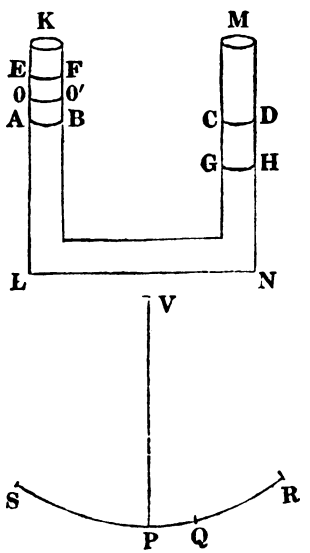
$$\frac{I\,L\,T}{O\,S} \times O\,R - I\,G\,H = 0.$$

$$\therefore \frac{O\,R \times I\,E\,F}{O\,Q} - I\,G\,H = Z$$

$$\therefore \text{Resistance : gravity :: } \frac{O\,R}{O\,Q}.\,I\,E\,F - I\,G\,H : P\,M\,N\,I.$$

SECTION VIII.

56. PROP. XLIV. The friction not being considered, suppose the mean



altitude of the water in the two arms of the vessel to be A B, C D. Then when the water in the arm K L has ascended to E F, the water in the arm M N will descend to G H, and the moving force of the water equals the excess of the water in one arm above the water in the other, equals twice A E F B. Let V P be a pendulum, R S a cycloid = ½ length of the canal, and P Q = A E. The accelerating force of the water : whole weight :: A E or P Q : P R.

Also, the accelerating force of P through the arc P Q : whole weight of P : : P Q : P R; therefore the accelerating force of the water and P $\propto$ the weights. Therefore if P equal the weight of the water in the canal, the vibration of the water in the canal will be similar and cotemporaneous with the oscillations of P in the cycloid.

Cor. 1. Hence the vibrations of the water are isochronous.

Cor. 2. If the length of the canal equal twice the length of the pendulum which oscillates in seconds; the vibrations will also be performed in seconds.

Cor. 3. The time of a vibration will $\propto \sqrt{L}$.

Let the length = L, A E = a,

then the accelerating force : whole weight : : 2 a : L,

$$\therefore \text{accelerating force} = \frac{2a}{L};$$

$$\therefore \text{when the surface is at 0, the accelerating force} = \frac{2 A 0}{L}.$$

Put E 0 = x,

A 0 = a — x,

$$\therefore \text{accelerating force} = \frac{2a - 2x}{L},$$

$$\therefore v\,dv = \frac{g.\overline{2a\,dx - 2x\,dx}}{L},$$

$$\therefore v^2 = \frac{2g}{L} \times \overline{2ax - x^2},$$

$$\therefore v = \sqrt{\frac{2g}{L}} \times \sqrt{2ax - x^2}$$

$$dt = \frac{dx}{v} = \sqrt{\frac{L}{2ga^2}} \times \frac{a\,dx}{\sqrt{2ax - x^2}},$$

$$\therefore t = \sqrt{\frac{L}{2ga^2}} \times \text{cir. arc rad.} = a, \text{ and vers.} = x$$

+ corn. and corn. = 0,

$\because t = 0,\ x = 0,$

$\therefore$ if p = 3. 14159, &c.

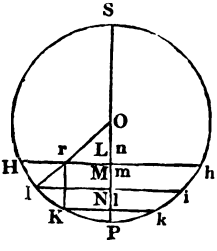
$$t = \sqrt{\frac{L}{2ga}} \times \frac{pa}{2} \{\text{when } (x) = (a)\} = \sqrt{\frac{L}{2g}} \times \frac{p}{2}$$

$\therefore$ time of one entire vibration $= p \times \sqrt{\frac{L}{2g}}$ = time of one entire vibration of a pendulum whose length $= \frac{L}{2}$.

57. COR. 1. Since the distance (a) above the quiescent surface does not enter into the expression. The time will be the same, whatever be the value of A E.

58. COR. 2. The greatest velocity is at A $= \sqrt{\frac{2g}{L}} \times a, \propto \frac{L}{L^{\frac{1}{2}}} \propto \sqrt{\frac{a^2}{I}}$
$\propto \sqrt{\frac{A E^2}{L}}$.

59. PROP. XLVII. Let E, F, G be three physical points in the line B C, which are equally distant; E e, F f, G g the spaces through which they move during the time of one vibration. Let $\varepsilon, \varphi, \gamma$ be their place at any time. Make P S = E e, and bisect it in O, and with center O and radius O P = O S, describe a circle. Let the circumference of this circle represent the time of one vibration, so that in the time P H or P H S h, if H L or h l be drawn perpendicular to P S and E ε be taken = P L or P l, E ε may be found in E; suppose this the nature of the medium. Take in the circumference P H S h, the arcs H I, I K, h i, i k which may bear the same ratio to the circumference of the circle as E F or F G to B C. Draw I M, K N or i m, k n perpendicular to P S. Hence P I, or P H S i will represent the motion of F. and P K or P H S k that of G. E ε, F φ, G γ = P L, P M, P N or P l, P m, P n respectively.



Hence $\varepsilon\gamma$ or E G + G γ — E ε = G E — L N = expansion at $\varepsilon\gamma$; or = E G + l n.

∴ in going, expansion : mean expansion : : G E — L N : E G
In returning, ——— : ——— : : E G + l n : E G

Now join I O, and draw K r perpendicular to H L, H K r, I O M are similar triangles, since the ∠ K H r = ½ K O k = ½ I O i = ∠ I O P and ∠ at r and M = 90°,
∴ L N : K H : : I M : I O or O P, and by supposition K H : E G : : circumference P S L P : B C : : O P : V = radius of the circle whose circumference = B C.
∴ by composition L N : G E : : I M : V.
∴ expansion : mean expansion : : V — I M : V,

$\therefore$ elasticity : mean elasticity : : $\frac{1}{V - IM} : \frac{1}{V}$. In the same way, for the points E and G, the ratio will be $\frac{1}{V - HL} : \frac{1}{V} \propto \frac{1}{V - KN} : \frac{1}{V}$

: : excess of elasticity of E : mean elasticity

$$:: \frac{HL - KN}{V^2 - HL \times V - KN \times V + HL \times KN} : \frac{1}{V}$$

: : H L — K N : V.

Now

$$V \propto 1.$$

$\therefore$ the excess of E's elasticity $\propto$ H L — K N, and since H L — K N = H r : H K : : O M : O P,

$\therefore$ H L — K N $\propto$ O M,

$\therefore$ excess of E's elasticity $\propto$ O M.

Since E and G exert themselves in opposite directions by the arc's tendency to dilate, this excess is the accelerating force of $\varepsilon\gamma$, $\therefore$ accelerating force $\propto$ O M.*

ON THE HARMONIC CURVE.

Since the ordinates in the harmonic curve drawn perpendicular to the axis are in a constant ratio, the subtenses of the angle of contact will be in the same given ratio. Now the subtenses $\propto \frac{\text{arcs}}{\text{rad. of curv.}}$, and when the curve performs very small vibrations, the arcs are nearly equal.

Now the curv. $\propto \frac{1}{\text{rad.}}$, $\therefore$ subtense $\propto$ curvature.

Hence the accelerating force on any point of the string $\propto$ curvature at that point.

---

* Now bisect F f in Ω,

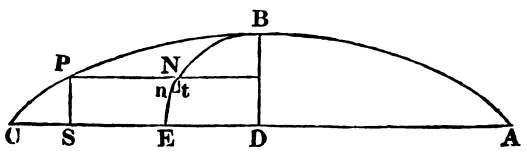
$\therefore$ O M = $\Omega\varphi$

For

O M = O P — P M = Ω F — F φ = $\Omega\varphi$

i. e. the accelerating force $\propto$ distance from Ω the middle point. Q. e. d.

To find the equation to the harmonic curve.



Let A C be the axis of the harmonic curve C B A, D the middle point, draw B D perpendicular cutting the curve in B; draw P M perpendicular to B D cutting the curve in P, and cutting the quadrant described with the center D and radius D B in N. Draw P S perpendicular to A C.

Put

$$B\,D = a,\ P\,M = y,\ B\,M = x,$$
$$\therefore D\,M = a - x = P\,S.$$
$$r = \text{rad. of curv. at B},\ B\,P = z,$$
$$\therefore \text{rad. of curv.} = \frac{-\,d\,z\,d\,x}{d^2 y} \text{ (if d z be constant).}$$

Now

$$B\,D : P\,S :: \text{curvature at B} : \text{curvature at P}$$
$$:: \text{rad. of cur. at P} : \text{rad. at B}$$

or

$$a : \overline{a - x} :: \frac{d\,z\,d\,x}{d^2 y} : r,$$
$$\therefore r\,a\,d^2 y + a\,d\,z\,d\,x - x\,d\,x\,d\,z = 0,$$
$$\therefore r\,a\,d\,y + a\,d\,z\,x - \frac{x^2 d\,z}{2} = 0 + C.$$

Now

$$x = 0,\ d\,y = d\,x,$$
$$\therefore r\,a\,d\,z = 0 + C = C,$$
$$\therefore r\,a\,d\,y + a\,x\,d\,z - \frac{x^2 d\,z}{2} = r\,a\,d\,z.$$

Put

$$a\,x - \frac{x^2}{2} = b^2,$$
$$\therefore r\,a\,d\,y = \overline{r\,a - b^2}\,d\,z,$$
$$\therefore r^2 a^2 d y^2 = (r\,a - b^2)^2 \times d\,x^2 + r^2 a^2 d y^2 - 2\,r\,a\,b^2 d y^2 + b^4 d y^2,$$

$$\therefore (ra - b^2)^2 \times dx^2 = 2rab^2dy^2 - b^4dy^2,$$

$$\therefore r^2a^2dx^2 = 2rab^2dy^2$$

if (b) be small compared to (a),

$$\therefore dy^2 = \frac{r\,a\,dx^2}{2b^2},$$

$$\therefore dy = \frac{\sqrt{ra} \times dx}{\sqrt{2ax - x^2}} = \sqrt{\frac{r}{a}} \times \frac{a\,dx}{\sqrt{2ax - x^2}},$$

$$\therefore y = \sqrt{\frac{r}{a}} \times \text{circular arc whose rad.} = a, \text{ and vers.} = x + C, \text{ and cor}^n. = 0,$$

because when $y = 0$, $x = 0$,

$$\therefore \text{arc} = 0.$$

$$\therefore CD = \sqrt{\frac{r}{a}} \times \text{quadrant BNE},$$

and therefore

$$\sqrt{\frac{r}{a}} = \frac{CD}{BNE},$$

$$\therefore y = \frac{BN \times \frac{1}{2}Z}{BNE}.$$

60. Prop. XLIX. Put A = attraction of a homogeneous atmosphere when the weight and density equal the weight and density of the medium through which the physical line E G is supposed to vibrate. Then every thing remaining as in Prop. XLVII. the vibration of the line E G will be performed in the same times as the vibrations in a cycloid, whose length = P S, since in each case they would move according to the same law, and through the same space. Also, if A be the length of a pendulum, since $T \propto \sqrt{L}$

The time of a vibration : time of oscillation of a pendulum A

$$:: \sqrt{PO} : \sqrt{A}.$$

Also (Prop. XLVII.), the accelerating force of E G in medium : accelerating force in cycloid

$$:: A \times HK : V \times EG;$$

since $HK : GE :: PO : V.$

$$:: PO \times A : V^2.$$

Now

$$T \propto \sqrt{\frac{1}{T.}} \text{ when L is given.}$$

$\therefore$ the time of vibration : time of oscillation of the pendulum A

:: V : A

:: B C : circumference of a circle rad. = A.

Now B C = space described in the time of one vibration, therefore the circumference of the circle of radius A = space described in the time of the oscillation of a pendulum whose length = A.

Since the time of vibration : time of describing a space = circumference of the circle whose rad. = A :: B C : that circumference.

COR. 1. The velocity equals that acquired down half the altitude of A. For in the same time, with this velocity uniform, the body would describe A; and since the time down half A : time of an oscillation :: r : circumference. In the time of an oscillation the body would describe the circumference.

COR. 2. Since the comparative force or weight $\propto$ density $\times$ attraction of a homogeneous atmosphere, $A \propto \frac{\text{elastic force}}{\text{density}}$, and the velocity $\propto \sqrt{A}$. $\propto \frac{\sqrt{\text{elastic force}}}{\sqrt{\text{density}}}$.

SCHOLIUM.

61. PROP. XLIX. Sound is produced by the pulses of air, which theory is confirmed, 1st, from the vibrations of solid bodies opposed to it. 2d. from the coincidence of theory with experiment, with respect to the velocity of sound.

The specific gravity of air : that of mercury :: 1 : 11890.

Now since the alt. $\propto \frac{1}{\text{sp. gr.}}$, $\therefore$ 1 : 11890 :: 30 inches : 29725 feet = altitude of the homogeneous atmosphere. Hence a pendulum whose length = 29725, will perform an oscillation in 190″, in which time by Prop. XLIX, sound will move over 186768 feet, therefore in 1″ sound will describe 979 feet. This computation does not take into consideration the solidity of the particles of air, through which sound is propagated instantly. Now suppose the particles of air to have the same density as the particles of water, then the diameter of each particle : dis-

tance between their centers : : 1 : 9, or 1 : 10 nearly. (For if there are two cubes of air and water equal to each other, D the diameter of the particles, S the interval between them, S + D = the side of the cube, and if N = N°. N S + N D = N°. in the side of the cube, N°. in the cube $\propto N^3$. Also, if M be the N°. in the cube of water, M D the side of the cube and the N°. in the cube $\propto M^3$.

$$\text{Put } 1 : A :: N^3 : M^3,$$

$$\therefore M = A^{\frac{1}{3}} N,$$

By Proposition

$$N S + N D = M D = N A^{\frac{1}{3}} D,$$

$$\therefore S = D \times \overline{A^{\frac{1}{3}} - 1},$$

$$\therefore S : D :: A^{\frac{1}{3}} - 1 : 1,$$

$$\therefore S + D : D :: A^{\frac{1}{3}} : 1 :: 9 : 1 \text{ if } A = 870$$

or 10 : 1 if A = 1000).

Now the space described by sound : space which the air occupies : : 9 : 11, $\therefore$ space to be added $= \frac{979}{9} = 108$ or the velocity of sound is 1088 feet per 1″.

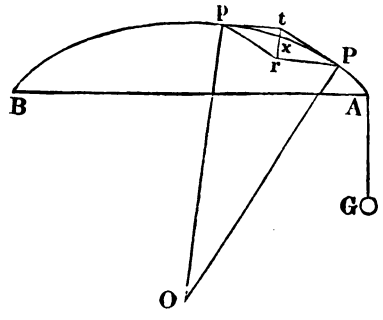
Again, also the elasticity of air is increased by vapours. Hence since the velocity $\propto \frac{\sqrt{\text{elasticity}}}{\sqrt{\text{density}}}$; if the density remain the same the velocity $\propto \sqrt{\text{elasticity}}$. Hence if the air be supposed to consist of 11 feet, 10 of air, and 1 of vapour, the elasticity will be increased in the ratio of 11 : 10, therefore the velocity will be increased in the ratio of $11\frac{1}{2} : 10\frac{1}{2}$ or 21 : 20, therefore the velocity of sound will altogether be 1142 feet per 1″, which is the same as found by experiment.

In summer the air being more elastic than in winter, sound will be propagated with a greater velocity than in winter. The above calculation relates to the mean elasticity of the air which is in spring and autumn. Hence may be found the intervals of pulses of the air.

By experiment, a tube whose length is five Paris feet, was observed to give the same sound as a chord which vibrated 100 times in 1″, and in the same time sound moves through 1070 feet, therefore the interval of the pulses of air = 10.7 or about twice the length of the pipe.

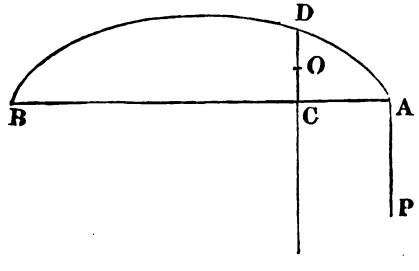
62. On the vibrations of a harmonic string.

The force with which a string tends to the center of the curve : force which stretches the string :: length : radius of curvature. Let P p be a



small portion of the string, O the center of the curve; join O P, O p, and draw P t, p t, tangents at P and p meeting in t, complete the parallelogram P t p r. Join t r, then P t, p t represent the stretching force of the string, which may be resolved into P x, t x and p x, t x of which P x, p x destroy each other, and 2 t x = force with which the string tends to the center O. Now the $\angle$ t P r = $\frac{1}{2}\angle$ P O p, $\therefore \angle$ t P x = $\angle$ P O p, $\therefore$ t r : P t : : P p : O P, i. e. the force with which any particle moves towards the center of the curve : force which stretches it : : length : radius.

63. To find the times of vibration of a harmonic string.



Let w = weight of the string. L = length.

D d : L : : weight D d : w

$\therefore$ weight of D d = $\frac{\text{D d} \times \text{w}}{\text{L}}$

Also

$$D\,d : \frac{L^2}{p^2 a^2} = \text{rad. of curve} : : \text{the moving force of } D\,d : P$$

$$\therefore \text{the moving force of } D\,d = \frac{P \times D\,d \times a\,p^2}{L\,w}$$

$$\therefore \text{accelerating force} = \frac{P \times D\,d \times a\,p^2}{L^2} \times \frac{L}{D\,d \times w}$$

$$= \frac{P \times a\,p^2}{L\,w.}$$

if $D\,O = x$, $D\,C = a$, $O\,C = a - x$,

$$\therefore \text{the accelerating force at } O = \frac{P\,p^2 . \overline{a - x}}{L\,w}$$

$$\therefore v\,d\,s = \frac{g\,.\,P\,p}{L\,w} \times \overline{a\,d\,x - x\,d\,x}$$

$$\therefore v^2 = \frac{g\,P\,p^2}{L\,w} \times \overline{2\,a\,x - z^2}$$

$$\therefore v = \sqrt{\frac{g\,P\,p^2}{L\,w}} \times \sqrt{2\,a\,x - x^2}.$$

$\therefore C$ and $l = 0$,

$$\therefore d\,t = \frac{d\,x}{v} = \sqrt{\frac{L\,w}{g\,P\,p^2}} \times \frac{d\,x}{\sqrt{2\,a\,x - x}}$$

$$\therefore t = \sqrt{\frac{L\,w}{g\,P\,p^2}} \times \text{cir. arc rad.} = l$$

and

$$\text{vers. sine} = \frac{x}{a},$$

when $x = a$,

$t = 0$.

$$\therefore \sqrt{\frac{L\,w}{g\,P\,p^2}} \times \text{quadrant} = \sqrt{\frac{L\,w}{g\,P\,p^2}} \times \frac{P}{2}$$

$$= \tfrac{1}{2} \times \sqrt{\frac{L\,w}{g\,P^2}}$$

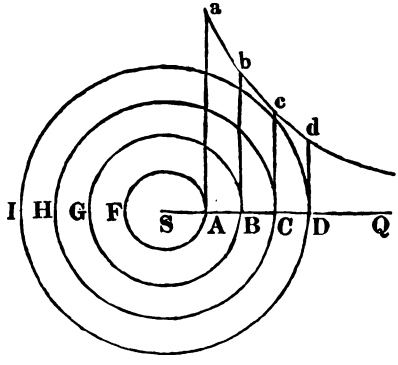
$$\therefore \text{time of a vibration} = \sqrt{\frac{L\,w}{g\,P}}\ l''$$

$$\therefore \text{number of vibrations in } 1'' = \sqrt{\frac{g\,P}{L\,w}}.$$

Cor. Time of vibration = time of the oscillation of a pendulum whose length $= \frac{L\,w}{P\,p^2}$.

For this time $= \sqrt{\frac{L\,w}{g\,P}}$.

64. PROP. LI. Let A F be a cylinder moving in a fluid round a fixed axis in S, and suppose the fluid divided into a great number of solid



orbs of the same thickness. Then the disturbing force $\propto$ translation of parts $\times$ surfaces. Now the disturbing forces are constant. $\therefore$ Translation of parts, from the defect of lubricity $\propto \frac{1}{\text{distance}}$. Now the difference of the angular motions $\propto \frac{\text{translation}}{\text{distance}} \propto \frac{1}{\text{distance}^2}$. On A Q draw A a, B b, C c, &c. : : $\frac{1}{\text{distance}^2}$, then the sum of the differences will $\propto$ hyperbolic area.

$\therefore$ periodic time $\propto \frac{1}{\text{angular motion}} \propto \frac{1}{\text{hyperbolic area}} \propto$ distance.

In the same way, if they were globes or spheres, the periodic time would vary as the distance$^{\frac{3}{2}}$.

END OF THE FIRST VOLUME.

A

COMMENTARY

ON

NEWTON'S PRINCIPIA.

WITH

A SUPPLEMENTARY VOLUME.

DESIGNED FOR THE USE OF STUDENTS AT THE UNIVERSITIES.

BY

J. M. F. WRIGHT, A. B.

LATE SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE, AUTHOR OF SOLUTIONS OF THE CAMBRIDGE PROBLEMS, &c. &c.

IN TWO VOLUMES.

VOL. II.

LONDON:
PRINTED FOR T. T. & J. TEGG, 73, CHEAPSIDE;
AND RICHARD GRIFFIN & CO., GLASGOW.

MDCCCXXXIII.

GLASGOW:
GEORGE BROOKMAN, PRINTER, VILLAFIELD.

INTRODUCTION

TO

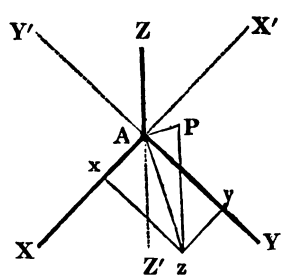
VOLUME II.

AND TO THE

MECANIQUE CELESTE.

ANALYTICAL GEOMETRY

1. *To determine the position of a point in fixed space.*
Assume any point A in fixed space as known and immoveable, and let



three fixed planes of indefinite extent, be taken at right angles to one another and passing through A. Then shall their intersections A X′, A Y′, A Z′ pass through A and be at right angles to one another.

This being premised, let P be any point in fixed space; from P draw P z parallel to A Z, and from z where it meets the plane X A Y, draw z x, z y parallel to A Y, A X respectively. Make

$$A\,x = x,\quad A\,y = y,\quad P\,z = z.$$

Then it is evident that if x, y, z are given, the point P can be found *practically* by taking A x = x, A y = y, drawing x z, y z parallel to A Y, A X; lastly, from their intersection, making z P parallel to A Z and equal to z. Hence x, y, z determine the position of the point P.

The lines x, y, z are called the rectangular coordinates of the point P; the point A the origin of coordinates; the lines A X, A Y, A Z the axes of coordinates, A X being further designated the axis of x, A Y the axis of y, and A Z the axis of z; and the planes X A Y, Z A X, Z A Y coordinate planes.

These coordinate planes are respectively denoted by

plane (x, y), plane (x, z), plane (y, z);

and in like manner, any point whose coordinates are x, y, z is denoted briefly by

point (x, y, z).

If the coordinates x, y, z when measured along A X, A Y, A Z be always considered positive; when measured in the opposite directions, viz. along A X′ A Y′, A Z′, they must be taken negatively. Thus accordingly as P is in the spaces

$$Z\,A\,X\,Y,\ Z\,A\,Y\,X',\ Z\,A\,X'\,Y',\ Z\,A\,Y'\,X;$$
$$Z'\,A\,X\,Y,\ Z'\,A\,Y\,X',\ Z'\,A\,X'\,Y',\ Z'\,A\,Y'\,X,$$

the point P will be denoted by

point (x, y, z), point (— x, y, z), point (— x, — y, z), point (x, — y, z);
point (x, y, – z), point (– x, y, – z), point (– x, – y, – z), point (x, – y, – z)

respectively.

2. *Given the position of two points* (α, β, γ), $(\alpha', \beta', \gamma')$ *in fixed space, to find the distance between them.*

The distance P P′ is evidently the diagonal of a rectangular parallelopiped whose three edges are parallel to A X, A Y, A Z and equal to

$$\alpha \sim \alpha',\ \beta \sim \beta',\ \gamma \sim \gamma'.$$

Hence

$$P\,P' = \sqrt{(\alpha - \alpha')^2 + (\beta - \beta')^2 + (\gamma - \gamma')^2} \quad \ldots \quad (1)$$

the distance required.

Hence if P′ coincides with A or α', β', γ' equal zero,

$$P\,A = \sqrt{\alpha^2 + \beta^2 + \gamma^2} \quad \ldots \quad (2)$$

3. Calling the distance of any point P (x, y, z) from the origin A of coordinates the *radius-vector*, and denoting it by ϱ, suppose it inclined to the axes A X, A Y, A Z or to the planes (y, z), (x, z), (x, y), by the angles X, Y, Z.

Then it is easily seen that

$$x = \varrho \cos. X,\ y = \varrho \cos. Y,\ z = \varrho \cos. Z \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence (see 2)

$$\cos. X = \frac{x}{\sqrt{(x^2 + y^2 + z^2)}},\ \cos. Y = \frac{y}{\sqrt{(x^2 + y^2 + z^2)}},$$

$$\cos. Z = \frac{z}{\sqrt{(x^2 + y^2 + z^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

so that *when the coordinates of a point are given, the angles which the radius-vector makes with each of the axes may hence be found.*

Again, adding together the squares of equations (3), we have

$$(x^2 + y^2 + z^2) = \varrho^2 (\cos.^2 X + \cos.^2 Y + \cos.^2 Z).$$

But

$$\varrho^2 = x^2 + y^2 + z^2 \text{ (see 2)},$$

$$\therefore \cos.^2 X + \cos.^2 Y + \cos.^2 Z = 1 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which shows that when two of these angles are given the other may be found.

4. *Given two points in space,* viz. (α, β, γ), $(\alpha', \beta', \gamma')$, *and one of the coordinates of any other point* (x, y, z) *in the straight line that passes through them, to determine this other point;* that is, *required the equations to a straight line given in space.*

The distances of the point (α, β, γ) from the points $(\alpha', \beta', \gamma')$, and (x, y, z) are respectively, (see 2)

$$P P' = \sqrt{(\alpha - \alpha')^2 + (\beta - \beta')^2 + (\gamma - \gamma')^2},$$

and

$$P Q = \sqrt{(\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2}.$$

But from similar triangles, we get

$$(\gamma - z)^2 : (P Q)^2 :: (\gamma - \gamma')^2 : (P P')^2$$

whence

$$(\gamma - z)^2 = \frac{(\gamma - \gamma')^2}{(\alpha - \alpha')^2 + (\beta - \beta')^2 + (\gamma - \gamma')^2} \cdot \{(\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2\}$$

which gives

$$\{(\alpha - \alpha')^2 + (\beta - \beta')^2\} (\gamma - z)^2 = (\gamma - \gamma')^2 \cdot \{(\alpha - x)^2 + (\beta - y)^2\}$$

But since α, α' are independent of β, β' and vice versa, the two first terms of the equation,

$$(\alpha - \alpha')^2 \cdot (\gamma - z)^2 - (\gamma - \gamma')^2 (\alpha - x)^2 - (\gamma - \gamma')^2 (\beta - y)^2 + (\beta - \beta')^2 (\gamma - z)^2 = 0$$

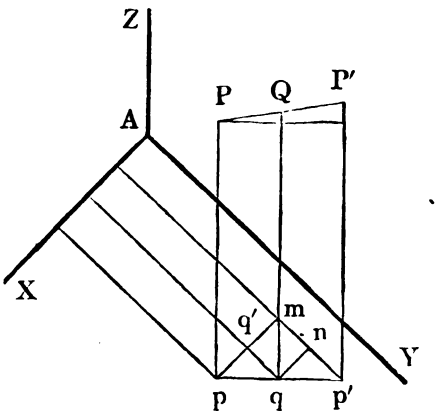
are essentially different from the last. Consequently by (6 vol. 1.)

$$(\alpha - \alpha')^2 (\gamma - z)^2 = (\gamma - \gamma')^2 (\alpha - x)^2$$
$$(\beta - \beta')^2 (\gamma - z)^2 = (\gamma - \gamma')^2 (\beta - y)^2$$

which give

$$\left.\begin{aligned} z - \gamma &= \pm \frac{\gamma - \gamma'}{\alpha - \alpha'} (\alpha - x') \\ z - \gamma &= + \frac{\gamma - \gamma'}{\beta - \beta'} (\beta - y) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6)$$

These results may be otherwise obtained; thus, p g p′,is the projection of the given line on the plane (x, y) &c. as in fig.



Hence

$$\begin{aligned} z - \gamma : \gamma' - \gamma &:: p\,q : p\,p' \\ &:: m\,n : m\,p' \\ &:: y - \beta : \beta' - \beta \end{aligned}$$

Also

$$\begin{aligned} z - \gamma : \gamma' - \gamma &:: p\,q : p\,p' :: p\,r : p\,m \\ &:: \alpha - x : \alpha - \alpha'. \end{aligned}$$

Hence the general forms of the equations to a straight line given in space, not considering signs, are

$$\left.\begin{aligned} z &= a\,x + b \\ z &= a'\,y + b' \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7)$$

To find where the straight line meets the planes, (x, y), (x, z), (y, z), we make

$$z = 0,\ y = 0,\ x = 0,$$

which give

$$x = -\frac{b}{a}$$

$$y = -\frac{b}{a'}$$
$$z = b'$$
$$x = \frac{b' - b}{a}$$
$$z = b$$
$$y = \frac{b - b'}{a'}$$

which determine the points required.

To find when the straight line is parallel to the planes, (x, y), (x, z), (y, z), we must make z, y, x, respectively constant, and the equations become of the form

$$\left.\begin{aligned} z &= c \\ a'y &= ax + b - b' \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8)$$

To find when the straight line is perpendicular to the planes, (x, y), (x, z) (y, z), or parallel to the axes of z, y, x, we must assume x, y; x, z; y, z; respectively constant, and z, y, x, will be any whatever.

To find the equations to a straight line passing through the origin of coordinates; we have, since $x = 0$, and $y = 0$, when $z = 0$,

$$\left.\begin{aligned} z &= ax \\ z &= a'y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (9)$$

5. *To find the conditions that two straight lines in fixed space may intersect one another; and also their point of intersection.*

Let their equations be

$$\left.\begin{aligned} z &= ax + A \\ z &= by + B \end{aligned}\right\}$$
$$\left.\begin{aligned} z &= a'x + A' \\ z &= b'y + B' \end{aligned}\right\}$$

from which eliminating x, y, z, we get the equation of condition

$$\frac{a'A - aA'}{a' - a} = \frac{b'B - bB'}{b' - b}.$$

Also when this condition is fulfilled, the point is found from

$$x = \frac{A - A'}{a' - a}, \quad y = \frac{B - B'}{b' - b}, \quad z = \frac{a'A - aA'}{a' - a}. \quad \ldots \quad (10)$$

6. *To find the angle I, at which these lines intersect.*

Take an isosceles triangle, whose equal sides measured along these lines equal 1, and let the side opposite the angle required be called i; then it is evident that

$$\cos. I = 1 - \tfrac{1}{2} i^2$$

But if at the extremities of the line i, the points in the intersecting lines be (x', y', z'), (x'', y'', z''), then (see 2)

$$i^2 = (x' - x'')^2 + (y' - y')^2 + (z' - z'')^2$$

$$\therefore 2\cos. I = 2 - \{(x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2\}$$

But by the equations to the straight lines, we have (5)

$$\left.\begin{aligned} z' &= a x' + A \\ z' &= b y' + B \end{aligned}\right\}$$

$$\left.\begin{aligned} z'' &= a' x'' + A' \\ z'' &= b' y'' + B' \end{aligned}\right\}$$

and by the construction, and Art. 2, if (x, y, z) be the point of intersection,

$$\left.\begin{aligned} (x - x')^2 + (y - y')^2 + (z - z')^2 &= 1 \\ (x - x'')^2 + (y - y'')^2 + (z - z'')^2 &= 1 \end{aligned}\right\}$$

Also at the point of intersection,

$$\left.\begin{aligned} z = a x + A &= a' x + A' \\ z = b y + B &= b' y + B' \end{aligned}\right\}$$

From these several equations we easily get

$$z - z' = a (x - a')$$

$$y - y' = \frac{a}{b} (x - x')$$

$$z - z'' = a' (x - x'')$$

$$y - y'' = \frac{a'}{b'} (x - x'')$$

whence by substitution,

$$(x - x')^2 + a^2 (x - x')^2 + \frac{a^2}{b^2} (x - x')^2 = 1$$

$$(x - x'')^2 + a'^2 (x - x'')^2 + \frac{a'^2}{b'^2} (x - x'')^2 = 1$$

which give

$$x - x' = \frac{1}{\sqrt{\left(1 + a^2 + \frac{a^2}{b^2}\right)}}$$

$$x - x'' = \frac{1}{\sqrt{\left(1 + a'^2 + \frac{a'^2}{b'^2}\right)}}$$

Hence

$$(x' - x'')^2 = \frac{1}{1 + a^2 + \frac{a^2}{b^2}} + \frac{1}{1 + a'^2 + \frac{a'^2}{b'^2}} - \frac{2}{\sqrt{\left(1 + a^2 + \frac{a^2}{b^2}\right)}\sqrt{\left(1 + a'^2 + \frac{a'^2}{b'^2}\right)}}$$

Also, since

$$y - y' = \frac{a}{b}(x - x')$$

$$y - y'' = \frac{a'}{b'}(x - x'')$$

and

$$z - z' = a\ (x - x')$$

$$z - z'' = a'\ (x - x'')$$

we have

$$(y'-y'')^2 = \frac{a^2}{b^2}\cdot\frac{1}{1+a^2+\frac{a^2}{b^2}} + \frac{a'^2}{b'^2}\cdot\frac{1}{1+a'^2+\frac{a'^2}{b'^2}} - \frac{aa'}{bb'}\cdot\frac{2}{\sqrt{\left(1+a^2+\frac{a^2}{b^2}\right)}\sqrt{\left(1+a'^2+\frac{a'^2}{b'^2}\right)}}$$

$$(z'-z'')^2 = \frac{a^2}{1+a^2+\frac{a^2}{b^2}} + \frac{a'^2}{1+a'^2+\frac{a'^2}{b'^2}} - aa'\,\frac{2}{\sqrt{\left(1+a^2+\frac{a^2}{b^2}\right)}\sqrt{\left(1+a'^2+\frac{a'^2}{b'^2}\right)}}$$

Hence by adding these squares together we get

$$2\cos. I = 2 - \left\{1+1 - \frac{2\left(1 + aa' + \frac{aa'}{bb'}\right)}{\sqrt{\left(1 + a^2 + \frac{a^2}{b^2}\right)}\sqrt{\left(1 + a'^2 + \frac{a'^2}{b'^2}\right)}}\right\}$$

which gives

$$\cos. I = \frac{1 + aa' + \frac{aa'}{bb'}}{\sqrt{\left(1 + a^2 + \frac{a^2}{b^2}\right)}\sqrt{\left(1 + a'^2 + \frac{a'^2}{b'^2}\right)}} \quad \ldots\ldots (11)$$

This result may be obtained with less trouble by drawing straight lines from the origin of coordinates, parallel to the intersecting lines; and then finding the cosine of the angle formed by these new lines. For the new angle is equal to the one sought, and the equations simplify into

$$\left.\begin{array}{l} z' = ax' = by',\ z'' = a'x'' = b'y'' \\ z = ax = by,\ z = a'x = b'y \\ x'^2 + y'^2 + z'^2 = 1 \\ x''^2 + y''^2 + z''^2 = 1 \end{array}\right\}$$

From the above general expression for the angle formed by two intersecting lines, many particular consequences may be deduced.

For instance, *required the conditions requisite that two straight lines given in space may intersect at right angles.*

That they intersect at all, this equation must be fulfilled. (see 5)

$$\frac{a'A - aA'}{a' - a} = \frac{b'B - bB'}{b' - b};$$

and that being the case, in order for them to intersect at right angles, we have

$$I = \frac{\pi}{2}, \quad \cos. I = 0$$

and therefore

$$1 + aa' + \frac{a\,a'}{b\,b'} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

7. In the preceding No. the angle between two intersecting lines is expressed in a function of the rectangular coordinates, which determine the positions of those lines. But since the lines themselves would be given in *parallel* position, if their inclinations to the planes, (x, y), (x, z), (y, z), were given, it may be required, from other *data*, to find the same angle.

Hence denoting generally the complements of the inclinations of a straight line to the planes, (x, y), (x, z), (y, z), by Z, Y, X, the problem may be stated and resolved, as follows:

Required the angle made by the two straight lines, whose angles of inclination are Z, Y, X; Z′, Y′, X′.

Let two lines be drawn, from the origin of the coordinates, parallel to given lines. These make the same angles with the coordinate planes, and with one another, as the given lines. Moreover, making an isosceles triangle, whose vertex is the origin, and equal sides equal unity, we have, as in (6),

$$\cos. I = 1 - \tfrac{1}{2} i^2 = 1 - \tfrac{1}{2}\{(x - x')^2 + (y - y')^2 + (z - z')^2\}$$

the points in the straight lines equally distant from the origin being (x, y, z), (x′, y′, z′).

But in this case,

$$x^2 + y^2 + z^2 = 1$$
$$x'^2 + y'^2 + z'^2 = 1$$

and

$$x = \cos. X, \; y = \cos. Y, \; z = \cos. Z$$
$$x' = \cos. X', \; y' = \cos. Y', \; z' = \cos. Z'$$

$$\therefore \cos. I = x\,x' + y\,y' + z\,z'$$
$$= \cos. X. \cos. X' + \cos. Y. \cos. Y' + \cos. Z. \cos. Z'. \quad . \quad (13)$$

Hence when the lines pass through the origin of coordinates, the same expression for their mutual inclination holds good; but at the same time X, Y, Z; X′, Y′, Z′, not only mean the *complements of the inclinations* to the planes as above described, but also the *inclinations* of the lines to the axes of coordinates of x, y, z, respectively.

8. *Given the length* (L) *of a straight line and the complements of its inclinations to the planes* (x, y), (x, z), (y z), viz. Z, Y, X, *to find the lengths of its projections upon those planes.*

By the figure in (4) it is easily seen that

$$\left.\begin{aligned} \text{L projected on the plane } (x, y) &= L . \sin. Z \\ \text{---------- } (x, z) &= L . \sin. Y \\ \text{---------- } (y, z) &= L . \sin. X \end{aligned}\right\} \quad . \ . \ . \quad (14)$$

9. Instead of determining the parallelism or direction of a straight line in space by the angles Z, Y, X, it is more concise to do it by means of Z (for instance) and the angle θ which its projection on the plane (x, y) makes with the axis of x.

For, drawing a line parallel to the given line from the origin of the coordinates, the projection of this line is parallel to that of the given line, and letting fall from any point (x, y, z) of the new line, perpendiculars upon the plane (x, y) and upon the axes of x and of y, it easily appears, that

$$x = L \cos. X = (L \sin. Z) \cos. \theta \text{ (see No. 8)}$$
$$y = L . \cos. Y = (L . \sin. Z) \sin. \theta$$

which give

$$\left.\begin{aligned} \cos. X &= \sin. Z . \cos. \theta \\ \cos. Y &= \sin. Z . \sin. \theta \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \quad (15)$$

Hence the expression (13) assumes this form,

$$\cos. I = \sin. Z . \sin. Z' (\cos. \theta \cos. \theta' + \sin. \theta \sin. \theta') + \cos. Z \cos. Z'$$
$$= \sin. Z . \sin. Z' \cos. (\theta - \theta') + \cos. Z . \cos. Z' \quad . \ . \ . \ . \quad (16)$$

which may easily be adapted to logarithmic computation.

The expression (5) is merely verified by the substitution.

10. *Given the angle of intersection* (I) *between two lines in space and their inclinations to the plane* (x, y), *to find the angle at which their projections upon that plane intersect one another.*

If, as above, Z, Z′ be the complements of the inclinations of the lines upon the plane, and θ, θ' the inclinations of the projections to the axis of x, we have from (16)

$$\cos. (\theta - \theta') = \frac{\cos. I - \cos. Z . \cos. Z'}{\sin. Z . \sin. Z'} \quad . \ . \ . \ . \ . \quad (17)$$

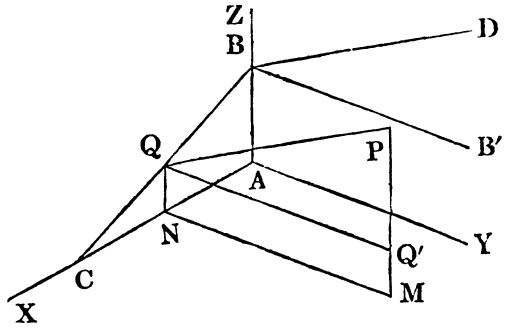
This result indicates that I, Z, Z′ are sides of a spherical triangle (radius = 1), $\theta - \theta'$ being the angle subtended by I. The form may at once indeed be obtained by taking the origin of coordinates as the center of the sphere, and radii to pass through the angles of the spherical triangle, measured along the axis of z, and along lines parallel to the given lines.

Having considered at some length the mode of determining the position and properties of points and straight lines in fixed space, we proceed to treat, in like manner, of planes.

It is evident that the position of a plane is fixed or determinate in position when three of its points are known. Hence is suggested the following problem.

11. *Having given the three points* (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ *in space, to find the equation to the plane passing through them;* that is, *to find the relation between the coordinates of any other point in the plane.*

Suppose the plane to make with the planes (z, y), (z, x) the intersec-



tions or traces B D, B C respectively, and let P be any point whatever in the plane; then through P draw P Q in that plane parallel to B D, &c. as above. Then

$$z - Q\,N = P\,Q' = Q\,Q' \cot. D\,B\,Z$$
$$= y \cot. D\,B\,Z.$$

But

$$Q\,N = A\,B - N\,A \cdot \cot. C\,B\,A$$
$$= A\,B + x \cot. C\,B\,Z,$$
$$\therefore z = A\,B + x \cot. C\,B\,Z + y \cot. D\,B\,Z.$$

Consequently if we put A B = g, and denote by (X, Z), (Y, Z) the inclinations to A Z of the traces in the planes of (x, z), (y, z) respectively, we have

$$z = g + x \cot. (X, Z) + y \cot. (Y, Z) \quad . \quad . \quad . \quad . \quad (18)$$

Hence the form of the equation to the plane is generally

$$z = A\,x + B\,y + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Now to find these constants there are given the coordinates of three points of the plane; that is

$$\gamma = A\alpha + B\beta + C$$
$$\gamma' = A\alpha' + B\beta' + C$$
$$\gamma'' = A\alpha'' + B\beta'' + C$$

from which we get

$$A = \frac{\gamma\beta' - \gamma'\beta + \gamma'\beta'' - \gamma''\beta' + \gamma''\beta - \gamma\beta''}{\alpha\beta' - \alpha'\beta + \alpha'\beta'' - \alpha''\beta' + \alpha''\beta - \alpha\beta''} = \cot.(X, Z) \quad \ldots (20)$$

$$B = \frac{\gamma\alpha' - \gamma'\alpha + \gamma'\alpha'' - \gamma''\alpha' + \gamma''\alpha - \gamma\alpha''}{\alpha\beta' - \alpha'\beta + \alpha'\beta'' - \alpha''\beta' + \alpha''\beta - \alpha\beta''} = \cot.(Y, Z) \quad \ldots (21)$$

$$C = \frac{\beta''(\gamma\alpha' - \gamma'\alpha) + \beta(\gamma'\alpha'' - \gamma''\alpha') + \beta'(\gamma''\alpha - \gamma\alpha'')}{\alpha\beta' - \alpha'\beta + \alpha'\beta'' - \alpha''\beta' + \alpha''\beta - \alpha\beta''} \quad \ldots (22)$$

Hence when the trace coincides with the axis of x, we have

$$C = 0,$$

and

$$A = \cot.\frac{\pi}{2} = 0$$

$$\left.\begin{aligned} \beta''(\gamma\alpha' - \gamma'\alpha) + \beta(\gamma'\alpha'' - \gamma''\alpha') + \beta'(\gamma''\alpha - \gamma\alpha'') &= 0 \\ \gamma\beta' - \gamma'\beta + \gamma'\beta'' - \gamma''\beta' + \gamma''\beta - \gamma\beta'' &= 0 \end{aligned}\right\} \quad \ldots (23)$$

$$B = \frac{1}{\beta''} \cdot \frac{(\beta - \beta'')\cdot(\gamma'\alpha'' - \gamma''\alpha') + (\beta' - \beta'')\cdot(\gamma''\alpha - \gamma\alpha'')}{\alpha\beta' - \alpha'\beta + \alpha'\beta'' - \alpha''\beta' + \alpha''\beta - \alpha\beta''} \quad \ldots\ldots (24)$$

and the equation to the plane becomes

$$z = By \quad \ldots\ldots\ldots\ldots (25)$$

When the plane is parallel to the plane (x, y),

$$A = 0, \quad B = 0,$$

and,

$$z = C \quad \ldots\ldots\ldots\ldots (26)$$

from which, by means of A = 0, B = 0, any two of the quantities $\gamma, \gamma', \gamma''$ being eliminated, the value of C will be somewhat simplified.

Hence also will easily be deduced a number of other particular results connected with the theory of the plane, the point, and the straight line, of which the following are some.

To find the projections on the planes (x, y), (x, z), (y, z) *of the intersection of the planes,*

$$z = Ax + By + C,$$
$$z = A'x + B'y + C'.$$

Eliminating z, we have

$$(A - A')x + (B - B')y + C - C' = 0 \quad \ldots\ldots (27)$$

which is the equation to the projection on (x, y).

Eliminating x, we get

$$(A' - A)z + (AB' - A'B)y + AC' - A'C = 0 \quad . \quad . \quad . \quad . \quad (28)$$

which is the equation to the projection on the plane (y, z).

And in like manner, we obtain

$$(B' - B)z + (A'B - AB')x + BC' - B'C = 0 \quad . \quad . \quad . \quad . \quad (29)$$

for the projection on the plane (x, z).

To find the conditions requisite that a plane and straight line shall be parallel or coincide.

Let the equations to the straight line and plane be

$$\left.\begin{aligned} x &= az + A \\ y &= bz + B \end{aligned}\right\}$$
$$z = A'x + B'y + C'.$$

Then by substitution in the latter, we get

$$z(A'a + B'b - 1) + A'A + B'B + C' = 0.$$

Now if the straight line and plane have only one point common, we should thus at once have the coordinates to that point.

Also if the straight line coincide with the plane in the above equation, z is indeterminate, and (Art. 6. vol. I.)

$$A'a + B'b - 1 = 0, \quad A'A + B'B + C' = 0 \quad . \quad . \quad . \quad (27)$$

But finally if the straight line is merely to be parallel to the plane, the above conditions ought to be fulfilled even when the plane and line are moved parallelly up to the origin or when A, B, C′ are zero. The only condition in this case is

$$A'a + B'b = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

To find the conditions that a straight line be perpendicular to a plane $z = Ax + By + C$.

Since the straight line is to be perpendicular to the given plane, the plane which projects it upon (x, y) is at right angles both to the plane (x, y) and to the given plane. The intersection, therefore, of the plane (x, y) and the given plane is perpendicular to the projecting plane. Hence the trace of the given plane upon (x, y) is perpendicular to the projection on (x, y) of the given straight line. But the equations of the traces of the plane on (x, z), (y, z), are

$$\left.\begin{aligned} z = Ax + C, \quad z = By + C \\ \text{or} \\ x = \frac{1}{A}z - \frac{C}{A}, \quad y = \frac{1}{B}z - \frac{C}{B} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and if those of the perpendicular be

$$\left.\begin{aligned} x &= az + A, \\ y &= bz + B, \end{aligned}\right\}$$

it is easily seen from (11) or at once, that the condition of these traces being at right angles to the projections, are

$$A + a = 0, \quad A + b = 0.$$

To draw a straight line passing through a given point (α, β, γ) *at right angles to a given plane.*

The equations to the straight line, are clearly

$$x - \alpha + A(z - \gamma) = 0, \quad y - \beta + B(z - \gamma) = 0. \ldots (30)$$

To find the distance of a given point (α, β, γ) *from a given plane.*

The distance is (30) evidently, when (x, y, z) is the common point in the plane and perpendicular

$$\sqrt{(z-\gamma)^2 + (y-\beta)^2 + (x-\alpha)^2} = (z-\gamma)\sqrt{1 + A^2 + B^2}.$$

But the equation to the plane then also subsists, viz.

$$z = Ax + By + C$$

from which, and the equations to the perpendicular, we have

$$z - \gamma = C - \gamma + A\alpha + B\beta,$$

therefore the distance required is

$$\frac{C - \gamma + A\alpha + B\beta}{A^2 + B^2} \ldots\ldots\ldots (31)$$

To find the angle I *formed by two planes*

$$z = Ax + By + C,$$
$$z = A'x + B'y + C'.$$

If from the origin perpendiculars be let fall upon the planes, the angle which they make is equal to that of the planes themselves. Hence, if generally, the equations to a line passing through the origin be

$$\left.\begin{aligned} x &= az \\ y &= bz \end{aligned}\right\}$$

the conditions that it shall be perpendicular to the first plane are

$$A + a = 0,$$
$$B + b = 0,$$

and for the other plane

$$A' + a = 0,$$
$$B' + b = 0.$$

Hence the equations to these perpendiculars are

$$\left.\begin{aligned} x + Az &= 0 \\ y + Bz &= 0 \end{aligned}\right\}$$
$$\left.\begin{aligned} x + A'z &= 0 \\ y + B'z &= 0, \end{aligned}\right\}$$

which may also be deduced from the forms (30).

Hence from (11) we get

$$\cos. I = \frac{1 + A A' + B B'}{\sqrt{(1 + A^2 + B^2)}\sqrt{(1 + A'^2 + B'^2)}} \quad . \quad . \quad . \quad (32)$$

Hence *to find the inclination* (ε) *of a plane with the plane* (x, y).

We make the second plane coincident with (x, y), which gives

$$A' = 0, \quad B' = 0,$$

and therefore

$$\cos. \varepsilon = \frac{1}{\sqrt{(1 + A^2 + B^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

In like manner may the inclinations (ζ), (η) of a plane

$$z = A x + B y + C$$

to the planes (x, z), (y, z) be expressed by

$$\left.\begin{aligned} \cos. \zeta &= \frac{B}{\sqrt{(1 + A^2 + B^2)}} \\ \cos. \eta &= \frac{A}{\sqrt{(1 + A^2 + B^2)}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Hence

$$\cos.^2 \varepsilon + \cos.^2 \zeta + \cos.^2 \eta = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Hence also, if ε', ζ', η' be the inclinations of another plane to (x, y), (x, z), (y, z).

$$\cos. I = \cos. \varepsilon \cos. \varepsilon' + \cos. \zeta \cos. \zeta' + \cos. \eta \cos. \eta' \quad . \quad . \quad . \quad (36)$$

To find the inclination υ *of a straight line* $x = a z + A'$, $y = b z + B'$, *to the plane* $z = A x + B y + C$.

The angle required is that which it makes with its projection upon the plane. If we let fall from any part of the straight line a perpendicular upon the plane, the angle of these two lines will be the complement of υ. From the origin, draw any straight line whatever, viz. $x = a' z$, $y = b' z$. Then in order that it may be perpendicular to the plane, we must have

$$a' = -A, \quad b' = -B.$$

The angle which this makes with the given line can be found from (11); consequently by that expression

$$\sin. \upsilon = \frac{1 - A a - B b}{\sqrt{(1 + a^2 + b^2)}\sqrt{(1 + A^2 + B^2)}} \quad . \quad . \quad . \quad (37)$$

Hence we easily find that the angles made by this line and the coordinate planes (x, y), (x, z), (y, z), viz. Z, Y, X are found from

$$\cos. Z = \frac{1}{\sqrt{(1 + a^2 + b^2)}},$$

$$\cos. Y = \frac{b}{\sqrt{(1 + a^2 + b^2)}},$$

$$\cos. X = \frac{a}{\sqrt{(1 + a^2 + b^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

which agrees with what is done in (3).

TRANSFORMATION OF COORDINATES.

12. *To transfer the origin of coordinates to the point* (α, β, γ) *without changing their direction.*

Let it be premised that instead of supposing the coordinate planes at right angles to one another, we shall here suppose them to make any angles whatever with each other. In this case the axes cease to be rectangular, but the coordinates x, y, z are still drawn parallel to the axes.

This being understood, assume

$$x = x' + \alpha, \quad y = y' + \beta, \quad z = z' + \gamma \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and substitute in the expression involving x, y, z. The result will contain x′, y′, z′ the coordinates referred to the origin (α, β, γ).

When the substitution is made, the signs of α, β, γ as explained in (1), must be attended to.

13. *To change the direction of the axes from rectangular, without affecting the origin.*

Conceive three new axes A x′, A y′, A z′, the first axes being supposed rectangular, and these having any given arbitrary direction whatever. Take any point; draw the coordinates x′, y′, z′ of this point, and project them upon the axis A X. The abscissa x will equal the sum, taken with their proper signs, of these three projections, (as is easily seen by drawing the figure); but if (x x′), (y, y′), (z, z′) denote the angles between the axes A x, A x′; A y, A y′; A z, A z′ respectively; these projections are

$$x' \cos. (x'\,x), \quad y' \cos. (y'\,x), \quad z' \cos. (z'\,x).$$

In like manner we proceed with the other axes, and therefore get

$$\left.\begin{aligned} x &= x' \cos. (x'\,x) + y' \cos. (y'\,x) + z' \cos. (z'\,x) \\ y &= y' \cos. (y'\,y) + z' \cos. (z'\,y) + x' \cos. (x'\,y) \\ z &= z' \cos. (z'\,z) + y' \cos. (y'\,z) + x' \cos. (x'\,z) \end{aligned}\right\} \quad . \quad . \quad . \quad (40)$$

Since (x' x), (x' y), (x' z) are the angles which the staight line A x', makes with the rectangular axes of x, y, z, we have (5)

$$\left.\begin{aligned} \cos.^2(x'\,x) + \cos.^2(x'\,y) + \cos.^2 x'\,z &= 1 \\ \cos.^2(y'\,x) + \cos.^2(y'\,y) + \cos.^2(y'\,z) &= 1 \\ \cos.^2(z'\,x) + \cos.^2(z'\,y) + \cos.^2(z'\,x) &= 1 \end{aligned}\right\} \quad . \; . \; . \quad (41)$$

We also have from (13) p.

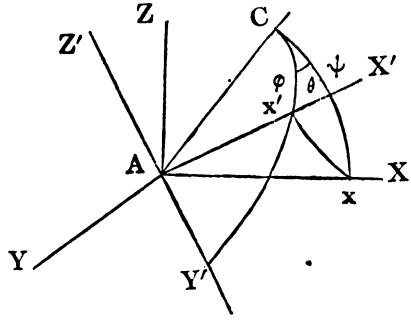
$$\left.\begin{aligned} \cos.(x'y') &= \cos.(x'x)\cos.(y'x) + \cos.(x'y)\cos.(y'y) + \cos.(x'z)\cos.(y'z) \\ \cos.(x'z') &= \cos.(x'x)\cos.(z'x) + \cos.(x'y)\cos.(z'y) + \cos.(x'z)\cos.(z'z) \\ \cos.(y'z') &= \cos.(y'x)\cos.(z'x) + \cos.)y'y)\cos.(z'y) + \cos.(y'z)\cos.(z'z) \end{aligned}\right\} . \; (42)$$

The equations (40) and (41), contain the nine angles which the axes of x', y', z' make with the axes of x, y, z.

Since the equations (41) determine three of these angles only, six of them remain arbitrary. Also when the new system is likewise rectangular, or cos. (x' y') = cos. (x' z') = cos. (y' z') = 1, three others of the arbitraries are determined by equations (42). Hence in that case there remain but three arbitrary angles.

14. *Required to transform the rectangular axe of coordinates to other rectangular axes, having the same origin, but two of which shall be situated in a given plane.*

Let the given plane be Y' A C, of which the trace in the plane (z, x) is



A C. At the distance A C describe the arcs C Y', C x, x x' in the planes C A Y', (z, x), and X' A X. Then if one of the new axes of the coordinates be A X', its position and that of the other two, A Y', A Z', will be determined by C x' = φ, C x = ψ, and the spherical angle x C x' = θ = inclination of the given plane to the plane (z, x).

Hence to transform the axes, it only remains to express the angles (x'x), (y'x), &c. which enter the equations (40) in terms of θ ψ and φ.

By spherics

$$\cos.(x'x) = \cos.\psi \cos.\varphi + \sin.\psi \sin.\varphi \cos.\theta.$$

In like manner forming other spherical triangles, we get

$$\cos.(y'x) = \cos.(90^\circ + \varphi) \cos.\psi + \sin.\psi \sin.(90^\circ + \varphi) \cos.\theta$$
$$\cos.(x'y) = \cos.(90^\circ + \psi) \cos.\varphi + \sin.(90^\circ + \psi) \sin.\varphi \cos.\theta$$
$$\cos.(y'y) = \cos.(90^\circ + \psi)\cos.(90^\circ + \varphi) + \sin.(90^\circ + \psi)\sin.(90^\circ + \varphi)\cos.\theta$$

So that we obtain these four equations

$$\left.\begin{aligned} \cos.(x'x) &= \cos.\psi \cos.\varphi + \sin.\psi \sin.\varphi \cos.\theta \\ \cos.(y'x) &= -\sin.\psi \sin.\varphi + \sin.\psi \cos.\varphi \cos.\theta \\ \cos.(x'y) &= -\sin.\psi \cos.\varphi + \cos.\psi \sin.\varphi \cos.\theta \\ \cos.(y'y) &= \sin.\psi \sin.\varphi + \cos.\psi \cos.\varphi \cos.\theta \end{aligned}\right\} \quad . \; . \quad (43)$$

Again by spherics, (since A Z′ is perpendicular to A C, and the inclination of the planes Z′A C, (x, y) is $90^\circ - \theta$) we have

$$\left.\begin{aligned} \cos.(z'x) &= \sin.\psi \sin.\theta \\ \cos.(z'y) &= \cos.\psi \sin.\theta \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (44)$$

And by considering that the angle between the planes Z A C, Z A X′, $= 90^\circ + \theta$, by spherics, we also get

$$\left.\begin{aligned} \cos.(x'z) &= -\sin.\varphi \sin.\theta \\ \cos.(y'z) &= -\cos.\varphi \sin.\theta \\ \cos.(z'z) &= \cos.\theta \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (45)$$

which give the nine coefficients of equations (40).

Equations (41), (42) will also hereby be satisfied when the systems are rectangular.

15. *To find the section of a surface made by a plane.*

The last transformation of axes is of great use in determining the nature of the section of a surface, made by a plane, or of the section made by any two surfaces, plane or not, provided the section lies in one plane; for having transformed the axes to others, A Z′, A X′, A Y′, the two latter lying in the plane of the section, by the equations (40), and the determinations of the last article, by putting $z' = 0$ in the equation to the surface, we have that of the section at once. It is better, however, to make $z = 0$ in the equations (40), and to seek directly the values of cos. (x′x), cos. (y′x), &c. The equations (40) thus become

$$\left.\begin{aligned} x &= x' \cos.\psi + y' \sin.\psi \cos.\theta \\ y &= x' \sin.\psi - y' \cos.\psi \cos.\theta \\ z &= y' \sin.\theta \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (46)$$

16. *To determine the nature and position of all surfaces of the second order*; or *to discuss the general equation of the second order*, viz.

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + gx + hy + iz = k \quad . \; . \quad (a)$$

First simplify it by such a transformation of coordinates as shall destroy

the terms in x y, x z, y z; the axes from rectangular will become oblique, by substituting the values (40), and the nine angles which enter these, being subjected to the conditions (41), there will remain six of them arbitrary, which we may dispose of in an infinity of ways. Equate to zero the coefficients of the terms in x′ y′, x′ z′, y′ z′.

But if it be required that the new axes shall be also rectangular, as this condition will be expressed by putting each of the equations (42) equal zero, the *six* arbitrary angles will be reduced to *three*, which the three coefficients to be destroyed will make known, and the problem will thus be determined.

This investigation will be rendered easier by the following process:

Let $x = \alpha z$, $y = \beta z$ be the equations of the axis of x′; then for brevity making

$$l = \frac{1}{\sqrt{(1 + \alpha^2 + \beta^2)}}$$

we find that (3)

$$\cos.(x'x = \alpha l, \quad \cos.(x'y) = \beta l, \quad \cos. x'z = l.$$

Reasoning thus also as to the equations $x = \alpha' z$, $y = \beta' z$ of the axis of y′, and the same for the axis of z′, we get

$$\cos.(y'x) = \alpha' l', \quad \cos.(y'y) = \beta' l', \quad \cos.(y'z) = l'$$
$$\cos.(z'x) = \alpha'' l'', \quad \cos.(z'y) = \beta'' l'', \quad \cos.(z'z) = l''.$$

Hence by substitution the equations (40) become

$$\left.\begin{aligned} x &= l\alpha x' + l'\alpha' y' + l''\alpha'' z' \\ y &= l\beta x' + l'\beta' y' + l''\beta'' z' \\ z &= l x' + l' y' + l'' z'. \end{aligned}\right\}$$

The nine angles of the problem are replaced by the six unknowns α, α', α'', β, β', β'', provided the equations (41) are thereby also satisfied.

Substitute therefore these values of x, y, z, in the general equation of the 2d degree, and equate to zero the coefficients of x′ y′, x′ z′, y′ z′, and we get

$$\left.\begin{aligned} &(a\alpha + d\beta + e)\alpha' + (d\alpha + b\beta + f)\beta' + e\alpha + f\beta + c = 0 \\ &(a\alpha + d\beta + e)\alpha'' + (d\alpha + b\beta + f)\beta'' + e\alpha + f\beta + c = 0 \\ &(a\alpha'' + d\beta'' + e)\alpha' + (d\alpha'' + b\beta'' + f)\beta' + e\alpha'' + f\beta'' + c = 0 \end{aligned}\right\}$$

One of these equations may be found without the others, and by making the substitution only in part. Moreover from the symmetry of the process the other two equations may be found from this one. Eliminate α', β' from the first of them, and the equations $x = \alpha' z$, $y = \beta' z$, of the axis of y′; the resulting equation

$$(a\alpha + d\beta + e)x + (d\alpha + b\beta + f)y + (e\alpha + f\beta + c)z = 0 \quad . \quad . \quad (b)$$

is that of a plane (19).

But the first equation is the condition which destroys the term $x'y'$: since if we only consider it, α, β, α', β', may be any whatever that will satisfy it; it suffices therefore that the axis of y' be traced in the plane above alluded to, in order that the transformed equations may not contain any term in $x'y'$.

In the same manner eliminating α'', β'', from the 2d equation by means of the equations of the axis of z', viz. $x = a''z$, $y = \beta''z$, we shall have a plane such, that if we take for the axis of z' every straight line which it will there trace out, the transformed equation will not contain the term in $x'z'$. But, from the form of the two first equations, it is evident that this second plane is the same as the first: therefore, if we there trace the axes of y' and z' at pleasure, this plane will be that of y' and z', and the transformed equation will have no terms involving $x'y'$ or xz'. The direction of these axes in the plane being any whatever, we have an infinity of systems which will serve this purpose; the equation (b) will be that of a plane parallel to the plane which bisects all the parallels to x, and which is therefore called the *Diametrical Plane.*

Again, if we wish to make the term in $y'z'$ disappear, the third equation will give α', β', and there are an infinity of oblique axes which will answer the three required conditions. But make x', y', z', rectangular. The axis of x' must be perpendicular to the plane $(y'z')$ whose equation we have just found; and that $x = \alpha z$, $y = \beta z$, may be the equations (see equations b) we must have

$$a\alpha + d\beta + e = (e\alpha + f\beta + c)\alpha \quad . \quad . \quad . \quad . \quad (c)$$

$$d\alpha + b\beta + f = (e\alpha + f\beta + c)\beta \quad . \quad . \quad . \quad . \quad (d)$$

Substituting in (c) the value of α found from (d) we get

$$\{(a-b)fe + (f^2-e^2)d\}\beta^3$$
$$+ \{(a-b)(c-b)e + (2d^2-f^2-e^2)e + (2c-a-b)fd\}\beta^2$$
$$+ \{(c-a)(c-b)d + (2e^2-f^2-d^2)d + (2b-a-c)fe\}\beta$$
$$+ (a-c)fd + (f^2-d^2)e = 0.$$

This equation of the 3d degree gives for β at least one real root; hence the equation (d) gives one for α; so that the axis of x' is determined so as to be perpendicular to the plane $(y', z',)$ and to be free from terms in $x'z'$, and $y'z'$. It remains to make in this plane $(y', z',)$ the axes at right angles and such that the term $x'y'$ may also disappear. But it is evident that we shall find at the same time a plane $(x', z',)$ such that the axis of y' is perpendicular to it, and also that the terms in $x'y'$, $z'y'$ are not involved. But it happens that the conditions for making the axis of y' perpendicular to this plane are both (c) and (d) so that the same equation of the 3d de-

gree must give also β'. The same holds good for the axis of z. Consequently the three roots of the equation β are all real, and are the values of β, β', β''. Therefore α, α', α'', are given by the equation (d). Hence, *There is only one system of rectangular axes which eliminates* $x'y'$, $x'z'$, $x'y'$; *and there exists one in all cases.* These axes are called the *Principal axes of the Surface.*

Let us analyze the case which the cubic in β presents.

1. If we make

$$(a - b)\,fe + (f^2 - e^2)\,d = 0$$

the equation is deprived of its first term. This shows that then one of the roots of β is infinite, as well as that α derived from equation (d) becomes $e\alpha + f\beta = 0$. The corresponding angles are right angles. One of the axes, that of z' for instance, falls upon the plane (x, y), and we obtain its equation by eliminating α and β from the equations $x = \alpha z$, $y = \beta z$, which equation is

$$ex + fy = 0$$

The directions of y', z' are given by the equation in β reduced to a quadrature.

2ndly. If besides this first coefficient the second is also $= 0$, by substituting b, found from the above equation, in the factor of β^2, it reduces to the last term of the equation in β, viz.

$$(a - c)\,fd + (f^2 - d^2)\,e = 0.$$

These two equations express the condition required. But the coefficient of β is deduced from that of β^2 by changing b into c and d into e, and the same holds for the first and last term of the equation in β. Therefore the cubic equation is also thus satisfied. There exists therefore an infinity of rectangular systems, which destroy the terms in $x'y'$, $x'z'$, $y'z'$. Eliminating a and b from equations (c) and (d) by aid of the above two equations of condition, we find that they are the product of $f\alpha - d$ and $e\beta - d$ by the common factor $ed\alpha + fd\beta + fe$. These factors are therefore nothing; and eliminating α and β, we find

$$fx = dz,\ ey = dz,\ edx + fdy + fez = 0.$$

The two first are the equations of one of the axes. The third that of a plane which is perpendicular to it, and in which are traced the two other axes under arbitrary directions. This plane will cut the surface in a curve wherein all the rectangular axes are principal, which curve is therefore a circle, the only one of curves of the second order which has that property. The surface is one then of revolution round the axis whose equations we have just given.

The equation once freed from the three rectangles, becomes of the form

$$kz^2 + my^2 + nx^2 + qx + q'y + q''z = h \quad . \quad . \quad . \quad . \quad (e)$$

The terms of the first dimension are evidently destroyed by removing the origin (39). It is clear this can be effected, except in the case where one of the squares x^2, y^2, z^2 is deficient. We shall examine these cases separately. First, let us discuss the equation

$$kz^2 + my^2 + nx^2 = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

Every straight line passing through the origin, cuts the surface in two points at equal distances on both sides, since the equation remains the same after having changed the signs of x, y, z. The origin being in the middle of all the chords drawn through this point is a *center*. *The surface therefore has the property of possessing a center whenever the transformed equation has the squares of all the variables.*

We shall always take *n* positive: it remains to examine the cases where k and m are both positive, both negative, or of different signs.

If in the equation (f) k, m, and n, are all positive, h is also positive; and if h is nothing, we have $x = 0$, $y = 0$, $z = 0$, and the surface has but one point.

But when h is positive by making x, y, or z, separately equal zero, we find the equations to an ellipse, curves which result from the section of the surface in question by the three coordinate planes. Every plane parallel to them gives also an ellipse, and it will be easy to show the same of all plane sections. Hence the surface is termed an *Ellipsoid.*

The lengths A, B, C, of the *three principal axes* are obtained by finding the sections of the surface through the axes of x, y, and z. These give

$$kC^2 = h, \quad mB^2 = h, \quad nA^2 = h.$$

from which eliminating k, m and n, and substituting in equation (f) we get

$$\left.\begin{aligned} \frac{z^2}{C^2} + \frac{y^2}{B^2} + \frac{x^2}{A^2} &= 1 \\ A^2B^2z^2 + A^2C^2y^2 + B^2C^2x^2 &= A^2B^2C^2 \end{aligned}\right\} \quad (47)$$

which is the equation to an *Ellipsoid* referred to its *center and principal axes.*

We may conceive this surface to be generated by an ellipse, traced in the plane (x, y), moving parallel to itself, whilst its two axes vary, the curve sliding along another ellipse, traced in the plane (x, z) as a direct-

rix. If two of the quantities A, B, C, are equal, we have an ellipsoid of revolution. If all three are equal, we have a sphere.

Now suppose k negative, and m and h positive or

$$k z^2 - m y^2 - a x^2 = - h.$$

Making x or y equal zero, we perceive that the sections by the planes (y z) and (x z), are hyperbolas, whose axis of z is the second axis. All planes passing through the axis of z, give this same curve. Hence the surface is called an *hyperboloid.* Sections parallel to the plane (x y) are always real ellipses, A, B, C $\sqrt{-1}$ designating the lengths upon the axes from the origin, the equation is the same as the above equation excepting the first term becoming negative.

Lastly, when k and h are negative

$$k z^2 + m y^2 + n x^2 = - h;$$

all the planes which pass through the axis of z cut the surface in hyperbolas, whose axis of z is the first axis. The plane (x y) does not meet the surface and its parallels passing through the opposite limits, give ellipses. This is a *hyperboloid* also, but having two vertexes about the axis of z. The equation in A, B, C is still the same as above, excepting that the term in z^2 is the only positive one.

When h = 0, we have, in these two cases,

$$k^2 = m y^2 + n x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

the equation to a cone, which cone is the same to these hyperboloids that an asymptote is to hyperbolas.

It remains to consider the case of k and m being negative. But by a simple inversion in the axes, this is referred to the two preceding ones. The hyperboloid in this case has one or two vertexes about the axis of x according as h is negative or positive.

When the equation (e) is deprived of one of the squares, of x^2 for instance, by transferring the origin, we may disengage that equation from the constant term and from those in y and z; thus it becomes

$$k z^2 + m y^2 = h x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

The sections due to the planes (x z), (x y) are parabolas in the same or opposite directions according to the signs of k, m, h; the planes parallel to these give also parabolas. The planes parallel to that of (y z) give ellipses or parabolas according to the sign of m. The surface is an *elliptic paraboloid* in the one case, and a *hyperbolic paraboloid* in the other case. When k = m, it is a *paraboloid of revolution.*

When h = 0, the equation takes the form

$$a^2 z^2 \pm b^2 y^2 = 0$$

according to the signs of k and m. In the one case we have

$$z = 0, \quad y = 0$$

whatever may be the value of x, and the surface reduces to the axis of x. In the other case.

$$(a\,z + b\,y)\,(a\,z - b\,y) = 0$$

shows that we make another factor equal zero; thus we have the system of two planes which increase along the axis of x.

When the equation (e) is deprived of two squares, for instance of x^2, y^2, by transferring the origin parallelly to z, we reduce the equation to

$$k\,z^2 + p\,y + q\,x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

The sections made by the planes drawn according to the axis of z, are parabolas. The plane (x y) and its parallels give straight lines parallel to them. The surface is, therefore, a cylinder whose base is a parabola, or a *parabolic cylinder*.

If the three squares in (e) are wanting, it will be that of a plane.

It is easy to recognise the case where the proposed equation is decomposable into rational factors; the case where it is formed of positive squares, which resolve into two equations representing the sector of two planes.

PARTIAL DIFFERENCES.

17. If $u = f(x, y, z, \&c.)$ denote any function of the variable x, y, z, &c. d u be taken on the supposition that y, z, &c. are constant, then is the result termed the partial difference of u relative to x, and is thus written

$$\left(\frac{d\,u}{d\,x}\right) d\,x.$$

Similarly

$$\left(\frac{d\,u}{d\,y}\right) d\,y, \quad \left(\frac{d\,u}{d\,z}\right) d\,z, \ \&c.$$

denote the partial differences of u relatively to y, z, &c. respectively.

Now since the total difference of u arises from the increase or decrease of its variables, it is evident that

$$d\,u = \left(\frac{d\,u}{d\,x}\right) d\,x + \left(\frac{d\,u}{d\,y}\right) d\,y + \left(\frac{d\,u}{d\,z}\right) d\,z + \&c. \quad . \quad . \quad . \quad (51)$$

But, by the general principle laid down in (6) Vol. I, this result may be demonstrated as follows; Let

$$u + du = A + A\,dx + B\,dy + C\,dz + \&c.$$
$$A'\,dx^2 + B'\,dy^2 + C'\,dz^2 + \&c.$$
$$+ M\,dx.dy + N\,dx.dz + \&c.$$
$$+ A''\,dx^3 + \&c.$$

Then equating quantities of the same nature, we have

$$du = A\,dx + B\,dy + C\,dz + \&c.$$

Hence when dy, dz, &c. $= 0$, or when y, z, &c. are considered constant

$$du = A\,dx$$

or according to the above notation

$$A = \left(\frac{du}{dx}\right).$$

In the same manner it is shown, that

$$B = \left(\frac{du}{dy}\right)$$

$$C = \left(\frac{du}{dz}\right)$$

&c.

Hence

$$du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy + \left(\frac{du}{dz}\right)dz + \&c.$$ as before.

Ex. 1. $u = xyz$, &c.

$$\left(\frac{du}{dx}\right) = yz,\ \left(\frac{du}{dy}\right) = xz,\ \left(\frac{du}{dz}\right) = xy,$$

$$\therefore du = yz\,dx + xz\,dy + xy\,dz$$

$$\text{or } \frac{du}{u} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}.$$

Ex. 2. $u = xyz$, &c. Here as above

$$\frac{du}{u} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} + \&c.$$

And in like manner the total difference of any function of any number of variables may be found, viz. by first taking the partial differences, as in the rules laid down in the Comments upon the first section of the first book of the Principia.

But this is not the only use of partial differences. They are frequently used to abbreviate expressions. Thus, in p. 13, and 14, Vol. II. we

learn that the actions of M, μ, μ'', &c. upon μ resolved parallel to x, amount to

$$\frac{d^2(\zeta+x)}{dt^2} = \frac{\mu'(x'-x)}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}} + \frac{\mu''(x''-x)}{[(x''-x)^2+(y''-y)^2+(z''-z)^2]^{\frac{3}{2}}}$$
$$+ \frac{\mu'''(x'''-x)}{[(x'''-x)^2+(y'''-y)^2+(z'''-z)^2]^{\frac{3}{2}}} + \&c. - \frac{Mx}{[(x^2+y^2+z^2)]^{\frac{3}{2}}}$$

retaining the notation there adopted.

But if we make

$$\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2} = \underset{0,1}{\rho}$$

and generally

$$\sqrt{(x''^{\ldots n}-x'^{\ldots m})^2+(y''^{\ldots n}-y'''^{\ldots m})^2+(z''^{\ldots n}-z'''^{\ldots m})^2} = \underset{n,m,}{\rho}$$

and then assume

$$\lambda = \frac{\mu\mu'}{\underset{0,1}{\rho}} + \frac{\mu\mu''}{\underset{0,2}{\rho}} + \&c. \quad \ldots\ldots\ldots\ldots \quad (A)$$

$$+\frac{\mu'\mu''}{2} + \frac{\mu'\mu'''}{\underset{1,3}{\rho}} + \&c. \quad \ldots\ldots\ldots\ldots \quad (B)$$

$$+\frac{\mu''\mu'''}{\underset{2,}{\rho}} + \frac{\mu''\mu''''}{\underset{2,4}{\rho}} + \&c. \quad \ldots\ldots\ldots\ldots \quad (C)$$

&c.

we get

$$\left(\frac{d\lambda}{dx}\right) = \left(\frac{dA}{dx}\right) = \frac{\mu\mu'(x'-x)}{\underset{0,1}{\rho^3}} + \frac{\mu\mu''(x''-x)}{\underset{0,2}{\rho^3}} + \&c.$$

$$\left(\frac{d\lambda}{dy}\right) = \left(\frac{dA}{dy}\right) = \frac{\mu\mu'(y'-y)}{\underset{0,1}{\rho^3}} + \frac{\mu\mu''(y''-y)}{\underset{0,2}{\rho^3}} + \&c.$$

$$\left(\frac{d\lambda}{dz}\right) = \left(\frac{dA}{dz}\right) = \frac{\mu\mu'(z'-z)}{\underset{0,1}{\rho^3}} + \frac{\mu\mu''(z''-z)}{\underset{0,2}{\rho^3}} + \&c.$$

We also get

$$\left(\frac{d\lambda}{dx'}\right) = -\frac{\mu\mu'(x'-x)}{\underset{0,1}{\rho^3}} + \left(\frac{dB}{dx'}\right)$$

$$\left(\frac{d\lambda}{dx''}\right) = -\frac{\mu\mu''(x''-x)}{\underset{0,2}{\rho^3}} - \frac{\mu'\mu''(x''-x')}{\underset{1,2}{\rho^3}} + \left(\frac{dC}{dx''}\right)$$

$$\left(\frac{d\lambda}{dx'''}\right) = -\frac{\mu\mu'''(x'''-x)}{\underset{0,3}{\rho^3}} - \frac{\mu'\mu'''(x'''-x')}{\underset{1,3}{\rho^3}} - \frac{\mu''\mu'''(x'''-x'')}{\underset{2,3}{\rho^3}} + \left(\frac{dD}{dx''}\right)$$

Hence since (B) has one term less than (A); (C) one term less than (B); and so on; it is evident that

$$\left(\frac{d\lambda}{dx'}\right) + \left(\frac{d\lambda}{dx''}\right) + \left(\frac{d\lambda}{dx'''}\right) + \&c. = -\left(\frac{d\lambda}{dx}\right) - \left(\frac{dB}{dx'}\right) - \left(\frac{dC}{dx}\right) - \&c.$$
$$+ \left(\frac{dB}{dx'}\right) + \left(\frac{dC}{dx''}\right) + \&c.$$

and therefore that

$$\Sigma.\left(\frac{d\lambda}{dx}\right) = \left(\frac{d\lambda}{dx}\right) + \left(\frac{d\lambda}{dx'}\right) + \left(\frac{d\lambda}{dx''}\right) + \&c. = 0$$

See p. 15, Vol. II.

Hence then λ is so assumed that the sum of its partial differences relative to x, x', x'' &c. shall equal zero, and at the same time abbreviate the expression for the forces upon μ along x from the above complex formula into

$$\frac{d^2(\zeta + x)}{dt^2} = \frac{1}{\mu}\left(\frac{d\lambda}{dx}\right) - \frac{Mx}{\rho^3};$$

and the same may be said relatively to the forces resolved parallel to y, z, &c. &c.

Another consequence of this assumption is

$$x\left(\frac{d\lambda}{dy}\right) + x'\left(\frac{d\lambda}{dy'}\right) + \&c. = y\left(\frac{d\lambda}{dx}\right) + y'\left(\frac{d\lambda}{dx'}\right) + \&c.$$

$$\text{or } \Sigma.x\left(\frac{d\lambda}{dy}\right) = \Sigma.y\left(\frac{d\lambda}{dx}\right).$$

For

$$y\left(\frac{d\lambda}{dx}\right) = \frac{\mu\mu'(x'-x)y}{\rho^3_{0,1}} + \frac{\mu\mu''(x''-x)y}{\rho^3_{0,2}} + \frac{\mu\mu'''(x'''-x)y}{\rho^3_{0,3}} + \&c.$$

$$y'\left(\frac{d\lambda}{dx'}\right) = \frac{\mu'\mu''(x''-x')y'}{\rho^3_{1,2}} + \frac{\mu'\mu'''(x'''-x')y'}{\rho^3} + \&c. - \frac{\mu\mu'(x'-x)y'}{\rho^3_{0,1}}$$

$$y''\left(\frac{d\lambda}{dx''}\right) = \frac{\mu''\mu'''(x'''-x'')y''}{\rho^3_{2,3}} + \frac{\mu''\mu''''(x''''-x'')y''}{\rho^3_{2,4}} + \&c. \frac{\mu\mu''(x''-x)y'}{\rho^3_{0,2}} - \frac{(\mu'\mu''(x''-x')y''}{\rho^3_{1,2}}$$

&c.

Hence it is evident that

$$\Sigma.y\left(\frac{d\lambda}{dx}\right) = \frac{\mu\mu'(x'-x)(y-y')}{\rho^3_{0,1}} + \frac{\mu\mu''(x''-x)(y-y'')}{\rho^3_{0,2}} + \&c.$$
$$+ \frac{\mu'\mu''(x''-x')(y'-y'')}{\rho^3_{1,2}} + \frac{\mu'\mu'''(x'''-x')(y-y''')}{\rho^3_{1,2}} + \&c.$$
$$+ \frac{\mu''\mu'''(x'''-x'')(y''-y''')}{\rho^3_{2,3}} + \frac{\mu''\mu''''(x''''-x'')(y''-y'''')}{\rho^3_{2,4}} + \&c.$$

&c.

In like manner it is found that

$$\Sigma . x \left(\frac{d\lambda}{dy}\right) = \frac{\mu\mu'(y'-y)(x-x')}{\rho^3_{0,1}} + \frac{\mu\mu''(y''-y)(x-x'')}{\rho^3_{0,2}} + \&c.$$
$$+ \frac{\mu'\mu''(y''-y')(x'-x'')}{\rho^3_{1,2}} + \frac{\mu'\mu'''(y'''-y')(x'-x''')}{\rho^3_{1,3}} + \&c.$$
$$\&c.$$

which is also perceptible from the substitution in the above equation of y for x, x for y; y′ for x′, x′ for y′; and so on.

But

$$(y'-y)(x-x') = (x'-x)(y-y')$$
$$(y''-y)(x-x'') = (x''-x)(y-y'')$$
$$\&c.$$

consequently

$$\Sigma . x \left(\frac{d\lambda}{dy}\right) = \Sigma . y \left(\frac{d\lambda}{dx}\right)$$

See p. 16. For similar uses of partial differences, see also pp. 22, and 105.

CHANGE OF THE INDEPENDENT VARIABLE.

When an expression is given containing differential coefficients, such as

$$\frac{dy}{dx}, \frac{d^2y}{dx^2} \&c.$$

in which the first differential only of x and its powers are to be found, it shows that the differential had been taken on the supposition that dx is constant, or that $d^2x = 0$, $d^3x = 0$, and so on. But it may be required to transform this expression to another in which d^2x, d^3x shall appear, and in which dy shall be constant, or from which d^2y, &c. shall be excluded. This is performed as follows:

For instance if we have the expression

$$\frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}} \frac{dy}{dx}$$

the differential coefficients

$$\frac{dy}{dx}, \frac{d^2y}{dx}$$

may be eliminated by means of the equation of the curve to which we mean to apply that expression. For instance, from the equation to a parabola $y = a x^2$, we derive the values of

$$\frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}$$

which being substituted in the above formula, these differential coefficients will disappear. If we consider

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}$$

unknown, we must in general have two equations to eliminate them from one formula, and these equations will be given by twice differentiating the equation to the curve.

When by algebriacal operations, dx ceases to be placed underneath dy, as in this form

$$\frac{y(dx^2 + dy^2)}{dx^2 + dy^2 - y\,d^2y} \quad \ldots\ldots\ldots \quad (52)$$

the substitution is effected by regarding dx, dy, d^2y as unknown; but then in order to eliminate them, there must be in general the same number of equations as of unknowns, and consequently it would seem the elimination cannot be accomplished, because by means of the equation to the curve, only two of the equations between dx, dy, d^2y can be obtained. It must be remarked, however, that when by means of these two equations we shall have eliminated dy and d^2y, there will remain a common factor dx^2, which will also vanish. For example, if the curve is always a parabola represented by the equation $y = ax$, by differentiating twice we obtain

$$dy = 2a\,x\,dx \quad 0\,d^2y = 2a\,dx^2$$

and these being substituted in the formula immediately above, we shall obtain, after suppressing the common factor dx^2,

$$\frac{y(1 + 4a^2x^2)}{4a^2x^2 - 2ay}.$$

The reason why dx^2 becomes a common factor is perceptible at once, for when from a formula which primitively contained

$$\frac{d^2y}{dx^2}, \frac{dy}{dx},$$

we have taken away the denominator of $\frac{d^2y}{dx^2}$ all the terms independent of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ must acquire the factor dx^2; then the terms which were affected by $\frac{d^2y}{dx^2}$, do not contain dx, whilst those affected by $\frac{dy}{dx}$

contain d x. When we afterwards differentiate the equation of the curve, and obtain results of the form $dy = M\,dx$, $d^2y = N\,dx^2$, these values being substituted in the terms in d^2y, and in $dy\,dx$, will change them, as likewise the other terms, into products of dx^2.

What has been said of a formula containing differentials of the two first orders applying equally to those in which these differentials rise to superior orders, it thence follows that by differentiating the equation of the curve as often as is necessary, we can always make disappear from the expression proposed, the differentials therein contained.

The same will also hold good if, beside these differentials which we have just been considering, the formula contain terms in d^2x, in d^3x, &c.; for suppose that there enter the formula these differentials d x, d y, d^2x, d^2y and that by twice differentiating the equation represented by $y = fx$, we obtain these equations

$$F(x, y, dy, dx) = 0$$
$$F(x, y, dx, dy, d^2x, d^2y) = 0,$$

we can only find two of the three differentials d y, d^2x, d^2y, and we see it will be impossible to eliminate all the differentials of the formula; there is therefore a condition tacitly expressed by the differential d^2x; it is that the variable x is itself considered a function of a third variable which does not enter the formula, and which we call the *independent variable.* This will become manifest if we observe, that the equation $y = fx$ may be derived from the system of two equations

$$x = Ft, \quad y = \varphi t$$

from which we may eliminate t. Thus the equation

$$y = a \cdot \frac{(x - c)^2}{b^2}$$

is derived from the system of two equations

$$x = bt + c, \quad y = at^2,$$

and we see that x and y must vary by virtue of the variation which t may undergo. But this hypothesis that x and y vary as t alters, supposes that there are relations between x and t, and between y and t. One of these relations is arbitrary, for the equation which we represent generally by $y = fx$, for example

$$y = \frac{a}{b}(x - c)^2,$$

if we substitute between x and t, the arbitrary relation,

$$x = \frac{t^3}{c^2},$$

this value being put in the equation

$$y = \frac{a}{b^2}(x - c)^2$$

will change it to

$$y = a\frac{(t^3 - c^3)^2}{b^2 c^4},$$

an equation which, being combined with this,

$$x = \frac{t^3}{c^2},$$

ought to reproduce by elimination,

$$y = a\frac{(x - c)^2}{b^2},$$

the only condition which we ought to regard in the selection of the variable t.

We may therefore determine the independent variable t at pleasure. For example, we may take the chord, the arc, the abscissa or ordinate for this independent variable; if t represent the arc of the curve, we have

$$t = \sqrt{(dx^2 + dy^2)};$$

if t denote the chord and the origin be at the vertex of the curve, we have

$$t = \sqrt{(x^2 + y^2)};$$

lastly, if t be the abscissa or ordinate of the curve, we shall have

$$t = x, \text{ or } t = y.$$

The choice of one of the three hypotheses or of any other, becoming indispensible in order that the formula which contains the differentials, may be delivered from them, if we do not always adopt it, it is even then tacitly supposed that the independent variable has been determined. For example, in the usual case where a formula contains only the differentials dx, dy, d^2y, d^3y, &c. the hypothesis is that the independent variable t has been taken for the abscissa, for then it results that

$$t = x, \frac{dx}{dt} = 1,$$

$$\frac{d^2x}{dt^2} = 0,$$

$$\frac{d^3x}{dt^3} = 0, \text{ \&c.}$$

and we see that the formula does not contain the second, third, &c. differentials.

To establish this formula, in all its generality, we must, as above, suppose x and y to be functions of a third variable t, and then we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt};$$

from which we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \ldots\ldots\ldots\ldots \quad (53)$$

taking the second differential of y and operating upon the second member as if a fraction, we shall get

$$\frac{d^2y}{dx} = \frac{\frac{dx}{dt} \cdot \frac{d^2y}{dt} - \frac{dy}{dt} \cdot \frac{d^2x}{dt}}{\frac{dx^2}{dt^2}}.$$

In this expression, d t has two uses; the one is to indicate that it is the independent variable, and the other to enter as a sign of algebra. In the second relation only will it be considered, if we keep in view that t is the independent variable. Then supposing dt^2 the common factor, the above expression simplifies into

$$\frac{d^2y}{dx} = \frac{dx\,d^2y - dy\,d^2x}{dx^2},$$

and dividing by d x, it will become

$$\frac{d^2y}{dx^2} = \frac{dx\,d^2y - dy\,d^2x}{dx^3}.$$

Operating in the same way upon the equation (53), we see that in taking t as the independent variable, the second member of the equation ought to become identical with the first; consequently we have only one change to make in the formula which contains the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, viz. to replace $\frac{d^2y}{dx^2}$ by

$$\frac{dx\,d^2y - dy\,d^2x}{dx^2} \quad \ldots\ldots\ldots \quad (54)$$

To apply these considerations to the radius of curvature which is given by the equation See p. 61. vol. I.)

$$R = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

if we wish to have the value of R, in the case where t shall be the independent variable, we must change the equation to

$$R = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{dx\,d^2y - dy\,d^2x}{dx^3}};$$

and observing that the numerator amounts to

$$\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx^3}$$

we shall have

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,d^2y - dy^2\,d^2x} \quad \ldots \quad (55)$$

This value of R supposes therefore that x and y are functions of a third independent variable. But if x be considered this variable, that is to say, if $t = x$, we shall have $d^2x = 0$, and the expression again reverts to the common one

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,d^2y} = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

But if, instead of x for the independent variable, we wish to have the ordinate y, this condition is expressed by $y = t$; and differentiating this equation twice, we have

$$\frac{dy}{dt} = 1, \quad \frac{d^2y}{dt^2} = 0.$$

The first of these equations merely indicates that y is the independent variable, which effects no change in the formula; but the second shows us that d^2y ought to be zero, and then the equation (55) becomes

$$R = -\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dy\,d^2x} \quad \ldots \quad (56)$$

We next remark, that when x is the independent variable, and consequently $d^2x = 0$, this equation indicates that dx is constant. Whence it follows, that generally the independent variable has always a constant differential.

Lastly, if we take the arc for the independent variable, we shall have

$$dt = \sqrt{(dx^2 + dy^2)};$$

Hence, we easily deduce

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = 1;$$

differentiating this equation, we shall regard d t as constant, since t is the independent variable; we get

$$\frac{2\,dx\,d^2x}{dt^2} + \frac{2\,dy\,d^2y}{dt^2} = 0;$$

which gives

$$dx\,d^2x = -dy\,d^2y.$$

Consequently, if we substitute the value of d^2x, or that of d^2y, in the equation (55), we shall have in the first case

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{(dx\ + dy^2)\,d^2y}\,dx = \frac{\surd(dx^2 + dy^2)}{d^2y}\,dx \quad . \quad (57)$$

and in the second case,

$$R = -\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{(dx\ + dy^2)\,d^2x}\,dy = -\frac{\surd(dx^2 + dy^2)}{d^2x}\,dy \quad . \quad (58)$$

In what precedes, we have only considered the two differential coefficients

$$\frac{dy}{dx}, \frac{d^2y}{dx^2};$$

but if the formula contain coefficients of a higher order, we must, by means analogous to those here used, determine the values of

$$\frac{d^3y}{dx^3} \text{ of } \frac{d^4y}{dx^4}, \text{ \&c.}$$

which will belong to the case where x and y are functions of a third independent variable.

PROPERTIES OF HOMOGENEOUS FUNCTIONS.

If $M\,dx + N\,dy + P\,dt + \text{\&c.} = dz$, *be a homogeneous function of any number of variables*, x, y, t, &c. *in which the dimension of each term is* n, *then is*

$$Mx + Ny + Pt + \text{\&c.} = nz.$$

For let $M\,dx + N\,dy$ be the differential of a homogeneous function z between two variables x and y; if we represent by n the sum of the exponents of the variables, in one of the terms which compose this function, we shall have therefore the equation

$$M\,dx + N\,dy = dz.$$

Making $\frac{y}{x} = q$, we shall find (vol. I.)

$$F(q) \times x^n = z;$$

and replacing, in the above equation, y by its value q x, and calling M' N', what M and N then become, that equation transforms to

$$M' dx + N' d.\ qx = dz;$$

and substituting the value of z, we shall have

$$M' dx + N' d(qz) = d(x^n F.q.)$$

But $d(qz) = q\,dx + x\,dq$. Therefore

$$(M' + N'q)\,dx + N'x\,dq = d(x^n F.q).$$

But, $(M' + N'q)\,dx$ being the differential of $x^n F q$ relatively to x, we have (Art. 6. vol. 1.)

$$M' + N'q = nx^{n-1} \times F.q.$$

If in this equation y be put for q x, it will become

$$M + N\frac{y}{x} = nx^{n-1} F.q,$$

or

$$Mx + Ny = nz.$$

This theorem is applicable to homogeneous functions of any number of variables; for if we have, for example, the equation

$$M\,dx + N\,dy + P\,dt = dz,$$

in which the dimension is n in every term, it will suffice to make

$$\frac{y}{x} = q, \frac{t}{x} = r$$

to prove, by reasoning analogous to the above, that we get $z = x^n F(q, r)$, and, consequently, that

$$Mx + Ny + Pt = nz \quad \ldots\ldots\ldots \quad (59)$$

and so on for more variables.

THEORY OF ARBITRARY CONSTANTS.

An equation $V = 0$ between x, y, and constants, may be considered as the complete integral of a certain differential equation, of which the order depends on the number of constants contained in $V = 0$. These constants are named *arbitrary constants*, because if one of them is represented by *a*, and V or one of its differentials is put under the form $f(x, y) = a$, we see that *a* will be nothing else than the arbitrary constant given by the integration of $d\,f(x, y)$. Hence, if the differential equation in question is of the order *n*, each integration introducing an arbitrary constant, we have $V = 0$, which is given by the last of three integrations, and contains, at

least, n arbitrary constants more than the given differential equation. Let therefore

$$F(x, y) = 0, F\left(x, y, \frac{dy}{dx}\right) = 0, F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \text{ \&c.} \quad (a)$$

be the primitive equation of a differential equation of the second order and its immediate differentials.

Hence we may eliminate from the two first of these three equations, the constants a and b, and obtain

$$\varphi\left(x, y, \frac{dy}{dx}, b\right) = 0, \varphi\left(x, y, \frac{dy}{dx}, a\right) = 0 \quad \ldots \quad (b)$$

If, without knowing $F(x, y) = 0$, we find these equations, it will be sufficient to eliminate from them $\frac{dy}{dx}$, to obtain $F(x, y) = 0$, which will be the complete integral, since it will contain the arbitrary constants a, b.

If, on the contrary, we eliminate these two constants between the above three equations, we shall arrive at an equation which, containing the same differential coefficients, may be denoted by

$$\varphi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \quad \ldots \quad (c)$$

But each of the equations (b) will give the same. In fact, by eliminating the constant contained in one of these equations and its immediate differential, we shall obtain separately two equations of the second order, which do not differ from equation (c) otherwise than the values of x and of y are not the same in both. Hence it follows, that a differential equation of the second order may result from two equations of the first order which are necessarily different, since the arbitrary constant of the one is different from that of the other. The equations (b) are what we call the first integrals of the equation (c), which is independent, and the equation $F(x, y) = 0$ is the second integral of it.

Take, for example, the equation $y = ax + b$, which, because of its two constants, may be regarded as the primitive equation of an equation of the second order. Hence, by differentiation, and then by elimination of a, we get

$$\frac{dy}{dx} = a, y = x\frac{dy}{dx} + b.$$

These two first integrals of the equation of the second order which we are seeking, being differentiated each in particular, conduct equally, by the elimination of a, b, to the independent equation $\frac{d^2y}{dx^2} = 0$. In the

case where the number of constants exceeds that of the required arbitrary constants, the surplus constants, being connected with the same equations, do not acquire any new relation. Required, for instance, the equation of the second order, whose primitive is

$$y = \tfrac{1}{2} a x^2 + b x + c = 0;$$

differentiating we get

$$\frac{dy}{dx} = ax + b.$$

The elimination of a, and then that of b, from these equations, give separately these two first integrals

$$\frac{dy}{dx} = ax + b, \quad y = x\frac{dy}{dx} - \tfrac{1}{2} a x^2 + c \quad . \quad . \quad . \quad (d)$$

Combining them each with their immediate differentials, we arrive, by two different ways, at $\frac{d^2y}{dx^2} = a$. If, on the contrary, we had eliminated the third constant a between the primitive equation and its immediate differential, that would not have produced a different result; for we should have arrived at the same result as that which would lead to the elimination of a from the equations (d), and we should then have fallen upon the equation $x\frac{d^2y}{dx^2} = \frac{dy}{dx} - b$, an equation which reduces to $\frac{d^2y}{dx^2} = a$ by combining it with the first of the equations (d).

Let us apply these considerations to a differential equation of the third order: differentiating three times successively the equation $F(x, y) = 0$, we shall have

$$F\left(x, y, \frac{dy}{dx}\right) = 0, \; F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0, \; F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}\right) = 0$$

These equations admitting the same values for each of the arbitrary constants contained by $F(x, y) = 0$, we may generally eliminate these constants between this latter equation and the three preceding ones, and obtain a result which we shall denote by

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

This will be the differential equation of the third order of $F(x, y) = 0$, and whose three arbitrary constants are eliminated; reciprocally, $F(x, y) = 0$, will be the third integral of the equation (e).

If we eliminate successively each of the arbitrary constants from the

equation $F(x, y) = 0$, and that which we have derived by differentiation, we shall obtain three equations of the first order, which will be the second integrals of the equation (e).

Finally, if we eliminate two of the three arbitrary constants by means of the equation $F(x, y) = 0$, and the equations which we deduce by two successive differentiations, that is to say, if we eliminate these constants from the equations

$$F\left(x, y\right) = 0,\ F\left(x, y, \frac{dy}{dx}\right) = 0,\ F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \quad . \quad . \quad (f)$$

we shall get, successively, in the equation which arises from the elimination, one of the three arbitrary constants; consequently, we shall have as many equations as arbitrary constants. Let a, b, c, be these arbitrary constants. Then the equations in question, considered only with regard to the arbitrary constants which they contain, may be represented by

$$\varphi c = 0,\ \varphi b = 0,\ \varphi a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

Since the equations (f) all aid in the elimination which gives us one of these last equations, it thence follows that the equations (g) will each be of the second order; we call them the first integrals of the equation (e).

Generally, a differential equation of an order n will have a number n of first integrals, which will contain therefore the differential coefficients from $\frac{dy}{dx}$ to $\frac{d^{n-1}y}{dx^{n-1}}$ inclusively; that is to say, a number $_{n-1}$ of differential coefficients; and we see that then, when these equations are all known, to obtain the primitive equation it will suffice to eliminate from these equations the several differential coefficients.

PARTICULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS.

It is easily seen that a particular integral may always be deduced from the complete integral, by giving a suitable value to the arbitrary constant.

For example, if we have given the equation

$$x\,dx + y\,dy = dy\sqrt{x^2 + y^2 - a^2},$$

whose complete integral is

$$y + c = \sqrt{(x^2 + y^2 - a^2)},$$

whence (for convenience, by rationalizing,) we get

$$(a^2 - x^2)\frac{dy^2}{dx^2} + 2xy\frac{dy}{dx} + x^2 = 0 \quad . \quad . \quad . \quad . \quad (h)$$

and the complete integral becomes

$$2cy + c^2 - x^2 + a^2 = 0 \quad . \quad . \quad . \quad . \quad (i)$$

Hence, in taking for c an arbitrary constant value $c = 2a$, we shall obtain this particular integral

$$2cy + 5a^2 - x^2 = 0,$$

which will have the property of satisfying the proposed equation (h) as well also as the complete integral. In fact, we shall derive from this particular integral

$$y = \frac{x^2 - 5a^2}{2c}, \quad \frac{dy}{dx} = \frac{x}{c};$$

these values reduce the proposed to

$$(x^2 - a^2)\frac{x^2}{c^2} = \frac{x^2}{c^2}(x^2 + c^2 - 5a^2),$$

an equation which becomes identical, by substituting in the second member, the value of c^2, which gives the relation $c = 2a$. Let

$$M\,dx + N\,dy = 0,$$

be a differential equation of the first order of a function of two variables x and y; we may conceive this equation as derived by the elimination of a constant c from a certain equation of the same order, which we shall represent by

$$m\,dx + n\,dy = 0,$$

and the complete integral

$$F(x, y, c) = 0,$$

which we shall designate by u. But, since every thing is reduced to taking the constant c such, that the equation

$$M\,dx + N\,dy = 0,$$

may be the result of elimination, we perceive that is at the same time permitted to vary the constant c, provided the equation

$$M\,dx + N\,dy = 0,$$

holds good; in this case, the complete integral

$$F(x, y, c) = 0$$

will take a greater generality, and will represent an infinity of curves of the same kind, differing from one another by a parameter, that is, by a constant.

Suppose therefore that the complete integral being differentiated, by considering c as the variable, we have obtained

$$dy = \left(\frac{dy}{dx}\right)dx + \left(\frac{dy}{dc}\right)dc$$

an equation which, for brevity, we shall write

$$d\,y = p\,d\,x + q\,d\,c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (k)$$

Hence it is clear, that if p remaining finite, q d c is nothing, the result of the elimination of c as a variable from

$$F\,(x, y, c) = 0,$$

and the equation (k), will be the same as that arising from c considered constant, from

$$F\,(x, y, c) = 0,$$

and the equation

$$d\,y = p\,d\,x$$

(this result is on the hypothesis

$$M\,d\,x + N\,d\,y = 0),$$

for the equation (k), since

$$q\,d\,c = 0,$$

does not differ from

$$d\,y = p\,d\,x;$$

but in order to have

$$q\,d\,c = 0,$$

one of the factors of this equation = constant, that is to say, that we have

$$d\,c = 0, \text{ or } q = 0.$$

In the first case, $d\,c = 0$, gives $c = constant$, since that takes place for particular integrals; the second case, only therefore conducts to a particular solution. But, q being the coefficient of d c of the equation (k), we see that $q = 0$, gives

$$\frac{d\,y}{d\,x} = 0.$$

This equation will contain c or be independent of it. If it contain c, there will be two cases; either the equation $q = 0$, will contain only c and constants, or this equation will contain c with variables. In the first case, the equation $q = 0$, will still give $c = constant$, and in the second case, it will give $c = f\,(x, y)$; this value being substituted in the equation $F\,(x, y, c) = 0$, will change it into another function of x, y, which will satisfy the proposed, without being comprised in its complete integral, and consequently will be a singular solution; but we shall have a particular integral if the equation $c = f\,(x, y)$, by means of the complete integral, is reduced to a constant.

When the factor $q = 0$ from the equation $q\,dc = 0$ not containing the arbitrary constant c, we shall perceive whether the equation $q = 0$ gives rise to a particular solution, by combining this equation with the complete integral. For example, if from $q = 0$, we get $x = M$, and put this value in the complete integral $F(x, y, c) = 0$, we shall obtain

$$c = \text{constant} = B \text{ or } c = fy;$$

In the first case, $q = 0$, gives a particular integral; for by changing c into B in the complete integral, we only give a particular value to the constant, which is the same as when we pass from the complete integral to a particular integral. In the second case, on the contrary, the value fy introduced instead of c in the complete integral, will establish between x and y a relation different from that which was found by merely replacing c by an arbitrary constant. In this case, therefore, we shall have a particular solution. What has been said of y, applies equally to x.

It happens sometimes that the value of c presents itself under the form $\frac{0}{0}$: this indicates a factor common to the equations u and U which is extraneous to them, and which must be made to disappear.

Let us apply this theory to the research of particular solutions, when the complete integral is given.

Let the equation be

$$y\,dx - x\,dy = a\sqrt{(dx^2 + dy^2)}$$

of which the complete integral is thus found.

Dividing the equation by dx, and making

$$\frac{dy}{dx} = p$$

we obtain

$$y - px = a\sqrt{(1 + p^2)}.$$

Then differentiating relatively to x and to p, we get

$$dy - p\,dx - x\,dp = \frac{a\,p\,dp}{\sqrt{(1 + p^2)}};$$

observing that

$$dy = p\,dx,$$

this equation reduces to

$$x\,dp + \frac{a\,p\,dp}{\sqrt{(1 + p^2)}} = 0$$

and this is satisfied by making $dp = 0$. This hypothesis gives $p = \text{constant} = c$, a value which being put in the above equation gives

$$y - cx = a\sqrt{(1 + c^2)} \quad \ldots\ldots\ldots \quad (l)$$

This equation containing an arbitrary constant c, which is not to be found in the proposed equation, is the complete integral of it.

This being accomplished, the part $q\,dc$ of the equation $dy = p\,dx + q\,dc$ will be obtained by differentiating the last equation relatively to c regarded as the only variable. Operating thus we shall have

$$x\,dc + \frac{a\,c\,dc}{\sqrt{(1 + c^2)}} = 0;$$

consequently the coefficients of dc, equated to zero, will give us

$$x = -\frac{ac}{\sqrt{(1 + c^2)}} \quad \ldots\ldots\ldots \quad (m)$$

To find the value of c, we have

$$(1 + c^2)\,x^2 = a^2 c^2,$$

which gives

$$c^2 = \frac{x^2}{a^2 - x^2}, \quad 1 + c^2 = \frac{a^2}{a^2 - x^2},$$

and

$$\sqrt{(1 + c^2)} = \frac{a}{\sqrt{(a^2 - x^2)}};$$

by means of this last equation, eliminating the radical of the equation (m) we shall thus obtain

$$c = -\frac{x}{\sqrt{(a^2 - x^2)}} \quad \ldots\ldots\ldots \quad (n)$$

This value and that of $\sqrt{(1 + c^2)}$ being substituted in the equation (l) will give us

$$y + \frac{x^2}{\sqrt{(a^2 - x^2)}} = \frac{a^2}{\sqrt{(a^2 - x^2)}}$$

whence is derived

$$y = \sqrt{(a^2 - x^2)},$$

an equation which, being squared, will give us

$$y^2 = a^2 - x^2;$$

and we see that this equation is a particular solution, for by differentiating it we obtain

$$dy = -\frac{x\,dx}{y};$$

this value and that of $\sqrt{(x^2 + y^2)}$, being substituted in the equation originally proposed, reduce it to

$$a^2 = a^2.$$

In the application which we have just given, we have determined the

value of c by equating to zero the differential coefficient $\left(\frac{dy}{dc}\right)$. This process may sometimes prove insufficient. In fact, the equation

$$dy = p\,dx + q\,dc$$

being put under this form

$$A\,dx + B\,dy + C\,dc = 0$$

where A, B, C, are functions of x and y, we shall thence obtain

$$dy = -\frac{A}{B}dx - \frac{C}{B}dc \quad \ldots\ldots\ldots \text{(o)}$$

$$dx = -\frac{B}{A}dy - \frac{C}{A}dc \quad \ldots\ldots\ldots \text{(p)}$$

and we perceive that if all that has been said of y considered a function of x, is applied to x considered a function of y, the value of the coefficient of d c will not be the same, and that it will suffice merely that any factor of B destroys in C another factor than that which may destroy a factor of A, in order that the value of the coefficient of d c, on both hypotheses, may appear entirely different. Thus although very often the equations

$$\frac{C}{B} = 0, \frac{C}{A} = 0$$

give for c the same value, that will not always happen; the reason of which is, that when we shall have determined c by means of the equation

$$\frac{dy}{dc} = 0,$$

it will not be useless to see whether the hypothesis of $\frac{dx}{dc}$ gives the same result.

Clairaut was the first to remark a general class of equations susceptible of a particular solution; these equations are contained in the form

$$y = \frac{dy}{dx}x + F.\frac{dy}{dx}$$

an equation which we shall represent by

$$y = px + Fp \quad \ldots\ldots\ldots \text{(r)}$$

By differentiating it, we shall find

$$dy = p\,dx + x\,dp + \left(\frac{dFp}{dp}\right)dp;$$

this equation, since $dy = p\,dx$, becomes

$$x\,dp + \left(\frac{dFp}{dp}\right)dp = 0;$$

and since d p is a common factor, it may be thus written:

$$\left\{x + \left(\frac{d\,Fp}{dp}\right)\right\} dp = 0$$

We satisfy this equation by making $dp = 0$, which gives $p = \text{const.} = c$; consequently, by substituting this value in the equation (r) we shall find

$$y = cx + Fc.$$

This equation is the complete integral of the equation proposed, since an arbitrary constant c has been introduced by integration. If we differentiate relatively to c we shall get

$$\left\{x + \left(\frac{d\,Fc}{dc}\right)\right\} dc.$$

Consequently, by equating to zero the coefficients of d c, we have

$$x + \frac{d\,Fc}{dc} = 0,$$

which being substituted in the complete integral, will give the particular solution.

THE INTEGRATION OF EQUATIONS OF PARTIAL DIFFERENCES.

An equation which subsists between the differential coefficients, combined with variables and constants, is, in general, a partial differential equation, or an equation of partial differences. These equations are thus named, because the notation of the differential coefficients which they contain indicates that the differentiation can only be effected partially; that is to say, by regarding certain variables as constant. This supposes, therefore, that the function proposed contains only one variable.

The first equation which we shall integrate is this; viz.

$$\left(\frac{dz}{dx}\right) = a.$$

If contrary to the hypothesis, z instead of being a function of two variables x, y, contains only x, we shall have an ordinary differential equation, which, being integrated, will give

$$z = ax + c$$

but, in the present case, z being a function of x and of y, the *y*s contained in z have been made to disappear by differentiation, since differen-

tiating relatively to x, we have considered y as constant. We ought, therefore, when integrating, to preserve the same hypothesis, and suppose that the arbitrary constant is in general a function of y; consequently, we shall have for the integral of the proposed equation

$$z = ax + \varphi y.$$

Required to integrate the equation

$$\left(\frac{dz}{dx}\right) = X,$$

in which X is any function of x. Multiplying by dx, and integrating, we get

$$z = \int X\,dx + \varphi y.$$

For example, if the function X were $x^2 + a^2$, the integral would be

$$z = \frac{x^3}{3} + a^2 x + \varphi y.$$

In like manner, it is found that the integral of

$$\left(\frac{dz}{dx}\right) = Y$$

is

$$z = xY + \varphi y.$$

Similarly, we shall integrate every equation in which $\left(\frac{dz}{dx}\right)$ is equal to a function of two variables x, y. If, for example,

$$\left(\frac{dz}{dx}\right) = \frac{x}{\sqrt{ay + x^2}},$$

considering y as constant, we integrate by the ordinary rules, making the arbitrary constant a function of y. This gives

$$z = \sqrt{(ay + x^2)} + \varphi y.$$

Finally, if we wish to integrate the equation

$$\left(\frac{dz}{dx}\right) = \frac{1}{\sqrt{(y^2 - x^2)}}$$

regarding y as constant, we get

$$z = \sin.^{-1}\frac{x}{y} + \varphi y.$$

Generally to integrate the equation

$$\left(\frac{dz}{dx}\right) dx = F(x, y)\,dx,$$

we shall take the integral relatively to x, and adding to it an arbitrary function of y, as the constant, to complete it, we shall find

$$z = \int F(x, y)\,dx + \varphi y.$$

Now let us consider the equations of partial differences which contain two differential coefficients of the first order; and let the equation be

$$M.\left(\frac{dz}{dx}\right) + N\left(\frac{dz}{dy}\right) = 0$$

in which M and N represent given functions of x, y. Hence

$$\left(\frac{dz}{dy}\right) = -\frac{M}{N}\left(\frac{dz}{dx}\right);$$

substituting this value in the formula

$$dz = \left(\frac{dz}{dx}\right)dx + \left(\frac{dz}{dy}\right)dy,$$

which has no other meaning than to express the condition that z is a function of x and of y, we obtain

$$dz = \left(\frac{dz}{dx}\right)\left\{dx - \frac{M}{N}dy\right\}$$

or

$$dz = \left(\frac{dz}{dx}\right)\frac{N\,dx - M\,dy}{N}.$$

Let λ be the factor proper to make $N\,dx - M\,dy$ a complete differential ds; we shall have

$$\lambda\,(N\,dx - M\,dy) = ds.$$

By means of this equation, we shall eliminate $N\,dx - M\,dy$ from the preceding equation, and we shall obtain

$$dz = \frac{1}{\lambda N}\cdot\left(\frac{dz}{dx}\right).ds.$$

Finally, if we remark that the value of $\left(\frac{dz}{dx}\right)$ is indeterminate, we may take it such that $\frac{1}{\lambda N}\cdot\left(\frac{dz}{dx}\right)ds$ may be integrable, which would make it a function of s; for we know that the differential of every given function of s must be of the form $Fs.ds$. It therefore follows, that we may assume

$$\frac{1}{\lambda N}\cdot\left(\frac{dz}{dx}\right) = Fs,$$

an equation which will change the preceding one into

$$dz = Fs.ds$$

which gives

$$z = \varphi s.$$

Integrating by this method the equation

$$x\left(\frac{d\,z}{d\,y}\right) - y\left(\frac{d\,z}{d\,x}\right) = 0$$

we have in this case

$$M = -y,$$

and

$$N = x;$$

consequently

$$d\,s = \lambda\,(x\,d\,x + y\,d\,y).$$

It is evident that the factor necessary to make this integrable is z. Substituting this for λ and integrating, we get

$$s = x^2 + y^2.$$

Hence the integral of the proposed equation is

$$z = \varphi\,(x^2 + y^2).$$

Now let us consider the equation

$$P\left(\frac{d\,z}{d\,x}\right) + Q\left(\frac{d\,z}{d\,y}\right) + R = 0;$$

in which P, Q, R are functions of the variables x, y, z; dividing it by P and making

$$\frac{Q}{P} = M,\ \frac{R}{P} = N,$$

we shall put it under this form:

$$\left(\frac{d\,z}{d\,x}\right) + M\left(\frac{d\,z}{d\,y}\right) + N = 0;$$

and again making

$$\left(\frac{d\,z}{d\,x}\right) = p,$$

and

$$\left(\frac{d\,z}{d\,y}\right) = q,$$

it becomes

$$p + M\,q + N = 0. \quad \ldots\ldots\ldots\ldots \quad (a)$$

This equation establishes a relation between the coefficients p and q of the general formula

$$d\,z = \left(\frac{d\,z}{d\,x}\right) d\,x + \left(\frac{d\,z}{d\,y}\right) d\,y$$
$$= p\,d\,x + q\,d\,y;$$

without which relation p and q would be perfectly arbitrary, for as it has been already observed, this formula has no other meaning than to indicate that z is a function of two variables x, y, and that function may be any

whatever; so that we ought to regard p and q as indeterminate in this last equation. Eliminating p from it, we shall obtain

$$dz + N\,dx = q\,(dy - M\,dx)$$

and q will remain always indeterminate. Hence the two members of this equation are heterogeneous (See Art. 6. vol. 1), and consequently

$$dz + N\,dx = 0,\quad dy - M\,dx = 0. \quad . \quad . \quad . \quad . \quad (b)$$

If P, Q, R do not contain the variable z, it will be the same of M and N; so that the second of these equations will be an equation of two variables x and y, and may become a complete differential by means of a factor λ. This gives

$$\lambda\,(dy - M\,dx) = 0.$$

The integral of this equation will be a function of x and of y, to which we must add an arbitrary constant s; so that we shall have

$$F(x, y) = s;$$

whence we derive

$$y = f(x, s).$$

Such will be the value of y given us by the second of the above equations; and to show that they subsist simultaneously we must substitute this value in the first of them. But although the variable y is not shown, it is contained in N. This substitution of the value of y just found, amounts to considering y in the first equation as a function of x and of the arbitrary constant s. Integrating therefore this first equation on that hypothesis we find

$$z = -\int N\,dx + \varphi s.$$

To give an example of this integration, take the equation

$$x\left(\frac{dz}{dx}\right) + y\left(\frac{dz}{dy}\right) = a\sqrt{(x^2 + y^2)};$$

and comparing it with the general equation (a), we have

$$M = \frac{y}{x},\quad N = -\frac{a}{x}\sqrt{(x^2 + y^2)}.$$

These values being substituted in the equations (b) will change them to

$$dz - \frac{a}{x}\sqrt{(x^2 + y^2)}\,dx = 0,\quad dy - \frac{y}{x}\,dx = 0. \quad . \quad . \quad . \quad . \quad (c)$$

Let λ be the factor necessary to make the last of these integrable, and we have

$$\lambda\left(dy - \frac{y}{x}\,dx\right) = 0,$$

or rather

$$\lambda . \frac{x\,dy - y\,dx}{x} = 0;$$

which is integrable when $\lambda = \frac{1}{x}$; for then the integral is $\frac{y}{x} =$ constant. Put therefore

$$\frac{y}{x} = s$$

and consequently

$$y = s\,x.$$

By means of this value of y, we change the first of the equations (c) into

$$dz - a\,\frac{\sqrt{x^2 - s^2 x^2}}{x}\,.\,dx = 0,$$

or rather into

$$dz = a\,dx\,\sqrt{(1 + s^2)}.$$

Integrating on the supposition that s is constant, we shall obtain

$$z = a\int dx\,\sqrt{(1 + s^2)} + \varphi s$$

and consequently

$$z = a\,x\,\sqrt{(1 + s^2)} + \varphi s.$$

Substituting for s its value we get

$$z = a\,x\,\sqrt{\left(1 + \frac{y^2}{x^2}\right)} + \varphi\left(\frac{y}{x}\right)$$

$$= a\,\sqrt{(x^2 + y^2)} + \varphi\left(\frac{y}{x}\right).$$

In the more general case where the coefficients P, Q, R of the equation contain the three variables x, y, z it may happen that the equations (b) contain only the variables which are visible, and which consequently we may put under the forms

$$dz = f(x, z)\,dx = 0,\quad dy = F(x, y)\,dx.$$

These equations may be treated distinctly, by writing them as above,

$$z = \int f(x, z)\,dx + \quad z,\quad y = \int F(x, y)\,dx + \Phi y$$

for then we see we may make z constant in the first equation and y in the second; contradictory hypotheses, since one of three coordinates x, y, z cannot be supposed constant in the first equation without its being not constant in the second.

Let us now see in what way the equations (b) may be integrated in the case where they only contain the variables which are seen in them.

Let μ and λ be the factors which make the equations (b) integrable. If their integrals thus obtained be denoted by U and by V, we have

$$\lambda\,(dz + N\,dx) = dU,\quad \mu\,(dy - M\,dx) = dV.$$

By means of these values the above equation will become

$$dU = q\frac{\lambda}{\mu}dV \quad \ldots\ldots\ldots \quad (d).$$

Since the first member of this equation is a complete differential the second is also a complete differential, which requires $q\frac{\lambda}{\mu}$ to be a function of V. Represent this function by φV. Then the equation (d) will become

$$dU = \varphi V . dV$$

which gives, by integrating,

$$U = \Phi V.$$

Take, for example, the equation

$$xy\left(\frac{dz}{dx}\right) + x^2\left(\frac{dz}{dy}\right) = yz;$$

which being written thus, viz.

$$\left(\frac{dz}{dx}\right) + \frac{x}{y}\left(\frac{dz}{dy}\right) - \frac{z}{x} = 0$$

we compare it with the equation

$$\left(\frac{dz}{dx}\right) + M\left(\frac{dz}{dy}\right) + N = 0$$

and obtain

$$M = \frac{x}{y}, \; N = -\frac{z}{x}$$

By means of these values the equations (b) becomes

$$dz - \frac{z}{x}.dx = 0, \; dy - \frac{x}{y}dx = 0;$$

which reduce to

$$x\,dz - z\,dx = 0, \; y\,dy - x\,dx = 0.$$

The factors necessary to make these integrable are evidently $\frac{1}{x^2}$ and 2. Substituting which and integrating, we find $\frac{z}{x}$ and $y^2 - x^2$ for the integrals. Putting, therefore, these values for U and V in the equation $U = \Phi V$, we shall obtain, for the integral of the proposed equation,

$$\frac{z}{x} = \Phi(y^2 - x^2)$$

It must be remarked, that, if we had eliminated q instead of p, the equations (b) would have been replaced by these

$$M\,dz + N\,dy = 0, \; dy - M\,dx = 0 \quad \ldots \quad (e)$$

and since all that has been said of equations (b) applies equally to these,

it follows that, in the case where the first of equations (b) was not integrable, we may replace those equations by the system of equations (e), which amounts to employing the first of the equations (e) instead of the first of the equations (b).

For instance, if we had

$$a z \left(\frac{d z}{d x}\right) - z x \left(\frac{d z}{d y}\right) + x y = 0;$$

this equation being divided by a z and compared with

$$\left(\frac{d z}{d x}\right) + M \left(\frac{d z}{d y}\right) + N = 0$$

will give us

$$M = -\frac{x}{a}, N = \frac{x y}{a z}$$

and the equations (b) will become

$$d z + \frac{x y}{a z} d x = 0, d y + \frac{x}{a} d x = 0;$$

which reduce to

$$a z d z + x y d x = 0, a d y + x d x = 0 \quad . \quad . \quad . \quad (f)$$

The first of these equations, which, containing three variables, is not immediately integrable, we replace by the first of the equations (e), and we shall have, instead of the equations (f), these

$$-\frac{x}{a} d z + \frac{x y}{a z} d y = 0, a d y + x d x = 0;$$

which reduce to

$$2 y d y - 2 z d z = 0, 2 a d y + 2 x d x = 0;$$

equations, whose integrals are

$$y^2 - z^2, 2 a y + x^2.$$

These values being substituted for U and V, will give us

$$y^2 - z^2 = \varphi (2 a y + x^2).$$

It may be remarked, that the first of equations (e) is nothing else than the result of the elimination of d x from the equations (b).

Generally we may eliminate every variable contained in the coefficients M, N, and in a word, combine these equations after any manner whatever; if after having performed these operations, and we obtain two integrals, represented by U = a, V = b, a and b being arbitrary constants, we can always conclude that the integral is U = Φ V. In fact, since a and b are two arbitrary constants, having taken b at pleasure, we may compose a in terms of b in any way whatsoever; which is tantamount to saying that we may take for *a* an arbitrary function of b. This condition will be expressed by the equations a = φ (b). Consequently, we shall

have the equations $U = \varphi b$, $V = b$, in which x, y, z represent the same coordinates. If we eliminate (b) from these equations, we shall obtain $U = \varphi V$.

This equation also shows us that in making $V = b$, we ought to have $U = \varphi b =$ *constant*; that is to say, that U and V are at the same time constant; without which a and b would depend upon one another, whereas the function φ is arbitrary. But this is precisely the condition expressed by the equations $U = a$, $V = b$.

To give an application of this theorem, let

$$z x \left(\frac{d z}{d x}\right) - z y \left(\frac{d z}{d y}\right) - y^2 = 0.$$

Dividing by z x and comparing it with the general equation we have

$$M = -\frac{y}{x}, \; N = -\frac{y^2}{z x};$$

and the equations (b) give us

$$d z - \frac{y^2}{z x} d x = 0, \; d y + \frac{y}{x} d x = 0$$

or

$$z x d z - y^2 d x = 0, \; x d y + y d z = 0.$$

The first of these equations containing three variables we shall not attempt its integration in that state; but if we substitute in it for y d x its value derived from the second equation, it will acquire a common factor x, which being suppressed, the equation becomes

$$z d z + y d y = 0,$$

and we perceive that by multiplying by 2 it becomes integrable. The other equation is already integrable, and by integrating we find

$$z^2 + y^2 = a, \; x y = b,$$

whence we conclude that

$$z^2 + y^2 = \varphi x y.$$

We shall conclude what we have to say upon equations of partial differences of the first order, by the solution of this problem.

Given an equation which contains an arbitrary function of one or more variables, to find the equation of partial differences which produced it.

Suppose we have

$$z = F (x^2 + y^2).$$

Make

$$x^2 + y^2 = u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

and the equation becomes

$$z = F u.$$

The differential of F u must be of the form φ u . d u. Consequently

$$d z = d u . \varphi u$$

If we take the differential of z relatively to x only, that is to say, in regarding y as constant, we ought to take also d u on the same hypothesis. Consequently, dividing the preceding equation by d x, we get

$$\left(\frac{d z}{d x}\right) = \left(\frac{d u}{d x}\right) \varphi u.$$

Again, considering x as constant and y as variable, we shall similarly find

$$\left(\frac{d z}{d y}\right) = \left(\frac{d u}{d y}\right) \varphi u.$$

But the values of these coefficients are found from the equation (f), which gives

$$\left(\frac{d u}{d x}\right) = 2 x, \left(\frac{d u}{d y}\right) = 2 y.$$

Hence our equations become

$$\left(\frac{d z}{d x}\right) = 2 x \varphi u, \left(\frac{d z}{d y}\right) = 2 y \varphi u;$$

and eliminating φ u from these, we get the equation required; viz.

$$y \left(\frac{d z}{d x}\right) = x \left(\frac{d z}{d y}\right).$$

As another example, take this equation

$$z^2 + 2 a x = F (x - y).$$

Making

$$x - y = u,$$

it becomes

$$z^2 + 2 a x = F u$$

and differentiating, we get

$$d (z^2 + 2 a x) = d u \varphi u.$$

Then taking the differential relatively to x, we have

$$2 z \left(\frac{d z}{d x}\right) + 2 a = \left(\frac{d u}{d x}\right) \varphi u.$$

and similarly, with regard to y, we get

$$2 z \left(\frac{d z}{d y}\right) = \left(\frac{d u}{d y}\right) \varphi u.$$

But since

$$x - y = u$$

$$\left(\frac{d\,u}{d\,x}\right) = 1, \left(\frac{d\,u}{d\,y}\right) = -1,$$

which, being substituted in the above equation, gives us

$$2\,z\left(\frac{d\,z}{d\,x}\right) + 2\,a = \varphi\,u, 2\,z\left(\frac{d\,z}{d\,y}\right) = -\varphi\,u$$

and eliminating $\varphi\,u$ from these, we have the equation required; viz.

$$\left(\frac{d\,z}{d\,x}\right) + \left(\frac{d\,z}{d\,y}\right) + \frac{a}{z} = 0.$$

We now come to

EQUATIONS OF PARTIAL DIFFERENCES OF THE SECOND ORDER.

An equation of Partial Differences of the second order in which z is a function of two variables x, y ought always to contain one or more of the differential coefficients

$$\left(\frac{d^2\,z}{d\,x^2}\right), \left(\frac{d^2\,z}{d\,y^2}\right), \left(\frac{d^2\,z}{d\,x.d\,y}\right)$$

independently of the differential coefficients which enter equations of the first order.

We shall merely integrate the simplest equations of this kind, and shall begin with this, viz.

$$\left(\frac{d^2\,z}{d\,x^2}\right) = 0.$$

Multiplying by d x and integrating relatively to x we add to the integral an arbitrary function of y; and we shall thus get

$$\left(\frac{d\,z}{d\,x}\right) = \varphi\,y\,,$$

Again multiplying by d x and integrating, the integral will be completed when we add another arbitrary function of y, viz. $\downarrow y$. We thus obtain

$$z = x\,\varphi\,y + \downarrow y.$$

Now let us integrate the equation.

$$\left(\frac{d^2\,z}{d\,x^2}\right) = P$$

in which P is any function of x, y. Operating as before we first obtain

$$\left(\frac{dz}{dx}\right) = \int P\,dx + \varphi y;$$

and the second integration gives us

$$z = \int \{\int P\,dx + \varphi y\}\,dx + \psi y.$$

In the same manner we integrate

$$\left(\frac{d^2 z}{dy^2}\right) = P$$

and find

$$z = \int \{\varphi x + \int P\,dy\}\,dy + \psi x.$$

The equation

$$\left(\frac{d^2 z}{dy\,dx}\right) = P$$

must be integrated first relatively to one of the variables, and then relatively to the other, which will give

$$z = \int \{\varphi y + \int P\,dx\}\,dy + \varphi x.$$

In general, similarly may be treated the several equations

$$\left(\frac{d^n z}{dy^n}\right) = P,\ \left(\frac{d^n z}{dx\,dy^{n-1}}\right) = Q,\ \left(\frac{d^n z}{dx^2\,dy^{n-2}}\right) = R,\ \&c.$$

in which P, Q, R, &c. are functions of x, y, which gives place to a series of integrations, introducing for each of them an arbitrary function.

One of the next easiest equations to integrate is this:

$$\left(\frac{d^2 z}{dy^2}\right) + P\left(\frac{dz}{dy}\right) = Q;$$

in which P and Q will always denote two functions of x and y.

Make

$$\left(\frac{dz}{dy}\right) = u,$$

and the proposed will transform to

$$\left(\frac{du}{dy}\right) + Pu = Q.$$

To integrate this, we consider x constant, and then it contains only two variables y and u, and it will be of the same form as the equation

$$dy + Py\,dx = Q\,dx$$

whose integral (see Vol. 1. p. 109) is

$$y = e^{-\int P dx}\{\int Q\,e^{\int P dx}\,dx + C\}.$$

Hence our equation gives

$$u = e^{-\int P dy}\{\int Q e^{\int P_{dy}}\,dy + \varphi x\}.$$

But

$$u = \left(\frac{dz}{dy}\right).$$

Hence by integration we get

$$z = \int \{ e^{-\int P dy} (\int Q e^{\int P dy} dy) + \varphi x \} dy + \psi x.$$

By the same method we may integrate

$$\left(\frac{d^2 z}{dx\,dy}\right) + P\left(\frac{dz}{dx}\right) = Q, \quad \frac{d^2 z}{dx\,dy} + P\left(\frac{dz}{dy}\right) = Q,$$

in which P, Q represent functions of x, and because of the divisor $dx\,dy$, we perceive that the value of z will not contain arbitrary functions of the same variable.

THE DETERMINATION OF THE ARBITRARY FUNCTIONS WHICH ENTER THE INTEGRALS OF EQUATIONS OF PARTIAL DIFFERENCES OF THE FIRST ORDER.

The arbitrary functions which complete the integrals of equations of partial differences, ought to be given by the conditions arising from the nature of the problems from which originated these equations; problems generally belonging to the physical branches of the Mathematics.

But in order to keep in view the subject we are discussing, we shall limit ourselves to considerations purely analytical, and we shall first seek what are the conditions contained in the equation

$$\left(\frac{dz}{dx}\right) = a.$$

Since z is a function of x, y, this equation may be considered as that of a surface. This surface, from the nature of its equation, has the following property, that $\left(\frac{dz}{dx}\right)$ must always be constant. Hence it follows that every section of this surface made by a plane parallel to that of x, y is a straight line. In fact, whatever may be the nature of this section, if we divide it into an infinity of parts, these, to a small extent, may be considered straight lines, and will represent the elements of the section, one of these elements making with a parallel to the axis of abscissæ, an angle whose tangent is $\left(\frac{dz}{dx}\right)$. Since this angle is constant, it follows that all the angles formed in like manner by the elements of the curve, with par-

allels to the axis of abscissæ will be equal. Which proves that the section in question is a straight line.

We might arrive at the same result by considering the integral of the equation

$$\left(\frac{d\,z}{d\,x}\right) = a$$

which we know to be

$$z = a\,x + \varphi\,y,$$

since for all the points of the surface which in the cutting plane, the ordinate is equal to a constant c. Replacing therefore $\varphi\,y$ by $\varphi\,c$, and making $\varphi\,c = C$, the above equation becomes

$$z = a\,x + C;$$

this equation being that of a straight line, shows that the section is a straight line.

The same holding good relatively to other cutting planes which may be drawn parallel to that of x, z, we conclude that all these planes will cut the surface in straight lines, which will be parallel, since they will each form with a parallel to the axis of x, an angle whose tangent is a.

If, however, we make $x = 0$, the equation $z = a\,x + \varphi\,y$ reduces to $z = \varphi\,y$, and will be that of a curve traced upon the plane of y, z; this curve containing all the points of the surface whose coordinates are $x = 0$, will meet the plane in a point whose coordinate is $x = 0$; and since we have also $y = c$, the third coordinate by means of the equation

$$z = a\,x + C$$

will be

$$z = C.$$

What has been said of this one plane, applies equally to all others which are parallel to it, and it thence results that through all the points of the curve whose equation is $z = \varphi\,y$, and which is traced in the plane of y, z, will pass straight lines parallel to the axis of x. This is expressed by the equations

$$\left(\frac{d\,z}{d\,x}\right) = a$$

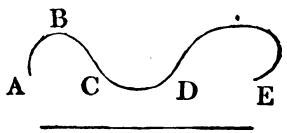
and

$$z = a\,x + \varphi\,y;$$

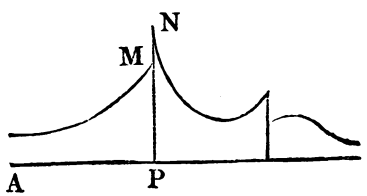
and since this condition is always fulfilled, whatever may be the figure of the curve whose equation is $z = \varphi\,y$, we see that this curve is arbitrary.

From what precedes, it follows that the curve whose equation is $z = \varphi y$,

may be composed of arcs of different curves, which unite at their extremities, as in this diagram



or which have a break off in their course, as in this figure.



In the first case the curve is *discontinuous*, and in the second it is *discontiguous*. We may remark that in this last case, two different ordinates P M, P N corresponding to the same abscissa A P; finally, it is possible, that without being discontiguous, the curve may be composed of an infinite series of arcs indefinitely small, which belong each of them to different curves; in this case, the curve is irregular, as will be, for instance, the flourishes of the pen made at random; but in whatever way it is formed, the curve whose equation is $z = \varphi y$, it will suffice, to construct the surface, to make a straight line move parallelly with this condition, that its general point shall trace out the curve whose equation is

$$z = \varphi y,$$

and which is traced at random upon the plane of y, z.

If instead of the equation

$$\left(\frac{dz}{dx}\right) = a,$$

we had

$$\left(\frac{dz}{dx}\right) = X,$$

in which X was a function of x, then in drawing a plane parallel to the plane (x, z), the surface will be cut by it no longer in a straight line, as in the preceding case. In fact, for every point taken in this section, the tangent of the angle formed by the element produced of the section, with a parallel to the axis of x, will be equal to a function X of the abscissa x of this point; and since the abscissa x is different for every point it fol-

lows that this angle will be different at each point of the section, which section, therefore, is no longer, as before, a straight line. The surface will be constructed, as before, by moving the section parallelly, so that its point may ride continually in the curve whose equation is $z = \varphi y$.

Suppose now that in the preceding equation, instead of X we have a function, P of x, and of y. The equation

$$\left(\frac{dz}{dx}\right) = P,$$

containing three variables will belong still to a curve surface. If we cut this surface by a plane parallel to that of x, z, we shall have a section in which y will be constant; and since in all its points $\left(\frac{dz}{dx}\right)$ will be equal to a function of the variable x, this section must be a curve, as in the preceding case. The equation

$$\left(\frac{dz}{dx}\right) = P$$

being integrated, we shall have for that of the surface

$$z = \int P\,dx + \varphi y;$$

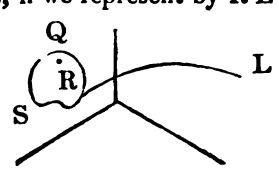
if in this equation we give successively to y the increasing values y′, y″, y‴, &c. and make P′, P″, P‴, &c. what the function P becomes in these cases, we shall have the equations

$$\left.\begin{array}{l} z = \int P'dx + \varphi y', \quad z = \int P''dx + \varphi y'' \\ z = \int P'''dx + \varphi y'', \quad z = \int P''''dx + \varphi y'''' \text{ &c.} \end{array}\right\}$$

and we see that these equations will belong to curves of the same nature, but different in form, since the values of the constant y will not be the same. These curves are nothing else than the sections of the surface made by planes parallel to the plane (x, z); and in meeting the plane (y, z) they will form a curve whose equation will be obtained by equating to zero, the value of x in that of the surface. Call the value of $\int P\,dx$, in this case, Y, and we shall have

$$z = Y + \varphi y;$$

and we perceive that by reason of φy, the curve determined by this equation must be arbitrary. Thus, having traced at pleasure a curve, Q R S, upon the plane (y, z), if we represent by R L the section whose equation



is $z = \int P'dx + \varphi y'$, we shall move this section, always keeping the ex-

tremity R applied to the curve Q R S; but so that this section as it moves, may assume the successive forms determined by the above group of equations, and we shall thus construct the surface to which will belong the equation

$$\left(\frac{d\,z}{d\,x}\right) = P.$$

Finally let us consider the general equation

$$\left(\frac{d\,z}{d\,x}\right) + M\left(\frac{d\,z}{d\,y}\right) + N = 0,$$

whose integral is $U = \varphi\, V$. Since $U = a$, $V = b$, each of these equations subsisting between three coordinates, we may regard them as belonging to two surfaces; and since the coordinates are common, they ought to belong to the curve of intersection of the two surfaces. This being shown, a and b being arbitrary constants, if in $U = a$, we give to x and y the values x', y' we shall obtain for z, a function of x', of y' and of a, which will determine a point of the surface whose equation is $U = a$. This point, which is any whatever, will vary in position if we give successively different values to the arbitrary constant a, which amounts to saying that by making a vary, we shall pass the surface whose equation is $U = a$, through a new system of points. This applies equally to $V = b$, and we conclude that the curve of intersection of the two surfaces will change continually in position, and consequently will describe a curved surface in which a, b may be considered as two coordinates; and since the relation $a = \varphi\, b$ which connects these two coordinates, is arbitrary we perceive that the determination of the function φ amounts to making a surface pass through a curve traced arbitrarily.

To show how this sort of problems may conduct to analytical conditions, let us examine what is the surface whose equation is

$$y\left(\frac{d\,z}{d\,x}\right) = x\left(\frac{d\,z}{d\,y}\right).$$

We have seen that this equation being integrated gives

$$z = \varphi\,(x^2 + y^2).$$

Reciprocally we hence derive

$$x^2 + y^2 = \Phi\, z.$$

If we cut the surface by a plane parallel to the plane (x, y) the equation of the section will be

$$x^2 + y^2 = \Phi\, c;$$

and representing by a^2 the constant $\Phi\, c$, we shall have

$$x^2 + y^2 = a^2.$$

This equation belongs to the circle. Consequently the surface will

have this property, viz. that every section made by a plane parallel to the plane (x, y) will be a circle.

This property is also indicated by the equation

$$y\left(\frac{d\,z}{d\,x}\right) = x\left(\frac{d\,z}{d\,y}\right)$$

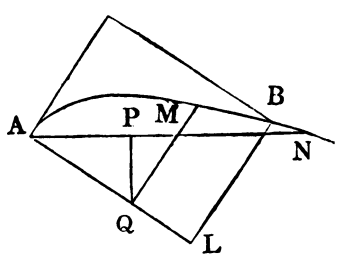
for this equation gives

$$x = y\,\frac{d\,y}{d\,x}.$$

This equation shows us that the subnormal ought to be always equal to the abscissa which is the property of the circle.

The equation $z = \varphi\,(x^2 + y^2)$ showing merely that all the sections parallel to the plane (x, y) are circles, it follows thence that the law according to which the radii of these sections ought to increase, is not comprised in this equation, and that consequently, every surface of revolution will satisfy the problem; for we know that in this sort of surfaces, the sections parallel to the plane (x, y) are always circles, and it is needless to say that the generatrix which, during a revolution, describes the surface, may be a curve discontinued, discontiguous, regular or irregular.

Let us therefore investigate the surface for which this generatrix will be a parabola A N, and suppose that, in this hypothesis, the surface is cut by a plane A B, which shall pass through the axis of z; the trace of



this plane upon the plane (x, y) will be a straight line A L, which, being drawn through the origin, will have the equation $y = a\,x$; if we represent by t the hypothenuse of the right angled triangle A P Q, constructed upon the plane (x, y) we shall have

$$t^2 = x^2 + y^2;$$

but t being the abscissa of the parabola A M, of which $Q\,M = z$ is the ordinate, we have, by the nature of the curve,

$$t^2 = b\,z.$$

Putting for t^2 its value $x^2 + y^2$, we get

$$z = \frac{1}{b}(y^2 + x^2), \text{ or } z = \frac{1}{b}(a^2\,x^2 + x^2) = \frac{1}{b}\,x^2\,(1 + a^2);$$

and making

$$\frac{1}{b}(a^2 + 1) = m,$$

we shall obtain

$$z = m x^2;$$

so that the condition prescribed in the hypothesis, where the generatrix is a parabola, is that we ought to have

$$z = m x^2, \text{ when } y = a x.$$

Let us now investigate, by means of these conditions, the arbitrary function which enters the equation $z = \varphi (x^2 + y^2)$. For that purpose, we shall represent by U the quantity $x^2 + y^2$, which is effected by the symbol φ, and the equation then becomes

$$z = \varphi U;$$

and we shall have the three equations

$$x^2 + y^2 = U, \quad y = a x, \quad z = m x^2.$$

By means of the two first we eliminate y and obtain the value of x^2 which being put into the third, will give

$$Z = m \cdot \frac{U}{1 + a^2},$$

an equation which reduces to

$$Z = \frac{1}{b} U;$$

the value of z being substituted in the equation $z = \varphi U$, will change it to

$$\varphi U = \frac{1}{b} U;$$

and putting the value of U in this equation, we shall find that

$$\varphi (x^2 + y^2) = \frac{1}{b}(x^2 + y^2),$$

and we see that the function is determined. Substituting this value of $\varphi (x^2 + y^2)$ in the equation $z = \varphi (x^2 + y^2)$, we get

$$z = \frac{1}{b}(x^2 + y^2),$$

for the integral sought, an equation which has the property required, since the hypothesis of $y = a x$ gives

$$z = m x^2.$$

This process is general; for, supposing the conditions which determine the arbitrary constant to be that the integral gives $F(x, y, z) = 0$, when we have $f(x, y, z) = 0$, we shall obtain a third equation by equating to

U the quantity which follows φ, and then by eliminating, successively, two of the variables x, y, z, we shall obtain each of these variables in a function of U; putting these values in the integral, we shall get an equation whose first member is φ U, and whose second member is a compound expression in terms of U; restoring the value of U in terms of the varibles, the arbitrary function will be determined.

THE ARBITRARY FUNCTIONS WHICH ENTER THE INTEGRALS OF THE EQUATIONS OF PARTIAL DIFFERENCES OF THE SECOND ORDER.

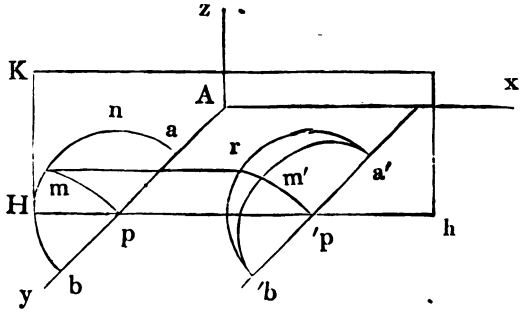
Equations of partial differences of the second order conduct to integrals which contain two arbitrary functions; the determination of these functions amounts to making the surface pass through two curves which may be discontinuous or discontiguous. For example, take the equation

$$\left(\frac{d^2 z}{dx^2}\right) = 0,$$

whose integral has been found to be

$$z = x \varphi y + \psi y$$

Let A x, A y, A z, be the axis of coordinates; if we draw a plane



K L parallel to the plane (x, z), the section of the surface by this plane will be a straight line; since, for all the points of this section, y being equal to A p, if we represent A p by a constant c, the quantities φ y, ψ y will become φ c, ψ c, and, consequently, may be replaced by two constants, a, b, so that the equation

$$z = x \varphi y + \psi y$$

will become

$$z = a x + b,$$

and this is the equation to the section made by the plane K L.

To find the point where this section meets the plane (y, z) make $x = 0$, and the equation above gives $z = \psi y$, which indicates a curve a m b, traced upon the plane (y, z). It will be easy to show that the section meets the curve a m b in a point m; and since this section is a straight line, it is only requisite, to find the position of it, to find a second point. For that purpose, observe that when $x = 0$, the first equation reduces to

$$z = \psi y,$$

whilst, when $x = 1$, the same equation reduces to

$$z = \varphi y + \psi y.$$

Making, as above, $y = A p = c$, these two values of z will become

$$z = b, \quad z = a + b,$$

and determining two points m and r, taken upon the same section, m r we know to be in a straight line. To construct these points we thus proceed: we shall arbitrarily trace upon the plane (y, z) the curve a m b, and through the point p, where the cutting plane K L meets the axis of y, raise the perpendicular p m = b, which will be an ordinate to the curve; we shall then take at the intersection H L of the cutting plane, and the plane (x, y), the part p p′ equal to unity, and through the point p′, we shall draw a plane parallel to the plane (y, z), and in this plane construct the curve a′ m′ b′, after the modulus of the curve a m b, and so as to be similarly disposed; then the ordinate m′ p′ will be equal to m p; and if we produce m′ p′ by m′ r, which will represent a, we shall determine the point r of the section.

If, by a second process, we then produce all the ordinates of the curve a′ m′ b′, we shall construct a new curve a′ r′ b′, which will be such, that drawing through this curve and through a m b, a plane parallel to the plane (x, z), the two points where the curves meet, will belong to the same section of the surface.

From what precedes, it follows that the surface may be constructed, by moving the straight line m r so as continually to touch the two curves, a m b, a′ m′ b′.

This example suffices to show that the determination of the arbitrary functions which complete the integrals of equations of partial differences of the second order, is the same as making the surface pass through two curves, which, as well as the functions themselves, may be discontinuous, discontiguous, regular or irregular.

CALCULUS OF VARIATIONS.

If we have given a function $Z = F, (x, y, y', y'')$, wherein y', y'' mean

$$\left(\frac{d\,y}{d\,x}\right), \left(\frac{d^2\,y}{d\,x^2}\right),$$

y itself being a function of x, it may be required to make L have certain properties, (such as that of being a maximum, for instance) whether by assigning to x, y numerical values, or by establishing relations between these variables, and connecting them by equations. When the equation $y = \varphi x$ is given, we may then deduce y, y', y''... in terms of x and substituting, we have the form

$$Z = f\,x.$$

By the known rules of the differential calculus, we may assign the values of x, when we make of x a *maximum* or *minimum*. Thus we determine what are the points of a given curve, for which the proposed function Z, is greater or less than for every other point of the same curve.

But if the equation $y = \varphi\,x$ is not given, then taking successively for $\varphi\,x$ different forms, the function $Z = f\,x$ will, at the same time, assume different functions of x. It may be proposed to assign to $\varphi\,x$ such a form as shall make Z greater or less than every other form of $\overset{\Delta}{\varphi}\,x$, *for the same numerical value of x whatever it may be in other respects.* This latter species of problem belongs to the calculus of variations. This theory relates not to maxima and minima only; but we shall confine ourselves to these considerations, because it will suffice to make known all the rules of the calculus. We must always bear in mind, that the variables x, y are not *independent*, but that the equation $y = \varphi\,x$ is unknown, and that we only suppose it given to facilitate the resolution of the prob lem. We must consider x as any quantity whatever which remains the same for all the differential forms of $\varphi\,x$; the forms of φ, φ', φ''.... are therefore variable, whilst x is constant.

In $Z = F\,(x, y, y', y''...)$ put $y + k$ for y, $y' + k'$, for $y'...$, k being an arbitrary function of x, and k', k,''... the quantities

$$\frac{d\,k}{d\,x}, \frac{d^2\,k}{d\,x^2}\cdots$$

But, Z will become

$$Z_{,} = F\,(x, y + k, y' + k', y'' + k,''\ldots)$$

Taylor's theorem holds good whether the quantities x, y, k be dependent or independent. Hence we have

$$Z_{,} = Z + \left\{ k \left(\frac{dZ}{dy}\right) + k' \left(\frac{dZ}{dy'}\right) + k'' \left(\frac{dZ}{dy''}\right) + \ldots \right\} + \&c.$$

so that we may consider x, y, y′, y″... as so many independent variables.

The nature of the question requires that the equation $y = \varphi x$ should be determined, so that for the same value of x, we may have always $Z_{,} > Z$, or $Z_{,} < Z$: reasoning as in the ordinary *maxima and minima*, we perceive that the terms of the first order must equal zero, or that we have

$$k \left(\frac{dZ}{dy}\right) + k' \left(\frac{dZ}{dy'}\right) + k'' \left(\frac{dZ}{dy''}\right) + \&c. = 0.$$

Since k is arbitrary for every value of x, and it is not necessary that its value or its form should remain the same, when x varies or is constant, k′, k″... is as well arbitrary as k. For we may suppose for any value $x = X$ that $k = a + b(x - X) + \frac{1}{2} c (x - X)^2 + \&c.$, X, a, b, c... being taken at pleasure; and since this equation, and its differentials, ought to hold good, whatever is x, they ought also to subsist when $x = X$, which gives $k = a$, $k' = b$, $k'' = c$, &c. Hence the equation $Z_{,} = Z + \ldots$ cannot be satisfied when a, b, c... are considered independent, unless (see 6, vol. I.)

$$\left(\frac{dZ}{dy}\right) = 0, \left(\frac{dZ}{dy'}\right) = 0, \left(\frac{dZ}{dy''}\right) = 0 \ldots \left(\frac{dZ}{dy^{(n)}}\right) = 0,$$

n being the highest order of y in Z. These different equations subsist simultaneously, whatever may be the value of x; and if so, there ought to be a maximum or minimum; and the relation which then subsists between x, y will be the equation sought, viz. $y = \varphi x$, which will have the property of making Z greater or less than every other relation between x and y can make it. We can distinguish the maximum from the minimum from the signs of the terms of the second order, as in vol. I. p. (31.)

But if all these equations give different relations between x, y, the problem will be impossible in the state of generality which we have ascribed to it; and if it happen that some only of these equations subsist mutually, then the function Z will have *maxima and minima*, relative to some of the quantities y, y′, y″... without their being common to them all. The equations which thus subsist, will give the relative *maxima and minima*. And if we wish to make X a maximum or minimum only relatively

to one of the quantities y, y', y''..., since then we have only one equation to satisfy, the problem will be always possible.

From the preceding considerations it follows, that first, the quantities x, y depend upon one another, and that, nevertheless, we ought to make them vary, as if they were independent, for this is but an artifice to get the more readily at the result.

Secondly, that these variations are not indefinitely small; and if we employ the differential calculus to obtain them, it is only an expeditious means of getting the second term of the developement, the only one which is here necessary.

Let us apply these general notions to some examples.

Ex. 1. Take, upon the axis of x of a curve, two abscissas m, n; and draw indefinite parallels to the axis of y. Let $y = \varphi x$ be the equation of this curve: if through any point whatever, we draw a tangent, it will cut the parallels in points whose ordinates are

$$l = y + y'(m - x), \quad h = y + y'(n - x).$$

If the form of φ is given, every thing else is known; but if it is not given, it may be asked, what is the curve which has the property of having for each point of tangency, the product of these two ordinates less than for every other curve.

Here we have $l \times h$; or

$$Z = \{y \times (m - x)\, y'\} + \{y + (n - x)\, y'\}.$$

From the enunciation of the problem, the curves *which pass through the same point* (x, y) have tangents taking different directions, and that which is required, ought to have a tangent, such that the condition $Z = maximum$ is fulfilled. We may consider x and y constant; whence

$$\left(\frac{dZ}{dy'}\right) = 0, \quad \frac{2y'}{y} = \frac{2x - m - n}{(x-m)(x-n)} = \frac{1}{x-m} + \frac{1}{x-n}.$$

Then integrating we get

$$y^2 = C(x - m)(x - n).$$

The curve is an ellipse or a hyperbola, according as C is positive or negative; the vertexes are given by $x = m$, $x = n$; in the first case, the product $h \times l$ or Z is a *maximum*, because y'' is negative; in the second, Z is a *minimum* or rather a negative *maximum*; this product is moreover constant, and $l\,h = -\frac{1}{4} C (m - n)^2$, the square of the semi-axis.

Ex. 2. *What is the curve for which, in each of its points, the square of the subnormal added to the abscissa is a minimum?*

We have in this case

$$Z = (y\,y' + x)^2$$

whence we get two equations subsisting mutually by making

$$y\,y' + x = 0$$

and thence

$$x^2 + y^2 = r^2.$$

Therefore all the circles described from the origin as a center will alone satisfy the equation.

The theory just expounded has not been greatly extended; but it serves as a preliminary developement of great use for the comprehension of a far more interesting problem which remains to be considered. This requires all the preceding reasonings to be applied to a function of the form $\int Z$: the sign $\int$ indicates the function Z to be a differential and that after having integrated it between prescribed limits it is required to endow it with the preceding properties. The difficulty here to be overcome is that of resolving the problem without integrating.

When a body is in motion, we may compare together either the different points of the body in one of its positions or the plane occupied successively by a given point. In the first case, the body is considered fixed, and the symbol d will relate to the change of the coordinates of its surface; in the second, we must express by a convenient symbol, *variations* altogether independent of the first, which shall be denoted by δ. When we consider a curve immoveable, or even variable, but taken in one of its positions, $d\,x$, $d\,y\ldots$ announce a comparison between its coordinates; but to consider the different planes which the same point of a curve occupies, the curve varying in form according to any law whatever, we shall write $\delta\,x$, $\delta\,y\ldots$ which denote the increments considered under this point of view, and are functions of x, $y\ldots$ In like manner, $d\,x$ becoming $d\,(x + \delta\,x)$ will increase by $d\,\delta\,x$; $d^2\,x$ will increase by $d^2\,\delta\,x$, &c.

Observe that the variations indicated by the symbol δ are finite, and wholly independent of those which d represents; the operations to which these symbols relate being equally independent, the order in which they are used must be equally a matter of indifference as to the result. So that we have

$$\delta.d\,x = d.\delta\,x$$
$$d^2.\delta\,x = \delta.d^2\,x$$
$$\&c.$$
$$\int \delta\,U = \delta \int U.$$

and so on.

It remains to establish relations between x, y, $z\ldots$ such that $\int Z$ may be a *maximum or a minimum between given limits.* That the calculus may be rendered the more symmetrical, we shall not suppose any differential

constant; moreover we shall only introduce three variables because it will be easy to generalise the result. To abridge the labour of the process, make

$$d\,x = x_{,},\quad d^{2}x = x_{,,},\ \&c.$$

so that

$$z = F\,(x,\ x_{,},\ x_{,,}\ldots y,\ y_{,},\ y_{,,}\ldots z,\ z_{,},\ z_{,,}\ldots).$$

Now x, y and z receiving the arbitrary and finite increments δx, δy, δz, $d\,x$ or $x_{,}$ becomes

$$d\,(x + \delta x) = d\,x + \delta\,d\,x \text{ or } x_{,} + \delta x_{,}.$$

In the same manner, $x_{,,}$ increases by $\delta x_{,,}$ and so on; so that developing $Z_{,}$ by Taylor's theorem, and integrating $\int Z$ becomes

$$\int Z_{,} = \int Z + \int \left\{ \left(\frac{d\,Z}{d\,x}\right)\delta x + \left(\frac{d\,Z}{d\,y}\right)\delta y + \left(\frac{d\,Z}{d\,z}\right)\delta z + \left(\frac{d\,Z}{d\,x_{,}}\right)\delta x_{,} + \left(\frac{d\,Z}{d\,y_{,}}\right)\delta y_{,} + \left(\frac{d\,Z}{d\,z_{,}}\right)\delta z_{,} + \left(\frac{d\,Z}{d\,x_{,,}}\right)\delta x_{,,} + \&c. \right\} + \int \&c.$$

The condition of a *maximum* or *minimum* requires the integral of the terms of the first order to be zero between given limits *whatever may be* δx, δy, δz as we have already seen. Take the differential of the known function Z considering $x,\ x_{,},\ x_{,,}\ldots y,\ y_{,},\ y_{,,}\ldots$ as so many independent variables; we shall have

$d\,Z = m\,d\,x + n\,d\,x_{,} + p\,d\,x_{,,} + \ldots M\,d\,y + N\,d\,y_{,} \ldots + \mu\,d\,z + \nu\,d\,z_{,}\ldots$

$m,\ n \ldots M,\ N \ldots \mu,\ \nu \ldots$ being the coefficients of the partial differences of Z relatively to $x,\ x_{,} \ldots y,\ y_{,} \ldots z,\ z_{,} \ldots$ considered as so many variables; these are therefore known functions for each proposed value of Z. Performing this differentiation exactly in the same manner by the symbol δ, we have

$$\left.\begin{aligned} \delta Z = {} & m\,\delta x + n\,\delta\,d\,x + p\,\delta\,d^{2}x + q\,\delta\,d^{3}x + \ldots \\ & + M\,\delta y + N\,\delta\,d\,y + P\,\delta\,d^{2}y + q\,\delta\,d^{3}y + \ldots \\ & + \mu\,\delta z + \nu\,\delta\,d\,z + \pi\,\delta\,d^{2}z + \chi\,\delta\,d^{3}y + \ldots \end{aligned}\right\} (A)$$

But this known quantity, whose number of terms is limited, is precisely that which is under the sign $\int$, in the terms of the first order of the developement: so that the required condition of *max.* or *min.* is that

$$\int \delta Z = 0,$$

between given limits, whatever may be the variations δx, δy, δz. Observe, that here, as before, the differential calculus is only employed as a means of obtaining easily the assemblage of terms to be equated to zero; so that the variations are still any whatever and finite.

We have said that $d \cdot \delta x$ may be put for $d \cdot \delta x$; thus the first line is equivalent to

$$m\,\delta x + n \cdot d\,\delta x + p \cdot d^2\,\delta x + q \cdot d^3\,\delta x + \&c.$$

m, n . . . contains differentials, so that the defect of homogeneity is here only apparent. To integrate this, we shall see that it is necessary to disengage from the symbol $\int$ as often as possible, the terms which contain $d\,\delta$. To effect this, we integrate *by parts* which gives

$$\begin{aligned}
&\int n\,d\,\delta x = n.\ \delta x - \int d\,n.\,\delta x \\
&\int p.d^2\delta x = p\,d\,\delta x - d\,p\ \ \delta x + \int d^2 p\,\delta x \\
&\int q\,d^3\,\delta x = q\,d^2\delta x - dq.d\,\delta x + d^2 q.d\,x - \int d^3 q.\,\delta x \\
&\qquad\qquad \&c.
\end{aligned}$$

Collecting these results, we have this series, the law of which is easily recognised; viz.

$$\begin{aligned}
&\int (m - d\,n + d^2 p - d^3 q + d^4 r - \ldots)\,\delta x \\
&+ (n - d\,p + d^2 q - d^3 r + d^4 s - \ldots)\,\delta x \\
&+ (p - d\,q + d^2 r - d^3 s + d^4 t - \ldots)\,d\,\delta x \\
&+ (q - d\,r + \ldots)\,d^2\,\delta x \\
&+ \&c.
\end{aligned}$$

The integral of (A) or $\int . \delta z = 0$, becomes therefore

$$(B)\ldots \int \{(m - d\,n + d^2 p - \ldots)\delta x + (M - d\,N + d^2 P - \ldots)\delta y + (\mu - d\,\nu - \ldots)\delta z\} = 0$$

$$(C)\ldots \begin{cases} (n - d\,p + d^2 q \ldots)\,\delta x + (N - d\,P + d^2 Q - \ldots)\,\delta y + (\nu - d\,\pi \ldots)\,d\,z \\ + (p - d\,q + d^2 r \ldots)\,d\,\delta x + (P - d\,Q + \ldots)d\,\delta y + (\pi - d\,\chi \ldots)d\,\delta z \\ + (q - d\,r \ldots)\,d^2\,\delta x \ldots + K = 0 \end{cases}$$

K being the arbitrary constant. The equation has been split into two, because the terms which remain under the sign $\int$ cannot be integrated, at least whilst δx, δy, δz are arbitrary. In the same manner, if the nature of the question does not establish some relation between δx, δy, δz, the independence of these variations requires also that equation (B) shall again make three others; viz.

$$\left.\begin{aligned}
0 &= m - d\,n + d^2 p - d^3 q + d^4 r - \ldots \\
0 &= M - d\,N + d^2 P - d^3 Q + d^4 R - \ldots \\
0 &= \mu - d\,\nu + d^2 \pi - d^3 \chi + d^4 \varrho - \ldots
\end{aligned}\right\} \quad . \quad . \quad (D)$$

Consequently, to find the relations between x, y, z, which make $\int Z$ a *maximum*, we must take the differential of the given function Z by considering x, y, z, d x, d y, d z, $d^2 x$, . . . as so many independent variables, and use the letter δ to signify their increase; this is what is termed *taking the variation* of Z. Comparing the result with the equation (A), we shall observe the values of m, M, μ, n, N . . . in terms of x, y, z, and

their differences expressed by d. We must then substitute these in the equations (C), (D); the first refers to the limits between which the *maximum* should subsist; the equations (D) constitute the relations required; they are the differentials of x, y, z, and, excepting a case of absurdity, may form distinct conditions, since they will determine numerical values for the variables. If the question proposed relate to Geometry, these equations are those of a curve or of a surface, to which belongs the required property.

As the integration is effected and should be taken between given limits, the terms which remain and compose the equation (C) belong to these limits: it is become of the form $K + L = 0$, L being a function of $x, y, z, \delta x, \delta y, \delta z \ldots$ Mark with one and two accents the numerical values of these variables at the first and second limit. Then, since the integral is to be taken between these limits, we must mark the different terms of L which compose the equation C, first with one, and then with two accents; take the first result from the second and equate the difference to zero; so that the equation

$$L_{//} - L_{/} = 0$$

contains no variables, because x, d x ... will have taken the values $x_{/}, \delta x_{/} \ldots x_{//}, \delta x_{//} \ldots$ assigned by the limits of the integration. We must remember that these accents merely belong to the limits of the integral.

There are to be considered four separate cases.

1. *If the limits are given and fixed*, that is to say, if the extreme values of x, y, z are constant, since $\delta x_{/}$, d $\delta x_{/} \ldots$ d $x_{//}$, d $\delta x_{//}$, &c. are zero, all the terms of $L_{/}$ and $L_{//}$ are zero, and the equation (C) is satisfied. Thus we determine the constants which integration introduces into the equations (D), by the conditions conferred by the limits.

2. *If the limits are arbitrary and independent*, then each of the coefficients $\delta x_{/}, \delta x_{//} \ldots$ in the equation (C) is zero in particular.

3. *If there exist equations of condition*, (which signifies geometrically that the curve required is terminated at points which are not fixed, but which are situated upon two given curves or surfaces,) for the limits, that is to say, if the nature of the question connects together by equations, some of the quantities $x_{/}, y_{/}, z_{/}, x_{//}, y_{//}, z_{//}$ we use the differentials of these equations to obtain more variations $\delta x_{/}, \delta y_{/}, \delta z_{/}$, d $x_{//}$, &c. in functions of the others; substituting in $L_{//} - L_{/} = 0$, these variations will be reduced to the least number possible: the last being absolutely independent, the equation will split again into many others by equating separately their coefficients to zero.

Instead of this process, we may adopt the following one, which is more elegant. Let

$$u = 0,\ v = 0,\ \&c.$$

be the given equations of condition; we shall multiply their variations $\delta u, \delta v \ldots$ by the indeterminates $\lambda, \lambda' \ldots$ This will give $\lambda\,\delta u + \lambda'\delta v + \ldots$ a known function of $\delta x_{,}, \delta x_{,,}, \delta y_{,} \ldots$ Adding this sum to $L_{,,} - L_{,}$, we shall get

$$L_{,,} - L_{,} + \lambda\,\delta u + \lambda'\,d v + \ldots = 0 \quad . \quad . \quad . \quad . \quad (E).$$

Consider all the variations $\delta x_{,}, \delta x_{,,} \ldots$ as independent, and equate their coefficients separately to zero. Then we shall eliminate the indeterminates $\lambda, \lambda' \ldots$ from these equations. By this process, we shall arrive at the same result as by the former one; for we have only made legitimate operations, and we shall obtain the same number of final equations.

It must be observed, that we are not to conclude from $u = 0$, $v = 0$, that at the limits we have $d u = 0$, $d v = 0$; these conditions are independent, and may easily not coexist. In the contrary case, we must consider $d u = 0$, $d v = 0$, as new conditions, and besides $\lambda\,\delta u$, we must also take $\lambda'\,\delta d u \ldots$

4. Nothing need be said as to the case where one of the limits is fixed and the other subject to certain conditions, or even altogether arbitrary, because it is included in the three preceding ones.

It may happen also that the nature of the question subjects the variations δx, δy, δz, to certain conditions, given by the equations

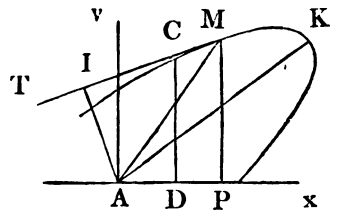
$$\varepsilon = 0,\ \theta = 0,$$

and independently of limits; thus, for example, when the required curve is to be traced upon a given curve surface. Then the equation (B) will not split into three equations, and the equations (D) will not subsist. We must first reduce, as follows, the variations to the smallest number possible in the formula (B), by means of the equations of condition, and equate to zero the coefficients of the variations that remain; or, which is tantamount, add to (B) the terms $\lambda\,\delta\varepsilon + \lambda'\,\delta\theta + \ldots$; then split this equation into others by considering δx, δy, δz as independent; and finally eliminate $\lambda, \lambda' \ldots$

It must be observed, that, in particular cases, it is often preferable to make, upon the given function Z, all the operations which have produced the equations (B), (C) instead of comparing each particular case with the general formulæ above given.

Such are the general principles of the calculus of variations: let us illustrate it with examples.

Ex. 1. *What is the curve* C M K *of which the length* M K, *comprised between the given radii-vectors* A M, A K *is the least possible.*



We have, (vol. I, p. 000), if r be the radius-vector,

$$s = \int (r^2 d\theta^2 + dr^2) = Z$$

it is required to find the relation $r = \varphi\theta$, which renders Z a minimum the variation is

$$\delta Z = \frac{r\,d\theta^2 . \delta + r^2 d\theta . \delta d\theta + dr . \delta dr}{\sqrt{(r^2 d\theta^2 + dr^2)}};$$

Comparing with equation (A), where we suppose $x = r$, $y = \theta$, we have

$$m = \frac{r\,d\theta^2}{ds}, n = \frac{dr}{ds}, M = 0, N = \frac{r^2 d\theta}{ds},$$

the equations (D) are

$$\frac{r\,d\theta^2}{ds} = d\left(\frac{dr}{ds}\right), \frac{r^2 d\theta}{ds} = c.$$

Eliminating $d\theta$, and then ds, from these equations, and $ds^2 = r^2 d\theta^2 + dr^2$, we perceive that they subsist mutually or agree; so that it is sufficient to integrate one of them. But the perpendicular A I let fall from the origin A upon any tangent whatever. T M is

$$AJ = AM + \text{sin. } AMT = r \text{ sin. } \beta,$$

which is equivalent, as we easily find, to

$$\frac{r \text{ tan. } \beta}{\sqrt{(1 + \text{tan.}^2 \beta)}}$$

which gives

$$\frac{r^2 d\theta}{\sqrt{(r^2 d\theta^2 + dr^2)}} = \frac{r^2 d\theta}{ds} = c;$$

and since this perpendicular is here constant, the required line is a straight line. The limits M and K being indeterminate, the equations (C) are unnecessary.

Ex. 2. *To find the shortest line between two given points, or two given curves.*

The length s of the line is

$$\int Z = \int \sqrt{(dx^2 + dy^2 + dz^2)}.$$

It is required to make this quantity a minimum; we have

$$\delta Z = \frac{dx}{ds}\delta dx + \frac{dy}{ds}\delta dy + \frac{dz}{ds}\delta dz,$$

and comparing with the formula (A), we find

$$m = 0, M = 0, \mu = 0, n = \frac{dx}{ds}, N = \frac{dy}{ds}, \nu = \frac{dz}{ds}:$$

the other coefficients P, p, π . . . are zero. The equations (D) become, therefore, in this case,

$$d\left(\frac{dx}{ds}\right) = 0, d\left(\frac{dy}{ds}\right) = 0, d\left(\frac{dz}{ds}\right) = 0;$$

whence, by integrating

$$dx = a\,ds, dy = b\,ds, dz = c\,ds.$$

Squaring and adding, we get

$$a^2 + b^2 + c^2 = 1,$$

a condition that the constants a, b, c must fulfil in order that these equations may simultaneously subsist. By division, we find

$$\frac{dy}{dx} = \frac{b}{a}, \frac{dz}{dx} = \frac{c}{a},$$

whence

$$bx = ay + a', cx = az + b';$$

the projections of the line required are therefore straight lines—the line is therefore itself a straight line.

To find the position of it, we must know the five constants a, b, c, a', b'. If it be required to find the shortest distance between two given fixed points $(x_,, y_,, z_,)$, $(x_{,,}, y_{,,} z_{,,})$, it is evident that $\delta_, x$, $\delta x_{,,}$ $\delta y_,$ . . . are zero, and that the equation (C) then holds good. Subjecting our two equations to the condition of being satisfied when we substitute therein $x_,, x_{,,}, y_,$ . . . for $x_,, y_,, z_,$ we shall obtain four equations, which, with $a^2 + b^2 + c^2 = 1$, determine the five necessary constants.

Suppose that the second limit is a fixed point $(x_{,,}, y_{,,} z_{,,})$, in the plane (x, y), and the first a curve passing through the point $(x_,, y_,, z_,)$, and also situated in this plane; the equation

$$bx = ay + a'$$

then suffices. Let $y_, = f x_,$ be the equation of the curve; hence

$$\delta y_, = A\,\delta x_,;$$

the equation (C) becomes

$$L = \left(\frac{dx}{ds}\right)\delta x + \left(\frac{dy}{ds}\right)\delta y;$$

and since the second limit is fixed it is sufficient to combine together the equations

$$\delta y_{,} = A \delta x_{,}$$
$$dx_{,} \delta x_{,} + dy_{,} \delta y_{,} = 0.$$

Eliminating δy, we get

$$dx_{,} + A dy_{,} = 0.$$

We might also have multiplied the equation of condition

$$\delta y_{,} - A \delta x_{,} = 0$$

by the indeterminate λ, and have added the result to $L_{,}$, which would have given

$$\left(\frac{dx_{,}}{ds_{,}}\right) \delta x_{,} + \left(\frac{dy'}{ds_{,}}\right) \delta y_{,} + \lambda \delta y_{,} - \lambda A \delta x_{,} = 0,$$

whence

$$\left(\frac{dx_{,}}{ds_{,}}\right) - \lambda A = 0, \left(\frac{dy_{,}}{ds_{,}}\right) + \lambda = 0.$$

Eliminating λ we get

$$dx_{,} + A dy_{,} = 0.$$

But then the point $(x_{,}, y_{,})$ is upon the straight line passing through the points $(x_{,}, y_{,}, z_{,})$, $(x_{,,}, y_{,,}, z_{,,})$, and we have also

$$b dx_{,} = a dy_{,}$$

whence

$$a = - b A$$

and

$$\frac{dy}{dx} = - \frac{1}{A} = \frac{b}{a};$$

which shows the straight line is a normal to the curve of condition. The constant a' is determined by the consideration of the second limit which is given and fixed.

It would be easy to apply the preceding reasoning to three dimensions, and we should arrive at similar conclusions; we may, therefore, infer generally that the shortest distance between two curves is the straight line which is a normal to them.

If the shortest line required were to be traced upon a curve surface whose equation is $u = 0$, then the equation (B) would not decompose into three others. We must add to it the term $\lambda \delta u$; then regarding δx, δy, δz as independent, we shall find the relations

$$d.\left(\frac{dx}{ds}\right) + \lambda \left(\frac{du}{dx}\right) = 0,$$

$$d\left(\frac{dy}{ds}\right) + \lambda\left(\frac{du}{dy}\right) = 0,$$

$$d\left(\frac{dz}{ds}\right) + \lambda\left(\frac{du}{dz}\right) = 0.$$

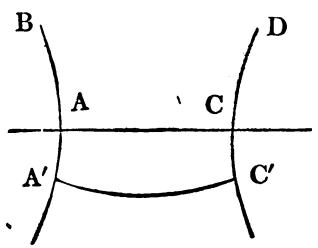
From these eliminating λ, we have the two equations

$$\left(\frac{du}{dz}\right) d\left(\frac{dx}{ds}\right) = \left(\frac{du}{dx}\right) d\left(\frac{dz}{ds}\right)$$

$$\left(\frac{du}{dy}\right) d\left(\frac{dz}{ds}\right) = \left(\frac{du}{dz}\right) d\left(\frac{dy}{ds}\right)$$

which are those of the curve required.

Take for example, the least distance measured upon the surface of a



sphere, whose center is at the origin of coordinates: hence

$$u = x^2 + y^2 + z^2 - r^2 = 0,$$

$$\left(\frac{du}{dx}\right) = 2x, \left(\frac{du}{dy}\right) = 2y, \left(\frac{du}{dz}\right) = 2z.$$

Our equations give, making d s constant,

$$z\,d^2x = x\,d^2z,\ z\,d^2y = y\,d^2z,$$

whence

$$y\,d^2x = x\,d^2y.$$

Integrating we have

$$z\,dx - x\,dz = a\,ds,\ z\,dy - y\,dz = b\,ds,\ y\,dx - x\,dy = c\,ds.$$

Multiplying the first of these equations by $-y$, the second by x, the third by z, and adding them, we get

$$ay = bx + cz$$

the equation of a plane passing through the origin of coordinates. Hence the curve required is a great circle which passes through the points A′ C′, or which is normal to the two curves A′ B and C′ D which are limits and are given upon the spherical surface.

When a body moves in a fluid it encounters a resistance which *ceteris*

paribus depends on its form (see vol. I.): if the body be one of revolution and moves in the direction of its axis, we can show by mechanics that the resistance is the least possible when the equation of the generating curve fulfils the condition

$$\int \frac{y\, d\, dy^3}{dx^2 + dy^2} = \textit{minimum},$$

or

$$Z = \frac{y\, y'^3\, dx}{1 + y'^2}.$$

Let us determine the generating curve of the *solid of least resistance* (see Principia, vol. II.).

Taking the variation of the above expression, we get

$$m = 0,\ n = \frac{-2\, y\, dy^3\, dx}{(dx^2 + dy^2)^2} = \frac{-2\, y\, y'^3}{(1 + y'^2)^2},\ p = 0, \&c.$$

$$M = \frac{dy^3}{dx^2 + dy^2} = \frac{y'^3\, dx}{1 + y'^2},\ N = \frac{y\, y'^2\, (3 + y'^2)}{(1 + y'^2)^2}, \&c.$$

the second equation (D) is

$$M - d\, N = 0;$$

and it follows from what we have done relatively to Z, that

$$d\left(\frac{y'^3 y}{1 + y'^2}\right) = M \frac{dy}{dx} + N\, dy' = y'\, dN + N\, dy',$$

because

$$M = d\, N.$$

Thus integrating, we have

$$a + \frac{y'^3 y}{1 + y'^2} = N y' = \frac{y\, y'^3\, (3 + y'^2)}{(1 + y'^2)^2}.$$

Therefore

$$a\, (1 + y'^2)^2 = 2\, y\, y'^3.$$

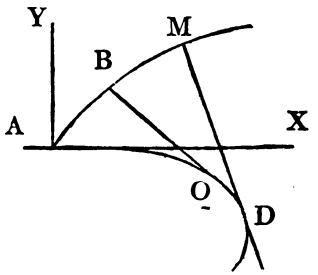
Observe that the first of the equations (D) or $m - d\, n = 0$, would have given the same result $-n = a$; so that these two equations conduct to the same result. We have

$$y = \frac{a\, (1 + y'^2)^2}{2\, y'^3},$$

$$x = \int \frac{dy}{y'} = \frac{y}{y'} + \int \frac{y\, dy'}{y'^2};$$

substituting for y its value, this integral may easily be obtained; it remains to eliminate y' from these values of x and y, and we shall obtain the equation of the required curve, containing two constants which we shall determine from the given conditions.

Ex. 3. *What is the curve* A B M *in which the area* B O D M *comprised*



between the arc B M *the radii of curvature* B O, D M *and the arc* O D *of the evolute, is a minimum?*

The element of the arc A M is

$$d s = d x \sqrt{1 + y'^2};$$

the radius of curvature M D is

$$\frac{(1 + y'^2)^{\frac{3}{2}}}{y''};$$

and their product is the element of the proposed area, or

$$Z = \frac{(1 + y'^2)\, d x}{y''} = \frac{(d x^2 + d y^2)}{d x\, d y}.$$

It is required to find the equation $y = f x$, which makes $\int Z$, a minimum.

Take the variation δ N, and consider only the second of the equations (D), which is sufficient for our object, and we get

$$M = 0, \quad N - d P = 4 a,$$

$$N = \frac{d x^2 + d y^2}{d x\, d^2 y} . 4 d y = \frac{1 + y'^2}{y''} 4 y',$$

$$P = - \frac{(1 + y'^2)^2}{y''^2 d x}.$$

But

$$d \left(\frac{(1 + y'^2)^2}{y''}\right) = N d y' + P d y'' d x$$

$$= 4 a d y + d P d y' + P d y'' d x,$$

putting $4 a + P$ for N. Moreover $y'' d x = d y'$, changes the last terms into

$$(y'' d P + P d y'') d x = d (P y'') . d x = - d \left(\frac{(1 + y'^2)^2}{y''}\right).$$

Integrating, therefore,

$$\frac{(1+y'^2)^2}{2y''} = a\,y' + b,$$

$$y'' = \frac{(1+y'^2)^2}{2(a\,y'+b)} = \frac{d\,y'}{d\,x},\quad d\,x = \frac{2(a\,y'+b)\,d\,y'}{(1+y'^2)^2},$$

finally,

$$x = c + \frac{b\,y' - a}{1+y'^2} + b\,\tan.^{-1} y';$$

On the other side we have

$$y = \int y'\,d\,x = y'\,x - \int x\,d\,y'$$

or

$$y = y'\,x - c\,y' - \int \frac{b\,y' - a}{1+y'^2}\,d\,y' - \int b\,d\,y'\,\tan.^{-1} y';$$

this last term integrates *by parts*, and we have

$$y = y'\,x - c\,y' - (b\,y' - a)\,\tan.^{-1} y' + f.$$

Eliminating the tangent from these values of x and y, we get

$$b\,y = a\,(x - c) + \frac{(b\,y' - a)^2}{1+y'^2} + b\,f,$$

$$\sqrt{(b\,y - a\,x + g)} = \frac{(b\,y' - a)\,d\,x}{d\,s},\quad d\,s = \frac{b\,d\,y - a\,d\,x}{\sqrt{(b\,y - a\,x + g)}};$$

finally,

$$s = 2\sqrt{(b\,y - a\,x + g)} + h.$$

This equation shows that the curve required is a cycloid, whose four constants will be determined from the same number of conditions.

Ex. 4. *What is the curve of a given length* s, *between two fixed points, for which* $\int y\,d\,s$ *is a maximum?*

We easily find

$$(y + \lambda)\left(\frac{d\,x}{d\,s}\right) = c,\ \text{whence}\ d\,x = \frac{c\,d\,y}{\sqrt{\{(y+\lambda)^2 - c^2\}}};$$

and it will be found that the curve required is a *catenary.*

Since $\int \frac{y\,d\,s}{s}$ is the vertical ordinate of the center of gravity of an arc whose length is s, we see that the center of gravity of any arc whatever of the catenary is lower than that of any other curve terminated by the same points.

Ex. 5. Reasoning in the same way for $\int y^2\,d\,x$ = *minimum*, and $\int y\,d\,x$ = const. we find $y^2 + \lambda\,y = c$, or rather $y = c$. We have here a straight line parallel to x. Since $\frac{\int y^2\,d\,x}{2\int y\,d\,x}$ is the vertical ordinate of the center of gravity of every plane area, that of a rectangle, whose side is horizontal, is the lowest possible; so that every mass of water

whose upper surface is horizontal, has its center of gravity the lowest possible.

FINITE DIFFERENCES.

If we have given a series a, b, c, d, . . . take each term of it from that which immediately follows it, and we shall form the *first differences*, viz.

$$a' = b - a,\ b' = c - b,\ c' = d - c,\ \&c.$$

In the same manner we find that this series a′, b′, c′, d′ . . . gives the *second differences*

$$a'' = b' - a',\ b'' = c' - b',\ c'' = d' - c',\ \&c.$$

which again give the third differences

$$a''' = b'' - a'',\ b''' = c'' - b'',\ c''' = d'' - c'',\ \&c.$$

These differences are indicated by Δ, and an exponent being given to it will denote the order of differences. Thus Δ^n is a term of the series of nth differences. Moreover we give to each difference the sign which belongs to it; this is —, when we take it from a decreasing series.

For example, the function

$$y = x^3 - 9x + 6$$

in making x successively equal to 0, 1, 2, 3, 4 . . . gives a series of numbers of which y is the general term, and from which we get the following differences,

for x =	0,	1,	2,	3,	4,	5,	6,	7 . . .
series y =	6,	— 2,	— 4,	6,	34,	86,	168,	286 . . .
first diff. Δ y =	— 8,		— 2,	10,	28,	52,	82,	118 . . .
second diff. Δ^2 y =			6,	12,	18,	24,	30,	36 . . .
third diff. Δ^3 y =			6,	6,	6,	6,	6,	. . .

We perceive that the third differences are here constant, and that the second difference is an arithmetic progression: we shall always arrive at constant differences, whenever y is a rational and integer function of x; which we now demonstrate.

In the monomial $k x^m$ make $x = \alpha, \beta, \gamma, \ldots \theta, \varkappa, \lambda$ (these numbers having h for a constant difference), and we get the series

$$k\alpha^m,\ k\beta^m,\ \ldots k\theta^m,\ k\varkappa^m,\ k\lambda^m.$$

Since $\varkappa = \lambda - h$, by developing $k\varkappa^m = k(\lambda - h)^m$, and designating by m, A′, A″ . . . the coefficients of the binomial, we find, that

$$k(\lambda^m - \varkappa^m) = k\,m\,h\,\lambda^{m-1} - k\,A'\,h^2\lambda^{m-2} + k\,A''\,^3\,h. \ .$$

Such is the first difference of any two terms whatever of the series

$$k\,\alpha^m,\ k\,\beta^m \ldots k\,\varkappa^m,\ \&c.$$

The difference which precedes it, or $k\,(\varkappa^m - \theta^m)$ is deduced by changing λ into $\varkappa$ and $\varkappa$ into θ' and since $\varkappa = \lambda - h$, we must put $\lambda - h$ for λ in the second member:

$$k\,m\,h(\lambda-h)^{m-1} - k\,A'\,h^2(\lambda-h^{m-2}) \ldots = k\,m\,h\,\lambda^{m-1} - \{A' + m(m-1)\}kh^2\lambda^{m-2} \ldots$$

Subtracting these differences, the two first terms will disappear, and we get for the second difference of an arbitrary rank

$$k\,m\,(m-1)\,h^2\lambda^{m-2} + k\,B'\,h^3\lambda^{m-3} + \ldots$$

In like manner, changing λ into $\lambda - h$, in this last developement, and subtracting, the two first terms disappear, and we have for the third difference

$$k\,m\,(m-1)\,(m-2)\,h^3\lambda^{m-3} + k\,B''\,h^4\lambda^{m-4} \ldots,$$

and so on continually.

Each of these differences has one term at least, in its developement, like the one above; the first has m terms; the second has m — 1 terms; third, m — 2 terms; and so on. From the form of the first term, which ends by remaining alone in the mth difference, we see this is reduced to the constant

$$1 \cdot 2 \cdot 3 \ldots m\,k\,h^m.$$

If in the functions M and N we take for x two numbers which give the results m, n; then M + N becomes m + n. In the same manner, let m′, n′ be the results given by two other values of x; the first difference, arising from M + N, is evidently

$$(m - m') + (n - n').$$

that is, *the difference of the sum is the sum of the differences.* The same may be shown of the 3d and 4th . . . differences.

Therefore, if we make

$$x = \alpha,\ \beta,\ \gamma \ldots$$

in

$$k\,x^m + p\,x^{m-1} + \ldots$$

the mth difference will be the same as if these were only the first term $k\,x^m$, for that of $p\,x^{m-1}$, $q\,x^{m-2}$ . . . is nothing. Therefore *the mth difference is constant, when for* x *we substitute numbers in arithmetic progression, in a rational and integer function of* x.

We perceive, therefore, that if it be required to substitute numbers in arithmetic progression, as is the case in the resolution of numerical equations, according to Newton's Method of Divisors, it will suffice to find the (m + 1) first results, to form the first, second, &c. differences. The

mth difference will have but one term; as we know it is constant and $= 1 . 2 . 3 \ldots m\, k\, h^m$, we can extend the series at pleasure. That of the (m — 1)th differences will then be extended to that of two known terms, since it is an arithmetic procession; that of the (m — 2)th differences will, in its turn, be extended; and so on of the rest.

This is perceptible in the preceding example, and also in this; viz.

x = 0.	1. 2. 3	3d Diff. 6.	6. 6. 6. 6. 6. 6. 6 . . .
Series 1.	−1. 1. 13	2nd . . 4.	10 . 16 . 22 . 28 . 34 . 40 . . .
1st. . . .	−2. 2. 12	1st . —2.	2 . 12 . 28 . 50 . 78 . 112 . . .
2nd . .	3. 10	Results 1. —	1. 1. 13 . 41 . 91 . 169 . . .
3d . . .	6	For x 0.	1. 2. 3. 4. 5. 6 . . .

These series are deduced from that which is constant

$$6 . 6 . 6 . 6 \ldots$$

and from the initial term already found for each of them: *any term is derived by adding the two terms on the left which immediately precede it.* They may also be continued in the contrary direction, in order to obtain the results of x = — 1, — 2, — 3, &c.

In resolving an equation it is not necessary to make the series of results extend farther than the term where we ought only to meet with numbers of the same sign, which is the case when all the terms of any column are positive on the right, and alternate in the opposite direction; for the additions and subtractions by which the series are extended as required, preserve constantly the same signs in the results. We learn, therefore, by this method, the limits of the roots of an equation, whether they be positive or negative.

Let y_x denote the function of x which is the general term, viz. the x + 1th, of a proposed series

$$y_0 + y_2 + y_1 + \ldots y_x + y_{x+1} + \ldots$$

which is formed by making

$$x = 0,\ 1,\ 2,\ 3 \ldots$$

For example, y_5 will designate that x has been made = 5, or, with regard to the place of the terms, that there are 5 before it (in the last example this is 91). Then

$$y_1 - y_0 = \Delta y_0,\quad y_2 - y_1 = \Delta y_1,\quad y_3 - y_2 = y_2 \ldots$$
$$\Delta y^1 - \Delta y_0 = \Delta^2 y_0,\ \Delta y^2 - \Delta^2 y_1 = \Delta^2 y_1,\ \Delta y_3 - \Delta y_2 = \Delta^2 y_2 \ldots$$
$$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0,\ \Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1,\ \Delta^2 y_3 - \Delta^2 y_2 = \Delta^3 y_2 \ldots$$

&c.

and generally we have

$$\begin{aligned} y_x - y_{x-1} &= \Delta y_{x-1} \\ \Delta y_x - \Delta y_{x-1} &= \Delta^2 y_{x-1} \\ \Delta^2 y_x - \Delta^2 y_{x-1} &= \Delta^3 y_{x-1} \\ &\&c. \end{aligned}$$

Now let us form the differences of any series a, b, c, d . . . in this manner. Make

$$\begin{aligned} b &= c + a' \\ c &= b + b' \\ d &= c + c' \\ &\&c. \\ b' &= a' + a'' \\ c' &= b' + b'' \\ d' &= c' + c'' \\ &\&c. \\ b'' &= a'' + a''' \\ c'' &= b'' + b''' \\ d'' &= c'' + c''' \\ &\&c. \end{aligned}$$

and so on continually. Then eliminating b, b', c, c', &c. from the first set of equations, we get

$$\begin{aligned} b &= a + a' \\ c &= a + 2a' + a'' \\ d &= a + 3a' + 3a'' + a''' \\ e &= a + 4a' + 6a'' + 4a''' + a'''' \\ f &= a + 5a' + 10a'' + \&c. \\ &\&c. \end{aligned}$$

Also we have

$$\begin{aligned} a' &= b - a \\ a'' &= c - 2b + a \\ a''' &= d - 3c + 3b - a \\ &\&c. \end{aligned}$$

But the letters a', a'', a''', &c. are nothing else than Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ . . . a, b, c . . . being y_0, y_1, y_2 . . . , consequently

$$\begin{aligned} y_1 &= y_0 + \Delta y^0 \\ y_2 &= y_0 + 2\Delta y_0 + \Delta^2 y \\ y_3 &= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0 \\ &\&c. \end{aligned}$$

And

$$\Delta y_0 = y_1 - y_0$$
$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$
$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$
$$\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 + 4y_1 + y_0$$
&c.

Hence, generally, we have

$$y_x = y_0 + x\Delta y_0 + x\frac{x-1}{2}\cdot\Delta^2 y_0 + x\cdot\frac{x-1}{2}\cdot\frac{x-2}{3}\Delta^3 y + \ldots (A)$$

$$\Delta_n y_0 = y_n - n y_{n-1} + n\cdot\frac{n-1}{2}\cdot y_{n-2} - n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} y_{n-3} + \ldots (B)$$

These equations, which are of great importance, give the general term of any series, from knowing its first term and the first term of all the orders of differences; and also the first term of the series of nth differences, from knowing all the terms of the series $y_0, y_1, y_2 \ldots$

To apply the former to the example in p. (81), we have

$$y_0 = 1$$
$$\Delta y_0 = -2$$
$$\Delta^2 y_0 = 4$$
$$\Delta^3 y_0 = 6$$
$$\Delta^4 y_0 = 0$$

whence

$$y_x = 1 - 2x + 2x(x-1) + x(x-1)(x-2) = x^3 - x^2 - 2x + 1$$

The equations (A), (B) will be better remembered by observing that

$$y_x = (1 + \Delta y_0)^x,$$
$$\Delta^n y_0 = (y - 1)^n,$$

provided that in the developements of these powers, we mean by the exponents of Δy_0, the orders of differences, and by those of y the place in the series.

It has been shown that a, b, c, d ... may be the values of y_x, when those of x are the progressional numbers

$$m, m + h, m + 2h \ldots m + ih$$

that is

$$a = y_m, b = y_{m+h}, c = \&c.$$

In the equation (A), we may, therefore, put y_{m+ih} for y_x, y_m for y_0, Δy_m for Δy_0, &c. and, finally, the coefficients of the ith power. Make $ih = z$, and write $\Delta, \Delta^2 \ldots$ for $\Delta y_m, \Delta^2 y_m \ldots$ and we shall get

$$y_{m+z} = y_m + \frac{z\Delta}{h} + \frac{z\cdot(z-h)\Delta^2}{2h^2} + \frac{z(z-h)(z-2h)\Delta^3}{2h^3} + \ldots (C)$$

This equation will give y_x when $x = m + z$, z being either integer or fractional. We get from the proposed series the differences of all orders, and the initial terms represented by Δ, Δ^2, &c.

But in order to apply this formula, so that it may be limited, we must arrive at constant differences; or, at least, this must be the case if we would have Δ, Δ^2... decreasing in value so as to form a converging series: the developement then gives an approximate value of a term corresponding to

$$x = m + z;$$

it being understood that the factors of Δ do not increase so as to destroy this convergency, a circumstance which prevents z from surpassing a certain limit.

For example, if the radius of a circle is 1000,

the arc of 60° has a chord	1000,0		
		$\Delta. = 74{,}6$	
65°	1074,6		$\Delta^2 = -2{,}0$
		72,6	
70°	1147,2		$-2{,}3$
75°	1217,5		

Since the difference is nearly constant from 60° to 75°, to this extent of the arc we may employ the equation (C); making $h = 5$, we get for the quantity to be added to $y_m = 1090$, this

$$\tfrac{1}{5}. 74{,}6. z - \tfrac{2}{50} z (z - 5) = 15{,}12. z - 0{,}04. z^2$$

So that, by taking $z = 1, 2, 3...$ then adding 1000, we shall obtain the chords of 61°, 62°, 63°; in the same manner, making z the necessary *fraction*, we shall get the chord of any arc whatever, that is intermediate to those, and to the limits 60° and 75°. It will be better, however, when it is necessary thus to employ great numbers for z, to change these limits.

Let us now take

log. 3100	$= y_m = 4913617$		
		$\Delta. = 13987$	
log. 3110	$= 4927604$		$\Delta.^2 = -45$
		13942	
log. 3120	$= 4941546$		-45
		13897	
log. 3130	$= 4955443$		

We shall here consider the decimal part of the logarithm as being an integer. By making $h = 10$, we get, for the part to be added to log. 3100, this

$$1400,95 \times z - 0,225 \times z^2.$$

To get the logarithms of 3101, 3102, 3103, &c. we make

$$z = 1, 2, 3....;$$

and in like manner, if we wish for the log. 3107, 58, we must make

$z - 7$, 58, whence the quantity to be added to the logarithm of 3100 is 10606. Hence

$$\log. 3107\overset{\cdot}{6}8 = 5,4924223.$$

The preceding methods may be usefully employed to abridge the labour of calculating tables of logarithms, tables of sines, chords, &c. Another use which we shall now consider, is that of inserting the intermediate terms in a given series, of which two distant terms are given. This is called

INTERPOLATION.

It is completely resolved by the equation (C).

When it happens that $\Delta^2 = 0$, or is very small, the series reduces to

$$\underset{m}{z} + \frac{y\,\Delta}{h}$$

whence we learn that the results have a difference which increases proportionally to z.

When Δ^2 is constant, which happens more frequently, by changing z into $z + 1$ in (C), and subtracting, we have the general value of the first difference of the new interpolated series; viz.

$$\text{First difference } \Delta' = \frac{\Delta.}{h} + \frac{2z - h + 1}{2h^2}\Delta.^2$$

$$\text{Second difference } \Delta'' = \frac{\Delta.^2}{h^2}.$$

If we wish to insert u terms between those of a given series, we must make

$$h = n + 1;$$

then making $z = 0$, we get the initial term of the differences

$$\Delta'' = \frac{\Delta.^2}{(n+1)^2}$$

$$\Delta' = \frac{\Delta.}{n+1} - \tfrac{1}{2}\,n\,\Delta'';$$

we calculate first Δ'', then Δ'; the initial term Δ' will serve to compose the series of first differences of the interpolated series, (Δ'' is the constant difference of it); and then finally the other terms are obtained by simple additions.

If we wish in the preceding example to find the log. of 3101,

3102, 3103 ... we shall interpolate 9 numbers between those which are given: whence

$$u = 9$$
$$\Delta'' = -0{,}45$$
$$\Delta' = 1400{,}725.$$

We first form the arithmetical progression whose first term is Δ', and $-0{,}45$ for the constant. The first differences are

1400,725; 1400,725; 1399,375; 1398,925, &c.

Successive additions, beginning with log. 3100, will give the consecutive logarithms required.

Suppose we have observed a physical phenomenon every twelve hours, and that the results ascertained by such observations have been

For 0 hours . . .	78		
12 . . .	300	$\Delta = 222$	$\Delta^2 = 144$
24 . . .	666	366	
36 . . .	1176	510	144.

&c.

If we are desirous of knowing the state corresponding to 4^h, 8^h, 12^h, &c., we must interpolate two terms; whence

$$u = z,\ \Delta'' = 16,\ \Delta' = 58$$

composing the arithmetic progression whose first term is 58, and common difference 16, we shall have the first differences of the new series, and then what follow

First differences	58, 74, 90, 106, 122, 138 . . .
Series	78, 136, 210, 300, 406, 528, 646 . . .
A	0^h, 4^h, 8^h, 16^h 20^h, 24^h.

The supposition of the second differences being constant, applies almost to all cases, because we may choose intervals of time which shall favour such an hypothesis. This method is of great use in astronomy; and even when observation or calculation gives results whose second differences are irregular, we impute the defect to errors which we correct by establishing a greater degree of regularity.

Astronomical, and geodesical tables are formed on these principles. We calculate directly different terms, which we take so near that their first or second differences may be constant; then we interpolate to obtain the intermediate numbers.

Thus, when a converging series gives the value of y by aid of that of a variable x; instead of calculating y for each known value of x, when the formula is of frequent use, we determine the results y for the continually

increasing values of x, in such a manner that y shall always be nearly of the same value: we then write in the form of a table every value by the side of that of x, which we call the *argument* of this table. For the numbers x which are intermediate to them, y is given by simple proportions, and by inspection alone we then find the results required.

When the series has two variables, or arguments x and z, the values of y are disposed in a table by a sort of *double entry*; taking for coordinates x and z, the result is thus obtained. For example, having made $z = 1$, we range upon the first line all the values of y corresponding to

$$x = 1, 2, 3 \ldots;$$

we then put upon the second line which $z = z$ gives; in a third line those which $z = 3$ gives, and so on. To obtain the result which corresponds to

$$x = 3, z = 5$$

we stop at the case which, in the third column, occupies the fifth place. The intermediate values are found analogously to what has been already shown.

So far we have supposed x to increase continually by the same difference. If this is not the case and we know the results

$$y = a, b, c, d \ldots$$

which are due to any suppositions

$$x = \alpha, \beta, \gamma, \delta \ldots$$

we may either use the theory which makes a parabolic curve pass through a series of given points, or we may adopt the following:

By means of the known corresponding values

$$a, \alpha; \; b\,\beta; \; \&c.$$

we form the consecutive functions

$$A = \frac{b - a}{\beta - \alpha}$$

$$A_1 = \frac{c - b}{\gamma - \beta}$$

$$A_2 = \frac{d - c}{\delta - \gamma}$$

&c.

$$B = \frac{A_1 - A}{\gamma - \alpha}$$

$$B_1 = \frac{A_2 - A}{\delta - \beta}$$

$$B_2 = \frac{A_3 - A}{\epsilon - \gamma}$$

&c.

$$C = \frac{B_1 - B}{\delta - \alpha}$$

$$C_1 = \frac{B_2 - B_1}{\varepsilon - \beta}$$

&c.

$$D = \frac{C_1 - C}{\nu - \alpha}$$

and so on.

By elimination we easily get

$b = a + A(\beta - \alpha)$

$c = a + A(\gamma - \alpha) + B(\gamma - \alpha)(\gamma - \beta)$

$d = a + A(\delta - \alpha) + B(\delta - \alpha)(\delta - \beta) + C(\delta - \alpha)(\delta - \beta)(\delta - \gamma)$

&c.

and generally

$$y_x = a + A(x - \alpha) + B(x - \alpha)(x - \beta) + C(x - \alpha)(x - \beta)(x - \gamma) + \&c.$$

We must seek therefore the first differences amongst the results

$$a, b, c \ldots$$

and divide by the differences of

$$\alpha, \beta, \gamma \ldots$$

which will give

$$A, A_1, A_2, \&c.$$

proceeding in the same manner with these numbers, we get

$$B, B_1, B_2, \&c.$$

which in like manner give

$$C, C_1, C_2, \&c.$$

and, finally substituting, we get the general term required.

By actually multiplying, the expression assumes the form

$$a + a' x + a'' x^2 + \ldots$$

of every rational and integer polynomial, which is the same as when we neglect the superior differences.

The chord of 60° = rad.	= 1000	35	$A = 15$	$B = -0,035$
62°.20′	= 1035	42	$A_1 = 14,82 - 0,18$	$B_1 = -0,031$
65°.10′	= 1077	56	$A_2 = 14,61 - 0,21$	
69°. 0′	= 1133			

We have

$$\alpha = 0, \ \beta = 2\tfrac{1}{3}, \ \gamma = 5\tfrac{7}{11}, \ \delta = 9.$$

We may neglect the third differences and put

$$y_x = 100 + 15,082\, x - 0,035\, x^2.$$

Considering every function of x, y_x, as being the general term of the series which gives

$$x = m, \ m + h, \ m + 2h, \ \&c.$$

if we take the differences of these results, to obtain a new series, the general term will be what is called the *first difference* of the proposed function y_x which is represented by Δy_x. Thus we obtain this difference by changing x into $x + h$ in y_x and taking y_x from the result; the remainder will give the series of first differences by making

$$x = m,\ m + h,\ m + 2h,\ \&c.$$

Thus if

$$y_x = \frac{x^2}{a + x}$$

$$\Delta y_x = \frac{(x + h)^2}{a + x + h} - \frac{x^2}{a + x}.$$

It will remain to reduce this expression, or to develope it according to the increasing powers of h.

Taylor's theorem gives *generally* (vol. I.)

$$\Delta y_x = \frac{dy}{dx} h + \frac{d^2 y}{dx^2} \cdot \frac{h^2}{1.2} + \&c.$$

To obtain the second difference we must operate upon Δy_x as upon the proposed y_x, and so on for the third, fourth, &c. differences.

INTEGRATION OF FINITE DIFFERENCES.

Integration here means the method of finding the quantity whose difference is the proposed quantity; that is to say the general term y_x of a

$$y_m,\ y_{m+h},\ y_{m+2h},\ \&c.$$

from knowing that of the series of a difference of any known order. This operation is indicated by the symbol Σ.

For example

$$\Sigma (3x^2 + x - 2)$$

ought to indicate that here

$$h = 1.$$

A function y_x generates a series by making

$$x = 0,\ 1,\ 2,\ 3 \ldots$$

the first differences which here ensue, form another series of which

$$3x^2 + x - 2$$

is the general term, and it is

$$-2,\ 2,\ 12,\ 28 \ldots$$

By integrating we here propose to find y_x such, that putting $x + 1$ for x, and subtracting, the remainder shall be

$$3x^2 + x - 2.$$

If

It is easy to perceive that, first the symbols Σ and Δ destroy one another as do f and d; thus

$$\Sigma \Delta f x = f x.$$

Secondly, that

$$\Delta (a y) = a \Delta y$$

gives

$$\Sigma a y = a \Sigma y.$$

Thirdly, that as

$$\Delta (A t - B u) = A \Delta t - B \Delta u$$

so is

$$\Sigma (A t - B u) = A \Sigma t - B \Sigma u,$$

t and u being the functions of x.

The problem of determining y_x by its first difference does not contain data sufficient completely to resolve it; for in order to recompose the series derived from y_x in beginning with

$$- 2, 2, 12, 28, \&c.$$

we must make the first term

$$y_0 = a$$

and by successive additions, we shall find

$$a, a - 2, a + 2, a + 12, \&c.$$

in which a remains arbitrary.

Every integral may be considered as comprised in the equation (A) p. 83; for by taking

$$x = 0, 1, 2, 3 \ldots$$

in the first difference given in terms of x, we shall form the series of first differences; subtracting these successively, we shall have the second differences; then in like manner, we shall get the third and fourth differences. The initial term of these series will be

$$\Delta y_0, \Delta^2 y_0 \ldots$$

and these values substituted in y_x will give y_x. Thus, in the example above, which is only that of page (81) when $a = 1$, we have

$$\Delta y_0 = - 2, \Delta^2 y_0 = 4, \Delta^3 y_0 = 6, \Delta^4 y_0 = 0, \&c.;$$

which give

$$y_x = y_0 - 2x - x^2 + x^3.$$

Generally, the first term y_0 of the equation (A) is an arbitrary constant, which is to be added to the integral. If the given function is a second difference, we must by a first integration reascend to the first difference and thence by another step to y_x; thus we shall have two arbitrary constants; and in fact, the equation (A) still gives y_x by finding Δ^2, Δ^3, the

only difference in the matter being that y_0 and Δy_0 are arbitrary. And so on for the superior orders.

Let us now find Σx^m, the exponent m being integer and positive. Represent this developement by

$$\Sigma x^m = p x + q x^b + r x^c + \&c.$$

a, b, c, &c. being decreasing exponents, which as well as the coefficients p, q, &c. must be determined. Take the first difference, by suppressing Σ in the first member, then changing x into $x + h$ in the second member and subtracting. Limiting ourselves to the two first terms, we get

$$x^m = p a h x^{a-1} + \tfrac{1}{2} p a (a - 1) h^2 x^{a-2} + \ldots q b h x^{b-1} + \ldots$$

But in order that the identity may be established, the exponents ought to give

$$a - 1 = m$$
$$a - 2 = b - 1$$

whence

$$a = m + 1, b = m.$$

Moreover the coefficients give

$$1 = p a h, -\tfrac{1}{2} p a (a - 1) h = q b;$$

whence

$$p = \frac{1}{(m + 1) h}, q = -\tfrac{1}{2}.$$

As to the other terms, it is evident, that the exponents are all integer and positive; and we may easily perceive that they fail in the alternate terms. Make therefore

$$\Sigma x^m = p x^{m+1} - \tfrac{1}{2} x^m + \alpha x^{m-1} + \beta x^{m-3} + \gamma x^{m-5} + \ldots$$

and determine α, β, γ ... &c.

Take, as before, the first difference by putting $x + h$ for x, and subtracting: and first transferring

$$p x^{m+1} - \tfrac{1}{2} x^m,$$

we find that the first member, by reason of

$$p h (m + 1) = 1,$$

reduces to

$$A' . \frac{h^2}{2.3} x^{m-2} + A'' . \frac{m - 3}{4} . \frac{3 h^4}{2.5} x^{m-4} + A'''' . \frac{m - 5}{6} . \frac{5 h^6}{2.7} x^{m-6} \ldots$$

To abridge the operation, we omit here the alternate terms of the developement; and we designate by

$$1, m, A', A'', \&c.$$

the coefficients of the binomial.

Making the same calculations upon

$$\alpha x^{m-1} + \beta x^{m-3} + \&c.$$

we shall have, with the same respective powers of x and of h,

$$(m-1)\,\alpha+(m-1).\frac{n-2}{2}.\frac{m-3}{3}\alpha+(m-1).\frac{m-2}{3}\ldots\frac{m-4}{5}\alpha+\ldots$$
$$+\quad (m-3)\,\beta+(m-3).\frac{m-4}{2}\ldots\frac{m-5}{3}\beta+\ldots$$
$$+\quad (m-4)\,\gamma+\ldots$$

Comparing them term by term, we easily derive

$$\alpha=\frac{m}{3.4},$$
$$\beta=\frac{-A''}{2.3.4.5},$$
$$\gamma=\frac{A''''}{6.6.7}$$
&c.

whence finally we get

$$\Sigma x^m=\frac{x^{m+1}}{(m+1)\,h}-\frac{x^m}{2}+m\,a\,h\,x^{m-1}+A''\,b\,h^3x^{m-3}$$
$$+A''''\,c\,h^5x^{m-5}+A^{vi}\,d\,h^7x^{m-7}+\ldots(D)$$

This developement has for its coefficients those of the binomial, taken from *two to two*, multiplied by certain numerical factors a, b, c . . ., which are called the *numbers* of Bernoulli, because James Bernoulli first determined them. These factors are of great and frequent use in the theory of series; we shall give an easy method of finding them presently. These are their values

$$a=\frac{1}{12}$$
$$b=-\frac{1}{120}$$
$$c=\frac{1}{252}$$
$$d=-\frac{1}{240}$$
$$e=\frac{1}{132}$$
$$f=-\frac{691}{32780}$$
$$g=\frac{1}{12}$$
$$h=-\frac{3617}{8160}$$
$$i=\frac{43867}{14364}$$
&c.

which it will be worth the trouble fully to commit to memory.

From the above we conclude that to obtain Σx^m, m being any number, integer and positive, we must besides the two first terms

$$\frac{x^{m+1}}{(m+1)h} - \frac{x^m}{2}$$

also take the developement of

$$(x + h)^m$$

reject the odd terms, the first, third, fifth, &c. and multiply the retained terms respectively by

$$a, b, c \ldots$$

Now x and h *have even exponents only when* m *is odd and reciprocally;* so that we must reject the last term h^m when it falls in a useless situation; the number of terms is $\frac{1}{2}m + 2$ when m is even, and it is $\frac{1}{2}(m+3)$ when m is odd; that is to say, it is the same for two consecutive values of m.

Required the integral of x^{10}.

Besides

$$\frac{x^{11}}{11h} - \tfrac{1}{2}x^{10}$$

we must develope $(x + h)^{10}$, retaining the second, fourth, sixth, &c. terms and we shall have

$$10\,x^9\,a\,h + 120\,x^7\,b\,h^3 + 252\,x^5\,c\,h^5 + \&c.$$

Therefore

$$\Sigma x^{10} = \frac{x^{11}}{11h} - \tfrac{1}{2}x^{10} + \frac{5}{6}x^9 h - x^7 h^3 + x^5 h^5 - \tfrac{1}{2}x^3 h^7 + \frac{5}{66}x h^9$$

In the same manner we obtain

$$\Sigma x^0 = \frac{x}{h}$$

$$\Sigma x^1 = \frac{x^2}{2h} - \frac{x}{2}$$

$$\Sigma x^2 = \frac{x^3}{3h} - \frac{x^2}{2} + \frac{hx}{6},$$

$$\Sigma x^3 = \frac{x^4}{4h} - \frac{x^3}{2} + \frac{hx^2}{4}$$

$$\Sigma x^4 = \frac{x^5}{5h} - \frac{x^4}{4} + \frac{hx^3}{3} - \frac{h^3 x}{30}$$

$$\Sigma x^5 = \frac{x^6}{6h} - \frac{x^5}{2} + \frac{5hx^4}{12} - \frac{h^3 x^2}{12},$$

$$\Sigma x^6 = \frac{x^7}{7h} - \frac{x^6}{2} + \frac{hx^5}{2} - \frac{h^3 x^3}{6} + \frac{h^5 x}{42},$$

$$\Sigma x^7 = \frac{x^8}{8h} - \frac{x^7}{2} + \frac{7hx^6}{12} - \frac{7h^3 x^4}{24} + \frac{h^5 x^2}{12}$$

$$\Sigma x^8 = \frac{x^9}{9h} - \frac{x^8}{2} + \frac{2hx^7}{3} - \frac{7h^3x^5}{15} + \frac{2h^5x^3}{9} - \frac{h^7x}{30}$$

$$\Sigma x^9 = \frac{x^{10}}{10h} - \frac{x^9}{2} + \frac{3hx^8}{4} - \frac{7h^3x^6}{10} + \frac{h^5x^4}{2} - \frac{3h^7x^2}{20}$$

$$\Sigma x^{10} = \frac{x^{11}}{11h} - \text{ \&c. as before,}$$

&c.

We shall now give an easy method of determining the *Number of Bernoulli* a, b, c... In the equation (D) make

$$x = h = 1;$$

Σx^m is the general term of the series whose first difference is x^m. We shall here consider $\Sigma . x^0 = 1$, and the corresponding series which is that of the natural numbers

$$0, 1, 2, 3 \ldots$$

Take zero for the first member and transpose

$$\frac{1}{m+1} - \tfrac{1}{2}$$

which equals

$$\frac{2(m+1)}{1-m}.$$

Then we get

$$\frac{m-1}{2(m+1)} = a\,m + b\,A'' + c\,A^{iv} + d\,A^{vi} + \ldots + k\,m.$$

By making $m = 2$, the second member is reduced to am, which gives

$$a = \frac{1}{12}.$$

Making $m = 4$, we get

$$\frac{3}{10} = 4a + b\,A''$$
$$= 4a + m.\frac{m-1}{2}.\frac{m-2}{3}b$$
$$= 4a + 4b$$
$$= \tfrac{1}{3} + 4b.$$

Whence

$$b = -\frac{1}{120}.$$

Again, making $m = 6$, we get

$$\frac{5}{14} = 6a + b\,A'' + c\,A^{iv}$$
$$= 6a + 20b + 6c$$
$$= \tfrac{1}{2} - \tfrac{1}{6} + 6c$$

which gives

$$c = \frac{1}{252};$$

and proceeding thus by making

$$m = 2, 4, 6, 8, \&c.$$

we obtain at each step a new equation which has one term more than the preceding one, which last terms, viz.

$$2\,a,\ 4\,b,\ 6\,c, \ldots\ m\,k$$

will hence successively be found, and consequently,

$$a, b, c \ldots k.$$

Take the difference of the product

$$y_x = (x - h)\,x\,(x + h)\,(x + 2h) \ldots (x + i\,h),$$

by $x + h$ for x and subtracting; it gives

$$\Delta\, y_x = x\,(x + h)\,(x + 2h) \ldots (x + i\,h) \times (i + 2)\,h;$$

dividing by the last constant factor, integrating, and substituting for y_x its value, we get

$$\Sigma\, x\,(x + h)\,(x + 2h) \ldots (x + i\,h) = \frac{x - h}{(i + 2)\,h} \times x.\,(x + h)\,(x + 2h) \ldots (x + i\,h)$$

This equation gives *the integral of a product of factors in arithmetic progression.*

Taking the difference of the second member, we verify the equation

$$\Sigma \frac{1}{x\,(x + h)\,(x + 2h) \ldots (x + i\,h)} = \frac{-1}{i\,h\,x\,(x + h) \ldots \{x + (i - 1)\,h\}}$$

which gives the integral of any inverse product.

Required the integral of a^x.

Let

$$y_x = a^x.$$

Then

$$\Delta\, y_x = a^x\,(a^h - 1)$$

whence

$$y_x = \Sigma\, a^x\,(a^h - 1) = a^x;$$

consequently

$$\Sigma\, a^x = \frac{a^x}{a^h - 1} + \text{constant.}$$

Required the integrals of sin. x, cos. x.

Since

$$\cos. B - \cos. A = 2 \sin. \tfrac{1}{2}(A + B).\ \sin. \tfrac{1}{2}(A - B)$$

$$\Delta \cos. x = \cos. (x + h) - \cos. x$$

$$= -2 \sin. \left(x + \frac{h}{2}\right) \sin. \frac{h}{2}$$

Integrating and changing $x + \frac{h}{2}$ into z, we have

$$\Sigma \sin. z = - \cos. \frac{\left(z - \frac{h}{2}\right)}{2 \sin. \frac{h}{2}} + \text{constant.}$$

In the same way we find

$$\Sigma \cos. z = \frac{\sin. \left(z - \frac{h}{2}\right)}{2 \sin. \frac{h}{2}} + \text{constant.}$$

When we wish to integrate the powers of sines and cosines, we transform them into sines and cosines of multiple arcs, and we get terms of the form

$$A \sin. q x, \; A \cos. q x.$$

Making

$$q x = x$$

the integration is performed as above.

Required the integral of a product, viz.

Assume

$$\Sigma (u z) = u \Sigma z + t$$

u, z and t being all functions of x, t being the only unknown one. By changing x into x + h in

$$u \Sigma z + t$$

u becomes $u + \Delta u$, z becomes $z + \Delta z$, &c. and we have

$$u \Sigma z + u z + \Delta u \Sigma (z + \Delta z) + t + \Delta t;$$

substituting from this the second member

$$u \Sigma z + t,$$

we obtain the difference, or u z; whence results the equation

$$0 = \Delta u' \Sigma (z + \Delta z) + \Delta t$$

which gives

$$t = - \Sigma \{\Delta u \Sigma (z + \Delta z)\}.$$

Therefore

$$\Sigma (u z) = u \Sigma z - \Sigma \{\Delta u . \Sigma (z + \Delta z)\}$$

which is analogous to integrating *by parts* in differential functions.

There are but few functions of which we can find the finite integral; when we cannot integrate them exactly, we must have recourse to series.

Taylor's theorem gives us

$$\Delta y_x = \frac{d y}{d x} h + \frac{d^2 y}{d x^2} \cdot \frac{h^2}{2} + \text{&c.}$$

$$= y' h + \frac{y''}{2} h^2 + \&c.$$

by supposition. Hence

$$y_x = h \Sigma y' + \frac{h^2}{2} \Sigma y'' + \&c.$$

Considering y' as a given function of x, viz. z, we have

$$y' = z$$
$$y'' = z'$$
$$y''' = z''$$
$$\&c.$$

and

$$y_x = \int y' \, dx = \int z \, dx$$

whence

$$\int z \, dx = h \Sigma z + \frac{h^2}{2} \Sigma z' + \&c.$$

which gives

$$\Sigma z = h^{-1} \int z \, dx - \frac{h}{2} \Sigma z' - \tfrac{1}{6} h^2 \Sigma z'' - \&c.$$

This equation gives Σz, when we know z', $\Sigma z'$, &c. Take the differentials of the two numbers. That of the first Σz will give, when divided by dx, $\Sigma z'$. Hence we get $\Sigma z''$, then $\Sigma z'''$, &c.; and even without making the calculations, it is easy to see, that the result of the substitution of these values, will be of the form

$$\Sigma z = h^{-1} \int z \, dx + A z + B h z' + C h^2 z'' + \&c.$$

It remains to determine the factors A, B, C, &c. But if

$$z = x^m$$

we get

$$\int z \, dx, \; z', \; z'', \; \&c.$$

and substituting, we obtain a series which should be identical with the equation (D), and consequently defective of the powers $m - 2$, $m - 4$, so that we shall have

$$\Sigma z = \frac{\int z \, dx}{h} \quad \frac{z}{2} + \frac{a h z'}{1} + \frac{b h^3 z'''}{1.2} + \frac{c h^5 z''''}{2.3.4} + \frac{d h^7 z''''''}{2 \ldots 6} + \&c.$$

a, b, c, &c. being the numbers of Bernoulli.

For example, if

$$z = l x$$
$$\int l x \, . \, dx = x \, l x - x$$
$$z' = x^{-1}$$
$$z'' = \&c.$$

consequently

$$\Sigma \, l\, x = C + x\, l\, x - x - \tfrac{1}{2}\, l\, x + a\, x^{-1} + b\, x^{-3} + c\, x^{-5} + \&c.$$

The series

$$a, b, c \ldots k, l,$$

having for first differences

$$a', b', c' \ldots k'$$

we have

$$b = a + a'$$
$$c = b + b$$
$$d = c + c'$$
$$\&c.$$
$$l = k + k'$$

equations whose sum is

$$l = a + a' + b' + c' + \ldots k'.$$

If the numbers a', b', c', &c. are known, we may consider them as being the first differences of another series a, b, c, &c. since it is easy to compose the latter by means of the first, and the first term a. By definition we know that any term whatever l', taken in the given series a', b', c', &c. is nothing else than $\Delta\, l$, for $l' = m - l$; integrating

$$l' = \Delta\, l$$

we have

$$\Sigma\, l' = l$$

or

$$\Sigma\, l' = a' + b' + c' \ldots + k',$$

supposing the initial a is comprised in the constant due to the integration. Consequently

The integral of any term whatever of a series, we obtain the sum of all the terms that precede it, and have

$$\Sigma\, y_x = y_0 + y_1 + y_2 + \cdots y_{x-1}.$$

In order to get the sum of a series, we must add y_x to the integral; or which is the same, in it must change x into $x + 1$, before we integrate. The arbitrary constant is determined by finding the value of the sum y_0 when

$$x = 1.$$

We know therefore how to find the summing term of every series whose general term is known in a rational and integer function of x.

Let

$$y_x = A\, x^m - B\, x^n + C$$

m and n being positive and integer, and we have

$$A\, \Sigma\, x^m - B\, \Sigma\, x^n + C\, \Sigma\, x^0$$

for the sum of the terms as far as y_x exclusively. This integral being once found by equation D, we shall change x into x + 1, and determine the constant agreeably.

For example, let

$$y_x = x\,(2x - 1);$$

changing x into x + 1, and integrating the result, we shall find

$$2\,\Sigma x^3 + 3\,\Sigma x + \Sigma x^0 = \frac{4x^3 + 3x^2 - x}{2.3}$$

$$= x.\frac{x+1}{2}.\frac{4x-1}{3};$$

there being no constant, because when $x = 0$, the sum $= 0$.

The series

$$1^m,\ 2^m,\ 3^m \ldots$$

of the mth powers of the natural numbers is found by taking Σx^m (equation D); but we must add afterwards the xth term which is x^m; that is to say, it is sufficient to change $-\frac{1}{2}x^m$, the second term of the equation (D), into $\frac{1}{2}x^m$; it then remains to determine the constant from the term we commence from.

For example, to find

$$S = 1 + 2^2 + 3^2 + 4^2 + \ldots x^2$$

we find Σx^2, changing the sign of the second term, and we have

$$S = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6} = x.\frac{x+1}{3}.\frac{2x+1}{2};$$

the constant is 0, because the sum is 0 when $x = 0$. But if we wish to find the sum

$$S' = (n+1)^2 + (n+2)^2 + \ldots x^2$$

$S' = 0$, whence $x = n - 1$, and the constant is

$$-n.\frac{n-1}{2}.\frac{2n-1}{3},$$

which of course must be added to the former; thus giving

$$S' = (n+1)^2 + (n+2)^2 + \ldots x^2$$

$$= x.\frac{x+1}{3}.\frac{2x+1}{2} - n.\frac{n-1}{2}.\frac{2n-1}{3}$$

$$= \frac{1}{6} \times \{x.(x+1).(2x+1) - n.(n-1)(2n-1)$$

$$= \frac{1}{6} \times \{2(x^3 - n^3) + 3(x^2 + n^1) + x - n\}.$$

This theory applies to the summation of *figurate numbers*, of the different orders:—

First order,	1 . 1 . 1 . 1 . 1 . 1 . 1 , &c.
Second order,	1 . 2 . 3 . 4 . 5 . 6 . 7 , &c.
Third order,	1 . 3 . 6 . 10 . 15 . 21 . 28 , &c.
Fourth order,	1 . 4 . 10 . 20 . 35 . 56 . 84 , &c.
Fifth order,	1 . 5 . 15 . 35 . 70 . 126 . 210, &c.

and so on.

The law which every term follows being the sum of the one immediate 'y over it added to the preceding one. The general terms are

First, 1

Second, x

Third, $\frac{x.(x+1)}{2}$

Fourth, $\frac{x\,(x+1)\,(x+2)}{2.3}$

&c.

p^{th} $\frac{x.(x+1)\,(x+2)\ldots x+p-2}{1.2.3\ldots p-1}$.

To sum the Pyramidal numbers, we have

$$S = 1 + 4 + 10 + 20 + \&c.$$

Now the general or x^{th} term in this is

$$y_x = \frac{1}{6} . x\,(x+1)\,(x+2).$$

But we find for the $(x-1)^{th}$ term of numbers of the next order

$$\frac{1}{24}(x-1)\,x\,(x+1)\,(x+2);$$

finally changing x into $x+1$, we have for the required form

$$S = \frac{1}{24} x.\,(x+1)\,(x+2)\,(x+3).$$

Since $S = 1$, when $x = 1$, we have

$1 = 1 +$ constant, consequently

$\therefore$ constant $= 0$.

Hence it appears that the sum of x terms of the fourth order, is the xth term or general term of the fifth order, and *vice versâ*; and in like manner, it may be shown that the xth term of the $(n+1)^{th}$ order is the sum of x terms of the nth order.

Inverse figurate numbers are fractions which have 1 for the numerator, and a figurate series for the denominator. Hence the xth term of the pth order is

$$\frac{1.2.3\ldots(p-1)}{x\,(x+1)\ldots x+p-2}$$

and the integral of this is

$$C - \frac{1.2.3\ldots(p-1)}{(p-2)\,x\,(x+1)\ldots(x+p-3)}.$$

Changing x into x + 1, then determining the constant by making x = 0, which gives the sum = 0, we shall have

$$c = \frac{p-1}{p-2};$$

and the sum of the x first terms of this general series is

$$\frac{p-1}{p-2} - \frac{1.2.3\ldots(p-1)}{(p-2)\,(x+1)\,(x+2)\ldots(x+p-2)}.$$

In this formula make

$$p = 3, 4, 5\ldots$$

and we shall get

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots \frac{1.2}{x\,(x+1)} = \frac{2}{1} - \frac{2}{x+1}$$

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \ldots \frac{1.2.3}{x\,(x+1)\,(x+2)} = \frac{3}{2} - \frac{3}{(x+1)\,(x+2)}$$

$$\frac{1}{1} + \frac{1}{5} + \frac{1}{10} + \frac{1}{35} + \ldots \frac{1.2.3.4}{x\,(x+1)\,(x+2)\,(x+3)} = \frac{4}{3} - \frac{2.4}{(x+1)\ldots(x+3)}$$

$$\frac{1}{1} + \frac{1}{6} + \frac{1}{21} + \frac{1}{56} + \ldots \frac{1.2.3.4.5}{x\,(x+1)\ldots(x+4)} = \frac{5}{4} - \frac{2.3.5}{(x+1)\ldots(x+4)}$$

and so on. To obtain the whole sum of these series continued to infinity, we must make

$$x = \infty$$

which gives for the sum required the general value

$$\frac{p-1}{p-2}$$

which in the above particular cases, becomes

$$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \text{ \&c.}$$

To sum the series

$$\text{sin. } a + \text{sin. } (a + h) + \text{sin. } (a + 2h) + \ldots \text{sin. } (a + x - 1h)$$

we have

$$\Sigma \text{ sin. } (a + x\,h) = C - \frac{\text{cos. } \left(a + h\,x - \frac{h}{2}\right)}{2 \text{ sin. } \frac{h}{2}}$$

changing x into x + 1, and determining C by the condition that x = − 1 makes the sum = zero, we find for the *summing-term*.

$$\frac{\text{cos. } \left(a - \frac{h}{2}\right) - \text{cos. } \left(a + h\,x + \frac{h}{2}\right)}{2 \text{ sin. } \frac{h}{2}}$$

or

$$\frac{\sin.\left(a+\frac{h}{2}x\right)\sin.\frac{h(x+1)}{2}}{\sin.\frac{h}{2}}$$

In a similar manner, if we wish to sum the series

$$\cos.a+\cos.(a+h)+\cos.(a+2h)+\ldots\cos.(a+\overline{x-1}h)$$

we easily find the *summing-term* to be

$$\frac{\sin.\left(a-\frac{h}{2}\right)-\sin.\left(a+hx+\frac{h}{2}\right)}{2\sin.\frac{h}{2}}$$

or

$$\frac{\cos.\left(a+\frac{h}{2}x\right)\sin.\frac{h(x+1)}{2}}{\sin.\frac{h}{2}}.$$

A COMMENTARY

ON

NEWTON'S PRINCIPIA.

SUPPLEMENT

TO

SECTION XI.

460 PROP. LVII, depends upon Cor. 4 to the Laws of Motion, which is

If any number of bodies mutually *attract each other, their center of gravity will either remain at rest or will move uniformly in a straight line.*

First let us prove this for two bodies.

Let them be referred to a fixed point by the rectangular coordinates

$$x,\ y;\ x',\ y',$$

and let their masses be

$$\mu,\ \mu'.$$

Also let their distance be ϱ, and $f(\varrho)$ denote the law according to which they attract each other.

Then

$$\mu f(\varrho),\ \mu' f(\varrho)$$

will be their respective actions, and resolving these parallel to the axes of abscissas and ordinates, we have (46)

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mu' f(\varrho)\frac{x'-x}{\varrho} \\ \frac{d^2y}{dt^2} &= \mu' f(\varrho)\frac{y'-y}{\varrho} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

$$\left.\begin{aligned}\frac{d^2x'}{dt^2} &= -\mu f(\rho)\frac{x'-x}{\rho}\\ \frac{d^2y'}{dt^2} &= -\mu f(\rho)\frac{y'-y}{\rho}\end{aligned}\right\} \quad \ldots\ldots\ldots (2)$$

Hence multiplying equations (1) by μ and those marked (2) by μ' and adding, &c. we get

$$\frac{\mu\, d^2x + \mu'\, d^2x'}{dt^2} = 0,$$

and

$$\frac{\mu\, d^2y + \mu'\, d^2y'}{dt^2} = 0$$

and integrating

$$\mu . \frac{dx}{dt} + \mu' . \frac{dx'}{dt} = c$$

$$\mu . \frac{dy}{dt} + \mu' . \frac{dy'}{dt} = c'.$$

Now if the coordinates of the center of gravity be denoted by

$$\bar{x},\ \bar{y},$$

we have by Statics

$$\bar{x} = \frac{\mu x + \mu' x'}{\mu + \mu'}$$

$$\bar{y} = \frac{\mu y + \mu' y'}{\mu + \mu'}$$

$$\therefore \frac{d\bar{x}}{dt} = \frac{1}{\mu + \mu'} \cdot \left(\mu . \frac{dx}{dt} + \mu' \frac{dx'}{dt}\right) = \frac{c}{\mu + \mu'}$$

and

$$\frac{d\bar{y}}{dt} = \frac{1}{\mu + \mu'} \left(\mu \frac{dy}{dt} + \mu' \frac{dy'}{dt}\right) = \frac{c'}{\mu + \mu'}.$$

But

$$\frac{d\bar{x}}{dt},\ \frac{d\bar{y}}{dt}$$

represent the velocity of the center of gravity resolved parallel to the axes of coordinates, and these resolved parts have been shown to be constant. Hence it easily appears by composition of motion, that the actual velocity of the center of gravity is uniform, and also that it moves in a straight line, viz. in that produced which is the diagonal of the rectangular parallelogram whose two sides are $d\bar{x}$, $d\bar{y}$.

If

$$c = 0,\ c' = 0$$

then the center of gravity remains quiescent.

461 The general proposition is similarly demonstrated, thus.

Let the bodies whose masses

$$\mu', \mu'', \mu''', \text{ \&c.}$$

be referred to three rectangular axes, issuing from a fixed point by the coordinates

$$\begin{array}{l} x', \ y', \ z' \\ x'', \ y'', \ z'' \\ x''', \ y''', \ z''' \\ \text{\&c.} \end{array}$$

Also let

$$\begin{array}{l} \varrho_{1,2} \text{ be the distance of } \mu', \ \mu'' \\ \varrho_{1,3} \text{ - - - - - - - - } \mu', \ \mu''' \\ \varrho_{2,3} \text{ - - - - - - - - } \mu'', \ \mu''' \\ \text{\&c. \&c.} \end{array}$$

and suppose the law of attraction to be denoted by

$$f.(\varrho_{1,2}), \ f(\varrho_{1,3}), \ f(\varrho_{2,3}), \text{ \&c.}$$

Now resolving the attractions or forces

$$\begin{array}{l} \mu' \ f(\varrho_{1,2}) \\ \mu'' \ f(\varrho_{1,2}) \\ \mu''' f(\varrho_{1,3}) \\ \text{\&c.} \end{array}$$

parallel to the axes, and collecting the parts we get

$$\begin{aligned} \frac{d^2x'}{dt^2} &= \mu'' f(\varrho_{1,2}) \frac{x' - x''}{\varrho_{1,2}} + \mu''' f(\varrho_{1,3}) \frac{x' - x'''}{\varrho_{1,3}} + \text{\&c.} \\ \frac{d^2x''}{dt^2} &= -\mu' f(\varrho_{1,2}) \frac{x' - x''}{\varrho_{1,2}} + \mu''' f(\varrho_{2,3}) \frac{x'' - x'''}{\varrho_{2,3}} + \text{\&c.} \\ \frac{d^2x'''}{dt^2} &= -\mu' f(\varrho_{1,3}) \frac{x' - x'''}{\varrho_{1,3}} - \mu'' f(\varrho_{2,3}) \frac{x'' - x'''}{\varrho_{2,3}} + \text{\&c.} \\ \text{\&c.} &= \text{\&c.} \end{aligned}$$

Hence multiplying the first of the above equations by μ', the second by μ'', and so on, and adding, we get

$$\frac{\mu' d^2x' + \mu'' d^2x'' + \mu''' d^2x''' + \text{\&c.}}{dt^2} = 0.$$

Again, since it is a matter of perfect indifference whether we collect the forces parallel to the other axes or this; or since all the circumstances are similar with regard to these independent axes, the results arising from similar operations must be similar, and we therefore have also

$$\frac{\mu' d^2y' + \mu'' d^2y'' + \mu''' d^2y''' + \text{\&c.}}{dt^2} = 0,$$

$$\frac{\mu' d^2z' + \mu'' d^2z'' + \mu''' d^2z''' + \text{\&c.}}{dt^2} = 0.$$

Hence by integration

$$\mu' \cdot \frac{dx'}{dt} + \mu'' \frac{dx''}{dt} + \mu''' \frac{dx'''}{dt} + \&c. = c$$
$$\mu' \cdot \frac{dy'}{dt} + \mu'' \frac{dy''}{dt} + \mu''' \frac{dy'''}{dt} + \&c = c'$$
$$\mu' \cdot \frac{dz'}{dt} + \mu'' \frac{dz''}{dt} + \mu''' \frac{dz'''}{dt} + \&c. = c''.$$

But $\bar{x}$, $\bar{y}$, $\bar{z}$ denoting the coordinates of the center of gravity, by statics we have

$$\bar{x} = \frac{\mu' x' + \mu'' x'' + \mu''' x''' + \&c.}{\mu' + \mu'' + \mu''' + \&c.}$$
$$\bar{y} = \frac{\mu' y' + \mu'' y'' + \mu''' y''' + \&c.}{\mu' + \mu'' + \mu''' + \&c.}$$
$$\bar{z} = \frac{\mu' z' + \mu'' z'' + \mu''' z''' + \&c.}{\mu' + \mu'' + \mu''' + \&c.}$$

and hence by taking the differentials, &c. we get

$$\frac{d\bar{x}}{dt} = \frac{c}{\mu' + \mu'' + \mu''' + \&c.}$$
$$\frac{d\bar{y}}{dt} = \frac{c'}{\mu' + \mu'' + \mu''' + \&c.}$$
$$\frac{d\bar{z}}{dt} = \frac{c''}{\mu' + \mu'' + \mu''' + \&c.}$$

that is, the velocity of the center of gravity resolved parallel to any three rectangular axes is constant. Hence by composition of motion the actual velocity of the center of gravity is constant and uniform, and it easily appears also that its path is a straight line, scil. the diagonal of the rectangular parallelopiped whose sides are $d\bar{x}$, $d\bar{y}$, $d\bar{z}$.

462. We will now give another demonstration of Prop. LXI. or that *Of two bodies the motion of each about the center of gravity, is the same as if that center was the center of force, and the law of force the same as that of their mutual attractions.*

Supposing the coordinates of the two bodies referred to the center of gravity to be

$$x_{,}, y_{,}; x_{,,}, y_{,,}$$

we have

$$\left.\begin{aligned} x &= \bar{x} + x_{,} \\ y &= \bar{y} + y_{,} \end{aligned}\right\} \qquad \left.\begin{aligned} x' &= \bar{x} + x_{,,} \\ y' &= \bar{y} + y_{,,} \end{aligned}\right\}$$

Hence since

$$\frac{d\bar{x}}{dt}, \frac{d\bar{y}}{dt}$$

are constant as it has been shown, and therefore

$$\frac{d^2\bar{x}}{dt^2} = 0, \quad \frac{d^2\bar{y}}{dt^2} = 0$$

we have

$$\frac{d^2x}{dt^2} = \frac{d^2x_{,}}{dt^2}$$

$$\frac{d^2y}{dt^2} = \frac{d^2y_{,}}{dt^2}$$

and we therefore get (46)

$$\frac{d^2x_{,,}}{dt^2} = -\mu f(\varrho)\frac{x_{,,} - x_{,}}{\varrho}$$

$$\frac{d^2y_{,,}}{dt^2} = -\mu f(\varrho)\frac{y_{,,} - y_{,}}{\varrho}.$$

But by the property of the center of gravity

$$\varrho = \frac{\mu + \mu'}{\mu}.\delta'$$

δ being the distance of μ' from the center of gravity. We also have

$$\frac{x_{,,} - x_{,}}{\varrho} = \frac{x_{,,}}{\delta'}.$$

Hence by substitution the equations become

$$\frac{d^2x_{,,}}{dt^2} = -\mu f\left(\frac{\mu + \mu'}{\mu}\delta'\right)\frac{x_{,,}}{\delta'},$$

$$\frac{d^2y_{,,}}{dt^2} = -\mu f\left(\frac{\mu + \mu'}{\mu}\delta'\right)\frac{y_{,,}}{\delta'}.$$

Similarly we should find

$$\frac{d^2x_{,}}{dt^2} = \mu' f\left(\frac{\mu + \mu'}{\mu'}\delta\right)\frac{x_{,}}{\delta}$$

and

$$\frac{d^2y_{,}}{dt^2} = \mu' f\left(\frac{\mu + \mu'}{\mu'}\delta\right)\frac{y_{,}}{\delta}.$$

Hence if the force represented by

$$\mu f\left(\frac{\mu + \mu'}{\mu}\delta'\right)$$

were placed in the center of gravity, it would cause μ' to move about it as a fixed point; and if

$$\mu' f\left(\frac{\mu + \mu'}{\mu'}\delta\right)$$

were there residing, it would cause μ to centripetate in like manner.

Moreover if

$$f(\varrho) = \varrho^n$$

then these forces vary as

$$\delta'^{n},\ \delta^{n};$$

so that the law of force &c. &c.

ANOTHER PROOF OF PROP. LXII.

463. Let μ, μ' denote the two bodies. Then since μ has no motion round G (G being the center of gravity), it will descend in a straight line to G. In like manner μ' will fall to G in a straight line.

Also since the accelerating forces on μ, μ' are inversely as μ, μ' or directly as G μ, G μ', the velocities will follow the same law and corresponding portions of G μ, G μ' will be described in the same times; that is, the whole will be described in the same time. Moreover after they meet at G, the bodies will go on together with the same constant velocity with which G moved before they met.

Since here

$$f(\varrho) = \frac{1}{\varrho^2}$$

u will move towards G as if a force

$$\mu' \left(\frac{\mu + \mu'}{\mu'}\,\delta\right)^{-2}$$

or

$$\frac{\mu'^3}{(\mu + \mu')^2} \times \frac{1}{\delta^2}.$$

Hence by the usual methods it will be found that if a be the distance at which μ begins to fall, the time to G is

$$\frac{(\mu + \mu')\, a^{\frac{5}{2}}}{\mu'^{\frac{5}{2}}} \cdot \frac{\pi}{2\sqrt{2}}$$

and if a′ be the original distance of μ', the time is

$$\frac{(\mu + \mu')\, a'^{\frac{5}{2}}}{\mu^{\frac{3}{2}}} \cdot \frac{\pi}{2\sqrt{2}}.$$

But

$$a : a' :: \mu' : \mu$$

therefore these times are equal, which has just been otherwise shown.

ANOTHER PROOF OF PROP. LXIII.

464. We know from (461) that the center of gravity moves uniformly in a straight line; and that (Prop. LVII,) μ and μ' will describe about G similar figures, μ moving as though actuated by the force

$$\frac{\mu'^3}{(\mu+\mu')^2}\cdot\frac{1}{\delta^2}$$

and Q as if by

$$\frac{\mu^3}{(\mu+\mu')^2}\cdot\frac{1}{\delta'^2}.$$

Hence the curves described will be similar ellipses, with the center of force G in the focus. Also if we knew the original velocities of μ and μ' about G, the ellipse would easily be determined.

The velocities of μ and μ' at any time are composed of two velocities, viz. the progressive one of the center of gravity and that of each round G. *Hence having given the whole original velocities required to find the separate parts of them,*

is a problem which we will now resolve.

Let

$$V,\ V'$$

be the original velocities of μ, μ', and suppose their directions to make with the straight line $\mu\,\mu'$ the angles

$$\alpha,\ \alpha'.$$

Also let the velocity of the center of gravity be

$$v$$

and the direction of its motion to make with $\mu\,\mu'$ the angle

$$\bar{\alpha}.$$

Moreover let

$$v,\ v'$$

be the velocities of μ, μ' around G and the common inclination of their directions to be

$$\theta.$$

Now V resolved parallel to $\mu\,\mu'$ is

$$V\cos.\,\alpha.$$

But since it is composed of $\bar{v}$ and of v it will also be

$$\bar{v}\cos.\,\bar{\alpha}+v\cos.\,\theta$$

$$\therefore V\cos.\,\alpha=\bar{v}\cos.\,\bar{\alpha}+v\cos.\,\theta.$$

In like manner we get

$$V\sin.\,\alpha=\bar{v}\sin.\,\bar{\alpha}+v\sin.\,\theta.$$

and also

$$V' \cos. \alpha' = \bar{v} \cos. \bar{\alpha} - v' \cos. \theta$$
$$V \sin. \alpha' = \bar{v} \sin. \bar{\alpha} - v' \sin. \theta.$$

Hence multiplying by μ, μ', adding and putting

$$\mu v = \mu' v'$$

we get

$$\left.\begin{aligned} &\mu V \cos. \alpha + \mu' V' \cos. \alpha' = (\mu + \mu') \bar{v} \cos. \bar{\alpha} \\ \text{and} \\ &\mu V \sin. \alpha + \mu' V' \sin. \alpha' = (\mu + \mu') \bar{v} \sin. \bar{\alpha} \end{aligned}\right\}$$

Squaring these and adding them, we get

$$\mu^2 V^2 + \mu'^2 V'^2 + 2 \mu \mu' V V' \cos. (\alpha - \alpha') = (\mu + \mu')^2 \bar{v}^2$$

which gives

$$\bar{v} = \frac{\sqrt{\{\mu^2 V^2 + \mu'^2 V'^2 + 2 \mu \mu' V V' \cos. (\alpha - \alpha')\}}}{\mu + \mu'}.$$

By division we also have

$$\tan. \bar{\alpha} = \frac{\mu V \sin. \alpha + \mu' V' \sin. \alpha'}{\mu V \cos. \alpha + \mu' V' \cos. \alpha'}.$$

Again, from the first four equations by subtraction we also have

$$V \cos. \alpha - V' \cos. \alpha' = (v + v') \cos. \theta = v . \frac{\mu + \mu'}{\mu'} \cos. \theta$$

$$V \sin. \alpha - V' \sin. \alpha' = (v + v') \sin. \theta = v . \frac{\mu + \mu'}{\mu'} \sin. \theta$$

and adding the squares of these

$$V^2 + V'^2 - 2 V V' \cos. (\alpha - \alpha') = v^2 \left(\frac{\mu + \mu'}{\mu'}\right)^2$$

whence

$$v = \frac{\mu'}{\mu + \mu'} . \sqrt{\{V^2 + V'^2 - 2 V V' \cos. (\alpha - \alpha')\}}$$

$$v' = \frac{\mu}{\mu + \mu'} \sqrt{\{V^2 + V'^2 - 2 V V' \cos. (\alpha - \alpha')\}}$$

and by division

$$\tan. \theta = \frac{V \sin. \alpha - V' \sin. \alpha'}{V \cos. \alpha - V' \cos. \alpha'}.$$

Whence are known the velocity and direction of projection of μ about G and (by Sect. III. or Com.) the conic section can therefore be found; and combining the motion in this orbit with that of the center of gravity, which is given above, we have also that of μ.

465. Hence since the orbit of μ round μ' is similar to the orbit of μ round G, if A be the semi-axis of the ellipse which μ describes round

G, and a that of the ellipse which it describes relatively to μ' which is also in motion; we shall have

$$A : a :: \mu' : \mu + \mu'.$$

466. Hence also since an ellipse whose semi-axis is A, is described by the force

$$\frac{\mu'^3}{(\mu + \mu')^2} \times \frac{1}{\delta^2}$$

we shall have (309) the periodic time, viz.

$$T = \frac{2 A^{\frac{3}{2}} \pi}{\sqrt{\frac{\mu'^3}{(\mu + \mu')^2}}} = \frac{2 \pi A^{\frac{3}{2}} (\mu + \mu')}{\mu'^{\frac{3}{2}}}$$

$$= \frac{2 \pi a^{\frac{3}{2}}}{\sqrt{(\mu + \mu')}}.$$

467. Hence we easily get Prop. LIX.

For if μ were to revolve round μ' at rest, its semi-axis would be a, and periodic time

$$T' = \frac{2 \pi a^{\frac{3}{2}}}{\sqrt{\mu'}}$$

$$\therefore T : T' :: \sqrt{\mu'} : \sqrt{(\mu + \mu')}.$$

468. Prop. LX is also hence deducible. For if μ revolve round μ' at rest, in an ellipse whose semi-axis is a', we have

$$T'' = \frac{2 \pi a'^{\frac{3}{2}}}{\sqrt{\mu'}}$$

and equating this with T in order to give it the same time about μ' at rest as about μ' in motion, we have

$$\frac{2 \pi a'^{\frac{3}{2}}}{\sqrt{\mu'}} = \frac{2 \pi a^{\frac{3}{2}}}{\sqrt{(\mu + \mu')}};$$

$$\therefore a : a' :: (\mu + \mu')^{\frac{1}{3}} : \mu'^{\frac{1}{3}}.$$

ANOTHER PROOF OF PROP. LXIV.

469. *Required the motions of the bodies whose masses are*

$$\mu, \mu', \mu'', \mu''', \&c.$$

and which mutually attract each other with forces varying directly as the distance.

Let the distance of any two of them as μ, μ', be ϱ; then the force of μ' on μ is

$$\mu' \varrho$$

and the part resolved parallel to x is

$$\mu' \varrho \cdot \frac{x - x'}{\varrho} = \mu' (x - x').$$

In like manner the force of μ'' on μ, resolved parallel to x, is

$$\mu'' (x - x'')$$

and so on for the rest of the bodies and for their respective forces resolved parallel to the other axes of coordinates.

Hence

$$\frac{d^2 x}{dt^2} = \mu' (x - x') + \mu'' (x - x'') + \&c.$$

$$\frac{d^2 x'}{dt^2} = \mu (x' - x) + \mu'' (x' - x'') + \&c.$$

$$\frac{d^2 x''}{dt^2} = \mu (x'' - x) + \mu' (x'' - x') + \&c.$$

$$\&c. = \&c.$$

which give

$$\frac{d^2 x}{dt^2} = (\mu + \mu' + \mu'' + \&c.)\, x - (\mu x + \mu' x' + \&c.)$$

$$\frac{d^2 x'}{dt^2} = (\mu + \mu' + \mu'' + \&c.)\, x' - (\mu x + \mu' x' + \&c.)$$

$$\frac{d^2 x''}{dt^2} = (\mu + \mu' + \mu'' + \&c.)\, x'' - (\mu x + \mu' x' + \&c.)$$

$$\&c. = \&c.$$

Or since

$$\mu x + \mu' x' + \&c. = (\mu + \mu' + \&c.)\, \bar{x}$$

making the coordinates of the center of gravity

$$\bar{x}, \bar{y}, \bar{z},$$

we have

$$\frac{d^2 x}{dt^2} = (\mu + \mu' + \&c.)\, (x - \bar{x})$$

$$\frac{d^2 x'}{dt^2} = (\mu + \mu' + \&c.)\, (x' - \bar{x})$$

$$\frac{d^2 x''}{dt^2} = (\mu + \mu' + \&c.)\, (x'' - \bar{x}).$$

$$\&c. = \&c.$$

In like manner, we easily get

$$\frac{d^2 y}{dt^2} = (\mu + \mu' + \&c.)\, (y - \bar{y})$$

$$\frac{d^2 y'}{dt^2} = (\mu + \mu' + \&c.)\, (y' - y)$$

$$\frac{d^2 y''}{dt^2} = (\mu + \mu' + \&c.)(y'' - \bar{y})$$
$$\&c. = \&c.$$

and also

$$\frac{d^2 z}{dt^2} = (\mu + \mu' + \&c.)(z - \bar{z})$$
$$\frac{d^2 z'}{dt^2} = (\mu + \mu' + \&c.)(z' - \bar{z})$$
$$\frac{d^2 z''}{dt^2} = (\mu + \mu' + \&c.)(z'' - \bar{z})$$
$$\&c. = \&c.$$

Again,

$$x - \bar{x}, \; y - \bar{y}, \; z - \bar{z}$$
$$x' - \bar{x}, \; y' - \bar{y}, \; z' - \bar{z}$$
$$\&c. \quad \&c. \quad \&c.$$

are the coordinates of μ, μ', μ'', &c. when measured from the center of gravity, and it has been shown already that

$$\frac{d^2(x - \bar{x})}{dt^2} = \frac{d^2 x}{dt^2}$$
$$\frac{d^2(y - \bar{y})}{dt^2} = \frac{d^2 y}{dt^2}$$
$$\frac{d^2(z - \bar{z})}{dt^2} = \frac{d^2 z}{dt^2}$$

and so on for the other bodies. Hence then it appears, that the motions of the bodies about the center of gravity, are the same as if there were but one force, scil.

$$(\mu + \mu' + \&c.) \times \text{distance}$$

and as if this force were placed in the center of gravity.

Hence the bodies will all describe ellipses about the center of gravity, as a center; and their periodic times will all be the same. But their magnitudes, excentricities, the positions of the planes of their orbits, and of the major axes, may be of all varieties.

Moreover the motion of any one body relative to any other, will be governed by the same laws as the motion of a body relative to a center of force, which force varies directly as the distance; for if we take the equations

$$\frac{d^2 x}{dt^2} = (\mu + \mu' + \&c.)(x - \bar{x})$$
$$\frac{d^2 x'}{dt^2} = (\mu + \mu' + \&c.)(x' - \bar{x})$$

and subtract them we get

$$\frac{d^2(x-x')}{dt^2} = (\mu + \mu' + \&c.)(x-x')$$

and similarly

$$\frac{d^2(y-y')}{dt^2} = (\mu + \mu' + \&c.)(y-y')$$

and

$$\frac{d^2(z-z')}{dt^2} = (\mu + \mu' + \&c.)(z-z').$$

Hence by composition and the general expression for force $\left(\frac{d^2\varrho}{dt^2}\right)$ it readily appears that the motion of μ about μ', is such as was asserted.

470. Thus far relates merely to the motions of two bodies; and these can be accurately determined. But the operations of Nature are on a grander scale, and she presents us with Systems composed of Three, and even more bodies, mutually attracting each other. In these cases the equations of motion cannot be integrated by any methods hitherto discovered, and we must therefore have recourse to methods of approximation.

In this portion of our labours we shall endeavour to lay before the reader such an exposition of the Lunar, Planetary and Cometary Theories, as may afford him a complete succedaneum to the discoveries of our author.

471. Since relative motions are such only as can be observed, we refer the motions of the Planets and Comets, to the center of the sun, and the motions of the Satellites to the center of their planets. Thus to compare theory with observations,

It is required to determine the relative motion of a system of bodies, about a body considered as the center of their motions.

Let M be this last body, μ, μ', μ'', &c. being the other bodies of which is required the relative motion about M. Also let

$$\zeta, \Pi, \gamma$$

be the rectangular coordinates of M;

$$\zeta + x, \Pi + y, \gamma + z;$$
$$\zeta + x' \Pi + y', \gamma + z';$$
$$\&c.$$

those of μ, μ', &c. Then it is evident that

$$x, y, z;$$
$$x', y', z'$$
$$\&c.$$

will be the coordinates of μ, μ', &c. referred to M.

Call

$$\varrho,\ \varrho',\ \&c.$$

the distances of μ, μ', &c. from M; then we have

$$\varrho = \sqrt{(x^2 + y^2 + z^2)}$$
$$\varrho' = \sqrt{(x'^2 + y'^2 + z'^2)}$$

ϱ, ϱ', &c. being the diagonals of rectangular parallelopipeds, whose sides are

$$x,\ y,\ z$$
$$x',\ y',\ z'$$
$$\&c.$$

Now the actions of μ, μ', μ'', &c. upon M are

$$\frac{\mu}{\varrho^2},\ \frac{\mu'}{\varrho'^2},\ \frac{\mu''}{\varrho''^2},\ \&c.$$

and these resolved parallel to the axis of x, are

$$\frac{\mu x}{\varrho^3},\ \frac{\mu' x'}{\varrho'^3},\ \frac{\mu'' x''}{\varrho''^3},\ \&c.$$

Therefore to determine ζ, we have

$$\frac{d^2\zeta}{dt^2} = \frac{\mu x}{\varrho^3} + \frac{\mu' x'}{\varrho'^3} + \frac{\mu'' x''}{\varrho''^3} + \&c.$$
$$= \Sigma . \frac{\mu x}{\varrho^3}$$

the symbol Σ denoting the sum of such expressions.

In like manner to determine Π, γ we have

$$\frac{d^2\Pi}{dt^2} = \Sigma . \frac{\mu y}{\varrho^3};$$
$$\frac{d^2\gamma}{dt^2} = \Sigma . \frac{\mu z}{\varrho^3}.$$

The action of M upon μ, resolved parallel to the axis of x, and in the contrary direction, is

$$-\frac{M x}{\varrho^3}.$$

Also the actions of μ', μ'', &c. upon μ resolved parallel to the axis of x are, in like manner,

$$\frac{\mu'(x' - x)}{\varrho^3_{0,1}},\ \frac{\mu''(x'' - x)}{\varrho^3_{0,2}},\ \frac{\mu'''(x''' - x)}{\varrho^3_{0,3}},\ \&c.$$

$\varrho_{n,m}$ generally denoting the distance between $\mu''' \ldots^n$ and $\mu''' \ldots^m$

But

$$\varrho_{0,1} = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}$$
$$\varrho_{0,2} = \sqrt{(x'' - x)^2 + (y'' - y)^2 + (z'' - z)^2}$$
$$\&c. = \&c.$$

$$\varrho_{1,2} = \sqrt{(x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2}$$

and so on.

Hence if we assume

$$\lambda = \frac{\mu \cdot \mu'}{\varrho_{0,1}} + \frac{\mu\,\mu''}{\varrho_{0,2}} + \&c.$$
$$+ \frac{\mu'\,\mu''}{\varrho_{1,2}} + \frac{\mu'\,\mu'''}{\varrho_{1,3}}$$
$$+ \frac{\mu''\,\mu'''}{\varrho_{2,3}} + \&c.$$
$$\&c.$$

and taking the Partial Difference upon the supposition that x is the only variable, we have

$$\frac{1}{\mu} \cdot \left(\frac{d\,\lambda}{d\,x}\right) = \frac{\mu' \times (x' - x)}{\varrho^3_{0,1}} + \frac{\mu''(x'' - x)}{\varrho^3_{0,2}} + \&c.$$

the parenthesis ( ) denoting the Partial Difference. Hence the sum of all the actions of μ', μ'', &c. on μ is

$$\frac{1}{\mu} \cdot \left(\frac{d\,\lambda}{d\,x}\right).$$

Hence then the whole action upon μ parallel to x is

$$\frac{d.^2(\zeta + x)}{d\,t^2} = \frac{1}{\mu} \cdot \left(\frac{d\,\lambda}{d\,x}\right) - \frac{M\,x}{\varrho^3};$$

But

$$\frac{d^2\,\xi}{d\,t^2} = \Sigma \frac{\mu\,x}{\varrho^3}$$

$$\therefore \frac{d^2\,x}{d\,t^2} = \frac{1}{\mu}\left(\frac{d\,\lambda}{d\,x}\right) - \frac{M\,x}{\varrho^3} - \Sigma.\frac{\mu\,x}{\varrho^3} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, we have

$$\frac{d^2\,y}{d\,t^2} = \frac{1}{\mu}\left(\frac{d\,\lambda}{d\,y}\right) - \frac{M\,y}{\varrho^3} - \Sigma.\frac{\mu\,y}{\varrho^3} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{d^2\,z}{d\,t^2} = \frac{1}{\mu}\left(\frac{d\,\lambda}{d\,z}\right) - \frac{M\,z}{\varrho^3} - \Sigma.\frac{\mu\,z}{\varrho^3} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we change successively in the equations (1), (2), (3) the quantities μ, x, y, z into

μ', x', y', z';
μ'', x'', y'', z'';
&c.

and reciprocally; we shall have all the equations of motion of the bodies μ', μ'', &c. round M.

If we multiply the equations involving ζ by $M + \Sigma.\mu$; that in x, by μ: that in x′, by μ', and so on; and add them together, we shall have

$$(M + \Sigma.\mu)\frac{d^2\zeta}{dt^2} = \left(\frac{d\lambda}{dx}\right) + \left(\frac{d\lambda}{dx'}\right) + \left(\frac{d\lambda}{dx''}\right) + \&c. - \Sigma.\mu\frac{d^2x}{dt^2}$$

But since

$$\left(\frac{d\lambda}{dx}\right) = \frac{\mu\mu'(x' - x)}{\rho^3_{0,1}} + \&c.$$

$$\left(\frac{d\lambda}{dx'}\right) = -\frac{\mu\mu'(x' - x)}{\rho^3_{0,1}} + \&c.$$

and so on in pairs, it will easily appear that

$$\left(\frac{d\lambda}{dx}\right) + \left(\frac{d\lambda}{dx'}\right) + \&c. = 0.$$

$$\therefore (M + \Sigma.\mu)\frac{d^2\zeta}{dt^2} = -\Sigma.\mu\frac{d^2x}{dt^2};$$

whence by integrating we get

$$(M + \Sigma.\mu)\frac{d\zeta}{dt} = c - \Sigma.\mu\frac{dx}{dt},$$

$$\therefore d\zeta = \frac{c}{M + \Sigma.\mu}.dt - \frac{\Sigma.\mu\,dx}{M + \Sigma.\mu}$$

and again integrating

$$\zeta = a + bt - \frac{\Sigma.\mu x}{M + \Sigma.\mu},$$

a and b being arbitrary constants.

Similarly, it is found that

$$\Pi = a' + b't - \frac{\Sigma.\mu y}{M + \Sigma.\mu}$$

$$\gamma = a'' + b''t - \frac{\Sigma.\mu z}{M + \Sigma.\mu}$$

These three equations, therefore, give the *absolute* motion of M in space, when the *relative* motions around it of μ, μ', μ'', &c. are known.

Again, if we multiply the equations in x and y by

$$-\mu y + \mu.\frac{\Sigma.\mu y}{M + \Sigma.\mu},$$

and

$$\mu x - \mu.\frac{\Sigma.\mu x}{M + \Sigma.\mu};$$

in like manner the equations in x′ and y′ by

$$-\mu' y' + \mu'.\frac{\Sigma.\mu y}{M + \Sigma.\mu},$$

and

$$\mu' x' - \mu' \cdot \frac{\Sigma . \mu x}{M + \Sigma . \mu};$$

and so on.

And if we add all these results together, observing that from the nature of λ, (which is easily shown)

$$\Sigma . x . \left(\frac{d\lambda}{dy}\right) = \Sigma . y \left(\frac{d\lambda}{dx}\right)$$

and that (as we already know)

$$\Sigma . \left(\frac{d\lambda}{dx}\right) = 0, \Sigma . \left(\frac{d\lambda}{dy}\right) = 0,$$

we have

$$\Sigma . \mu . \frac{x\, d^2 y - y\, d^2 x}{dt^2} = \frac{\Sigma . \mu x}{M + \Sigma \mu} . \Sigma . \mu . \frac{d^2 y}{dt^2} - \frac{\Sigma . \mu y}{M + \Sigma \mu} . \Sigma . \mu . \frac{d^2 x}{dt^2}$$

and integrating, since

$$\begin{aligned} \int (x\, d^2 y - y\, d^2 x) &= \int x\, d^2 y - \int y\, d^2 x \\ &= x\, dy - \int dx\, dy - (y\, dx - \int dx\, dy) \\ &= x\, dy - y\, dx, \end{aligned}$$

we have

$$\Sigma . \mu . \frac{x\, dy - y\, dx}{dt} = \text{const.} + \frac{\Sigma . \mu x}{M + \Sigma . \mu} . \Sigma . \mu . \frac{dy}{dt} - \frac{\Sigma . \mu y}{M + \Sigma . \mu} . \Sigma . \mu . \frac{dx}{dt}$$

Hence

$$\begin{aligned} c &= M . \Sigma . \mu . \frac{x\, dy - y\, dx}{dt} + \Sigma . \mu x \times \Sigma \mu . \frac{x\, dy - y\, dx}{dt} + \Sigma . \mu y \times \Sigma . \mu \frac{dx}{dt} \\ &\qquad - \Sigma . \mu x \times \Sigma . \mu \frac{dy}{dt} \\ &= M . \Sigma . \mu . \frac{x\, dy - y\, dx}{dt} + \Sigma . \mu \mu' . \left\{ \frac{(x' - x)(dy' - dy) - (y' - y)(dx' - dx)}{dt} \right\} \quad . . (4) \end{aligned}$$

c being an arbitrary constant.

In the same manner we arrive at these two integrals,

$$c' = M . \Sigma . \mu . \frac{x\, dz - z\, dx}{dt} + \Sigma . \mu \mu' \left\{ \frac{(x' - x)(dz' - dz) - (z' - z)(dx' - dx)}{dt} \right\} \quad . . (5)$$

$$c'' = M . \Sigma . \mu . \frac{y\, dz - z\, dy}{dt} + \Sigma . \mu \mu' \left\{ \frac{(y' - y)(dz' - dz) - (z' - z)(dy' - dy)}{dt} \right\} \quad . . (6)$$

c' and c'' being two other arbitrary constants.

Again, if we multiply the equation in x by

$$2\mu\,dx - 2\mu\,\frac{\Sigma.\mu\,dx}{M+\Sigma.\mu};$$

the equation in y by

$$2\mu\,dy - 2\mu.\,\frac{\Sigma.\mu\,dy}{M+\Sigma.\mu};$$

the equation in z by

$$2\mu\,dz - 2\mu.\,\frac{\Sigma.\mu\,dz}{M+\Sigma.\mu};$$

if in like manner we multiply the equations in x′, y′, z′ by

$$2\mu'\,dx' - 2\mu'.\,\frac{\Sigma.\mu\,dx}{M+\Sigma.\mu}$$

$$2\mu'\,dy' - 2\mu'.\,\frac{\Sigma.\mu\,dy}{M+\Sigma.\mu}$$

$$2\mu'\,dz' - 2\mu'.\,\frac{\Sigma.\mu\,dz}{M+\Sigma.\mu};$$

respectively, and so on for the rest; and add the several results, observing that

$$\Sigma.\left(\frac{d\lambda}{dx}\right) = 0;$$

$$\Sigma.\left(\frac{d\lambda}{dy}\right) = 0;$$

$$\Sigma.\left(\frac{d\lambda}{dz}\right) = 0;$$

we get

$$2.\Sigma.\mu\frac{dx\,d^2x+dy\,d^2y+dz\,d^2z}{dt^2} = \frac{2\Sigma.\mu\,dx}{M+\Sigma\mu}.\Sigma.\frac{\mu\,d^2x}{dt^2}$$

$$+\frac{2\Sigma.\mu\,dy}{M+\Sigma\mu}.\Sigma.\frac{\mu\,d^2y}{dt^2}+\frac{2\Sigma.\mu\,dz}{M+\Sigma\mu}.\Sigma.\frac{\mu\,d^2z}{dt^2}-$$

$$2M.\Sigma.\frac{\mu\,d\varrho}{\varrho^2}+2\,d\lambda;$$

and integrating, we have

$$\Sigma.\mu\frac{dx^2+dy^2+dz^2}{dt^2} = \text{const.}+\frac{(\Sigma.\mu\,dx)^2+(\Sigma.\mu\,dy)^2+(\Sigma.\mu\,dz)^2}{(M+\Sigma\mu)\,dt^2}$$

$$+2M\Sigma\frac{\mu}{\varrho}+2\lambda,$$

which gives

$$h = M.\ \Sigma.\mu\frac{dx^2+dy^2+dz^2}{dt^2}+\Sigma.\mu\mu'.\left\{\frac{(dx'-dx)^2+(dy'-dy)^2+(dz'-dz)^2}{dt^2}\right\}$$

$$-\left\{2M\ \Sigma.\frac{\mu}{\varrho}+2\lambda\right\}(M+\Sigma\mu)\quad\ldots\ldots\ldots\quad(7)$$

h being an arbitrary constant.

These integrals being the only ones attainable by the present state of analysis, we are obliged to have recourse to Methods of Approximation, and for this object to take advantage of the facilities afforded us by the constitution of the system of the World. One of the principal of these is due to the fact, that the Solar System is composed of Partial Systems, formed by the Planets and their Satellites: which systems are such, that the distances of the Satellites from their Planet, are small in comparison with the distance of the Planet from the Sun: whence it results, that the action of the Sun being nearly the same upon the Planet as upon its Satellites, these latter move nearly the same as if they obeyed no other action than that of the Planet. Hence we have this remarkable property, namely,

472. *The motion of the Center of Gravity of a Planet and its Satellites, is very nearly the same as if all the bodies formed one in that Center.*

Let the mutual distances of the bodies μ, μ', μ'', &c. be very small compared with that of their center of gravity from the body M. Let also

$$x = \bar{x} + x_{,}; \; y = \bar{y} + y_{,}; \; z = \bar{z} + z_{,}.$$
$$x' = \bar{x} + x_{,}'; \; y' = y + y_{,}'; \; z' = \bar{z} + z_{,}';$$
&c.

$\bar{x}$, $\bar{y}$, $\bar{z}$ being the coordinates of the center of gravity of the system of bodies μ, μ', μ'', &c.; the origin of these and of the coordinates x, y, z; x', y', z', &c. being at the center of M. It is evident that $x_{,}$, $y_{,}$, $z_{,}$; $x_{,}'$, $y_{,}'$, $z_{,}'$, &c. are the coordinates of μ, μ', &c. relatively to their center of gravity; we will suppose these, compared with $\bar{x}$, $\bar{y}$, $\bar{z}$, as small quantities of the first order. This being done, we shall have, as we know by Mechanics, the force which sollicits the center of gravity of the system parallel to any straight line, by taking the sum of the forces which act upon the bodies parallel to the given straight line, multiplied respectively by their masses, and by dividing this sum by the sum of the masses. We also know (by Mech.) that the mutual action of the bodies upon one another, does not alter the motion of the center of gravity of the system; nor does their mutual attraction. It is sufficient, therefore, in estimating the forces which animate the center of gravity of a system, merely to regard the action of the body M which forms no part of the system.

The action of M upon μ, resolved parallel to the axis of x is

$$-\frac{M x}{\rho^3};$$

the whole force which sollicits the center of gravity parallel to this straight line is, therefore,

$$-\frac{M \cdot \Sigma \cdot \frac{\mu x}{\varrho^3}}{\Sigma \mu}.$$

Substituting for x and ϱ their values

$$\frac{x}{\varrho^3} = \frac{\bar{x} + x_{,}}{\{(\bar{x} + x_{,})^2 + (\bar{y} + y_{,})^2 + (\bar{z} + z_{,})^2\}^{\frac{3}{2}}}.$$

If we neglect small quantities of the second order, scil. the squares and products of

$$x_{,}, y_{,}, z_{,}; x_{,}', y_{,}', z_{,}'; \text{ \&c.}$$

and put

$$\bar{\varrho} = \sqrt{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)}$$

the distance of the center of gravity from M, we have

$$\frac{x}{\varrho^3} = \frac{\bar{x}}{\bar{\varrho}^3} + \frac{x_{,}}{\bar{\varrho}^3} - \frac{3x(x x_{,} + \bar{y} y_{,} + z z_{,})}{\bar{\varrho}^3}$$

for omitting x^2, y^2 &c., we have

$$\begin{aligned}\frac{x}{\varrho^3} &= (\bar{x} + x_{,}) \times \{(\bar{\varrho})^2 + 2(\bar{x} x_{,} + \bar{y} y_{,} + \bar{z} z_{,})\}^{-\frac{3}{2}} \text{ nearly} \\ &= (\bar{x} + x_{,}) \times \{(\bar{\varrho})^{-3} - 3(\bar{\varrho})^{-5}(\bar{x} x_{,} + y y_{,} + \bar{z} z_{,}\} \text{ nearly} \\ &= \frac{\bar{x} + x_{,}}{(\bar{\varrho})^3} - \frac{3\bar{x}}{(\bar{\varrho})^5} \cdot (\bar{x} x_{,} + \bar{y} y_{,} + \bar{z} z_{,}) \text{ nearly.}\end{aligned}$$

Again, marking successively the letters $x_{,}, y_{,}, z_{,}$ with one, two, three, &c. dashes or accents, we shall have the values of

$$\frac{x'}{\varrho'^3}, \frac{x''}{\varrho''^3}, \text{ \&c.}$$

But from the nature of the center of gravity

$$\Sigma \cdot \mu x = 0, \quad \Sigma \cdot \mu y = 0, \quad \Sigma \cdot \mu z = 0$$

we shall therefore have

$$-\frac{M \cdot \Sigma \cdot \frac{\mu x}{\varrho^3}}{\Sigma \mu} = -\frac{M\bar{x}}{\bar{\varrho}^3} \text{ nearly.}$$

Thus the center of gravity of the system is sollicited parallel to the axis of x, by the action of the body M, very nearly as if all the bodies of the system were collected into one at the center. The same result evidently takes place relatively to the axes of y and z; so that the forces, by

which the center of gravity of the system is animated parallel to these axes, by the action of M, are respectively

$$-\frac{M\bar{y}}{(\bar{\varrho})^3} \text{ and } -\frac{M\bar{z}}{(\bar{\varrho})^3}.$$

When we consider the relative motion of the center of gravity of the system about M, the direction of the force which sollicits M must be changed. This force resulting from the action of μ, μ', &c. upon M, and resolved parallel to x, in the contrary direction from the origin, is

$$\Sigma.\frac{\mu x}{\varrho^3};$$

if we neglect small quantities of the second order, this function becomes, after what has been shown, equal to

$$\frac{\bar{x}\,\Sigma.\mu}{\bar{\varrho}^3}.$$

In like manner, the forces by which M is actuated arising from the system, parallel to the axes of y, and of z, in the contrary direction, are

$$\frac{\bar{y}\,\Sigma.\mu}{(\bar{\varrho})^3}, \text{ and } \frac{\bar{z}\,\Sigma.\mu}{(\bar{\varrho})^3}.$$

It is thus perceptible, that the action of the system upon the body M, is very nearly the same as if all the bodies were collected at their common center of gravity. Transferring to this center, and with a contrary sign, the three preceding forces; this point will be sollicited parallel to the axes of x, y and z, in its relative motion about M, by the three following forces, scil.

$$-(M+\Sigma\mu)\frac{\bar{x}}{(\bar{\varrho})^3}, -(M+\Sigma\mu)\frac{\bar{y}}{(\bar{\varrho})^3}, -(M+\Sigma\mu)\frac{\bar{z}}{(\bar{\varrho})^3}.$$

These forces are the same as if all the bodies μ, μ', μ'', &c. were collected at their common center of gravity; *which center, therefore, moves nearly (to small quantities of the second order) as if all the bodies were collected at that center.*

Hence it follows, that if there are many systems, whose centers of gravity are very distant from each other, relatively to the respective distances of the bodies of each system; these centers will be moved very nearly, as if the bodies of each system were there collected; for the action of the first system upon each body of the second system, is the same very nearly as if the bodies of the first system were collected at their common center of gravity; the action of the first system upon the center of gravity of the second, will be therefore, by what has preceded, the same as on this hypothesis; whence we may conclude generally *that the reciprocal action of*

different systems upon their respective centers of gravity, is the same as if all the bodies of each system were there collected, and also that these centers move as on that supposition.

It is clear that this result subsists equally, whether the bodies of each system be free, or connected together in any way whatever; for their mutual action has no influence upon the motion of their common center of gravity.

The system of a planet acts, therefore, upon the other bodies of the Solar system, very nearly the same as if the Planet and its Satellites, were collected at their common center of gravity; and this center itself is attracted by the different bodies of the Solar system, as it would be on that hypothesis.

Having given the equations of motion of a system of bodies submitted to their mutual attraction, it remains to integrate them by successive approximations. In the solar system, the celestial bodies move *nearly* as if they obeyed only the principal force which actuates them, and the perturbing forces are inconsiderable; we may, therefore, in a first approximation consider only the mutual action of two bodies, scil. that of a planet or of a comet and of the sun, in the theory of planets and comets; and the mutual action of a satellite and of its planet, in the theory of satellites. We shall begin by giving a rigorous determination of the motion of two attracting bodies: this first approximation will conduct us to a second in which we shall include the first powers of small quantities or the perturbing forces; next we shall consider the squares and products of these forces; and continuing the process, we shall determine the motions of the heavenly bodies with all the accuracy that observations will admit of.

FIRST APPROXIMATION.

473. We know already that a body attracted towards a fixed point, by a force varying reciprocally as the square of the distance, describes a conic section; or in the relative motion of the body μ, round M, this latter body being considered as fixed, we must transfer in a direction contrary to that of μ, the action of μ upon M; so that in this relative motion, μ is sollicited towards M, by a force equal to the sum of the masses M, and μ divided by the square of their distance. All this has been ascertained already. But the importance of the subject in the Theory of the system of the world, will be a sufficient excuse for representing it under another form.

First transform the variables x, y, z into others more commodious for astronomical purposes. ϱ being the distance of the centers of μ and M, call (v) the angle which the projection of ϱ upon the plane of x, y makes with the axis of x; and (θ) the inclination of ϱ to the same plane; we shall have

$$\left.\begin{aligned} x &= \varrho \cos.\,\theta \cos.\,v;\\ y &= \varrho \cos.\,\theta \sin.\,v;\\ z &= \varrho \sin.\,\theta. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

Next putting

$$Q = \frac{M+\mu}{\varrho} - \Sigma.\frac{\mu'(x x' + y y' + z z')}{\varrho'^3} + \frac{\lambda}{\mu}$$

we have

$$\left(\frac{d\,Q}{d\,x}\right) = \frac{1}{\mu}\left(\frac{d\,\lambda}{d\,x}\right) - \frac{M+\mu}{\varrho^3} - \Sigma.\frac{\mu' x'}{\varrho'^3}$$

$$= \frac{1}{\mu}\left(\frac{d\,\lambda}{d\,x}\right) - \frac{M}{\varrho^3} - \Sigma.\frac{\mu x}{\varrho^3}.$$

Similarly

$$\left(\frac{d\,Q}{d\,y}\right) = \frac{1}{\mu}\left(\frac{d\,\lambda}{d\,y}\right) - \frac{M}{\varrho^3} - \Sigma.\frac{\mu y}{\varrho^3}$$

$$\left(\frac{d\,Q}{d\,z}\right) = \frac{1}{\mu}\left(\frac{d\,\lambda}{d\,z}\right) - \frac{M}{\varrho^3} - \Sigma.\frac{\mu z}{\varrho^3}.$$

Hence equations (1), (2), (3) of number 471, become

$$\frac{d^2 x}{d\,t^2} = \left(\frac{d\,Q}{d\,x}\right);\ \frac{d^2 y}{d\,t^2} = \left(\frac{d\,Q}{d\,y}\right);\ \frac{d^2 z}{d\,t^2} = \left(\frac{d\,Q}{d\,z}\right).$$

Now multiplying the first of these equations by cos. θ. cos. v; the second by cos. θ. sin. v; the third by sin. θ, we get, by adding them

$$\frac{d^2\varrho}{d\,t^2} - \frac{\varrho\,d\,v^2}{d\,t^2}.\cos.^2\theta - \frac{\varrho\,d\,\theta^2}{d\,t^2} = \left(\frac{d\,Q}{d\,\varrho}\right) \quad \ldots\ldots \quad (2)$$

In like manner, multiplying the first of the above equations by $-\varrho$ cos.$\theta\times$ sin. v; the second by ϱ cos. θ cos. v and adding them, &c. we have

$$\frac{d\left(\varrho^2 \frac{d\,v}{d\,t}\cos.^2\theta\right)}{d\,t} = \left(\frac{d\,Q}{d\,v}\right) \quad \ldots\ldots\ldots \quad (3)$$

And lastly multiplying the first by $-\varrho$ sin. θ. cos. v; the second by $-\varrho$ sin. θ. cos. v and adding them to the third multiplied by cos. θ. we have

$$\varrho^2.\frac{d^2\theta}{d\,t^2} + \varrho^2\frac{d\,v^2}{d\,t^2}.\sin.\theta\cos.\theta + \frac{2\varrho\,d\,\varrho\,d\,\theta}{d\,t^2} = \left(\frac{d\,Q}{d\,\theta}\right) \quad \ldots \quad (4)$$

To render the equations (2), (3), (4), still better adapted for use, let

$$u = \frac{1}{\varrho\cos.\theta}$$

and

$$s = \tan.\ \theta$$

u being unity divided by the projection of the radius ϱ upon the plane of x, y; and s the tangent of the latitude of μ from that same plane.

If we multiply equation (3) by $\varrho^2\, d\, v \cos.^2 \theta$ and integrate, we get

$$\left(\frac{d\, v}{u^2\, d\, t}\right)^2 = h^2 + 2\int\left(\frac{d\, Q}{d\, v}\right)\cdot\frac{d\, v}{u^2};$$

h being the arbitrary constant.

Hence

$$d\, t = \frac{d\, v}{u^2\sqrt{\left(h^2 + 2\int\left(\frac{d\, Q}{d\, v}\right)\cdot\frac{d\, v}{u^2}\right.}} \quad \ldots\ldots (5)$$

If we add equation (2) multiplied by $-\cos.\ \theta$ to equation (4) multiplied by $\frac{\sin.\ \theta}{\varrho}$, we shall have

$$-\frac{d^2\frac{1}{u}}{d\, t^2} + \frac{1}{u}\cdot\frac{d\, v^2}{d\, t^2} = u^2\left(\frac{d\, Q}{d\, u}\right) + u\, s\left(\frac{d\, Q}{d\, s}\right)$$

whence

$$d\cdot\left(\frac{d\, u}{u^2\, d\, t}\right) + \frac{d\, v^2}{u\, d\, t} = u^2\, d\, t\left\{\left(\frac{d\, Q}{d\, u}\right) + \frac{s}{u}\left(\frac{d\, Q}{d\, s}\right)\right\}.$$

Substituting for d t, its foregoing value, and making d v constant, we shall have

$$0 = \frac{d^2 u}{d\, v^2} + u + \frac{\left(\frac{d\, Q}{d\, v}\right)\frac{d\, u}{u^2 d\, v} - \left(\frac{d\, Q}{d\, u}\right) - \frac{s}{u}\left(\frac{d\, Q}{d\, s}\right)}{h^2 + 2\int\left(\frac{d\, Q}{d\, v}\right)\frac{d\, v}{u^2}} \quad \ldots\ldots (6)$$

In the same way making d v constant, equation (4) will become

$$0 = \frac{d^2 s}{d\, v^2} + s + \frac{\frac{d\, s}{d\, v}\left(\frac{d\, Q}{d\, v}\right) - u\, s\left(\frac{d\, Q}{d\, u}\right) - (1+s^2)\left(\frac{d\, Q}{d\, s}\right)}{u^2\left\{h^2 + 2\int\left(\frac{d\, Q}{d\, v}\right)\cdot\frac{d\, v}{u^2}\right\}} \quad \ldots (7)$$

Now making $M + \mu = m$, we have (in this case)

$$Q = \frac{m}{\varrho} \text{ or } = \frac{m\, u}{\sqrt{(1 + s^2)}}$$

and the equations (5), (6), (7) will become

$$\left.\begin{aligned} d\, t &= \frac{d\, v}{h\,.\,u^2};\\ 0 &= \frac{d^2 u}{d\, v^2} + u - \frac{\mu}{h^2\,(1 + s^2)^{\frac{3}{2}}};\\ 0 &= \frac{d^2 s}{d\, v^2} + s. \end{aligned}\right\} \quad \ldots\ldots\ldots (8)$$

(These equations may be more simply deduced directly 124 and Woodhouse's Phys. Astron.)

The area described during the element of time d t, by the projection of the radius-vector is $\frac{1}{2}\frac{d v}{u^2}$; the first of equations (8) show that this area is proportional to that element, and also that in a finite time it is proportional to the time.

Moreover integrating the last of them (by 122) or by multiplying by 2 d s, we get

$$s = \gamma \sin. (v - \theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

γ and θ being two arbitrary constants.

Finally, the second equation gives by integration

$$u = \frac{\mu}{h^2 (1 + \gamma^2)} \{\sqrt{1 + s^2} + e \cos. (v - \varpi)\} = \frac{\sqrt{1 + s^2}}{\varrho}; \quad . \quad . \quad . \quad (10)$$

e and ϖ being two new arbitraries.

Substituting for s in this expression, its value in terms of v, and then this expression in the equation

$$d t = \frac{d v}{h u^2};$$

the integral of this equation will give t in terms of v; thus we shall have v, u and s in functions of the time.

This process may be considerably simplified, by observing that the value of s indicates the orbit to lie wholly in one plane, the tangent of whose inclination to a fixed plane is γ, the longitude of the node θ being reckoned from the origin of the angle v. In referring, therefore, to this plane the motion of μ; we shall have

$$s = 0 \text{ and } \gamma = 0,$$

which give

$$u = \frac{1}{\varrho} = \frac{\mu}{h^2} \{1 + e \cos. (v - \varpi)\}.$$

This equation is that of an ellipse in which the origin of ϱ is at the focus:

$$\frac{h^2}{\mu (1 - e^2)}$$

is the semi-axis-major which we shall designate by a; e is the ratio of the excentricity to the semi-axis-major; and lastly ϖ is the longitude of the perihelion. The equation

$$d t = \frac{d v}{h u^2}$$

hence becomes

$$dt = \frac{a^{\frac{5}{2}}(1-e^2)^{\frac{5}{2}}}{\sqrt{\mu}} \times \frac{dv}{\{1+e\cos.(v-\varpi)\}^2}.$$

Develope the second member of this equation, in a series of the angle $v - \varpi$ and of its multiples. For that purpose, we will commence by developing

$$\frac{1}{1+e\cos.(v-\varpi)}$$

in a similar series. If we make

$$\lambda = \frac{e}{1+\sqrt{(1-e^2)}};$$

we shall have

$$\frac{1}{1+e\cos.(v-\varpi)} = \frac{1}{\sqrt{1-e^2}}\left\{\frac{1}{1+\lambda c^{(v-\varpi)\sqrt{-1}}} + \frac{\lambda c^{-(v-\varpi)\sqrt{-1}}}{1+\lambda c^{-(v-\varpi)\sqrt{-1}}}\right\};$$

c being the number whose hyperbolic is unity. Developing the second member of this equation, in a series; namely the first term relatively to powers of $c^{(v-\varpi)\sqrt{-1}}$, and the second term relatively to powers of $c^{-(v-\varpi)\sqrt{-1}}$ and then substituting, instead of imaginary exponentials, their expressions in terms of sine and cosine; we shall find

$$\frac{1}{1+e\cos.(v-\varpi)} = \frac{1}{\sqrt{1-e^2}} \times$$

$$\{1 - 2\lambda\cos.(v-\varpi) + 2\lambda^2\cos.2(v-\varpi) - 2\lambda^3\cos.3(v-\varpi) + \&c.\}$$

Calling φ the second member of this equation, and making $q = \frac{1}{e}$; we shall have generally

$$\frac{1}{\{1+e\cos.(v-\varpi)\}^{m+1}} = \frac{\pm e^{-m-1} d^m.\left(\frac{\varphi}{q}\right)}{1.2.3.....m.\,dq^m},$$

for putting

$$\frac{\varphi}{q} = \frac{1}{q+R}$$

R being $= \cos.(v-\varpi)$

$$\frac{d.\left(\frac{\varphi}{q}\right)}{dq} = -\frac{1}{(q+R)^2}$$

$$\frac{d^2.\left(\frac{\varphi}{q}\right)}{dq^2} = \frac{2}{(q+R)^3}$$

$$\&c. = \&c.$$

$$\frac{d^m\left(\frac{\varphi}{q}\right)}{dq^m} = \pm\frac{2.3....m}{(q+R)^{m+1}}$$

$$\therefore \pm\frac{d^m\left(\frac{\varphi}{q}\right)}{dq^m} \times \frac{e^{-m-1}}{2.3...m} = \frac{q^{m+1}}{(q+R)^{m+1}} = \varphi^{m+1}$$

$$= \frac{1}{\{1+e\cos.(v-\varpi)\}^{m+1}}.$$

Hence it is easy to conclude that if we make

$$\frac{1}{\{1+e\cos.(v-\varpi)\}^2} = (1-e^2)^{-\frac{3}{2}} \times$$

$$\{1 + E^{(1)}.\cos.(v-\varpi) + E^{(2)}\cos.2(v-\varpi) + \&c.\}$$

we shall have generally whatever be the number (i)

$$E^{(i)} = \pm\frac{2e^i\{1+i\sqrt{1-e^2}\}}{(1+\sqrt{1-e^2})^i};$$

the signs $\pm$ being used according as i is even or odd; supposing therefore that $u = a^{-\frac{3}{2}}\sqrt{m}$, we have

$$n\,dt = dv\{1 + E^{(1)}\cos.(v-\varpi) + E^{(2)}\cos.2(v-\varpi) + \&c.\}$$

and integrating

$$nt + \varepsilon = v + E^{(1)}\sin.(v-\varpi) + \tfrac{1}{2}E^{(2)}\sin.2(v-\varpi) + \&c.$$

ε being an arbitrary constant. This expression for $nt + \varepsilon$ is very convergent when the orbits are of small excentricity, such as are those of the Planets and of the Satellites; and by the Reversion of Series we can find v in terms of t: we shall proceed to this presently.

474. When the Planet comes again to the same point of its orbit, v is augmented by the circumference 2π; naming therefore T the time of the whole revolution, we have (see also 159)

$$T = \frac{2\pi}{n} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{m}}.$$

This could be obtained immediately from the expression

$$T = \frac{\int \xi^2\,dv}{h}$$

$$= \frac{2 \text{ area of Ellipse}}{h} = \frac{2\pi ab}{h}$$

But by 157

$$h^2 = m\,a\,(1-e^2)$$

$$\therefore T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{m}}.$$

If we neglect the masses of the planets relatively to that of the sun we have

$$\sqrt{\mu} = \sqrt{M}$$

which will be the same for all the planets; T is therefore proportional in that hypothesis to $a^{\frac{3}{2}}$, and consequently the squares of the Periods are as the cubes of the major axes of the orbits. We see also that the same law holds with regard to the motion of the satellites around their planet, provided their masses are also deemed inconsiderable compared with that of the planet.

475. The equations of motion of the two bodies M and μ may also be integrated in this manner.

Resuming the equations (1), (2), (3), of 471, and putting $M+\mu=m$, we have for these two bodies

$$\left.\begin{aligned} 0 &= \frac{d^2x}{dt^2} + \frac{mx}{\varrho^3} \\ 0 &= \frac{d^2y}{dt^2} + \frac{my}{\varrho^3} \\ 0 &= \frac{d^2z}{dt^2} + \frac{mz}{\varrho^3} \end{aligned}\right\} \quad (0)$$

The integrals of these equations will give in functions of the time t, the three coordinates x, y, z of the body μ referred to the center of M; we shall then have (471) the coordinates ζ, Π, γ of the body M, referred to a fixed point by means of the equations

$$\left.\begin{aligned} \zeta &= a + bt - \frac{\mu x}{m}; \\ \Pi &= a' + b't - \frac{\mu y}{m}; \\ \gamma &= a'' + b''t - \frac{\mu z}{m}. \end{aligned}\right\}$$

Lastly, we shall have the coordinates of μ, referred to the same fixed point, by adding x to ζ, y to Π, and z to γ: We shall also have the relative motion of the bodies M and μ, and their absolute motion in space.

476. To integrate the equations (0) we shall observe that if amongst the (n) variables $x^{(1)}$, $x^{(2)}$ ..... $x^{(n)}$ and the variable t, whose difference is supposed constant, a number n of equations of the following form

$$0 = \frac{d^i x^{(s)}}{dt^i} + A\frac{d^{i-1}x^{(s)}}{dt^{i-1}} + B\frac{d^{i-2}x^{(s)}}{dt^{i-2}} ..... H.x^{(s)}$$

in which we suppose s successively equal to 1, 2, 3 ..... n; A, B ..... H being functions of the variables $x^{(1)}$, $x^{(2)}$, &c. and of t symmetrical

with regard to the variables $x^{(1)}$, $x^{(2)}$, &c. that is to say, such that they remain the same, when we change any one of these variables to any other and reciprocally; suppose

$$x^{(1)} = a^{(1)} x^{(n-i+1)} + b^{(1)} x^{(n-i+2)} + \ldots\ldots h^{(1)} x^{(n)},$$
$$x^{(2)} = a^{(2)} x^{(n-i+1)} + b^{(2)} x^{(n-i+2)} + \ldots\ldots h^{(2)} x^{n}.$$
$$- - - - - - - - - - - - - - - - - -$$
$$x^{(n-i)} = a^{(n-i)} x^{(n-i+1)} + b^{(n-i)} x^{(n-i+2)} \ldots\ldots + h^{(n-i)} x^{(n)}$$

$a^{(1)}$, $b^{(1)}$, $\ldots\ldots h^{(1)}$; $a^{(2)}$, $b^{(2)}$, &c. being the arbitraries of which the number is i (n — i). It is clear that these values satisfy the proposed system of equations: Moreover these equations are thereby reduced to i equations involving the i variables $x^{(n-i+1)} \ldots\ldots x^{(n)}$. Their integrals will introduce i^2 new arbitraries, which together with the i (n — i) preceding ones will form i n arbitraries which ought to give the integration of the equations proposed.

477. To apply the above Theorem to equations (0); we have

$$z = a x + b y$$

a and b being two arbitrary constants, this equation being that of a plane passing through the origin of coordinates; also the orbit of μ is wholly in one plane.

The equations (0) give

$$\left.\begin{aligned} 0 &= d\left(\varrho^3 . \frac{d^2 x}{d t^2}\right) + m\, d x \\ 0 &= d\left(\varrho^3 . \frac{d^2 y}{d t^2}\right) + m\, d y \\ 0 &= d\left(\varrho^3 . \frac{d^2 z}{d t^2}\right) + m\, d z \end{aligned}\right\}; \quad (0')$$

Also since

$$\varrho^2 = x^2 + y^2 + z^2$$

and

$$\therefore \varrho\, d\varrho = x\, dx + y\, dy + z\, dz$$

and differentiating twice more, we have

$$\varrho\, d^3\varrho + 3\, d\varrho\, d^2\varrho = x\, d^3 x + y\, d^3 y + z\, d^3 z + 3\,(dx\, d^2 x + dy\, d^2 y + dz\, d^2 z),$$

and consequently

$$d.\left(\varrho^3 \frac{d^2\varrho}{d t^2}\right) = \varrho^2 \left\{ x \frac{d^3 x}{d t^2} + y \frac{d^3 y}{d t^2} + z \frac{d^3 z}{d t^2} \right\} + 3\,\varrho^2 \left\{ dx \frac{d^2 x}{d t^2} + dy \frac{d^2 y}{d t^2} + dz \frac{d^2 z}{d t^2} \right\}.$$

Substituting in the second member of this equation for $d^3 x$, $d^3 y$, $d^3 z$

their values given by equations (0'), and for d^2x, d^2y, d^2z their values given by equations (0); we shall find

$$0 = d\left(\varrho^3 \frac{d^2\varrho}{dt^2} + m\,d\varrho\right).$$

If we compare this equation with equations (0'), we shall have in virtue of the preceding Theorem, by considering $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \frac{d\varrho}{dt}$, as so many particular variables $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}$, and ϱ as a function of the time t;

$$d\varrho = \lambda\,dx + \gamma\,dy;$$

λ and γ being constants; and integrating

$$= \frac{h^2}{m} + \lambda x + \gamma y,$$

$\frac{h^2}{m}$ being a constant. This equation combined with

$$z = ax + by;\ \varrho^2 = x^2 + y^2 + z^2$$

gives an equation of the second degree in terms of x, y, or in terms of x, z, or of y, z; whence it follows that the three projections of the curve described by μ about M, are lines of the second order, and therefore that the curve itself (lying in one plane) is a line of the second order or a conic section. It is easy to perceive from the nature of conic sections that, the radius-vector ϱ being expressed by a linear function of x, y, the origin of x, y ought to be in the focus. But the equation

$$\varrho = \frac{h^2}{m} + \lambda x + \gamma y$$

gives by means of equations (0)

$$0 = \frac{d^2\varrho}{dt^2} + \mu\frac{\left(\varrho - \frac{h^2}{m}\right)}{\varrho^3}.$$

Multiplying this by $d\varrho$ and integrating we get

$$\varrho^2\frac{d\varrho^2}{dt^2} - 2m\varrho + \frac{m\varrho^2}{a'} + h^2 = 0,$$

a' being an arbitrary constant. Hence

$$dt = \frac{\varrho\,d\varrho}{m\sqrt{\left(2\varrho - \frac{\varrho^2}{a'} - \frac{h^2}{m}\right)}};$$

which will give ϱ in terms of t; and since x, y, z are given above in terms of ϱ, we shall have the coordinates of μ in functions of the times.

478. We can obtain these results by the following method, which has the advantage of giving the arbitrary constants in terms of the coordinates x, y, z and of their first differences; which will presently be of great use to us.

Let V = constant, be an integral of the first order of equations (0), V being a function of $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. Call the three last quantities x', y', z'. Then V = constant will give, by taking the differential,

$$0 = \left(\frac{dV}{dx}\right) \cdot \frac{dx}{dt} + \left(\frac{dV}{dy}\right) \cdot \frac{dy}{dt} + \left(\frac{dV}{dz}\right) \cdot \frac{dz}{dt}$$
$$+ \left(\frac{dV}{dx'}\right) \cdot \frac{dx'}{dt} + \left(\frac{dV}{dy'}\right) \cdot \frac{dy'}{dt} + \left(\frac{dV}{dz'}\right) \cdot \frac{dz'}{dt}$$

But equations (0) give

$$\frac{dx'}{dt} = -\frac{mx}{\varrho^3}, \quad \frac{dy'}{dt} = -\frac{my}{\varrho^3}, \quad \frac{dz'}{dt} = -\frac{mz}{\varrho^3};$$

we have therefore the equation of Partial Differences

$$0 = x'\left(\frac{dV}{dx}\right) + y'\left(\frac{dV}{dy}\right) + z'\left(\frac{dV}{dz}\right)$$
$$- \frac{m}{\varrho^3}\left\{ x\left(\frac{dV}{dx'}\right) + y\left(\frac{dV}{dy'}\right) + z\left(\frac{dV}{dz'}\right) \right. \quad . \quad . \quad . \quad (I)$$

It is evident that every function of x, y, z, x', y', z' which, when substituted for V in this equation, satisfies it, becomes, by putting it equal to an arbitrary constant, an integral of the first order of the equations (0).

Suppose

$$V = U + U' + U'' + \&c.$$

U being a function of x, y, z; U′ a function of x, y, z, x′, y′, z′ but of the first order relatively to x′, y′, z′; U″ a function of x, y, z, x′, y′, z′ and of the second order relatively to x′, y′, z′, and so on. Substitute this value of V in the equation (I) and compare separately 1. the terms without x′, y′, z′; 2. those which contain their first powers; 3. those involving their squares and products, and so on; and we shall have

$$0 = x\left(\frac{dU'}{dx'}\right) + y\left(\frac{dU'}{dy'}\right) + z\left(\frac{dU'}{dz'}\right);$$

$$x'\left(\frac{dU}{dx}\right) + y'\left(\frac{dU}{dy}\right) + z'\left(\frac{dU}{dz}\right) = \frac{m}{\varrho^3}\left\{ x\left(\frac{dU''}{dx'}\right) + y\left(\frac{dU''}{dy'}\right) + z\left(\frac{dU''}{dz'}\right) \right.$$

$$x'\left(\frac{dU'}{dx}\right) + y'\left(\frac{dU'}{dx}\right) + z'\left(\frac{dU'}{dz}\right) = \frac{m}{\varrho^3}\left\{ x\left(\frac{dU'''}{dx'}\right) + y\left(\frac{dU'''}{dy'}\right) + z\left(\frac{dU'''}{dz'}\right) \right\}$$

$$x'\left(\frac{dU''}{dx}\right) + y'\left(\frac{dU''}{dx}\right) + z'\left(\frac{dU''}{dz}\right) = \frac{m}{\varrho^3}\left\{ x\left(\frac{dU''''}{dx'}\right) + y\left(\frac{dU''''}{dy'}\right) + z\left(\frac{dU''''}{dz'}\right) \right\}$$

&c.

which four equations call (I′).

The integral of the first of them is

$$U' = \text{funct.}\{x y' - y x', x z' - z x', y z' - z y', x, y, z\}$$

The value of U' is linear with regard to x', y', z'; suppose it of this form

$$U' = A(x y' - y x') + B(x z' - z x') + C(y z' - z y');$$

A, B, C being arbitrary constants. Make

$$U''', \&c. = 0;$$

then the third of the equations (I') will become

$$0 = x'\left(\frac{d U'}{d x}\right) + y'\left(\frac{d U'}{d y}\right) + z'\left(\frac{d U'}{d z}\right)$$

The preceding value of U' satisfies also this equation.

Again, the fourth of the equations (I') becomes

$$0 = x'\left(\frac{d U''}{d x}\right) + y'\left(\frac{d U''}{d y}\right) + z'\left(\frac{d U''}{d z}\right);$$

of which the integral is

$$U'' = \text{funct. } \{x y' - y x', x z' - z x', y z' - z y', x', y', z'\}.$$

This function ought to satisfy the second of equations (I'), and the first member of this equation multiplied by $d t$ is evidently equal to $d U$. The second member ought therefore to be an exact differential of a function of x, y, z; and it is easy to perceive that we shall satisfy at once this condition, the nature of the function U'', and the supposition that this function ought to be of the second order, by making

$$U'' = (D y' - E x') \cdot (x y' - y x') + (D z' - F x')(x z' - z x') + (E z' - F y')(y z' - z y') + G(x'^2 + y'^2 + z'^2);$$

D, E, F, G being arbitrary constants; and then ϱ being $= \sqrt{x^2+y^2+z^2}$, we have

$$U = -\frac{m}{\varrho}(D x + E y + F z + 2 G);$$

Thus we have the values of

$$U, U', U'';$$

and the equation $V =$ constant will become

$$\text{const.} = -\frac{m}{\varrho}\{D x + E y + F z + 2 G\} + (A + D y' - E x')(x y' - y x') + (B + D z' - F x')(x z' - z x') + (C + E z' - F y')(y z' - z y') + G(x'^2 + y'^2 + z'^2).$$

This equation satisfies equation (I) and consequently the equations (0) whatever may be the arbitrary constants A, B, C, D, E, F, G. Supposing all these $= 0$, 1. except A, 2. except B, 3. except C, &c. and putting

$$\frac{d x}{d t}, \frac{d y}{d t}, \frac{d z}{d t} \text{ for } x', y', z',$$

we shall have the integrals

$$(P)\quad\begin{cases} c=\dfrac{x\,dy-y\,dx}{dt},\ c'=\dfrac{x\,dz-z\,dx}{dt},\ c''=\dfrac{y\,dz-z\,dy}{dt} \\ 0=f+x\left\{\dfrac{m}{\varrho}-\dfrac{dy^2+dz^2}{dt^2}\right\}+\dfrac{y\,dy.dx}{dt^2}+\dfrac{z\,dz.dx}{dt^2} \\ 0=f'+y\left\{\dfrac{m}{\varrho}-\dfrac{dx^2+dz^2}{dt^2}\right\}+\dfrac{x\,dx.dy}{dt^2}+\dfrac{z\,dz.dy}{dt^2} \\ 0=f''+z\left\{\dfrac{m}{\varrho}-\dfrac{dx^2+dy^2}{dt^2}\right\}+\dfrac{x\,dx.dz}{dt^2}+\dfrac{y\,dy.dz}{dt^2} \\ 0=\dfrac{m}{a}-\dfrac{2m}{\varrho}+\dfrac{dx^2+dy^2+dz^2}{dt^2} \end{cases}$$

c, c', c'', f, f', f'' and a being arbitrary constants.

The equations (0) can have but six distinct integrals of the first order, by means of which, if we eliminate d x, d y, d z, we shall have the three variables x, y, z in functions of the time t; we must therefore have at least one of the seven integrals (P) contained in the six others. We also perceive *à priori*, that two of these integrals ought to enter into the five others. In fact, since it is the element only of the time which enters these integrals, they cannot give the variables x, y, z in functions of the time, and therefore are insufficient to determine completely the motion of μ about M. Let us examine how it is that these integrals make but five distinct integrals.

If we multiply the fourth of the equations (P) by $\dfrac{z\,dy-y\,dz}{dt}$, and add the product to the fifth multiplied by $\dfrac{x\,dz-z\,dx}{dt}$, we shall have

$$0=f.\frac{z\,dy-y\,dz}{dt}+f'.\frac{x\,dz-z\,dx}{dt}+z.\frac{x\,dy-y\,dx}{dt}\left\{\frac{m}{\varrho}-\frac{dx^2+dy^2}{dt^2}\right\}$$
$$+\frac{x\,dy-y\,dx}{dt}\left\{\frac{x\,dx.dz}{dt^2}+\frac{y\,dy.dz}{dt^2}\right\}.$$

Substituting for $\dfrac{x\,dy-y\,dx}{dt}$, $\dfrac{x\,dz-z\,dx}{dt}$, $\dfrac{y\,dz-z\,dy}{dt}$, their values given by the three first of the equations (P), we shall have

$$0=\frac{f'c'-fc''}{c}+z\left\{\frac{m}{\varrho}-\frac{dx^2+dy^2}{dt^2}\right\}+\frac{x\,dx.dz}{dt^2}+\frac{y\,dy.dz}{dt^2}.$$

This equation enters into the sixth of the integrals P, by making $f''=\dfrac{f'c'-fc''}{c}$ or $0=fc''-f'c'+f''c$. Also the sixth of these integrals results from the five first, and the six arbitraries c, c', c'', f, f', f'' are connected by the preceding equation.

If we take the squares of f, f′, f″ given by the equations (P), then add them together, and make $f^2 + f'^2 + f''^2 = l^2$, we shall have

$$l^2 - m^2 = \left\{\varrho^2 . \frac{dx^2+dy^2+dz^2}{dt^2} - \left(\frac{\varrho\, d\varrho}{dt}\right)^2\right\} . \left\{\frac{dx^2+dy^2+dz^2}{dt^2} - \frac{2m}{\varrho}\right\};$$

but if we square the values of c, c′, c″, given by the same equations, and make $c^2 + c'^2 + c''^2 = h^2$; we get

$$\varrho^2 . \frac{dx^2 + dy^2 + dz^2}{dt^2} - \left(\frac{\varrho\, d\varrho}{dt}\right)^2 = h^2;$$

the equation above thus becomes

$$0 = \frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2m}{\varrho} + \frac{m^2 - l^2}{h^2}.$$

Comparing this equation with the last of equations (P), we shall have the equation of condition,

$$\frac{m^2 - l^2}{h^2} = \frac{m}{a}.$$

The last of equations (P) consequently enters the six first, which are themselves equivalent only to five distinct integrals, the seven arbitrary constants, c, c′, c″, f, f′, f″, and a being connected by the two preceding equations of condition. Whence it results that we shall have the most general expression of V, which will satisfy equation (I) by taking for this expression an arbitrary function of the values of c, c′, c″, f, and f′, given by the five first of the equations (P).

479. Although these integrals are insufficient for the determination of x, y, z in functions of the time; yet they determine the nature of the curve described by μ about M. In fact, if we multiply the first of the equations (P) by z, the second by $-y$, and the third by x, and add the results, we shall have

$$0 = cz - c'y + c''x,$$

the equation to a plane whose position depends upon the constants c, c′, c″.

If we multiply the fourth of the equations (P) by x, the fifth by y, and the sixth by z, we shall have

$$0 = fx + f'y + f''z + m\varrho - \varrho^2 . \frac{dx^2+dy^2+dz^2}{dt^2} + \frac{\varrho^2 d\varrho^2}{dt^2};$$

but by the preceding number

$$\varrho^2 . \frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{\varrho^2 d\varrho^2}{dt^2} = h^2;$$

$$\therefore 0 = m\varrho - h^2 + fx + f'y + f''z.$$

This equation combined with

$$0 = c''x - c'y + cz$$

and

$$\varrho^2 = x^2 + y^2 + z^2$$

gives the equation to conic sections, the origin of ϱ being at the focus. The planets and comets describe therefore round the sun very nearly conic sections, the sun being in one of the foci; and these stars so move that their radius-vectors describe areas proportional to the times. In fact, if d v denote the elemental angle included by ϱ, $\varrho + d\varrho$, we have

$$dx^2 + dy^2 + dz^2 = \varrho^2 dv^2 + d\varrho^2$$

and the equation

$$\varrho^2 . \frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{\varrho^2 d\varrho^2}{dt^2} = h^2$$

becomes

$$\varrho^4 dv^2 = h^2 dt^2;$$

$$\therefore dv = \frac{h\,dt}{\varrho^2}.$$

Hence we see that the elemental area $\frac{1}{2}\varrho^2 dv$, described by ϱ, is proportional to the element of time d t; and the area described in a finite time is therefore also proportional to that time. We see also that the angular motion of μ about M, is at every point of the orbit, as $\frac{1}{\varrho^2}$; and *since without sensible error we may take very short times for those indefinitely small, we shall have, by means of the above equation, the horary motions of the planets and comets, in the different points of their orbits.*

The elements of the section described by μ, are the arbitrary constants of its motion; these are functions of the arbitraries c, c′, c″, f, f′, f″, and $\frac{m}{n}$. Let us determine these functions.

Let θ be the angle which the intersection of the planes of the orbit and of (x, y) makes with the axis of x, this intersection being called the *line of the nodes*; also let φ be the inclination of the planes. If x′, y′ be the coordinates of μ referred to the line of the nodes as the axis of abscissas, then we have

$$x' = x \cos.\theta + y \sin.\theta$$
$$y' = y \cos.\theta - x \sin.\theta.$$

Moreover

$$z = y' \tan.\varphi$$
$$\therefore z = y \cos.\theta \tan.\varphi - x \sin.\theta \tan.\varphi.$$

Comparing this equation with the following one

$$0 = c'' x - c' y + c z$$

we shall have

$$c' = c \cos. \theta . \tan. \varphi$$
$$c'' = c \sin. \theta \tan. \varphi$$

whence

$$\tan. \theta = \frac{c''}{c'}$$

and

$$\tan. \varphi = \frac{\sqrt{(c'^2 + c''^2)}}{c}.$$

Thus are determined the position of the nodes and the inclination of the orbit, in functions of the arbitrary constants c, c', c''.

At the perihelion, we have

$$\varrho \, d \varrho = 0, \text{ or } x \, dx + y \, dy + z \, dz = 0.$$

Let X, Y, Z be the coordinates of the planet at this point; the fourth and the fifth of the equations (P) will give

$$\frac{Y}{X} = \frac{f'}{f}.$$

But if I be called the longitude of the projection of the perihelion upon the plane of x, y this longitude being reckoned from the axis of x, we have

$$\frac{Y}{X} = \tan. I;$$
$$\therefore \tan. I = \frac{f'}{f},$$

which determines the position of the major axis of the conic section.

If from the equation

$$\varrho^2 . \frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{\varrho^2 d\varrho^2}{dt^2} = h^2$$

we eliminate $\frac{dx^2 + dy^2 + dz^2}{dt^2}$, by means of the last of the equations (P), we shall have

$$2m\varrho - \frac{m\varrho^2}{a} - \frac{\varrho^2 d\varrho^2}{dt^2} = h^2$$

but $d\varrho$ is 0 at the extremities of the axis major; we therefore have at these points

$$0 = \varrho^2 - 2a\varrho + \frac{a h^2}{m}.$$

The sum of the two values of ϱ in this equation, is the axis major, and their difference is double the excentricity; thus a is the semi-axis major of the orbit, or the mean distance of μ from M; and

$$\sqrt{\left(1 - \frac{h^2}{m a}\right)}$$

is the ratio of the excentricity to the semi-axis major. Let

$$e = \sqrt{\left(1 - \frac{h^2}{m\,a}\right)}$$

and having by the above

$$\frac{m}{a} = \frac{m^2 - l^2}{h^2};$$

we shall get

$$m\,e = l.$$

Thus we know all the elements which determine the nature of the conic section and its position in space.

480. The three finite equations found above between x, y, z and ϱ give x, y, z in functions of ϱ; and to get these coordinates in functions of the time it is sufficient to obtain ϱ in a similar function; which will require a new integration. For that purpose take the equation

$$2\,m\,\varrho - \frac{m\,\varrho^2}{a} - \frac{\varrho^2\,d\,\varrho^2}{d\,t^2} = h^2.$$

But we have above

$$h^2 = \frac{a}{m}\,(m^2 - l^2) = a\,m\,(1 - e^2);$$

$$\therefore d\,t = \frac{\varrho\,d\,\varrho}{\sqrt{m}\,\sqrt{\left\{2\,\varrho - \frac{\varrho^2}{a} - a\,(1 - e^2)\right\}}}.$$

whose integral (237) is

$$t + T = \frac{a^{\frac{3}{2}}}{\sqrt{m}}\,(u - e\,\sin.\,u) \quad \ldots\ldots\ldots \quad (S)$$

u being $= \cos.^{-1}\left(\frac{1}{e} - \frac{\varrho}{a\,e}\right)$, and T an arbitrary constant.

This equation gives u and therefore ϱ in terms of t; and since x, y, z are given in functions of ϱ, we shall have the values of the coordinates for any instants whatever.

We have therefore completely integrated the equations (0) of 475, and thereby introduced the six arbitrary constants a, e, I, θ, φ, and T. The two first depend upon the nature of the orbit; the three next depend upon its position in space, and the last relates to the position of the body μ at any given epoch; or which amounts to the same, depends upon the instant of its passing the perihelion.

Referring the coordinates of the body μ, to such as are more commodious for astronomical uses, and for that, naming v the angle which the radius-

vector makes with the major axis setting out from the perihelion, the equation to the ellipse is

$$\varrho = \frac{a\,(1 - e^2)}{1 + e \cos. v}.$$

The equation

$$\varrho = a\,(1 - e \cos. u)$$

indicates that u is 0 at the perihelion, so that this point is the origin of two angles u and v; and it is easy hence to conclude that the angle u is formed by the axis major, and by the radius drawn from its center to the point where the circumference described upon the axis major as a diameter, is met by the ordinate passing through the body μ at right angles to the axis major. Hence as in (237) we have

$$\tan. \frac{v}{2} = \sqrt{\frac{1 + e}{1 - e}} . \tan. \frac{u}{2}.$$

We therefore have (making T = 0, &c.)

$$\left.\begin{aligned} n\,t &= u - e \sin. u \\ \varrho &= a\,(1 - e \cos. u) \\ \text{and} \\ \tan. \frac{v}{2} &= \sqrt{\frac{1 + e}{1 - e}} . \tan. \frac{u}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (f)$$

n t being the *Mean Anomaly*,

n the *Excentric Anomaly*,

v the *True Anomaly*.

The first of these equations gives u in terms of t, and the two others will give ϱ and v when u shall be determined. The equation between u and t is transcendental, and can only be resolved by approximation. Happily the circumstances attending the motions of the heavenly bodies present us with rapid approximations. In fact the orbits of the stars are either nearly circular or nearly parabolical, and in both cases, we can determine u in terms of t by series very convergent, which we now proceed to develope. For this purpose we shall give some general Theorems upon the reduction of functions into series, which will be found very useful hereafter.

481. Let u be any function whatever of α, which we propose to develope into a series proceeding by the powers of α. Representing this series by

$$u = u' + \alpha . q_1 + \alpha^2 . q_2 + \ldots\ldots \alpha^n . q_n + \alpha^{n+1} . q_{n+} + \&c.$$

u, q_1, q_2, &c. being quantities independent of α, it is evident that u is what u will become when we suppose $\alpha = 0$; and that whatever n may be

$$\left(\frac{d^n u}{d \alpha^n}\right) = 1.2....n.q_n + 2.3....(n+1).\alpha.q_{n+1} + \&c.$$

the difference $\left(\frac{d^n u}{d x^n}\right)$ being taken on the supposition that every thing in u varies with α. Hence if we suppose after the differentiations, that $\alpha = 0$, in the expression $\left(\frac{d^n u}{d \alpha^n}\right)$ we have

$$q_n = \left(\frac{d^n u}{d a^n}\right) \times \frac{1}{1.2....n}.$$

This is Maclaurin's Theorem (see 32) for one variable.

Again, if u be a function of two quantities α, α', let it be put

$$\begin{aligned} u = u &+ \alpha.q_{1,0} + \alpha^2.q_{2,0} + \&c. \\ &+ \alpha'.q_{0,1} + \alpha\alpha'.q_{1,1} + \&c. \\ &+ \alpha'^2.q_{0,2} + \&c. \end{aligned}$$

the general term being

$$\alpha^n \alpha'^{n'} q_{n,n'}.$$

Then if generally

$$\left(\frac{d^{n+n'} u}{d \alpha^n . d \alpha'^{n'}}\right)$$

denotes the $(n + n')^{th}$ difference of u, the operation being performed (n) times, on the supposition that α is the only variable, and then n' times on that of α' being the only variable, we have

$$\begin{aligned} \left(\frac{d u}{d \alpha}\right) &= q_{1,0} + 2\alpha.q_{2,0} + 3\alpha^2.q_{3,0} + 4\alpha^3 q_{4,0} + 5\alpha^4.q_{5,0} + \&c. \\ &+ \alpha' q_{1,1} + 2\alpha.\alpha' q_{2,1} + 3\alpha^2\alpha' q_{3,1} + 4\alpha^3\alpha' q_{4,1} + \&c. \\ &+ \alpha'^2 q_{1,2} + 2\alpha\alpha'^2 q_{2,2} + 3\alpha^2\alpha'^2 q_{3,2} + \&c. \\ &+ \alpha'^3 q_{1,3} + 2\alpha\alpha'^3 q_{2,3} + \&c. \\ &+ \alpha'^4 q_{1,4} + \&c. \end{aligned}$$

$$\begin{aligned} \left(\frac{d^2 \alpha}{d \alpha^2}\right) &= 2q_{2,0} + 3.2\alpha q_{3,0} + 4.3\alpha^2 q_{4,0} + 5.4\alpha^3 q_{5,0} + \&c. \\ &+ 2\acute{\alpha} q_{2,1} + 3.2\alpha\acute{\alpha} q_{3,1} + 4.3\alpha^2\acute{\alpha} q_{4,1} + \&c. \\ &+ 2\acute{\alpha}^2 q_{2,2} + 3.2\alpha\acute{\alpha}^2 q_{3,2} + \&c. \\ &+ 2\acute{\alpha}^3 q_{2,3} + \&c. \end{aligned}$$

$$\begin{aligned} \left(\frac{d^3 \alpha}{d \alpha^2 . d \acute{\alpha}}\right) &= 2q_{2,1} + 3.2\alpha q_{3,1} + \&c. \\ &+ 2\acute{\alpha} q_{2,2} + \&c. \end{aligned}$$

and continuing the process it will be found that

$$\left(\frac{d^{n+n'} u}{d a^n . d \acute{\alpha}^n}\right) = 2.3....n \times 2.3....n' \times q_{n,n'} + \left\{\begin{matrix} Q \times \alpha \\ Q' \times \alpha' \end{matrix}\right\}$$

so that when α, $\acute{\alpha}$ both equal 0, we have

$$q_{n,n'} = \frac{\left(\frac{d^{n+n'} u}{d\alpha^n . d\acute{\alpha}^{n'}}\right)}{2.3....n \times 2.3....n'} \quad \ldots\ldots\ldots (1)$$

And generally, if u be a function of α, $\acute{\alpha}$, α'', &c. and in developing it into a series, if the coefficient of $\alpha^n . \acute{\alpha}^{n'} . \alpha''^{n''}$. &c. be denoted by $q_{n, n', n''}$, &c. we shall have, in making α, $\acute{\alpha}$, α'', &c. all equal 0,

$$q_{n,n',n''}, \text{ \&c.} = \frac{\left(\frac{d^{n+n'+n''+\text{\&c.}} u}{d\alpha^n . d\acute{\alpha}^{n'} . d\alpha''^{n''}, \text{\&c.}}\right)}{2.3....n \times 2.3....n' \times 2.3....n'' \times \text{\&c.}} \quad \ldots (2)$$

This is Maclaurin's Theorem made general.

482. Again let u be any function of $t+\alpha$, $t'+\acute{\alpha}$, $t''+\alpha''$, &c. and put

$$u = \varphi (t + a, t' + \acute{\alpha}, t'' + \alpha'', \text{\&c.})$$

then since t and α are similarly involved it is evident that

$$\left(\frac{d^{n+n'+n''+\text{\&c.}} . u}{d\alpha^n . d\acute{\alpha}^{n'} . d\alpha''^{n''} \text{\&c.}}\right) = \left(\frac{d^{n+n'+n''+\text{\&c.}} . u}{dt^n . dt'^{n'} . dt''^{n''} . \text{\&c.}}\right)$$

and making

$$\alpha, \acute{\alpha}, \alpha'', \text{\&c.} = 0,$$

or

$$u = \varphi (t, t', t'', \text{\&c.})$$

by (2) of the preceding article we have

$$q_{n,n',n'',\text{\&c.}} = \frac{\left(\frac{d^{n+n'+n'',\text{\&c.}} . \varphi (t, t', t'', \text{\&c.})}{dt^n . dt'^{n'} . dt''^{n''} \text{\&c}}\right)}{2.3....n \times 2.3....n' \times 2.3....n'' \times \text{\&c.}} \quad \ldots (1)$$

which gives Taylor's Theorem in all its generality (see 32).

Hence when

$$u = \varphi . (t + \alpha)$$

$$q_n = \frac{d^n . \varphi (t)}{2.3....n . dt^n}$$

and we thence get

$$\varphi (t + \alpha) = \varphi (t) + \alpha \frac{d . \varphi (t)}{dt} + \frac{\alpha^2}{2} . \frac{d^2 \varphi (t)}{dt^2} + \text{\&c.} \ldots\ldots (i)$$

483. Generally, suppose that u is a function of α, $\acute{\alpha}$, α'', &c. and of t, t', t'', &c. Then, if by the nature of the function or by an equation of Partial Differences which represents it, we can obtain

$$\left(\frac{d^{n+n'+\text{\&c.}} . u}{d\alpha^n . d\acute{\alpha}^{n'} . \text{\&c.}}\right)$$

in a function of u, and of its Differences taken with regard to t, t', &c.

calling it F when for u we put u or make α, δ, α'', &c. $= 0$; it is evident we have

$$q_{n,n',n'',\&c.} = \frac{F}{2.3\ldots n \times 2.3\ldots n' \times 2.3\ldots n'', \times \&c.}$$

and therefore the law of the series into which u is developed.

For instance, let u, instead of being given immediately in terms of α, and t, be a function of x, x itself being deducible from the equation of Partial Differences

$$\left(\frac{d\,x}{d\,\alpha}\right) = X\left(\frac{d\,x}{d\,t}\right)$$

in which X is any function whatever of x. That is

Given

$$u = \text{function } (x)$$

$$\left(\frac{d\,x}{d\,\alpha}\right) = X.\left(\frac{d\,x}{d\,t}\right)$$

to develope u *into a series ascending by the powers of* α.

First, since

$$\left(\frac{d\,u}{d\,\alpha}\right) = \left(\frac{d\,u}{d\,x}\right).\left(\frac{d\,x}{d\,\alpha}\right) = X\left(\frac{d\,u}{d\,x}\right).\left(\frac{d\,x}{d\,t}\right)$$

$$\therefore \left(\frac{d\,u}{d\,\alpha}\right) = \left(\frac{d \int X\,d\,u}{d\,t}\right) \quad \ldots\ldots\ldots\ldots \quad (k)$$

Hence

$$\left(\frac{d^2\,u}{d\,\alpha^2}\right) = \left(\frac{d^2 \int X\,d\,u}{d\,\alpha\,.\,d\,t}\right);$$

But by equation (k), changing u into $\int X\,d\,u$

$$\left(\frac{d.\int X\,d\,u}{d\,\alpha}\right) = \left(\frac{d.\int X^2\,d\,u}{d\,t}\right);$$

$$\therefore \left(\frac{d^2\,u}{d\,\alpha^2}\right) = \left(\frac{d^2 \int X^2\,d\,u}{d\,t^2}\right).$$

Again

$$\left(\frac{d^3\,u}{d\,\alpha^3}\right) = \left(\frac{d^3 \int X^2\,d\,u}{d\,\alpha\,.\,d\,t^2}\right).$$

But by equation k, and changing u into $\int X^2\,d\,u$

$$\left(\frac{d \int X^2\,d\,u}{d\,\alpha}\right) = \left(\frac{d \int X^3\,d\,u}{d\,t}\right)$$

$$\therefore \left(\frac{d^3\,u}{d\,\alpha^3}\right) = \left(\frac{d^3.\int X^3\,d\,u}{d\,t^3}\right).$$

Thus proceeding we easily conclude generally that

$$\left(\frac{d^n\,u}{d\,\alpha^n}\right) = \left(\frac{d^n \int X^n\,d\,u}{d\,t^n}\right) = \left(\frac{d^{n-1}.\,X^n\left(\frac{d\,u}{d\,t}\right)}{d\,t^{n-1}}\right) \quad \ldots \quad (l)$$

Now, when $\alpha = 0$, let

$$x = \text{function of } t = T$$

and substitute this value of x in X and u; and let these then become X and u respectively. Then we shall have

$$\left(\frac{d^n u}{d\alpha^n}\right) = \frac{d^{n-1}.X^n.\frac{du}{dt}}{dt^{n-1}},$$

and

$$\therefore q_n = \frac{d^{n-1}.X^n.\frac{du}{dt}}{2.3....n\,dt^{n-1}} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

which gives

$$u = u + \alpha X.\frac{du}{dt} + \frac{\alpha^2}{2}.\frac{d\left(X^2.\frac{du}{dt}\right)}{dt} + \frac{\alpha^3}{2.3}.\frac{d^2\left(X^3.\frac{du}{dt}\right)}{dt^2} + \text{\&c.} \ldots (p)$$

which is Lagrange's Theorem.

To determine the value of x in terms of t and α, we must integrate

$$\left(\frac{dx}{d\alpha}\right) = X.\left(\frac{dx}{dt}\right).$$

In order to accomplish this object, we have

$$dx = \left(\frac{dx}{dt}\right)dt + \left(\frac{dx}{d\alpha}\right)d\alpha,$$

and substituting

$$\left(\frac{dx}{d\alpha}\right) \text{for } X.\left(\frac{dx}{dt}\right)$$

we shall have

$$dx = \left(\frac{dx}{dt}\right)\{dt + X\,d\alpha\}$$

$$= \frac{dx}{dt}\left\{d(t + \alpha X) - \alpha\left(\frac{dX}{dx}\right)dx\right\};$$

$$\therefore dx = \frac{\left(\frac{dx}{dt}\right).d.(t + \alpha X)}{1 + \alpha\left(\frac{dX}{dx}\right).\left(\frac{dx}{dt}\right)};$$

which by integration, gives

$$x = \varphi(t + \alpha X) \quad \ldots\ldots\ldots \quad (2)$$

φ denoting an arbitrary function.

Hence whenever we have an equation reducible to this form $x = \varphi(t + \alpha X)$, the value of u will be given by the formula (p), in a series of the powers of α.

By an extension of the process, the Theorem may be generalized to the case, when

$$u = \text{function } (x, x', x'', \text{\&c.})$$

and

$$x = \varphi(t + \alpha X)$$
$$x' = \varphi'(t' + \alpha' X')$$
$$x'' = \varphi''(t'' + \alpha'' X'')$$
$$\&c. = \&c.$$

484. *Given* (237)

$$u = n t + e \sin. u$$

required to develope u *or any function of it according to the powers of* e.

Comparing the above form with

$$x = \varphi(t + \alpha X)$$

x, t, α, X become respectively

u, n t, e, sin. u.

Hence the formula (p) 483. gives

$$\psi(u) = \psi(n t) + e\psi'(n t)\sin. n t + \frac{e^2}{2}\cdot\frac{d\{\psi'(n t)\sin.^2 n t\}}{n\,d\,t}$$
$$+ \frac{e^3}{2.3}\cdot\frac{d^2\{\psi'(n t)\sin.^3 n t\}}{n^2 d t^2} + \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (q)$$

$$\psi'(n t) \text{ being } = \frac{d.\psi(n t)}{n\,d\,t}.$$

To farther develope this formula we have generally (see Woodhouse's Trig.)

$$\sin.^i(n t) = \left(\frac{c^{n t\sqrt{-1}} - c^{-n t\sqrt{-1}}}{2\sqrt{-1}}\right)^i ; \quad \cos.^i(n t) = \left(\frac{c^{n t\sqrt{-1}} + c^{-n t\sqrt{-1}}}{2}\right)^i;$$

c being the hyperbolic base, and i any number whatever. Developing the second members of these equations, and then substituting

$$\cos. r n t + \sqrt{-1}\sin. r n t, \text{ and } \cos. r n t - \sqrt{-1}\sin. r n t$$

for $c^{r n t\sqrt{-1}}$, and $c^{-r n t\sqrt{-1}}$, r being any number whatever, we shall have the powers i of sin. n t, and of cos. n t expressed in sines and cosines of n t and its multiples; hence we find

$$P = \sin. n t + \frac{e}{2}\sin.^2 n t + \frac{e^2}{2.3}\sin.^3 n t + \&c.$$
$$= \sin. n t - \frac{e}{2.2}\cdot\{\cos. 2 n t - 1\}$$
$$- \frac{e^2}{2.3.2^2}\cdot\{\sin. 3 n t - 3\sin. n t\}$$
$$+ \frac{e^3}{2.3.4.2^3}\cdot\left\{\cos. 4 n t - 4\cos. 2 n t + \frac{1}{2}\cdot\frac{4.3}{1.2}\right\}$$
$$+ \frac{e^4}{2.3.4.5.2^4}\left\{\sin. 5 n t - 5\sin. 3 n t + \frac{5.4}{1.2}\sin. n t\right\}$$

$$-\frac{e^5}{2.3.4.5.6.2^5}\cdot\left\{\cos.6nt-6\cos.4nt+\frac{6.5}{1.2}\cos.2nt-\frac{1}{2}\cdot\frac{6.5.4}{1.2.3}\right\}$$

— &c.

Now multiply this function by $\psi'(n\,t)$, and differentiate each of its terms relatively to t a number of times indicated by the power of e which multiplies it, d t being supposed constant; and divide these differentials by the corresponding power of n d t. Then if P′ be the sum of the quotients, the formula (q) will become

$$\psi(u) = \psi(n\,t) + e\,P'.$$

By this method it is easy to obtain the values of the angle u, and of the sine and cosine of its multiples. Supposing for example, that

$$\psi\,u = \text{sin. i u}$$

we have

$$\psi'(n\,t) = i\cos.\,i\,n\,t.$$

Multiply therefore the preceding value of P, by i. cos. i n t, and develope the product into sines and cosines of n t and its multiples. The terms multiplied by the even powers of e, are sines, and those multiplied by the odd powers of e, are cosines. We change therefore any term of the form, $K\,e^{2r}$ sin. s n t, into $\pm K\,e^{2r}s^{2r}$ sin. s n t, + or — obtaining according as r is even or odd. In like manner, we change any term of the form, $K\,e^{2r+1}$ cos. s n t, into $\mp K\,e^{2r+1}.s^{2r+1}$. sin. s n t, — or + obtaining according as r is even or odd. The sum of all these terms will be P′ and we shall have

$$\text{sin. i u} = \text{sin.. i n t} + e\,P'.$$

But if we suppose

$$\psi(u) = u;$$

then

$$\psi(n\,t) = 1$$

and we find by the same process

$$u = n\,t + e\,\text{sin. n t} + \frac{e^2}{2.2}\cdot 2\,\text{sin. 2 n t}$$

$$+\frac{e^3}{2.3.2^2}\cdot\{3^2\,\text{sin. 3 n t} - 3\,\text{sin. n t}\}$$

$$+\frac{e^4}{2.3.4.2^3}\cdot\{4^3\,\text{sin. 4 n t} - 4.2^3\,\text{sin. 2 n t}\}$$

$$+\frac{e^5}{2.3.4.5.2^4}\cdot\left\{5^4\,\text{sin. 5 n t} - 5.3^4\,\text{sin. 3 n t} + \frac{5.4}{1.2}\,\text{sin. n t}\right\}$$

+ &c.

a formula which expresses the Excentric Anomaly in terms of the Mean Anomaly.

This series is very convergent for the Planets. Having thus determined u for any instant, we could thence obtain by means of (237), the corresponding values of ϱ and v. But these may be found directly as follows, also in convergent series.

485. *Required to express ϱ in terms of the Mean Anomaly.*

By (237) we have

$$\varrho = a\,(1 - e\cos.\,u).$$

Therefore if in formula (q) we put

$$\psi\,(u) = 1 - e\cos.\,u$$

we have

$$\psi'\,(n\,t) = e\sin.\,n\,t,$$

and consequently

$$1 - e\cos.\,u = 1 - e\cos.\,n\,t + e^2\sin.^2 n\,t + \frac{e^3}{2}\cdot\frac{d.\sin.^3 n\,t}{n\,d\,t} + \&c.$$

Hence, by the above process, we shall find

$$\frac{\varrho}{a} = 1 + \frac{e^2}{2} - e\cos.\,n\,t - \frac{e^2}{2}\cos.\,2\,n\,t$$
$$- \frac{e^3}{2.\,2^2}\cdot\{3\cos.\,3\,n\,t - 3\cos.\,n\,t\}$$
$$- \frac{e^4}{2.\,3.\,2^3}\cdot\{4^2\cos.\,4\,n\,t - 4.\,2^2.\cos.\,2\,n\,t\}$$
$$- \frac{e^5}{2.\,3.\,4.\,2^4}\cdot\left\{5^3\cos.\,5\,n\,t - 5.\,3^3\cos.\,3\,n\,t + \frac{5.\,4}{1.\,2}\cdot\cos.\,n\,t\right\}$$
$$- \frac{e^6}{2.3.4.5.2^5}\left\{6^4\cos.\,6\,n\,t - 6.\,4^4\cos.\,4\,n\,t + \frac{6.\,5}{1.2}\cdot 2^4\cos.\,2\,n\,t\right\}$$
$$- \&c.$$

486. *To express the True Anomaly in terms of the Mean.*

First we have (237)

$$\frac{\sin.\frac{v}{2}}{\cos.\frac{v}{2}} = \sqrt{\frac{1+e}{1-e}}\cdot\frac{\sin.\frac{u}{2}}{\cos.\frac{u}{2}}$$

$\therefore$ substituting the imaginary expressions

$$\frac{c^{v\sqrt{-1}} - 1}{c^{v\sqrt{-1}} + 1} = \sqrt{\frac{1+e}{1-e}} \times \frac{c^{u\sqrt{-1}} - 1}{c^{u\sqrt{-1}} + 1};$$

and making

$$\lambda = \frac{e}{1 + \sqrt{(1 - e^2)}}$$

we shall have

$$c^{v\sqrt{-1}} = c^{u\sqrt{-1}} \times \frac{1 - \lambda c^{-u\sqrt{-1}}}{1 - \lambda c^{u\sqrt{-1}}};$$

and therefore

$$v = u + \frac{\log.(1-\lambda.c^{-u\sqrt{-1}}) - \log.(1-\lambda c^{u\sqrt{-1}})}{\sqrt{-1}}$$

whence expanding the logarithms into series (see p. 28), and putting sines and cosines for their imaginary values, we have

$$v = u + 2\lambda \text{ sin. } u + \frac{2\lambda^2}{2} \text{ sin. } 2u + \frac{2\lambda^3}{3} \text{ sin. } 3u + \&c.$$

But by the foregoing process we have u, sin. u, sin. 2 u, &c. in series ordered by the powers of e, and developed into sines and cosines of n t and its multiples. There is nothing else then to be done, in order to express v in a similar series, but to expand λ into a like series.

The equation, (putting $u = 1 + \sqrt{1 - e^2}$)

$$u = 2 - \frac{e^2}{u}$$

will give by the formula (p) of No. (483)

$$\frac{1}{u^i} = \frac{1}{2^i} + \frac{i.e^2}{2^{i+2}} + \frac{i(i+3)}{2} \cdot \frac{e^4}{2^{i+4}} + \frac{i(i+3)(i+5)}{2.3} \cdot \frac{e^6}{2^{i+6}} \&c.$$

and since

$$u = 1 + \sqrt{1 - e^2}$$

we have

$$\lambda^i = \frac{e^i}{2^i}\left\{1 + i\left(\frac{e}{2}\right)^2 + \frac{i(i+3)}{1.2} \cdot \left(\frac{e}{2}\right)^4 + \&c.\right\}$$

These operations being performed we shall find

$$\begin{aligned} v = nt &+ \left\{2e - \frac{1}{4}e^3 + \frac{5}{96}e^5\right\} \text{sin. } nt \\ &+ \left\{\frac{5}{4}e^2 - \frac{11}{24}e^4 + \frac{17}{192}e^6\right\} \text{sin. } 2nt \\ &+ \left\{\frac{13}{12}e^3 - \frac{43}{64}e^5\right\} \text{sin. } 3nt \\ &+ \left\{\frac{103}{96}e^4 - \frac{451}{480}e^6\right\} \text{sin. } 4nt \\ &+ \frac{1097}{960}e^5 \text{ sin. } 5nt \\ &+ \frac{1223}{960}e^6 \text{ sin. } 6nt, \end{aligned}$$

the approximation being carried on to quantities of the order e^6 inclusively.

487. The angles v and n t are here reckoned from the Perihelion; but if we wish to compute from the Aphelion, we have only to make e negative. It would, therefore, be sufficient to augment the angle n t by π, in order to render negative the sines and cosines of the odd multiples of n t; then to make the results of these two methods identical; we have only in the expressions for ϱ and v, to multiply the sines and cosines of odd multiples of n t by odd powers of e; and the even multiples by the even powers. This is confirmed, in fact, by the process, *a posteriori*.

488. Suppose that instead of reckoning v from the perihelion, we fix its origin at any point whatever; then it is evident that this angle will be augmented by a constant, which we shall call ϖ, and which will express the Longitude of the Perihelion. If instead of fixing the origin of t at the instant of the passage over the perihelion, we make it begin at any point, the angle n t will be augmented by a constant which we will call $\varepsilon - \varpi$; and then the foregoing expressions for $\frac{\varrho}{a}$ and v, will become

$$\frac{\varrho}{a} = 1 + \frac{1}{2}e^2 - (e - \frac{3}{8}e^3)\cos.(nt + \varepsilon - \varpi) - (\frac{1}{2}e^2 - \frac{1}{3}e^4)\cos.2(nt + \varepsilon - \varpi) + \&c.$$

$$v = nt + \varepsilon + (2e - \frac{1}{4}e^3)\sin.(nt + \varepsilon - \varpi) + (\frac{5}{4}e^2 - \frac{11}{24}e^4)\sin.2(nt + \varepsilon - \varpi) + \&c.$$

where v is the true longitude of the planet and n t + ε its mean longitude, these being measured on the plane of the orbit.

Let, however, the motion of the planet be referred to a fixed plane a little inclined to that of the orbit, and φ be the mutual inclination of the two planes, and θ the longitude of the Ascending Node of the orbit, measured upon the fixed plane; also let β be this longitude measured upon the plane of the orbit, so that θ is the projection of β, and lastly let $v_{,}$ be the projection of v upon the fixed plane. Then we shall have

$$v_{,} - \theta, \quad v - \beta,$$

making the two sides of a right angled spherical triangle, $v - \beta$ being opposite the right angle, and φ the angle included between them, and therefore by Napier's Rules

$$\tan.(v_{,} - \theta) = \cos.\varphi \tan.(v - \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation gives $v_{,}$ in terms of v and reciprocally; but we can express either of them in terms of the other by a series very convergent after this manner.

By what has preceded, we have the series

$$\frac{1}{2}v = \frac{1}{2}u + \lambda \sin.u + \frac{\lambda^2}{2}\sin.2u + \frac{\lambda^3}{3}\sin.3u + \&c.$$

from

$$\tan.\ \frac{1}{2}\ v = \sqrt{\frac{1+e}{1-e}}.\ \tan.\ \frac{1}{2}\ u,$$

by making

$$\lambda = \frac{\sqrt{\frac{1+e}{1-e}} - 1}{\sqrt{\frac{1+e}{1-e}} + 1}.$$

If we change $\frac{1}{2}$ v into $v_{,} - \theta$, and $\frac{1}{2}$ u into $v - \beta$, and $\sqrt{\frac{1+e}{1-e}}$ into cos. φ, we have

$$\lambda = \frac{\cos.\ \varphi - 1}{\cos.\ \varphi + 1} = -\tan.^{2}\frac{\varphi}{2}; \quad \ldots\ldots\ (1)$$

The equation between $\frac{1}{2}$ v and $\frac{1}{2}$ u will change into the equation between $v_{,} - \theta$ and $v - \beta$, and the above series will give

$$v_{,} - \theta = v - \beta - \tan^{2}\frac{1}{2}\ \varphi.\ \sin.\ 2\ (v - \beta) + \frac{1}{2}\ \tan.^{4}\frac{1}{2}\ \varphi.\ \sin.\ 4\ (v - \beta)$$

$$- \frac{1}{3}\ \tan.^{6}\frac{1}{2}\ \varphi\ \sin.\ 6\ (v - \beta) + \&c. \quad \ldots\ldots\ (2)$$

If in the equation between $\frac{v}{2}$ and $\frac{u}{2}$, we change $\frac{1}{2}$ v into $v - \beta$ and $\frac{1}{2}$ u into $v_{,} - \theta$, and $\sqrt{\frac{1+e}{1-e}}$ into $\frac{1}{\cos.\ \varphi}$, we shall have

$$\lambda = \tan.^{2}\frac{1}{2}\ \varphi, \quad \ldots\ldots\ (3)$$

and

$$v - \beta = v_{,} - \theta + \tan.^{2}\frac{1}{2}\ \varphi.\ \sin.\ 2\ (v_{,} - \theta)$$

$$+ \frac{1}{2}\ \tan.^{4}\frac{1}{2}\ \varphi.\ \sin.\ 4\ (v_{,} - \theta)$$

$$+ \frac{1}{3}\ \tan.^{6}\frac{1}{2}\ \varphi.\ \sin.\ 6\ (v_{,} - \theta) \quad \ldots\ldots\ (4)$$

Thus we see that the two preceding series reciprocally interchange, by changing the sign of $\tan.^{2}\frac{1}{2}\ \varphi$, and by changing $v_{,} - \theta$, $v - \beta$ the one for the other. We shall have $v_{,} - \theta$ in terms of the sine and cosine of n't and its multiples, by observing that we have, by what precedes

$$v = n\ t + \varepsilon + e\ Q,$$

Q being a function of the sine of the angle $n\ t + \varepsilon - \varpi$, and its multiples; and that the formula (i) of number (482) gives, whatever is i,

$$\sin.\ i\ (v - \beta) = \sin.\ i\ (n\ t + \varepsilon - \beta + e\ Q)$$

$$= \left\{1 - \frac{i^2 e^2 Q^2}{1.2} + \frac{i^4 e^4 Q^4}{1.2.3.4.} - \&c.\right\} \sin.\, i\,(n\,t + \varepsilon - \beta)$$

$$+ \left\{i\,e\,Q - \frac{i^3 e^3 Q}{1.2.3} + \frac{i^5 e^5 Q^5}{2.3.4.5.} - \&c.\right\} \cos.\, i\,(n\,t + \varepsilon - \beta).$$

Lastly, s being the tangent of the latitude of the planet above the fixed plane, we have

$$s = \tan.\, \varphi \sin.\,(v, - \theta);$$

and if we call ϱ, the radius-vector projected upon the fixed plane, we shall have

$$\varrho, = \varrho\,(1 + s^2)^{-\frac{1}{2}} = \varrho\left\{1 - \tfrac{1}{2}\,s^2 + \frac{3}{8}\,s^4 - \&c.\right\},$$

we shall therefore be able to determine $v_,$, s and $\varrho,$ in converging series of the sines and cosines of the angle n t and of its multiples.

489. Let us now consider very excentric orbits or such as are those of the Comets.

For this purpose resume the equations of No. (237), scil.

$$\varrho = \frac{a\,(1 - e^2)}{1 + e\cos.\, v}$$

$$n\,t = u - e \sin.\, u$$

$$\tan.\, \tfrac{1}{2}\,v = \sqrt{\frac{1+e}{1-e}}\,.\,\tan.\, \tfrac{1}{2}\,u.$$

In this case e differs very little from unity; we shall therefore suppose

$$1 - e = \alpha$$

α being very small compared with unity.

Calling D the perihelion distance of the Comet, we shall have

$$D = u\,(1 - e) = \alpha\,a;$$

and the expression for ϱ will become

$$\varrho = \frac{(2 - \alpha)\,D}{2\cos.^2 \frac{1}{2}\,v - \alpha \cos.\, v} = \frac{D}{\cos.^2 \frac{1}{2}\,v \left\{1 + \frac{\alpha}{2-\alpha} \tan.^2 \frac{1}{2}\,v\right\}},$$

which gives, by reduction into a series

$$\varrho = \frac{D}{\cos.^2 \frac{1}{2}\,v}\left\{1 - \frac{\alpha}{2-\alpha} \tan.^2 \frac{1}{2}\,v + \left(\frac{\alpha}{2-\alpha}\right)^2 \tan.^4 \frac{1}{2}\,v - \&c.\right\}$$

To get the relation of v to the time t, we shall observe that the expression of the arc in terms of the tangent gives

$$u = 2 \tan.\, \frac{1}{2}\,u \left\{1 - \frac{1}{3} \tan.^2 \frac{1}{2}\,u + \frac{1}{5} \tan.^4 \frac{1}{2}\,u - \&c.\right\}$$

But

$$\tan.\, \frac{1}{2}\,u = \sqrt{\frac{\alpha}{2-\alpha}} \tan.\, \frac{1}{2}\,u;$$

we therefore have

$$u = 2\sqrt{\frac{\alpha}{2-\alpha}}\tan.\frac{1}{2}v\left\{1-\frac{1}{3}\left(\frac{\alpha}{2-\alpha}\right)\tan.^2\frac{1}{2}v+\frac{1}{5}\left(\frac{\alpha}{2-\alpha}\right)^2\tan.^4\frac{1}{2}v-\&c.\right\}$$

Next we have

$$\sin. u = \frac{2\tan.\frac{1}{2}u}{1+\tan.^2\frac{1}{2}u} = 2\tan.\frac{1}{2}u\left\{1-\tan.^2\frac{u}{2}+\tan.^4\frac{u}{2}-\&c.\right\}$$

Whence we get

$$e\sin. u = 2(1-\alpha)\sqrt{\frac{\alpha}{2-\alpha}}\tan.\frac{1}{2}v.\left\{1-\frac{\alpha}{2-\alpha}\tan.^2\frac{1}{2}v + \left(\frac{\alpha}{2-\alpha}\right)^2.\tan.^4\frac{1}{2}v-\&c.\right\}.$$

Substituting these values of u, and e sin. u in the equation $n t = u - e \sin. u$, we shall have the time t in a function of the anomaly v, by a series very convergent; but before we make this substitution, we shall observe that (237)

$$n = a^{-\frac{3}{2}}.\sqrt{m},$$

and since

$$D = \alpha a,$$

we have

$$\frac{1}{n} = \frac{D^{\frac{3}{2}}}{\alpha^{\frac{3}{2}}\sqrt{m}}.$$

Hence we find

$$t = \frac{2D^{\frac{3}{2}}}{\sqrt{(2-\alpha)m}}\tan.\frac{1}{2}v\left\{1+\frac{\frac{2}{3}-\alpha}{2-\alpha}\tan.^2\frac{1}{2}v-\frac{\frac{4}{5}-\alpha}{(2-\alpha)^2}\tan.^4\frac{1}{2}v+\&c.\right\}.$$

If the orbit is parabolic

$$\alpha = 0$$

and consequently

$$\varrho = \frac{D}{\cos.\frac{1}{2}v};$$

$$t = \frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{m}}\left\{\tan.\frac{v}{2}+\frac{1}{3}\tan.^3\frac{1}{2}v\right\}$$

which expression may also be got at once from (237).

The time t, the distance D and sum m of the masses of the sun and comet, are heterogeneous quantities, to compare which, we must divide each by the units of their species. We shall suppose therefore that the mean distance of the sun from the Earth is the unit of distance, so that D is expressed in parts of that distance. We may next observe that if T

represent the time of a sidereal revolution of the Earth, setting off from the perihelion; we shall have in the equation

$$n\,t = u - e \sin. u$$

$u = 0$ at the beginning of the revolution, and $u = 2\pi$ at the end of it.

Hence

$$n\,T = 2\pi.$$

But we have

$$n = a^{-\frac{3}{2}} \sqrt{m} = \sqrt{m},$$

$$\therefore \sqrt{m} = \frac{2\pi}{T}.$$

The value of m is not exactly the same for the Earth as for the Comet, for in the first case it expresses the sum of the masses of the sun and earth; whereas in the second it implies the sum of the masses of the sun and comet: but the masses of the Earth and Comet being much smaller than that of the sun, we may neglect them, and suppose that m is the same for all Planets and all Comets and that it expresses the mass of the sun merely. Substituting therefore for $\sqrt{m}$ its value $\frac{2\pi}{T}$ in the preceding expression for t; we shall have

$$t = \frac{D^{\frac{3}{2}} . T}{\pi \sqrt{2}} \left\{ \tan. \frac{1}{2} v + \frac{1}{3} \tan.^3 \frac{1}{2} v \right\}.$$

This equation contains none but quantities comparable with each other; it will give t very readily when v is known; but to obtain v by means of t, we must resolve a Cubic Equation, which contains only one real root. We may dispense with this resolution, by making a table of the values of v corresponding to those of t, in a parabola of which the perihelion distance is unity, or equal to the mean distance of the earth from the sun. This table will give the time corresponding to the anomaly v, in any parabola of which D is the perihelion distance, by multiplying by $D^{\frac{3}{2}}$, the time which corresponds to the same anomaly in the Table. We also get the anomaly v corresponding to the time t, by dividing t by $D^{\frac{3}{2}}$, and seeking in the table, the anomaly which corresponds to the quotient arising from this division.

490. Let us now investigate the anomaly, corresponding to the time t, in an ellipse of great excentricity.

If we neglect quantities of the order α^2, and put $1 - e$ for α, the above expression of t in terms of v in an ellipse, will give

$$t = \frac{D^{\frac{3}{2}} \sqrt{2}}{\sqrt{m}} \left\{ \begin{array}{l} \tan. \frac{1}{2} v + \frac{1}{3} \tan.^3 \frac{1}{2} v \\ + (1 - e) \tan.^2 \frac{1}{2} v \left\{ \frac{1}{4} - \frac{1}{4} \tan.^2 \frac{1}{2} v - \frac{1}{3} \tan.^4 \frac{1}{2} v \right\} \end{array} \right\}.$$

Then, find by the table of the motions of the comets, the anomaly cor-

responding to the time t, in a parabola of which D is the perihelion distance. Let U be this anomaly and U + x the true anomaly in an ellipse corresponding to the same time, x being a very small angle. Then if we substitute in the above equation U + x for v, and then transform the second member into a series of powers of x, we shall have, neglecting the square of x, and the product of x by 1 — e,

$$t = \frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{m}} \cdot \left\{ \begin{array}{l} \tan.\frac{1}{2}U + \frac{1}{3}\tan.^3\frac{1}{2}U + \frac{x}{2\cos.^4\frac{1}{2}U} \\ + \frac{1-e}{4}\tan.\frac{1}{2}U\{1 - \tan.^2\frac{1}{2}U - \frac{4}{5}\tan.^4\frac{1}{2}U\} \end{array} \right\}.$$

But by supposition

$$t = \frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{m}}\{\tan.\tfrac{1}{2}U + \tfrac{1}{3}\tan.^3\tfrac{1}{2}U\}.$$

Therefore, substituting for x its sine and substituting for $\sin.^4\frac{1}{2}U$ its value $(1 - \cos.^2\frac{1}{2}U)^2$, &c.

$$\sin. x = \tfrac{1}{10}(1 - e)\tan.\tfrac{1}{2}U\{4 - 3\cos.^2\tfrac{1}{2}U - 6\cos.^4\tfrac{1}{2}U\}.$$

Hence, in forming a table of logarithms of the quantity

$$\tfrac{1}{10}\tan.\tfrac{1}{2}U\{4 - 3\cos.^2\tfrac{1}{2}U - 6\cos.^4\tfrac{1}{2}U\}$$

it will be sufficient to add the logarithm of 1 — e, in order to have that of sin. x; consequently we have the correction of the anomaly U, estimated from the parabola, to obtain the corresponding anomaly in a very excentric ellipse.

491. *To find the masses of such planets as have satellites.*

The equation

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{m}}$$

gives a very simple method of comparing the mass of a planet, having satellites, with that of the sun. In fact, M representing the mass of the sun, if μ the mass of the planet be neglected, we have

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{M}}$$

If we next consider a satellite of any planet μ', and call its mass p, and mean distance from the center of μ', h, and T' its periodic time, we shall have

$$T' = \frac{2\pi h^{\frac{3}{2}}}{\sqrt{\mu' + p}}$$

$$\therefore \frac{\mu' + p}{M} = \frac{h^3}{a^3} \times \frac{T^2}{T'^2}.$$

This equation gives the ratio of the sum of the masses of the planet μ' and its satellite to that of the sun. Neglecting therefore the mass of the

satellite, as small compared with that of the planet, or supposing their ratio known, we have the ratio of the mass of the planet to that of the sun.

492. *To determine the Elements of Elliptical Motion.*

After having exposed the General Theory of Elliptical Motion and Method of Calculating by converging series, in the two cases of nature, that of orbits almost circular, and the case of orbits greatly excentric, it remains to determine the *Elements* of those orbits. In fact if we call V the velocity of μ in its relative motion about M, we have

$$V^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2}$$

and the last of the equations (P) of No. 478, gives

$$V^2 = m\left\{\frac{2}{\varrho} - \frac{1}{a}\right\}.$$

To make m disappear from this expression, we shall designate by U the velocity which μ would have, if it described about M, a circle whose radius is equal to the unity of distance. In this hypothesis, we have

$$\varrho = a = 1,$$

and consequently

$$U^2 = m.$$

Hence

$$V^2 = U^2\left\{\frac{2}{\varrho} - \frac{1}{a}\right\}.$$

This equation will give the semi-axis major a of the orbit, by means of the primitive velocity of μ and of its primitive distance from M. But a is positive in the ellipse, and infinite in the parabola, and negative in the hyperbola. Thus *the orbit described by μ is an ellipse, a parabola, or hyperbola, according as* V *is* $<$ $=$ *or* $>$ *than* $U\sqrt{\frac{2}{\varrho}}$. It is remarkable that *the direction of primitive motion has no influence upon the species of conic section.*

To find the excentricity of the orbit, we shall observe that if ε represents the angle made by the direction of the relative motion of μ with the radius-vector, we have

$$\frac{d\varrho^2}{dt^2} = V^2 \cos.^2 \varepsilon.$$

Substituting for V^2 its value $m\left\{\frac{2}{\varrho} - \frac{1}{a}\right\}$, we have

$$\frac{d\varrho^2}{dt^2} = m\left(\frac{2}{\varrho} - \frac{1}{a}\right) \cos.^2 \varepsilon;$$

But by 480

$$2\,m\,\varrho - \frac{m\,\varrho^2}{a} - \frac{\varrho^2\,d\,\varrho^2}{d\,t^2} = m\,a\,(1 - e^2):$$

$$\therefore a\,(1 - e)^2 = \varrho^2 \sin.^2 \varepsilon \left(\frac{2}{\varrho} - \frac{1}{a}\right);$$

whence we know the excentricity a e of the orbit.

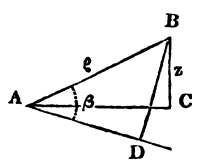
To find v or the true anomaly, we have

$$\varrho = \frac{a\,(1 - e^2)}{1 + e \cos.\, v}$$

$$\therefore \cos.\, v = \frac{a\,(1 - e^2) - \varrho}{e\,\varrho}.$$

This gives the position of the Perihelion. Equations (f) of No. 480 will then give u and by its means the instant of the Planet's passing its perihelion.

To get the position of the orbit, referred to a fixed plane passing through the center of M, supposed immoveable, let φ be the inclination of the orbit to this plane, and β the angle which the radius ϱ makes with the Line of the Nodes. Let, Moreover, z be the primitive elevation of μ above the fixed plane, supposed known. Then we shall have, C A D being the fixed plane, A D the line of the nodes, A B $= \varrho$, &c. &c.



$$z = B\,D\,.\,\sin.\,\varphi = \varrho \sin.\,\beta \sin.\,\varphi;$$

so that the inclination of the orbit will be known when we shall have determined β. For this purpose, let λ be the known angle which the primitive direction of the relative motion of μ makes with the fixed plane; then if we consider the triangle formed by this direction produced to meet the line of the nodes, by this last line and by the radius ϱ, calling l the side of the triangle opposite to β, we have

$$l = \frac{\varrho \sin.\,\beta}{\sin.\,(\beta + \varepsilon)}.$$

Next we have

$$\frac{z}{l} = \sin.\,\lambda$$

consequently

$$\tan.\,\beta = \frac{z \sin.\,\varepsilon}{\varrho \sin.\,\lambda - z \cos.\,\varepsilon}.$$

The elements of the Planetary Orbit being determined by these formulas, in terms of ϱ and z, of the velocity of the planet, and of the direction of its motion, we can find the variation of these elements corresponding

to the supposed variations in the velocity and its direction; and it will be easy, by methods about to be explained, from hence to obtain the differential variations of the Elements, due to the action of perturbing forces.

Taking the equation

$$V^2 = U^2\left\{\frac{2}{\varrho} - \frac{1}{a}\right\}.$$

In the circle $a = \varrho$ and $\therefore$

$$V = U\sqrt{\frac{1}{\varrho}}$$

so that the velocities of the planets in different circles are reciprocally as the squares of their radii (see Prop. IV of Princip.)

In the parabola, $a = \infty$,

$$\therefore V = U\sqrt{\frac{2}{\varrho}};$$

the velocities in the different points of the orbit, are therefore in this case reciprocally as the squares of the radius-vectors; and the velocity at each point, is to that which the body would have if it described a circle whose radius = the radius-vector ϱ, as $\sqrt{2} : 1$ (see 160)

An ellipse indefinitely diminished in breadth becomes a straight line, and in this case V expresses the velocity of μ, supposing it to descend in a straight line towards M. Let μ fall from rest, and its primitive distance be ϱ; also let its velocity at the distance ϱ' be V'; the above expression will give

$$0 = \frac{2}{\varrho} - \frac{1}{a}; \quad V'^2 = U^2\left\{\frac{2}{\varrho'} - \frac{1}{a}\right\};$$

whence

$$V' = U\sqrt{\frac{2(\varrho - \varrho')}{\varrho\varrho'}}.$$

Many other results, which have already been determined after another manner, may likewise be obtained from the above formula.

493. The equation

$$0 = \frac{dx^2 + dy^2 + dz^2}{dt^2} - m\left(\frac{2}{\varrho} - \frac{1}{a}\right)$$

is remarkable from its giving the velocity independently of the excentricity. It is also shown from a more general equation which subsists between the axis-major of the orbit, the chord of the elliptic arc, the sum of the extreme radius-vectors, and the time of describing this arc.

To obtain this equation, we have

$$\varrho = \frac{a(1 - e^2)}{1 + e\cos. v}$$

$$\varrho = a\,(1 - e \cos.\, u)$$
$$t = a^{\frac{3}{2}}\,(u - e \sin.\, a);$$

in which suppose ϱ, v, u, and t to correspond to the first extremity of the elliptic arc, and that ϱ', v', u', t' belong to the other extremity; so that we also have

$$\varrho' = \frac{a\,(1 - e^2)}{1 + e \cos.\, v'}$$
$$\varrho' = a\,(1 - e \cos.\, u')$$
$$t' = a^{\frac{3}{2}}\,(u' - e \sin.\, u').$$

Let now

$$t - t' = T;\ \frac{u' - u}{2} = \beta;$$
$$\frac{u' + u}{2} = \beta';\ \varrho' + \varrho = R;$$

then, if we take the expression of t from that of t', and observe that

$$\sin.\, u' - \sin.\, u = 2 \sin.\, \beta \cos.\, \beta'$$

we shall have

$$T = 2\,a^{\frac{3}{2}}\,\{\beta - e \sin.\, \beta \cos.\, \beta'\}.$$

If we add them together taking notice that

$$\cos.\, u' + \cos.\, u = 2 \cos.\, \beta.\ \cos.\, \beta'$$

we shall get

$$R = 2\,a\,(1 - e \cos.\, \beta \cos.\, \beta').$$

Again, if c be the chord of the elliptic arc, we have

$$c^2 = \varrho^2 + \varrho'^2 - 2\,\varrho\,\varrho' \cos.\,(v - v')$$

but the two equations

$$\varrho = \frac{a\,(1 - e^2)}{1 + e \cos.\, v};\ \varrho = a\,(1 - e \cos.\, u)$$

give these

$$\cos.\, v = a\,\frac{\cos.\, u - e}{\varrho};\ \sin.\, v = \frac{a\sqrt{1 - e^2}.\ \sin.\, u}{\varrho};$$

and in like manner we have

$$\cos.\, v' = a\,.\,\frac{\cos.\, u' - e}{\varrho'};\ \sin.\, v' = \frac{a\,\sqrt{1 - e^2}\,\sin.\, u'}{\varrho'};$$

whence, we get

$$\varrho\,\varrho' \cos.\,(v - v') = a^2\,(e - \cos.\, u)\,(e - \cos.\, u') + a^2\,(1 - e^2) \sin.\, u \sin.\, u';$$

and consequently

$$c^2 = 2\,a^2\,(1 - e^2)\,\{1 - \sin.\, u \sin.\, u' - \cos.\, u \cos.\, u'\}$$
$$+ a^2\,e^2\,(\cos.\, u - \cos.\, u')^2;$$

But

$$\text{sin. u sin. u}' + \text{cos. u cos. u}' = 2\,\text{cos.}^2\beta - 1$$
$$\text{cos. u} - \text{cos. u}' = 2\,\text{sin.}\,\beta\,\text{sin.}\,\beta'$$
$$\therefore c^2 = 4\,a^2\,\text{sin.}^2\beta\,(1 - e^2\,\text{cos.}^2\beta').$$

We therefore have these three equations, scil.

$$R = 2\,a\,\{1 - e\,\text{cos.}\,\beta\,\text{cos.}\,\beta'\};$$
$$T = 2\,a^{\frac{3}{2}}\,\{\beta - e\,\text{sin.}\,\beta\,\text{cos.}\,\beta'\},$$
$$c^2 = 4\,a^2\,\text{sin.}^2\beta\,(1 - e^2\,\text{cos.}^2\beta).$$

The first of them gives

$$e\,\text{cos.}\,\beta' = \frac{2\,a - R}{2\,a\,\text{cos.}\,\beta}$$

and substituting this value of e cos. β' in the two others, we shall have

$$T = 2\,a^{\frac{3}{2}}\left\{\beta + \frac{R-2\,a}{2\,a}\,\text{tan.}\,\beta\right\};$$
$$c^2 = 4\,a^2\,\text{tan.}^2\beta\left\{\text{cos.}^2\beta - \left(\frac{2\,a - R}{2\,a}\right)^2\right\}.$$

These two equations do not involve the excentricity e, and if in the first we substitute for β its value given by the second, we shall get T in a function c, R, and a. Thus we see that the time T depends only on the semi-axis major, the chord c and the sum R of the extreme radius-vectors.

If we make

$$z = \frac{2\,a - R + c}{2\,a};\quad z' = \frac{2\,a - R - c}{2\,a};$$

the last of the preceding equations will give

$$\text{cos. } 2\beta = z\,z' + \sqrt{(1 - z^2).(1 - z'^2)};$$

whence

$$2\beta = \text{cos.}^{-1} z' - \text{cos.}^{-1} z$$

(for cos. (A — B) = cos. A cos. B + sin. A sin. B).

Consequently

$$\text{tan.}\,\beta = \frac{\text{sin. (cos.}^{-1} z') - \text{sin. (cos.}^{-1} z)}{z + z'};$$

we have also

$$z + z' = \frac{2\,a - R}{a}.$$

Hence the expression of T will become, observing that if T is the duration of the sidereal revolution, whose mean distance from the sun is taken for unity, we have

$$T = 2\pi,$$

$$T = \frac{a^{\frac{3}{2}} T}{2\pi} \{\cos.^{-1} z' - \cos.^{-1} z - \sin.(\cos.^{-1} z') + \sin.(\cos^{-1} z)\} \ldots (a)$$

Since the same cosines may belong to many arcs, this expression is ambiguous, and we must take care to distinguish the arcs which correspond to z, z'.

In the parabola, the semi-axis major is infinite, and we have

$$\cos.^{-1} z' - \sin.(\cos.^{-1} z') = \frac{1}{6}\left(\frac{R + c}{a}\right)^{\frac{3}{2}}.$$

And making c negative we shall have the value of

$$\cos.^{-1} z - \sin.(\cos.^{-1} z);$$

hence the formula (a) will give the time T employed to describe the arc subtending the chord c, scil.

$$T = \frac{T}{12\pi}\left\{(\varrho + \varrho' + c)^{\frac{3}{2}} \mp (\varrho + \varrho' - c)^{\frac{3}{2}}\right\};$$

the sign — being taken, when the two extremities of the parabolic arc are situated on the same side of the axis of the parabola.

Now T being = 365.25638 days, we have

$$\frac{T}{12\pi} = 9.688754 \text{ days.}$$

The formula (a) gives the time of a body's descent in a straight line towards the focus, beginning from a given distance; for this, it is sufficient to suppose the axis-minor of the ellipse indefinitely diminished. If we suppose, for example, that the body falls from rest at the distance 2 a from the focus and that it is required to find the time (T) of falling to the distance c, we shall have

$$R = 2a + \varrho, \quad \varrho = 2a - c$$

whence

$$z' = -1, \quad z = \frac{c - a}{a}$$

and the formula gives

$$T = \frac{a^{\frac{3}{2}} T}{2\pi}\left\{\pi - \cos.^{-1}\frac{c - a}{a} + \sqrt{\frac{2ac - c}{a^2}}\right\}.$$

There is, however, an essential difference between elliptical motion towards the focus, and the motion in an ellipse whose breadth is indefinitely small. In the first case, the body having arrived at the focus, passes beyond it, and again returns to the same distance at which it departed; but in the second case, the body having arrived at the focus immediately returns to the point of departure. A tangential velocity at the aphelion,

however small, suffices to produce this difference which has no influence upon the time of the body's descent to the center, nor upon the velocity resolved parallel to the axis-major. Hence the principles of the 7th Section of Newton give accurately the Times and Velocities, although they do not explain all the circumstances of motion. For it is clear that if there be absolutely no tangential velocity, the body having reached the center of force, will proceed beyond it to the same distance from which it commenced its motion, and then return to the center, pass through it, and proceed to its first point of departure, the whole being performed in just double the time as would be required to return by moving in the indefinitely small ellipse.

494. Observations not conducting us to the circumstances of the primitive motion of the heavenly bodies; by the formulas of No. 492 we cannot determine the elements of their orbits. It is necessary for this end to compare together their respective positions observed at different epochs, which is the more difficult from not observing them from the center of their motions. Relatively to the planets, we can obtain, by means of their oppositions and conjunctions, their Heliocentric Longitude. This consideration, together with that of the smallness of the excentricity and inclination of their orbits to the ecliptic, affords a very simple method of determining their elements. But in the present state of astronomy, the elements of these orbits need but very slight corrections; and as the variations of the distances of the planets from the earth are never so great as to elude observation, we can rectify, by a great number of observations, the elements of their orbits, and even the errors of which the observations themselves are susceptible. But with regard to the Comets, this is not feasible; we see them only near their perihelion: if the observations we make on their appearance prove insufficient for the determination of their elements, we have then no means of pursuing them, even by thought, through the immensity of space, and when after the lapse of ages, they again approach the sun, it is impossible for us to recognise them. It becomes therefore important to find a method of determining, by observations alone during the appearance of one Comet, the elements of its orbit. But this problem considered rigorously surpasses the powers of analysis, and we are obliged to have recourse to approximations, in order to obtain the first values of the elements, these being afterwards to be corrected to any degree of accuracy which the observations permit.

If we use observations made at remote intervals, the eliminations will lead to impracticable calculations; we must therefore be content to con-

sider only near observations; and with this restriction, the problem is abundantly difficult.

It appears, that instead of directly making use of observations, it is better to get from them the data which conduct to exact and simple results. Those in the present instance, which best fulfil that condition, are the geocentric longitude and latitude of the Comet at a given instant, and their first and second differences divided by the corresponding powers of the element of time; for by means of these data, we can determine rigorously and with ease, the elements, without having recourse to a single integration, and by the sole consideration of the differential equations of the orbit. This way of viewing the problem, permits us moreover, to employ a great number of near observations, and to comprise also a considerable interval between the extreme observations, which will be found of great use in diminishing the influence of such errors, as are due to observations from the nebulosity by which Comets are enveloped. Let us first present the formulas necessary to obtain the first differences, of the longitude and latitude of any number of near observations; and then determine the elements of the orbit of a Comet by means of these differences; and lastly expose the method which appears the simplest, of correcting these elements by three observations made at remote intervals.

495. At a given epoch, let α be the geocentric longitude of a Comet, and θ its north geocentric latitude, the south latitudes being supposed negative. If we denote by s, the number of days elapsed from this epoch, the longitude and latitude of the Comet, after that interval, will, by using Taylor's Theorem (481), be expressed by these two series

$$\alpha + s.\left(\frac{d\,\alpha}{d\,s}\right) + \frac{s^2}{1.2}\left(\frac{d^2\alpha}{d\,s^2}\right) + \&c.$$

$$\theta + s.\left(\frac{d\,\theta}{d\,s}\right) + \frac{s^2}{1.2}\left(\frac{d^2\theta}{d\,s^2}\right) + \&c.$$

We must determine the values of

$$\alpha,\ \left(\frac{d\,\alpha}{d\,s}\right),\ \left(\frac{d^2\alpha}{d\,s^2}\right),\ \&c.,\ \theta,\ \left(\frac{d\,\theta}{d\,s}\right),\ \&c.$$

by means of several observed geocentric longitudes and latitudes. To do this most simply, consider the infinite series which expresses the geocentric longitude. The coefficients of the powers of s, in this series, ought to be determined by the condition, that by it is represented each observed longitude; we shall thus have as many equations as observations; and if their number is n, we shall be able to find from them, in series, the n

quantities α, $\left(\frac{d\,\alpha}{d\,s}\right)$, &c. But it ought to be observed that s being supposed very small, we may neglect all terms multiplied by s^n, s^{n+1}, &c. which will reduce the infinite series to its n first terms; which by n observations we shall be able to determine. These are only approximations, and their accuracy will depend upon the smallness of the terms which are omitted. They will be more exact in proportion as s is more diminutive, and as we employ a greater number of observations. The theory of interpolations is used therefore *To find a rational and integer function of* s *such, that in substituting therein for* s *the number of days which correspond to each observation, it shall become the observed longitude.*

Let β, β', β'', &c. be the observed longitudes of the comet, and by i, i', i'', &c. the corresponding numbers of days from the given epoch, the numbers of the days prior to the given epoch being supposed negative. If we make

$$\frac{\beta'-\beta}{i'-i}=\delta\beta;\ \frac{\beta''-\beta'}{i''-i'}=\delta\beta';\ \frac{\beta'''-\beta''}{i'''-i''}=\delta\beta'';\ \&c.$$

$$\frac{\delta\beta'-\delta\beta}{i''-i}=\delta^2\beta;\ \frac{\delta\beta''-\delta\beta'}{i'''-i'}\delta^2\beta';\ \&c.$$

$$\frac{\delta^2\beta'-d^2\beta}{i'''-i}=\delta^3\beta;\ \&c.$$

&c.;

the required functions will be

$\beta+(s-i).\delta\beta+(s-i)(s-i').\delta^2\beta+(s-i)(s-i')(s-i'')\delta^3\beta$, &c.

for it is easy to perceive that if we make successively $s=i$, $s=i'$, $s=i''$, &c. it will change itself into β, β', β'', &c.

Again, if we compare the preceding function with this

$$\alpha+s.\left(\frac{d\,\alpha}{d\,s}\right)+\frac{s^2}{2}.\left(\frac{d^2\alpha}{d\,s^2}\right)+\&c.$$

we shall have by equating coefficients of homogeneous terms.

$$\alpha=\beta-i\,\delta\beta+i.i'.\delta^2\beta-i.i'.i''\,\delta^3\beta+\&c.$$

$$\left(\frac{d\,\alpha}{d\,s}\right)=\delta\beta-(i+i')\,\delta^2\beta+(i\,i'+i\,i''+i'\,i'')\,\delta^3\beta-\&c.$$

$$\frac{1}{2}\left(\frac{d^2\alpha}{d\,s^2}\right)=\delta^2\beta-(i+i'+i'')\,\delta^3\beta+\&c.$$

The higher differences of α will be useless. The coefficients of these expressions are alternately positive and negative; the coefficient of $\delta^r\beta$ is, disregarding the sign, the product of r and r together of r quantities i, i', ... $i^{(r-1)}$ in the value of α; it is the sum of the products of the

same quantities, r — 1 together in the value of $\left(\frac{d\,\alpha}{d\,s}\right)$; lastly it is the sum of the products of these quantities r — 2, together in the value of $\frac{1}{2}\left(\frac{d^2\alpha}{d\,s^2}\right)$.

If γ, γ', γ'', &c. be the observed geocentric latitudes, we shall have the values of θ, $\left(\frac{d\,\theta}{d\,s}\right)$, $\left(\frac{d^2\theta}{d\,s^2}\right)$, &c. by changing in the preceding expressions for α $\left(\frac{d\,\alpha}{d\,s}\right)$, $\left(\frac{d^2\alpha}{d\,s^2}\right)$, &c. the quantities β, β', β'' into γ, γ', γ''.

These expressions are the more exact, the greater the number of observations and the smaller the intervals between them. We might, therefore, employ all the near observations made at a given epoch, provided they were accurate; but the errors of which they are always susceptible will conduct to imperfect results. So that, in order to lessen the influence of these errors, we must augment the interval between the extreme observations, employing in the investigation a greater number of them. In this way with five observations we may include an interval of thirty-five or forty degrees, which would give us very near approximations to the geocentric longitude and latitude, and to their first and second differences.

If the epoch selected were such, that there were an equal number of observations before and after it, so that each successive longitude may have a corresponding one which succeeds the epoch. This condition will give values still more correct of α, $\left(\frac{d\,\alpha}{d\,s}\right)$ and $\left(\frac{d^2\alpha}{d\,s^2}\right)$, and it easily appears that new observations taken at equal distances from either side of the epoch, would only add to these values, quantities which, with regard to their last terms, would be as $s^2\left(\frac{d^2\alpha}{d\,t^2}\right)$ to α. This symmetrical arrangement takes place, when all the observations being equidistant, we fix the epoch at the middle of the interval which they comprise. It is therefore advantageous to employ observations of this kind.

In general, it will be advantageous to fix the epoch near the middle of this interval; because the number of days included between the extreme observations being less considerable, the approximations will be more convergent. We can simplify the calculus still more by fixing the epoch at the instant of one of the observations; which gives immediately the values of α, and θ.

When we shall have determined as above the values of

$$\left(\frac{d\alpha}{ds}\right),\ \left(\frac{d^2\alpha}{ds^2}\right),\ \left(\frac{d\theta}{ds}\right),\ \text{and}\ \left(\frac{d^2\theta}{ds^2}\right)$$

we shall then obtain as follows the first and second differences of α, and θ divided by the corresponding powers of the elements of time. If we neglect the masses of the planets and comets, that of the sun being the unit of mass; if, moreover, we take the distance of the sun from the earth for the unit of distance; the mean motion of the earth round the sun will be the measure of the time t. Let therefore λ be the number of seconds which the earth describes in a day, by reason of its mean sidereal motion; the time t corresponding to the number of days will be λ s; we shall, therefore, have

$$\left(\frac{d\alpha}{dt}\right) = \frac{1}{\lambda}\left(\frac{d\alpha}{ds}\right)$$

$$\left(\frac{d^2\alpha}{dt^2}\right) = \frac{1}{\lambda^2}\left(\frac{d^2\alpha}{ds^2}\right).$$

Observations give by the Logarithmic Tables,

$$\log.\lambda = 4.0394622$$

and also

$$\log.\lambda^2 = \log.\lambda + \log.\frac{\lambda}{R}$$

R being the radius of the circle reduced to seconds; whence

$$\log.\lambda^2 = 2.2750444;$$

$\therefore$ if we reduce to seconds, the values of $\left(\frac{d\alpha}{ds}\right)$, and of $\left(\frac{d^2\alpha}{ds^2}\right)$, we shall have the logarithms of $\left(\frac{d\alpha}{dt}\right)$, and of $\left(\frac{d^2\alpha}{dt^2}\right)$ by taking from the logarithms of these values the logarithms of 4.039422, and 2.2750444. In like manner we get the logarithms of $\left(\frac{d\theta}{dt}\right)$, $\left(\frac{d^2\theta}{dt^2}\right)$, after subtracting the same logarithms, from the logarithms of their values reduced to seconds.

On the accuracy of the values of

$$\alpha,\ \left(\frac{d\alpha}{dt}\right),\ \left(\frac{d^2\alpha}{dt^2}\right),\ \theta,\ \left(\frac{d\theta}{dt}\right)\ \text{and}\ \left(\frac{d^2\theta}{dt^2}\right)$$

depends that of the following results; and since their formation is very simple, we must select and multiply observations so as to obtain them with the greatest exactness possible. We shall determine presently, by means of these values, the elements of the orbit of a Comet, and to generalize these results, we shall

496. *Investigate the motion of a system of bodies sollicited by any forces whatever.*

Let x, y, z be the rectangular coordinates of the first body; x′, y′, z′ those of the second body, and so on. Also let the first body be sollicited parallel to the axes of x, y, z by the forces X, Y, Z, which we shall suppose tend to diminish these variables. In like manner suppose the second body sollicited parallel to the same axes by the forces X′, Y′, Z′, and so on. The motions of all the bodies will be given by differential equations of the second order

$$0 = \frac{d^2 x}{dt^2} + X;\quad 0 = \frac{d^2 y}{dt^2} + Y;\quad 0 = \frac{d^2 z}{dt^2} + Z;$$

$$0 = \frac{d^2 x'}{dt^2} + X';\quad 0 = \frac{d^2 y'}{dt^2} + Y';\quad 0 = \frac{d^2 z'}{dt^2} + Z'.$$

&c. = &c.

If the number of the bodies is n, that of the equations will be 3 n; and their finite integrals will contain 6 n arbitrary constants, which will be the elements of the orbits of the different bodies.

To determine these elements by observations, we shall transform the coordinates of each body into others whose origin is at the place of the observer. Supposing, therefore, a plane to pass through the eye of the observer, and of which the situation is always parallel to itself, whilst the observer moves along a given curve, call r, r′ r″, &c. the distances of the observer from the different bodies, projected upon the plane; α, α', α'', &c. the apparent longitudes of the bodies, referred to the same plane, and θ, θ', θ'', &c. their apparent latitudes. The variables x, y, z will be given in terms of r, α, θ, and of the coordinates of the observer. In like manner, x′, y′, z′ will be given in functions of r′, α', θ', and of the coordinates of the observer, and so on. Moreover, if we suppose that the forces X, Y, Z; X′, Y′, Z′, &c. are due to the reciprocal action of the bodies of the system, and independent of attractions; they will be given in functions of r, r′, r″, &c.; α, α', α'', &c.; θ, θ', θ'', &c. and of known quantities. The preceding differential equations will thus involve these new variables and their first and second differences. But observations make known, for a given instant, the values of

$$\alpha,\ \left(\frac{d\alpha}{dt}\right),\ \left(\frac{d^2\alpha}{dt^2}\right),\ \theta,\ \left(\frac{d\theta}{dt}\right),\ \left(\frac{d^2\theta}{dt^2}\right);\ \alpha',\ \left(\frac{d\alpha'}{dt}\right),\ \&c.$$

There will hence of the unknown quantities only remain r, r′, r″, &c. and their first and second differences. These unknowns are in number 3 n, and since we have 3 n differential equations, we can determine them.

At the same time we shall have the advantage of presenting the first and second differences of r, r′, r″, &c. under a linear form.

The quantities α, θ, r, α', θ', r′, &c. and their first differences divided by d t, being known; we shall have, for any given instant, the values of x, y, z, x′, y′, z′, &c. and of their first differences divided by d t. If we substitute these values in the 3 n finite integrals of the preceding equations, and in the first differences of these integrals; we shall have 6 n equations, by means of which we shall be able to determine the 6 n arbitrary constants of the integrals, or the elements of the orbits of the different bodies.

497. To apply this method to the motion of the Comets,

We first observe that the principal force which actuates them is the attraction of the sun; compared with which all other forces may be neglected. If, however, the Comet should approach one of the greater planets so as to experience a sensible perturbation, the preceding method will still make known its velocity and distance from the earth; but this case happening but very seldom, in the following researches, we shall abstain from noticing any other than the action of the sun.

If the sun's mass be the unit, and its mean distance from the earth the unit of distance; if, moreover, we fix the origin of the coordinates x, y, z of a Comet, whose radius-vector is ϱ; the equations (0) of No. 475 will become, neglecting the mass of the Comet,

$$\left.\begin{aligned} 0 &= \frac{d^2x}{dt^2} + \frac{x}{\varrho^3}; \\ 0 &= \frac{d^2y}{dt^2} + \frac{y}{\varrho^3}; \\ 0 &= \frac{d^2z}{dt^2} + \frac{z}{\varrho^3}. \end{aligned}\right\} \quad (k)$$

Let the plane of x, y be the plane of the ecliptic. Also let the axis of x be the line drawn from the center of the sun to the first point of aries, at a given epoch; the axis of y the line drawn from the center of the sun to the first point of cancer, at the same epoch; and finally the positive values of z be on the same side as the north pole of the ecliptic. Next call x′, y′ the coordinates of the earth and R its radius-vector. This being supposed, transfer the coordinates x, y, z to others relative to the observer; and to do this let α be the geocentric longitude, and r its distance from the center of the earth projected upon the ecliptic; then we shall have

$$x = x' + r \cos. \alpha; \quad y = y + r \sin. \alpha; \quad z = r \tan. \theta.$$

If we multiply the first of equations (k) by sin. α, and take from the result the second multiplied by cos. α, we shall have

$$0 = \text{sin. } \alpha \frac{d^2 x}{dt^2} - \text{cos. } \alpha . \frac{d^2 y}{dt^2} + \frac{x \text{ sin. } \alpha - y \text{ cos. } \alpha}{\varrho^3};$$

whence we derive, by substituting for x, y their values given above,

$$0 = \text{sin. } \alpha . \frac{d^2 x'}{dt^2} - \text{cos. } \alpha . \frac{d^2 y'}{dt^2} + \frac{x' \text{ sin. } \alpha - y \text{ cos. } \alpha}{\varrho^3} - z . \left(\frac{dr}{dt}\right)\left(\frac{d\alpha}{dt}\right) - r\left(\frac{d^2\alpha}{dt^2}\right).$$

The earth being retained in its orbit like a comet, by the attraction of the sun, we have

$$0 = \frac{d^2 x'}{dt^2} + \frac{x'}{R^3}, \quad 0 = \frac{d^2 y'}{dt^2} + \frac{y'}{R^3};$$

which give

$$\text{sin. } \alpha \frac{d^2 x'}{dt^2} - \text{cos. } \alpha . \frac{d^2 y'}{dt^2} = \frac{y' \text{ cos. } \alpha - x' \text{ sin. } \alpha}{R^3}$$

We shall, therefore, have

$$0 = (y' \text{ cos. } \alpha - x' \text{ sin. } \alpha)\left\{\frac{1}{R^2} - \frac{1}{\varrho^3}\right\} - z\left(\frac{dr}{dt}\right).\left(\frac{d\alpha}{dt}\right) - r\left(\frac{d^2\alpha}{dt^2}\right).$$

Let A be the longitude of the earth seen from the sun; we shall have

$$x' = R \text{ cos. } A; \quad y' = R \text{ sin. } A;$$

therefore

$$y' \text{ cos. } \alpha - x' \text{ sin. } \alpha = R \text{ sin. } (A - \alpha);$$

and the preceding equation will give

$$\left(\frac{dr}{dt}\right) = \frac{R \text{ sin. } (A - \alpha)}{2\left(\frac{d\alpha}{dt}\right)} . \left\{\frac{1}{R^3} - \frac{1}{\varrho^3}\right\} - \frac{r . \left(\frac{d^2\alpha}{dt^2}\right)}{2\left(\frac{d\alpha}{dt}\right)} \quad . \quad . \quad . \quad (1)$$

Now let us seek a second expression for $\left(\frac{dr}{dt}\right)$. For this purpose we will multiply the first of equations (k) by tan. θ. cos. α, the second by tan. θ sin. α, and take the third equation from the sum of these two products; we shall thence obtain

$$0 = \text{tan. } \theta \left\{\text{cos. } \alpha \frac{d^2 x}{dt^2} + \text{sin. } \alpha \frac{d^2 y}{dt^2}\right\} + \text{tan. } \theta . \frac{x \text{ cos. } \alpha + y \text{ sin. } \alpha}{\varrho^3} - \frac{d^2 z}{dt^2} - \frac{z}{\varrho^3}.$$

This equation will become by substitution for x, y, z

$$0 = \text{tan. } \theta \left\{\left(\frac{d^2 x'}{dt^2} + \frac{x'}{\varrho^3}\right)\text{cos. } \alpha + \left(\frac{d^2 y'}{dt^2} + \frac{y'}{\varrho^3}\right)\text{sin. } \alpha\right\}$$

$$-\frac{2\left(\frac{d\theta}{dt}\right)\left(\frac{dr}{dt}\right)}{\cos.^2\theta}-r\left\{\frac{\left(\frac{d^2\theta}{dt^2}\right)}{\cos.^2\theta}+\frac{2\left(\frac{d\theta}{dt}\right)^2\sin.\theta}{\cos.^3\theta}+\left(\frac{d\alpha}{dt}\right)^2\tan.\theta\right\}$$

But

$$\left(\frac{d^2x'}{dt^2}+\frac{x'}{\varrho^3}\right)\cos.\alpha+\left(\frac{d^2y'}{dt^2}+\frac{y'}{\varrho^3}\right)\sin.\alpha=(x'\cos.\alpha+y'\sin.\alpha)\left(\frac{1}{\varrho^3}-\frac{1}{R^3}\right)$$

$$=R\cos.(A-\alpha)\left\{\frac{1}{\varrho^3}-\frac{1}{R^3}\right\};$$

Therefore,

$$\left(\frac{dr}{dt}\right)=-\tfrac{1}{2}r\left\{\frac{\left(\frac{d^2\theta}{dt^2}\right)}{\left(\frac{d\theta}{dt}\right)}+2\left(\frac{d\theta}{dt}\right)\tan.\theta+\frac{\left(\frac{d\alpha}{dt}\right)^2\sin.\theta\cos.\theta}{\left(\frac{d\theta}{dt}\right)}\right\}$$

$$+\frac{R\sin.\theta\cos.\theta\cos.(A-\alpha)}{2\left(\frac{d\theta}{dt}\right)}\cdot\left\{\frac{1}{\varrho^3}-\frac{1}{R^3}\right\}\cdot\ \ldots\ (2)$$

If we take this value of $\left(\frac{d\varrho}{dt}\right)$ from the first and suppose

$$\mu'=\frac{\left(\frac{d\alpha}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right)-\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\alpha}{dt^2}\right)+2\left(\frac{d\alpha}{dt}\right)\left(\frac{d\theta}{dt}\right)^2\tan.\theta+\left(\frac{d\alpha}{dt}\right)^3\sin.\theta\cos.\theta}{\left(\frac{d\alpha}{dt}\right)\sin.\theta\cos.\theta\cos.(A-\alpha)+\left(\frac{d\theta}{dt}\right)\sin.(A-\alpha)}$$

we shall have

$$r=\frac{R}{\mu'}\cdot\left\{\frac{1}{\varrho^3}-\frac{1}{R^3}\right\}\ \ldots\ (3)$$

The projected distance r of the comet from the earth, being always positive, this equation shows that the distance ϱ of the comet from the sun, is less or greater than the distance R of the sun from the earth, according as μ' is positive or negative; the two distances are equal if $\mu'=0$.

By inspection alone of a celestial globe, we can determine the sign of μ'; and consequently whether the comet is nearer to or farther from the Earth. For that purpose imagine a great circle which passes through two Geocentric positions of the Comet infinitely near to one another. Let γ be the inclination of this circle to the ecliptic, and λ the longitude of its ascending node; we shall have

$$\tan.\gamma\sin.(\alpha-\lambda)=\tan.\theta;$$

whence

$$d\theta\sin.(\alpha-\lambda)=d\alpha\sin.\theta\cos.\theta\cos.(\alpha-\lambda).$$

Differentiating, we have, also

$$0 = \left(\frac{d\alpha}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) - \left(\frac{d\theta}{dt}\right)\left(\frac{d^2\alpha}{dt^2}\right) + 2\left(\frac{d\alpha}{dt}\right)\left(\frac{d\theta}{dt}\right)^2 \tan.\theta + \left(\frac{d\alpha}{dt}\right)^3 \sin.\theta \cos.\theta;$$

$d^2\theta'$ being the value of $d^2\theta$, which would take place, if the apparent motion of the Comet continued in the great circle. The value of μ' thus becomes, by substituting for $d\theta$ its value

$$\frac{d\alpha \sin.\theta \cos.\theta \cos.(\alpha - \lambda)}{\sin.(\alpha - \lambda)},$$

$$\mu' = \frac{\left\{\left(\frac{d^2\theta}{dt^2}\right) - \left(\frac{d^2\theta'}{dt^2}\right)\right\} \sin.(\alpha - \lambda)}{\sin.\theta \cos.\theta \sin.(A - \lambda)}.$$

The function $\frac{\sin.(\alpha - \lambda)}{\sin.\theta \cos.\theta}$ is constantly positive; the value of μ is therefore positive or negative, according as $\left(\frac{d^2\theta}{dt^2}\right) - \left(\frac{d^2\theta'}{dt^2}\right)$ has the same or a different sign from that of sin. $(A - \lambda)$. But $A - \lambda$ is equal to two right angles plus the distance of the sun from the ascending node of the great circle. Whence it is easy to conclude that μ' will be positive or negative, according as in a third geocentric position of the comet, indefinitely near to the two first, the comet departs from the great circle on the same or the opposite side on which is the sun. Conceive, therefore, that we make a great circle of the sphere pass through the two geocentric positions of the comet; then according as, in a third consecutive geocentric position, the comet departs from this great circle, on the same side as the sun or on the opposite one, it will be nearer to or farther from the sun than the Earth. If it continues to appear in this great circle, it will be equally distant from both; so that the different deflections of its apparent path points out to us the variations of its distance from the sun.

To eliminate ϱ from equation (3), and to reduce this equation so as to contain no other than the unknown r, we observe that $\varrho^2 = x^2 + y^2 + z^2$ in substituting for x, y, z, their values in terms of

r, α, and θ;

and we have

$$\varrho^2 = x'^2 + y'^2 + 2r\{x' \cos.\alpha + y' \sin.\alpha\} + \frac{r^2}{\cos.^2\theta};$$

but we have

$$x' = R \cos.A,\ y' = R \sin.A;$$

$$\therefore \varrho^2 = \frac{r^2}{\cos.^2\theta} + 2Rr \cos.(A - \alpha) + R^2;$$

But

$$x' = R \cos. A ; \quad y' = R \sin. A$$

$$\therefore \varrho^2 = \frac{r^2}{\cos.^2 \theta} + 2 R r \cos. (A - \alpha) + R^2.$$

If we square the two members of equation (3) put under this form

$$\varrho^3 \{\mu' R^2 r + 1\} = R^3$$

we shall get, by substituting for ϱ^2,

$$\left\{\frac{r^2}{\cos.^2 \theta} + 2 R r \cos. (A - \alpha) + R^2\right\}^3 . \{\mu' R^2 r + 1\}^2 = R^6 \ldots (4)$$

an equation in which the only unknown quantity is r, and which will rise to the seventh degree, because a term of the first member being equal to R^6, the whole equation is divisible by r. Having thence determined r, we shall have $\left(\frac{dr}{dt}\right)$ by means of equations (1) and (2). Substituting, for example, in equation (1), for $\frac{1}{\varrho^3} - \frac{1}{R^3}$ its value $\frac{\mu' r}{R}$, given by equation (3); we shall have

$$\left(\frac{dr}{dt}\right) = -\frac{r}{2\left(\frac{d\alpha}{dt}\right)} . \left\{\left(\frac{d^2\alpha}{dt^2}\right) + \mu' \sin. (A - \alpha)\right\} .$$

The equation (4) is often susceptible of many real and positive roots; reducing it and dividing by r, its last term will be

$$2 R^5 \cos.^6 \theta \{\mu' R^3 + 3 \cos. (A - \alpha)\}.$$

Hence the equation in r being of the seventh degree or of an odd degree, it will have at least two real positive roots if $\mu' R^3 + 3 \cos. (A - \alpha)$ is positive; for it ought always, by the nature of the problem, to have one positive root, and it cannot then have an odd number of positive roots. Each real and positive value of r gives a different conic section, for the orbit of the comet; we shall, therefore, have as many curves which satisfy three near observations, as r has real and positive values; and to determine the true orbit of the comet, we must have recourse to a new observation.

498. The value of r, derived from equation (4) would be rigorously exact, if

$$\alpha, \left(\frac{d\alpha}{dt}\right), \left(\frac{d^2\alpha}{dt^2}\right), \theta, \left(\frac{d\theta}{dt}\right), \left(\frac{d^2\theta}{dt^2}\right)$$

were exactly known; but these quantities are only approximate. In fact, by the method above exposed, we can approximate more and more, merely by making use of a great number of observations, which presents the advantage of considering intervals sufficiently great, and of making the errors arising from observations compensate one another. But this

method has the analytical inconvenience of employing more than three observations, in a problem where three are sufficient. This may be obviated, and the solution rendered as approximate as can be wished by three observations only, after the following manner.

Let α and θ, representing the geocentric longitude and latitude of the intermediate; if we substitute in the equations (k) of the preceding No. instead of x, y, z their values $x' + r\cos.\alpha$; $y' + r\sin.\alpha$; and $r\tan.\theta$; they will give $\left(\frac{d^2 r}{dt^2}\right)$, $\left(\frac{d^2\alpha}{dt^2}\right)$ and $\left(\frac{d^2\theta}{dt^2}\right)$ in functions of r, α, and θ, of their first differences and known quantities. If we differentiate these, we shall have $\left(\frac{d^3 r}{dt^3}\right)$, $\left(\frac{d^3\alpha}{dt^3}\right)$ and $\left(\frac{d^3\theta}{dt^3}\right)$ in terms of r, α, θ, and of their first and second differences. Hence by equation (2) of 497 we may eliminate the second difference of r by means of its value and its first difference. Continuing to differentiate successively the values of $\left(\frac{d^3\alpha}{dt^3}\right)$, $\left(\frac{d^3\theta}{dt^3}\right)$, and eliminating the differences of α, and of θ superior to second differences, and all the differences of r, we shall have the values of

$$\left(\frac{d^3\alpha}{dt^3}\right),\quad \left(\frac{d^4\alpha}{dt^4}\right),\ \&c.$$

$$\left(\frac{d^3\theta}{dt^3}\right),\quad \left(\frac{d^4\theta}{dt^4}\right),\ \&c.\ \text{in terms of}$$

$$r,\ \alpha,\ \left(\frac{d\alpha}{dt}\right),\quad \left(\frac{d^2\alpha}{dt^2}\right),\ \theta,\ \left(\frac{d\theta}{dt}\right),\quad \left(\frac{d^2\theta}{dt^2}\right);$$

this being supposed, let

$$\alpha_{,},\ \alpha,\ \alpha',$$

be the three geocentric observed longitudes of the Comet; $\theta_{,}$, θ, θ' its three corresponding geocentric latitudes; let i be the number of days which separate the first from the second observation, and i' the interval between the second and third observation; lastly let λ be the arc which the earth describes in a day, by its mean sidereal motion; then by (481) we have

$$\alpha_{,} = \alpha - i.\lambda\left(\frac{d\alpha}{dt}\right) + \frac{i^2.\lambda^2}{1.2}\left(\frac{d^2\alpha}{dt^2}\right) - \frac{i^3\lambda^3}{1.2.3}\left(\frac{d^3\alpha}{dt^3}\right) + \&c.;$$

$$\alpha' = \alpha + i'.\lambda\left(\frac{d\alpha}{dt}\right) + \frac{i'^2.\lambda^2}{1.2}\left(\frac{d^2\alpha}{dt^2}\right) + \frac{i'^3.\lambda^3}{1.2.3}\left(\frac{d^3\alpha}{dt^3}\right) + \&c.,$$

$$\theta_{,} = \theta - i.\lambda\left(\frac{d\theta}{dt}\right) + \frac{i'^2.\lambda^2}{1.2}\left(\frac{d^2\theta}{dt^2}\right) - \frac{i^3\lambda^3}{1.2.3}\left(\frac{d^3\alpha}{dt^3}\right) + \&c.,$$

$$\theta' = \theta + i'.\lambda\left(\frac{d\theta}{dt}\right) + \frac{i'^2.\lambda^2}{1.2}\left(\frac{d^2\theta}{dt^2}\right) + \frac{i'^3.\lambda^3}{1.2.3}\left(\frac{d^3\theta}{dt^3}\right) + \&c.$$

If we substitute in these series for

$$\left(\frac{d^3\alpha}{dt^3}\right),\ \left(\frac{d^4\alpha}{dt^4}\right)\text{ &c. }\left(\frac{d^3\theta}{dt^3}\right),\ \left(\frac{d^4\theta}{dt^4}\right),\text{ &c.}$$

their values obtained above, we shall have four equations between the five unknown quantities

$$r, \left(\frac{d\alpha}{dt}\right),\ \left(\frac{d^2\alpha}{dt^2}\right),\ \left(\frac{d\theta}{dt}\right),\ \left(\frac{d^2\theta}{dt^2}\right).$$

These equations will be the more exact in proportion as we consider a greater number of terms in the series. We shall thus have

$$\left(\frac{d\alpha}{dt}\right),\ \left(\frac{d^2\alpha}{dt^2}\right),\ \left(\frac{d\theta}{dt}\right),\ \left(\frac{d^2\theta}{dt^2}\right)$$

in terms of r and known quantities; and substituting in equation (4) of the preceding No. it will contain the unknown r only. As to the rest, this method, which shows how to approximate to r by employing three observations only, would require in practice, laborious calculations, and it is a more exact and simple process to consider a greater number of observations by the method of No. 495.

499. When the values of r and $\left(\frac{dr}{dt}\right)$ shall be determined, we shall have those of

$$x, y, z, \left(\frac{dx}{dt}\right),\ \left(\frac{dy}{dt}\right)\text{ and }\left(\frac{dz}{dt}\right),$$

by means of the equations

$$x = R\cos. A + r\cos.\alpha$$
$$y = R\sin. A + r\sin.\alpha$$
$$z = r\tan.\theta$$

and of their differentials divided by d t, viz.

$$\left(\frac{dx}{dt}\right) = \left(\frac{dR}{dt}\right)\cos. A - R\left(\frac{dA}{dt}\right)\sin. A + \left(\frac{dr}{dt}\right)\cos.\alpha - r\left(\frac{d\alpha}{dt}\right)\sin.\alpha;$$

$$\left(\frac{dy}{dt}\right) = \left(\frac{dR}{dt}\right)\sin. A + R\left(\frac{dA}{dt}\right)\cos. A + \left(\frac{dr}{dt}\right)\sin.\alpha + r\left(\frac{d\alpha}{dt}\right)\cos.\alpha;$$

$$\left(\frac{dz}{dt}\right) = \left(\frac{dr}{dt}\right)\tan.\theta + \frac{r\left(\frac{d\theta}{dt}\right)}{\cos.^2\theta}.$$

The values of $\left(\frac{dA}{dt}\right)$ and of $\left(\frac{dR}{dt}\right)$ are given by the Theory of the motion of the Earth:

To facilitate the investigation let E be the excentricity of the earth's

orbit, and H the longitude of its perihelion; then by the nature of elliptical motion we have

$$\left(\frac{d\,A}{d\,t}\right) = \frac{\sqrt{(1-E^2)}}{R^2}; \quad -R = \frac{1-E^2}{1+E\cos.(A-H)}.$$

These two equations give

$$\left(\frac{d\,R}{d\,t}\right) = \frac{E\sin.(A-H)}{\sqrt{(1-E^2)}}.$$

Let R′ be the radius-vector of the earth corresponding to the longitude A of this planet augmented by a right angle; we shall have

$$R' = \frac{1-E^2}{1-E\sin.(A-H)};$$

whence is derived

$$E\sin.(A-H) = \frac{R'-1+E^2}{R'};$$

$$\therefore \left(\frac{d\,R}{d\,t}\right) = \frac{R'+E^2-1}{R'-\sqrt{(1-E^2)}}.$$

If we neglect the square of the excentricity of the earth's orbit, which is very small, we shall have

$$\left(\frac{d\,A}{d\,t}\right) = \frac{1}{R^2}; \quad \left(\frac{d\,R}{d\,t}\right) = R'-1;$$

the preceding values of $\left(\frac{d\,x}{d\,t}\right)$ and $\left(\frac{d\,y}{d\,t}\right)$ will hence become

$$\left(\frac{d\,x}{d\,t}\right) = (R'-1)\cos.A - \frac{\sin.A}{R} + \left(\frac{d\,r}{d\,t}\right)\cos.\alpha - r\left(\frac{d\,\alpha}{d\,t}\right)\sin.\alpha;$$

$$\left(\frac{d\,y}{d\,t}\right) = (R'-1)\sin.A + \frac{\cos.A}{R} + \left(\frac{d\,r}{d\,t}\right)\sin.\alpha + r\left(\frac{d\,\alpha}{d\,t}\right)\cos.\alpha;$$

R, R′, and A being given immediately by the tables of the sun, the estimate of the six quantities x, y, z, $\left(\frac{d\,x}{d\,t}\right)$, $\left(\frac{d\,y}{d\,t}\right)$, $\left(\frac{d\,z}{d\,t}\right)$ will be easy when r and $\left(\frac{d\,r}{d\,t}\right)$ shall be known. Hence we derive the elements of the orbit of the comet after this mode.

The indefinitely small sector, which the projection of the radius-vector and the comet upon the plane of the ecliptic describes during the element of time d t, is $\frac{x\,d\,y - y\,d\,x}{2}$; and it is evident that this sector is positive or negative, according as the motion of the comet is direct or retrograde. Thus in forming the quantity $x\left(\frac{d\,y}{d\,t}\right) - y\left(\frac{d\,x}{d\,t}\right)$, it will indicate by its sign, the direction of the motion of the comet.

To determine the position of the orbit, call φ its inclination to the ecliptic, and I the longitude of the node, which would be ascending if the motion of the comet were direct or progressive. We shall have

$$z = y \cos. I \tan. \varphi - x \sin. I \tan. \varphi$$

These two equations give

$$\tan. I = \frac{y\left(\frac{dz}{dt}\right) - z\left(\frac{dy}{dt}\right)}{x\left(\frac{dz}{dt}\right) - z\left(\frac{dx}{dt}\right)};$$

$$\tan. \varphi = \frac{y\left(\frac{dz}{dt}\right) - z\left(\frac{dy}{dt}\right)}{\sin. I\left\{x\left(\frac{dy}{dt}\right) - y\left(\frac{dx}{dt}\right)\right\}}.$$

Wherein since φ ought always to be positive and less than a right angle, the sign of sin. I is known. But the tangent of I and the sign of its sine being determined, the angle I is found completely. This angle is the longitude of the ascending node of the orbit, if the motion is progressive; but to this we must add two right angles, in order to get the longitude of the node when the motion is retrograde. It would be more simple to consider only progressive motions, by making vary φ, the inclination of the orbits, from zero to two right angles; for it is evident that then the retrograde motions correspond to an inclination greater than a right angle.

In this case, tan. φ has the same sign as $x\left(\frac{dy}{dt}\right) - y\left(\frac{dx}{dt}\right)$, which will determine sin. I, and consequently the angle I, which always expresses the longitude of the ascending node.

If a, a e be the semi-axis major and the excentricity of the orbit, we have (by 492) in making m = 1,

$$\frac{1}{a} = \frac{2}{\varrho} - \left(\frac{dx}{dt}\right) - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2;$$

$$a(1 - e^2) = 2\varrho - \frac{\varrho^2}{a} - \left\{x\left(\frac{dx}{dt}\right) + y\left(\frac{dy}{dt}\right) + z\left(\frac{dz}{dt}\right)\right\}^2.$$

The first of these equations gives the semi-axis major, and the second the excentricity. The sign of the function $x\left(\frac{dx}{dt}\right) + y\left(\frac{dy}{dt}\right) + z\left(\frac{dz}{dt}\right)$ shows whether the comet has already passed its perihelion; for it approaches if this function is negative; and in the contrary case, the comet recedes from that point.

Let T be the interval of time comprised between the epoch and passage of the comet over the perihelion; the two first of equations (f) (480) will give, observing that m being supposed unity we have $n = a^{-\frac{3}{2}}$,

$$\varrho = a\,(1 - e\cos.\,u)$$

$$T = a^{\frac{3}{2}}\,(u - e\cos.\,u).$$

The first of these equations gives the angle u, and the second T. This time added to or subtracted from the epoch, according as the comet approaches or leaves its perihelion, will give the instant of its passage over this point. The values of x, y, determine the angle which the projection of the radius-vector ϱ makes with the axis of x; and since we know the angle I, formed by this axis and by the line of the nodes, we shall have the angle which this last line forms with the projection of ϱ; whence we derive by means of the inclination φ of the orbit, the angle formed by the line of the nodes and the radius ϱ. But the angle u being known, we shall have by means of the third of the equations (f), the angle v which this radius forms with the line of the apsides; we shall therefore have the angle comprised between the two lines of the apsides and of the nodes, and consequently, the position of the perihelion. All the elements of the orbit will thus be determined.

500. These elements are given, by the preceding investigations, in terms of r, $\left(\frac{dr}{dt}\right)$ and known quantities; and since $\left(\frac{dr}{dt}\right)$ is given in terms of r by No. 497, the elements of the orbit will be functions of r and known quantities. If one of them were given, we should have a new equation, by means of which we might determine r; this equation would have a common divisor with equation (4) of No. 497; and seeking this divisor by the ordinary methods, we shall obtain an equation of the first degree in terms of r; we should have, moreover, an equation of condition between the data of the observations, and this equation would be that which ought to subsist, in order that the given element may belong to the orbit of the comet.

Let us apply this consideration to the case of nature. First suppose that the orbits of the comets are ellipses of great excentricity, and are nearly parabolas, in the parts of their orbits in which these stars are visible. We may therefore without sensible error suppose $a = \infty$, and consequently $\frac{1}{a} = 0$; the expression for $\frac{1}{a}$ of the preceding No. will therefore give

$$0 = \frac{2}{\varrho} - \frac{dx^2 + dy^2 + dz^2}{dt^2}.$$

If we then substitute for $\left(\frac{dx}{dt}\right)$, $\left(\frac{dy}{dt}\right)$ and $\left(\frac{dz}{dt}\right)$ their values found in the same No., we shall have after all the reductions and neglecting the square of $R' - 1$,

$$\begin{aligned} 0 = {} & \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\alpha}{dt}\right)^2 + \left\{\frac{dr}{dt}\tan.\theta + \frac{r\left(\frac{d\theta}{dt}\right)}{\cos.^2\theta}\right\}^2 \\ & + 2\left(\frac{dr}{dt}\right).\left\{(R'-1)\cos.(A-\alpha)\frac{\sin.(A-\alpha)}{R}\right\} \cdots (5) \\ & + 2r\left(\frac{d\alpha}{dt}\right)\left\{(R'-1)\sin.(A-\alpha) + \frac{\cos.(A-\alpha)}{R}\right\} \\ & + \frac{1}{R^2} - \frac{2}{\varrho}. \end{aligned}$$

Substituting in this equation for $\left(\frac{dr}{dt}\right)$ its value

$$-\frac{r}{2\left(\frac{d\alpha}{dt}\right)}.\left\{\left(\frac{d^2\alpha}{dt^2}\right) + \mu'\sin.(A-\alpha)\right\},$$

found in No. 497, and then making

$$4\left(\frac{d\alpha}{dt}\right)^2.B = 4\left(\frac{d\alpha}{dt}\right)^4 + \left\{\left(\frac{d^2\alpha}{dt^2}\right) + \mu'\sin.(A-\alpha)\right\}^2$$

$$+\left\{\tan.\theta.\left(\frac{d^2\alpha}{dt^2}\right) + \mu'\tan.\theta\sin.(A-\alpha) - \frac{2\left(\frac{d\alpha}{dt}\right).\left(\frac{d\theta}{dt}\right)}{\cos.^2\theta}\right\}^2,$$

and

$$C = \frac{\left\{\left(\frac{d^2\alpha}{dt^2}\right) + \mu'\sin.(A-\alpha)\right\}}{\left(\frac{d\alpha}{dt}\right)}.\left\{\frac{\sin.(A-\alpha)}{R} - (R'-1)\cos.(A-\alpha)\right\}$$

$$+ 2\left(\frac{d\alpha}{dt}\right)\left\{(R'-1)\sin.(A-\alpha) + \frac{\cos.(A-\alpha)}{R}\right\},$$

we shall have

$$0 = Br^2 + Cr + \frac{1}{R^2} - \frac{2}{\varrho}$$

and consequently

$$\varrho^2\left\{Br^2 + Cr + \frac{1}{R^2}\right\}^2 = 4.$$

This equation rising only to the sixth degree, is in that respect, more

simple than equation (4) of No. (497); but it belongs to the parabola alone, whereas the equation (4) equally regards every species of conic section.

501. We perceive by the foregoing investigation, that the determination of the parabolic orbits of the comets, leads to more equations than unknown quantities; and that, therefore, in combining these equations in different ways, we can form as many different methods of calculating the orbits. Let us examine those which appear to give the most exact results, or which seem least susceptible of the errors of observations.

It is principally upon the values of the second differences $\left(\frac{d^2\alpha}{dt^2}\right)$ and $\left(\frac{d^2\theta}{dt^2}\right)$, that these errors have a sensible influence. In fact, to determine them, we must take the finite differences of the geocentric longitudes and latitudes of the comet, observed during a short interval of time. But these differences being less than the first differences, the errors of observations are a greater aliquot part of them; besides this, the formulas of No. 495 which determine, by the comparison of observations, the values of α, θ, $\left(\frac{d\alpha}{dt}\right)$, $\left(\frac{d\theta}{dt}\right)$, $\left(\frac{d^2\alpha}{dt^2}\right)$ and $\left(\frac{d^2\theta}{dt^2}\right)$ give with greater precision the four first of these quantities than the two last. It is, therefore, desirable to rest as little as possible upon the second differences of α and θ; and since we cannot reject both of them together, the method which employs the greater, ought to give the more accurate results. This being granted let us resume the equations found in Nos. 497, &c.

$$\varrho^2 = \frac{r^2}{\cos.^2\theta} + 2\,R\,r\cos.(A-\alpha) + R^2;$$

$$\left(\frac{dr}{dt}\right) = \frac{R\sin.(A-\alpha)}{2\left(\frac{d\alpha}{dt}\right)}\cdot\left\{\frac{1}{R^3} - \frac{1}{\varrho^3}\right\} - \frac{r\cdot\left(\frac{d^2\alpha}{dt^2}\right)}{2\left(\frac{d\alpha}{dt}\right)}, \quad \text{(L)}$$

$$\left(\frac{dr}{dt}\right) = -\frac{1}{2}r\left\{\frac{\left(\frac{d^2\theta}{dt^2}\right)}{\left(\frac{d\theta}{dt}\right)} + 2\left(\frac{d\theta}{dt}\right)\tan.\theta + \frac{\left(\frac{d\alpha}{dt}\right)^2\sin.\theta\cos.\theta}{\left(\frac{d\theta}{dt}\right)}\right\}$$
$$+ \frac{R\sin.\theta\cos.\theta\cos.(A-\alpha)}{2\left(\frac{d\theta}{dt}\right)}\cdot\left\{\frac{1}{\varrho^3} - \frac{1}{R^3}\right\}$$

$$0 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\alpha}{dt}\right)^2 + \left\{\left(\frac{dr}{dt}\right)\tan.\theta + \frac{r\left(\frac{d\theta}{dt}\right)}{\cos.^2\theta}\right\}^2$$

$$+ 2\left(\frac{dr}{dt}\right).\left\{(R' - 1)\cos.(A - \alpha) - \frac{\sin.(A - \alpha)}{R}\right\}$$

$$+ 2r\left(\frac{d\alpha}{dt}\right)\left\{(R' - 1)\sin.(A - \alpha) + \frac{\cos.(A - \alpha)}{R}\right\}$$

$$+ \frac{1}{R^2} - \frac{2}{\varrho}.$$

If we wish to reject $\left(\frac{d^2\theta}{dt^2}\right)$, we consider only the first, second and fourth of those equations. Eliminating $\left(\frac{dr}{dt}\right)$ from the last by means of the second, we shall form an equation which cleared of fractions, will contain a term multiplied by $\varrho^6 r^2$, and other terms affected with even and odd powers of r and ϱ. If we put into one side of the equation all the terms affected with even powers of ϱ, and into the other all those which involve its odd powers, and square both sides, in order to have none but even powers of ϱ, the term multiplied by $\varrho^6 r^2$ will produce one multiplied by $\varrho^{12} r^4$. Substituting, therefore, instead of ϱ^2, its value given by the first of equations (L), we shall have a final equation of the sixteenth degree in r. But instead of forming this equation in order afterwards to resolve it, it will be more simple to satisfy by trial the three preceding ones.

If we wish to reject $\left(\frac{d^2a}{dt^2}\right)$, we must consider the first, third and fourth of equations (L). These three equations conduct us also to a final equation of the sixteenth degree in r; and we can easily satisfy by trial.

The two preceding methods appear to be the most exact, which we can employ in the determination of the parabolic orbits of the comets. It is at the same time necessary to have recourse to them, if the motion of the comet in longitude or latitude is insensible, or too small for the errors of observations sensibly to alter its second difference. In this case, we must reject that of the equations (L), which contains this difference. But although in these methods, we employ only three equations, yet the fourth is useful to determine amongst all the real and positive values of r, which satisfy the system of three equations, that which ought to be selected.

502. The elements of the orbit of a comet, determined by the above process, would be exact, if the values of α, θ and their first and second differences, were rigorous; for we have regarded, after a very simple manner, the excentricity of the terrestrial orbit, by means of the radius-vector R′ of the earth, corresponding to its true anomaly + a right angle; we are therefore permitted only to neglect the square of this excen-

tricity, as too small a fraction to produce by its omission a sensible influence upon the results. But θ, α and their differences, are always susceptible of any degree of inaccuracy, both because of the errors of observations, and because these differences are only obtained approximately. It is therefore necessary to correct the elements, by means of three distant observations, which can be done in many ways; for if we know nearly, two quantities relative to the motion of a comet, such that the radius-vector corresponding to two observations, or the position of the node, and the inclination of the orbit; calculating the observations, first with these quantities and afterwards with others differing but little from them, the law of the differences between the results, will easily show the necessary corrections. But amongst the combinations taken two and two, of the quantities relative to the motion of comets, there is one which ought to produce greatest simplicity, and which for that reason should be selected. It is of importance, in fact, in a problem so intricate, and complicated, to spare the calculator all superfluous operations. The two elements which appear to present this advantage, are the perihelion distance, and the instant when the comet passes this point. They are not only easy to be derived from the values of r and $\left(\frac{d\,r}{d\,t}\right)$; but it is very easy to correct them by observations, without being obliged for every variation which they undergo, to determine the other corresponding elements of the orbit.

Resuming the equation found in No. 492

$$a\,(1 - e^2) = 2\,\varrho - \frac{\varrho^2}{a} - \frac{\varrho^2 . d\,\varrho^2}{d\,t^2};$$

$a\,(1 - e^2)$ is the semi-parameter of the conic section of which a is the semi axis-major, and a e the excentricity. In the parabola, where a is infinite, and e equal to unity, $a\,(1 - e^2)$ is double the perihelion distance: let D be this distance: the preceding equation becomes relatively to this curve

$$D = \varrho - \frac{1}{2}\left(\frac{\varrho\,d\,\varrho}{d\,t}\right)^2.$$

$\frac{\varrho\,d\,\varrho}{d\,t}$ is equal to $\frac{\frac{1}{2}\,d\,\varrho^2}{d\,t^2}$; in substituting for ϱ^2 its value $\frac{r^2}{\cos.^2\theta} + 2\,R\,r \times \cos.\,(A - \alpha) + R^2$, and for $\left(\frac{d\,R}{d\,t}\right)$ and $\left(\frac{d\,A}{d\,t}\right)$, their values found in No. 499, we shall have

$$\frac{\varrho\,d\,\varrho}{d\,t} = \frac{r}{\cos.^2\theta}\left\{\left(\frac{d\,r}{d\,t}\right) + r\left(\frac{d\,\theta}{d\,t}\right)\tan.\,\theta\right\} + R\left(\frac{d\,r}{d\,t}\right)\cos.\,(A - \alpha)$$

$$+ r\left\{(R' - 1)\cos.(A - \alpha) - \frac{\sin.(A - \alpha)}{R}\right\}$$

$$+ r R \left(\frac{d\alpha}{dt}\right) \sin.(A - \alpha) + R(R' - 1).$$

Let P represent this quantity; if it is negative, the radius-vector decreases, and consequently, the comet tends towards its perihelion. But it goes off into the distance, if P is negative. We have then

$$D = \varrho - \frac{1}{2} P^2;$$

the angular distance v of the comet from its perihelion, will be determined from the polar equation to the parabola,

$$\text{cor.}^2 \frac{1}{2} v = \frac{D}{\varrho};$$

and finally we shall have the time employed to describe the angle v, by the table of the motion of the comets. This time added to or subtracted from that of the epoch, according as P is negative or positive, will give the instant when the comet passes its perihelion.

503. Recapitulating these different results, we shall have the following method to determine the parabolic orbits of the comets.

General method of determining the orbits of the comets.

This method will be divided into two parts; in the first, we shall give the means of obtaining approximately, the perihelion distance of the comet and the instant of its passage over the perihelion; in the second, we shall determine all the elements of the orbit on the supposition that the former are known.

Approximate determination of the Perihelion distance of the comet, and the instant of its passage over the perihelion.

We shall select three, four, five, &c. observations of the comet equally distant from one another as nearly as possible; with four observations we shall be able to consider an interval of 30°; with five, an interval of 36°, or 40° and so on for the rest; but to diminish the influence of their errors, the interval comprised between the observations must be greater, in proportion as their number is greater. This being supposed,

Let β, β', β'', &c. be the successive geocentric longitudes of the comet, γ, γ', γ'' the corresponding latitudes, these latitudes being supposed positive or negative according as they are north or south. We shall divide the difference $\beta' - \beta$, by the number of days between the first and second observation; we shall divide in like manner the difference $\beta'' - \beta'$ by the

number of days between the second and third observation; and so on. Let $\delta\beta$, $\delta\beta'$, $\delta\beta''$, &c. be these quotients.

We next divide the difference $\delta\beta' - \delta\beta$ by the number of days between the first observation and the third; we divide, in like manner, the difference $\delta\beta'' - \delta\beta'$ by the number of days between the second and fourth observations; similarly we divide the difference $\delta\beta''' - \delta\beta''$ by the number of days between the third and fifth observation, and so on. Let $\delta^2\beta$, $\delta^2\beta'$, $\delta^2\beta''$, &c. denote these quotients.

Again, we divide the difference $\delta^2\beta' - \delta^2\beta$ by the number of days which separate the first observation from the fourth; we divide in like manner $\delta^2\beta'' - \delta^2\beta'$ by the number of days between the second observation and the fifth, and so on. Make $\delta^3\beta$, $\delta^3\beta'$, &c. these quotients. Thus proceeding, we shall arrive at $\delta^{n-1}\beta$, n being the number of observations employed.

This being done, we proceed to take as near as may be a mean epoch between the instants of the two extreme observations, and calling i, i', i'', &c. the number of days, distant from each observation, i, i', i'', &c. ought to be supposed negative for the observations made prior to this epoch; the longitude of the comet, after a small number z of days reckoned from the Epoch will be expressed by the following formula:

$$\left\{\begin{array}{l}\beta - \mathrm{i}\,\delta\beta + \mathrm{i}\,\mathrm{i}'\,\delta^2\beta - \mathrm{i}\,\mathrm{i}'\,\mathrm{i}''\,\delta^3\beta + \&c. \\ +z\{\delta\beta - (\mathrm{i}+\mathrm{i}')\delta^2\beta + (\mathrm{i}\,\mathrm{i}'+\mathrm{i}\,\mathrm{i}''+\mathrm{i}'\,\mathrm{i}'')\delta^3\beta - (\mathrm{i}\,\mathrm{i}'\,\mathrm{i}''+\mathrm{i}\,\mathrm{i}'\,\mathrm{i}'''+\mathrm{i}\,\mathrm{i}''\,\mathrm{i}'''+\ldots\ (p) \\ \mathrm{i}'\,\mathrm{i}''\,\mathrm{i}''')\,\delta^4\beta + \&c.\} \\ +z^2\{\delta^2\beta - (\mathrm{i}+\mathrm{i}'+\mathrm{i}'')\,\delta^3\beta + (\mathrm{i}\,\mathrm{i}'+\mathrm{i}\,\mathrm{i}''+\mathrm{i}\,\mathrm{i}'''+\mathrm{i}'\,\mathrm{i}'''+\mathrm{i}'\,\mathrm{i}''+\mathrm{i}'\,\mathrm{i}''')\,\delta^4\beta - \&c.\}\end{array}\right.$$

The coefficients of $-\delta\beta$, $+\delta^2\beta$, $-\delta^3\beta$, &c. in the part independent of z are 1st the numbers i and i', secondly the sum of the products two and two of the three numbers i, i', i''; thirdly the sum of the products three and three, of the four numbers i, i', i'', i''', &c.

The coefficients of $-\delta^3\beta$, $+\delta^4\beta$, $-\delta^5\beta$, &c. in the part multiplied by z^2, are first, the sum of the three numbers i, i', i''; secondly of the products two and two of the four numbers i, i', i'', i'''; thirdly the sum of the products three and three of the five numbers i, i', i'', i''', i'''', &c.

Instead of forming these products, it is as simple to develope the function $\beta + (z - \mathrm{i})\,\delta\beta + (z - \mathrm{i})(z - \mathrm{i}')\,\delta^2\beta + (z - \mathrm{i})(z - \mathrm{i}')(z - \mathrm{i}'') \times \delta^3\beta + \&c.$ rejecting the powers of z superior to the square. This gives the preceding formula.

If we operate in a similar manner upon the observed geocentric latitudes of the comet; its geocentric latitude, after the number z of days from the epoch, will be expressed by the formula (p) in changing β into γ. Call (q) the equation (p) thus altered. This being done,

α will be the part independent of z in the formula (p); and θ that in the formula (q).

Reducing into seconds the coefficient of z in the formula (p), and taking from the tabular logarithm of this number of seconds, the logarithm 4,0394622, we shall have the logarithm of a number which we shall denote by a.

Reducing into seconds the coefficients of z^2 in the same formula, and taking from the logarithm of this number of seconds, the logarithm 1.9740144, we shall have the logarithm of a number, which we shall denote by b.

Reducing in like manner into seconds the coefficients of z and z^2 in the formula (q) and taking away respectively from the logarithms of these numbers of seconds, the logarithms, 4,0394622 and 1,9740144, we shall have the logarithms of two numbers, which we shall name h and l.

Upon the accuracy of the values of a, b, h, l, depends that of the method; and since their formation is very simple, we must select and multiply observations, so as to obtain them with all the exactness which the observations will admit of. It is perceptible that these values are only the quantities $\left(\frac{d\alpha}{dt}\right)$, $\left(\frac{d^2\alpha}{dt^2}\right)$, $\left(\frac{d\theta}{dt}\right)$, $\left(\frac{d^2\theta}{dt^2}\right)$, which we have expressed more simply by the above letters.

If the number of observations is odd, we can fix the Epoch at the instant of the mean observation; which will dispense with calculating the parts independent of z in the two preceding formulas; for it is evident, that then these parts are respectively equal to the longitude and latitude of the mean observation.

Having thus determined the values of α, a, b, θ, h, and l, we shall determine the longitude of the sun, at the instant we have selected for the epoch, R the corresponding distance of the Earth from the sun, and R′ the distance which answers to E augmented by a right angle. We shall have the following equations

$$\varrho^2 = \frac{x^2}{\cos.^2\theta} - 2\,R\,x\cos.(E-\alpha) + R^2 \quad \ldots\ldots\ldots \quad (1)$$

$$y = R\,\frac{\sin.(E-\alpha)}{2a}\left\{\frac{1}{\varrho^3} - \frac{1}{R^3}\right\} - \frac{b\,x}{2a} \quad \ldots\ldots\ldots \quad (2)$$

$$\left.\begin{aligned} y = -x\left\{h\tan.\theta + \frac{1}{2h} + \frac{a^2\sin.\theta.\cos.\theta}{2h}\right\} \\ + \frac{R\sin.\theta\cos.\theta}{2h}\cos.(E-\alpha)\left\{\frac{1}{R^3} - \frac{1}{\varrho^3}\right\} \end{aligned}\right\} \quad \ldots\ldots \quad (3)$$

$$0 = y^2 + a^2x^2 + \left(y\tan.\theta + \frac{h\,x}{\cos.^2\theta}\right)^2 + 2\,y\left\{\frac{\sin.(E-\alpha)}{R}\right.$$

$$- (R' - 1) \cos. (E - \alpha)\Big\} - 2 a x \Big\{(R' - 1) \sin. (E - \alpha) + \frac{\cos. (E - \alpha)}{R}\Big\} + \frac{1}{R^2} - \frac{2}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To derive from these equations the values of the unknown quantities x, y, ϱ, we must consider, signs being neglected, whether b is greater or less than l. In the first case we shall make use of equation (1), (2), and (4). We shall form a first hypothesis for x, supposing it for instance equal to unity; and we then derive by means of equations (1), (2), the values of ϱ and of y. Next we substitute these values in the equation (4); and if the result is 0, this will be a proof that the value of x has been rightly chosen. But if it be negative we must augment the value of x, and diminish it if the contrary. We shall thus obtain, by means of a small number of trials the values of x, y and ϱ. But since these unknown quantities may be susceptible of many real and positive values, we must seek that which satisfies exactly or nearly so the equation (3).

In the second case, that is to say, if l be greater than b, we shall use the equations (1), (3), (4), and then equation (2) will give the verification.

Having thus the values of x, y, ϱ, we shall have the quantity

$$P = \frac{x}{\cos.^2 \theta} \cdot \{y + h x \tan. \theta\} - R y \cos. (E - \alpha) + x \Big\{\frac{\sin. (E - \alpha)}{R} - (R' - 1) \cos. (E - \alpha)\Big\} - R a x \sin (E - \alpha) + R . (R' - 1).$$

The Perihelion distance D of the comet will be

$$D = \varrho - \frac{1}{2} P^2;$$

the cosine of its anomaly v will be given by the equation

$$\cos^2 \frac{1}{2} v = \frac{D}{\varrho};$$

and hence we obtain, by the table of the motion of the comets, the time employed to describe the angle v. To obtain the instant when the comet passes the perihelion, we must add this time to, or subtract it from the epoch according as P is negative or positive. For in the first case the comet approaches, and in the second recedes from, the perihelion.

Having thus nearly obtained the perihelion distance of the comet, and the instant of its passage over the perihelion; we are enabled to correct them by the following method, which has the advantage of being independent of the approximate values of the other elements of the orbit.

An exact Determination of the elements of the orbit, when we know approximate values of the perihelion distance of the comet, and of the instant of its passage over the perihelion.

We shall first select three distant observations of the comet; then taking the perihelion distance of the comet, and the instant of its crossing the perihelion, determined as above, we shall calculate the three anomalies of the comet and the corresponding radius-vectors corresponding to the instants of the three observations. Let v, v′, v″ be these anomalies, those which precede the passage over the perihelion being supposed negative. Also let ϱ, ϱ' ϱ'' be the corresponding radius-vectors of the comet; then v′ — v, v″ — v will be the angles comprised by ϱ and ϱ' and by ϱ, ϱ''. Let U be the first of these angles, U′ the second. Again, call α, α' α'' the three observed geocentric longitudes of the comet, referred to a fixed equinox; θ, θ', θ'' its three geocentric latitudes, the south latitudes being negative. Let β, β', β'' be the three corresponding heliocentric longitudes and ϖ, ϖ', ϖ'', its three heliocentric latitudes. Lastly call E, E′, E″ the three corresponding longitudes of the sun, and R, R′, R″ its three distances to the center of the earth.

Conceive that the letter S indicates the center of the sun, T that of the earth, and C that of the comet, C′ that of its projection upon the plane of the ecliptic. The angle S T C′ is the difference of the geocentric longitudes of the sun and of the comet. Adding the logarithm of the cosine of this angle, to the logarithm of the cosine of the geocentric latitude of the comet, we shall have the logarithm of the cosine of the angle S T C. We know, therefore, in the triangle S T C, the side S T or R, the side S C or ϱ, and the angle S T C, to find the angle C S T. Next we shall have the heliocentric latitude ϖ of the comet, by means of the equation

$$\sin. \varpi = \frac{\sin. \theta \sin. \text{C S T}}{\sin. \text{C T S}}.$$

The angle T S C′ is the side of a spherical right angled triangle, of which the hypothenuse is the angle T S C, and of which one of the sides is the angle ϖ. Whence we shall easily derive the angle T S C′, and consequently the heliocentric longitude β of the comet.

We shall have after the same manner ϖ', β'; ϖ'', β''; and the values of β, β', β'' will show whether the motion of the comet be direct or retrograde.

If we imagine the two arcs of latitude ϖ, ϖ', to meet at the pole of the ecliptic, they would make there an angle equal to $\beta' - \beta$; and in the

spherical triangle formed by this angle, and by the sides $\frac{\pi}{2} - \varpi$, $\frac{\pi}{2} - \varpi'$ π being the semi-circumference, the side opposite to the angle $\beta' - \beta$ will be the angle at the sun comprised between the radius-vectors ϱ, and ϱ'. We shall easily determine this by spherical Trigonometry, or by the formula

$$\sin.^2 \frac{1}{2} V = \cos.^2 \frac{1}{2} (\varpi + \varpi') - \cos^2 \frac{1}{2} (\beta' - \beta) \cos. \varpi \cos. \varpi',$$

in which V represents this angle; so that if we call A the angle of which the sine squared is

$$\cos^2 \frac{1}{2} (\beta' - \beta) \cos. \varpi . \cos. \varpi',$$

and which we shall easily find by the tables, we shall have

$$\sin.^2 \frac{1}{2} V = \cos. \left(\frac{1}{2} \varpi + \frac{1}{2} \varpi' + A\right) \cos. \left(\frac{1}{2} \varpi + \frac{1}{2} \varpi' - A\right).$$

If in like manner we call V′ the angle formed by the two radius-vectors ϱ, ϱ'', we have

$$\sin.^2 \frac{1}{2} V' = \cos. \left(\frac{1}{2} \varpi + \frac{1}{2} \varpi' + A'\right) \cos. \left(\frac{1}{2} \varpi + \frac{1}{2} \varpi' - A'\right)$$

A′ being what A becomes, when ϖ', β' are changed into ϖ'', β''.

If, however, the perihelion distance and the instant of the comet's crossing the perihelion, were exactly determined, and if the observations were rigorously exact, we should have

$$V = U, \quad V' = U';$$

But since that is hardly ever the case, we shall suppose

$$m = U - V; \quad m' = U' - V'.$$

We shall here observe that the revolution of the triangle S T C, gives for the angle C S T two different values: for the most part the nature of the motion of the comets, will show that which we ought to use, and the more plainly if the two values are very different; for then the one will place the comet more distant from the earth, than the other, and it will be easy to judge, by the apparent motion of the comet at the instant of observation, which ought to be preferred. But if there remains any uncertainty, we can always remove it, by selecting the value which renders V and V′ least different from U and U′.

We next make a second hypothesis in which, retaining the same passage over the perihelion as before, we shall suppose the perihelion distance to vary by a small quantity; for instance, by the fiftieth part of

its value, and we shall investigate on this hypothesis, the values of U—V, U′ — V′. Let then

$$n = U - V;\ n' = U' - V'.$$

Lastly, we shall frame a third hypothesis, in which, retaining the same perihelion distance as in the first, we shall suppose the instant of the passage over the perihelion to vary by a half-day, or a day more or less. In this new hypothesis we must find the values of

$$U - V \text{ and of } U' - V';$$

which suppose to be

$$p = U - V,\ p' = U' - V'.$$

Again, if we suppose u the number by which we ought to multiply the supposed variation in the perihelion distance in order to make it the true one, and t the number by which we ought to multiply the supposed variation of the instant when the comet passes over the perihelion in order to make it the true instant, we shall have the two following equations:

$$(m - n)\,u + (m - p)\,t = m;$$
$$(m' - n')\,u + (m' - p')\,t = m';$$

whence we derive u and t and consequently the perihelion distance corrected, and the true instant of the comet's passing its perihelion.

The preceding corrections suppose the elements determined by the first approximation, to be sufficiently near the truth for their errors to be regarded as infinitely small. But if the second approximation should not even suffice, we can have recourse to a third, by operating upon the elements already corrected as we did upon the first; provided care be taken to make them undergo smaller variations. It will also be sufficient to calculate by these corrected elements the values of U — V, and of U′ — V′. Calling them M, M′, we shall substitute them for m, m′ in the second members of the two preceding equations. We shall thus have two new equations which will give the values of u and t, relative to the corrections of these new elements.

Thus having obtained the true perihelion distance and the true instant of the comet's passing its perihelion, we obtain the other elements of the orbit in this manner.

Let j be the longitude of the node which would be ascending if the motion of the comet were direct, and φ the inclination of the orbit. We shall have by comparison of the first and last observation,

$$\text{tan. } j = \frac{\text{tan. } \varpi \text{ sin. } \beta' - \text{tan. } \varpi' \text{ sin. } \beta}{\text{tan. } \varpi \text{ cos. } \beta'' - \text{tan. } \varpi'' \text{ cos. } \beta};$$

$$\tan.\ \varphi = \frac{\tan.\ \varpi''}{\sin.\ (\beta'' - j)}\ .\ .$$

Since we can compare thus two and two together, the three observations, it will be more correct to select those which give to the above fractions, the greatest numerators and the greatest denominators.

Since tan. j may equally belong to j and $\pi + j$, j being the smallest of the positive angles containing its value, in order to find that which we ought to fix upon, we shall observe that φ is positive and less than a right angle; and that sin. $(\beta'' - j)$ ought to have the same sign as tan. ϖ''. This condition will determine the angle j, and this will be the position of the ascending node, if the motion of the comet is direct; but if retrograde we must add two right angles to the angle j to get the position of the node.

The hypothenuse of the spherical triangle whose sides are $\beta'' - j$ and ϖ'', is the distance of the comet from its ascending node in the third observation; and the difference between v'' and this hypothenuse is the interval between the node and the perihelion computed along the orbit.

If we wish to give to the theory of a comet all the precision which observations will admit of, we must establish it upon an *aggregate* of the best observations; which may be thus done. Mark with one, two, &c. dashes or strokes the letters m, n, p relative to the second observation, the third, &c. all being compared with the first observation. Hence we shall form the equations

$$\begin{aligned}
(m - n)\, u + (m - p)\, t &= m \\
(m' - n')\, u + (m' - p')\, t &= m' \\
(m'' - n'')\, u + (m'' - p'')\, t &= m'' \\
\&c. &= \&c.
\end{aligned}$$

Again, combining these equations so as to make it easier to determine u and t, we shall have the corrections of the perihelion distance and of the instant of the comet's passing its perihelion, founded upon the *aggregate* of these observations. We shall have the values of

$$\beta,\ \beta',\ \beta'',\ \&c.\ \varpi,\ \varpi',\ \varpi'',\ \&c.,$$

and obtain

$$\tan.\ j = \frac{\tan.\ \varpi\,(\sin.\ \beta' + \sin.\ \beta'' + \&c.) - \sin.\ \beta\,(\tan.\ \varpi' + \tan.\ \varpi'' + \&c.)}{\tan.\ \varpi\,(\cos.\ \beta' + \cos.\ \beta'' + \&c.) - \cos.\ \beta\,(\tan.\ \varpi' + \tan.\ \varpi'' + \&c.)}$$

$$\tan.\ \varphi = \frac{\tan.\ \varpi' + \tan.\ \varpi'' + \&c.}{\sin.\ (\beta' - j) + \sin.\ (\beta'' - j) + \&c.}\ .$$

504. There is a case, very rare indeed, in which the orbit of a comet can be determined rigorously and simply; it is that where the comet has been observed in its two nodes. The straight line which joins these

two observed positions, passes through the center of the sun and coincides with the line of the nodes. The length of this straight line is determined by the time elapsed between the two observations. Calling T this time reduced into decimals of a day, and denoting by c the straight line in question, we shall have (No. 493)

$$c = \frac{1}{2}\sqrt[3]{\frac{T^2}{(9.688724)^2}}.$$

Let β be the heliocentric longitude of the comet, at the moment of the first observation; ϱ its radius-vector; r its distance from the earth; and α its geocentric longitude. Let, moreover, R be the radius of the terrestrial orbit, at the same instant, and E the corresponding longitude of the sun. Then we shall have

$$\varrho \sin. \beta = r \sin. \alpha - R \sin. E;$$
$$\varrho \cos. \beta = r \cos. \alpha - R \cos. E.$$

Now $\pi + \beta$ will be the heliocentric longitude of the comet at the instant of the second observation; and if we distinguish the quantities ϱ, α, r, R, and E relative to this instant by a dash, we shall have

$$\varrho' \sin. \beta = R' \sin. E' - r' \sin. \alpha';$$
$$\varrho' \cos. \beta = R' \cos. E' - r' \cos. \alpha'.$$

These four equations give

$$\tan. \beta = \frac{r \sin \alpha - R \sin. E}{r \cos. \alpha - R \cos. E} = \frac{r' \sin. \alpha' - R' \sin. E'}{r' \cos. \alpha' - R' \cos. E'};$$

whence we obtain

$$r' = \frac{R\,R' \sin. (E - E') - R\,r \sin. (\alpha - E')}{r \sin. (\alpha' - \alpha) - R \sin. (\alpha' - E)}.$$

We have also

$$(\varrho + \varrho') \sin. \beta = r \sin. \alpha - r' \sin. \alpha' - R \sin. E + R' \sin. E'$$
$$(\varrho + \varrho') \cos. \beta = r \cos. \alpha - r' \cos. \alpha' - R \cos. E + R' \cos. E'.$$

Squaring these two equations, and adding them together, and substituting c for $\varrho + \varrho'$, we shall have

$$\begin{aligned} c^2 = {} & R^2 - 2\,R\,R' \cos. (E' - E) + R'^2 \\ & + 2\,r \{R' \cos. (\alpha - E') - R \cos. (\alpha - E)\} \\ & + 2\,r' \{R \cos. (\alpha' - E) - R' \cos. (\alpha' - E')\} \\ & + r^2 - 2\,r\,r' \cos. (\alpha' - \alpha) + r'^2. \end{aligned}$$

If we substitute in this equation for r' its preceding value in terms of r, we shall have an equation in r of the fourth degree, which can be resolved by the usual methods. But it will be more simple to find values of r, r' by trial such as will satisfy the equation. A few trials will suffice for that purpose.

By means of these quantities we shall have β, ϱ and ϱ'. If v be the angle which the radius ϱ makes with the perihelion distance called D; π — v will be the angle formed by this same distance, and by the radius ϱ'. We shall thus have by the equation to the parabola

$$\varrho = \frac{D}{\cos.^2 \frac{1}{2} v}; \quad \varrho' = \frac{D}{\sin.^2 \frac{1}{2} v};$$

which give

$$\tan.^2 \frac{1}{2} v = \frac{\varrho}{\varrho'}; \quad D = \frac{\varrho \varrho'}{\varrho + \varrho'}.$$

We shall therefore have the anomaly v of the comet, at the instant of the first observation, and its perihelion distance D, whence it is easy to find the position of the perihelion, at the instant of the passage of the comet over that point. Thus, of the five elements of the orbit of the comet, four are known, namely, the perihelion distance, the position of the perihelion, the instant of the comet's passing the perihelion, and the position of the node. It remains to learn the inclination of the orbit; but for that purpose it will be necessary to have recourse to a third observation, which will also serve to select from amongst the real and positive roots of the equation in r, that which we ought to make use of.

505. The supposition of the parabolic motion of comets is not rigorous; it is, at the same time, not at all probable, since compared with the cases that give the parabolic motion, there is an infinity of those which give the elliptic or hyperbolic motions. Besides, a comet moving in either a parabolic or hyperbolic orbit, will only once be visible; thus we may with reason suppose these bodies, if ever they existed, long since to have disappeared; so that we shall now observe those only which, moving in orbits returning into themselves, shall, after greater or less incursions into the regions of space, again approach their center the sun. By the following method, we shall be able to determine, within a few years, the period of their revolutions, when we have given a great number of very exact observations, made before and after the passage over the perihelion.

Let us suppose we have four or a greater number of good observations, which embrace all the visible part of the orbit, and that we have determined, by the preceding method, the parabola, which nearly satisfies these observations. Let v, v', v'', v''', &c. be the corresponding anomalies; ϱ, ϱ', ϱ'', ϱ''', &c. the radius-vectors. Let also

$$v' - v = U, \quad v'' - v = U', \quad v''' - v = U'', \text{ &c.}$$

Then we shall estimate, by the preceding method with the parabola already found, the values of U, U′, U″, &c., V, V′, V″, &c. Make

$$m = U - V,\quad m' = U' - V',\quad m'' = U'' - V'',\ \&c.$$

Next, let the perihelion distance in this parabola vary by a very small quantity, and on this hypothesis suppose

$$n = U - V;\quad n' = U' - V';\quad n'' = U'' - V'',\ \&c.$$

We will form a third hypothesis, in which the perihelion distance remaining the same as in the first, we shall make the instant of the comet's passing its perihelion vary by a very small quantity; in this case let

$$p = U - V;\quad p' = U' - V';\quad p'' = U'' - V'';\ \&c.$$

Lastly, we shall calculate the angle v and radius ϱ, with the perihelion distance, and instant over the perihelion on the first hypothesis, supposing the orbit an ellipse, and the difference 1 — e between its excentricity and unity a very small quantity, for instance $\frac{1}{50}$. To get the angle v, in this hypothesis, it will suffice (489) to add to the anomaly v, calculated in the parabola of the first hypothesis, a small angle whose sine is

$$\frac{1}{10}(1 - e)\tan.\frac{1}{2}v\left\{4 - 3\cos.^2\frac{1}{2}v - 6\cos.^4\frac{1}{2}v\right\}.$$

Substituting afterwards in the equation

$$\varrho = \frac{D}{\cos.^2\frac{1}{2}v}\left\{1 - \frac{1-e}{2}\tan.^2\frac{1}{2}v\right\};$$

for v, this anomaly, as calculated in the ellipse, we shall have the corresponding radius-vector ϱ. After the same manner, we shall obtain v′, ϱ′, v″, ϱ″, &c. Whence we shall derive the values of U, U′, U″, &c. and (by 503) of V, V′, V″, &c.

In this case let

$$q = U - V;\quad q' = U' - V';\quad q'' = U'' - V'',\ \&c.$$

Finally, call u the number by which we ought to multiply the supposed variation in the perihelion distance, to make it the true one; t the number by which we ought to multiply the supposed variation in the instant over the perihelion, to make it the true instant; and s that by which we should multiply the supposed value of 1 — e, in order to get the true one; and we shall obtain these equations:

$$\begin{aligned}
&(m - n)\,u + (m - p)\,t + (m - q)\,s = m;\\
&(m' - n')\,u + (m' - p')\,t + (m' - q')\,s = m';\\
&(m'' - n'')\,u + (m'' - p'')\,t + (m'' - q'')\,s = m'';\\
&(m''' - n''')\,u + (m''' - p''')\,t + (m''' - q''')\,s = m''';\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \&c.
\end{aligned}$$

We shall determine, by means of these equations, the values of u, t, s; whence will be derived the true perihelion distance, the true instant over the perihelion, and the true value of 1 — e. Let D be the perihelion distance, and a the semi-axis major of the orbit; then we shall have $a = \frac{D}{1 - e}$; the time of a sidereal revolution of the comet, will be expressed by a number of sidereal years equal to $a^{\frac{3}{2}}$ or to $\left(\frac{D}{1 - e}\right)^{\frac{3}{2}}$, the mean distance of the sun from the earth being unity. We shall then have (by 503) the inclination of the orbit and the position of the node.

Whatever accuracy we may attribute to the observations, they will always leave us in uncertainty as to the periodic times of the comets. To determine this, the most exact method is that of comparing the observations of a comet in two consecutive revolutions. But this is practicable, only when the lapse of time shall bring the comet back towards its perihelion.

Thus much for the motions of the planets and comets as caused by the action of the principal body of the system. We now come to

506. *General methods of determining by successive approximations, the motions of the heavenly bodies.*

In the preceding researches we have merely dwelt upon the elliptic motion of the heavenly bodies, but in what follows we shall estimate them as deranged by perturbing forces. The action of these forces requires only to be added to the differential equations of elliptic motion, whose integrals in finite terms we have already given, certain small terms. We must determine, however, by successive approximations, the integrals of these same equations when thus augmented. For this purpose here is a general method, let the number and degree of the equations be what they may.

Suppose that we have between the n variables y, y′, y″, &c. and the time t whose element d t is constant, the n differential equations

$$0 = \frac{d^i y}{d t^i} + P + \alpha Q;$$

$$0 = \frac{d^i y'}{d t^i} + P' + \alpha . Q',$$

$$\&c. = \&c.$$

P, Q, P′, Q′, &c. being functions of t, y, y′, &c. and of the differences to the order i — 1 inclusively, and α being a very small constant coefficient, which, in the theory of celestial motions, is of the order of the perturbing forces. Then let us suppose we have the finite integrals of those

equations when Q, Q′, &c. are nothing. Differentiating each i — 1 times successively, we shall form with their differentials $i\,n$ equations by means of which we shall determine by elimination, the arbitrary constants c, c′, c″, &c. in functions of t, y, y′, y″, &c. and of their differences to the order i — 1. Designating therefore by V, V′, V″, &c. these functions we shall have

$$c = V;\quad c' = V';\quad c'' = V'';\ \&c.$$

These equations are the $i\,n$ integrals of the $(i-1)^{th}$ order, which the equations ought to have, and which, by the elimination of the differences of the variables, give their finite integrals.

But if we differentiate the preceding integrals of the order i — 1, we shall have

$$0 = d\,V;\quad 0 = d\,V';\quad 0 = d\,V'';\ \&c.$$

and it is clear that these last equations being differentials of the order i without arbitrary constants, they can only be the sums of the equations

$$0 = \frac{d^i y}{d\,t^i} + P$$

$$0 = \frac{d^i y'}{d\,t^i} + P'$$

$$0 = \&c.$$

each multiplied by proper factors, in order to make these sums exact differences. Calling, therefore, F d t, F′ d t′, &c. the factors which ought respectively to multiply them in order to make 0 = d V; also in like manner making H d t, H′ d t′, &c. the factors which would make 0 = d V′, and so on for the rest, we shall have

$$d\,V = F\,d\,t\left\{\frac{d^i y}{d\,t^i} + P\right\} + F'\,d\,t\left\{\frac{d^i y'}{d\,t^i} + P'\right\} + \&c.$$

$$d\,V' = H\,d\,t\left\{\frac{d^i y}{d\,t^i} + P\right\} + H'\,d\,t\left\{\frac{d^i y'}{d\,t^i} + P'\right\} + \&c.$$

&c.

F, F′, &c. H, H′, &c. are functions of t, y, y′, y″, &c. and of their differences to the order i — 1. It is easy to determine them when V, V′, &c. are known. For F is evidently the coefficient of $\frac{d^i y}{d\,t^i}$ in the differential of V; F′ is the coefficient of $\frac{d^i y'}{d\,t^i}$ in the same differential, and so on. In like manner, H, H′, &c. are the coefficients of $\frac{d^i y}{d\,t^i}$, $\frac{d^i y'}{d\,t^i}$, &c. in the differential of V′. Thus, since we may suppose V, V′, &c. known, by dif-

ferentiating with regard to $\frac{d^{i-1}y}{dt^{i-1}}$, $\frac{d^{i-1}y'}{dt^{i-1}}$, &c. we shall have the factors by which we ought to multiply the differential equations

$$0 = \frac{d^i y}{dt^i} + P, \quad 0 = \frac{d^i y'}{dt^i} + P', \text{ &c.}$$

in order to make them exact differences.

Now resume the differential equations

$$0 = \frac{d^i y}{dt^i} + P + \alpha . Q; \quad 0 = \frac{d^i y'}{dt^i} + P' + \alpha . Q', \text{ &c.}$$

If we multiply the first by $F\,dt$, the second by $F'\,dt$, and so on, we shall have by adding the results

$$0 = dV + \alpha\, dt \{F Q + F' Q' + \text{&c.}\},$$

In the same manner, we shall have

$$0 = dV' + \alpha\, dt \{H Q + H' Q' + \text{&c.}\}$$
$$\text{&c.}$$

whence by integration

$$c - \alpha \int dt \{F Q + F' Q' + \text{&c.}\} = V;$$
$$c' - \alpha \int dt \{H Q + H' Q' + \text{&c.}\} = V';$$
$$\text{&c.}$$

We shall thus have $i\,n$ differential equations, which will be of the same form as in the case when Q, Q', &c. are nothing, with this only difference, that the arbitrary constants c, c', c'', &c. must be changed into

$$c - \alpha \int dt \{F Q + F' Q' + \text{&c.}\}, \quad c - \alpha \int dt \{H Q + H' Q' + \text{&c.}\} \text{&c.}$$

But if, in the supposition of Q, Q', &c. being equal to zero, we eliminate from the $i\,n$ integrals of the order $i - 1$, the differences of the variables y, y', &c. we shall have n finite integrals of the proposed equations. We shall therefore have these same integrals when Q, Q', &c. are not zero, by changing in the first integrals, c, c', &c. into

$$c - \alpha \int dt \{F Q + \text{&c.}\}, \quad c' - \alpha \int dt \{H Q + \text{&c.}\} \text{&c.}$$

507. If the differentials

$$dt \{F Q + F' Q' + \text{&c.}\}, \quad dt \{H Q + H' Q' + \text{&c.}\} \text{&c.}$$

are exact, we shall have, by the preceding method, finite integrals of the proposed differentials. But this is not so, except in some particular cases, of which the most extensive and interesting is that in which they are linear. Thus let P, P', &c. be linear functions of y, y', &c. and of their differences up to the order $i - 1$, without any term independent of these variables, and let us first consider the case in which Q, Q', &c. are nothing. The differential equations being linear, their successive integrals

are likewise linear, so that $c = V$, $c' = V'$, &c. being the $i\ n$ integrals of the order $i - 1$, of the linear differential equations

$$0 = \frac{d^i y}{d t^i} + P; \quad 0 = \frac{d^i y'}{d t^i} + P'; \text{ &c.}$$

V, V', &c. may be supposed linear functions of y y', &c. and of their differences to the order $i - 1$. To make this evident, suppose that in the expressions for y, y', &c. the arbitrary constant c is equal to a determinate quantity plus an indeterminate δc; the arbitrary constant c' equal to a determinate quantity plus an indeterminate $\delta c'$ &c.; then reducing these expressions according to the powers and products of δc, $\delta c'$, &c. we shall have by the formulas of No. 487

$$y = Y + \delta c \left(\frac{d Y}{d c}\right) + \delta c' \left(\frac{d Y}{d c'}\right) + \text{&c.}$$
$$+ \frac{\delta c^2}{1.2} \cdot \left(\frac{d^2 Y}{d c^2}\right) + \text{&c.}$$
$$y' = Y' + \delta c \left(\frac{d Y'}{d c}\right) + \delta c' \left(\frac{d Y'}{d c'}\right) + \text{&c.}$$
$$+ \frac{\delta c^2}{1.2} \left(\frac{d^2 Y'}{d c^2}\right) + \text{&c.}$$
&c.

Y, Y', $\left(\frac{d Y}{d c}\right)$, &c. being functions of t without arbitrary constants. Substituting those values, in the proposed differential equations, it is evident that δc, $\delta c'$, &c. being indeterminate, the coefficients of the first powers of such of them ought to be nothing in the several equations. But these equations being linear, we shall evidently have the terms affected with the first powers of δc, $\delta c'$, &c. by substituting for y, y', &c. these quantities respectively

$$\left(\frac{d Y}{d c}\right) \delta c + \left(\frac{d Y}{d c'}\right) \delta c' + \text{&c.}$$
$$\left(\frac{d Y'}{d c}\right) \delta c + \left(\frac{d Y'}{d c'}\right) \delta c' + \text{&c.}$$

These expressions of y, y', &c. satisfy therefore separately the proposed equations; and since they contain the $i\ n$ arbitraries δc, $\delta c'$, &c. they are complete integrals. Thus we perceive, that the arbitraries are under a linear form in the expressions of y, y', &c. and consequently also in their differentials. Whence it is easy to conclude that the variables y, y', &c. and their differences, may be supposed to be linear in the successive integrals of the proposed differential equations.

Hence it follows, that F, F', &c. being the coefficients of $\frac{d^i y}{d t^i}$, $\frac{d^i y'}{d t^i}$,

&c. in the differential of V; H, H', &c. being the coefficients of the same differences in the differential of V', &c. these quantities are functions of variable t only. Therefore, if we suppose Q, Q', &c. functions of t alone, the differentials

$$d\,t\,\{F\,Q + F'\,Q' + \&c.\};\quad d\,t\,\{H\,Q + H'\,Q' + \&c.\};\ \&c.$$

will be exact.

Hence there results a simple means of obtaining the integrals of any number whatever n of linear differential equations of the order i, and which contain any terms $\alpha\,Q$, $\alpha\,Q'$, &c. functions of one variable t, having known the integrals of the same equations in the case where Q, Q', &c. are supposed nothing. For then if we differentiate their n finite integrals $i - 1$ times successively, we shall have $i\,n$ equations which will give, by elimination, the values of the $i\,n$ arbitrary constants c, c', &c. in functions of t, y, y', &c. and of their differences to the $i - 1^{th}$ order. We shall thus form the $i\,n$ equations $c = V$, $c' = V'$, &c. This being done, F, F', &c. will be the coefficients of $\frac{d^{i-1}y}{dt^{i-1}}$, $\frac{d^{i-1}y'}{dt^{i-1}}$, &c. in V; H, H', &c. will be the coefficients of the same differences in V', and so on. We shall, therefore, have the finite integrals of the linear differential equations

$$0 = \frac{d^i y}{dt^i} + P + \alpha Q;\quad 0 = \frac{d^i y'}{dt^i} + P' + \alpha Q';\ \&c.$$

by changing, in the finite integrals of these equations deprived of their last terms $\alpha\,Q$, $\alpha\,Q'$, &c. the arbitrary constants c, c', &c. into

$$c - \alpha \int d\,t\,\{F\,Q + F'\,Q' + \&c.\},\quad c' - \alpha \int d\,t\,\{H\,Q + H'\,Q' + \&c.\}\ \&c.$$

Let us take, for example, the linear equation

$$0 = \frac{d^2 y}{dt^2} + P^2 y + \alpha Q.$$

The finite integral of the equation

$$0 = \frac{d^2 y}{dt^2} + a^2 y$$

is (found by multiplying by cos. a t, and then *by parts* getting $\int \cos.\,a\,t \cdot \frac{d^2 y}{dt} = \cos.\,a\,t\,\frac{dy}{dt} + a \int \sin.\,a\,t\,\frac{dy}{dt} \cdot dt = \cos.\,a\,t \cdot \frac{dy}{dt} +$ a sin. a t. y $- a^2 \int \cos.\,a\,t \cdot y$ $\therefore$ $c = a \cos.\,a\,t \cdot \frac{dy}{dt} + a \sin.\,a\,t \cdot y$, &c.)

$$y = \frac{c}{a} \sin.\,a\,t + \frac{c'}{a} \cos.\,a\,t,$$

c, c' being arbitrary constants.

This integral gives by differentiation

$$\frac{dy}{dt} = c \cos. at - c' \sin. at.$$

If we combine this with the integral itself, we shall form two integrals of the first order

$$c = a y \sin. at + \frac{dy}{dt} \cos. at;$$

$$c' = a y \cos. at - \frac{dy}{dt} \sin. at;$$

and therefore shall have in this case

$$F = \cos. at; \quad H = -\sin. at,$$

and the complete integral of the proposed equation will therefore be

$$y = \frac{c}{a} \sin. at + \frac{c'}{a} \cos. at - \frac{\alpha \sin. at}{a} \int Q\, dt \cos. at + \frac{\alpha \cos. at}{a} \int Q\, dt \sin. at.$$

Hence it is easy to conclude that if Q is composed of terms of the form $K. \frac{\sin.}{\cos.} (mt + \varepsilon)$ each of these terms will produce in the value of y the corresponding term

$$\frac{\alpha K}{m^2 - a^2} \cdot \frac{\sin.}{\cos.} (mt + \varepsilon).$$

If m be equal to a, the term $K \frac{\sin.}{\cos.} (mt + \varepsilon)$ will produce in y, 1st. the term $-\frac{\alpha K}{4a^2} \cdot \frac{\sin.}{\cos.} (at + \varepsilon)$ which being comprised by the two terms $\frac{c}{a} \sin. at + \frac{c'}{a} \cos. at$, may be neglected; 2dly. the term $\pm \frac{\alpha K t}{2a} \cdot \frac{\cos.}{\sin.} (at + \varepsilon)$, + or — being used according as the term of Q is a sine or cosine. We thus perceive how the arc t produces itself in the values of y, y′, &c. without sines and cosines, by successive integrations, although the differentials do not contain it in that form. It is evident this will take place whenever the functions F Q, F′, Q′, &c. H Q, H′ Q′, &c. shall contain constant terms.

508. If the differences

$$dt\{F Q + \&c.\}, \quad dt\{H Q + \&c.\}$$

are not exact, the preceding analysis will not give their rigorous integrals. But it affords a simple process for obtaining them more and more nearly by approximation when α is very small, and when we have the values of

y, y′, &c. on the supposition of α being zero. Differentiating these values, i — 1 times successively, we shall form the differential equations of the order i — 1, viz.

$$c = V;\ c' = V', \text{ &c.}$$

The coefficients of $\frac{d^i y}{d t^i}$, $\frac{d^i y'}{d t^i}$, &c. in the differentials of V, V′, &c. being the values of F, F′, &c. H, H′, &c. we shall substitute them in the differential functions

$$d t (F Q + F' Q' + \text{&c.});\ d t (H Q + H' Q' + \text{&c}); \text{ &c.}$$

Then, we shall substitute in these functions, for y, y′, &c. their first approximate values, which will make these differences functions of t and of the arbitrary constants c, c′, &c.

Let T d t, T′ d t, &c. be these functions. If we change in the first approximate values of y, y′, &c. the arbitrary constants c, c′, &c. respectively into $c - \alpha \int T d t$, $c' - \alpha \int T' d t$, &c. we shall have the second approximate values of those variables.

Again substitute these second values in the differential functions

$$d t . (F Q + \text{&c.});\ d t (H Q + \text{&c.}) \text{ &c.}$$

But it is evident that these functions are then what T d t, T′ d t, &c. become when we change the arbitrary constants c, c′, &c. into $c - \alpha \int T d t$, $c' - \alpha \int T' d t$, &c. Let therefore T,, T′,, &c. denote what T, T′, &c. become by these changes. We shall get the third approximate values of y, y′, &c. by changing in the first c, c′, &c. respectively into $c - \alpha \int T, d t$, $c' - \int T,' d t$, &c.

Calling T,,, T,,′, in like manner, what T, T′, &c. become when we change c, c′, &c. into $c - \alpha \int T, d t$, $c' - \alpha \int T,' d t$, &c. we shall have the fourth approximate values of y, y′, &c. by changing in the first approximate values of these variables into $c - \alpha \int T,, d t$, $c' - \alpha \int T,,' d t$, &c. and so on.

We shall see presently that the determination of the celestial motions, depends almost always upon differential equations of the form

$$0 = \frac{d^2 y}{d t^2} + a^2 y + \alpha Q,$$

Q being a rational and integer function of y, of the sine and cosine of angles increasing proportionally with the time represented by t. The following is the easiest way of integrating this equation.

First suppose α nothing, and we shall have by the preceding No. a first value of y.

Next substitute this value in Q, which will thus become a rational and

entire function of sines and cosines of angles proportional to the time. Then integrating the differential equation, we shall have a second value of y approximate up to quantities of the order α inclusively.

Again substitute this value in Q, and, integrating the differential equation, we shall have a third approximation of y, and so on.

This way of integrating by approximation the differential equations of the celestial motions, although the most simple of all, possesses the disadvantage of giving in the expressions of the variables y, y', &c. the arcs of a circle (symbols *sine* and *cosine*) in the very case where these arcs do not enter the rigorous values of these variables. We perceive, in fact, that if these values contain sines or cosines of angles of the order α t, these sines or cosines ought to present themselves in the form of series, in the approximate values found by the preceding method; for these last values are ordered according to the powers of α. This developement into series of the sine and cosine of angles of the order α t, ceases to be exact when, by lapse of time, the arc α t becomes considerable. The approximate values of y, y', &c. cannot extend to the case of an unlimited interval of time. It being important to obtain values which include both past and future ages, the reversion of arcs of a circle contained by the approximate values, into functions which produce them by their developement into series, is a delicate and interesting problem of analysis. Here follows a general and very simple method of solution.

509. Let us consider the differential equation of the order i,

$$0 = \frac{d^i y}{dt^i} + P + \alpha Q$$

α being very small, and P and Q algebraic functions of $y, \frac{dy}{dt}, \ldots \frac{d^{i-1}y}{dt^{i-1}}$, and of sines and cosines of angles increasing proportionally with the time. Suppose we have the complete integral of this differential, in the case of $\alpha = 0$, and that the value of y given by this integral, does not contain the arc t, without the symbols *sine* and *cosine*. Also suppose that in integrating this equation by the preceding method of approximation, when α is not nothing, we have

$$y = X + tY + t^2 Z + t^3 S + \&c.$$

X, Y, Z, &c. being periodic functions of t, which contain the i arbitraries c, c', c'', &c. and the powers of t in this expression of y, going on to infinity by the successive approximations. It is evident the coefficients of these powers will decrease with the greater rapidity, the less is α. In the theory of the motions of the heavenly bodies, α expresses the order of perturbing forces, relative to the principal forces which animate them.

If we substitute the preceding value of y in the function $\frac{d^i y}{dt^i} + P + \alpha Q$, it will take the form $k + k' t + k'' t^2 +$ &c., k, k', k'', &c. being periodic functions of t; but by the supposition, the value of y satisfies the differential equation

$$0 = \frac{d^i y}{dt^i} + P + \alpha Q;$$

we ought therefore to have identically

$$0 = k + k' t + k'' t^2 + \text{&c.}$$

If k, k', k'', &c. be not zero this equation will give by the reversion of series, the arc t in functions of sines and cosines of angles proportional to the time t. Supposing therefore α to be infinitely small, we shall have t equal to a finite function of sines and cosines of similar angles, which is impossible. Hence the functions k, k', &c. are identically nothing.

Again, if the arc t is only raised to the first power under the symbols sine and cosine, since that takes place in the theory of celestial motions, the arc will not be produced by the successive differences of y. Substituting, therefore, the preceding value of y, in the function $\frac{d^i y}{dt^i} + P + \alpha . Q$, the function of $k + k' t +$ &c. to which it transforms, will not contain the arc t out of the symbols *sine* and *cosine*, inasmuch as it is already contained in y. Thus changing in the expression of y, the arc t, without the periodic symbols, into $t - \theta$, θ being any constant whatever, the function $k + k' t +$ &c. will become $k + k' (t - \theta) +$ &c. and since this last function is identically nothing by reason of the identical equations $k = 0$ $k' = 0$, it results that the expression

$$y = X + (t - \theta) Y + (t - \theta)^2 Z + \text{&c.}$$

also satisfies the differential equation

$$0 = \frac{d^i y}{dt^i} + P + \alpha Q.$$

Although this second value of y seems to contain i + 1 arbitrary constants, namely, the i arbitraries c, c', c'', &c. and θ, yet it can only have i distinct ones. It is therefore necessary that by a proper change in the constants c, c', &c. the arbitrary θ be made to disappear, and thus the second value of y will coincide with the first. This consideration will furnish us with the means of making disappear the arc of a circle out of the periodic symbols.

Give the following form to the second expression for y:

$$y = X + (t - \theta) . R.$$

Then supposing θ to disappear from y, we have

$$\left(\frac{d\,y}{d\,\theta}\right) = 0$$

and consequently

$$R = \left(\frac{d\,X}{d\,\theta}\right) + (t - \theta)\left(\frac{d\,R}{d\,\theta}\right).$$

Differentiating successively this equation we shall have

$$2\left(\frac{d\,R}{d\,\theta}\right) = \left(\frac{d^2\,X}{d\,\theta^2}\right) + (t - \theta)\left(\frac{d^2\,R}{d\,\theta^2}\right)$$

$$3\left(\frac{d^2\,R}{d\,\theta^2}\right) = \left(\frac{d^3\,X}{d\,\theta^3}\right) + (t - \theta)\left(\frac{d^3\,R}{d\,\theta^3}\right),$$

whence it is easy to obtain, by eliminating R and its differentials, from the preceding expression of y,

$$y = X + (t - \theta)\left(\frac{d\,X}{d\,\theta}\right) + \frac{(t-\theta)^2}{1.\,2}\cdot\left(\frac{d^2\,X}{d\,\theta^2}\right) + \frac{(t-\theta)^3}{2.\,3}\cdot\left(\frac{d^3\,X}{d\,t^3}\right) + \&c.$$

X is a function of t, and of the constants, c, c′, c″, &c. and since these constants are functions of θ, X is a function of t and of θ, which we can represent by φ (t, θ). The expression of y is by Taylor's Theorem the developement of the function φ (t, θ + t − θ), according to the powers of t − θ. We have therefore $y = \varphi$ (t, t). Whence we shall have y by changing in X, θ into t. The problem thus reduces itself to determine X in a function of t and θ, and consequently to determine c, c′, c″, &c. in functions of θ.

To solve this problem, let us resume the equation

$$y = X + (t - \theta).\,Y + (t - \theta)^2.\,Z + \&c.$$

Since the constant θ is supposed to disappear from this expression of y, we shall have the identical equation

$$0 = \left(\frac{d\,X}{d\,\theta}\right) - Y + (t-\theta)\left\{\left(\frac{dY}{d\,\theta}\right) - 2Z\right\} + (t-\theta)^2\left\{\left(\frac{dZ}{d\,\theta}\right) - 3S\right\} + \&c. \ldots (a)$$

Applying to this equation the reasoning which we employed upon

$$0 = k + k'\,t + k''\,t^2 + \&c.$$

we perceive that the coefficients of the successive powers of t − θ ought to be each zero. The functions X, Y, Z, &c. do not contain θ, inasmuch as it is contained in c, c′, &c. so that to form the partial differences $\left(\frac{d\,X}{d\,\theta}\right)$, $\left(\frac{d\,Y}{d\,\theta}\right)$, $\left(\frac{d\,Z}{d\,\theta}\right)$, &c. it is sufficient to make c, c′, &c. vary in these functions, which gives

$$\left(\frac{d\,X}{d\,\theta}\right) = \left(\frac{d\,X}{d\,c}\right)\frac{d\,c}{d\,\theta} + \left(\frac{d\,X}{d\,c'}\right)\frac{d\,c'}{d\,\theta} + \left(\frac{d\,X}{d\,c''}\right)\frac{d\,c''}{d\,\theta} + \&c.$$

$$\left(\frac{d\,Y}{d\,\theta}\right) = \left(\frac{d\,Y}{d\,c}\right)\frac{d\,c}{d\,\theta} + \left(\frac{d\,Y}{d\,c'}\right)\frac{d\,c'}{d\,\theta} + \left(\frac{d\,Y}{d\,c''}\right)\frac{d\,c''}{d\,\theta} + \&c.$$

$$\&c. = \&c.$$

Again, it may happen that some of the arbitrary constants c, c', c'', &c. multiply the arc t in the periodic functions X, Y, Z, &c. The differentiation of these functions relatively to θ, or, which is the same thing, relatively to these arbitrary constants, will develope this arc, and bring it from without the symbols of the periodic functions. The differences $\left(\frac{d\,X}{d\,\theta}\right)$, $\left(\frac{d\,Y}{d\,\theta}\right)$, $\left(\frac{d\,Z}{d\,\theta}\right)$, &c. will be then of this form:

$$\left(\frac{d\,X}{d\,\theta}\right) = X' + t\,X'';$$

$$\left(\frac{d\,Y}{d\,\theta}\right) = Y' + t\,Y'';$$

$$\left(\frac{d\,Z}{d\,\theta}\right) = Z' + t\,Z'';$$

$$\&c.$$

X', X'', Y', Y'', Z', Z'', &c. being periodic functions of t, and containing moreover the arbitrary constants c, c', c'', &c. and their first differences divided by d θ, differences which enter into these functions only under a linear form; we shall have therefore

$$\left(\frac{d\,X}{d\,\theta}\right) = X' + \theta\,X'' + (t - \theta)\,X''$$

$$\left(\frac{d\,Y}{d\,\theta}\right) = Y' + \theta\,Y'' + (t - \theta)\,Y''$$

$$\left(\frac{d\,Z}{d\,\theta}\right) = Z' + \theta\,Z'' + (t - \theta)\,Z''$$

$$\&c.$$

Substituting these values in the equation (a) we shall have

$$0 = X' + \theta\,X'' - Y$$
$$+ (t - \theta)\,\{Y' + \theta\,Y'' + X'' - 2\,Z\}$$
$$+ (t - \theta)^2\,\{Z' + \theta\,Z'' + Y'' - 3\,S\} + \&c.;$$

whence we derive, in equalling separately to zero, the coefficients of the powers of $t - \theta$,

$$0 = X' + \theta\,X'' - Y$$
$$0 = Y' + \theta\,Y'' + X'' - 2\,Z$$
$$0 = Z' + \theta\,Z'' + Y'' - 3\,S;$$
$$\&c.$$

If we differentiate the first of these equations, i — 1 times successively relatively to t, we shall thence derive as many equations between the quantities c, c′, c″, &c. and their first differences divided by d θ. Then integrating these new equations relatively to θ, we shall obtain the constants in terms of θ.

Inspection alone of the first of the above equations will almost always suffice to get the differential equations in c, c′, c″, &c. by comparing separately the coefficients of the sines and cosines which it contains. For it is evident that the values of c, c′, &c. being independent of t, the differential equations which determine them, ought, in like manner, to be independent of it. The simplicity which this consideration gives to the process, is one of its principal advantages. For the most part these equations will not be integrable except by successive approximations, which will introduce the arc θ out of the periodic symbols, in the values of c, c′, &c. at the same time that this arc does not enter the rigorous integrals. But we can make it disappear by the following method.

It may happen that the first of the preceding equations, and its i — 1 differentials in t, do not give a number i of distinct equations between the quantities c, c′, c″, &c. and their differences. In this case we must have recourse to the second and following equations.

When we shall have thus determined c, c′, c″, &c. in functions of θ, we shall substitute them in X, and changing afterwards θ into t, we shall obtain the value of y, without arcs of a circle or free from periodic symbols, when that is possible.

510. Let us now consider any number n of differential equations.

$$0 = \frac{d^i y}{d t^i} + P + \alpha Q;$$

$$0 = \frac{d^i y'}{d t^i} + P' + \alpha Q';$$

&c.

P, Q, P′, Q′ being functions of y, y′, &c. of their differentials to the order i — 1, and of the sines and cosines of angles increasing proportionally with the variable t, whose difference is constant. Suppose the approximate integrals of these equations to be

$$y = X + t Y + t^2 Z + t^3 S + \&c.$$

$$y' = X_, + t Y_, + t^2 Z_, + t^3 S_, + \&c.$$

X, Y, Z, &c. X$_,$, Y$_,$, Z$_,$, &c. being periodic functions of t and containing $i\,n$ arbitrary constants c, c′, c″, &c. We shall have as in the preceding No.

$$0 = X' + \theta X'' - Y;$$
$$0 = Y' + \theta Y'' + X'' - 2Z;$$
$$0 = Z' + \theta Z'' + Y'' - 3S;$$
&c.

The value of y′ will give, in like manner, equations of this form

$$0 = X_{,}' + \theta X_{,}'' - Y_{,};$$
$$0 = Y_{,}' + \theta Y_{,}'' + X_{,}'' - 2Z_{,};$$
&c.

The values of y″, y‴, &c. will furnish similar equations. We shall determine by these different equations, selecting the most simple and approximable, the values of c, c′, c″, &c. in functions of θ. Substituting these values in X, X′, &c. and then changing θ into t, we shall have the values of y, y′, &c. independent of arcs free from periodic symbols when that is possible.

511. Let us resume the method already exposed in No. 506. It thence results that, if instead of supposing the parameters c, c′, c″, &c. constant, we make them vary so that we have

$$dc = -\alpha dt \{FQ + F'Q' + \&c\};$$
$$dc' = -\alpha dt \{HQ + H'Q' + \&c.\};$$

we shall always have the i n integrals of the order i — 1,

$$c = V;\ c' = V';\ c'' = V''; \&c.$$

as in the case of $\alpha = 0$. Whence it follows that not only the finite integrals, but also all the equations in which these enter the differences inferior to the order i, will preserve the same form, in the case of $\alpha = 0$, and in that where it is any quantity whatever; for these equations may result from the comparison alone of the preceding integrals of the order i — 1. We can, therefore, in the two cases equally differentiate i — 1 times successively the finite integrals, without causing c, c′, &c. to vary; and since we are at liberty to make all vary together, there will thence result the equations of condition between the parameters c, c′, &c. and their differences.

In the two cases where $\alpha = 0$, and $\alpha =$ any quantity whatever, the values of y, y′, &c. and of their differences to the order i — 1 inclusively, are the same functions of t and of the parameters c, c′, &c. Let Y be any function of the variables y, y′, y″, &c. and of their differentials inferior to the order i — 1, and call T the function of t, which it becomes, when we substitute for these variables and their differences their values in t. We can differentiate the equation Y = T, regarding the parameters c, c′, &c. constant; we can only, however, take the partial difference of Y relatively

to one only or to many of the variables y, y', &c. provided we suppose what varies with these, to vary also in T. In all these differentiations, the parameters c, c', c'', &c. may always be treated as constants; since by substituting for y, y', &c. and their differences, their values in t, we shall have equations identically zero in the two cases of α nothing and of α any quantity whatever.

When the differential equations are of the order i.— 1, it is no longer allowed, in differentiating them, to treat the parameters c, c', &c. as constants. To differentiate these equations, consider the equation $\varphi = 0$, φ being a differential function of the order i — 1, and which contains the parameters c, c', c'', &c. Let d φ be the difference of this function taken in regarding c, c', &c. constant, as also the differences $d^{i-1} y$, $d^{i-1} y'$, &c. Let S be the coefficient of $\frac{d^i y}{d t^{i-1}}$ in the entire difference of φ. Let S' be the coefficient of $\frac{d^i y'}{d t^{i-1}}$ in this same difference, and so on. The equation $\varphi = 0$ when differentiated will give

$$0 = \delta \varphi + \left(\frac{d \varphi}{d c}\right) d c + \left(\frac{d \varphi}{d c'}\right) d c' + \&c.$$
$$+ S \frac{d^i y}{d t^{i-1}} + S' \frac{d^i y'}{d t^{i-1}} + \&c.;$$

Substituting for $\frac{d^i y}{d t^{i-1}}$ its value $- d t \{P + \alpha Q\}$; for $\frac{d^i y'}{d t^{i-1}}$ its value $- d t \{P' + \alpha Q'\}$ &c. we shall have

$$0 = \delta \varphi + \left(\frac{d \varphi}{d c}\right) d c + \left(\frac{d \varphi}{d c'}\right) d c' + \&c.$$
$$- d t \{S P + S' P' + \&c.\} - \alpha d t \{S Q + S' Q' + \&c.\} \quad . \quad (t)$$

In the supposition of $\alpha = 0$, the parameters c, c', c'', &c. are constant. We have thus

$$0 = \delta \varphi - d t \{S P + S' P' + \&c.\}$$

If we substitute in this equation for c, c', c'', &c. their values V, V', V'', &c. we shall have differential equations of the order i — 1, without arbitraries, which is impossible, at least if this equation is to be identically nothing. The function

$$\delta \varphi - d t \{S P + S' P' + \&c.\}$$

becoming therefore identically nothing by reason of equations c = V, c' = V', &c. and since these equations hold still, when the parameters c, c', c'', &c. are variable, it is evident, that in this case, the preceding

function is still identically nothing. The equation (t) therefore will become

$$0 = \left(\frac{d\varphi}{dc}\right) dc + \left(\frac{d\varphi}{dc'}\right) dc' + \&c.$$
$$- \alpha\, dt \{S Q + S' Q' + \&c.\} \quad \ldots\ldots\ldots \quad (x)$$

Thus we perceive that to differentiate the equation $\varphi = 0$, it suffices to vary the parameters c, c', &c. in φ and the differences $d^{i-1} y$, $d^{i-1} y'$, &c. and to substitute after the differentiations, for $-\alpha Q$, $\alpha Q'$, &c. the quantities $\frac{d^i y}{dt^i}$, $\frac{d^i y'}{dt^i}$, &c.

Let $\psi = 0$, be a finite equation between y, y', &c. and the variable t. If we designate by $\delta\psi$, $\delta^2\psi$, &c. the successive differences of ψ, taken in regarding c, c', &c. as constant, we shall have, by what precedes, in that case where c, c', &c. are variable, these equations:

$$\psi = 0;\ \delta\psi = 0;\ \delta^2\psi = 0 \ldots\ldots \delta^{i-1}\psi = 0;$$

changing therefore successively in the equation (x) the function φ into ψ, $\delta\psi$, $\delta^2\psi$, &c. we shall have

$$0 = \left(\frac{d\psi}{dc}\right) dc + \left(\frac{d\psi}{dc'}\right) . dc' + \&c.$$

$$0 = \left(\frac{d . \delta\psi}{dc}\right) dc + \left(\frac{d . \delta\psi}{dc'}\right) dc' + \&c.$$

$$\ldots\ldots\ldots\ldots$$

$$0 = \left(\frac{d . \delta^{i-1}\psi}{dc}\right) . dc + \left(\frac{d . \delta^{i-1}\psi}{dc'}\right) dc' + \&c.$$
$$- \alpha\, dt \left\{ Q\left(\frac{d\psi}{dy}\right) + Q'\left(\frac{d\psi}{dy'}\right) + \&c. \right\}$$

Thus the equations $\psi = 0$, $\psi' = 0$, &c. being supposed to be the n finite integrals of the differential equations

$$0 = \frac{d^i y}{dt^i} + P$$
$$0 = \frac{d^i y'}{dt^i} + P'$$
&c.

we shall have $i\ n$ equations, by means of which we shall be able to determine the parameters c, c', c'', &c. without which it would be necessary for that purpose to form the equations $c = V$, $c' = V'$, &c. But when the integrals are under this last form, the determination will be more simple.

512. This method of making the parameters vary, is one of great utility

in analysis and in its applications. To exhibit a new use of it, let us take the differential equation

$$0 = \frac{d^i y}{d t^i} + P,$$

P being a function of t, y, of their differences to the order i — 1, and of the quantities q, q′, &c. which are functions of t. Suppose we have the finite integral of this differential equation of the supposition of q, q′, &c. being constant, and represent by $\varphi = 0$, this integral, which shall contain i arbitraries c, c′, &c. Designate by $\delta\varphi$, $\delta^2\varphi$, $\delta^3\varphi$, &c. the successive differences of φ taken in regarding q, q′, &c. constant, as also the parameters c, c′, c″, &c. If we suppose all these quantities to vary, the differences of φ will be

$$\delta\varphi + \left(\frac{d\varphi}{dc}\right) dc + \left(\frac{d\varphi}{dc'}\right) dc' + \&c. + \left(\frac{d\varphi}{dq}\right) dq + \left(\frac{d\varphi}{dq'}\right) dq' + \&c.$$

making therefore

$$0 = \left(\frac{d\varphi}{dc}\right) dc + \left(\frac{d\varphi}{dc'}\right) dc' + \&c. + \left(\frac{d\varphi}{dq}\right) dq + \left(\frac{d\varphi}{dq'}\right) dq' + \&c.$$

$\delta\varphi$ will be still the first difference of φ in the case of c, c′, &c. q, q′, &c. being variable. If we make, in like manner,

$$0 = \left(\frac{d\delta\varphi}{dc}\right) dc + \left(\frac{d\delta\varphi}{dc'}\right) dc' + \&c. + \left(\frac{d\delta\varphi}{dq}\right) dq + \left(\frac{d\delta\varphi}{dq'}\right) dq' + \&c.$$

- - - - - - - - - - - - - - - - - -

$$0 = \left(\frac{d\delta^{i-1}\varphi}{dc}\right) dc + \left(\frac{d\delta^{i-1}\varphi}{dc'} dc'\right) + \&c. + \left(\frac{d\delta^{i-1}\varphi}{dq}\right) dq + \left(\frac{d\delta^{i-1}\varphi}{dq'}\right) dq' + \&c.$$

$\delta^2\varphi$, $\delta^3\varphi$, ... $\delta^i\varphi$ will likewise be the second, third, &c. differences of φ when c, c′, &c. q, q′, &c. are supposed variable.

Again in the case of c, c′, &c. q, q′, &c. being constant, the differential equation

$$0 = \frac{d^i y}{d t^i} + P,$$

is the result of the elimination of the parameters c, c′, &c. by means of the equations $\varphi = 0$, $\delta\varphi = 0$, $\delta^2\varphi = 0$, ... $d^i\varphi = 0$. Thus, these last equations still holding good when q, q′, &c. are supposed variable, the equation $\varphi = 0$ will also satisfy, in this case, the proposed differential equation, provided the parameters c, c′, &c. are determined by means of the i preceding differential equations; and since their integration gives i arbitrary constants, the function φ will contain these arbitraries, and the equation $\varphi = 0$ will be the complete integral of the proposed equation.

This method, the variation of parameters, may be employed with advantage when the quantities q, q', &c. vary very slowly. Because this consideration renders the integration by approximation of the differential equations which determine the variables c, c', c'', &c. in general much easier.

513. *Second Approximation of Celestial Motions.*

Let us apply the preceding method to the perturbations of celestial motions, in order thence to obtain the most simple expressions of their periodical and secular inequalities. For that purpose let us resume the differential equations (1), (2), (3) of No. 471, which determine the relative motion of μ about M. If we make

$$R = \frac{\mu'(x x' + y y' + z z')}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} + \frac{\mu''(x x'' + y y'' + z z'')}{(x''^2 + y''^2 + z''^2)^{\frac{3}{2}}} + \&c. - \frac{\lambda}{\mu}$$

λ being by the No. cited equal to

$$\frac{\mu\mu'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}} + \frac{\mu\mu''}{\{(x''-x)^2+(y''-y)^2+(z''-z)^2\}^{\frac{1}{2}}} + \frac{\mu'\mu''}{\{(x''-x')^2+(y''-y')^2+(z''-z')^2\}^{\frac{1}{2}}} + \&c.$$

If, moreover, we suppose $M + \mu = m$ and

$$\varrho = \sqrt{x^2 + y^2 + z^2}$$
$$\varrho' = \sqrt{x'^2 + y'^2 + z'^2}$$

we shall have

$$\left.\begin{aligned} 0 &= \frac{d^2x}{dt^2} + \frac{m x}{\varrho^3} + \left(\frac{dR}{dx}\right) \\ 0 &= \frac{d^2y}{dt^2} + \frac{m y}{\varrho^3} + \left(\frac{dR}{dy}\right) \\ 0 &= \frac{d^2z}{dt^2} + \frac{m z}{\varrho^3} + \left(\frac{dR}{dz}\right) \end{aligned}\right\}; \quad (P)$$

The sum of these three equations multiplied respectively by d x, d y, d z gives by integration

$$0 = \frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2m}{\varrho} + \frac{m}{a} + 2\int dR \quad . \quad . \quad . \quad (Q)$$

the differential d R being only relative to the coordinates x, y, z of the body μ, and a being an arbitrary constant, which, when R = 0, becomes by No. 499, the semi-axis major of the ellipse described by μ about M.

The equations (P) multiplied respectively by x, y, z and added to the integral (Q) will give

$$0=\tfrac{1}{2}\frac{d^2\varrho^2}{dt^2}-\frac{m}{\varrho}+\frac{m}{a}+2\int dR+x\left(\frac{dR}{dx}\right)+y\left(\frac{dR}{dy}\right)+z\left(\frac{dR}{dz}\right); \quad (R)$$

We may conceive, however, the perturbing masses μ', μ'', &c. multiplied by a coefficient α, and then the value of ϱ will be a function of the time t and of α. If we develope this function according to the powers of α, and afterwards make $\alpha = 1$, it will be ordered according to the powers and products of the perturbing masses. Designate by the characteristic δ when placed before a quantity, this differential of it taken relatively to α, and divided by $d\alpha$. When we shall have determined $\delta\varrho$ in a series ordered according to the powers of α, we shall have the radius ϱ by multiplying this series by $d\alpha$, then integrating it relatively to α, and adding to the integral a function of t independent of α, a function which is evidently the value of ϱ in the case where the perturbing forces are nothing, and where the body μ describes a conic section. The determination of ϱ reduces itself, therefore, to forming and integrating the differential equation which determines $\delta\varrho$.

For that purpose, resume the differential equation (R) and make for the greater simplicity

$$x\left(\frac{dR}{dx}\right)+y\left(\frac{dR}{dy}\right)+z\left(\frac{dR}{dz}\right)=\varrho R';$$

differentiating this relatively to α, we shall have

$$0=\frac{d^2\varrho\,\delta\varrho}{dt^2}+\frac{m\varrho\,\delta\varrho}{\varrho^3}+2\int\delta d R+\delta.\varrho R' \quad . \quad . \quad . \quad . \quad . \quad (S)$$

Call dv the indefinitely small arc intercepted between the two radius-vectors ϱ and $\varrho+d\varrho$; the element of the curve described by μ around M will be $\sqrt{d\varrho^2+\varrho^2 dv^2}$. We shall thus have

$$dx^2+dy^2+dz^2=d\varrho^2+\varrho^2 dv^2,$$

and the equation (Q) will become

$$0=\frac{\varrho^2 dv^2+d\varrho^2}{dt^2}-\frac{2m}{\varrho}+\frac{m}{a}+2\int dR.$$

Eliminating $\frac{m}{a}$ from this equation by means of equation (R) we shall have

$$\frac{\varrho^2 dv^2}{dt^2}=\frac{\varrho\, d^2\varrho}{dt^2}+\frac{m}{\varrho}+\varrho R'$$

whence we derive, by differentiating relatively to α,

$$\frac{2\varrho^2 dv.d\delta v}{dt^2}=\frac{\varrho\, d^2\delta\varrho-\delta\varrho\, d^2\varrho}{dt^2}-\frac{3m\varrho\,\delta\varrho}{\varrho^3}+\varrho\,\delta R'-R'\delta\varrho.$$

If we substitute in this equation for $\frac{m \varrho \delta \varrho}{\varrho^3}$ its value derived from equation (S), we shall have

$$d\delta v = \frac{d(d\varrho \delta \varrho + 2\varrho d\delta\varrho) + dt^2 \{3\int \delta d R + 2\varrho \delta R' + R' \delta \varrho\}}{\varrho^2 d v} \quad (T)$$

By means of the equations (S), (T), we can get as exactly as we wish the values of $\delta \varrho$ and of δv. But we must observe that dv being the angle intercepted between the radii ϱ and $\varrho + d\varrho$, the integral v of these angles is not wholly in one plane. To obtain the value of the angle described round M, by the projection of the radius-vector ϱ upon a fixed plane, denote by $v_{,}$, this last angle, and name s the tangent of the latitude of μ above this plane; then $\varrho (1 + s^2)^{-\frac{1}{2}}$ will be the expression of the projected radius-vector, and the square of the element of the curve described by μ will be

$$\frac{\varrho^2 dv_{,}^2}{1 + s^2} + d\varrho^2 + \frac{\varrho^2 ds^2}{(1 + s^2)^2};$$

But the square of this element is also $\varrho^2 dv^2 + d\varrho^2$; therefore we have, by equating these two expressions

$$dv_{,} = \frac{dv \sqrt{(1+s^2)^2 - \frac{ds^2}{dv^2}}}{\sqrt{1 + s^2}}.$$

We shall thus determine $dv_{,}$ by means of dv, when s is known.

If we take for the fixed plane, that of the orbit of μ at a given epoch, s and $\frac{ds}{dv}$ will evidently be of the order of perturbing forces. Neglecting therefore the squares and the products of these forces, we shall have $v = v_{,}$. In the Theory of the planets and of the comets, we may neglect these squares and products with the exception of some terms of that order, which particular circumstances render of sensible magnitude, and which it will be easy to determine by means of the equations (S) and (T). These last equations take a very simple form, when we take into account the first power only of the disturbing forces. In fact, we may then consider $\delta \varrho$ and δv as the parts of ϱ and v due to these forces; δR, $\delta . \varrho R'$ are what R and $\varrho R'$ become, when we substitute for the coordinates of the bodies their values relative to the elliptic motion: We may designate them by these last quantities when subjected to that condition. The equation (S) thus becomes,

$$0 = \frac{d^2 . \varrho \delta \varrho}{dt^2} + \frac{m \varrho \delta \varrho}{\varrho^3} + 2\int d R + \varrho R'.$$

The fixed plane of x, y being supposed that of the orbit of μ at a given epoch, z will be of the order of perturbing forces: and since we may neglect the square of these forces, we can also neglect the quantity $z\left(\frac{d\,R}{d\,z}\right)$. Moreover, the radius ϱ differs only from its projection by quantities of the order z^2. The angle which this radius makes with the axis of x, differs only from its projection by quantities of the same order. This angle may therefore be supposed equal to v and to quantities nearly of the same order

$$x = \varrho \cos. v\,;\ y = \varrho \sin. v\,;$$

whence we get

$$x\left(\frac{d\,R}{d\,x}\right) + y\left(\frac{d\,R}{d\,y}\right) = \varrho\left(\frac{d\,R}{d\,\varrho}\right);$$

and consequently $\varrho\,.\,R' = \varrho\left(\frac{d\,R}{d\,\varrho}\right)$. It is easy to perceive by differentiation, that if we neglect the square of the perturbing force, the preceding differential equation will become, by means of the two first equations (P)

$$\varrho\,\delta\varrho = \frac{x\int y\,d\,t\left\{2\int d\,R + \varrho\left(\frac{d\,R}{d\,\varrho}\right)\right\} - y\int x\,d\,t\left\{2\int d\,R + \varrho\left(\frac{d\,R}{d\,\varrho}\right)\right\}}{\left(\frac{x\,d\,y - y\,d\,x}{d\,t}\right)}.$$

In the second member of this equation the coordinates may belong to elliptic motion; this gives $\frac{x\,d\,y - y\,d\,x}{d\,t}$ constant and equal to $\sqrt{m\,a(1-e^2)}$, a e being the excentricity of the orbit of μ. If we substitute in the expression of $\varrho\,\delta\,\varrho$ for x and y, their values ϱ cos. v and ϱ sin. v, and for $\frac{x\,d\,y - y\,d\,x}{d\,t}$, the quantity $\sqrt{\mu\,a\,(1-e^2)}$; finally, if we observe that by No. (480)

$$m = n^2\,a^3,$$

we shall have

$$\delta\varrho = \frac{\left\{\begin{array}{l} a\cos. v\int n\,d\,t\,.\,\varrho\sin. v\left\{2\int d\,R + \varrho\left(\frac{d\,R}{d\,\varrho}\right)\right\} \\ -a\sin. v\int n\,d\,t\,.\,\varrho\cos. v\left\{2\int d\,R + \varrho\left(\frac{d\,R}{d\,\varrho}\right)\right\} \end{array}\right\}}{m\sqrt{1-e^2}}. \quad (X)$$

The equation (T) gives by integration and neglecting the square of perturbing forces,

$$d\,v = \frac{\frac{2\,\varrho\,d\,.\,\delta\varrho + d\,\varrho\,.\,\delta\varrho}{a^2\,n\,d\,t} + \frac{3\,a}{m}\int\int n\,d\,t\,.\,d\,R + \frac{2\,a}{m}\int n\,d\,t\,.\,\varrho\left(\frac{d\,R}{d\,\varrho}\right)}{\sqrt{1-e^2}} \quad (Y)$$

This expression, when the perturbations of the radius-vector are known, will easily give those of the motion of μ in longitude.

It remains for us to determine the perturbations of the motion in latitude. For that purpose let us resume the third of the equations (P): integrating this in the same manner as we have integrated the equation (S), and making $z = \varrho\, \delta s$, we shall have

$$\delta s = \frac{a \cos. v \int n\, dt . \varrho \sin. v \left(\frac{dR}{dz}\right) - a \sin. v \int n\, dt . \varrho \cos. v \left(\frac{dR}{dz}\right)}{m\sqrt{1-e^2}}; \quad (Z)$$

δs is the latitude of μ above the plane of its primitive orbit: if we wish to refer the motion of μ to a plane somewhat inclined to this orbit, by calling s its latitude, when it is supposed not to quit the plane of the orbit, $s + \delta s$ will be very nearly the latitude of μ above the proposed plane.

514. The formulas (X), (Y), (Z) have the advantage of presenting the perturbations under a finite form. This is very useful in the Cometary Theory, in which these perturbations can only be determined by quadratures. But the excentricity and inclination of the respective orbits of the planets being small, permits a developement of their perturbations into converging series of the sines and cosines of angles increasing proportionally to the time, and thence to make tables of them to serve for any times whatever. Then, instead of the preceding expressions of $\delta \varrho$, δs, it is more commodious to make use of differential equations which determine these variables. Ordering these equations according to the powers and products of the excentricities and inclinations of the orbits, we may always reduce the determination of the values of $\delta \varrho$, and of δs to the integration of equations of the form

$$0 = \frac{d^2 y}{dt^2} + n^2 y + Q$$

equations whose integrals we have already given in No. 509. But we can immediately reduce the preceding differential equations to this simple form, by the following method.

Let us resume the equation (R) of the preceding No., and abridge it by making

$$Q = 2\int d R + \varrho \left(\frac{dR}{d\varrho}\right).$$

It thus becomes

$$0 = \frac{1}{2}\frac{d^2 . \varrho^2}{dt^2} - \frac{m}{\varrho} + \frac{m}{a} + Q \ldots\ldots (R')$$

In the case of elliptic motion, where $Q = 0$, ϱ^2 is by No. (488) a function of e cos. $(n\,t + \varepsilon - \varpi)$, a e being the excentricity of the orbit, and $n\,t + \varepsilon - \varpi$ the mean anomaly of the planet μ. Let e cos. $(n\,t + \varepsilon - \varpi) = u$, and suppose $\varrho^2 = \varphi(u)$; we shall have

$$0 = \frac{d^2 u}{d t^2} + n^2 u.$$

In the case of disturbed motion, we can still suppose $\varrho^2 = \varphi(u)$, but u will no longer be equal to e cos. $(n\,t + \varepsilon - \varpi)$. It will be given by the preceding differential equation augmented by a term depending upon the perturbing forces. To determine this term, we shall observe that if we make $u = \psi(\varrho^2)$ we shall have

$$\frac{d^2 u}{d t^2} + n^2 u = \frac{d^2.\varrho^2}{d t^2}\psi'(\varrho^2) + \frac{4\varrho^2 d\varrho^2}{d t^2}\psi''(\varrho^2) + n^2.\psi(\varrho^2),$$

$\psi'(\varrho^2)$ being the differential of $\psi(\varrho^2)$ divided by $d.\varrho^2$ and $\psi''(\varrho^2)$ the differential of $\psi'(\varrho^2)$ divided by $d.\varrho^2$. The equation (R′) gives $\frac{d^2.\varrho^2}{d t^2}$ equal to a function of ϱ plus a function depending upon the perturbing force. If we multiply this equation by $2\varrho\,d\varrho$, and then integrate it, we shall have $\frac{\varrho^2 d\varrho^2}{d t^2}$ equal to a function of ϱ plus a function depending upon the perturbing force. Substituting these values of $\frac{d^2.\varrho^2}{d t^2}$ and of $\frac{\varrho^2 d\varrho^2}{d t^2}$ in the preceding expression of $\frac{d^2 u}{d t^2} + n^2 u$, the function of ϱ, which is independent of the perturbing force will disappear of itself, because it is identically nothing when that force is nothing. We shall therefore have the value of $\frac{d^2 u}{d t^2} + n^2 u$ by substituting for $\frac{d^2.\varrho^2}{d t^2}$, and $\frac{\varrho^2 d\varrho^2}{d t^2}$, the parts of their expressions which depend upon the perturbing force. But regarding these parts only, the equation (R′) and its integral give

$$\frac{d^2.\varrho^2}{d t^2} = -2Q;$$

$$\frac{4\varrho^2 d\varrho^2}{d t^2} = -8\int Q\varrho\,d\varrho.$$

Wherefore

$$\frac{d^2 u}{d t^2} + n^2 u = -2Q\psi'(\varrho^2) - 8\psi''(\varrho^2)\int Q.\varrho\,d\varrho.$$

Again, from the equation $u = \varphi(\varrho^2)$, we derive $d\,u = 2\varrho\,d\varrho\,\psi'(\varrho^2)$; this $\varrho^2 = \varphi(u)$ gives $2\varrho\,d\varrho = d\,u.\,\varphi'(u)$ and consequently

$$\Psi'(\varrho^2) = \frac{1}{\varphi'(u)}.$$

Differentiating this last equation and substituting $\varphi'(u)$ for $\frac{2\varrho\, d\varrho}{du}$, we shall have

$$\Psi''(\varrho^2) = -\frac{\varphi''(u)}{\varphi'(u)^3},$$

$\varphi''(u)$ being equal to $\frac{d.\varphi'(u)}{du}$, in the same way as $\varphi'(u)$ is equal to $\frac{d.\varphi(u)}{du}$. This being done; if we make

$$u = e\cos.(nt + \varepsilon - \varpi) + \delta u,$$

the differential equation in u will become

$$0 = \frac{d^2.\delta u}{dt^2} + n^2.\delta u - \frac{4.\varphi''(u)}{\varphi'(u)^3}\int Q\, du.\varphi'(u) + \frac{2Q}{\varphi'(u)};$$

and if we neglect the square of the perturbing force, u may be supposed equal to e cos. $(nt + \varepsilon - \varpi)$, in the terms depending upon Q.

The value of $\frac{\varrho}{a}$ found in No. (485) gives, including quantities of the order e^3

$$\varrho = a\left\{1 + e^2 - u\left(1 - \frac{3}{2}e^2\right) - u^2 - \frac{3}{2}u^3\right\}$$

whence we derive

$$\varrho^2 = a^2\left\{1 + 2e^2 - 2u\left(1 - \frac{1}{2}e^2\right) - u^2 - u^3\right\} = \varphi(u).$$

If we substitute this value of $\varphi(u)$ in the differential equation in δu, and restore to Q its value $2\int dR + \varrho\left(\frac{dR}{d\varrho}\right)$, and e cos. $(nt + \varepsilon - \varpi)$ for u, we shall have including quantities of the order e^3,

$$0 = \frac{d^2.\delta u}{dt^2} + n^2\delta u$$

$$-\frac{1}{a^2}\left\{1 + \frac{1}{4}e^2 - e\cos.(nt + \varepsilon - \varpi) - \frac{1}{4}e^2\cos.(2nt + 2\varepsilon - 2\varpi)\right\}$$

$$\times\left\{2\int dR + \varrho\left(\frac{dR}{d\varrho}\right)\right\}$$

$$-\frac{2e}{a^2}\int ndt\left[\sin.(nt+\varepsilon-\varpi)\{1+e\cos.(nt+\varepsilon-\varpi)\}\left\{2\int dR+\varrho\left(\frac{dR}{d\varrho}\right)\right\}\right] \quad (X')$$

When we shall have determined δu by means of this differential equa-

tion, we shall have $\delta\varrho$ by differentiating the expression of ϱ, relative to the characteristic δ, which gives

$$\delta\varrho = -a\,\delta u\left\{1+\frac{3}{4}e^2+2e\cos.(nt+\varepsilon-\varpi)+\frac{9}{4}e^2\cos.(2nt+2\varepsilon-2\varpi)\right\}.$$

This value of $\delta\varrho$ will give that of δv by means of formula (Y) of the preceding number.

It remains for us to determine δs; but if we compare the formulas (X) and (Z) of the preceding No. we perceive that $\delta\varrho$ changes itself into δs by substituting $\left(\frac{dR}{dz}\right)$ for $2\int dR+\varrho\left(\frac{dR}{d\varrho}\right)$ in its expression. Whence it follows that to get δs, it suffices to make this change in the differential equation in δu, and then to substitute the value of δu given by this equation, and which we shall designate by $\delta u'$, in the expression of $\delta\varrho$. Thus we get

$$0=\frac{d^2\delta u'}{dt^2}+n^2\delta u'$$

$$-\frac{1}{a^2}\left\{1+\frac{1}{4}e^2-e\cos.(nt+\varepsilon-\varpi)-\frac{1}{4}e^2\cos.(2nt+2\varepsilon-2\varpi)\right\}\left(\frac{dR}{dz}\right)$$

$$-\frac{2e}{a^2}\int n\,dt\left\{\sin.(nt+\varepsilon-\varpi)\{1+e\cos.(nt+\varepsilon-\varpi)\}.\left(\frac{dR}{dz}\right)\right\};\ (Z')$$

$$\delta s=-a\delta u'\left\{1+\frac{3}{4}e^2+2e\cos.(nt+\varepsilon-\varpi)+\frac{9}{4}e^2\cos.(2nt+2e-2\varpi)\right\}$$

The system of equations (X'), (Y), (Z') will give, in a very simple manner, the perturbed motion of μ in taking into account only the first power of the perturbing force. The consideration of terms due to this power being in the Theory of Planets very nearly sufficient to determine their motions, we proceed to derive from them formulas for that purpose.

515. It is first necessary to develope the function R into a series. If we disregard all other actions than that of μ upon μ', we shall have by (513)

$$R=\frac{\mu'(xx'+yy'+zz')}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}-\frac{\mu'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}}.$$

This function is wholly independent of the position of the plane of x, y; for the radical $\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$, expressing the distance of μ, μ', is independent of the position; the function $x^2+y^2+z^2+x'^2+y'^2+z'^2-2xx'-2yy'-2zz'$ is in like manner independent of it. But the squares $x^2+y^2+z^2$ and $x'^2+y'^2+z'^2$ of the radius-vectors, do not depend upon the position; and therefore the quantity $xx'+yy'+zz'$ does not depend upon it, and consequently

R is independent of the position of the plane of x, y. Suppose in this function

$$x = \varrho \cos. v; \quad y = \varrho \sin. v;$$
$$x' = \varrho' \cos. v'; \quad y' = \varrho' \sin. v';$$

we shall then have

$$R = \frac{\mu' \{\varrho \varrho' \cos.(v'-v) + z z'\}}{(\varrho'^2 + z'^2)^{\frac{3}{2}}} - \frac{\omega'}{\{\varrho^2 - 2\varrho\varrho' \cos.(v'-v) + \varrho'^2 + (z'-z)^2\}^{\frac{1}{2}}}.$$

The orbits of the planets being almost circular and but little inclined to one another, we may select the plane of x, y, so that z and z' may be very small. In this case ϱ and ϱ' are very little different from the semi-axis-majors a, a' of the elliptic orbits, we will therefore suppose

$$\varrho = a(1 + u_{,}); \quad \varrho' = a'(1 + u_{,}');$$

$u_{,}$ and $u_{,}'$ being small quantities. The angles $v_{,}$ v' differing but little from the mean longitudes $nt + \varepsilon$, $n't + \varepsilon'$, we shall suppose

$$v = nt + \varepsilon + v_{,}; \quad v' = n't + \varepsilon' + v_{,}';$$

v' and $v_{,}'$ being inconsiderable. Thus, reducing R into a series ordered according to the powers and products of $u_{,}$, $v_{,}$, z, $u_{,}'$, $v_{,}'$, and z', this series will be very convergent. Let

$$\frac{a}{a'^2} \cos.(n't - nt + \varepsilon' - \varepsilon) - \{a^2 - 2aa' \cos.(n't - nt + \varepsilon' - \varepsilon) + a'^2\}^{-\frac{1}{2}}$$
$$= \frac{1}{2} A^{(0)} + A^{(1)} \cos.(n't - nt + \varepsilon' - \varepsilon) + A^{(2)} \cos. 2(n't - nt + \varepsilon' - \varepsilon)$$
$$+ A^{(3)} \cos. 3(n't - nt + \varepsilon' - \varepsilon) + \&c.;$$

We may give to this series the form $\frac{1}{2} \Sigma A^{(i)} \cos. i(n't - nt + \varepsilon' - \varepsilon)$, the characteristic Σ of finite integrals, being relative to the number i, and extending itself to all whole numbers from $i = -\infty$ to $i = \infty$; the value $i = 0$, being comprised in this infinite number of values. But then we must observe that $A^{(-i)} = A^{(i)}$. This form has the advantage of serving to express after a very simple manner, not only the preceding series, but also the product of this series, by the sine or the cosine of any angle $ft + \varpi$; for it is perceptible that this product is equal to

$$\frac{1}{2} \Sigma A^{(i)} \frac{\sin.}{\cos.} \{i(n't - nt + \varepsilon' - \varepsilon) + ft + \varpi\}.$$

This property will furnish us with very commodious expressions for the perturbations of the planets. Let in like manner

$$\{a^2 - 2aa' \cos.(n't - nt + \varepsilon' - \varepsilon) + a'^2\}^{-\frac{3}{2}}$$
$$= \frac{1}{2} \Sigma B^{i} \cos. i(nt - nt + \varepsilon - \varepsilon);$$

$B^{(-i)}$ being equal to $B^{(i)}$. This being done, we shall have by (483)

$$R = \frac{\mu'}{2} . \Sigma A^{(i)} \cos. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$+ \frac{\mu'}{2} u_{,} \Sigma a \left(\frac{d A^{(i)}}{d a}\right) \cos. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$+ \frac{\mu'}{2} u_{,}' \Sigma a' \left(\frac{d A^{(i)}}{d a'}\right) \cos. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$- \frac{\mu'}{2} (v_{,}' - v_{,}) \Sigma . i A^{(i)} \sin. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$+ \frac{\mu'}{4} . u_{,}^{2} . \Sigma . a^{2} \left(\frac{d^{2} A^{(i)}}{d a^{2}}\right) \cos. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$+ \frac{\mu'}{2} u_{,} u_{,}' \Sigma a a' \left(\frac{d^{2} A^{(i)}}{d a \, d a'}\right) \cos. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$+ \frac{\mu'}{4} u_{,}'^{2} . \Sigma a'^{2} \left(\frac{d^{2} A^{(i)}}{d a'^{2}}\right) \cos. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$- \frac{\mu'}{2} (v_{,}' - v_{,}) u_{,} \Sigma . i a \left(\frac{d A^{(i)}}{d a}\right) \sin. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$- \frac{\mu'}{2} (v_{,}' - v_{,}) u_{,}' \Sigma . i a' \left(\frac{d A^{(i)}}{d a'}\right) \sin. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$- \frac{\mu'}{4} (v_{,}' - v_{,})^{2} . \Sigma . i^{2} A^{(i)} \cos. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$+ \frac{\mu' z z'}{a'^{3}} - \frac{3 \mu' a z'^{2}}{2 a'^{4}} \cos. (n' t - n t + \varepsilon' - \varepsilon)$$

$$+ \frac{\mu' (z' - z)^{2}}{4} \Sigma B^{(i)} \cos. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$+ \&c.$$

If we substitute in this expression of R, instead of $u_{,}$, $u_{,}'$, $v_{,}$, $v_{,}'$, z and z', their values relative to elliptic motion, values which are functions of sines and cosines of the angles $n t + \varepsilon$, $n' t + \varepsilon'$ and of their multiples, R will be expressed by an infinite series of cosines of the form $\mu' k \cos. (i n' t - i n t + A)$, i and i' being whole numbers.

It is evident that the action of μ'', μ''', &c. upon μ will produce in R terms analogous to those which result from the action of μ', and we shall obtain them by changing in the preceding expression of R, all that relates to μ', in the same quantities relative to μ'', μ''', &c.

Let us consider any term $\mu' k \cos. (i' n' t - i n t + A)$ of the expression of R. If the orbits were circular, and in one plane we should have $i' = i$. Therefore i' cannot surpass i or be exceeded by it, except by means of the sines or cosines of the expression for $u_{,}$, $v_{,}$, z, $u_{,}'$, $v_{,}'$, z' which combined with the sines and cosines of the angle $n' t - n t + \varepsilon' - \varepsilon$

and of its multiples, produce the sines and cosines of angles in which i' is different from i.

If we regard the excentricities and inclinations of the orbits as very small quantities of the first order, it will result from the theorems of (481) that in the expressions of $u_{,}$, $v_{,}$, z or ϱ s, s being the tangent of the latitude of μ, the coefficient of the sine or of the cosine of an angle such as f. (n t + ε), is expressed by a series whose first term is of the order f; second term of the order f + 2; third term of the order f + 4 and so on. The same takes place with regard to the coefficient of the sine or of the cosine of the angle f' (n' t + ε') in the expressions of $u'_{,}$, $v'_{,}$, z'. Hence it follows that i, and i' being supposed positive and i' greater than i, the coefficient k in the term m' k cos. (i' n' t — i n t + A) is of the order i' — i, and that in the series which expresses it, the first term is of the order i' — i the second of the order i' — i + 2 and so on; so that the series is very convergent. If i be greater than i', the terms of the series will be successively of the orders i — i', i — i' + 2, &c.

Call ϖ the longitude of the perihelion of the orbit of μ and θ that of its node, in like manner call ϖ' the longitude of the perihelion of μ', and θ' that of its node, these longitudes being reckoned upon a plane inclined to that of the orbits. It results from the Theorems of (481), that in the expressions of $u_{,}$, $v_{,}$, and z, the angle n t + ε is always accompanied by $-\varpi$ or by $-\theta$; and that in the expressions of $u'_{,}$, $v'_{,}$, and z', the angle n' t + ε' is always accompanied by $-\varpi'$, or by $-\theta'$; whence it follows that the term μ' k cos. (i' n' t — i n t + A) is of the form

$$\mu' \text{ k cos. } (\text{i n' t} - \text{i n t} + \text{i'} \varepsilon' - \text{i} \varepsilon - \text{g} \varpi - \text{g'} \varpi' - \text{g''} \theta - \text{g'''} \theta'),$$

g, g', g'', g''' being whole positive or negative numbers, and such that we have

$$0 = \text{i'} - \text{i} - \text{g} - \text{g'} - \text{g''} - \text{g'''}.$$

It results also from this that the value of R, and its different terms are independent of the position of the straight line from which the longitudes are measured. Moreover in the Theorems of (No. 481) the coefficient of the sine and cosine of the angle ϖ, has always for a factor the excentricity e of the orbit of μ; the coefficient of the sine and of the cosine of the angle 2 ϖ, has for a factor the square e^2 of this excentricity, and so on. In like manner, the coefficient of the sine and cosine of the angle θ, has for its factor tan. $\frac{1}{2} \varphi$, φ being the inclination of the orbit of μ upon the fixed plane. The coefficient of the sine, and of the cosine of the angle 2 θ, has for its factor tan.$^2 \frac{1}{2} \varphi$, and so on. Whence it results that the coefficient k has for its factor, e^{g}. $e'^{g'}$. tan.$^{g''}$ $(\frac{1}{2} \varphi)$ tan.$^{g'''}$ $(\frac{1}{2} \varphi')$; the numbers g, g', g'', g''' being

taken positively in the exponents of this factor. If all these numbers are positive, this factor will be of the order $i' - i$, by virtue of the equation

$$0 = i' - i - g - g' - g'' - g''';$$

but if one of them such as g, is negative and equal to $-g$, this factor will be of the order $i' - i + 2g$. Preserving, therefore, amongst the terms of R, only those which depending upon the angle $i' n' t - i n t$ are of the order $i' - i$, and rejecting all those which depending upon the same angle, are of the order $i' - i + 2$, $i' - i + 4$, &c.; the expression of R will be composed of terms of the form

$$H e^{g} e'^{g'} \tan.^{g''} \left(\frac{1}{2}\varphi\right) \tan.^{g'''} \left(\frac{1}{2}\varphi'\right) \cos. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - g.\varpi - g'.\varpi' - g''.\theta - g'''.\theta'),$$

H being a coefficient independent of the excentricities, and inclinations of the orbits, and the numbers g, g', g'', g''' being all positive, and such that their sum is equal to $i' - i$.

If we substitute in R, $a(1 + u_{,})$, instead of ϱ, we shall have

$$\varrho\left(\frac{dR}{d\varrho}\right) = a\left(\frac{dR}{da}\right).$$

If in this same function, we substitute instead of u', v' and z, their values given by the theorems of (481), we shall have

$$\left(\frac{dR}{dv}\right) = \left(\frac{dR}{d\varepsilon}\right),$$

provided that we suppose $\varepsilon - \varpi$, and $\varepsilon - \theta$ constant in the differential of R, taken relatively to ε; for then $u_{,}$, $v_{,}$ and z are constant in this differential, and since we have $v = nt + \varepsilon + v_{,}$, it is evident that the preceding equation still holds. We shall, therefore, easily obtain the values of $\varrho\left(\frac{dR}{d\varrho}\right)$, and of $\left(\frac{dR}{dv}\right)$, which enter into the differential equations of the preceding numbers, when we shall have the value of R developed into a series of angles increasing proportionally to the time t. The differential d R it will be in like manner easy to determine, observing to vary in R the angle n t, and to suppose n' t constant; for d R is the difference of R, taken in supposing constant, the coordinates of μ', which are functions of n' t.

516. The difficulty of the developement of R into a series, may be reduced to that of forming the quantities $A^{(i)}$, $B^{(i)}$, and their differences taken relatively to a and to a'. For that purpose consider generally the function

$$(a^2 - 2aa' \cos.\theta + a'^2)^{-s}$$

and develope it according to the cosine of the angle θ and its multiples.

If we make $\frac{a}{a'} = \alpha$, it will become

$$a'^{-2s}.\{1 - 2\alpha\cos.\theta + \alpha^2\}^{-s}.$$

Let

$$(1 - 2\alpha\cos.\theta + \alpha^2)^{-s} = \tfrac{1}{2}\,{}_{s}b^{(0)} + {}_{s}b^{(1)}\cos.\theta + {}_{s}b^{(2)}\cos.2\theta + {}_{s}b^{(3)}\cos.3\theta + \&c.$$

${}_{s}b^{(0)}$, ${}_{s}b^{(1)}$, ${}_{s}b^{(2)}$, &c. being functions of α and of s. If we take the logarithmic differences of the two members of this equation, relative to the variable θ, we shall have

$$\frac{-2s\alpha\sin.\theta}{1 - 2\alpha\cos.\theta + \alpha^2} = \frac{-{}_{s}b^{(1)}\sin.\theta - 2\,{}_{s}b^{(2)}\sin.2\theta - \&c.}{\tfrac{1}{2}\,{}_{s}b^{(0)} + {}_{s}b^{(1)}\cos.\theta + {}_{s}b^{(2)}\cos.2\theta + \&c.}.$$

Multiplying this equation crosswise, and comparing similar cosines, we find generally

$${}_{s}b^{(i)} = \frac{(i-1)(1+\alpha^2)\,{}_{s}b^{(i-1)} - (i+s-2)\,a\,{}_{s}b^{(i-2)}}{(i-s).\alpha} \quad \ldots \text{(a)}$$

We shall thus have ${}_{s}b^{(2)}$, ${}_{s}b^{(3)}$, &c. when ${}_{s}b^{(0)}$ and ${}_{s}b^{(1)}$ are known.

If we change s into s + 1, in the preceding expression of $(1 - 2\alpha\cos.\theta + \alpha^2)^{-s}$, we shall have

$$(1-2\alpha\cos.\theta+\alpha^2)^{-s-1} = \tfrac{1}{2}\,{}_{s+1}b^{(0)} + {}_{s+1}b^{(1)}\cos.\theta + {}_{s+1}b^{(2)}\cos.2\theta + {}_{s+1}b^{(3)}\cos.3\theta + \&c.$$

Multiplying the two members of this equation, by $1 - 2\alpha\cos.\theta + \alpha^2$, and substituting for $(1 - 2\alpha\cos.\theta + \alpha^2)^{-s}$ its value in series, we shall have

$$\tfrac{1}{2}\,{}_{s}b^{(0)} + {}_{s}b^{(1)}\cos.\theta + {}_{s}b^{(2)}\cos.2\theta + \&c.$$
$$= (1 - 2\alpha\cos.\theta + \alpha^2)\{\,{}_{s+1}b^{(0)} + {}_{s+1}b^{(1)}\cos.\theta + {}_{s+1}b^{(2)}\cos.2\theta + \&c.\}$$

whence by comparing homogeneous terms, we derive

$${}_{s}b^{(i)} = (1+\alpha^2)\,{}_{s+1}b^{(i)} - \alpha\,{}_{s+1}b^{(i-1)} - \alpha\,{}_{s+1}b^{(i+1)}.$$

The formula (a) gives

$${}_{s+1}b^{(i+1)} = \frac{i(1+\alpha^2)\,{}_{s+1}b^{(i)} - (i+s)\,\alpha\,{}_{s+1}b^{(i-1)}}{(i-s).\alpha};$$

The preceding expression of ${}_{s}b^{(i)}$ will thus become

$${}_{s}b^{(i)} = \frac{2s.\alpha\,{}_{s+1}b^{(i-1)} - s(1+\alpha^2)\,{}_{s+1}b^{(i)}}{i-s}.$$

Changing i into i + 1 in this equation we shall have

$$b^{(i+1)}_{s} = \frac{2 s \alpha b^{(i)}_{s+1} - s(1+\alpha^2) b^{(i+1)}_{s+1}}{i - s + 1}$$

and if we substitute for $b^{(i+1)}_{s+1}$ its preceding value, we shall have

$$b^{(i+1)}_{s} = \frac{s(i+s)\alpha(1+\alpha^2) b^{(i-1)}_{s+1} + s\{2(i-s)\alpha^2 - i(1+\alpha^2)^2\} b^{(i)}_{s+1}}{(i-s)(i-s+1)\alpha}$$

These two expressions of $b^{(i)}_{s}$ and $b^{(i+1)}_{s}$ give

$$b^{(i)}_{s+1} = \frac{\frac{(i+s)}{s} \cdot (1+\alpha^2) b^{(i)}_{s} - 2 \cdot \frac{i-s+1}{s} \alpha b^{(i+1)}_{s}}{(1-\alpha^2)^2}; \quad \text{(b)}$$

substituting for $b^{(i+1)}_{s}$ its value derived from equation (a), we shall have

$$b^{(i)}_{s+1} = \frac{\frac{(s-i)}{s}(1+\alpha^2) b^{(i)}_{s} + \frac{2(i+s-1)}{s} \cdot \alpha \cdot b^{(i-1)}_{s}}{(1-\alpha^2)^2}; \quad \text{(c)}$$

an expression which may be derived from the preceding by changing i into — i, and observing that $b^{(i)} = b^{(-i)}$. We shall therefore have by means of this formula, the values of $b^{(0)}_{s+1}$, $b^{(1)}_{s+1}$, $b^{(2)}_{s+1}$, &c. when those of $b^{(0)}_{s}$, $b^{(1)}_{s}$, $b^{(2)}_{s}$, &c. are known.

Let λ, for brevity, denote the function $1 - 2\alpha \cos.\theta + \alpha^2$. If we differentiate relatively to α, the equation

$$\lambda^{-s} = \tfrac{1}{2} b^{(0)}_{s} + b^{(1)}_{s} \cos.\theta + b^{(2)}_{s} \cos.2\theta + \&c.$$

we shall have

$$-2s(\alpha - \cos.\theta)\lambda^{-s-1} = \tfrac{1}{2} \cdot \frac{d b^{(0)}_{s}}{d\alpha} + \frac{d b^{(1)}_{s}}{d\alpha} \cos.\theta + \frac{d b^{(2)}_{s}}{d\alpha} \cos.2\theta + \&c.$$

But we have

$$-\alpha + \cos.\theta = \frac{1-\alpha^2-\lambda}{2\alpha};$$

We shall, therefore, have

$$\frac{s(1-\alpha^2)}{\alpha}\lambda^{-s-1} - \frac{s\lambda^{-s}}{\alpha} = \tfrac{1}{2}\frac{d b^{(0)}_{s}}{d\alpha} + \frac{d b^{(1)}_{s}}{d\alpha} \cos.\theta + \&c.$$

whence generally we get

$$\frac{d b^{(i)}_{s}}{d\alpha} = \frac{s(1-\alpha^2)}{\alpha} b^{(i)}_{s+1} - \frac{s b^{(i)}_{s}}{\alpha}.$$

Substituting for $b^{(i)}_{s+1}$ its value given by the formula (b), we shall have

$$\frac{d b^{(i)}_{s}}{d\alpha} = \frac{i + (i+2s)\alpha^2}{\alpha(1-\alpha^2)} \cdot b^{(i)}_{s} - \frac{2(i-s+1)}{1-\alpha^2} \cdot b^{(i+1)}_{s}.$$

If we differentiate this equation, we shall have

$$\frac{d^2 b^{(i)}_{s}}{d\alpha^2} = \frac{i+(i+2s)\alpha^2}{\alpha(1-\alpha^2)} \cdot \frac{d b^{(i)}_{s}}{d\alpha} + \left\{ \frac{2(i+s)(1+\alpha^2)}{(1-\alpha^2)^2} - \frac{i}{\alpha^2} \right\} b^{(i)}_{s}$$
$$- \frac{2(i-s+1)}{1-\alpha^2} \cdot \frac{d b^{(i+1)}_{s}}{d\alpha} - \frac{4(i-s+1)\alpha}{(1-\alpha^2)^2} b^{(i+1)}_{s}.$$

Again differentiating, we shall get

$$\frac{d^3 b^{(i)}_{s}}{d\alpha^3} = \frac{i+(i+2.s)\alpha^2}{\alpha(1-\alpha^2)} \cdot \frac{d^2 b^{(i)}_{s}}{d\alpha^2} + 2\left\{ \frac{(i+s)(1+\alpha^2)}{(1-\alpha^2)^2} - \frac{i}{\alpha^2} \right\} \frac{d b^{(i)}_{s}}{d\alpha}$$
$$+ \left\{ \frac{4(i+s)\alpha(3+\alpha^2)}{(1-\alpha^2)^3} + \frac{2i}{\alpha^3} \right\} b^{(i)}_{s} - \frac{2(i-s+1)}{1-\alpha^2} \cdot \frac{d^2 b^{(i+1}_{s}}{d\alpha^2}$$
$$- \frac{8(i-s+1)\alpha}{(1-\alpha^2)^2} \cdot \frac{d b^{(i+1)}_{s}}{d\alpha} - \frac{4(i-s+1)(1+3\alpha^2)}{(1-\alpha^2)^3} b^{(i+1)}_{s}.$$

Thus we perceive that in order to determine the values of $b^{(i)}_{s}$ and of its successive differences, it is sufficient to know those of $b^{(0)}_{s}$ and of $b^{(1)}_{s}$. We shall determine these two as follows:

If we call c the hyperbolic base, we can put the expression of λ^{-s} under this form

$$\lambda^{-s} = (1 - \alpha c^{\theta\sqrt{-1}})^{-s} (1 - \alpha c^{-\theta\sqrt{-1}})^{-s}.$$

Developing the second member of this equation relatively to the powers of $c^{\theta\sqrt{-1}}$, and $c^{-\theta\sqrt{-1}}$, it is evident the two exponentials $c^{i\theta\sqrt{-1}}$, $c^{-i\theta\sqrt{-1}}$ will have the same coefficient which we denote by k. The sum of the two terms $k \cdot c^{i\theta\sqrt{-1}}$ and $k\, c^{-i\theta\sqrt{-1}}$ is $2k \cos. i\theta$. This will be the value of $b^{(i)}_{s} \cos. i\theta$. We have, therefore, $b^{(i)}_{s} = 2k$. Again the expression of λ^{-s} is equal to the product of the two series

$$1 + s\alpha c^{\theta\sqrt{-1}} + \frac{s(s+1)}{1.2}\alpha^2 c^{2\theta\sqrt{-1}} + \&c.$$

$$1 + s\alpha c^{-\theta\sqrt{-1}} + \frac{s(s+1)}{1.2}\alpha^2 c^{-2\theta\sqrt{-1}} + \&c.;$$

multiplying therefore these two together, we shall have when $i = 0$

$$k = 1 + s^2\alpha^2 + \left(\frac{s(s+1)}{1.2}\right)^2 \alpha^4 + \&c.;$$

and in the case of $i = 1$,

$$k = \alpha\left\{ s + s \cdot \frac{s(s+1)}{1.2} \cdot \alpha^2 + \frac{s(s+1)}{1.2} \cdot \frac{s(s+1)(s+2)}{1.2.3}\alpha^4 + \&c. \right\}$$

wherefore

$$b^{(0)}_{s} = 2\left\{ 1 + s^2\alpha^2 + \left(\frac{s(s+1)}{1.2}\right)^2 \alpha^4 + \left(\frac{s(s+1)(s+2)}{1.2.3}\right)^2 \alpha^6 + \&c. \right\}$$

$$b^{(1)}_{s} = 2\alpha\left\{s + s.\frac{s(s+1)}{1.2}\alpha^2 + \frac{s(s+1)}{1.2}.\frac{s(s+1)(s+2)}{1.2.3}\alpha^4 + \&c.\right\}.$$

That these series may be convergent, we must have α less than unity, which can always be made so, unless $a = a'$; α being $= \frac{a}{a'}$, we have only to take the greater for the denominator.

In the theory of the motion of the bodies μ, μ', μ'', &c. we have occasion to know the values of $b^{(0)}_{s}$ and of $b^{(1)}_{s}$ when $s = \frac{1}{2}$ and $s = \frac{3}{2}$. In these two cases, these values have but little convergency unless α is a small fraction.

The series converge with greater rapidity when $s = -\frac{1}{2}$, and we have

$$\frac{1}{2}\,b^{(0)}_{-\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)^2\alpha^2 + \left(\frac{1.1}{2.4}\right)^2\alpha^4 + \left(\frac{1.1.3}{2.4.6}\right)^2\alpha^6 + \left(\frac{1.1.3.5}{2.4.6.8}\right)^2\alpha^8 + \&c.$$

$$b^{(1)}_{-\frac{1}{2}} = -\alpha\left\{1 - \frac{1.1}{2.4}\alpha^2 - \frac{1}{4}.\frac{1.1.3}{2.4.6}\alpha^4 - \frac{1.3}{4.6}.\frac{1.1.3.5}{2.4.6.8}\alpha^6 - \frac{1.3.5}{4.6.8}.\frac{1.1.3\ldots7}{2.3\ldots10}\right\}\alpha^8 + \&c.$$

In the Theory of the planets and satellites, it will be sufficient to take the sum of eleven or a dozen first terms, in neglecting the following terms or more exactly in summing them as a geometric progression whose common ratio is $1 - \alpha^2$. When we shall have thus determined $b^{(0)}_{-\frac{1}{2}}$ and $b^{(1)}_{-\frac{1}{2}}$, we shall have $b^{(0)}_{\frac{1}{2}}$ in making $i = 0$, and $s = -\frac{1}{2}$ in the formula (b), and we shall find

$$b^{(0)}_{\frac{1}{2}} = \frac{(1+\alpha^2)\,b^{(0)}_{-\frac{1}{2}} + 6\alpha\,b^{(1)}_{-\frac{1}{2}}}{(1-\alpha^2)^2}$$

If in the formula (c) we suppose $i = 1$ and $s = -\frac{1}{2}$ we shall have

$$b^{(1)}_{\frac{1}{2}} = \frac{2\alpha\,b^{(0)}_{-\frac{1}{2}} + 3(1+\alpha^2)\,b^{(1)}_{-\frac{1}{2}}}{(1-\alpha^2)^2}.$$

By means of these values of $b^{(0)}_{\frac{1}{2}}$ and of $b^{(1)}_{\frac{1}{2}}$ we shall have by the preceding forms the values of $b^{(i)}_{\frac{1}{2}}$ and of its partial differences whatever may be the number i; and thence we derive the values of $b^{(i)}_{\frac{3}{2}}$ and of its differences. The values of $b^{(0)}_{\frac{3}{2}}$ and of $b^{(1)}_{\frac{3}{2}}$ may be determined very simply,

by the following formulæ

$$b^{(0)}_{\frac{1}{2}} = \frac{-\frac{1}{2}}{(1-\alpha^2)^2}; \quad b^{(1)}_{\frac{3}{2}} = -3 \cdot \frac{b^{(1)}_{\frac{1}{2}}}{(1-\alpha^2)^2}.$$

Again to get the quantities $A^{(0)}$, $A^{(1)}$, &c. and their differences, we must observe that by the preceding No., the series

$$\tfrac{1}{2} A^{(0)} + A^{(1)} \cos.\theta + A^{(2)} \cos. 2\theta + \&c.$$

results from the developement of the function

$$\frac{a \cos.\theta}{a'^2} - (a^2 - 2aa' \cos.\theta + a'^2)^{-\frac{1}{2}},$$

into a series of cosines of the angle θ and of its multiples. Making $\frac{a}{a'} = \alpha$, this same function becomes

$$-\frac{1}{2a'} b^{(0)}_{\frac{1}{2}} + \left(\frac{a}{a'^2} - \frac{1}{a'} b^{(1)}_{\frac{1}{2}}\right) \cos.\theta - \frac{1}{a'} b^{(2)}_{\frac{1}{2}} \cos. 2\theta - \&c.$$

which gives generally

$$A^{(i)} = -\frac{1}{a'} \cdot b^{(i)}_{\frac{1}{2}};$$

when i is zero, or greater than 1, abstraction being made of the sign.

In the case of $i = 1$, we have

$$A^{(i)} = \frac{a}{a'^2} - \frac{1}{a'} b^{(1)}_{\frac{1}{2}}.$$

We have next

$$\left(\frac{dA^{(i)}}{da}\right) = -\frac{1}{a'} \cdot \frac{db^{(i)}_{\frac{1}{2}}}{d\alpha} \left(\frac{d\alpha}{da}\right);$$

But we have $\frac{d\alpha}{da} = \frac{1}{a'}$; therefore

$$\left(\frac{dA^{(i)}}{da}\right) = -\frac{1}{a'^2} \cdot \frac{db^{(i)}_{\frac{1}{2}}}{d\alpha};$$

and in the case of $i = 1$, we have

$$\left(\frac{dA^{(1)}}{d\alpha}\right) = \frac{1}{a'^2} \left\{1 - \frac{db^{(1)}_{\frac{1}{2}}}{d\alpha}\right\}.$$

Finally, we have, in the same case of $i = 1$

$$\left(\frac{d^2 A^{(1)}}{da^2}\right) = -\frac{1}{a'^3} \cdot \frac{d^2 b^{(i)}_{\frac{1}{2}}}{d\alpha^2};$$

$$\left(\frac{d^3 A^{(i)}}{d a^3}\right) = -\frac{1}{a'^4}\cdot\frac{d^3 b^{(i)}}{d\alpha^3}_{\frac{1}{2}};$$

&c.

To get the differences of $A^{(i)}$ relative to a', we shall observe that $A^{(i)}$ being a homogeneous function in a and a', of the dimension — 1, we have by the nature of such functions,

$$a\left(\frac{d A^{(i)}}{d a}\right) + a'\left(\frac{d A^{(i)}}{d a'}\right) = -A^{(i)};$$

whence we get

$$a'\left(\frac{d A^{(i)}}{d a'}\right) = -A^{(i)} - a\left(\frac{d A^{(i)}}{d a}\right);$$

$$a'\left(\frac{d^2 A^{(i)}}{d a\, d a'}\right) = -2\left(\frac{d A^{(i)}}{d a}\right) - a\left(\frac{d^2 A^{(i)}}{d a^2}\right);$$

$$a'^2\left(\frac{d^2 A^{(i)}}{d a'^2}\right) = 2A^{(i)} + 4a\left(\frac{d A^{(i)}}{d a}\right) + a^2\left(\frac{d^2 A^{(i)}}{d a^2}\right);$$

$$a'^2\left(\frac{d^3 A^{(i)}}{d a'^2 d a}\right) = 6\left(\frac{d A^{(i)}}{d a}\right) + 6a\left(\frac{d^2 A^{(i)}}{d a^2}\right) + a^2\left(\frac{d^3 A^{(i)}}{d a^3}\right);$$

$$a'^3\left(\frac{d^3 A^{(i)}}{d a'^3}\right) = -6A^{(i)} - 18a\left(\frac{dA^{(i)}}{d a}\right) - 9a^2\left(\frac{d^2 A^{(i)}}{d a^2}\right) - a^3\left(\frac{d^3 A^{(i)}}{d a^3}\right);$$

&c.

We shall get $B^{(i)}$ and its differences, by observing that by the No. preceding, the series

$$\tfrac{1}{2} B^{(0)} + B^{(1)} \cos.\theta + B^{(2)} \cos. 2\theta + \&c.$$

is the developement of the function

$$a'^{-3}(1 - 2\alpha\cos.\theta + \alpha^2)^{-\frac{3}{2}}$$

according to the cosine of the angle θ and its multiples. But this function thus developed is equal to

$$a'^{-3}\left\{\tfrac{1}{2} b^{(0)}_{\frac{3}{2}} + b^{(1)}_{\frac{3}{2}} \cos.\theta + b^{(2)}_{\frac{3}{2}} \cos. 2\theta + \&c.\right\};$$

therefore we have generally

$$B^{(i)} = \frac{1}{a'^3} b^{(i)}_{\frac{3}{2}};$$

Whence we derive

$$\left(\frac{d B^{(i)}}{d a}\right) = \frac{1}{a'^4}\cdot\frac{d b^{(i)}}{d a}_{\frac{3}{2}}; \quad \left(\frac{d^2 B^{(i)}}{d a^2}\right) = \frac{1}{a'^5}\cdot\frac{d^2 b^{(i)}}{d a^2}_{\frac{3}{2}}; \quad \&c.$$

Moreover, $B^{(i)}$ being a homogeneous function of a and of a', of the dimension — 3 we have

$$a\left(\frac{d B^{(i)}}{d a}\right) + a'\left(\frac{d B^{(i)}}{d a'}\right) = -3 B^{(i)};$$

whence it is easy to get the partial differences of $B^{(i)}$ taken relatively to a' by means of those in a.

In the theory of the Perturbations of μ', by the action of μ, the values of $A^{(i)}$ and of $B^{(i)}$, are the same as above with the exception of $A^{(i)}$ which in this theory becomes $\frac{a'}{a^2} - \frac{1}{a'} b^{(i)}_{\frac{1}{2}}$. Thus the estimate of the values of $A^{(i)}$, $B^{(i)}$, and their differences will serve also for the theories of the two bodies μ and μ'.

517. After this digression upon the developement of R into series, let us resume the differential equations (X'), (Y), (Z') of Nos. 513, 514; and find by means of them, the values of $\delta \varrho$, δv, and δs true to quantities of the order of the excentricities and inclinations of orbits.

If in the elliptic orbits, we suppose

$$\varrho = a(1 + u_,); \quad \varrho' = a'(1 + u_,');$$
$$v = n t + \varepsilon + v_,; \quad v' = n' t - \varepsilon' + v_,';$$

we shall have by No. (488)

$$u_, = -e \cos.(n t + \varepsilon - \varpi); \quad u_,' = -e' \cos.(n' t + \varepsilon' - \varpi');$$
$$v_, = 2 e \sin.(n t + \varepsilon - \varpi); \quad v_,' = 2 e' \sin.(n' t + \varepsilon' - \varpi');$$

$n t + \varepsilon$, $n' t + \varepsilon'$ being the mean longitudes of μ, μ'; a, a' being the semi-axis-majors of their orbits; e, e' the ratios of the excentricity to the semi-axis-major; , and lastly ϖ, ϖ' being the longitudes of their perihelions. All these longitudes may be referred indifferently to the planes of the orbits, or to a plane which is but very little inclined to the orbits; since we neglect quantities of the order of the squares and products of the excentricities and inclinations. Substituting the preceding values in the expression of R in No. 515, we shall have

$$R = \frac{\mu'}{2} \Sigma A^{(i)} \cos. i(n' t - n t + \varepsilon' - \varepsilon)$$
$$- \frac{\mu'}{2} . \Sigma . \left\{ a\left(\frac{d A^{(i)}}{d a}\right) + 2 i A^{(i)} \right\} \times$$
$$e \cos.\{i(n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi\}$$
$$- \frac{\mu'}{2} \Sigma \left\{ a'\left(\frac{d A^{(i-1)}}{d a'}\right) - 2(i - 1) A^{(i-1)} \right\} \times$$
$$e' \cos.\{i(n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi'\};$$

the symbol Σ of finite integrals, extending to all the whole positive and negative values of i, not omitting the value $i = 0$.

Hence we obtain

$$2 \int d R + \varrho\left(\frac{d R}{d \varrho}\right) = 2 \mu' g + \frac{\mu'}{2} a\left(\frac{d A^{(0)}}{d a}\right)$$

$$+\frac{\mu'}{2}\Sigma\left\{a\left(\frac{d\,A^{(i)}}{d\,a}\right)+\frac{2\,n}{n-n'}A^{(i)}\right\}\cos.\,i\,(n'\,t-n\,t+\varepsilon'-\varepsilon)$$

$$-\frac{\mu'}{2}\left\{a^2\left(\frac{d^2\,A^{(0)}}{d\,a^2}\right)+3\,a\left(\frac{d\,A^{(0)}}{d\,a}\right)\right\}e\cos.\,(n\,t+\varepsilon-\varpi)$$

$$-\frac{\mu'}{2}\left\{a\,a'\left(\frac{d^2A^{(1)}}{da\,da'}\right)+2a\left(\frac{dA^{(1)}}{d\,a}\right)+2a'\left(\frac{dA^{(1)}}{d\,a'}\right)+4A^{(1)}\right\}.e'\cos.(nt+\varepsilon-\varpi')$$

$$-\frac{\mu}{2}\Sigma\left\{\begin{matrix}a^2\left(\frac{d^2A^{(i)}}{d\,a^2}\right)+(2\,i+1)\,a\left(\frac{d\,A^{(i)}}{d\,a}\right)\\+\frac{2(i-1)\,n}{i(n-n')-n}\left\{a\left(\frac{dA^{(i)}}{d\,a}\right)+2iA^{(i)}\right\}\end{matrix}\right\}e\cos.\{i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}$$

$$-\frac{\mu'}{2}\Sigma\left\{\begin{matrix}a\,a'\left(\frac{d^2A^{(i-1)}}{d\,a\,d\,a'}\right)-2\,(i-1)\,a\left(\frac{d\,A^{(i-1)}}{d\,a}\right)\\+\frac{2\,(i-1)\,n}{i(n-n')-n}\left\{a'\left(\frac{dA^{(i-1)}}{d\,a'}\right)-2(i-1)A^{(i-1)}\right\}\end{matrix}\right\}e'\cos.\{i\,(n'\,t-n\,t+\varepsilon'-\varepsilon)$$

$$+\,n\,t+\varepsilon-\varpi'\};$$

the integral sign Σ extending, as in what follows, to all integer positive and negative values of i, the value $i=0$ being alone excepted, because we have brought from without this symbol, the terms in which $i=0$: $\mu'\,g$ is a constant added to the integral $\int d\,R$. Making therefore

$$C=\tfrac{1}{2}\,a^3\left(\frac{d^2\,A^{(0)}}{d\,a^2}\right)+3\,a^2\left(\frac{d\,A^{(0)}}{d\,a'}\right)+6\,a\,g;$$

$$D=\tfrac{1}{2}\,a^2\,a'\left(\frac{d^2\,A^{(1)}}{d\,a\,d\,a'}\right)+a^2\left(\frac{d\,A^{(1)}}{d\,a}\right)+a\,a'\left(\frac{d\,A^{(1)}}{d\,a'}\right)+2\,a\,A^{(1)}$$

$$C^{(i)}=\tfrac{1}{2}\,a^3\left(\frac{d^2\,A^{(i)}}{d\,a^2}\right)+\frac{2\,i+1}{2}\,a^2\left(\frac{d\,A^{(i)}}{d\,a}\right)$$
$$+\frac{i\,(n-n')-3\,n}{2\{i\,(n-n')-n\}}\cdot\left\{a^2\left(\frac{d\,A^{(i)}}{d\,a}\right)+\frac{2\,n}{n-n'}\,a\,A^{(i)}\right\}$$
$$+\frac{(i-1)\,n}{i\,(n-n')-n}\left\{a^2\left(\frac{d\,A^{(i)}}{d\,a}\right)+2\,i\,a\,A^{(i)}\right\};$$

$$D^{(i)}=\tfrac{1}{2}\,a^2\,a'\left(\frac{d^2\,A^{(i-1)}}{d\,a\,d\,a'}\right)-(i-1)\,a^2\left(\frac{d\,A^{(i-1)}}{d\,a}\right)$$
$$+\frac{(i-1)\,n}{i\,(n-n')-n}\left\{a\,a'\left(\frac{d\,A^{(i-1)}}{d\,a'}\right)-2\,(i-1)\,a\,A^{(i-1)}\right\};$$

taking then for unity the sum of the masses $M+\mu$, and observing that (237) $\dfrac{M+\mu}{a^3}=n^2$, the equation (X′) will become

$$0=\frac{d^2.\delta\,u}{d\,t^2}+n^2\,\delta\,u-2\,n^2\,\mu'\,a\,g-\frac{n^2\,\mu'}{2}\,a^2\left(\frac{d\,A^{(0)}}{d\,a}\right)$$
$$-\frac{n^2\,\mu'}{2}\,\Sigma\left\{a^2\left(\frac{d\,A^{(i)}}{d\,a}\right)+\frac{2\,n}{n-n'}\,a\,A^{(i)}\right\}\cos.\,i\,(n'\,t-n\,t+\varepsilon'-\varepsilon)$$

$$+ n^2 \mu' C e \cos.(n t + \varepsilon - \varpi)$$
$$+ n^2 \mu' D e' \cos.(n t + \varepsilon - \varpi')$$
$$+ n^2 \mu' \Sigma C^{(i)} e \cos.\{i(n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi\}$$
$$+ n^2 \mu' \Sigma D^{(i)} e' \cos.\{i(n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi'\};$$

and integrating

$$\delta u = 2\mu a g + \frac{\mu'}{2} a^2 \left(\frac{d A^{(0)}}{d a}\right)$$
$$- \frac{\mu'}{2} n^2 \Sigma \cdot \frac{\left\{a^2 \left(\frac{d A^{(i)}}{d a}\right) + \frac{2n}{n - n'} a A^{(i)}\right\}}{i^2 (n - n')^2 - n^2} \cos.\, i(n' t - n t + \varepsilon' - \varepsilon)$$
$$+ \mu' f_{,} e \cos.(n t + \varepsilon - \varpi) + \mu' f_{,}' e' \sin.(n t + \varepsilon - \varpi')$$
$$- \frac{\mu'}{2} C \cdot n t \cdot e \sin.(n t + \varepsilon - \varpi) - \frac{\mu'}{2} D \cdot n t \cdot e' \sin.(n t + \varepsilon - \varpi')$$
$$+ \mu' \Sigma \cdot \frac{C^{(i)} n^2}{\{i(n - n') - n\}^2 - n^2} e \cos.\{i(n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi\}$$
$$+ \mu' \Sigma \frac{D^{(i)} n^2}{\{i(n - n') - n\}^2 - n^2} \cdot e' \cos.\{i(n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi'\},$$

$f_{,}$ and $f_{,}'$ being two arbitraries. The expression of $\delta \varrho$ in terms δu, found in No. 514 will give

$$\frac{\delta \varrho}{a} = - 2\mu' \cdot a g - \frac{\mu'}{2} a^2 \left(\frac{d A^{(0)}}{d a}\right)$$
$$+ \frac{\mu'}{2} n^2 \cdot \Sigma \left\{\frac{a^2 \left(\frac{d A^{(i)}}{d a}\right) + \frac{2n}{n - n'} a A^{(i)}}{i^2 (n - n')^2 - n^2}\right\} \cos.\, i(n' t - n t + \varepsilon' - \varepsilon)$$
$$- \mu' f e \cos.(n t + \varepsilon - \varpi) - \mu' f' e' \cos.(n t + \varepsilon - \varpi')$$
$$+ \tfrac{1}{2} \mu' C n t e \sin.(n t + \varepsilon - \varpi) + \tfrac{1}{2} \mu' D n t e' \sin.(n t + \varepsilon - \varpi')$$
$$+ \mu' n^2 \Sigma \cdot \left\{ \begin{array}{l} \left\{\frac{a^2 \left(\frac{d A^{(i)}}{d a}\right) + \frac{2n}{n - n'} a A^{(i)}}{i^2 (n - n')^2 - n^2} - \frac{C^{(i)}}{\{i(n - n') - n\}^2 - n^2}\right\} \\ \times e \cos.\{i(n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi\} \end{array} \right\}$$
$$- \mu' \cdot n^2 \Sigma \cdot \frac{D^{(i)}}{\{i(n - n') - n\}^2 - n^2} e' \cos.\{i(n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi'\},$$

f and f′ being arbitrary constants independent of $f_{,}$, $f_{,}'$.

This value of $\delta \varrho$, substituted in the formula (Y) of No. 513 will give δv or the perturbations of the planet in longitude. But we must observe that n t expressing the mean motion of μ, the term proportional to the time, ought to disappear from the expression of δv. This condition determines the constant (g) and we find

$$g = - \frac{1}{3} a \left(\frac{d A^{(0)}}{d a}\right).$$

We might have dispensed with introducing into the value of $\delta\varrho$ the arbitraries f, f', for they may be considered as comprised in the elements e and ϖ of elliptic motion. But then the expression of δv would include terms depending upon the mean anomaly, and which would not have been comprised in those which the elliptic motion gives: that is, it is more commodious to make these terms in the expression of the longitude disappear in order to introduce them into the expression of the radius-vector; we shall thus determine f, and f' so as to fulfil this condition. Then if we substitute for $a\left(\frac{dA^{(i-1)}}{da'}\right)$ its value $-A^{(i-1)} - a\left(\frac{dA^{(i-1)}}{da}\right)$, we shall have

$$C = a^2\left(\frac{dA^{(0)}}{da}\right) + \tfrac{1}{2}a^3\left(\frac{d^2A^{(0)}}{da^2}\right);$$

$$D = aA^{(1)} - a^2\left(\frac{dA^{(1)}}{da}\right) - \tfrac{1}{2}a^3\left(\frac{d^2A^{(1)}}{da^2}\right);$$

$$D^{(i)} = \frac{(i-1)(2i-1)n}{n - i(n-n')} \cdot aA^{(i-1)} + \frac{i^2(n-n') - n}{n - i(n-n')}a^2\left(\frac{dA^{(i-1)}}{da}\right) - \tfrac{1}{2}a^3\left(\frac{d^2A^{(i-1)}}{da^2}\right);$$

$$f = \tfrac{2}{3}a^2\left(\frac{dA^{(0)}}{da}\right) + \tfrac{1}{4}a^3\left(\frac{d^2A^{(0)}}{da^2}\right);$$

$$f' = \tfrac{1}{4}\left\{aA^{(1)} - a^2\left(\frac{dA^{(1)}}{da}\right) + a^3\left(\frac{d^2A^{(1)}}{da^2}\right)\right\};$$

Moreover let

$$E^{(i)} = -\frac{3n}{n-n'}aA^{(i)} + \frac{i^2(n-n')\cdot\{n + i(n-n')\} - 3n^2}{i^2(n-n')^2 - n^2} \times \left\{a^2\left(\frac{dA^{(i)}}{da}\right) + \frac{2n}{n-n'}aA^{(i)}\right\} + \tfrac{1}{2}a^3\left(\frac{d^2A^{(i)}}{da^2}\right);$$

$$F^{(i)} = \frac{(i-1)n}{n-n'}aA^{(i)} + \frac{\frac{in}{2}\{n + i(n-n')\} - 3n^2}{i^2(n-n')^2 - n^2} \times \left\{a^2\left(\frac{dA^{(i)}}{da}\right) + \frac{2n}{n-n'}aA^{(i)}\right\} - \frac{2n^2E^{(i)}}{n^2 - \{n - i(n-n')\}^2};$$

$$G^{(i)} = \frac{(i-1)(2i-1)naA^{(i-1)} + (i-1)na^2\left(\frac{dA^{(i-1)}}{da}\right)}{2\{n - i(n-n')\}} - \frac{2n^2D^{(i)}}{n^2 - \{n - i(n-n')\}^2};$$

and we shall have

$$\frac{\delta \varrho}{a} = \frac{\mu'}{6} a^2 \left(\frac{d A^{(0)}}{d a}\right) + \frac{\mu' n^2}{2} \Sigma . \frac{a^2 \left(\frac{d A^{(i)}}{d a}\right) + \frac{2n}{n - n'} a A^{(i)}}{i^2 (n - n')^2 - n^2} \times$$

$$\cos. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$- \mu' f e \cos. (n t + \varepsilon + \varpi) - \mu' f' e' \cos. (n t + \varepsilon - \varpi')$$

$$+ \tfrac{1}{2} \mu' C . n t e \sin. (n t + \varepsilon - \varpi) + \tfrac{1}{2} \mu' D n t e' \sin. (n t + \varepsilon - \varpi')$$

$$+ n^2 \mu' \Sigma . \left\{ \begin{array}{l} \frac{E^{(i)}}{n^2 - \{n - i(n - n')\}^2} e \cos. \{i (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi\} \\ \frac{D^{(i)}}{n^2 - \{n - i(n - n')\}^2} e' \cos. \{i (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi'\} \end{array} \right\} ;$$

$$\delta v = \frac{\mu'}{2} \Sigma \left\{ \frac{n^2}{i (n - n')^2} a A^{(i)} + \frac{2 n^3 \left\{ a^2 \left(\frac{d A^{(i)}}{d a}\right) + \frac{2n}{n - n'} a A^{(i)} \right\}}{i (n - n') . \{i^2 . (n - n')^2 - n^2\}} \right\} \sin. i$$

$$(n' t - n t + \varepsilon' - \varepsilon)$$

$$+ \mu' . C . n t . e \cos. (n t + \varepsilon - \varpi) + \mu' D . n t . e' \cos. (n t + \varepsilon - \varpi')$$

$$+ n \mu' \Sigma \left\{ \begin{array}{l} \frac{F^{(i)}}{n - i (n - n')} e \sin. \{i (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi\} \\ \frac{G^{(i)}}{n - i (n - n')} e' \sin. \{i (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi'\} \end{array} \right\} ;$$

the integral sign Σ extending in these expressions to all the whole positive and negative values of i, with the value $i = 0$ alone excepted.

Here we may observe, that even in the case where the series represented by

$$\Sigma . A^{(i)} \cos.^i (n' t - n t + \varepsilon' - \varepsilon)$$

is but little convergent, these expressions of $\frac{\delta \varrho}{a}$ and of δv, become convergent by the divisors which they acquire. This remark is the more important, because, did this not take place, it would have been impossible to express analytically the mutual perturbations of the planets, of which the ratios of their distances from the sun are nearly unity.

These expressions may take the following form, which will be useful to us hereafter. Let

$$h = e \sin. \varpi ; \quad h' = e' \sin. \varpi' ;$$

$$l = e \cos. \varpi ; \quad l' = e' \cos. \varpi' ;$$

then we shall have

$$\frac{\delta \varrho}{a} = \frac{\mu'}{6} . a^2 \left(\frac{d A^{(0)}}{d a}\right) + \frac{\mu' n^2}{2} \Sigma \left\{ \frac{a^2 \left(\frac{d A^{(i)}}{d a}\right) + \frac{2n}{n - n'} a A^{(i)}}{i^2 (n - n')^2 - n^2} \right\} \cos. i (n' t - n t + \varepsilon' - \varepsilon)$$

$$- \mu' (h f + h' f') \cos. (n t + \varepsilon) - \mu' (l f + l' f') \sin. (n t + \varepsilon)$$

$$+\frac{\mu'}{2}\{l\,C+l'\,D\}\,n\,t\sin.(n\,t+\varepsilon)-\frac{\mu'}{2}\{h\,C+h'\,D\}n\,t\cos.(n\,t+\varepsilon)$$

$$+n^2\mu'\,\Sigma.\begin{Bmatrix}\frac{h\,E^{(i)}+h'\,D^{(i)}}{n^2-\{n-i(n-n')\}^2}\sin.\{i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\}\\+\frac{l\,E^{(i)}+l'\,D^{(i)}}{n^2-\{n-i(n-n')\}^2}\cos.\{i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\}\end{Bmatrix}.$$

$$\delta v=\frac{\mu'}{2}\Sigma.\left\{\frac{n^2}{i(n-n')^2}a\,A^{(i)}+2n^3\,\frac{a^2\left(\frac{d\,A^{(i)}}{d\,a}\right)+\frac{2n}{n-n'}a\,A^{(i)}}{i(n-n')\{i^2.(n-n')^2-n^2\}}\right\}\times$$

$$\sin.i(n't-nt+\varepsilon'-\varepsilon)$$

$$+\mu'\{h\,C+h'\,D\}.n\,t.\sin.(n\,t+\varepsilon)+\mu'\{l.C+l'.D\}\,n\,t.\cos.(n\,t+\varepsilon)$$

$$+n\mu'.\Sigma\begin{Bmatrix}\frac{l\,F^{(i)}+l'\,G^{(i)}}{n-i(n-n')}\sin.\{i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\}\\-\frac{h\,F^{(i)}+h'\,G^{(i)}}{n-i(n-n')}\cos.\{i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\}\end{Bmatrix};$$

Connecting these expressions of $\delta\varrho$ and δv with the values of ϱ and v relative to elliptic motion, we shall have the entire values of the radius-vector of μ, and of its motion in longitude.

518. Now let us consider the motion of μ in latitude. For that purpose let us resume the formula (Z') of No. 514. If we neglect the product of the inclinations by the excentricities of the orbits it will become

$$0=\frac{d^2\delta u'}{dt^2}+n^2\delta u'-\frac{1}{a^2}\left(\frac{dR}{dz}\right);$$

the expression of R of No. 515 gives, in taking for the fixed plane that of the primitive orbit of μ,

$$\left(\frac{dR}{dz}\right)=\frac{\mu'z'}{a^3}-\frac{\mu'z'}{2}\Sigma.B^{(i)}\cos.i(n't-nt+\varepsilon'-\varepsilon);$$

the value of i belonging to all whole positive and negative numbers including also $i=0$. Let γ be the tangent of the inclination of the orbit of μ', to the primitive orbit of μ, and Π the longitude of the ascending node of the first of these orbits upon the second; we shall have very nearly

$$z'=a'\gamma\sin.(n't+\varepsilon'-\Pi);$$

which gives

$$\left(\frac{dR}{dz}\right)=\frac{\mu'}{a^2}.\gamma.\sin.(n't+\varepsilon'-\Pi)-\frac{\mu'}{2}.a'B^{(i)}\gamma\sin.(nt+\varepsilon-\Pi)$$

$$\frac{\mu'}{2}a'\Sigma B^{(i-1)}\gamma\sin.\{i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\Pi\}$$

the value here, as in what follows, extending to all whole positive and negative numbers, $i=0$ being alone excepted. The differential equation

in $\delta a'$ will become, therefore, when the value of $\left(\frac{dR}{dz}\right)$ is multiplied by $n^2 a^3$, which is equal to unity,

$$0 = \frac{d^2\delta u'}{dt^2} + n^2 \delta u' - \mu' n^2 . \frac{a}{a'^2} \gamma \sin. (n' t + \varepsilon' - \Pi)$$
$$+ \frac{\mu' n^2}{2} a a' B^{(i)} \gamma \sin. (n t + \varepsilon - \Pi)$$
$$+ \frac{\mu' n^2}{2} a a' \Sigma B^{(i-1)} \gamma \sin. \{i (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \Pi)\};$$

whence by integrating and observing that by 514

$$\delta s = - a \delta u',$$

$$\delta s = -\frac{\mu' n^2}{n^2 - n'^2} . \frac{a^2}{a'^2} \gamma \sin. (n' t + \varepsilon' - \Pi)$$
$$- \frac{\mu' . a^2 a'}{4} B^{(i)} . n t . \gamma \cos. (n t + \varepsilon - \Pi)$$
$$+ \frac{\mu' n^2 a^2 a'}{2} \Sigma . \frac{B^{(i-1)}}{n^2 - \{n - i(n - n')\}^2} \gamma \sin. \{i(n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \Pi\}.$$

To find the latitude of μ above a fixed plane a little inclined to that of its primitive orbit, by naming φ the inclination of this orbit to the fixed plane, and θ the longitude of its ascending node upon the same plane; it will suffice to add to δs the quantity tan. φ sin. $(v - \theta)$, or tan. φ sin. $(n t + \varepsilon - \theta)$, neglecting the excentricity of the orbit. Call φ' and θ' what φ and θ become relatively to μ'. If μ were in motion upon the primitive orbits of μ', the tangent of its latitude would be tan. φ' sin. $(n t + \varepsilon - \theta')$; this tangent would be tan. φ sin. $(n t + \varepsilon - \theta)$, if μ continued to move in its own primitive orbit. The difference of these two tangents is very nearly the tangent of the latitude of μ, above the plane of its primitive orbit, supposing it moved upon the primitive orbit of μ'; we have therefore

tan. φ' sin. $(n t + \varepsilon - \theta')$ — tan. φ sin. $(n t + \varepsilon - \theta) = \gamma$ sin. $(n t + \varepsilon - \Pi)$.

Let

$$\text{tan. } \varphi \text{ sin. } \theta = p; \quad \text{tan. } \varphi \text{ sin. } \theta' = p';$$
$$\text{tan. } \varphi \text{ cos. } \theta = q; \quad \text{tan. } \varphi' \text{ cos. } \theta' = q';$$

we shall have

$$\gamma \text{ sin. } \Pi = p' - p; \quad \gamma \text{ cos. } \Pi = q' - q$$

and consequently if we denote by s the latitude of μ above the fixed plane, we shall very nearly have

$$s = q \text{ sin. } (n t + \varepsilon) - p \text{ cos. } (n t + \varepsilon)$$
$$- \frac{\mu' a^2 a'}{4} (p' - p) B^{(i)} n t \text{ sin. } (n t + \varepsilon)$$

$$-\frac{\mu' a^2 a'}{4}(q'-q)\,B^{(i)}\,n\,t\cos.(n\,t+\varepsilon)$$

$$-\frac{\mu' n^2}{n^2-n'^2}\cdot\frac{a^2}{a'^2}\{(q'-q)\sin.(n't+\varepsilon')-(p'-p)\cos.(n't+\varepsilon')\}$$

$$+\frac{\mu' n^2 . a^2 a'}{2}\Sigma.\begin{Bmatrix}\frac{(q'-q)\,B^{(i-1)}}{n^2-\{n-i(n-n')\}^2}\sin.\{i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\}\\ \frac{-(p'-p)\,B^{(i-1)}}{n^2-\{n-i(n-n')\}^2}\cdot\cos.\{i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\}\end{Bmatrix}.$$

519. Now let us recapitulate. Call (ϱ) and (v) the parts of the radius-vector and longitude v upon the orbit, which depend upon the elliptic motion, we shall have

$$\varrho=(\varrho)+\delta\varrho;\quad v=(v)+\delta v.$$

The preceding value of s, will be the latitude of μ above the fixed plane. But it will be more exact to employ, instead of its two first terms, which are independent of μ', the value of the latitude, which takes place in the case where μ quits not the plane of its primitive orbit. These expressions contain all the theory of the planets, when we neglect the squares and the products of the excentricities and inclinations of the orbits, which is in most cases allowable. They moreover possess the advantage of being under a very simple form, and which shows the law of their different terms.

Sometimes we shall have occasion to recur to terms depending on the squares and products of the excentricities and inclinations, and even to the superior powers and products. We can find these terms by the preceding analysis, the consideration which renders them necessary will always facilitate their determination. The approximations in which we must notice them, would introduce new terms which would depend upon new arguments. They would reproduce again the arguments, which the preceding approximations afford, but with coefficients still smaller and smaller, following that law which it is easy to perceive from the developement of R into a series, which was given in No. 515; *an argument which, in the successive approximations, is found for the first time among the quantities of any order whatever* r, *and is reproduced only by quantities of the orders* r+2, r+4, &c.

Hence it follows that the coefficients of the terms of the form $t.\begin{matrix}\sin.\\ \cos.\end{matrix}.(n\,t+\varepsilon)$, which enter into the expressions of ϱ, v, and s, are approximated up to quantities of the third order, that is to say, that the approximation in which we should have regard to the squares and pro-

ducts of the excentricities and inclinations of the orbits would add nothing to their values; they have therefore all the exactness that can be desired. This it is the more essential to observe, because the secular variations of the orbits depend upon these same coefficients.

The several terms of the perturbations of ϱ, v, s are comprised in the form

$$k \cdot \begin{matrix}\sin.\\ \cos.\end{matrix} \{i\,(n'\,t - n\,t + \varepsilon' - \varepsilon) + r\,n\,t + r\,\varepsilon\},$$

r being a whole positive number or zero, and k being a function of the excentricities and inclinations of the orbits of the order r, or of a superior order. Hence we may judge of what order is a term depending upon a given angle.

It is evident that the motion of the bodies μ'', μ''', &c. make it necessary to add to the preceding values of ϱ, v, and s, terms analogous to those which result from the action of μ'; and that neglecting the square of the perturbing force, the sums of all these terms will give the whole values of ϱ, v and s. This follows from the nature of the formulas (X'), (Y), (Z'), which are linear relatively to quantities depending on the disturbing force.

Lastly, we shall have the perturbations of μ', produced by the action of μ by changing in the preceding formulas, a, n, h, l, ε, ϖ, p, q, and μ' into a', n', h', l', ε', ϖ', p', q', and μ and reciprocally.

THE SECULAR INEQUALITIES OF THE CELESTIAL MOTIONS.

520. The perturbing forces of elliptical motion introduce into the expressions of ϱ, $\frac{d\,v}{d\,t}$, and s of the preceding Nos. the time t free from the symbols *sine* and *cosine*, or under the form of arcs of a circle, which by increasing indefinitely, must at length render the expressions defective. It is therefore essential to make these arcs disappear, and to obtain the functions which produce them by their developement into series. We have already given, for this purpose, a general method, from which it results that these arcs arise from the variations of elliptic motion, which are then functions of the time. These variations taking place very slowly have been denominated *Secular Inequalities*. Their theory is one of the most interesting subjects of the system of the world. We now proceed to expound it to the extent which its importance demands.

By what has preceded we have

$$\varrho = a \left\{ \begin{array}{l} 1 - h \sin.(n t + \varepsilon) - l \cos.(n t + \varepsilon) - \&c. \\ + \frac{\mu'}{2} \{l . C + l' . D\} . n t . \sin.(n t + \varepsilon) \\ - \frac{\mu'}{2} \{h . C + h' . D\} . n t . \cos.(n t + \varepsilon) + \mu' S. \end{array} \right\}$$

$$\frac{dv}{dt} = n + 2 n h \sin.(n t + \varepsilon) + 2 n l \cos.(n t + \varepsilon) + \&c.$$
$$- \mu' \{l C + l' D\} n^2 t \sin.(n t + \varepsilon)$$
$$+ \mu' \{h C + h' D\} n^2 t \cos.(n t + \varepsilon) + \mu' T;$$

$$s = q \sin.(n t + \varepsilon) - p \cos.(n t + \varepsilon) + \&c.$$
$$- \frac{\mu'}{4} a^2 a' (p' - p) B^{(1)} . n t . \sin.(n t + \varepsilon)$$
$$- \frac{\mu'}{4} a^2 a' (q' - q) B^{(1)} . n t . \cos.(n t + \varepsilon) + \mu' \chi;$$

S, T, χ being periodic functions of the time t. Consider first the expression of $\frac{dv}{dt}$, and compare it with the expression of y in 510. The arbitrary n multiplying the arc t, under the periodic symbols, in the expression of $\frac{dv}{dt}$; we ought then to make use of the following equations found in No. 510,

$$0 = X' + \theta . X'' - Y;$$
$$0 = Y' + \theta . Y'' + X'' - 2 Z;$$

Let us see what these X, X', X'', Y, &c. become. By comparing the expression of $\frac{dv}{dt}$ with that of y cited above, we find

$$X = n + 2 n h \sin.(n t + \varepsilon) + 2 n l \cos.(n t + \varepsilon) + \mu' T$$
$$Y = \mu' n^2 \{h C + h' D\} \cos.(n t + \varepsilon) - \mu' n^2 \{l C + l' D\} \sin.(n t + \varepsilon).$$

If we neglect the product of the partial differences of the constants by the perturbing masses, which is allowed, since these differences are of the order of the masses, we shall have by No. 510,

$$X' = \left(\frac{dn}{d\theta}\right) \{1 + 2 h \sin.(n t + \varepsilon) + 2 l \cos.(n t + \varepsilon)\}$$
$$+ 2 n \left(\frac{d\varepsilon}{d\theta}\right) \{h \cos.(n t + \varepsilon) - l \sin.(n t + \varepsilon)\}$$
$$+ 2 n \left(\frac{dh}{d\theta}\right) \sin.(n t + \varepsilon) + 2 n \left(\frac{dl}{d\theta}\right) \cos.(n t + \varepsilon);$$

$$X'' = 2 n \left(\frac{dn}{d\theta}\right) \{h \cos.(n t + \varepsilon) - l \sin.(n t + \varepsilon)\}$$

The equation $0 = X' + \theta X'' - Y$ will thus become

$$0 = \left(\frac{d\,n}{d\,\theta}\right)\{1 + 2\,h \sin.\,(n\,t + \varepsilon) + 2\,l \cos.\,(n\,t + \varepsilon)\}$$
$$+ 2\,n\left(\frac{d\,h}{d\,\theta}\right)\sin.\,(n\,t + \varepsilon) + 2\,n\left(\frac{d\,l}{d\,\theta}\right)\cos.\,(n\,t + \varepsilon)$$
$$+ 2\,n\left\{\theta\left(\frac{d\,n}{d\,\theta}\right) + \left(\frac{d\,\varepsilon}{d\,\theta}\right)\right\}.\{h \cos.\,(n\,t+\varepsilon) - l \sin.\,(n\,t+\varepsilon)\}$$
$$-\mu' n^2\{h\,C + h'\,D\}\cos.\,(n\,t+\varepsilon) + \mu' n^2\{l\,C + l'\,D\}\sin.(n\,t+\varepsilon).$$

Equating separately to zero, the coefficients of like sines and cosines, we shall have

$$0 = \left(\frac{d\,n}{d\,\theta}\right)$$
$$0 = \left(\frac{d\,h}{d\,\theta}\right) - l\left(\frac{d\,\varepsilon}{d\,\theta}\right) + \frac{\mu' n}{2}\{l.\,C + l'\,D\};$$
$$0 = \left(\frac{d\,l}{d\,\theta}\right) + h\left(\frac{d\,\varepsilon}{d\,\theta}\right) - \frac{\mu' n}{2}\{h\,C + h'\,D\}.$$

If we integrate these equations, and if in their integrals we change θ into t, we shall have by No. 510, the values of the arbitraries in functions of t, and we shall be able to efface the circular arcs from the expressions of $\frac{d\,v}{d\,t}$ and of ϱ. But instead of this change, we can immediately change θ into t in these differential equations. The first of the equations shows us that n is constant, and since the arbitrary a of the expression for ϱ depends upon it, by reason of $n^2 = \frac{1}{a^3}$, a is likewise constant. The two other equations do not suffice to determine h, l, ε. We shall have a new equation in observing that the expression of $\frac{d\,v}{d\,t}$, gives, in integrating, $\int n\,d\,t$ for the value of the mean longitude of μ. But we have supposed this longitude equal to $n\,t+\varepsilon$; we therefore have $n\,t+\varepsilon = \int n\,d\,t$, which gives

$$t.\frac{d\,n}{d\,t} + \frac{d\,\varepsilon}{d\,t} = 0;$$

and as we have $\frac{d\,n}{d\,t} = 0$, we have in like manner $\frac{d\,\varepsilon}{d\,t} = 0$. Thus the two arbitraries n and ε are constants; the arbitraries h, l, will consequently be determined by means of the differential equations,

$$\frac{d\,h}{d\,t} = -\frac{\mu' n}{2}\{l\,C + l'\,D\}; \quad (1)$$
$$\frac{d\,l}{d\,t} = \frac{\mu' n}{2}\{h\,C + h'\,D\}; \quad (2)$$

The consideration of the expression of $\frac{dv}{dt}$ having enabled us to determine the values of n, a, h, l, and ε, we perceive *a priori*, that the differential equations between the same quantities, which result from the expression of ϱ, ought to coincide with those preceding. This may easily be shown *a posteriori*, by applying to this expression the method of 510.

Now let us consider the expression of s. Comparing it with that of y cited above, we shall have

$$X = q \sin.(nt + \varepsilon) - p \cos.(nt + \varepsilon) + \mu' \chi$$

$$Y = \frac{\mu' n}{4} . a^2 a' B^{(1)} (p - p') \sin.(nt + \varepsilon)$$

$$+ \frac{\mu' n}{4} . a^2 a' B^{(1)} (q - q') \cos.(nt + \varepsilon),$$

n and ε, by what precedes, being constants; we shall have by No. 510,

$$X' = \left(\frac{dq}{d\theta}\right) \sin.(nt + \varepsilon) - \left(\frac{dp}{d\theta}\right) \cos.(nt + \varepsilon)$$

$$X'' = 0.$$

The equation $0 = X' + \theta X'' - Y$ hence becomes

$$0 = \left(\frac{dq}{d\theta}\right) \sin.(nt + \varepsilon) - \frac{dp}{d\theta} \cos.(nt + \varepsilon)$$

$$- \frac{\mu' n}{4} a^2 a' B^{(1)} (p - p') \sin.(nt + \varepsilon)$$

$$- \frac{\mu' n}{4} a^2 a' B^{(1)} (q - q') \cos.(nt + \varepsilon);$$

whence we derive, by comparing the coefficients of the like sines and cosines, and changing θ into t, in order to obtain directly p and q in functions of t,

$$\frac{dp}{dt} = - \frac{\mu' n}{4} . a^2 a' B^{(1)} . (q - q'); \quad (3)$$

$$\frac{dq}{dt} = \frac{\mu' n}{4} . a^2 a' B^{(1)} (p - p'); \quad (4)$$

When we shall have determined p and q by these equations, we shall substitute them in the preceding expression of s, effacing the terms which contain circular arcs, and we shall have

$$s = q \sin.(nt + \varepsilon) - p \cos.(nt + \varepsilon) + \mu' \chi.$$

521. The equation $\frac{dn}{dt} = 0$, found above, is one of great importance in the theory of the system of the world, inasmuch as it shows that the mean motions of the celestial bodies and the major-axes of their orbits are unalterable. But this equation is approximate to quantities of the order

μ' h inclusively. If quantities of the order $\mu' h^2$, and following orders, produce in $\frac{dv}{dt}$, a term of the form 2 k t, k being a function of the elements of the orbits of μ and μ'; there will thence result in the expression of v, the term $k t^2$, which by altering the longitude of μ, proportionally to the time, must at length become extremely sensible. We shall then no longer have

$$\frac{dn}{dt} = 0;$$

but instead of this equation we shall have by the preceding No.

$$\frac{dn}{dt} = 2k;$$

It is therefore very important to know whether there are terms of the form $k.t^2$ in the expression of v. We now demonstrate, that *if we retain only the first power of the perturbing masses, however far may proceed the approximation, relatively to the powers of the excentricities and inclinations of the orbits, the expression* v *will not contain such terms.*

For this object we will resume the formula (X) of No. 513,

$$\delta\varrho = \frac{a\cos.v\int ndt.\varrho\sin.v\left\{2\int dR+\varrho\left(\frac{dR}{d\varrho}\right)\right\}-a\sin.v\int ndt.\varrho\cos.v\left\{2\int dR+\varrho\left(\frac{dR}{d\varrho}\right)\right\}}{m\sqrt{1-e^2}}$$

Let us consider that part of $\delta\varrho$ which contains the terms multiplied by t^2, or for the greater generality, the terms which being multiplied by the sine or cosine of an angle $\alpha t + \beta$, in which α is very small, have at the same time a^2 for a divisor. It is clear that in supposing $\alpha = 0$, there will result a term multiplied by t^2, so that the second case shall include the first. The terms which have the divisor α^2, can evidently only result from a double integration; they can only therefore be produced by that part of $\delta\varrho$ which contains the double integral sign $\int$. Examine first the term

$$\frac{2a\cos.v\int ndt(\varrho\sin.v\int dR)}{m\sqrt{(1-e^2)}}.$$

If we fix the origin of the angle v at the perihelion, we have

$$\varrho = \frac{a(1-e^2)}{1+e\cos.v},$$

and consequently

$$\cos.v = \frac{a(1-e^2)-\varrho}{e\varrho};$$

whence we derive by differentiating,

$$\varrho^2 dv.\sin.v = \frac{a(1-e^2)}{e}.d\varrho;$$

but we have,

$$\varrho^2 \, d\, v = d\, t \sqrt{m\, a\, (1 - e^2)} = a^2 . n\, d\, t \sqrt{1 - e^2};$$

we shall, therefore, have

$$\frac{a\, n\, d\, t\, \varrho \sin. v}{\sqrt{1 - e^2}} = \frac{\varrho\, d\, \varrho}{e}.$$

The term

$$\frac{2\, a \cos. v \int n\, d\, t . \{\varrho \sin. v \int d\, R\}}{m \sqrt{1 - e^2}}$$

will therefore become

$$\frac{2 \cos. v}{m\, e} \int (\varrho\, d\, \varrho \int d\, R), \text{ or } \frac{\cos. v}{m\, e} \{\varrho^2 \int d\, R - \int \varrho^2 . d\, R\}.$$

It is evident, this last function, no longer containing double integrals, there cannot result from it any term having the divisor α^2.

Now let us consider the term

$$- \frac{2\, a \sin. v \int n\, d\, t \{\varrho \cos. v \int d\, R\}}{m \sqrt{1 - e^2}}$$

of the expression of $\delta\, \varrho$. Substituting for cos. v, its preceding value in ϱ, this term becomes

$$\frac{2\, a \sin. v \int n\, d\, t . \{\varrho - a\, (1 - e^2)\} . \int d\, R}{m\, e \sqrt{1 - e^2}}.$$

We have

$$\varrho = a \{1 + \tfrac{1}{2} e^2 + e\, \chi'\},$$

χ' being an infinite series of cosines of the angle $n\, t + \epsilon$, and of its multiples; we shall therefore have

$$\frac{\int n\, d\, t}{e} \{\varrho - a\, (1 - e^2)\} \int d\, R = a \int n\, d\, t \{\tfrac{3}{2} e + \chi'\} \int d\, R.$$

Call χ'' the integral $\int \chi'\, n\, d\, t$; we shall have

$$a \int n\, d\, t . \{\tfrac{3}{2} e + \chi' . \} \int d\, R = \tfrac{3}{2}\, a\, e \int n\, d\, t \int d\, R + a\, \chi'' \int d\, R - a \int \chi'' . d\, R.$$

These two last terms not containing a double integral sign, there cannot thence result any term having α^2 for a divisor; reckoning only terms of this kind, we shall have

$$- \frac{2\, a \sin. v \int n\, d\, t \{\varrho \cos. v \int d\, R\}}{m \sqrt{1 - e^2}} = \frac{3\, a^2\, e \sin. v \int n\, d\, t \int d\, R}{m \sqrt{1 - e^2}}$$

$$= \frac{d\, \varrho}{n\, d\, t} \cdot \frac{3\, a}{m} \int n\, d\, t \int d\, R;$$

and the radius ϱ will become

$$(\varrho) + \left(\frac{d\, \varrho}{n\, d\, t}\right) \cdot \frac{3\, a}{m} \int n\, d\, t . \int d\, R;$$

(ϱ) and $\left(\frac{d\varrho}{n\,d\,t}\right)$ being the expressions of ϱ and of $\frac{d\varrho}{n\,d\,t}$, relative to the elliptic motion. Thus, to estimate in the expression of the radius-vector, that part of the perturbations, which is divided by α^2, it is sufficient to augment by the quantity $\frac{3\,a}{m}.\ \times \int n\,d\,t\,.\int d\,R$, the mean longitude $n\,t + \varepsilon$, of this expression relative to the elliptic motion.

Let us see how we ought to estimate this part of the perturbations in the expression of the longitude v. The formula (Y) of No. 516 gives by substituting $\frac{3\,a}{m}.\frac{d\varrho}{n\,d\,t}.\int n\,d\,t \int d\,R$ for $\delta\varrho$ and retaining only the terms divided by α^2,

$$\delta v = \frac{\left\{\frac{2\varrho\,d^2\varrho + d\varrho^2}{a^2 n^2 d\,t^2} + 1\right\}}{\sqrt{1-e^2}}.\frac{3\,a}{m}\int n\,d\,t \int d\,R;$$

But we have by what precedes

$$d\varrho = \frac{a\,e.\,n\,d\,t.\,\sin.\,v}{\sqrt{1-e^2}};\quad \varrho^2\,d\,v = a^2\,n\,d\,t\,\sqrt{1-e^2};$$

whence it is easy to obtain, by substituting for cos. v its preceding value in ϱ,

$$\frac{\frac{2\varrho\,d^2\varrho + d\varrho^2}{a^2 n^2 d\,t^2} + 1}{\sqrt{1-e^2}} = \frac{d\,v}{n\,d\,t};$$

in estimating therefore only that part of the perturbations, which has the divisor α^2, the longitude v will become

$$(v) + \left(\frac{d\,v}{n\,d\,t}\right).\frac{3\,a}{m}\int n\,d\,t \int d\,R;$$

(v) and $\left(\frac{d\,v}{n\,d\,t}\right)$ being the parts of v and $\frac{d\,v}{n\,d\,t}$, relative to the elliptic motion. Thus, in order to estimate that part of the perturbations in the expression of the longitude of μ, we ought to follow the same rule which we have given with regard to the same in the expression of the radius-vector, that is to say, we must augment in the elliptic expression of the true longitude, the mean longitude $n\,t + \varepsilon$ by the quantity $\frac{3\,a}{m}\int n\,d\,t \int d\,R$.

The constant part of the expression of $\left(\frac{d\,v}{n\,d\,t}\right)$ developed into a series of cosines of the angle $n\,t + \varepsilon$ and of its multiples, being reduced (see 488) to unity, there thence results, in the expression of the longi-

tude, the term $\frac{3a}{m} \int n\,dt \int d\,R$. If $d\,R$ contain a constant term $k\,\mu' . n\,dt$, this term will produce in the expression of the longitude v, the following one, $\frac{3}{2} . \frac{a\,\mu'}{m} k\,n^2\,t^2$. To ascertain the existence of such terms in this expression, we must therefore find whether $d\,R$ contains a constant term.

When the orbits are but little excentric and little inclined to one another, we have seen, No. 518, that R can always be developed into an infinite series of sines and cosines of angles increasing proportionally to the time. We can represent them generally by the term

$$k\,\mu' . \cos. \{i'\,n'\,t + i\,n\,t + A\},$$

i and i' being whole positive or negative numbers or zero. The differential of this term, taken solely relatively to the mean motion of μ, is

$$-i\,k . \mu' . n\,dt . \sin. \{i'\,n'\,t + i\,n\,t + A\};$$

this cannot be constant unless we have $0 = i'\,n' + i\,n$, which supposes the mean motions of the bodies μ and μ' to be parts of one another; and since that does not take place in the solar system, we ought thence to conclude that the value of $d\,R$ does not contain constant terms, and that in considering only the first power of the perturbing masses, the mean motions of the heavenly bodies, are uniform, or which comes to the same thing, $\frac{dn}{dt} = 0$. The value of a being connected to n by means of the equation $n^2 = \frac{m}{a^3}$, it thence results that if we neglect the periodical quantities, the major-axes of the orbits are constant.

If the mean motions of the bodies μ and μ', without being exactly commensurable, approach, however, very nearly to that condition, there will exist in the theory of their motions, inequalities of a long period, and which, by reason of the smallness of the divisor α^2, will become very sensible. We shall see hereafter this is the case with regard to Jupiter and Saturn. The preceding analysis will give, in a very simple manner, that part of the perturbations which depend upon this divisor. It hence results that in this case it is sufficient to vary the mean longitude $n\,t + \varepsilon$ or $\int n\,dt$ by the quantity $\frac{3a}{m} \int n\,dt \int d\,R$; or, which is the same, to augment n in the integral $\int n\,dt$ by the quantity $\frac{3\,a\,n}{m} \int d\,R$; but consider-

ing the orbit of μ as a variable ellipse, we have $n^2 = \frac{m}{a^3}$; the preceding variation of n introduces, therefore, in the semi-axis-major a of the orbit, the variation $-\frac{2a^2 \int d R}{m}$.

If we carry the approximation of the value $\frac{dv}{dt}$, to quantities of the order of the squares of the perturbing masses, we shall find terms proportional to the time; but considering attentively the differential equations of the motion of the bodies μ, μ', &c. we shall easily perceive that these terms are at the same time of the order of the squares and products of the excentricities and inclinations of the orbits. Since, however, every thing which affects the mean motion, may at length become very sensible, we shall now notice these terms, and perceive that they produce the secular equations observed in the motion of the moon.

522. Let us resume the equations (1) and (2) of No. 520, and suppose

$$(0, 1) = -\frac{\mu' . n . C}{2}; \quad \boxed{0, 1} = \frac{\mu' . n . D}{2};$$

they will become

$$\frac{dh}{dt} = (0, 1)\, l - \boxed{0, 1}\, l';$$

$$\frac{dl}{dt} = -(0, 1)\, h + \boxed{0, 1}\, h'.$$

The expression of (0, 1) and of $\boxed{0, 1}$ may be very simply determined in this way. Substituting, instead of C and D, their values determined in No. 517, we shall have

$$(0, 1) = -\frac{\mu' n}{2}\left\{a^2\left(\frac{dA^{(0)}}{da}\right) + \tfrac{1}{2}a^3\left(\frac{d^2 A^{(0)}}{da^2}\right)\right\};$$

$$\boxed{0, 1} = \frac{\mu' n}{2}\left\{a A^{(1)} - a^2\left(\frac{dA^{(1)}}{da}\right) - \tfrac{1}{2}a^3\left(\frac{d^2 A^{(1)}}{da^2}\right)\right\}.$$

We have by No. 516,

$$a^2\left(\frac{dA^{(0)}}{da}\right) + \tfrac{1}{2}a^3\left(\frac{d^2 A^{(0)}}{da^2}\right) = -\alpha^2 \frac{d\, b^{(0)}_{\frac{1}{2}}}{d\alpha} - \tfrac{1}{2}\alpha^3 . \frac{d^2 b^{(0)}_{\frac{1}{2}}}{d\alpha^2};$$

and we shall easily obtain, by the same No. $\frac{d\, b^{(0)}_{\frac{1}{2}}}{d\alpha}$ and $\frac{d^2 b^{(0)}_{\frac{1}{2}}}{d\alpha^2}$ in functions of $b^{(0)}_{\frac{1}{2}}$ and $b^{(1)}_{\frac{1}{2}}$; and these quantities are given in linear functions of $b^{(0)}_{-\frac{1}{2}}$

and of $b^{(1)}_{-\frac{1}{2}}$; this being done, we shall find

$$a^2\left(\frac{d\,A^{(0)}}{d\,a}\right)+\tfrac{1}{2}a^3\left(\frac{d^2\,A^{(0)}}{d\,a^2}\right)=\frac{3\,\alpha^2\,b^{(1)}_{-\frac{1}{2}}}{2\,(1-\alpha^2)^2};$$

wherefore

$$(0,1)=-\frac{3\,\mu'.\,n\,.\,\alpha^2.\,b^{(1)}_{-\frac{1}{2}}}{4\,(1-\alpha^2)^2}.$$

Let

$$(a^2-2\,a\,a'\cos.\theta+a'^2)^{\frac{1}{2}}=(a,a')+(a,a')'\cos.\theta+(a,a')''\cos.2\theta+\&c.$$

we shall have by No. 516.

$$(a,a')=\tfrac{1}{2}\,a'.\,b^{(0)}_{-\frac{1}{2}};\quad (a,a')'=a'.\,b^{(1)}_{-\frac{1}{2}},\ \&c.$$

We shall, therefore, have

$$(0,1)=-\frac{3\,\mu'.\,n\,a^2\,a'.\,(a,a')'}{4\,(a'^2-a^2)^2}.$$

Next we have, by 516,

$$a\,A^{(1)}-a^2\left(\frac{d\,A^{(1)}}{d\,a}\right)-\tfrac{1}{2}a^3\left(\frac{d^2\,A^{(1)}}{d\,a^2}\right)=-\alpha\left\{b^{(1)}_{\frac{1}{2}}-\alpha.\frac{d\,b^{(1)}_{\frac{1}{2}}}{d\,\alpha}-\tfrac{1}{2}\alpha^2.\frac{d^2\,b^{(1)}_{\frac{1}{2}}}{d\,\alpha^2}\right\}.$$

Substituting for $b^{(1)}_{\frac{1}{2}}$ and its differences, their values in $b^{(0)}_{-\frac{1}{2}}$ and $b^{(1)}_{-\frac{1}{2}}$, we shall find the preceding function equal to

$$-\frac{3\,\alpha\left\{(1+\alpha^2)\,b^{(1)}_{-\frac{1}{2}}+\tfrac{1}{2}\,\alpha\,b^{(0)}_{-\frac{1}{2}}\right\}}{(1-\alpha^2)^2}.$$

therefore

$$\boxed{0,1}=-\frac{3\,\alpha.\,\mu'\,n\left\{(1+\alpha^2)\,b^{(1)}_{-\frac{1}{2}}+\tfrac{1}{2}\,\alpha.\,b^{(0)}_{-\frac{1}{2}}\right\}}{2\,(1-\alpha^2)^2}$$

or

$$\boxed{0,1}=-\frac{3\,\mu'.\,a\,n\{(a^2+a'^2)\,(a,a')'+a\,a'\,(a,a')\}}{2\,(a'^2-a^2)^2}.$$

We shall, therefore, thus obtain very simple expressions of (0, 1) and of $\boxed{0,1}$, and it is easy to perceive from the values in the series of $b^{(0)}_{-\frac{1}{2}}$ and of $b^{(1)}_{-\frac{1}{2}}$, given in the No. 516, that these expressions are positive, if n is positive, and negative if n is negative.

Call (0, 2) and $\boxed{0,2}$, what (0, 1) and $\boxed{0,1}$ become, when we change a'

and μ' into a'' and μ''. In like manner let (0, 3), and (0, 3) be what the same quantities become, when we change a' and μ' into a''' and μ'''; and so on. Moreover let h'', l''; h''', l''', &c. denote the values of h and l relative to the bodies μ'', μ''', &c. Then, in virtue of the united actions of the different bodies μ', μ'', μ''', &c. upon μ, we shall have

$$\frac{dh}{dt} = \{(0,1) + (0,2) + (0,3) + \&c.\}l - \boxed{0,1}.l' - \boxed{0,2}.l'' - \&c.;$$

$$\frac{dl}{dt} = -\{(0,1) + (0,2) + (0,3) + \&c.\}h + \boxed{0,1}.h' + \boxed{0,2}.h'' + \&c.$$

It is evident that $\frac{dh'}{dt}$, $\frac{dl'}{dt}$; $\frac{dh''}{dt}$, $\frac{dl''}{dt}$; &c. will be determined by expressions similar to those of $\frac{dh}{dt}$ and of $\frac{dl}{dt}$; and they are easily obtained by changing successively what is relative to μ into that which relates to μ', μ'', &c. and reciprocally. Let therefore

$$(1,0), \boxed{1,0}; \quad (1,2), \boxed{1,2}; \quad \&c.$$

be what

$$(0,1), \boxed{0,1}; \quad (0,2), \boxed{0,2}; \quad \&c.$$

become, when we change that which is relative to μ, into what is relative to μ' and reciprocally. Let moreover

$$(2,0), \boxed{2,0}; \quad (2,1), \boxed{2,1}; \quad \&c.$$

be what

$$(0,2), \boxed{0,2}; \quad (0,1), \boxed{0,1}; \quad \&c.$$

become, when we change what is relative to μ into what is relative to μ'' and reciprocally; and so on. The preceding differential equations referred successively to the bodies μ, μ', μ'', &c. will give for determining h, l, h', l', h'', l'', &c. the following system of equations,

$$\left.\begin{aligned}
\frac{dh}{dt} &= \{(0,1) + (0,2) + \&c.\}l - \boxed{0,1}.l' - \boxed{0,2}.l'' - \&c.\\
\frac{dl}{dt} &= -\{(0,1) + (0,2) + \&c.\}h + \boxed{0,1}.h' + \boxed{0,2}.h'' + \&c.\\
\frac{dh'}{dt} &= \{(1,0) + (1,2) + \&c.\}l' - \boxed{1,0}.l - \boxed{1,2}.l'' - \&c.\\
\frac{dl'}{dt} &= -\{(1,0) + (1,2) + \&c.\}h' + \boxed{1,0}.h + \boxed{1,2}.h'' + \&c.\\
\frac{dh''}{dt} &= \{(2,0) + (2,1) + \&c.\}l'' - \boxed{2,0}.l - \boxed{2,1}.l' - \&c.\\
\frac{dl''}{dt} &= -\{(2,0) + (2,1) + \&c.\}.h'' + \boxed{2,0}.h + \boxed{2,1}.h' + \&c.\\
&\&c.
\end{aligned}\right\} \quad (A)$$

The quantities (0, 1) and (1, 0), $\boxed{0,1}$ and $\boxed{1,0}$ have remarkable relations, which facilitate the operations, and will be useful hereafter. By what precedes we have

$$(0,1) = -\frac{3\mu'.\,n\,a^2.\,a'\,(a,\,a')'}{4\,(a'^2-a^2)^2}.$$

If in this expression of (0, 1) we change μ' into μ, n into n', a into a' and reciprocally, we shall have the expression of (1, 0), which will consequently be

$$(1,0) = -\frac{3\mu.\,n'\,a'^2.\,a\,(a,'\,a)'}{4\,(a'^2-a^2)^2};$$

but we have $(a, a')' = (a', a)'$, since both these quantities result from th developement of the function $(a^2 - 2\,a\,a'\cos\theta + a'^2)^{\frac{1}{2}}$ into a series ordered according to the cosine of θ and of its multiples. We shall, therefore, have

$$(0,1).\,\mu.\,n'\,a' = (1,0).\,\mu'.\,n\,a.$$

But, neglecting the masses μ, μ', &c. in comparison with M, we have

$$n^2 = \frac{M}{a^3};\quad n'^2 = \frac{M}{a'^3};\quad \&c.$$

Therefore

$$(0,1)\,\mu\,\sqrt{a} = (1,0)\,\mu'\,\sqrt{a'};$$

an equation from which we easily derive (1, 0) when (0, 1) is determined.

In the same manner we shall find,

$$\boxed{0,1}\,\mu\,\sqrt{a} = \boxed{1,0}\,\mu'\,\sqrt{a'}.$$

These two equations will also subsist in the case where n and n' have different signs; that is to say, if the two bodies μ, μ' circulated in different directions; but then we must give the sign of n to the radical $\sqrt{a}$, and the sign of n' to the radical $\sqrt{a'}$.

From the two preceding equations evidently result these

$$(0,2)\,\mu\,\sqrt{a} = (2,0)\,\mu''\,\sqrt{a''};\quad \boxed{0,2}\,\mu\,\sqrt{a} = \boxed{2,0}.\,\mu''\,\sqrt{a''};\ \&c.$$

$$(1,2)\,\mu'\,\sqrt{a'} = (2,1)\,\mu''\,\sqrt{a''};\quad \boxed{1,2}\,\mu'\,\sqrt{a'} = \boxed{2,1}.\,\mu''.\,\sqrt{a''};\ \&c.$$

523. To integrate the equations (A) of the preceding No., we shall make

$$h = N.\sin.(g\,t+\beta);\quad l = N.\cos.(g\,t+\beta);$$
$$h' = N'.\sin.(g\,t+\beta);\quad l' = N'\cos.(g\,t+\beta);$$
$$\&c.$$

Then substituting these values in the equations (A), we shall have

$$\left.\begin{aligned} N\,g &= \{(0,1)+(0,2)+\&c.\}N - \boxed{0,1}.\,N' - \boxed{0,2}\,N'' - \&c.\\ N'\,g &= \{(1,0)+(1,2)+\&c.\}N' - \boxed{1,0}.\,N - \boxed{1,2}\,N'' - \&c.\\ N''g &= \{(2,0)+(2,1)+\&c.\}N'' - \boxed{2,0}.\,N - \boxed{2,1}\,N' - \&c. \end{aligned}\right\};\quad (B)$$

If we suppose the number of the bodies μ, μ', μ'', &c. equal to i; these equations will be in number i, and eliminating from them the constants N, N', &c., we shall have a final equation in g, of the degree i, which we easily obtain as follows:

Let φ be the function

$$\begin{aligned}
&N^2.\,\mu\,\surd\,a\{g - (0,1) - (0,2) - \&c.\}\\
&+ N'^2\,\mu'\,\surd\,a'\{g - (1,0) - (1,2) - \&c.\}\\
&+ \&c.\\
&+ 2\,N\,\mu\,\surd\,a\,\{\boxed{0,1}\,N' + \boxed{0,2}\,N'' + \&c.\}\\
&+ 2\,N'\,\mu'\,\surd\,a'\,\{\boxed{1,2}\,N'' + \boxed{1,3}\,N''' + \&c.\}\\
&+ 2\,N''\mu''\,\surd\,a''\{\boxed{2,3}\,N''' + \&c.\}\\
&+ \&c.
\end{aligned}$$

The equations (B) are reducible from the relations given in the preceding No. to these

$$\left(\frac{d\,\varphi}{d\,N}\right) = 0;\ \left(\frac{d\,\varphi}{d\,N'}\right) = 0;\ \left(\frac{d\,\varphi}{d\,N''}\right) = 0;\ \&c.$$

Considering therefore, N, N', N'', &c. as so many variables, φ will be a *maximum*. Moreover, φ being a homogeneous function of these variables, of the second dimension; we have

$$N\left(\frac{d\,\varphi}{d\,N}\right) + N'\left(\frac{d\,\varphi}{d\,N'}\right) + \&c. = 2\,\varphi;$$

we have, therefore, $\varphi = 0$, in virtue of the preceding equations.

Thus we can determine the *maximum* of the function φ. We shall first differentiate this function relatively to N, and then substitute in φ, for N, its value derived from the equation $\left(\frac{d\,\varphi}{d\,N}\right) = 0$, a value which will be a linear function of the quantities N', N'', &c. In this manner we shall have a rational function whole and homogeneous of the second dimension in terms of N', N'', &c.: let $\varphi^{(1)}$ be this function. We shall differentiate $\varphi^{(1)}$ relatively to N', and we shall substitute in $\varphi^{(1)}$ for N' its value derived from the equation $\left(\frac{d\,\varphi^{(0)}}{d\,N'}\right) = 0$: we shall have a homogeneous function of the second dimension in N'', N''', &c.: let $\varphi^{(2)}$ be this function. Continuing thus, we shall arrive at a function $\varphi^{(i-1)}$ of the second dimension, in $N^{(i-1)}$ and which will consequently be of the form $(N^{(i-1)})^2.\,k$, k being a function of g and constants. If we equal to zero, the differential of $\varphi^{(i-1)}$ taken relatively to $N^{(i-1)}$, we shall have $k = 0$; which will give an equation in g of the degree i, and whose different roots will give as many different systems for the indeterminates N, N', N'', &c.: the inde-

terminate $N^{(i-1)}$ will be the arbitrary of each system; and we shall immediately obtain, the relation of the other indeterminates N, N′, &c. of the same system, to this one, by means of the preceding equations taken in an inverse order, viz.,

$$\left(\frac{d\,\varphi^{(i-1)}}{d\,N^{(i-2)}}\right) = 0;\ \left(\frac{d\,\varphi^{(i-2)}}{d\,N^{(i-3)}}\right) = 0;\ \&c.$$

Let g, g_1, g_2, &c. be the i roots of the equation in g: let N, N′, N″, &c. be the system of indeterminates, relative to the root g: let $N_,$ $N_,'$, $N_,''$, &c. be the system of indeterminates relative to the root g_1, and so on: by the known theory of linear differential equations, we shall have

$$h = N \sin.(g\,t + \beta) + N_1 \sin.(g_1\,t + \beta_1) + N_2\,(g_2\,t + \beta_2) + \&c.;$$
$$h' = N' \sin.(g\,t + \beta) + N_1' \sin.(g_1\,t + \beta_1) + N_2'\,(g_2\,t + \beta_2) + \&c.;$$
$$h'' = N'' \sin.(g\,t + \beta) + N_1'' \sin.(g_1\,t + \beta_1) + N_2''(g_2\,t + \beta_2) + \&c.;$$
$$\&c.$$

β, β_1, β_2, &c. being arbitrary constants. Changing in these values of h, h′, h″, &c. the sines into cosines; we shall have the values of l, l′, l″, &c. These different values contain twice as many arbitraries as there are roots g, g_1, g_2, &c.; for each system of indeterminates contains an arbitrary, and moreover, it has i arbitraries β, β_1, β_2, &c.; these values are consequently the complete integrals of the equations (A) of the preceding No.

It is necessary, however, to determine only the constants N, N_1, &c.; N,′ N_1', &c.; β, β_1, &c. Observations will not give immediately the constants, but they make known at a given epoch, the excentricities e, e′, &c. of the orbits, and the longitudes ϖ, ϖ', &c. of their perihelions, and consequently, the values of h, h′, &c., l, l′, &c.: we shall thus derive the values of the preceding constants. For that purpose, we shall observe that if we multiply the first, third, fifth, &c. of the differential equations (A) of the preceding No., respectively by $N.\,\mu.\,\sqrt{a}$, $N'.\,\mu'.\,\sqrt{a'}$, &c.; we shall have in virtue of equations (B), and the relations found in the preceding No. between (0, 1) and (1, 0), (0, 2), and (2, 0), &c.

$$N.\frac{d\,h}{d\,t}\mu\sqrt{a} + N'.\frac{d\,h'}{d\,t}\mu'\sqrt{a'} + N''.\frac{d\,h''}{d\,t}\mu''\sqrt{a''} + \&c.$$
$$= g\{N.\,l.\,\mu.\sqrt{a} + N'.\,l'.\,\mu'.\sqrt{a'} + N''.\,l''.\,\mu''.\sqrt{a''} + \&c.\}$$

If we substitute in this equation for h, h′, &c. l, l′, &c. their preceding values; we shall have by comparing the coefficients of the same cosines

$$0 = N.\,N_1.\,\mu\sqrt{a} + N'.\,N_1'.\,\mu'\sqrt{a'} + N''.\,N_1''.\,\mu''.\sqrt{a''} + \&c.;$$
$$0 = N.\,N_2.\,\mu\sqrt{a} + N'.\,N_2'.\,\mu'\sqrt{a'} + N''.\,N_2''.\,\mu''.\sqrt{a''} + \&c.$$

Again, if we multiply the preceding values of h, h', &c. respectively by

$$N . \mu . \surd a,\ N' . \mu' . \surd a',\ \&c.$$

we shall have, in virtue of these last equations,

$$N . \mu\, h . \surd a + N' \mu' . h' . \surd a' + N'' . \mu'' h'' . \surd a'' + \&c.$$
$$= \{N^2 . \mu . \surd a + N'^2 . \mu' . \surd a' + N''^2 . \mu'' . \surd a'' + \&c.\} \sin (g t + \beta)$$

In like manner, we have

$$N . \mu\, l . \surd a + N' . \mu' l' . \surd a' + N'' . \mu'' l'' . \surd a'' + \&c.$$
$$= \{N^2 . \mu . \surd a + N'^2 . \mu' . \surd a' + N''^2 . \mu'' . \surd a'' + \&c.\} \cos . (g t + \beta).$$

By fixing the origin of the time t at the epoch for which the values of h, l, h', l', &c. are supposed known; the two preceding equations give

$$\tan . \beta = \frac{N . h \mu \surd a + N' . h' \mu' . \surd a' + N'' . h'' \mu'' . \surd a'' + \&c.}{N . l \mu . \surd a + N' . l' \mu' . \surd a' + N'' . l'' \mu'' . \surd a'' + \&c.}.$$

This expression of tan. β contains no indeterminate; for although the constants N, N', N'', &c. depend upon the indeterminate $N^{(i-1)}$, yet, as their relations to this indeterminate are known by what precedes, it will disappear from the expression of tan. β. Having thus determined β, we shall have $N^{(i-1)}$, by means of one of the two equations which give tan. β; and we thence obtain the system of indeterminates, N, N', N'', &c. relative to the root g. Changing, in the preceding expressions, this root into g_1, g_2, g_3, &c. we shall have the values of the arbitraries relative to each of these roots.

If we substitute these values in the expressions of h, l, h', l', &c.; we hence derive the values of the excentricities e, e', &c. of the orbits, and the longitudes of their perihelions, by means of the equations

$$e^2 = h^2 + l^2;\ e'^2 = h'^2 + l'^2;\ \&c.$$

$$\tan . \varpi = \frac{h}{l};\ \tan . \varpi' = \frac{h'}{l'};\ \&c.$$

we shall thus have

$$e^2 = N^2 + N_1^2 + N_2^2 + \&c. + 2 N N;\ \cos . \{(g_1 - g) t + \beta_1 - \beta\}$$
$$+ 2 N N_2 \cos . \{(g_2 - g) t + \beta_2 - \beta)\} + 2 N_1 N_2 \cos . \{(g_2 - g_1) t + \beta_2 - \beta_1\} + \&c.$$

This quantity is always less than $(N + N_1 + N_2 + \&c.)^2$, when the roots g, g_1, &c. are all real and unequal, by taking positively the quantities N, N_1, &c. In like manner, we shall have

$$\tan . \varpi = \frac{N \sin . (g t + \beta) + N_1 \sin . (g_1 t + \beta_1) + N_2 \sin . (g_2 t + \beta_2) + \&c.}{N \cos . (g t + \beta) + N_1 \cos . (g_1 t + \beta_1) + N_2 \cos . (g_2 t + \beta_2) + \&c.}$$

whence it is easy to get,

$$\tan . (\varpi - g t - \beta) = \frac{N_1 \sin . \{(g_1 - g) t + \beta_1 - \beta\} + N_2 \sin . \{(g_2 - g) t + \beta_2 - \beta\} + \&c.}{N + N_1 \cos . \{(g_1 - g) t + \beta_1 - \beta\} + N_2 \cos . \{(g_2 - g) t + \beta_2 - \beta\} + \&c.}$$

Whilst the sum $N_1 + N_2 +$ &c. of the coefficients of the cosines of the denominator, all taken positively, is less than N, tan. $(\varpi - g\,t - \beta)$ can never become infinite; the angle $\varpi - g\,t - \beta$ can never reach the quarter of the circumference; so that in this case the true mean motion of the perihelion is equal to g t.

524. From what has been shown it follows, that the excentricities of the orbits and the positions of their axis-majors, are subject to considerable variations, which at length change the nature of the orbits, and whose periods depending on the roots g, g_1, g_2, &c., embrace with regard to the planets, a great number of ages. We may thus consider the excentricities as variably elliptic, and the motions of the perihelions as not uniform. These variations are very sensible in the satellites of Jupiter, and we shall see hereafter, that they explain the singular inequalities, observed in the motion of the third satellite.

But it is of importance to examine whether the variations of the excentricities have limits, and whether the orbits are constantly almost circular. We know that if the roots of the equation in g are all real and unequal, the excentricity e of the orbit of μ is always less than the sum $N + N_1 + N_2 +$ &c. of the coefficients of the sines of the expression of h taken positively; and since the coefficients are supposed very small, the value of e will always be inconsiderable. By taking notice, therefore, of the secular variations only, we see that the orbits of the bodies μ, μ', μ'', &c. will only flatten more or less in departing a little from the circular form; but the positions of their axis-majors will undergo considerable variations. These axes will be constantly of the same length, and the mean motions which depend upon them will always be uniform, as we have seen in No. 521. The preceding results, founded upon the smallness of the excentricity of the orbits, will subsist without ceasing, and will extend to all ages past and future; so that we may affirm that at any time, the orbits of the planets and satellites have never been nor ever will be very excentric, at least whilst we only consider their mutual actions. But it would not be the same if any of the roots g, g_1, g_2, &c. were equal or imaginary: the sines and cosines of the expressions of h, l, h′, l′, &c. corresponding to these roots, would then change into circular arcs or exponentials, and since these quantities increase indefinitely with the time, the orbits would at length become very excentric; the stability of the planetary system would then be destroyed, and the results found above would cease to take place. It is therefore highly important to show that g, g_1, g_2, &c. are all real and unequal. This we will now demonstrate in a very simple

manner, for the case of nature, in which the bodies μ, μ', μ'', &c. of the system, all circulate in the same direction.

Let us resume the equations (A) of No. 528. If we multiply the first by $\mu . \sqrt{a} . h$; the second, by $\mu . \sqrt{a} . l$; the third by $\mu' . \sqrt{a'} . h'$; the fourth by $\mu' . \sqrt{a'} . l'$, &c. and afterwards add the results together; the coefficients of h l, h' l', h'' l'', &c. will be nothing in this sum, the coefficients of h' l — h l' will be $\overline{|0, 1|} . \mu . \sqrt{a} - \overline{|1, 0|} . \mu' . \sqrt{a'}$, and this will be nothing in virtue of the equation $\overline{|0, 1|} . \mu . \sqrt{a} = \overline{|1, 0|} . \mu' . \sqrt{a'}$ found in No. 522. The coefficients of h'' l — h l'', h'' l' — h' l'', &c. will be nothing for the same reason; the sum of the equations (A) thus prepared will therefore be reduced to

$$\frac{h\,dh + l\,dl}{dt} . \mu . \sqrt{a} + \frac{h'\,dh' + l'\,dl'}{dt} . \mu' . \sqrt{a'} + \&c. = 0;$$

and consequently to

$$0 = e\,de . \mu . \sqrt{a} + e'\,de' . \mu' . \sqrt{a'} + \&c.$$

Integrating this equation and observing that (No. 521) the semi-axis-majors are constant, we shall have

$$e^2 . \mu \sqrt{a} + e'^2 . \mu' . \sqrt{a'} + e''^2 . \mu'' . \sqrt{a''} + \&c. = \text{constant}; \quad (a)$$

The bodies μ, μ', μ'', &c. however being supposed to circulate in the same direction, the radicals $\sqrt{a}$, $\sqrt{a'}$, $\sqrt{a''}$, &c. ought to be taken positively in the preceding equation, as we have seen in No. 522; all the terms of the first member of this equation are therefore positive, and consequently, each of them is less than the constant of the second member. But by supposing at any epoch the excentricities to be very small, this constant will be very small; each of the terms of the equation will, therefore, remain always very small and cannot increase indefinitely; the orbits will always be very nearly circular.

The case which we have thus examined, is that of the planets and satellites of the solar system; since all these bodies circulate in the same direction, and at the present epoch their orbits have little excentricity. That no doubt may exist as to a result so important, we shall observe that if the equation which determines g, contained imaginary roots, some of the sines and cosines of the expressions of h, l, h', l', &c. would transform into exponentials; thus the expression of h would contain a finite number of terms of the form $P . c^{ft}$, c being the number of which the hyperbolic logarithm is unity, and P being a real quantity, because h or e sin. ϖ is a real quantity. Let

$$Q . c^{ft}, P' . c^{ft}, Q' . c^{ft}, P'' . c^{ft}, \&c.$$

be the corresponding terms of l, h', l', h'', &c.; Q, P', Q', P'', &c. being

also real quantities: the expression of e^2 will contain the term $(P^2+Q^2)c^{2ft}$; the expression of e'^2 will contain the term $(P'^2+Q'^2)c^{2ft}$, and so on; the first member of the equation (u) will therefore contain the term

$$\{(P^2+Q^2)\mu.\sqrt{a}+(P'^2+Q'^2)\mu'\sqrt{a'}+(P''^2+Q''^2)\mu''\sqrt{a''}+\&c.\}e^{2ft}.$$

If, therefore, we suppose c^{ft} to be the greatest of the exponentials which contain h, l, h', l', &c. that is to say, that in which f is the most considerable, c^{2ft} will be the greatest of the exponentials which contain the first member of the preceding equation: the preceding term cannot therefore be destroyed by any other term of this first member; so that for this member to be reduced to a constant, the coefficient of c^{2ft} must be nothing, which gives

$$0=(P^2+Q^2)\mu\sqrt{a}+(P'^2+Q'^2)\mu'\sqrt{a'}+(P''^2+Q''^2)\mu''\sqrt{a''}+\&c.$$

When $\sqrt{a}$, $\sqrt{a'}$, $\sqrt{a''}$, &c. have the same sign, or which is tantamount, when the bodies μ, μ', μ'', &c. circulate in the same direction, this equation is impossible, provided we do not suppose $P=0$, $Q=0$, $P'=0$, &c.; whence it follows that the quantities h, l, h' l', &c. do not contain exponentials, and that the equation in g does not contain imaginary roots.

If this equation had equal roots, the expressions of h, l, h', l', &c. would contain as we know, circular arcs and in the expression of h, we should have a finite number of terms of the form Pt^r. Let Qt^r, $P't^r$, $Q't^r$, &c. be the corresponding terms of l, h', l', &c. P, Q, P', Q', &c. being real quantities; the first member of the equation (u) will contain the term

$$\{(P^2+Q^2)\mu\sqrt{a}+(P'^2+Q'^2)\mu'\sqrt{a'}+(P''^2+Q''^2)\mu''\sqrt{a''}+\&c.\}.t^{2r}.$$

If t^r is the highest power of t, contained by the values of h, l, h' l', &c.; t^{2r} will be the highest power of t contained in the first member of the equation (u); thus, that this member may be reduced to a constant, we must have

$$0=(P^2+Q^2)\mu\sqrt{a}+(P'^2+Q'^2)\mu'\sqrt{a'}+\&c.$$

which gives

$$P=0,\ Q=0,\ P'=0,\ Q'=0,\ \&c.$$

The expressions of h, l, h', l', &c. contain therefore, neither exponentials nor circular arcs, and consequently all the roots of the equation in g are real and unequal.

The system of the orbits of μ, μ', μ'', &c. is therefore perfectly stable relatively to their excentricities; these orbits merely oscillate about a mean state of ellipticity, which they depart from but little by preserving the same major-axis: their excentricities are always subject to this condi-

tion, viz. *that the sum of their squares multiplied respectively by the masses of the bodies and by the square roots of the major-axes is always the same.*

525. When we shall have determined, by what precedes, the values of e and of ϖ; we shall substitute in all the terms of the expressions of ξ, and $\frac{dv}{dt}$, given in the preceding Nos., effacing the terms which contain the time t without the symbols *sine* and *cosine*. The elliptic part of these expressions will be the same as in the case of an orbit not disturbed, with this only difference, that the excentricity and the position of the perihelion are variable; but the periods of these variations being very long, by reason of the smallness of the masses μ, μ', μ'', &c. relatively to M; we may suppose these variations proportional to the time, during a great interval, which, for the planets, may extend to many ages before and after the given epoch.

It is useful, for astronomical purposes, to obtain under this form, the secular variations of the excentricities and perihelions of the orbits: we may easily get them from the preceding formulæ. In fact, the equation $e^2 = h^2 + l^2$, gives $e\,de = h\,dh + l\,dl$; but in considering only the action of μ', we have by No. 522,

$$\frac{dh}{dt} = (0,1)\,l - \boxed{0,1}\,l';$$

$$\frac{dl}{dt} = -(0,1)\,h + \boxed{0,1}.\,h';$$

wherefore

$$\frac{e\,de}{dt} = \boxed{0,1}.\,\{h'\,l - h\,l'\};$$

but we have $h'\,l - h\,l' = e\,e'\sin.(\varpi' - \varpi)$; we, therefore, have

$$\frac{de}{dt} = \boxed{0,1}.\,e'\sin.(\varpi' - \varpi);$$

thus, with regard to the reciprocal action of the different bodies μ', μ'', &c. we shall have

$$\frac{de}{dt} = \boxed{0,1}.\,e'\sin.(\varpi' - \varpi) + \boxed{0,2}.\,e''\sin.(\varpi'' - \varpi) + \&c.$$

$$\frac{de'}{dt} = \boxed{1,0}\,e\sin.(\varpi - \varpi') + \boxed{1,2}\,e''\sin.(\varpi'' - \varpi') + \&c.$$

$$\frac{de''}{dt} = \boxed{2,0}\,e\sin.(\varpi - \varpi'') + \boxed{2\ 1}\,e'\sin.(\varpi' - \varpi'') + \&c.$$

&c.

The equation $\tan. \varpi = \frac{h}{l}$, gives by differentiating

$$e^2 d\varpi = l\,dh - h\,dl.$$

With respect only to the action of μ', by substituting for dh and dl their values, we shall have

$$\frac{e^2 d\varpi}{dt} = (0,1)(h^2 + l^2) - \boxed{0,1}.\{h h' + l l'\};$$

which gives

$$\frac{d\varpi}{dt} = (0,1) - \boxed{0,1}.\frac{e'}{e}\cos.(\varpi' - \varpi);$$

we shall, therefore, have, through the reciprocal actions of the bodies μ, μ', μ'', &c.

$$\frac{d\varpi}{dt} = (0,1)+(0,2)+\&c.-\boxed{0,1}.\frac{e'}{e}\cos.(\varpi'-\varpi)-\boxed{0,2}.\frac{e''}{e}\cos.(\varpi''-\varpi)-\&c.$$

$$\frac{d\varpi'}{dt} = (1,0)+(1,2)+\&c.-\boxed{1,0}.\frac{e}{e'}\cos.(\varpi-\varpi')-\boxed{1,2}.\frac{e''}{e'}\cos.(\varpi''-\varpi')-\&c.$$

$$\frac{d\varpi''}{dt} = (2,0)+(2,1)+\&c.-\boxed{2,0}.\frac{e}{e''}\cos.(\varpi-\varpi'')-\boxed{2,1}.\frac{e'}{e''}\cos.(\varpi'-\varpi'')-\&c.$$

&c.

If we multiply these values of $\frac{de}{dt}$, $\frac{de'}{dt}$, &c. $\frac{d\varpi}{dt}$, $\frac{d\varpi'}{dt}$, &c. by the time t; we shall have the differential expressions of the secular variations of the excentricities and of the perihelions, and these expressions which are only rigorous whilst t is indefinitely small, will however serve for a long interval relatively to the planets. Their comparison with precise and distant observations, affords the most exact mode of determining the masses of the planets which have no satellites. For any time t we have the excentricity e, equal to

$$e + t.\frac{de}{dt} + \frac{t^2}{1.2}.\frac{d^2e}{dt^2} + \&c.$$

e, $\frac{de}{dt}$, $\frac{d^2e}{dt^2}$, &c. being relative to the origin of the time t or to the given epoch. The preceding value of $\frac{de}{dt}$ will give, by differentiating it, and observing that a, a', &c. are constant, the values of $\frac{d^2e}{dt^2}$, $\frac{d^3e}{dt^3}$, &c.; we can, therefore, thus continue as far as we wish, the preceding series, and by the same process, the series also relative to ϖ: but relatively to the planets, it will be sufficient, in comparing the most ancient observations

which have come down to us, to take into account the square of the time, in the expressions of the series of e, e', &c. ϖ, ϖ', &c.

526. We will now consider the equations relative to the position of the orbits. For this purpose let us resume the equations (3) and (4) of No. 520,

$$\frac{dp}{dt} = -\frac{\mu' n}{4} . a^2 a' B^{(1)} . (q - q');$$

$$\frac{dq}{dt} = \frac{\mu' n}{4} . a^2 a' B^{(1)} . (p - p').$$

By No. 516, we have

$$a^2 a' . B^{(1)} = \alpha^2 . b^{(1)}_{\frac{3}{2}};$$

and by the same No.,

$$b^{(1)}_{\frac{5}{2}} = -\frac{3 b^{(1)}_{-\frac{1}{2}}}{(1 - \alpha^2)^2};$$

We shall therefore have

$$\frac{\mu' n}{4} a^2 a' B^{(1)} = -\frac{3 \mu' . n . \alpha^2 b^{(1)}_{-\frac{1}{2}}}{4 (1 - \alpha^2)^2}.$$

The second member of this equation is what we have denoted by (0, 1) in 522; we shall hence have

$$\frac{dp}{dt} = (0, 1) (q' - q);$$

$$\frac{dq}{dt} = (0, 1) (p - p');$$

Hence, it is easy to conclude that the values of q, p, q', p', &c. will be determined by the following system of differential equations:

$$\left.\begin{aligned}
\frac{dq}{dt} &= \{(0,1) + (0,2) + \&c.\} . p - (0,1) p' - (0,2) p'' - \&c. \\
\frac{dp}{dt} &= -\{(0,1) + (0,2) + \&c.\} . q + (0,1) q' + (0,2) q'' + \&c. \\
\frac{dq'}{dt} &= \{(1,0) + (1,2) + \&c.\} . p' - (1,0) p - (1,2) p'' - \&c. \\
\frac{dp'}{dt} &= -\{(1,0) + (1,2) + \&c.\} . q' + (1,0) q + (1,2) q'' + \&c. \\
\frac{dq''}{dt} &= \{(2,0) + (2,1) + \&c.\} . p'' - (2,0) p - (2,1) p' - \&c. \\
\frac{dp''}{dt} &= -\{(2,0) + (2,1) + \&c.\} . q'' + (2,0) q + (2,1) q' + \&c. \\
&\&c.
\end{aligned}\right\}; \quad (C)$$

This system of equations is similar to that of the equations (A) of No. 522: it would entirely coincide with it, if in the equations of (A) we were to change h, l, h', l', &c. into q, p, q', p', &c. and if we were to suppose

$$\boxed{0, 1} = (0, 1);$$

$$\boxed{1, 0} = (1, 0);$$

&c.

Hence, the process which we have used in No. 523 to integrate the equations (A) applies also to the equations (C). We shall therefore suppose

$$q = N \cos.(gt+\beta) + N_1 \cos.(g_1 t+\beta_1) + N_2 \cos.(g_2 t+\beta_2) + \&c.$$

$$p = N \sin.(gt+\beta) + N_1 \sin.(g_1 t+\beta_1) + N_2 \sin.(g_2 t+\beta_2) + \&c.$$

$$q' = N' \cos.(gt+\beta) + N_1' \cos.(g_1 t+\beta_1) + N_2' \cos.(g_2 t+\beta_2) + \&c.$$

$$p' = N' \sin.(gt+\beta) + N_1' \sin.(g_1 t+\beta_1) + N_2' \sin.(g_2 t+\beta_2) + \&c.$$

&c.

and by No. 523, we shall have an equation in g of the degree i, and whose different roots will be g, g_1, g_2, &c. It is easy to perceive that one of these roots is nothing; for it is clear we satisfy the equations (C) by supposing p, p', p'', &c. equal and constant, as also q, q', q'', &c. This requires one of the roots of the equation in g to be zero, and we can thence depress the equation to the degree i — 1. The arbitraries N, N_1, N', &c. β, β_1, &c. will be determined by the method exposed in No. 523. Finally, we shall find by the process employed in No. 524.

$$\text{const.} = (p^2 + q^2)\,\mu\sqrt{a} + (p'^2 + q'^2)\,\mu'\sqrt{a'} + \&c.$$

Whence we conclude, as in the No. cited, that the expressions of p, q, p', q', &c. contain neither circular arcs nor exponentials, when the bodies μ, μ', μ'', &c. circulate in the same direction: and that therefore the equation in g has all its roots real and unequal.

We may obtain two other integrals of the equations (C). In fact, if we multiply the first of these equations by $\mu\sqrt{a}$, the third by $\mu'\sqrt{a'}$, the fifth by $\mu''\sqrt{a''}$, &c. we shall have, because of the relations found in No. 522,

$$0 = \frac{dq}{dt}\mu\sqrt{a} + \frac{dq'}{dt}\mu'\sqrt{a'} + \&c.;$$

which by integration gives

$$\text{constant} = q\mu\sqrt{a} + q'\mu'\sqrt{a'} + \&c. \quad . \quad . \quad . \quad . \quad (1)$$

In the same manner we find

$$\text{constant} = p\mu\sqrt{a} + p'\mu'\sqrt{a'} + \&c. \quad . \quad . \quad . \quad . \quad (2)$$

Call φ the inclination of the orbit of μ to the fixed plane, and θ the lon-

gitude of the ascending node of this orbit upon the same plane; the latitude of μ will be very nearly tan. φ sin. $(n\,t + \varepsilon - \theta)$: Comparing this value with q sin. $(n\,t + \varepsilon)$ — p cos. $(n\,t + \varepsilon)$, we shall have

$$p = \text{tan. } \varphi \text{ sin. } \theta;\ q = \text{tan. } \varphi \text{ cos. } \theta;$$

whence we obtain

$$\text{tan. } \varphi = \sqrt{(p^2 + q^2)};\ \text{tan. } \theta = \frac{p}{q};$$

We shall, therefore, have the inclination of the orbit of μ, and the position of its node, by means of the values of p and q. By marking successively with one dash, two dashes, &c. relatively to μ', μ'', &c. the values of tan. φ, tan. θ, we shall have the inclinations of the orbits of μ' μ'', &c. and the positions of their nodes by means of p′, q′, p″, q″, &c.

The quantity $\sqrt{p^2 + q^2}$ is less than the sum $N + N_1 + N_2 +$ &c. of the coefficients of the sines in the expression of q; thus, the coefficients being very small since the orbit is supposed but little inclined to the fixed plane, its inclination will always be inconsiderable; whence it follows, that the system of orbits is also stable, relatively to their inclinations as also to their excentricities. We may therefore consider the inclinations of the orbits, as variable quantities comprised within determinate limits, and the motion of the nodes as not uniform. These variations are very sensible in the satellites of Jupiter, and we shall see hereafter, that they explain the singular phenomena observed in the inclination of the orbit of the fourth satellite.

From the preceding expressions of p and q results this theorem:

Let us imagine a circle whose inclination to a fixed plane is N, *and of which the longitude of the ascending node is* $g\,t + \beta$; *also let us imagine upon this first circle, a second circle inclined by the angle* N_1, *the longitude of whose intersection with the former circle is* $g_1\,t + \beta_1$; *upon this second circle let there be a third inclined to it by the angle* N_2, *the longitude of whose intersection with the second circle is* $g_2\,t + \beta_2$, *and so on; the position of the last circle will be that of the orbit of* μ.

Applying the same construction to the expressions of h and l of No. 523, we see that the tangent of the inclination of the last circle upon the fixed plane, is equal to the excentricity of μ's orbit, and that the longitude of the intersection of this circle with the same plane, is equal to that of the perihelion of μ's orbit.

527. It is useful for astronomical purposes, to have the differential variations of the nodes and inclinations of the orbits. For this purpose, let us resume the equations of the preceding No.

$$\tan.\varphi = \sqrt{(p^2 + q^2)},\ \tan.\theta = \frac{p}{q}.$$

Differentiating these, we shall have

$$d\varphi = d\,p \sin.\theta + d\,q \cos.\theta;$$
$$d\theta = \frac{d\,p \cos.\theta - d\,q \sin.\theta}{\tan.\varphi}.$$

If we substitute for d p and d q, their values given by the equations (C) of the preceding No. we shall have

$$\frac{d\varphi}{dt} = (0,1) \tan.\varphi' \sin.(\theta - \theta') + (0,2) \tan.\varphi'' . \sin.(\theta - \theta'') + \&c.$$

$$\frac{d\theta}{dt} = -\{(0,1) + (0,2) + \&c.\} + (0,1) \frac{\tan.\varphi'}{\tan.\varphi} \cos.(\theta - \theta')$$
$$+ (0,2) \frac{\tan.\varphi''}{\tan.\varphi} \cos.(\theta - \theta'') + \&c.$$

In like manner, we shall have

$$\frac{d\varphi'}{dt} = (1,0) \tan.\varphi \sin.(\theta' - \theta) + (1,2) \tan.\varphi'' \sin.(\theta' - \theta'') + \&c.$$

$$\frac{d\theta'}{dt} = -\{(1,0) + (1,2) + \&c.\} + (1,0) . \frac{\tan.\varphi}{\tan.\varphi'} . \cos.(\theta' - \theta)$$
$$+ (1,2) . \frac{\tan.\varphi''}{\tan.\varphi'} \cos.(\theta' - \theta'') + \&c.$$

&c.

Astronomers refer the celestial motions to the moveable orbit of the earth; it is in fact from the plane of this orbit that we observe them; it is therefore important to know the variations of the nodes and the inclinations of the orbits, relatively to the orbit of one of the bodies μ, μ', μ'', &c. for example to the orbit of μ. It is clear that

$$q \sin.(n' t + \varepsilon') - p \cos.(n' t + \varepsilon')$$

would be the latitude of μ' above the fixed plane if it were in motion upon the orbit of μ. The latitude of this moveable plane above the same plane is

$$q' \sin.(n' t + \varepsilon') - p' \cos.(n' t + \varepsilon');$$

but the difference of these two latitudes is very nearly the latitude of μ' above the orbit of μ; calling therefore $\varphi_{,}'$ the inclination, and $\theta_{,}'$ the longitude of the node of μ' upon the orbit of μ, we shall have, by what precedes,

$$\tan.\varphi_{,}' = \sqrt{(p' - p)^2 + (q' - q)^2};\ \tan.\theta_{,}' = \frac{p' - p}{q' - q}.$$

If we take for the fixed plane, that of μ's orbit at a given epoch; we

shall have at that epoch p = 0, q = 0; but the differentials d p and d q will not be zero; thus we shall have.

$$d\varphi' = (dp' - dp)\sin.\theta + (dq' - dq)\cos.\theta;$$

$$d\theta' = \frac{(dp' - dp)\cos.\theta - (dq' - dq)\sin.\theta}{\tan.\varphi'}.$$

Substituting for d p, d q, d p', d q', &c. their values given by the equations (C) of the preceding No., we shall have

$$\frac{d\varphi'}{dt} = \{(1,2) - (0,2)\}\tan.\varphi''\sin.(\theta - \theta'') + \{(1,3) - (0,3)\}\tan.\varphi'''\sin.(\theta - \theta''') + \&c.$$

$$\frac{d\theta'}{dt} = -\{(1,0) + (1,2) + (1,3) + \&c.\} - (0,1) + \{(1,2) - (0,2)\}.\frac{\tan.\varphi''}{\tan.\varphi'}\cos.(\theta - \theta') + \{(1,3) - (0,3)\}.\frac{\tan.\varphi'''}{\tan.\varphi'}\cos.\theta - \theta''') + \&c.$$

It is easy to obtain from these expressions the variations of the nodes, and inclinations of the orbits of the other bodies μ'', μ''', &c. upon the moveable orbit of μ.

528. The integrals found above, of the differential equations which determine the variations of the elements of the orbits, are only approximate, and the relations which they give among the elements, only take place on the supposition that the excentricities of the orbits and their inclinations are very small. But the integrals (4), (5), (6), (7), which are given in No. 471, give the same relations, whatever may be the excentricities and inclinations. For this, we shall observe that $\frac{x\,dy - y\,dx}{dt}$ is double the area described during the instant d t, by the projection of the radius-vector of the planet μ upon the plane of x, y. In the elliptic motion, if we neglect the mass of the planet as nothing compared with that of the sun, taken for unity, we shall have, by the Nos. 157, 237, relatively to the plane of μ's orbit,

$$\frac{x\,dy - y\,dx}{dt} = \sqrt{a(1 - e^2)}.$$

In order to refer the area upon the orbit to the fixed plane, we must multiply by the cosine of the inclination φ of the orbit to this plane; we shall, therefore, have, with reference to this plane,

$$\frac{x\,dy - y\,dx}{dt} = \cos.\varphi\sqrt{a(1 - e^2)} = \sqrt{\frac{a.(1 - e^2)}{1 + \tan.^2\varphi}}.$$

In like manner

$$\frac{x'\,dy' - y'\,dx'}{dt} = \sqrt{\frac{a'(1-e'^2)}{1+\tan.^2\varphi'}};$$

&c.

These values of $x\,dy - y\,dx$, $x'\,dy' - y'\,dx'$, &c. may be used, abstraction being made of the inequalities of the motion of the planets, provided we consider the elements e, e', &c. φ, φ', &c. as variables, in virtue of the secular inequalities; the equation (4) of No. 471 will therefore give in that case,

$$c = \mu\sqrt{\frac{a(1-e^2)}{1+\tan.^2\varphi}} + \mu'\sqrt{\frac{a'(1-e'^2)}{1+\tan.^2\varphi'}} + \&c.$$
$$+\Sigma.\,\mu\mu'\left\{\frac{(x'-x)(dy'-dy)-(y'-y)(dx'-dx)}{dt}\right\}.$$

Neglecting this last term, which always remains of the order $\mu\mu'$, we shall have

$$c = \mu\sqrt{\frac{a.(1-e^2)}{1+\tan.^2\varphi}} + \mu'\sqrt{\frac{a'(1-e'^2)}{1+\tan.^2\varphi'}} + \&c.$$

Thus, whatever may be the changes which the lapse of time produces in the values of e, e', &c. φ, φ', &c. by reason of the secular variations, these values ought always to satisfy the preceding equation.

If we neglect the small quantities of the order e^4, or $e^2\varphi^2$, this equation will give

$$c = \mu\sqrt{a} + \mu'\sqrt{a'} + \&c. - \tfrac{1}{2}\mu\sqrt{a}\{e^2 + \tan.^2\varphi\}$$
$$- \tfrac{1}{2}\mu'\sqrt{a'}\{e'^2 + \tan.^2\varphi'\} - \&c.;$$

and consequently, if we neglect the squares of e, e', φ, &c. we shall have $\mu\sqrt{a} + \mu'\sqrt{a'} + \&c.$ constant. We have seen above, that if we only retain the first power of the perturbing force, a, a', &c. will be separately constant; the preceding equation will therefore give, neglecting small quantities of the order e^4 or $e^2\varphi^2$,

$$\text{const.} = \mu\sqrt{a}\{e^2 + \tan.^2\varphi\} + \mu'\sqrt{a'}\{e'^2 + \tan.^2\varphi'\} + \&c.$$

On the supposition that the orbits are nearly circular, and but little inclined to one another, the secular variations are determined (No. 522) by means of differential equations independent of the inclinations, and which consequently are the same as though the orbits were in one plane. But in this hypothesis we have

$$\varphi = 0,\ \varphi' = 0,\ \&c.$$

the preceding equation thus becoming

$$\text{constant} = e^2\mu\sqrt{a} + e'^2\mu'\sqrt{a'} + e''^2\mu''\sqrt{a''} + \&c.$$

an equation already given in No. 524.

In like manner the secular variations of the inclinations of the orbits, are (No. 526) determined by means of differential equations, independent of excentricities, and which consequently are the same as though the orbits were circular. But in this hypothesis we have $e = 0$, $e' = 0$, &c. Wherefore

$$\text{const.} = \mu\sqrt{a} \cdot \tan.^2\varphi + \mu'\sqrt{a'} \cdot \tan.^2\varphi' + \mu''\sqrt{a''} \cdot \tan.^2\varphi'' + \&c.$$

an equation which has already been given in No. 526.

If we suppose, as in the last No.

$$p = \tan.\varphi \sin.\theta;\quad q = \tan.\varphi \cos.\theta;$$

it is easy to prove that, the inclination of the orbit of μ to the plane of x, y being φ, and the longitude of its ascending node reckoned from the axis of x being θ, the cosine of the inclination of this orbit to the plane of x, z, will be

$$\frac{q}{\sqrt{(1 + \tan.^2\varphi)}}.$$

Multiplying this quantity by $\frac{x\,dy - y\,dx}{dt}$, or by its value $\sqrt{a(1 - e^2)}$, we shall have the value of $\frac{x\,dz - z\,dx}{dt}$; the equation (5) of No. 471, will therefore give us, neglecting quantities of the order μ^2,

$$c' = \mu \cdot q\sqrt{\frac{a(1 - e^2)}{1 + \tan.^2\varphi}} + \mu' \cdot q'\sqrt{\frac{a' \cdot (1 - e'^2)}{1 + \tan.^2\varphi'}} + \&c.$$

We shall find, in like manner, that the equation (6) of No. 471, gives

$$c'' = \mu \cdot p \cdot \sqrt{\frac{a(1 - e^2)}{1 + \tan.^2\varphi}} + \mu' \cdot p'\sqrt{\frac{a' \cdot (1 - e'^2)}{1 + \tan.^2\varphi'}} + \&c.$$

If in these two equations we neglect quantities of the order e^2 or $e^2\varphi$; they will become

$$\text{const.} = \mu\, q \cdot \sqrt{a} + \mu'\, q'\sqrt{a'} + \&c.$$
$$\text{const.} = \mu\, p\sqrt{a} + \mu'\, p'\sqrt{a'} + \&c.$$

equations already found in No. 526.

Finally, the equation (7) of No. 471, will give, observing that by 478,

$$\frac{m}{a} = \frac{2m}{\varrho} - \frac{dx^2 + dy^2 + dz^2}{dt^2}$$

and neglecting quantities of the order $\mu\ \mu'$,

$$\text{const.} = \frac{\mu}{a} + \frac{\mu'}{a'} + \frac{\mu''}{a''} + \&c.$$

These different equations subsist, when we regard inequalities due to very long periods, which affect the elements of the orbits of μ, μ', &c. We have observed in No. 521, that the relation of the mean motions of these bodies may introduce into the expressions of the axis-majors of the

orbits considered variable, inequalities whose arguments proportional to the time increase very slowly, and which having for divisors the coefficients of the time t, in these arguments, may become sensible. But it is evident that, retaining the terms only which have like divisors, and considering the orbits as ellipses whose elements vary by reason of those terms, the integrals (4), (5), (6), (7), of No. 471, will always give the relations between these elements already found; because the terms of the order $\mu\,\mu'$ which have been neglected in these integrals, to obtain the relations, have not for divisors the very small coefficients above mentioned, or at least they contain them only when multiplied by a power of the perturbing forces superior to that which we are considering.

529. We have observed already, that in the motion of a system of bodies, there exists an invariable plane, or such as always is of a parallel situation, which it is easy to find at all times by this condition, that the sum of the masses of the system, multiplied respectively by the projections of the areas described by the radius-vectors in a given time is a *maximum*. It is principally in the theory of the solar system, that the research of this plane is important, when viewed with reference to the proper motions of the stars and of the ecliptic, which make it so difficult to astronomers to determine precisely the celestial motions. If we call γ the inclination of this invariable plane to that of x, y, and Π the longitude of its ascending node, it is easily found that

$$\tan.\gamma \sin.\Pi = \frac{c''}{c};\quad \tan.\gamma\cos.\Pi = \frac{c'}{c};$$

and consequently that

$$\tan.\gamma \sin.\Pi = \frac{\mu\sqrt{a(1-e^2)}\sin.\varphi\sin.\theta + \mu'\sqrt{a'(1-e'^2)}\sin.\varphi'\sin.\theta' + \&c.}{\mu\sqrt{a(1-e^2)}\cos.\varphi + \mu'\sqrt{a'(1-e'^2)}\cos.\varphi' + \&c.}$$

$$\tan.\gamma\,.\cos.\Pi = \frac{\mu\sqrt{a(1-e^2)}.\sin.\varphi\cos.\theta + \mu'\sqrt{a'(1-e'^2)}\sin.\varphi'\cos.\theta' + \&c.}{\mu\sqrt{a(1-e^2)}.\cos.\varphi + \mu'\sqrt{a'(1-e'^2)}.\cos.\varphi' + \&c.}$$

We shall determine very easily, by means of these values, the angles γ and Π. We see that to determine the invariable plane we ought to know the masses of the comets, and the elements of their orbits; fortunately these masses appear to be so very small that we may, without sensible error, neglect their action upon the planets; but time alone can clear up this point to us. We may observe here, that relatively to this invariable plane the values of p, q, p', q', &c. contain no constant terms; for it is evident by the equations (C) of No. 526, that these terms are the same for p, p', p'', &c. and that they are also the same for q, q', q'', &c.; and since relatively to the invariable plane, the constants of the first members of the

equations (1) and (2) of No. 526 are nothing: the constant terms disappear, by reason of these equations, from the expressions p, p', &c. q, q', &c.

Let us consider the motion of the two orbits, supposing them inclined to one another, by any angle whatever: we shall have by No. 528,

$$c' = \sin.\varphi \cos.\theta . \mu \sqrt{a(1-e^2)} + \sin.\varphi' . \cos.\theta' . \mu' \sqrt{a'(1-e'^2)};$$

$$c'' = \sin.\varphi \sin.\theta . \mu \sqrt{a(1-e^2)} + \sin.\varphi' . \sin.\theta' . \mu' \sqrt{a'(1-e'^2)}.$$

Let us suppose that the fixed plane to which we refer the motion of the orbits, is the invariable plane of which we have spoken, and by reference to which the constants of the first members of these equations, are nothing, as may easily be shown. The angles φ and φ' being positive, the preceding equations give the following ones:

$$\mu \sqrt{a(1-e^2)} . \sin.\varphi = \mu' \sqrt{a'(1-e'^2)} . \sin.\varphi';$$

$$\sin.\theta = -\sin.\theta'; \quad \cos.\theta = -\cos.\theta';$$

whence we derive $\theta' = \theta +$ the semi circumference; the nodes of the orbits are consequently upon the same line; but the ascending node of the one coincides with the decending node of the other; so that the mutual inclination of the two orbits is equal to $\varphi + \varphi'$.

We have by No. 528,

$$c = \mu \sqrt{a(1-e^2)} . \cos.\varphi + \mu' \sqrt{a'(1-e'^2)} \cos.\varphi';$$

by combining this equation with the preceding one between sin. φ and sin. φ', we shall have

$$2\mu c . \cos.\varphi . \sqrt{a(1-e^2)} = c^2 + \mu^2 a(1-e^2) - \mu'^2 . a'(1-e'^2).$$

If we suppose the orbits circular, or at least having excentricity so small that we may neglect the squares of their excentricities, the preceding equation will give φ constant: for the same reason φ' will be constant; the inclinations of the planes of the orbits to the fixed plane, and to one another, will therefore be constant, and these three planes will always have a common intersection. It thence results that the mean instantaneous variation of this intersection is always the same; because it can only be a function of these inclinations. When they are very small, we shall easily find by No. 528, and in virtue of the preceding relation between sin. φ and sin. φ', that for the time t, the motion of this intersection is

$$-\{(0,1) + (1,0)\} . t.$$

The position of the invariable plane to which we refer the motion of the orbits, may easily be determined for any instant whatever; for we have only to divide the angle of the mutual inclination of the orbits into two angles, φ and φ', such as that we have in the preceding equation be-

tween sin. φ and sin. φ'. Designating, therefore, this mutual inclination by ϖ, we shall have

$$\tan.\ \varphi = \frac{\mu' \sqrt{a'(1-e'^2)}.\ \sin.\ \varpi}{\mu \sqrt{a(1-e^2)} + \mu' \sqrt{a'(1-e'^2)}.\ \cos.\ \varpi}$$

SECOND METHOD OF APPROXIMATION OF THE CELESTIAL MOTIONS.

530. We have already seen that the coordinates of the celestial bodies, referred to the foci of the principal forces which animate them, are determined by differential equations of the second order. We have integrated these equations in retaining only the principal forces, and we have shown that in this case, the orbits are conic sections whose elements are the arbitrary constants introduced by integration.

The perturbing forces adding only small inequalities to the elliptic motion, it is natural to seek to reduce to the laws of this motion the troubled motion of the celestial bodies. If we apply to the differential equations of elliptic motion, augmented by the small terms due to the perturbing forces, the method exposed in No. 512, we can also consider the celestial motions in orbits which turn into themselves, as being elliptic; but the elements of this motion will be variable, and by this method we shall obtain their variations. Hence it results that the equations of motion, being differentials of the second order, not only their finite integrals, but also their infinitely small integrals of the first order, are the same as in the case of invariable ellipses; so that we may differentiate the finite equations of elliptic motion, in treating the elements of this motion as constant. It also results from the same method that the differential equations of the first order may be differentiated, by making vary only the elements of the orbits, and the first differences of the coordinates; provided that instead of the second differences of these coordinates, we substitute only that part of their values which is due to their perturbing forces. These results can be derived immediately from the consideration of elliptic motion.

For that purpose, conceive an ellipse passing through a planet, and through the element of the curve which it describes, and whose focus is occupied by the sun. This ellipse is that which the planet would invariably describe, if the perturbing forces were to cease to act upon it. Its elements are constant during the instant d t; but they vary from one instant to another. Let therefore $V = 0$, be a finite equation to an invariable ellipse, V being a function of the rectangular coordinates x, y, z

and the parameters c, c′, &c. which are functions of the elements of elliptic motion. Since, however, this ellipse belongs to the element of the curve described by the planet during the instant d t; the equation V = 0 will still hold good for the first and last point of this element, by regarding c, c′, &c. as constant. We may, therefore, differentiate this equation once in only supposing x, y, z, to vary, which gives

$$0 = \left(\frac{d\,V}{d\,x}\right) d\,x + \left(\frac{d\,V}{d\,y}\right) d\,y + \left(\frac{d\,V}{d\,z}\right) d\,z; \quad \text{(i)}$$

We also see the reason why the finite equations of the invariable ellipse, may, in the case of the variable ellipse, be differentiated *once* in treating the parameters as constant. For the same reason, every differential equation of the first order relative to the invariable ellipse, equally holds good for the variable ellipse; for let V′ = 0 be an equation of this order, V′ being a function of x, y, z, $\frac{d\,x}{d\,t}$, $\frac{d\,y}{d\,t}$, $\frac{d\,z}{d\,t}$, and the parameters c, c′, &c. It is clear that all these quantities are the same for the variable ellipse as well as for the invariable ellipse, which for the instant d t coincides with it.

Now if we consider the planet, at the end of the instant d t, or at the commencement of the following one; the function V will vary from the ellipse relative to the instant d t to the consecutive ellipse only by the variation of the parameters, since the coordinates x, y, z, relative to the end of the first instant are the same for the two ellipses; thus the function V being nothing, we have

$$0 = \left(\frac{d\,V}{d\,c}\right).\,d\,c + \left(\frac{d\,V}{d\,c'}\right) d\,c' + \&c. \quad \text{(i′)}$$

This equation may be deduced from the equation V = 0, by making x, y, z, c, c′, &c. vary together; for if we take the differential equation (i) from this differential, we shall have the equation (i′).

Differentiating the equation (i), we shall have a new equation in d c, d c′, &c. which with the equation (i′) will serve to determine the parameters c, c′, &c. Thus it is that the geometers, who were first occupied in the theory of celestial perturbations, have determined the variations of the nodes and the inclinations of the orbits: but we may simplify this differentiation in the following manner.

Consider generally the differential equation of the first order V′ = 0, an equation which belongs equally to the variable ellipse, and to the invariable ellipse which, in the instant d t, coincides with it. In the following instant, this equation belongs also to the two ellipses, but with this

difference, that c, c′, &c. remain the same in the case of the invariable ellipse, but vary with the variable ellipse. Let V″ be what V′ becomes, when the ellipse is supposed invariable, and V′′′ what this same function becomes in the case of the variable ellipse. It is clear that in order to have V″ we must change in V′, the coordinates x, y, z, which are relative to the commencement of the first instant d t, in those which are relative to the commencement of the second instant; we must then augment the first differences d x, d y, d z respectively by the quantities $d^2 x$, $d^2 y$, $d^2 z$, relative to the invariable ellipse, the element d t of the time, being supposed constant.

In like manner, to get V′′′, we must change in V′, the coordinates x, y, z, in those which are relative to the commencement of the second instant, and which are also the same in the two ellipses; we must then augment d x, d y, d z respectively by the quantities $d^2 x$, $d^2 y$, $d^2 z$; finally, we must change the parameters c, c′, &c. into c + d c, c′ + d c′; &c.

The values of $d^2 x$, $d^2 y$, $d^2 z$ are not the same in the two ellipses; they are augmented, in the case of the variable ellipse, by the quantities due to the perturbing forces. We see also that the two functions V″ and V′′′, differing only in this that in the second the parameters c, c′, &c. increase by d c, d c′, &c.; and the values of $d^2 x$, $d^2 y$, $d^2 z$ relative to the invariable ellipse, are augmented by quantities due to the perturbing forces. We shall, therefore, form V′′′ — V″, by differentiating V′ in the supposition that x, y, z are constant, and that d x, d y, d z, c, c′, &c. are variable, provided that in this differential we substitute for $d^2 x$, $d^2 y$, $d^2 z$, &c. the parts of their values due solely to the disturbing forces.

If, however, in the function V″ — V′ we substitute for $d^2 x$, $d^2 y$, $d^2 z$ their values relative to elliptic motion, we shall have a function of x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, c, c′, &c., which in the case of the invariable ellipse, is nothing; this function is therefore also nothing in the case of the variable ellipse. We evidently have in this last case, V′′′ — V′ = 0, since this equation is the differential of the equation V′ = 0: taking it from the equation V′′′ — V′ = 0, we have V′′′ — V″ = 0. Thus, we may, in this case, differentiate the equation V′ = 0, supposing d x, d y, d z, c, c′, &c. alone to vary, provided that we substitute for $d^2 x$, $d^2 y$, $d^2 z$, the parts of their values relative to the disturbing forces. These results are exactly the same as those which we obtained in No. 512, by considerations purely analytical; but as is due to their importance, we shall here again present them, deduced from the consideration of elliptic motion.

531. Let us resume the equations (P) of No. 513,

$$\left.\begin{aligned} 0 &= \frac{d^2x}{dt^2} + \frac{mx}{\varrho^3} + \left(\frac{dR}{dx}\right); \\ 0 &= \frac{d^2y}{dt^2} + \frac{my}{\varrho^3} + \left(\frac{dR}{dy}\right); \\ 0 &= \frac{d^2z}{dt^2} + \frac{mz}{\varrho^3} + \left(\frac{dR}{dz}\right). \end{aligned}\right\} \text{(P)}$$

If we suppose R = 0, we shall have the equations of elliptic motion, which we have integrated in (478). We have there obtained the seven following integrals

$$c = \frac{x\,dy - y\,dx}{dt};$$

$$c' = \frac{x\,dz - z\,dx}{dt};$$

$$c'' = \frac{y\,dz - z\,dy}{dt};$$

$$\left.\begin{aligned} 0 &= f + x\left\{\frac{m}{\varrho} - \frac{dy^2 + dz^2}{dt^2}\right\} + \frac{y\,dy.dx}{dt^2} + \frac{z\,dz.dx}{dt^2}; \\ 0 &= f' + y\left\{\frac{m}{\varrho} - \frac{dx^2 + dz^2}{dt^2}\right\} + \frac{x\,dx.dy}{dt^2} + \frac{z\,dz.dy}{dt^2}; \\ 0 &= f'' + z\left\{\frac{m}{\varrho} - \frac{dx^2 + dy^2}{dt^2}\right\} + \frac{x\,dx.dz}{dt^2} + \frac{y\,dy.dz}{dt^2}; \\ 0 &= \frac{m}{a} - \frac{2m}{\varrho} + \frac{dx^2 + dy^2 + dz^2}{dt^2}. \end{aligned}\right\} \text{(p)}$$

These integrals give the arbitraries in functions of their first differences; they are under a very commodious form for determining the variations of these arbitraries. The three first integrals give, by differentiating them, and making vary by the preceding No. the parameters c, c′, c″, and the first differences of the coordinates,

$$dc = \frac{x\,d^2y - y\,d^2x}{dt}$$

$$dc' = \frac{x\,d^2z - z\,d^2x}{dt}$$

$$dc'' = \frac{y\,d^2z - z\,d^2y}{dt}.$$

Substituting for d^2x, d^2y, d^2z, the parts of their values due to the perturbing forces, and which by the differential equations (P) are

$$-dt^2\left(\frac{dR}{dx}\right), -dt^2\left(\frac{dR}{dy}\right), -dt^2\left(\frac{dR}{dz}\right);$$

we shall have

$$d\,c = d\,t\left\{y\left(\frac{d\,R}{d\,x}\right) - x\left(\frac{d\,R}{d\,y}\right)\right\};$$

$$d\,c' = d\,t\left\{z\left(\frac{d\,R}{d\,x}\right) - x\left(\frac{d\,R}{d\,z}\right)\right\};$$

$$d\,c'' = d\,t\left\{z\left(\frac{d\,R}{d\,y}\right) - y\left(\frac{d\,R}{d\,z}\right)\right\}.$$

We know from 478, 479 that the parameters c, c', c'' determine three elements of the elliptic orbit, viz., the inclination φ of the orbit to the plane of x, y, and the longitude θ of its ascending node, by means of the equations

$$\text{tan. } \varphi = \frac{\sqrt{(c'^2 + c''^2)}}{c}; \quad \text{tan. } \theta = \frac{c''}{c'};$$

and the semi-parameter $a\,(1 - e^2)$ of the ellipse by means of the equation

$$m\,a\,(1 - e^2) = c^2 + c'^2 + c''^2.$$

The same equations subsist also in the case of the variable ellipse, provided we determine c, c', c'' by means of the preceding differential equations. We shall thus have the parameter of the variable ellipse, its inclination to the fixed plane of x, y and the position of its node.

The three first of the equations (p) have given us in No. (479) the finite integral

$$0 = c''\,x - c'\,y + c\,z:$$

this equation subsists in the case of the troubled ellipse, as also its first difference

$$0 = c''\,d\,x - c'\,d\,y + c\,d\,z$$

taken in considering c, c', c'' constant.

If we differentiate the fourth, the fifth and the sixth of the integrals (p), making only the parameters f, f', f'', and the differences d x, d y, d z vary; if moreover, we substitute then for $d^2\,x$, $d^2\,y$, $d^2\,z$, the quantities $-d\,t^2\left(\frac{d\,R}{d\,x}\right)$, $-d\,t^2\left(\frac{d\,R}{d\,y}\right)$, $-d\,t^2\left(\frac{d\,R}{d\,z}\right)$, we shall have

$$d\,f = d\,y\left\{y\left(\frac{d\,R}{d\,x}\right) - x\left(\frac{d\,R}{d\,y}\right)\right\} + d\,z\left\{z\left(\frac{d\,R}{d\,x}\right) - x\left(\frac{d\,R}{d\,z}\right)\right\}$$
$$+ (y\,d\,x - x\,d\,y)\left(\frac{d\,R}{d\,y}\right) + (z\,d\,x - x\,d\,z)\left(\frac{d\,R}{d\,z}\right);$$

$$d\,f' = d\,x\left\{x\left(\frac{d\,R}{d\,y}\right) - y\left(\frac{d\,R}{d\,x}\right)\right\} + d\,z\left\{z\left(\frac{d\,R}{d\,y}\right) - y\left(\frac{d\,R}{d\,z}\right)\right\}$$
$$+ (x\,d\,y - y\,d\,x)\left(\frac{d\,R}{d\,x}\right) + (z\,d\,y - y\,d\,z)\left(\frac{d\,R}{d\,z}\right);$$

$$df'' = dx\left\{x\left(\frac{dR}{dz}\right) - z\left(\frac{dR}{dx}\right)\right\} + dy\left\{y\left(\frac{dR}{dz}\right) - z\left(\frac{dR}{dy}\right)\right\}$$
$$+ (x\,dz - z\,dx)\left(\frac{dR}{dx}\right) + (y\,dz - z\,dy)\left(\frac{dR}{dy}\right).$$

Finally, the seventh of the integrals (p), differentiated in the same manner, will give the variation of the semi-axis-major a, by means of the equation

$$d.\frac{m}{a} = 2\,d\,R,$$

the differential $d\,R$ being taken relatively to the coordinates x, y, z alone of the body μ.

The values of f, f′, f″ determine the longitude of the projection of the perihelion of the orbit, upon the fixed plane, and the relation of the excentricity to the semi-axis-major; for I being the longitude of this projection by (479) we have

$$\text{tan. } I = \frac{f'}{f};$$

and e being the ratio of the excentricity to the semi-axis-major, we have

$$m\,e = \sqrt{(f^2 + f'^2 + f''^2)}.$$

This ratio may also be determined by dividing the semi-parameter $a(1 - e^2)$, by the semi-axis-major a: the quotient taken from unity will give the value of e^2.

The integrals (p) have given by elimination (479) the finite integral

$$0 = m\varrho - h^2 + fx + f'y + f''z:$$

this equation subsists in the case of the troubled ellipse, and it determines at each instant, the nature of the variable ellipse. We may differentiate it, considering f, f′, f″ as constant; which gives

$$0 = m\,d\varrho + f\,dx + f'\,dy + f''\,dz.$$

The semi-axis-major a gives the mean motion of μ, or more exactly, that which in the troubled orbit, corresponds to the mean motion in the invariable orbit; for we have (479) $n = a^{-\frac{3}{2}}\sqrt{m}$; moreover, if we denote by ζ the mean motion of μ, we have in the invariable elliptic orbit $d\zeta = n\,dt$: this equation equally holds good in the variable ellipse, since it is a differential of the first order. Differentiating we shall have $d^2\zeta = d\,n.\,dt$; but we have

$$dn = \frac{3an}{2m}.d.\frac{m}{a} = \frac{3\,a\,n\,d\,R}{m},$$

therefore

$$d^2\zeta = \frac{3\,a\,n\,dt.\,d\,R}{m};$$

and integrating

$$\zeta = \frac{3}{m} . \iint a\, n\, d\, t . d\, R.$$

Finally we have seen in (No. 473) that the integrals (p) are equivalent to but five distinct integrals, and that they give between the seven parameters c, c′, c″, f, f′, f″ e, the two equations of condition

$$0 = f\, c'' - f'\, c' + f''\, c;$$

$$0 = \frac{m}{a} + \frac{f^2 + f'^2 + f''^2 - m^2}{c^2 + c'^2 + c''^2};$$

these equations subsist therefore in the case of the variable ellipse provided that the parameters are determined as above.

We can easily verify these statements a *posteriori*.

We have determined five elements of the variable orbit, viz., its inclination, position of the nodes, its semi-axis-major which gives its mean motion, its excentricity and the position of the perihelion. It remains for us to find the sixth element of elliptic motion,—that which in the invariable ellipse corresponds to the position of μ at a given epoch. For this purpose let us resume the expression of d t (473)

$$\frac{d\, t \sqrt{m}}{a^{\frac{3}{2}}} = \frac{d\, v\, (1 - e^2)^{\frac{3}{2}}}{\{1 + e \cos.(v - \varpi)\}^2}.$$

This equation developed into series gives (473)

$$n\, d\, t = d\, v \{1 + E^{(1)} \cos.(v - \varpi) + E^{(2)} \cos. 2\,(v - \varpi) + \&c.\}.$$

Integrating this equation on the supposition of e and ϖ being constant, we shall have

$$\int n\, d\, t + \varepsilon = v + E^{(1)} \sin.(v - \varpi) + \frac{E^{(1)}}{2} \sin. 2.(v - \varpi) + \&c.$$

ε being an arbitrary. This integral is relative to the invariable ellipse: to extend it to the variable ellipse, in making every thing vary even to the arbitraries, ε, e, ϖ which it contains, its differential must coincide with the preceding one; which gives

$$d\, \varepsilon = d\, e \left\{ \left(\frac{d\, E^{(1)}}{d\, e}\right) \sin.(v - \varpi) + \tfrac{1}{2} \left(\frac{d\, E^{(2)}}{d\, e}\right) \sin. 2\,(v - \varpi) + \&c. \right\}$$

$$- d\, \varpi \{E^{(1)} \cos.(v - \varpi) + E^{(2)} \cos. 2\,(v - \varpi) + \&c.\}$$

$v - \varpi$ being the true anomaly of μ measured upon the orbit, and ϖ the longitude of the perihelion also measured upon the orbit. We have determined above, the longitude I of the projection of the perihelion upon a fixed plane. But by (488) we have, in changing v into ϖ and $v_{,}$ into I in the expression of $v - \beta$ of this No.

$$\varpi - \theta = I - \theta + \tan.^2 \tfrac{1}{2} \varphi \sin. 2\,(I - \theta) + \&c.$$

Supposing next that v, $v_{,}$, are zero in this same expression, we have

$$\beta = \theta + \tan.^{2}\tfrac{1}{2}\varphi \sin. 2\theta + \&c.$$

wherefore,

$$\varpi = \mathrm{I} + \tan.^{2}\tfrac{1}{2}\varphi . \{\sin. 2\theta + \sin. 2(\mathrm{I} - \theta) + \&c.\}$$

which gives

$$\begin{aligned} \mathrm{d}\varpi = {} & \mathrm{d}\,\mathrm{I} . \{1 + 2\tan.^{2}\tfrac{1}{2}\varphi \cos. 2(\mathrm{I} - \theta) + \&c.\} \\ & + 2\,\mathrm{d}\theta \tan.^{2}\tfrac{1}{2}\varphi \{\cos. 2\theta - \cos. 2(\mathrm{I} - \theta) + \&c.\} \\ & + \frac{\mathrm{d}\varphi \tan.\tfrac{1}{2}\varphi}{\cos.^{2}\tfrac{1}{2}\varphi} . \{\sin. 2\theta + \sin. 2(\mathrm{I} - \theta) + \&c.\}. \end{aligned}$$

Thus the values of d I, d θ, and d φ being determined by the above, we shall have that of d ϖ; whence we shall obtain the value of d ε.

It follows from thence that the expressions in series, of the radius-vector, of its projection upon the fixed plane, of the longitude whether referred to the fixed plane or to the orbit, and of the latitude which we have given in (No. 488) for the case of the invariable ellipse, subsist equally in the case of the troubled ellipse, provided we change n t into $\int$n d t, and we determine the elements of the variable ellipse by the preceding formulas. For since the finite equations between ϱ, v, s, x, y, z, and $\int$n d t, are the same in the two cases, and because the series of No. 488 result from these equations, by analytical operations entirely independent of the constancy or variability of the elements, it is evident these expressions subsist in the case of variable elements.

When the ellipses are very excentric, as is the case with the orbits of the comets, we must make a slight change in the preceding analysis. The inclination φ of the orbit to the fixed plane, the longitude θ of its ascending node, the semi-axis-major a, the semi-parameter a $(1 - e^{2})$, the excentricity e, and the longitude I of the perihelion upon the fixed plane may be determined by what precedes. But the values of ϖ and of d ϖ being given in series ordered according to the powers of tan. $\frac{1}{2}\varphi$, in order to render them convergent, we must choose the fixed plane, so as to make tan. $\frac{1}{2}\varphi$ inconsiderable; and to effect this most simply is to take, for the fixed plane, that of the orbit of μ at a given epoch.

The preceding value of d ε is expressed by a series which is convergent only in the case where the excentricity of the orbit is inconsiderable, we cannot therefore make use of it in this case. Instead, let us resume the equation

$$\frac{\mathrm{d}\,\mathrm{t}\sqrt{\mathrm{m}}}{\mathrm{a}^{\frac{3}{2}}} = \frac{\mathrm{d}\,\mathrm{v}\,(1 - \mathrm{e}^{2})^{\frac{3}{2}}}{\{1 + \mathrm{e}\cos.(\mathrm{v} - \varpi)\}^{2}}.$$

If we make $1 - e = \alpha$, we have by (489) in the case of the invariable ellipse,

$$t + T = \frac{2\,a^{\frac{3}{2}}\,(1-e^2)^{\frac{3}{2}}}{(2-\alpha)^2.\sqrt{m}}.\tan.\tfrac{1}{2}(v-\varpi)\left\{1 + \frac{\frac{2}{3}-\alpha}{2-\alpha}.\tan.^2\tfrac{1}{2}(v-\varpi) + \&c.\right\}$$

T being an arbitrary. To extend this equation to the variable ellipse, we must differentiate it by making vary T, the semi parameter $a\,(1-e^2)$, α, and ϖ. We shall thence obtain a differential equation which will determine T, and the finite equations which subsist in the case of the invariable ellipse, will still hold good in that of the variable ellipse.

532. Let us consider more particularly the variations of the elements of μ's orbit, in the case of the orbits being of small excentricity and but little inclined to one another. We have given in No. 515. the manner of developing R in a series of sines and cosines of the form

$$\mu'\,k\cos.\,(i'\,n'\,t - i\,n\,t + A)$$

k and A being functions of the excentricity and inclinations of the orbits, the positions of their nodes and perihelions, the longitudes of the bodies at a given epoch, and the major-axes. When the ellipses are variable all these quantities must be supposed to vary conformably to what precedes. We must moreover change in the preceding term, the angle $i'\,n'\,t - i\,n\,t$ into $i'\int n'\,d\,t - i\int n\,d\,t$, or which is tantamount, into $i'\,\zeta' - i\,\zeta$.

However, by the preceding No., we have

$$\frac{m}{a} = 2\int d\,R;$$

$$\zeta = \int n\,d\,t = \frac{3}{m}.\iint a\,n\,d\,t.\,d\,R.$$

The difference $d\,R$ being taken relatively to the coordinates x, y, z, of the body μ, we must only make vary, in the term

$$\mu'\,k\cos.\,(i'\,\zeta' - i\,\zeta + A)$$

of the expression of R developed into a series, what depends upon the motion of this body; moreover, R being a finite function of x, y, z, x', y', z' we may by No. 530, suppose the elements of the orbit constant in the difference $d\,R$; it suffices therefore to make ζ vary in the preceding term, and since the difference of ζ is $n\,d\,t$, we have

$$i\,\mu'.\,k\,n\,d\,t.\sin.\,(i'\,\zeta' - i\,\zeta + A)$$

for the term of $d\,R$ which corresponds to the preceding term of R. Thus, with respect to this term only, we have

$$\frac{1}{a} = \frac{2\,i\,\mu'}{m}\int k\,n\,d\,t.\sin.\,(i'\,\zeta' - i\,\zeta + A);$$

$$\zeta = \frac{3\,i\,\mu'}{m} \iint a\,k\,n^2\,dt^2 \sin.\,(i'\,\zeta' - i\,\zeta + A).$$

If we neglect the squares and products of the perturbing masses, we may, in the integrals of the above terms, suppose the elements of elliptic motion constant. Hence ζ becomes $n\,t$ and ζ', $n'\,t$; whence we get

$$\frac{1}{a} = -\frac{2\,i\,\mu'\,n\,k}{m\,(i'\,n' - i\,n)} \cos.\,(i'\,n'\,t - i\,n\,t + A)$$

$$\zeta = -\frac{3\,i\,\mu'\,a\,n^2\,k}{m\,(i'\,n' - i\,n)^2} \sin.\,(i'\,n'\,t - i\,n\,t + A).$$

Hence we perceive that if $i'\,n' - i\,n$ is not zero, the quantities a and ζ only contain periodic inequalities, retaining only the first power of the perturbing force; but i and i' being whole numbers, the equation $i\,n' - i\,n = 0$ cannot subsist when the mean motions of μ and μ' are incommensurable, which is the case with the planets, and which can be admitted generally, since n and n' being arbitrary constants susceptible of all possible values, their exact relation of number to number is not at all probable.

We are, therefore, conducted to this remarkable result, viz., *that the principal axes of the planets, and their mean motions, are only subject to periodic inequalities depending on their configuration, and that thus in neglecting these quantities, their principal axes are constant and their mean motions uniform, a result agreeing with what has otherwise been found by* No. 521.

If the mean motions $n\,t$ and $n'\,t$, without being exactly commensurable, approach very nearly to the ratio $i' : i$; the divisor $i'\,n' - i\,n$ is very small, and there may result in ζ and ζ' inequalities, which increasing very slowly, may give reason for observers to suppose that the mean motions of the two bodies μ, μ' are not uniform. We shall see, in the theory of Jupiter and Saturn, that this is actually the case with regard to these two planets: their mean motions are such that twice that of Jupiter is very nearly equal to five times that of Saturn; so that $5\,n' - 2\,n$ is hardly the sixty-fourth part of n. The smallness of this divisor, renders very sensible the term of the expression for ζ, depending upon the angle $5\,n'\,t - 2\,n\,t$, although it is of the order $i' - i$, or of the third order, relatively to the excentricities and inclinations of the orbits, as we have seen in No. 515. The preceding analysis gives the most sensible part of these inequalities; for the variation of the mean longitude depends on two integrations, whilst the variations of the other elements of elliptic motion depend only on one integration; only terms of the expression of the mean longitude can therefore have the divisor $(i'\,n' - i\,n)^2$; consequently with regard only

to these terms, which, considering the smallness of the divisor ought to be the more considerable, it will suffice, in the expressions of the radius-vector, the longitude and latitude, to derive from these terms, the mean longitude.

When we have inequalities of this kind, which the action of μ' produces in the mean motion of μ, it is easy thence to get the corresponding inequalities which the action of μ produces in the mean motion of μ' In fact, if we have regard only to the mutual action of three bodies M, μ and μ'; the formula (7) of (471) gives

$$\text{const.} = \mu\frac{dx^2+dy^2+dz^2}{dt^2}+\mu'.\frac{dx'^2+dy'^2+dz'^2}{dt^2}$$

$$-\frac{(\mu dx+\mu' dx')^2+(\mu dy+\mu' dy')^2+(\mu dz+\mu' dz')^2}{(M+\mu+\mu')^2 dt^2} \quad \text{(a)}$$

$$-\frac{2M\mu}{\sqrt{x^2+y^2+z^2}}-\frac{2M\mu'}{\sqrt{x'^2+y'^2+z'^2}}-\frac{2\mu\mu'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}.$$

The last of the integrals (p) of the preceding No. gives, by substituting for $\frac{m}{a}$ the integral $2\int d\,R$,

$$\frac{dx^2+dy^2+dz^2}{dt^2}=\frac{2(M+\mu)}{\sqrt{x^2+y^2+z^2}}-2\int d\,R.$$

If we then call R', what R becomes when we consider the action of μ upon μ', we shall have

$$R'=\frac{\mu(xx'+yy'+zz')}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\frac{\mu}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}$$

$$\frac{dx'^2+dy'^2+dz'^2}{dt^2}=\frac{2(M+\mu')}{\sqrt{x'^2+y'^2+z'^2}}-2\int d'R';$$

the differential characteristic d' only belonging to the coordinates of the body μ'. Substituting for $\frac{dx^2+dy^2+dz^2}{dt^2}$ and $\frac{dx'^2+dy'^2+dz'^2}{dt^2}$ the values in the equation (a), we shall have

$$\mu\int dR+\mu'\int d'R'=\text{const.}-\frac{(\mu dx+\mu' dx')^2+(\mu dy+\mu' dy')^2+(\mu dz+\mu' dz')^2}{2(M+\mu+\mu')dt^2}$$

$$+\frac{\mu^2}{\sqrt{x^2+y^2+z^2}}+\frac{\mu'^2}{\sqrt{x'^2+y'^2+z'^2}}.$$

It is evident that the second member of this equation contains no terms of the order of squares and products of the μ, μ', which have the divisor $i'n'-in$; relative, therefore, only to these terms, we shall have

$$\mu\int d\,R+\mu'\int d'\,R'=0;$$

thus, by only considering the terms which have the divisor $(i' n' - i n)^2$, we shall have

$$\frac{3\int\int a' n' dt . d' R'}{M + \mu'} = -\frac{\mu (M + \mu) . a' n'}{\mu' (M + \mu') a n} . \frac{3\int\int a n dt . d R}{M + \mu};$$

but we have

$$\zeta = \frac{3\int\int a n dt . d R}{M + \mu}; \zeta' = \frac{3\int\int a' n' dt . d' R'}{M + \mu'};$$

we therefore get

$$\mu' (M + \mu') a n \zeta' + \mu (M + \mu) a' n' \zeta = 0.$$

Again, we have

$$n = \frac{\sqrt{(M + \mu)}}{a^{\frac{3}{2}}}; n = \frac{\sqrt{(M + \mu')}}{a'^{\frac{3}{2}}};$$

neglecting therefore μ, μ', in comparison with M, we shall have

$$\mu \sqrt{a} . \zeta + \mu' \sqrt{a'} . \zeta' = 0;$$

or

$$\zeta' = -\frac{\mu \sqrt{a}}{\mu' \sqrt{a'}} . \zeta.$$

Thus the inequalities of ζ, which have the divisor $(i' n' - i n)^2$, give us those of ζ', which have the same divisor. These inequalities are, as we see, affected with the contrary sign, if n and n' have the same sign, or which amounts to the same, if the two bodies μ and μ' circulate in the same direction; they are, moreover, in a constant ratio; whence it follows that if they seem to accelerate the mean motion of μ, they appear to retard that of μ' according to the same law, and the apparent acceleration of μ, will be to the apparent retardation of μ', as $\mu' \sqrt{a'}$ is to $\mu \sqrt{a}$. The acceleration of the mean motion of Jupiter and the retardation of that of Saturn, which the comparison of modern with ancient observations made known to Halley, being very nearly in this ratio; it may be concluded from the preceding theorem, that they are due to the mutual action of the two planets; and, since it is constant, that this action cannot produce in the mean motions any alteration independent of the configuration of the planets, it is very probable that there exists in the theory of Jupiter and Saturn a great periodic inequality, of a very long period. Next, considering that five times the mean motion of Saturn, minus twice that of Jupiter is very nearly equal to nothing, it seems very probable that the phenomenon observed by Halley, was due to an inequality depending upon this argument. The determination of this inequality will verify the conjecture.

The period of the argument $i' n' t - i n t$ being supposed very long,

the elements of the orbits of μ, and μ' undergo, in this interval sensible variations, which must be taken into account in the double integral

$$\int\int a\, k\, n^2\, d\, t^2 \sin.\, (i'\, n'\, t - i\, n\, t + A).$$

For that purpose we shall give to the function k sin. (i' n' t — i n t + A), the form

$$Q \sin.\, (i'\, n'\, t - i\, n\, t + i'\, \varepsilon' - i\, \varepsilon) + Q' \cos.\, (i'\, n'\, t - i\, n\, t + i'\, \varepsilon' - i\, \varepsilon)$$

Q and Q' being functions of the elements of the orbits: thus we shall have

$$\int\int a\, k\, n^2\, d\, t^2 \sin.\, (i'\, n'\, t - i\, n\, t + A) =$$

$$-\frac{n^2 a \sin.\, (i'n't - int + i'\varepsilon - i\varepsilon)}{(i'\, n' - i\, n)^2} \cdot \left\{ Q - \frac{2\, d\, Q'}{(i'n' - in)dt} - \frac{3\, d^2\, Q}{(i'n' - in)^2 dt^2} + \&c. \right\}$$

$$-\frac{n^2 a \cos.\, (i'n't - int + i'\varepsilon - i\varepsilon)}{(i'\, n' - i\, n)^2} \cdot \left\{ Q' - \frac{2\, d\, Q}{(i'n' - in)dt} - \frac{3\, d^2\, Q'}{(i'n' - in)^2 dt^2} + \&c. \right\}.$$

The terms of these two series decreasing very rapidly, with regard to the slowness of the secular variations of the elliptic elements, we may, in each series, stop at the two first terms. Then substituting for the elements of the orbits their values ordered according to the powers of the time, and only retaining the first power, the double integral above may be transformed in one term to the form

$$(F + E\, t) \sin.\, (i'\, n'\, t - i\, n\, t + A + H\, t).$$

Relatively to Jupiter and Saturn, this expression may serve for many ages before and after the instant from which we date the given epoch.

The great inequalities above referred to, become sensible amongst the terms depending upon the second power of the perturbing forces. In fact, if in the formula

$$\zeta = \frac{3\, i\, \mu'}{m} \int\int a\, k\, n^2.\, d\, t^2.\, \sin.\, (i'\, \zeta' - i\, \zeta + A),$$

we substitute for ζ, ζ' their values

$$n\, t - \frac{3\, i\, \mu'\, a\, n^2\, k}{m\, (i'n' - in)^2} \sin.\, (i'\, n'\, t - i\, n\, t + A);$$

$$n'\, t - \frac{3\, i\, \mu\, a\, n^2\, k}{m\, (i'\, n' - i\, n)^2} \sqrt{\frac{a}{a'}} \cdot \sin.\, (i'\, n'\, t - i\, n\, t + A),$$

there will result among the terms of the order μ^2, the following

$$-\frac{9\, i^2\, \mu'^2\, a^2\, n^4\, k^2}{8\, m^2\, (i'\, n' - i\, n)^4} \cdot \frac{i\, \mu' \sqrt{a'} + i'\, \mu \sqrt{a}}{\mu' \sqrt{a}} \sin.\, 2\, (i'\, n'\, t - i\, n\, t + A).$$

The value of ζ' contains the corresponding term, which is to the one preceding in the ratio $\mu \sqrt{a} : -\mu' \sqrt{a'}$, viz.

$$\frac{9\, i^2 \mu'^2\, a^2\, n^4\, k^2}{8\, m^2\, (i'\, n' - i\, n)^4} \{ i\, \mu' \sqrt{a'} + i'\, \mu \sqrt{a} \} \cdot \frac{\mu \sqrt{a}}{\mu'^2 \sqrt{a'}} \sin.\, 2\, (i'\, n'\, t - i\, n\, t + A).$$

533. It may happen that the inequalities of the mean motion which are the

most sensible, are only to be found among terms of the order of the squares of the perturbing masses. If we consider three bodies, μ, μ', μ'' circulating around M, the expression of d R relative to terms of this order, will contain inequalities of the form

$$k \sin. (i\, n\, t - i'\, n'\, t + i''\, n''\, t + A)$$

but if we suppose the mean motions $n\, t$, $n'\, t$, $n''\, t$ such that $i\, n - i'\, n' + i''\, n''$ is an extremely small fraction of n, there will result a very sensible inequality in the value of ζ. This inequality may render rigorously equal to zero, the quantity $i\, n - i'\, n' + i''\, n''$, and thus establish an equation of condition between the mean motions and the mean longitudes of the three bodies μ, μ', μ''. This very singular case exists in the system of Jupiter's satellites. We will give the analysis of it.

If we take M for the mass-unit, and neglect μ, μ', μ'' in comparison with it, we shall have

$$n^2 = \frac{1}{a^3};\ n'^2 = \frac{1}{a'^3},\ n''^2 = \frac{1}{a''^3},$$

we have then

$$d\,\zeta = n\, d\, t;\ d\,\zeta' = n'\, d\, t;\ d\,\zeta'' = n''\, d\, t;$$

wherefore

$$\frac{d^2\zeta}{d\,t} = -\frac{3}{2}\, n^{\frac{1}{3}} . \frac{d\, a}{a^2};$$

$$\frac{d^2\zeta'}{d\,t} = -\frac{3}{2}\, n'^{\frac{1}{3}} . \frac{d\, a'}{a'^2}$$

$$\frac{d^2\zeta''}{d\,t} = -\frac{3}{2}\, n''^{\frac{1}{3}} . \frac{d\, a''}{a''^2}.$$

We have seen in No. 528, that if we neglect the squares of the excentricities and inclinations of the orbits, we have

$$\text{const.} = \mu \sqrt{a} + \mu' . \sqrt{a'} + \mu'' \sqrt{a''};$$

which gives

$$0 = \mu . \frac{d\, a'}{\sqrt{a}} + \mu' . \frac{d\, a'}{\sqrt{a'}} + \mu'' . \frac{d\, a''}{\sqrt{a''}}.$$

From these several equations, it is easy to get

$$\frac{d^2\zeta}{d\,t} = -\frac{3}{2} . n^{\frac{1}{3}} . \frac{d\, a}{a^2}$$

$$\frac{d^2\zeta'}{d\,t} = \frac{3}{2} . \frac{\mu . n'^{\frac{4}{3}}}{\mu' . n} . \frac{n - n''}{n' - n''} . \frac{d\, a}{a^2};$$

$$\frac{d^2\zeta''}{d\,t} = -\frac{3}{2} . \frac{\mu . n''^{\frac{4}{3}}}{\mu'' . n} \frac{n - n'}{n' - n''} . \frac{d\, a}{a^2}.$$

Finally the equation

$$\frac{m}{a} = 2 \int d\,R$$

of No. 531, gives

$$-\frac{d\,a}{a^2} = 2.\,d\,R.$$

We have therefore only to determine $d\,R$.

By No. 513, neglecting the squares and products of the inclinations of the orbits, we have

$$R = \frac{\mu'\,\varrho}{\varrho'^2} \cos.\,(v' - v) - \mu' (\varrho^2 - 2\,\varrho\,\varrho' \cos.\,(v' - v) + \varrho'^2)^{-\frac{1}{2}}$$
$$+ \frac{\mu''\,\varrho}{\varrho''^2} \cos.\,(v'' - v) - \mu'' (\varrho^2 - 2\,\varrho\,\varrho'' \cos.\,(v'' - v) + \varrho''^2)^{-\frac{1}{2}}$$

If we develope this function in a series ordered according to the cosines of $v' - v$, $v'' - v$ and their multiples; we shall have an expression of this form

$$R = \frac{\mu'}{2} (\varrho, \varrho')^{(0)} + \mu' (\varrho, \varrho')^{(1)} \cos.\,(v' - v) + \mu' (\varrho, \varrho')^{(2)} \cos.\,2\,(v' - v)$$
$$+ \mu' (\varrho, \varrho')^{(3)} \cos.\,3\,(v' - v) + \&c.$$
$$+ \frac{\mu'}{2} (\varrho, \varrho'')^{(0)} + \mu'' (\varrho, \varrho'')^{(1)} \cos.\,(v'' - v) + \mu'' (\varrho, \varrho'')^{(2)} \cos.\,2\,(v'' - v)$$
$$+ \mu'' (\varrho, \varrho'')^{(3)} \cos.\,3\,(v'' - v) + \&c.;$$

whence we derive

$$dR = \left\{ \begin{array}{l} d\varrho \left\{ \begin{array}{l} \frac{\mu'}{2} \left(\frac{d\,(\varrho, \varrho')^{(0)}}{d\,\varrho} \right) + \mu' \left(\frac{d\,(\varrho, \varrho')^{(1)}}{d\,\varrho} \right) \cos.(v' - v) + \mu' \left(\frac{d(\varrho, \varrho')^{(2)}}{d\,\varrho} \right) \\ \cos.\,2\,(v' - v) + \&c. \\ + \frac{\mu''}{2} \left(\frac{d(\varrho, \varrho'')^{(0)}}{d\,\varrho} \right) + \mu'' \left(\frac{d(\varrho, \varrho'')^{(1)}}{d\,\varrho} \right) \cos.(v'' - v) + \mu'' \left(\frac{d(\varrho, \varrho'')^{(2)}}{d\,\varrho} \right) \\ \cos.\,2\,(v'' - v) + \&c. \end{array} \right. \\ + d\,v \left\{ \begin{array}{l} \mu' (\varrho, \varrho')^{(1)} \sin.\,(v' - v) + 2\,\mu' (\varrho, \varrho')^{(2)} \sin.\,2(v' - v) + \&c. \\ + \mu'' (\varrho, \varrho'')^{(1)} \sin.(v'' - v) + 2\mu'' (\varrho, \varrho'')^{(2)} \sin.2(v'' - v) + \&c. \end{array} \right. \end{array} \right\}$$

Suppose, conformably to what observations indicate in the system of the three first satellites of Jupiter, that $n - 2\,n'$ and $n' - 2\,n''$ are very small fractions of n, and that their difference $n - 2\,n' - (n' - 2\,n')$ or $n - 3\,n' + 2\,n''$ is incomparably smaller than each of them. It results from the expressions of $\frac{\delta}{a}\varrho$, and of $\delta\,v$ of No. 517, that the action of μ' produces in the radius-vector and in the longitude of μ, a very sensible inequality depending on the argument $2\,(n'\,t - n\,t + \varepsilon' - \varepsilon)$. The terms relative to this inequality have the divisor $4\,(n' - n)^2 - n^2$,

or $(n - 2n')(3n - 2n')$, and this divisor is very small, because of the smallness of the factor $n - 2n'$. We also perceive, by the consideration of the same expressions, that the action of μ produces in the radius-vector, and in the longitude of μ', an inequality depending on the argument $(n't - nt + \varepsilon' - \varepsilon)$, and which having the divisor $(n' - n)^2 - n'^2$, or $n(n - 2n')$, is very sensible. We see, in like manner, that the action of μ'' upon μ' produces in the same quantities a considerable inequality depending upon the argument $2(n''t - n't + \varepsilon'' - \varepsilon')$. Finally, we perceive that the action of μ' produces in the radius-vector and in the longitude of μ'' a considerable inequality depending upon the argument $n''t - n't + \varepsilon'' - \varepsilon$. These inequalities were first recognised by observations; we shall develope them at length in the Theory of Jupiter's Satellites. In the present question we may neglect them, relatively to other inequalities. We shall suppose, therefore,

$$\delta\varrho = \mu' E' \cos. 2(n't - nt + \varepsilon' - \varepsilon);$$
$$\delta v = \mu' F' \sin. 2(n't - nt + \varepsilon' - \varepsilon);$$
$$\delta\varrho' = \mu'' E'' \cos. 2(n''t - n't + \varepsilon'' - \varepsilon') + \mu G \cos.(n't - nt + \varepsilon' - \varepsilon)$$
$$\delta v' = \mu'' F'' \sin. 2(n''t - n't + \varepsilon'' - \varepsilon') + \mu H \sin.(n't - nt + \varepsilon' - \varepsilon)$$
$$\delta\varrho'' = \mu' G' \cos.(n''t - n't + \varepsilon'' - \varepsilon')$$
$$dv'' = \mu'' H' \sin.(n''t - n't + \varepsilon'' - \varepsilon').$$

We must, however, substitute in the preceding expression of d R for ϱ, v, ϱ', v', ϱ'', v'', the values of $a\,\delta\varrho$, $nt + \varepsilon + \delta v$, $a' + \delta\varrho'$, $n't + \varepsilon' + \delta v'$, $a'' + \delta\varrho''$, $n''t + \varepsilon'' + \delta v''$, and retain only the terms which depend upon the argument $nt - 3n't + 2n''t + \varepsilon - 3\varepsilon' + 2\varepsilon''$. But it is easy to see that the substitution of the values of $\delta\varrho$, δv, $\delta\varrho''$, $\delta v''$ cannot produce any such term. This is not the case with the substitution of the values of $\delta\varrho'$ and $\delta v'$: the term $\mu'(\varrho, \varrho')^{(1)} dv \sin.(v' - v)$ of the expression of d R, produces the following,

$$-\frac{\mu'\mu''.n\,dt}{2}.\left\{E''\left(\frac{d(a, a')^{(1)}}{da'}\right) - F''(a, a')^{(1)}\right\} \times$$
$$\sin.(nt - 3n't + 2n''t + \varepsilon' - 3\varepsilon' + 2\varepsilon'').$$

This is the only expression of the kind which the expression of d R contains. The expressions of $\frac{\delta\varrho}{a}$, and of δv of No. 517, applied to the action of μ'' upon μ', give, retaining only the terms which have the divisor $n' - 2n''$, and observing that n'' is very nearly equal to $\frac{1}{2}n'$,

$$\frac{E''}{a'} = \tfrac{1}{2}n'^2 \times \frac{a'^2\left(\frac{d.(a', a'')^{(2)}}{da'}\right) + \frac{2n''}{n' - n''}.a'(a', a'')^{(2)}}{(n' - 2n'')(3n' - 2n'')}$$

$$F' = \frac{2\,E''}{a'};$$

we therefore have

$$d\,R = \frac{\mu'\,\mu''\,n\,d\,t}{2} E''.\left\{\frac{2\,(a, a')^{(1)}}{a'} - \left(\frac{d.(a, a')^{(1)}}{d\,a'}\right)\right\}$$

$$\times \sin.\,(n\,t - 3\,n'\,t + 2\,n''\,t + \varepsilon - 3\,\varepsilon' + 2\,\varepsilon'') = -\tfrac{1}{2}.\frac{d\,a}{a^2}.$$

Substituting this value of $\frac{d\,a}{a^2}$ in the values of $\frac{d^2\zeta}{d\,t}$, $\frac{d^2\zeta'}{d\,t}$, $\frac{d^2\zeta''}{d\,t}$, and making for brevity's sake

$$\beta = \tfrac{3}{2}\,E''\left\{2\,(a, a')^{(1)} - a'\left(\frac{d\,(a, a')^{(1)}}{d\,a'}\right)\right\}.\left\{\frac{a}{a'}\,\mu'\,\mu'' + \frac{9}{4}\,\mu\,\mu'' + \frac{a''}{4\,a'}\,\mu\,\mu'\right\}$$

we shall have, since n is very nearly equal to 2 n', and n' to 2 n'',

$$\frac{d^2\zeta}{d\,t^2} - 3.\frac{d^2\zeta'}{d\,t^2} + 2.\frac{d^2\zeta''}{d\,t^2} = \beta\,n^2 \sin.\,(n\,t - 3\,n'\,t + 2\,n''\,t + \varepsilon - 3\,\varepsilon' + 2\,\varepsilon'');$$

or more exactly

$$\frac{d^2\zeta}{d\,t^2} - 3.\frac{d^2\zeta'}{d\,t^2} + 2.\frac{d^2\zeta''}{d\,t^2} = \beta\,n^2 \sin.\,(\zeta - 3\,\zeta' + 2\,\zeta'' + \varepsilon - 3\,\varepsilon' + 2\,\varepsilon'');$$

so that if we suppose

$$V = \zeta - 3\,\zeta' + 2\,\zeta'' + \varepsilon - 3\,\varepsilon' + 2\,\varepsilon'',$$

we shall have

$$\frac{d^2\,V}{d\,t^2} = \beta.\,n^2.\,\sin.\,V.$$

The mean distances a, a', a'', varying but little as also the quantity n, we may in this equation consider $\beta\,n^2$, as a constant quantity. Integrating, we have

$$d\,t = \frac{\pm\,d\,V}{\sqrt{c - 2\,\beta\,n^2 \cos.\,V}},$$

c being an arbitrary constant. The different values of which this constant is susceptible, give rise to the three following cases.

If c is positive and greater than $\pm\,2\,\beta\,n^2$, the angle V will increase continually, and this ought to take place, if at the origin of the motion, $(n - 3\,n' + 2\,n'')^2$ is greater than $\pm\,2\,\beta\,n^2\,(1 \mp \cos.\,V)$, the upper or lower signs being taken according as β is positive or negative. It is easy to assure ourselves of this, and we shall see particularly in the theory of the satellites of Jupiter, that β is a positive quantity relatively to the three first satellites. Supposing therefore $\mp\,\varpi = \pi - V$, π being the semi circumference, we shall have

$$d\,t = \frac{d\,\varpi}{\sqrt{c + 2\,\beta\,n^2 \cos.}}.$$

In the interval from $\varpi = 0$ to $\varpi = \frac{\pi}{2}$, the radical $\sqrt{c + 2\beta n^2 \cos. \varpi}$ is greater than $\sqrt{2\beta n^2}$, when c is equal or greater than $2\beta n^2$; we have therefore in this interval $\varpi > n t \sqrt{2\beta}$. Thus, the time t which the angle ϖ employs in arriving from zero to a right angle is less than $\frac{\pi}{2 n \sqrt{2\beta}}$. The value of β depends upon the masses, μ, μ', μ''; the inequalities observed in the three first satellites of Jupiter, and of which we spoke above, give, between their masses and that of Jupiter, relations from whence it results that $\frac{\pi}{2 n \sqrt{2\beta}}$ is under two years, as we shall see in the theory of these satellites; thus the angle ϖ would employ less than two years to increase from zero to a right angle; but the observations made upon Jupiter's satellites, give since their discovery, ϖ constantly nothing or insensible; the case which we are examining is not therefore that of the three first satellites of Jupiter.

If the constant c is less than $\pm 2\beta n^2$, the angle V will not oscillate; it will never reach two right angles, if β is negative, because then the radical $\sqrt{c - 2\beta n^2 \cos. V}$, becomes imaginary; it will never be nothing if β is positive. In the first case its value will be alternately greater and less than zero; in the second case it will be alternately greater and less than two right angles. All observations of the three first satellites of Jupiter, prove to us that this second case belongs to these stars; thus the value of β ought to be positive relatively to them; and since the theory of gravitation gives β positive, we may regard the phenomenon as a new confirmation of that theory.

Let us resume the equation

$$d t = \frac{d \varpi}{\sqrt{c + 2\beta n^2 \cos. \varpi}}.$$

The angle ϖ being always very small, according to the observations, we may suppose $\cos. \varpi = 1 - \frac{1}{2}\varpi^2$; the preceding equation will give by integration

$$\varpi = \lambda \sin. (n t \sqrt{\beta} + \gamma)$$

λ and γ being two arbitrary constants which observation alone can determine. Hitherto, it has not been recognised, a circumstance which proves it to be very small.

From the preceding analysis result the following consequences. Since the angle $n t + 3 n' t + 2 n'' t + \varepsilon - 3\varepsilon' + 1\varepsilon''$ oscillates being sometimes less and sometimes greater than two right angles, its mean value is

equal to two right angles; we shall therefore have, regarding only mean quantities

$$n - 3n' + 2n'' = 0$$

that is to say, *that the mean motion of the first satellite, minus three times that of the second, plus twice that of the third, is exactly and constantly equal to zero.* It is not necessary that this equality should subsist exactly at the origin, which would not in the least be probable; it is sufficient that it did very nearly so, and that $n - 3n' + 2n''$ has been less, abstraction being made of the sign, than $\lambda n \sqrt{\beta}$; and then that the mutual attraction has rendered the equality rigorous.

We have next $\varepsilon - 3\varepsilon' + 2\varepsilon''$ equal to two right angles; thus *the mean longitude of the first satellite, minus three times that of the second, plus twice that of the third, is exactly and constantly equal to two right angles.*

From this theorem, the preceding values of $\delta\varsigma'$, and of $\delta v'$ are reducible to the two following

$$\delta\varsigma' = (\mu G - \mu'' E'') \cos.(n't - nt + \varepsilon' - \varepsilon)$$

$$\delta v' = (\mu H - \mu'' F'') \sin.(n't - nt + \varepsilon' - \varepsilon).$$

The two inequalities of the motion of μ' due to the actions of μ and of μ'', merge consequently into one, and constantly remain so.

It also results from this theorem, that the three first satellites can never be eclipsed at the same time. They cannot be seen together from Jupiter neither in opposition nor in conjunction with the sun; for the preceding theorems subsist equally relative to the synodic mean motions, and to the synodic mean longitudes of the three satellites, as we may easily satisfy ourselves. These two theorems subsist, notwithstanding the alterations which the mean motions of the satellites undergo, whether they arise from a cause similar to that which alters the mean motion of the moon, or whether from the resistance of a very rare medium. It is evident that these several causes only require that there should be added to the value of $\frac{d^2 V}{dt^2}$, a quantity of the form of $\frac{d^2 \psi}{dt^2}$, and which shall only become sensible by integrations; supposing therefore $V = \pi - \varpi$, and ϖ very small, the differential equation in V will become

$$0 = \frac{d^2 \varpi}{dt^2} + \beta n^2 \varpi + \frac{d^2 \psi}{dt^2}.$$

The period of the angle $nt\sqrt{\beta}$ being a very small number of years, whilst the quantities contained in $\frac{d^2 \psi}{dt^2}$ are, either constant, or embrace many ages; by integrating the above equation we shall have

$$\varpi = \lambda \sin.(n\,t\,\sqrt{\beta} + \gamma) - \frac{\partial^2 \psi}{\beta\, n^2\, dt^2}.$$

Thus the value of ϖ will always be very small, and the secular equations of the mean motions of the three first satellites will always be ordered by the mutual action of these stars, so, that the secular equation of the first, plus twice that of the third, may be equal to three times that of the second.

The preceding theorems give between the six constants n, n′, n″, ε, ε′, ε″ two equations of condition which reduce these arbitraries to four; but the two arbitraries λ and γ of the value of ϖ replace them. This value is distributed among the three satellites, so, that calling p, p′, p″ the coefficients of sin. $(n\,t\,\sqrt{\beta} + \gamma)$ in the expressions of v, v′, v″, these coefficients are as the preceding values of $\frac{d^2\zeta}{dt^2}$, $\frac{d^2\zeta'}{dt^2}$, $\frac{d^2\zeta''}{dt^2}$; and moreover we have $p - 3p' + 2p'' = \lambda$. Hence results, in the mean motions of the three first satellites of Jupiter, an inequality which differs for each only by its coefficients, and which forms in these motions a sort of libration whose extent is arbitrary. Observations show it to be insensible.

534. Let us now consider the variations of the excentricities and of the perihelions of the orbits. For this purpose, resume the expressions of d f, d f′, d f″ found in 531: calling ϱ the radius-vector of μ projected upon the plane of x, y; v the angle which this projection makes with the axis of x; and s the tangent of the latitude of μ above the same plane, we shall have

$$x = \varrho \cos. v;\ y = \varrho \sin. v;\ z = \varrho\, s$$

whence it is easy to obtain

$$x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right) = \left(\frac{dR}{dv}\right)$$

$$x\left(\frac{dR}{dz}\right) - z\left(\frac{dR}{dx}\right) = (1 + s^2)\cos. v\left(\frac{dR}{ds}\right) - \varrho\, s \cos. v\left(\frac{dR}{d\varrho}\right) + s \sin. v\left(\frac{dR}{dv}\right)$$

$$y\left(\frac{dR}{dz}\right) - z\left(\frac{dR}{ds}\right) = (1 + s^2)\sin. v\left(\frac{dR}{ds}\right) - \varrho\, s \sin. v\left(\frac{dR}{d\varrho}\right) - s \cos. v\left(\frac{dR}{dv}\right)$$

By 531, we also have

$$x\,dy - y\,dx = c\,dt;\ x\,dz - z\,dx = c'\,dt;\ y\,dz - z\,dy = c\ dt;$$

the differential equations in f, f′, f″ will thus become

$$df = -dy\left(\frac{dR}{dv}\right) - dz\left\{(1+s^2)\cos.v\left(\frac{dR}{ds}\right) - \varrho\, s.\cos.v\left(\frac{dR}{d\varrho}\right) + s\sin.v\left(\frac{dR}{dv}\right)\right\}$$
$$-cdt\left\{\sin.v\left(\frac{dR}{d\varrho}\right) + \frac{\cos.v}{\varrho}\left(\frac{dR}{dv}\right) - \frac{s.\sin.v}{\varrho}\left(\frac{dR}{ds}\right)\right\} - \frac{c'dt}{\varrho}\left(\frac{dR}{ds}\right);$$

$$df' = dx\left(\frac{dR}{dv}\right) - dz\left\{(1+s^2)\sin.v\left(\frac{dR}{ds}\right) - \varrho\, s.\sin.v\left(\frac{dR}{d\varrho}\right) - s\cos.v\left(\frac{dR}{dv}\right)\right\}$$
$$+cdt\left\{\cos.v\left(\frac{dR}{d\varrho}\right) - \frac{\sin.v}{\varrho}\left(\frac{dR}{dv}\right) - \frac{s.\sin.v}{\varrho}\left(\frac{dR}{ds}\right)\right\} - \frac{c''dt}{\varrho}\left(\frac{dR}{ds}\right);$$

$$df'' = dx\left\{(1+s^2)\cos.v\left(\frac{dR}{ds}\right) - \varrho\, s\cos.v\left(\frac{dR}{d\varrho}\right) + s\sin.v\left(\frac{dR}{dv}\right)\right\}$$
$$+dy\left\{(1+s^2)\sin.v\left(\frac{dR}{ds}\right) - \varrho\, s.\sin.v\left(\frac{dR}{d\varrho}\right) - s\cos.v\left(\frac{dR}{dv}\right)\right\}$$
$$+c'.dt\left\{\cos.v\left(\frac{dR}{d\varrho}\right) - \frac{\sin.v}{\varrho}\left(\frac{dR}{dv}\right) - \frac{s.\cos.v}{\varrho}.\left(\frac{dR}{ds}\right)\right\}$$
$$+c''.dt\left\{\sin.v\left(\frac{dR}{d\varrho}\right) + \frac{\cos.v}{\varrho}\left(\frac{dR}{dv}\right) - \frac{s.\sin.v}{\varrho}.\left(\frac{dR}{ds}\right)\right\}.$$

The quantities c′, c″ depend, as we have seen in No. 531, upon the inclination of the orbit of μ to the fixed plane, in such a manner that they become zero when the inclination $= 0$; moreover it is easy to see by the nature of R that $\left(\frac{dR}{ds}\right)$ is of the order of the inclinations of the orbits; neglecting therefore the squares and products of these inclinations, the preceding expressions of d f and of d f′, will become

$$df = -dy\left(\frac{dR}{dv}\right) - cdt\left\{\sin.v\left(\frac{dR}{d\varrho}\right) + \frac{\cos.v}{\varrho}.\left(\frac{dR}{dv}\right)\right\};$$

$$df' = dx\left(\frac{dR}{dv}\right) + cdt\left\{\cos.v\left(\frac{dR}{d\varrho}\right) - \frac{\sin.v}{\varrho}\left(\frac{dR}{dv}\right)\right\};$$

but we have

$dx = d(\varrho\cos.v)$; $dy = d(\varrho\sin.v)$; $cdt = x\,dy - y\,dx = \varrho^2\,dv$, we therefore get

$$df = -\{d\varrho\sin.v + 2\varrho\,dv\cos.v\}\left(\frac{dR}{dv}\right) - \varrho^2\,dv\sin.v\left(\frac{dR}{d\varrho}\right);$$

$$df' = \{d\varrho\cos.v - 2\varrho\,dv\sin.v\}\left(\frac{dR}{dv}\right) + \varrho^2\,dv\cos.v\left(\frac{dR}{d\varrho}\right).$$

These equations are more exact, if we take for the fixed plane of x, y,

that of the orbit of μ, at a given epoch; for then c', c'' and s are of the order of the perturbing forces; thus the quantities which we neglect, are of the order of the squares of the perturbing forces, multiplied by the square of the respective inclination of the two orbits of μ and of μ'.

The values of ϱ, $d\varrho$, dv, $\left(\frac{dR}{d\varrho}\right)$, $\left(\frac{dR}{dv}\right)$, remain clearly the same whatever is the position of the point from which we reckon the longitudes; but in diminishing v by a right angle, sin. v becomes — cos. v, and cos. v becomes sin. v; the expression of df changes consequently to that of df'; whence it follows that having developed, into a series of sines and cosines of angles increasing proportionally with the times, the value of df, we shall have the value of df', by diminishing in the first the angles ϵ, ϵ', ϖ, ϖ', θ and θ' by a right angle.

The quantities f and f′ determine the position of the perihelion, and the excentricity of the orbit; in fact we learn from 531, that

$$\text{tan. } I = \frac{f'}{f};$$

I being the longitude of the perihelion referred to the fixed plane. When this plane is that of the primitive orbit of μ, we have up to quantities of the order of the squares of the perturbing forces multiplied by the square of the respective inclinations of the orbits, $I = \varpi$, ϖ being the longitude of the perihelion upon the orbit; we shall therefore then have

$$\text{tan. } \varpi = \frac{f'}{f};$$

which gives

$$\text{sin. } \varpi = \frac{f'}{\sqrt{f^2 + f'^2}}; \quad \text{cos. } \varpi = \frac{f}{\sqrt{f^2 + f'^2}}.$$

By 531, we then get

$$me = \sqrt{f^2 + f'^2 + f''^2}, \quad f'' = \frac{f'c' - fc''}{c};$$

thus c' and c'' being in the preceding supposition of the order of the perturbing forces, f'' is of the same order, and neglecting the terms of the square of these forces, we have

$$me = \sqrt{f^2 + f'^2}.$$

If we substitute for $\sqrt{f^2 + f'^2}$, its value me, in the expressions of sin. ϖ, and of cos. ϖ, we shall have

$$me \text{ sin. } \varpi = f'; \quad me \text{ cos. } \varpi = f;$$

these two equations will determine the excentricity and the position of the perihelion, and we thence easily obtain

$$m^2.\, e\, de = f\, df + f'\, df'; \quad m^2 e^2\, d\varpi = f\, df' - f'\, df.$$

Taking for the plane of x, y that of the orbit of μ; we have for the cases of the invariable ellipses,

$$\varrho = \frac{a(1-e^2)}{1+e\cos.(v-\varpi)}; \quad d\varrho = \frac{\varrho^2 dv.e.\sin.(v-\varpi)}{a(1-e^2)};$$

$$\varrho^2 dv = a^2 n dt \sqrt{1-e^2};$$

and by No. 530, these equations also subsist in the case of the variable ellipses; the expressions of d f and of d f' will thus become

$$df = -\frac{a n dt}{\sqrt{1-e^2}}.\{2\cos. v+\tfrac{5}{4}e\cos.\varpi+\tfrac{1}{4}e\cos.(2v-\varpi)\}.\left(\frac{dR}{dv}\right)$$
$$-a^2 n dt\sqrt{1-e^2}.\sin. v.\left(\frac{dR}{d\varrho}\right)$$

$$df' = -\frac{a n dt}{\sqrt{1-e^2}}.\{2\sin. v+\tfrac{5}{4}e\sin.\varpi+\tfrac{1}{4}e\sin.(2v-\varpi)\}\left(\frac{dR}{dv}\right)$$
$$+a^2 n dt\sqrt{1-e^2}.\cos. v\left(\frac{dR}{d\varrho}\right);$$

wherefore

$$e d\varpi = -\frac{a n dt}{m\sqrt{1-e^2}}\sin.(v-\varpi)\{2+e\cos.(v-\varpi)\}\left(\frac{dR}{dv}\right)$$
$$+\frac{a^2.n dt\sqrt{1-e^2}}{m}\cos.(v-\varpi).\left(\frac{dR}{d\varrho}\right)$$

$$de = -\frac{a n dt}{m\sqrt{1-e^2}}.\{2\cos.(v-\varpi)+e+e\cos.^2(v-\varpi)\}\left(\frac{dR}{dv}\right)$$
$$-\frac{a^2 n dt}{m}\sqrt{1-e^2}.\sin.(v-\varpi).\left(\frac{dR}{d\varrho}\right).$$

This expression of d e may be put into a more commodious form in some circumstances. For that purpose, we shall observe that

$$d\varrho\left(\frac{dR}{d\varrho}\right) = dR - dv\left(\frac{dR}{dv}\right)$$

substituting for ϱ and d ϱ their preceding values, we shall have

$$\varrho^2 dv.e.\sin.(v-\varpi)\left(\frac{dR}{d\varrho}\right) = a(1-e^2).dR - a(1-e^2)dv\left(\frac{dR}{dv}\right);$$

but we have

$$\varrho^2 dv = a^2 n dt\sqrt{1-e^2};$$
$$dv = \frac{n dt\{1+e\cos.(v-\varpi)\}^2}{(1-e^2)^{\frac{3}{2}}};$$

wherefore

$$a^2 n dt\sqrt{1-e^2}.\sin.(v-\varpi).\left(\frac{dR}{d\varrho}\right)$$
$$= \frac{a(1-e^2)}{e}dR - \frac{a n dt}{e\sqrt{1-e^2}}.\{1+e\cos.(v-\varpi)\}^2.\left(\frac{dR}{dv}\right);$$

the preceding expression of d e, will thus give

$$e\,d\,e = \frac{a\,n\,d\,t\,\sqrt{1-e^2}}{m}\cdot\left(\frac{d\,R}{d\,v}\right) - \frac{a\,(1-e^2)}{m}\,d\,R.$$

We can arrive very simply at this formula, in the following manner We have by No. 531,

$$\frac{d\,c}{d\,t} = y\left(\frac{d\,R}{d\,x}\right) - x\left(\frac{d\,R}{d\,v}\right) = -\left(\frac{d\,R}{d\,v}\right);$$

but by the same No. $c = \sqrt{m\,a\,(1-e^2)}$ which gives

$$d\,c = \frac{d\,a\,\sqrt{m\,a\,(1-e^2)}}{2\,a} - \frac{e\,d\,e\,\sqrt{m\,a}}{\sqrt{1-e^2}};$$

therefore

$$e\,d\,e = \frac{a\,n\,d\,t\,\sqrt{1-e^2}}{m}\left(\frac{d\,R}{d\,v}\right) + a\,(1-e^2)\,\frac{d\,a}{2\,a^2};$$

and then we have by No. 531

$$\frac{m\,d\,a}{2\,a^2} = -\,d\,R.$$

We thus obtain for e d e the same expression as before.

535. We have seen in 532, that if we neglect the squares of the perturbing forces, the variations of the principal axis and of the mean motion contain only periodic quantities, depending on the configuration of the bodies μ, μ', μ'', &c. This is not the case with respect to the variations of the excentricities and inclinations: their differential expressions contain terms independent of this configuration and which, if they were rigorously constant, would produce by integration, terms proportional to the time, which at length would render the orbits very excentric and greatly inclined to one another; thus the preceding approximations, founded upon the smallness of the excentricity and inclination of the orbits, would become insufficient and even faulty. But the terms apparently constant, which enter the differential expressions of the excentricities and inclinations, are functions of the elements of the orbits; so that they vary with an extreme slowness, because of the changes they there introduce. We conceive there ought to result in these elements, considerable inequalities independent of the mutual configuration of the bodies of the system, and whose periods depend upon the ratios of the masses μ, μ', &c. to the mass M. These inequalities are those which we have named *secular* inequalities, and which have been considered in (520). To determine them by this method we resume the value of d f of the preceding No.

$$d\,f = -\frac{a\,n\,d\,t}{\sqrt{1-e^2}}\left\{2\cos v + \tfrac{5}{2}\,e\cos.\,\varpi + \tfrac{1}{2}\,e\cos.\,(2\,v-\varpi)\right\}\left(\frac{d\,R}{d\,v}\right)$$

$$-a^2 n\,dt\,\sqrt{1-e^2}.\sin v.\left(\frac{dR}{d\varrho}\right).$$

We shall neglect in the developement of this equation the square and products of the excentricities and inclinations of the orbits; and amongst the terms depending upon the excentricities and inclinations, we shall retain those only which are constant: we shall then suppose, as in No. 515.

$$\varrho = a(1+u_{,}); \quad \varrho' = a'(1+u_{,}');$$
$$v = nt + \varepsilon + v_{,}; \quad v' = n't + \varepsilon' + v_{,}'.$$

Again, if we substitute for R, its value found in 515; if we next consider that by the same No. we have,

$$\left(\frac{dR}{d\varrho}\right) = \frac{a}{\varrho}\left(\frac{dR}{da}\right) = (1-u_{,})\left(\frac{dR}{da}\right),$$

and lastly if we substitute for $u_{,}, u_{,}', v_{,}, v_{,}'$ their values $-e\cos.(nt+\varepsilon-\varpi)$, $-e'\cos.(n't+\varepsilon'-\varpi')$, $2e\sin.(nt+\varepsilon-\varpi)$, $2e'\sin.(n't+\varepsilon'-\varpi')$ given in No. 484, &c. by retaining only the constant terms of those which depend upon the first power of the excentricities of the orbits, and neglecting the squares of the excentricities and inclinations, we shall find that

$$df = \frac{a\mu' n\,dt}{2}.e.\sin.\varpi\left\{a\left(\frac{dA^{(0)}}{da}\right) + \tfrac{1}{2}a^2\left(\frac{d^2A^{(0)}}{da^2}\right)\right\}$$
$$+ a\mu' n\,dt.e'.\sin.\varpi'\left\{A^{(1)} + \tfrac{1}{2}\left(\frac{dA^{(1)}}{da}\right) + \tfrac{1}{2}a'\left(\frac{dA^{(1)}}{da'}\right) + \tfrac{1}{4}a\,a'\left(\frac{d^2A^{(1)}}{da\,da'}\right)\right\}$$
$$- a\mu' n\,dt.\Sigma\left\{iA^{(i)} + \tfrac{1}{2}a\left(\frac{dA^{(i)}}{da}\right)\right\}\sin.\{i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\};$$

the integral sign belonging as in the value of R of 515, to all the whole positive and negative values of i, including also the value of $i = 0$.

We shall have by the preceding No. the value of df', by diminishing in that of df the angles $\varepsilon, \varepsilon', \varpi, \varpi'$ by a right angle; whence we get

$$df' = -\frac{a\mu' n\,dt}{2}.e.\cos.\varpi\left\{a\left(\frac{dA^{(0)}}{da}\right) + \tfrac{1}{2}a^2\left(\frac{d^2A^{(0)}}{da^2}\right)\right\}$$
$$- a\mu' n\,dt.e'.\cos.\varpi'\left\{A^{(1)} + \tfrac{1}{2}a\left(\frac{dA^{(1)}}{da}\right) + \tfrac{1}{2}a'\left(\frac{dA^{(1)}}{da'}\right) + \tfrac{1}{4}aa'\left(\frac{d^2A^{(1)}}{da\,da'}\right)\right\}$$
$$+ a\mu' n\,dt.\Sigma\left\{iA^{(i)} + \tfrac{1}{2}a\left(\frac{dA^{(i)}}{da}\right)\right\}\cos.\{i(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\}.$$

Let X, for the greater brevity, denote that part of df, which is contained under the sign Σ, and Y the corresponding part of df'. Make also, as in No. 522,

$$0, 1) = -\frac{\mu' n}{2}\left\{a^2.\left(\frac{dA^{(0)}}{da}\right) + \tfrac{1}{2}a^3\left(\frac{d^2A^{(0)}}{da^2}\right)\right\};$$

$$\boxed{1, 1} = \frac{\mu' n}{2}\left\{a A^{(1)} - a^2\left(\frac{d A^{(1)}}{d a}\right) - \tfrac{1}{2} a^3\left(\frac{d^2 A^{(1)}}{d a^2}\right)\right\};$$

then observe that the coefficient of $e' d t \sin. \pi'$, in the expression of $d f$, is reducible to $\boxed{0, 1}$ when we substitute for the partial differences in a', their values in partial differences relative to a; finally suppose, as in 517, that

$$e \sin. \varpi = h; \; e' \sin. \varpi' = h'$$
$$e \cos. \varpi = l; \; e' \cos. \varpi' = l'$$

which gives by the preceding No. $f = m l$, $f' = m h$ or simply $f = l$, $f' = h$, by taking M for the mass-unit, and neglecting μ with regard to M; we shall obtain

$$\frac{d h}{d t} = (0, 1). l - \boxed{0, 1}. l' + a \mu' n Y;$$

$$\frac{d l}{d t} = - (0, 1). h + \boxed{0, 1}. h' - a \mu' n. X.$$

Hence, it is easy to conclude that if we name (Y) the sum of the terms analogous to $a \mu' n Y$, due to the motion of each of the bodies μ', μ'', &c. upon μ; that if we name in like manner (X) the sum of the terms analogous to $- a \mu' n X$ due to the same actions, and finally if we mark successively with one dash, two dashes, &c. what the quantities (X), (Y), h, and l become relatively to the bodies μ', μ'', &c.; we shall have the following differential equations,

$$\frac{d h}{d t} = \{(0, 1) + (0, 2) + \&c.\} l - \boxed{0, 1} l' - \boxed{0, 2} l'' - \&c. + (Y);$$

$$\frac{d l}{d t} = - \{(0, 1) + (0, 2) + \&c.\} h + \boxed{0, 1} h' + \boxed{0, 2} h'' + \&c + (X);$$

$$\frac{d h'}{d t} = \{(1, 0) + (1, 2) + \&c.\} l' - \boxed{1, 0} l - \boxed{1, 2} l'' - \&c. + (Y').$$

$$\frac{d l'}{d t} = - \{(1, 0) + (1, 2) + \&c.\} h' + \boxed{1, 0} h + \boxed{1, 2} h'' + \&c. + (X')$$

&c.

To integrate these equations, we shall observe that each of the quantities h, l, h', l', &c. consists of two parts; the one depending upon the mutual configuration of the bodies μ, μ', &c.; the other independent of this configuration, and which contains the secular variations of these quantities. We shall obtain the first part by considering that if we regard hat alone, h, l, h', l', &c. are of the order of the perturbing masses, and consequently, (0, 1). h, (0, 1). l, &c. are of the order of the squares of

these masses. By neglecting therefore quantities of this order, we shall have

$$\frac{d\,h}{d\,t} = (Y)\,; \frac{d\,l}{d\,t} = (X)\,;$$

$$\frac{d\,h'}{d\,t} = (Y')\,; \frac{d\,l'}{d\,t} = (X')\,;$$

wherefore,

$$h = \int (Y)\,d\,t\,;\ l = \int (X)\,d\,t\,;\ h' = \int (Y')\,d\,t\,;\ \&c.$$

If we take these integrals, not considering the variability of the elements of the orbits and name Q what $\int (Y)\,d\,t$ becomes; by calling $\delta\,Q$ the variation of Q due to that of the elements we shall have

$$\int (Y)\,d\,t = Q - \int \delta\,Q\,;$$

but Q being of the order of the perturbing masses, and the variations of the elements of the orbits being of the same order, $\delta\,Q$ is of the order of the squares of the masses; thus, neglecting quantities of this order, we shall have

$$\int (Y)\,d\,t = Q.$$

We may, therefore, take the integrals $\int (Y)\,d\,t$, $\int (X)\,d\,t$, $\int (Y')\,d\,t$, &c. by supposing the elements of the orbits constant, and afterwards consider the elements variable in the integrals; we shall after a very simple method, obtain the periodic portions of the expressions of h, l, h', &c.

To get those parts of the expressions which contain the secular inequalities, we observe that they are given by the integration of the preceding differential equations deprived of their last terms, (Y), (X), &c.; for it is clear that the substitution of the periodic parts of h, l, h', &c. will cause these terms to disappear. But in taking away from the equations their last terms, they will become the same as those of (A) of No. 522, which we have already considered at great length.

536. We have observed in No. 532, that if the mean motions n t and n' t of the two bodies μ and μ', are very nearly in the ratio of i' to i so that $i'\,n' - i\,n$ may be a very small quantity; there may result in the mean motions of these bodies very sensible inequalities. This relation of the mean motions may also produce sensible variations in the excentricities of the orbits, and in the positions of their perihelions. To determine them, we shall resume the equation found in 534,

$$e\,d\,e = \frac{a\,n\,d\,t.\sqrt{1-e^2}}{m}.\left(\frac{d\,R}{d\,v}\right) - \frac{a\,(1-e^2)}{m}\,d\,R.$$

It results from what has been asserted in 515, that if we take for the fixed plane, that of the orbit of μ, at a given epoch, which allows us to

neglect in R the inclination φ of the orbit of μ to this plane; all the terms of the expression of R depending upon the angle $i' n' t - i n t$, will be comprised in the following form,

$$\mu' k \cos. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - g \varpi - g' \varpi' - g'' \theta'),$$

i, i', g, g', g'' being whole numbers and such that we have $0 = i' - i - g - g' - g''$. The coefficient k has the factor $e^{g} . e'^{g'} (\tan. \frac{1}{2} \varphi')^{g''}$; g, g', g'' being taken positively in the exponents: moreover, if we suppose i and i' positive, and i' greater than i; we have seen in No. 515, that the terms of R which depend upon the angle $i' n' t - i n t$ are of the order $i' - i$, or of a superior order of two, of four, &c. units; taking into account therefore only terms of the order $i' - i$, k will be of the form $e^{g} . e'^{g'} (\tan. \frac{1}{2} \varphi')^{g''}$. Q, Q being a function independent of the excentricities and the inclination of the orbits. The numbers g, g', g'' comprehended under the symbol *cos.*, are then positive; for if one of them, g for instance, be negative and equal to $-f$, k will be of the order $f + g' + g''$; but the equation $0 = i' - i - g - g' - g''$ gives $f + g' + g'' = i' - i + 2f$; thus k will be of an order superior to $i' - i$, which is contrary to the supposition. Hence by No. 515, we have $\left(\frac{d R}{d v}\right) = \left(\frac{d R}{d \varepsilon}\right)$ provided that in this last partial difference, we make $\varepsilon - \varpi$ constant; the term of $\left(\frac{d R}{d v}\right)$ corresponding to the preceding term of R, is therefore

$$u' (i + g) k \sin. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - g \varpi - g' \varpi' - g'' \theta').$$

The corresponding term of $d R$ is

$$u' \beta i n k d t \sin. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - g \varpi - g' \varpi' - g'' \theta').$$

Hence only regarding these terms and neglecting e^2 in comparison with unity, the preceding expression of $e d e$, will give

$$d e = \frac{\mu' a n d t}{m} . \frac{g k}{e} \sin. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - g \varpi - g' \varpi' - g'' \theta'),$$

but we have

$$\frac{g k}{e} = g e^{g-1} . e'^{g'} . (\tan. \tfrac{1}{2} \varphi')^{g''} . Q = \left(\frac{d k}{d e}\right);$$

integrating therefore we get

$$e = -\frac{\mu' a n}{m (i' n - i n)} \left(\frac{d k}{d e}\right) \cos. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - g \varpi - g' \varpi' - g'' \theta').$$

The sum of all the terms of R, however, which depend on the angle $i' n' t - i n t$, being represented by the following quantity

$$\mu' . P \sin. (i' n' t - i n t + i' \varepsilon' - i \varepsilon) + u' P' \cos. (i' n' t - i n t + i' \varepsilon' - i \varepsilon)$$

the corresponding part of e will be

$$\frac{-\mu' a n}{m (i' n' - i n)} \left\{ \left(\frac{d P}{d e}\right) \sin. (i' n' t - int + i' \varepsilon' - i \varepsilon) + \left(\frac{d P'}{d e}\right) \cos. (i' n' t - int + i' \varepsilon' - i \varepsilon) \right\}.$$

This inequality may become very sensible, if the coefficient $i' n' - i n$ is very small, for it actually takes place in the theory of Jupiter and Saturn. In fact, it has for a divisor only the first power of $i' n' - i n$, whilst the corresponding inequality of the mean motion, has for a divisor the second power of this quantity, as we see in No. 532; but $\left(\frac{d P}{d e}\right)$ and $\left(\frac{d P'}{d e}\right)$ being of an order inferior to P and P', the inequality of the excentricity may be considerable, and even surpass that of the mean motion, if the excentricities e and e' are very small; this will be exemplified in the theory of Jupiter's satellites.

Let us now determine this corresponding inequality of the motion of the perihelion. For that purpose, resume the two equations

$$e d e = \frac{f d f + f' d f'}{m^2}, \quad e^2 d \varpi = \frac{f d f' - f' d f}{m^2};$$

which we found in No. 534. These equations give

$$d f = m d e \cos. \varpi - m e d \varpi . \sin. \varpi;$$

thus with regard only to the angle

$$i' n' t - i n t + i' \varepsilon' - i \varepsilon - g \varpi - g' \varpi' - g'' \theta',$$

we shall have

$$d f = \mu' . a n d t \left(\frac{d k}{d e}\right) \cos. \varpi \sin. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - g \varpi - g' \varpi' - g'' \theta') - m e d \varpi . \sin. \varpi.$$

Representing by

$$-\mu' . a n d t \left\{ \left(\frac{d k}{d e}\right) + k' \right\} \cos. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - g \varpi - g' \varpi' - g'' \theta'),$$

the part of $m e d \varpi$, which depends upon the same angle, we shall have

$$d f = \mu' . a n d t \left\{ \left(\frac{d k}{d e}\right) + \tfrac{1}{2} k' \right\} \sin. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - (g-1) \varpi - g' \varpi' - g'' \theta') - \frac{\mu' . a n d t}{2} k' \sin. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - (g+1) \varpi - g' \varpi' - g'' \theta').$$

It is easy to see by the last of the expressions of d f, given in the No. 534, that the coefficient of this last sine has the factor $e^{g+1} . e'^{g'} (\tan. \tfrac{1}{2} \varphi)^{g''}$; k' is therefore of an order superior to that of $\left(\frac{d k}{d e}\right)$ by two units; thus, in neglecting it in comparison to $\left(\frac{d k}{d e}\right)$, we shall have

$$-\frac{\mu' . a n d t}{m} \left(\frac{d k}{d e}\right) \cos. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - g \varpi - g' \varpi' - g'' \theta')$$

for the term of $e d \varpi$, which corresponds to the term

$$\mu' k \cos. (i' n' t - i n t + i' \varepsilon' - i \varepsilon - g \varpi - g' \varpi' - g'' \theta'),$$

of the expression of R. Hence it follows that the part of ϖ, which corresponds to the part of R expressed by
$\mu' P \sin.(i' n' t - int + i' \varepsilon' - i\varepsilon) + \mu' P' \cos.(i' n' t - int + i' \varepsilon' - i\varepsilon)$,
is equal to

$$\frac{\mu'.an}{m(i'n'-in)e}\cdot\left\{\left(\frac{dP}{de}\right)\cos.(i'n't-int+i'\varepsilon'-i\varepsilon)-\left(\frac{dP'}{de}\right)\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)\right\};$$

we shall therefore, thus, after a very simple manner, find the variations of the excentricity and of the perihelion, depending upon the angle $i' n' t - int + i' e' - ie$. They are connected with the variation ζ of the corresponding mean motion, in such a way that the variation of the excentricity is

$$\frac{1}{3in}\cdot\left(\frac{d^2\zeta}{de.dt}\right);$$

and the variation of the longitude of the perihelion is

$$\frac{i'n'-in}{3ine}\cdot\left(\frac{d\zeta}{de}\right).$$

The corresponding variation of the excentricity of the orbit of μ', due to the action of μ, will be

$$-\frac{1}{3i'n'.e'}\cdot\left(\frac{d^2\zeta'}{de'.dt}\right),$$

and the variation of the longitude of its perihelion, will be

$$-\frac{i'n'-in}{3i'n'e'}\left(\frac{d\zeta'}{de'}\right),$$

and since by No. 532, $\zeta' = -\frac{\mu\sqrt{a}}{\mu'\sqrt{a'}}.\zeta$, the variations will be

$$\frac{\mu\sqrt{a}}{3i'.n'.\mu'\sqrt{a'}}\left(\frac{d^2\zeta}{de'.dt}\right), \text{ and } \frac{(i'n'-in)\mu\sqrt{a}}{3i'n'.e'\mu'\sqrt{a'}}\cdot\frac{d\zeta}{de'}.$$

When the quantity $i' n' - in$ is very small, the inequality depending upon the angle $i' n' t - int$, produces a sensible one in the expression of the mean motion, amongst the terms depending on the squares of the perturbing masses; we have given the analysis of this in No. 532. This same inequality produces in the expression of de and of $d\varpi$, terms of the order of the squares of the masses, and which, being only functions of the elements of the orbits, have a sensible influence upon the secular variations of these elements. Let us consider, in fact, the expression of de, depending on the angle $i' n' t - int$.

By what precedes, we have

$$de = -\frac{\mu'.an.dt}{m}\left\{\left(\frac{dP}{de}\right)\cos.(i'n't-int+i'\varepsilon'-i\varepsilon)\right.$$
$$\left.-\left(\frac{dP'}{de}\right).\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)\right\}.$$

By No. 532 the mean motion n t, ought to be augmented by

$$\frac{3\mu' a n^2 i}{(i' n' - i n)^2 . m} \cdot \left\{ P \cos.(i' n' t - i n t + i' \varepsilon' - i \varepsilon) - P' \sin.(i' n' t - i n t + i' \varepsilon' - i \varepsilon) \right\}$$

and the mean motion n' t, ought to be augmented by

$$-\frac{3\mu' a n^2 . i}{(i' n' - i n)^2 . m} \cdot \frac{\mu \sqrt{a}}{\mu' \sqrt{a'}} . \{ P \cos.(i' n' t - i n t + i' \varepsilon' - i \varepsilon) - P' \sin.(i' n' t - i n t + i' \varepsilon' - i \varepsilon) \}.$$

In virtue of these augments, the value of d e will be augmented by the function

$$-\frac{3\mu' a^2 . i n^3 . d t}{2 m^2 \sqrt{a' . (i' n' - i n)^2}} \{ i . \mu' \sqrt{a'} + i' \mu \sqrt{a} \} \left\{ P . \left(\frac{d P}{d e}\right) + P' \left(\frac{d P}{d e}\right) \right\};$$

and the value of d ϖ will be augmented by the function

$$\frac{3\mu' a^2 . i n^3 . d t}{2 m^2 \sqrt{a' (i' n' - i n)^2 . e}} . \{ i \mu' \sqrt{a'} + i' \mu \sqrt{a} \} . \left\{ P . \left(\frac{d P}{d e}\right) + P' \left(\frac{d P'}{d e}\right) \right\}.$$

In like manner we find that the value of d e' will be augmented by the function

$$-\frac{3\mu a^2 . \sqrt{a} . i n^3 . d t}{2 m^2 . a' . (i' n' - i n)^2} . \{ i . \mu' . \sqrt{a'} + i' . \mu . \sqrt{a} \} \left\{ P . \left(\frac{d P'}{d e'}\right) - P' . \left(\frac{d P}{d e}\right) \right\};$$

and that the value of d e' will be augmented by the function

$$\frac{3\mu a^2 . \sqrt{a} . i n^3 . d t}{2 m^2 a' . (i' n' - i n)^2 . e'} . \{ i \mu' \sqrt{a'} + i' \mu \sqrt{a} \} \left\{ P . \left(\frac{d P}{d e'}\right) + P' \left(\frac{d P'}{d e'}\right) \right\}.$$

These different terms are sensible in the theory of Jupiter and Saturn, and in that of Jupiter's satellites. The variations of e, e', ϖ, ϖ' relative to the angle i' n' t — i n t may also introduce some constant terms of the order of the square of the perturbing masses in the differentials d e, d e', d ϖ, and d ϖ', and depending on the variations of e, e', ϖ, ϖ' relative to the same angle. This may easily be discussed by the preceding analysis. Finally it will be easy, by our analysis, to determine the terms of the expressions of e, ϖ, e', ϖ' which depending upon the angle i' n' t — i n t + i' ε' — i ε have not i' n' — i n for a divisor, and those which, depending on the same angle and the double of this angle, are of the order of the square of the perturbing forces. These different terms are sufficiently considerable in the theory of Jupiter and Saturn, for us to notice them: we shall develope them to the extent they merit when we come to that theory.

537. Let us determine the variations of the nodes and inclinations of the orbits, and for that purpose resume the equations of 531,

$$d c = d t \left\{ y \left(\frac{d R}{d x}\right) - x \left(\frac{d R}{d y}\right) \right\};$$

$$d c' = d t \left\{ z \left(\frac{d R}{d x}\right) - x \left(\frac{d R}{d z}\right) \right\};$$

$$d c'' = d t \left\{ z \left(\frac{d R}{d y}\right) - y \left(\frac{d R}{d z}\right) \right\}.$$

If we only notice the action of μ', the value of R of No. 513, gives

$$y \left(\frac{d R}{d x}\right) - x \left(\frac{d R}{d y}\right) = \mu' (x' y - x y') \times$$

$$\left\{ \frac{1}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} - \frac{1}{\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\}^{\frac{3}{2}}} \right\};$$

$$z \left(\frac{d R}{d x}\right) - x \left(\frac{d R}{d z}\right) = \mu' (x' z - x z') \times$$

$$\left\{ \frac{1}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} - \frac{1}{\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\}^{\frac{3}{2}}} \right\};$$

$$z \left(\frac{d R}{d y}\right) - y \left(\frac{d R}{d z}\right) = \mu' (y' z - y z') \times$$

$$\left\{ \frac{1}{(x'^2 + y'^2 + z'^2)^2} - \frac{1}{\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\}^{\frac{3}{2}}} \right\}.$$

Let however,

$$\frac{c''}{c} = p; \quad \frac{c'}{c} = q;$$

the two variables p and q will determine, by No. 531, the tangent of the inclination φ of the orbit of μ, and the longitude θ of its node by means of the equations

$$\text{tan. } \varphi = \sqrt{p^2 + q^2}; \quad \text{tan. } \theta = \frac{p}{q}.$$

Call p', q', p'', q'', &c. what p and q become relatively to the bodies μ', μ'', &c.: we shall have by 531,

$$z = q y - p x; \quad z' = q' y' - p' x', \text{ &c.}$$

The preceding value of p differentiated gives

$$\frac{d p}{d t} = \frac{1}{c} \cdot \frac{d c'' - p d c}{d t};$$

substituting for d c, and d c'' their values we get

$$\frac{d p}{d t} = \frac{\mu'}{c} \{(q - q') y y' + (p' - p) x' y\} \times$$

$$\left\{ \frac{1}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} - \frac{1}{\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\}^{\frac{3}{2}}} \right\}.$$

In like manner we find

$$\frac{d q}{d t} = \frac{\mu'}{c} \{(p' - p) x x' + (q - q') x y'\} \times$$

$$\left\{\frac{1}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}-\frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}\right\}.$$

If we substitute for x, y, x', y' their values ϱ cos. v, ϱ sin. v, ϱ' cos. v', ϱ' sin. v', we shall have

$$(q-q')\,y\,y' + (p'-p)\,x'\,y = \frac{q'-q}{2}.\varrho\,\varrho'.\{\cos.(v'+v) - \cos.(v'-v)\}$$
$$+ \frac{p'-p}{2}.\varrho\,\varrho'.\{\sin.(v'+v) - \sin.(v'-v)\};$$

$$(p'-p)\,x\,x' + (q-q')\,x\,y' = \frac{p'-p}{2}.\varrho\,\varrho'.\{\cos.(v'+v) + \cos.(v'-v)\}$$
$$+ \frac{q-q'}{2}.\varrho\,\varrho'.\{\sin.(v'+v) + \sin.(v'-v)\}.$$

Neglecting the excentricities and inclinations of the orbits, we have

$$\varrho = a;\ v = n\,t + \varepsilon;\ \varrho' = a';\ v' = n'\,t + \varepsilon';$$

which give

$$\frac{1}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}} - \frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}} = \frac{1}{a'^3} -$$
$$\frac{1}{\{a^2 - 2\,a\,a'\cos.(n'\,t - n\,t + \varepsilon' - \varepsilon) + a'^2\}^{\frac{3}{2}}};$$

moreover by No. 516,

$$\frac{1}{\{a^2-2\,a\,a'\cos.(n'\,t-n\,t+\varepsilon'-\varepsilon)+a'^2\}^{\frac{3}{2}}} = \tfrac{1}{2}\,\Sigma.\,B^{(i)}.\cos.i\,(n'\,t-n\,t+\varepsilon'-\varepsilon)$$

the integral sign Σ belonging to all whole positive and negative values of i, including the value i = 0; we shall thus have, neglecting terms of the order of the squares and products of the excentricities and inclinations of the orbits,

$$\frac{d\,p}{d\,t} = \frac{q'-q}{2\,c}.\frac{\mu'\,a}{a'^2}.\{\cos.(n'\,t+n\,t+\varepsilon'+\varepsilon) - \cos.(n'\,t-n\,t+\varepsilon'-\varepsilon)\}$$
$$+ \frac{p'-p}{2\,c}.\frac{\mu'\,a}{a'^2}.\{\sin.(n'\,t+n\,t+\varepsilon'+\varepsilon) - \sin.(n'\,t-n\,t+\varepsilon'-\varepsilon)\}$$
$$+ \frac{q'-q}{4\,c}.\mu'.a\,a'.\Sigma.B^{(i)}\{\cos.[(i+1)(n'\,t-n\,t+\varepsilon'-\varepsilon)]$$
$$- \cos.[(i+1)(n'\,t-n\,t+\varepsilon'-\varepsilon)+2\,n\,t+2\,\varepsilon]\}$$
$$+ \frac{p'-p}{4\,c}.\mu'.a\,a'.\Sigma.B^{(i)}\{\sin.[(i+1)(n'\,t-n\,t+\varepsilon'-\varepsilon)]$$
$$- \sin.[(i+1)(n't-n\,t+\varepsilon'-\varepsilon)+2\,n\,t+2\,\varepsilon]\}.$$

$$\frac{d\,q}{d\,t} = \frac{p'-p}{2\,c}.\frac{\mu'\,a}{a'^2}.\{\cos.(n'\,t+n\,t+\varepsilon'+\varepsilon) + \cos.(n'\,t-n\,t+\varepsilon'-\varepsilon)\}$$

$$+\frac{q'-q}{2c}\cdot\frac{\mu' a}{a'^2}\cdot\{\sin.(n't+nt+\varepsilon'+\varepsilon)+\sin.(n't-nt+\varepsilon'-\varepsilon)\}$$

$$+\frac{p-p'}{4c}\cdot\mu'.aa'.\Sigma.B^{(i)}.\{\cos.[(i+1)(n't-nt+\varepsilon'-\varepsilon)]$$
$$+\cos.[(i+1)(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon]\}$$

$$+\frac{q'-q}{4c}\cdot\mu'.aa'.\Sigma.B^{(i)}.\{\sin.[(i+1)(n't-nt+\varepsilon'-\varepsilon)]$$
$$+\sin.[(i+1)(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon]\}.$$

The value $i = -1$ gives in the expression of $\frac{dp}{dt}$, the constant quantity $\frac{q'-q}{4c}\cdot\mu'.aa'B^{(-1)}$; all the other terms of the expression of $\frac{dp}{dt}$ are periodic: denoting their sum by P, and observing that $B^{(-1)} = B^{(1)}$ by 516, we shall have

$$\frac{dp}{dt} = \frac{q'-q}{4c}\cdot\mu'.aa'.B^{(1)} + P.$$

By the same process we shall find, that if we denote by Q the sum of all the periodic terms of the expression of $\frac{dq}{dt}$, we shall have

$$\frac{dq}{dt} = \frac{p-p'}{4c}\cdot\mu'.aa'.B^{(1)} + Q.$$

If we neglect the squares of the excentricities and inclinations of the orbits, by 531, we have $c = \sqrt{ma}$, and then supposing $m = 1$, we have $n^2a^3 = 1$ which gives $c = \frac{1}{an}$; the quantity $\frac{\mu'.aa'.B^{(1)}}{4c}$ thus becomes $\frac{\mu'.a^2a'.nB^{(1)}}{4}$ which by 526, is equal to (0, 1); hence we get

$$\frac{dp}{dt} = (0,1).(q'-q) + P;$$

$$\frac{dq}{dt} = (0,1).(p-p') + Q.$$

Hence it follows that, if we denote by (P) and (Q) the sum of all the functions P and Q relative to the action of the different bodies μ', μ'', &c. upon μ; if in like manner we denote by (P'), (Q'), (P''), (Q''), &c. what (P) and (Q) become when we change successively the quantities relative to μ into those which are relative to μ', μ'', &c. and reciprocally; we shall have for determining the variables p, q, p', q', p'', q'', &c. the following system of differential equations,

$$\frac{dp}{dt} = -\{(0,1)+(0,2)+\&c.\}q+(0,1).q'+(0,2)q''+\&c.+(P);$$

$$\frac{dq}{dt} = \{(0,1) + (0,2) + \&c.\}\, p - (0,1)\, p' - (0,2)\, p'' - \&c. + (Q);$$

$$\frac{dp'}{dt} = -\{(1,0) + (1,2) + \&c.\}\, q' + (1,0)\, q + (1,2)\, q'' + \&c. + (P');$$

$$\frac{dq'}{dt} = \{(1,0) + (1,2) + \&c.\}\, p' - (1,0)\, p - (1,2)\, p'' - \&c. + (Q');$$

&c.

The analysis of 535, gives for the periodic parts of p, q, p', q', &c.

$$p = \int (P).\, dt;\quad q = \int (Q).\, dt;$$
$$p' = \int (P').\, dt;\quad q' = \int (Q').\, dt;$$
&c.

We shall then have the secular parts of the same quantities, by integrating the preceding differential equations deprived of their last terms (P), (Q), (P'), &c.; and then we shall again hit upon the equations (C) of No. 526, which have been sufficiently treated of already to render it unnecessary again to discuss them.

538. Let us resume the equations of No. 531,

$$\text{tan.}\, \varphi = \frac{\sqrt{c'^2 + c''^2}}{c};\quad \text{tan.}\, \theta = \frac{c''}{c}$$

whence result these

$$\frac{c'}{c} = \text{tan.}\, \varphi \cos.\, \theta;\quad \frac{c''}{c} = \text{tan.}\, \varphi \sin.\, \theta.$$

Differentiating, we shall have

$$d\, \text{tan.}\, \varphi = \frac{1}{c}\, \{d\, c' \cos.\, \theta + d\, c'' \sin.\, \theta - d\, c\, \text{tan.}\, \varphi\}$$

$$d\, \theta\, \text{tan.}\, \varphi = \frac{1}{c}\, \{d\, c'' \cos.\, \theta - d\, c' \sin.\, \theta\}.$$

If we substitute in these equations for $\frac{dc}{dt}$, $\frac{dc'}{dt}$, $\frac{dc''}{dt}$, their values $y\left(\frac{dR}{dx}\right) - x\left(\frac{dR}{dy}\right)$, $z\left(\frac{dR}{dx}\right) - x\left(\frac{dR}{dz}\right)$, $z\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dz}\right)$, and for these last quantities their values given in 534; if moreover we observe that $s = \text{tan.}\, \varphi \sin.\, (v - \theta)$, we shall have

$$d.\, \text{tan.}\, \varphi = \frac{dt\, \text{tan.}\, \varphi \cos.\, (v-\theta)}{c} \cdot \left\{\varrho\left(\frac{dR}{d\varrho}\right) \sin.(v-\theta) + \left(\frac{dR}{dv}\right) \cos.(v-\theta)\right\}$$
$$- \frac{(1+s^2)\, dt}{c} \cos.\, (v-\theta) \cdot \left(\frac{dR}{ds}\right);$$

$$d\, \theta .\, \text{tan.}\, \varphi = \frac{dt\, \text{tan.}\, \varphi \sin.\, (v-\theta)}{c} \left\{\varrho \cdot \left(\frac{dR}{d\varrho}\right) \sin.(v-\theta) + \left(\frac{dR}{dv}\right) \cos.(v\theta-)\right\}$$
$$- \frac{(1+s^2)\, dt}{c} \sin.\, (v-\theta)\left(\frac{dR}{ds}\right).$$

These two differential equations will determine directly the inclination of the orbit and the motion of the nodes.

They give

$$\sin.(v - \theta)\, d \tan.\varphi - d\theta \cos.(v - \theta) \tan.\varphi = 0;$$

an equation which may be deduced from this

$$s = \tan.\varphi \sin.(v - \theta);$$

in fact, this last equation being finite, we may (530) differentiate it whether we consider φ and θ constant or variable; so that its differential, taken by only making φ and θ vary, is nothing; whence results the preceding differential equation.

Suppose, however, that the fixed plane is inclined extremely little to the orbit of μ, so that we may neglect the squares of s and tan. φ, we shall have

$$d.\tan.\varphi = -\frac{dt}{c}\cos.(v - \theta).\left(\frac{dR}{ds}\right);$$

$$d\theta \tan.\varphi = -\frac{dt}{c}\sin.(v - \theta)\left(\frac{dR}{ds}\right);$$

by making therefore as before

$$p = \tan.\varphi \sin.\theta;\quad q = \tan.\varphi \cos.\theta;$$

we shall have, instead of the preceding differential equations, the following ones,

$$dq = -\frac{dt}{c}\cos.v.\left(\frac{dR}{ds}\right);$$

$$dp = -\frac{dt}{c}\sin.v.\left(\frac{dR}{ds}\right);$$

But we have also

$$s = q \sin.v - p \cos.v$$

which gives

$$\left(\frac{dR}{ds}\right) = \frac{1}{\sin.v}.\left(\frac{dR}{dq}\right),\quad \left(\frac{dR}{ds}\right) = -\frac{1}{\cos.v}.\left(\frac{dR}{dp}\right),$$

wherefore

$$dq = \frac{dt}{c}\left(\frac{dR}{dp}\right);$$

$$dp = \frac{dt}{c}\left(\frac{dR}{dq}\right).$$

We have seen in 515 that the function R is independent of the position of the fixed plane of x, y; supposing, therefore, all the angles of that function referred to the orbit of μ, it is evident that R will be a function of these angles and the respective inclination of two orbits, an

inclination we denote by φ'. Let θ' be the longitude of the node of the orbit of μ' upon the orbit of μ; and supposing that

$$\mu' k (\tan. \varphi')^g \cos. (i' n' t - i n t + A - g \theta')$$

is a term of R depending on the angle $i' n' t - i n t$, we shall have, by 527,

$$\tan. \varphi' . \sin. \theta' = p' - p; \quad \tan. \varphi' \cos. \theta' = q' - q;$$

whence we get

$$(\tan. \varphi')^g \sin. g \theta' = \frac{\{q' - q + (p' - p)\sqrt{-1}\}^g - \{q' - q - (p' - p)\sqrt{-1}\}^g}{2\sqrt{-1}}$$

$$(\tan. \varphi')^g \cos. g \theta' = \frac{\{q' - q + (p' - p)\sqrt{-1}\}^g + \{q' - q - (p' - p)\sqrt{-1}\}^g}{2}.$$

With respect to the preceding term of R, we shall have

$$\left(\frac{dR}{dp}\right) = -g (\tan. \varphi')^{g-1} \mu' k . \sin. \{i' n' t - i n t + A - (g-1) \theta'\};$$

$$\left(\frac{dR}{dq}\right) = -g (\tan. \varphi')^{g-1} \mu' k \cos. \{i' n' t - i n t + A - (g-1) \theta'\}.$$

If we substitute these values in the preceding expressions of d p and d q, and observe that very nearly $c = \frac{m}{a n}$, we shall have

$$p = \frac{g \mu' k . a n}{m (i' n' - i n)} . (\tan. \varphi')^{g-1} . \sin. \{i' n' t - i n t + A - (g-1) \theta'\}$$

$$q = \frac{g \mu' k . a n}{\mu (i' n' - i n)} . (\tan. \varphi')^{g-1} \cos. \{i' n' t - i n t + A - (g-1) \theta'\}.$$

Substituting these values in the equation

$$s = q \sin. v - p \cos. v$$

we shall have

$$s = -\frac{g . \mu' k . a n}{m (i' n' - i n)} (\tan. \varphi')^{g-1} \sin. \{i' n' t - i n t - v + A - (g-1) \theta'\}.$$

This expression of s is the variation of the latitude corresponding to the preceding term of R: it is evident that it is the same whatever may be the fixed plane to which we refer the motions of μ and μ', provided that it is but little inclined to the plane of the orbits; we shall therefore thus have that part of the expression of the latitude, which the smallness of the divisor $i' n' - i n$ may make sensible. Indeed the inequality of the latitude, containing only the first power of this divisor, is in that degree less sensible than the corresponding inequality of the mean longitude, which contains the square of the same divisor; but, on the other hand, tan. φ' is then raised to a power less by one; a remark analogous to that which was made in No. 536, upon the corresponding inequality of the excentricities of the orbits. We thus see that all these inequalities are

connected with one another, and with the corresponding part of R, by very simple relations.

If we differentiate the preceding expressions of p and q, and if in the values of $\frac{dp}{dt}$ and $\frac{dq}{dt}$ we augment the angles n t and n′ t by the inequalities of the mean motions, depending on the angle i′ n′ t — i n t, there will result in these differentials, quantities which are functions only of the elements of the orbits, and which may influence, in a sensible manner, the secular variations of the inclinations and nodes although of the order of the squares of the masses. This is analogous to what was advanced in No. 536 upon the secular variations of the excentricities and aphelions.

539. It remains to consider the variation of the longitude ε of the epoch. By No. 531 we have

$$d\varepsilon = de\left\{\left(\frac{dE^{(1)}}{de}\right)\sin.(v-\varpi)+\tfrac{1}{2}\left(\frac{dE^{(2)}}{de}\right)\sin.2(v-\varpi)+\&c.\right\}$$
$$-d\varpi\{E^{(1)}\cos.(v-\varpi)+E^{(2)}\cos.2(v-\varpi)+\&c.\};$$

substituting for $E^{(1)}$, $E^{(2)}$, &c. their values in series ordered according to the powers of e, series which it is easy to form from the general expression of $E^{(i)}$ (473) we shall have

$$d\varepsilon = -2de\sin.(v-\varpi)+2ed\varpi\cos.(v-\varpi)$$
$$+ede\{\tfrac{3}{2}+\tfrac{1}{2}e^2+\&c.\}\sin.2(v-\varpi)-e^2d\varpi\{\tfrac{5}{2}+\tfrac{1}{4}e^2+\&c.\}\cos.2(v-\varpi)$$
$$-e^2de\{1+\&c.\}\sin.3(v-\varpi)+e^3d\varpi\{1+\&c.\}\cos.3(v-\varpi)$$
$$+\&c.$$

If we substitute for d e and e d ϖ their values given in 534, we shall find, carrying the approximation to quantities of the order e^2 inclusively,

$$d\varepsilon = \frac{a^2.n\,dt}{m}\sqrt{1-e^2}\{2-\tfrac{5}{2}e\cos.(v-\varpi)+e^2\cos.2(v-\varpi)\}.\left(\frac{dR}{d\varrho}\right)$$
$$-\frac{a\,n\,dt}{m\sqrt{1-e^2}}.e.\sin.(v-\varpi)\{1+\tfrac{1}{2}e\cos.(v-\varpi)\}\left(\frac{dR}{d\varrho}\right).$$

The general expression of d ε contains terms of the form

$$\mu' k.n\,dt.\cos.(i'n't-int+A)$$

and consequently the expression of ε contains terms of the form

$$\frac{\mu' k n}{i'n'-in}\sin.(i'n't-int+A);$$

but it is easy to be convinced that the coefficient k in these terms is of the order i′ — i, and that therefore these terms are of the same order as those of the mean longitude, which depend upon the same angle. These having the divisor $(i'n'-in)^2$, we see that we may neglect the corresponding terms of ε, when i′ n′ — i n is a very small quantity.

If in the terms of the expression of d ε, which are solely functions of the elements of the orbits, we substitute for these elements the secular parts of their values; it is evident that there will result constant terms, and others affected with the sines and cosines of angles, upon which depend the secular variations of the excentricities and inclinations of the orbits. The constant terms will produce, in the expression of ε, terms proportional to the time, and which will merge into the mean motion μ. As to the terms affected with sines and cosines, they will acquire by integration, in the expression of ε, very small divisors of the same order as the perturbing forces; so that these terms being at the same time multiplied and divided by the forces, may become sensible, although of the order of the squares and products of the excentricities and inclinations. We shall see in the theory of the planets, that these terms are there insensible; but in the theory of the moon and of the satellites of Jupiter, they are very sensible, and upon them depend the secular equations.

We have seen in No. 532, that the mean motion of μ, is expressed by

$$\frac{3}{m} \int\int a\, n\, dt\, .\, d\, R,$$

and that if we retain only the first power of the perturbing masses, $d\,R$ will contain none but periodic quantities. But if we consider the squares and products of the masses, this differential may contain terms which are functions only of the elements of the orbits. Substituting for the elements the secular parts of their values, there will thence result terms affected with sines and cosines of angles depending upon the secular variations of the orbits. These terms will acquire, by the double integration, in the expression of the mean motion, small divisors, which will be of the order of the squares and products of the perturbing masses; so that being both multiplied and divided by the squares and products of the masses, they become sensible, although of the order of the squares and products of the excentricities and inclinations of the orbits. We shall see that these terms are insensible in the theory of the planets.

540. The elements of μ's orbit being determined by what precedes, by substituting them in the expressions of the radius-vector, of the longitude and latitude which we have given in 484, we shall get the values of these three variables, by means of which astronomers determine the position of the celestial bodies. Then reducing them into series of sines and cosines, we shall have a series of inequalities, whence tables being formed, we may easily calculate the position of μ at any given instant.

This method, founded on the variation of the parameters, is very useful

in the research of inequalities, which, by the relations of the mean motions of the bodies of the system, will acquire great divisors, and thence become very sensible. This sort of inequality principally affects the elliptic elements of the orbits; determining, therefore, the variations which result in these elements, and substituting them in the expression of elliptic motion, we shall obtain, in the simplest manner, all the inequalities made sensible by these divisors.

The preceding method is moreover useful in the theory of the comets. We perceive these stars in but a very small part of their courses, and observations only give that part of the ellipse which coincides with the arc of the orbit described during their apparitions; thus, in determining the nature of the orbit considered a variable ellipse, we shall see the changes undergone by this ellipse in the interval between two consecutive apparitions of the same comet. We may therefore announce its return, and when it reappears, compare theory with observation.

Having given the methods and formulas for determining, by successive approximations, the motions of the centers of gravity of the celestial bodies, we have yet to apply them to the different bodies of the solar system: but the ellipticity of these bodies having a sensible influence upon the motions of many of them, before we come to numerical applications, we must treat of the figure of the celestial bodies, the consideration of which is as interesting in itself as that of their motions.

SUPPLEMENT

TO

SECTIONS XII. AND XIII.

ON ATTRACTIONS AND THE FIGURE OF THE CELESTIAL BODIES.

541. The figure of the celestial bodies depends upon the law of gravitation at their surface, and the gravitation itself being the result of the attractions of all their parts, depends upon their figure; the law of gravity at the surface of the celestial bodies, and their figure have, therefore, a reciprocal connexion, which renders the knowledge of the one necessary to the determination of the other. The research is thus very intricate,

and seems to require a very particular sort of analysis. If the planets were entirely solid, they might have any figure whatever; but if, like the earth, they are covered with a fluid, all the parts of this fluid ought to be disposed so as to be in equilibrium, and the figure of its exterior surface depends upon that of the fluid which covers it, and the forces which act upon it. We shall suppose generally that the celestial bodies are covered with a fluid, and on that hypothesis, which subsists in the case of the earth, and which it seems natural to extend to the other bodies of the system of the world, we shall determine their figure and the law of gravity at their surface. The analysis which we propose to use is a singular application of the Calculus of Partial Differences, which by simple differentiation, will conduct us to very extensive results, and which with difficulty we should obtain by the method of integrations.

THE ATTRACTIONS OF HOMOGENEOUS SPHEROIDS BOUNDED BY SURFACES OF THE SECOND ORDER.

542. The different bodies of the solar system may be considered as formed of shells very nearly spherical, of a density varying according to any law whatever; and we shall show that the action of a spherical shell upon a body exterior to it, is the same as if its mass were collected at its center. For that purpose we shall establish upon the attractions of spheroids, some general propositions which will be of great use hereafter.

Let x, y, z be the three coordinates of the point attracted which we call μ; let also d M be the element or molecule of the spheroid, and x', y', z' the coordinates of this element; if we call ϱ its density, ϱ being a function of x', y', z' independent of x, y, z, we shall have

$$d\,M = \varrho\,.\,dx'\,.\,dy'\,.\,dz'.$$

The action of d M upon μ decomposed parallel to the axis of x and directed towards their origin, will be

$$\frac{\varrho\,dx'\,.\,dy'\,.\,dz'\,(x-x')}{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{\frac{3}{2}}}$$

and consequently it will be equal to

$$-\left(\frac{d\,.\,\frac{\varrho\,dx'\,.\,dy'\,.\,dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{dx}\right);$$

calling therefore V the integral

$$\int\frac{\varrho\,dx'\,.\,dy'\,.\,dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

extended to the entire mass of the spheroid, we shall have $-\left(\frac{dV}{dx}\right)$

for the total action of the spheroid upon the point μ, resolved parallel to the axis of x and directed towards its origin.

V is the sum of the elements of the spheroid, divided by their respective distances from the point attracted; to get the attraction of the spheroid upon this point, parallel to any straight line, we must consider V as a function of three rectangular coordinates, one of which is parallel to this straight line, and differentiate this function relatively to this coordinate; the coefficient of this differential taken with a contrary sign, will be the expression of the attraction of the spheroid, parallel to the given straight line, and directed towards the origin of the coordinate which is parallel to it.

If we represent by β, the function $\{(x-x')^2+(y-y')^2+(z-z')^2\}^{-\frac{1}{2}}$; we shall have

$$V = \int \beta . \varrho . dx' dy' dz'.$$

The integration being only relative to the variables x', y', z', it is evident that we shall have

$$\left(\frac{d^2V}{dx^2}\right) + \left(\frac{d^2V}{dy^2}\right) + \left(\frac{d^2V}{dz^2}\right) = \int \varrho\, dx' dy' dz' \left\{ \left(\frac{d^2\beta}{dx^2}\right) + \left(\frac{d^2\beta}{dy^2}\right) + \left(\frac{d^2\beta}{dz^2}\right) \right\};$$

But we have

$$0 = \left(\frac{d^2\beta}{dx^2}\right) + \left(\frac{d^2\beta}{dy^2}\right) + \left(\frac{d^2\beta}{dz^2}\right);$$

in like manner we get

$$0 = \left(\frac{d^2V}{dx^2}\right) + \left(\frac{d^2V}{dy^2}\right) + \left(\frac{d^2V}{dz^2}\right); \quad \ldots\ldots \text{(A)}$$

This remarkable equation will be of the greatest use in the theory of the figure of the celestial bodies. We may present it under more commodious forms in different circumstances; conceive, for example, from the origin of coordinates we draw to the point attracted a radius which we call ϱ; let θ be the angle which this radius makes with the axis of x, and ϖ the angle which the plane formed by ϱ and this axis makes with the plane of x, y; we shall have

$$x = \varrho \cos.\theta;\ y = \varrho \sin.\theta \cos.\varpi;\ z = \varrho \sin.\theta \sin.\varpi;$$

whence we derive

$$\varrho = \sqrt{x^2+y^2+z^2};\ \cos.\theta = \frac{x}{\sqrt{x^2+y^2+z^2}};\ \tan.\varpi = \frac{z}{y};$$

thus we can obtain the partial differences of ϱ, θ, ϖ, relative to the variables x, y, z, and thence get the values of $\left(\frac{d^2V}{dx^2}\right)$, $\left(\frac{d^2V}{dy^2}\right)$, $\left(\frac{d^2V}{dz^2}\right)$

in partial differences of V relative to the variables ϱ, θ, ϖ. Since we shall often use these transformations of partial differences, it is useful here to lay down the principle of it. Considering V as a function of the variables x, y, z, and then of the variables ϱ, θ, ϖ, we have

$$\left(\frac{dV}{dx}\right) = \left(\frac{dV}{d\varrho}\right)\left(\frac{d\varrho}{dx}\right) + \left(\frac{dV}{d\theta}\right)\left(\frac{d\theta}{dx}\right) + \left(\frac{dV}{d\varpi}\right)\left(\frac{d\varpi}{dx}\right).$$

To get the partial differences $\left(\frac{d\varrho}{dx}\right)$, $\left(\frac{d\theta}{dx}\right)$, $\left(\frac{d\varpi}{dx}\right)$, we must make x alone vary in the preceding expressions of ϱ, cos. θ, tan. ϖ; differentiating therefore these expressions, we shall have

$$\left(\frac{d\varrho}{dx}\right) = \text{cos. } \theta; \quad \left(\frac{d\theta}{dx}\right) = -\frac{\text{sin. } \theta}{\varrho}; \quad \left(\frac{d\varpi}{dx}\right) = 0;$$

which gives

$$\left(\frac{dV}{dx}\right) = \text{cos. } \theta\left(\frac{dV}{d\varrho}\right) - \frac{\text{sin. } \theta}{\varrho}\left(\frac{dV}{d\theta}\right).$$

Thus we therefore get the partial difference $\left(\frac{dV}{dx}\right)$, in partial differences of the function V, taken relatively to the variables ϱ, θ, ϖ. Differentiating again this value of $\left(\frac{dV}{dx}\right)$, we shall have the partial difference $\left(\frac{d^2V}{dx^2}\right)$ in partial differences of V taken relatively to the variables ϱ, θ, ϖ. By the same process the values of $\left(\frac{d^2V}{dy^2}\right)$ and $\left(\frac{d^2V}{dz^2}\right)$ may be found.

In this way we shall transform equation (A) into the following one:

$$0 = \left(\frac{d^2V}{d\theta^2}\right) + \frac{\text{cos. } \theta.}{\text{sin. } \theta}\left(\frac{dV}{d\theta}\right) + \frac{\left(\frac{d^2V}{d\varpi^2}\right)}{\text{sin.}^2\theta} + \varrho\left(\frac{d^2.\varrho V}{d\varrho^2}\right); \quad \text{. . (B)}$$

And if we make cos. θ = m, this last equation will become

$$0 = \left(\frac{d.\left\{(1-m^2)\left(\frac{dV}{d\mu}\right)\right\}}{dm}\right) + \frac{\left(\frac{d^2V}{d\varpi^2}\right)}{1-m^2} + \varrho\left(\frac{d^2.\varrho V}{d\varrho^2}\right). \quad \text{(C)}$$

543. Suppose, however, that the spheroid is a spherical shell whose origin of coordinates is at the center; it is evident that V will only depend upon ϱ, and contain neither m nor ϖ; the equation (C) will therefore give

$$0 = \left(\frac{d^2.\varrho V}{d\varrho^2}\right);$$

whence by integration we get

$$V = A + \frac{B}{\varrho},$$

A and B being two arbitrary constants. We therefore have

$$-\left(\frac{dV}{d\varrho}\right) = \frac{B}{\varrho^2}.$$

$-\frac{dV}{d\varrho}$ expresses, by what precedes, the action of the spherical shell upon the point μ, decomposed along the radius ϱ and directed towards the center of the shell; but it is evident that the total action of the shell ought to be directed along this radius; $-\left(\frac{dV}{d\varrho}\right)$ expresses therefore the total action of the spherical shell upon the point μ.

First suppose this point placed within the shell. If it were at the center itself, the action of the shell would be nothing; we have therefore,

$$-\left(\frac{dV}{d\varrho}\right) = 0, \text{ or } \frac{B}{\varrho^2} = 0,$$

when $\varrho = 0$, which gives $B = 0$, and consequently $-\left(\frac{dV}{d\varrho}\right) = 0$, whatever ϱ may be; whence it follows that a point placed in the interior of the shell, suffers no action, or which comes to the same thing, it is equally attracted on all sides.

If the point μ is situated without the spherical shell, it is evident, supposing it infinitely distant from the center, that the action of the shell upon the point will be the same, as if all the mass of the shell were condensed at this center; calling, therefore M the mass of the shell, $-\left(\frac{dV}{d\varrho}\right)$ or $\frac{B}{\varrho^2}$ will become in this case equal to $\frac{M}{\varrho^2}$, which gives $B=M$; we have therefore generally relatively to exterior points,

$$-\left(\frac{dV}{d\varrho}\right) = \frac{M}{\varrho^2}$$

that is to say, the shell attracts them as if all its mass were collected at its center.

A sphere being a spherical shell, the radius of whose interior surface is nothing, we see that its attraction, upon a point placed at or above its surface, is the same as if its mass were collected at its center.

This result obtains for globes formed of concentric shells, varying in density from the center to the circumference according to any law whatever, for it is true for each of the shells: thus since the sun, the planets, and satellites may be considered nearly as globes of this nature, they attract exterior bodies very nearly as if their masses were collected into their centers of gravity. This is conformable with what has been found by

observations. Indeed the figure of the celestial bodies departs a little from the sphere, but the difference is very little, and the error which results from the preceding supposition is of the same order as this supposition relatively to points near the surface; and relatively to distant points, the error is of the same order as the product of this difference by the square of the ratio of the radii of the attracting bodies to their distances from the points attracted; for we know that the consideration alone of the distance of the points attracted, renders the error of the preceding supposition of the same order as the square of this ratio. The celestial bodies, therefore, attract one another very nearly as if their masses were collected at their centers of gravity, not only because they are very distant from one another relatively to their respective dimensions, but also because their figures differ very little from the sphere.

The property of spheres, by the law of Nature, of attracting as if their masses were condensed into their centers, is very remarkable, and we may be curious to learn whether it also obtains in other laws of attraction. For that purpose we shall observe, that if the law of gravity is such, that a homogeneous sphere attracts a point placed without it as if all its mass were collected at its center, the same result ought to obtain for a spherical shell of a constant thickness; for if we take from a sphere a spherical shell of a constant thickness, we form a new sphere of a smaller radius with the remainder, but which, like the former, shall have the property of attracting as if all its mass were collected at its center; but it is evident, that these two spheres can only have this common property, unless it also belongs to the spherical shell which forms their difference. The problem, therefore, is reduced to determine the laws of attraction according to which a spherical shell, of an infinitely small and constant thickness, attracts an exterior point as if all its mass were condensed into its center.

Let ϱ be the distance of the point attracted to the center of the spherical shell, u the radius of the shell, and d u its thickness. Let θ be the angle which the radius u makes with the straight line ϱ, ϖ the angle which the plane passing through the straight lines ϱ, u, makes with a fixed plane passing through ϱ, the element of the spherical shell will be $u^2 \, d\,u \, . \, d\,\varpi \, . \, d\,\theta \sin.\,\theta$. If we then call f the distance of this element from the point attracted, we shall have

$$f^2 = \varrho^2 - 2\,\varrho\,u \cos.\,\theta + u^2.$$

Represent by $\varphi'(f)$ the law of attraction to the distance f; the action of the shell's element upon the point attracted, decomposed parallel to ϱ and directed towards the center of the shell, will be

$$u^2 \, d\,u \,.\, d\,\varpi \sin.\,\theta \, \frac{\varrho - u \cos.\,\theta}{f} \, \varphi\,(f)\,;$$

but we have

$$\frac{\varrho - u \cos.\,\theta}{f} = \left(\frac{d\,f}{d\,\varrho}\right);$$

which gives to the preceding quantity this form

$$u^2 \, d\,u \,.\, d\,\varpi \sin.\,\theta \left(\frac{d\,f}{d\,\varrho}\right) \varphi\,(f)\,;$$

wherefore if we denote $\int d\,f\,\varphi\,(f)$ by $\varphi_{,}\,(f)$ we shall have the whole action of the spherical shell upon the point attracted, by means of the integral $u^2 \, d\,u \int d\,\varpi \, d\,\theta \sin.\,\theta \,.\, \varphi_{,}\,(f)$, differentiated relatively to ϱ, and divided by $d\,\varrho$.

This integral ought to be taken relatively to ϖ, from $\varpi = 0$ to ϖ equal to the circumference, and after this integration it becomes

$$2\,\pi\,u^2 \int d\,\theta \sin.\,\theta\,\varphi_{,}\,(f)\,;$$

If we differentiate the value of f relatively to θ, we shall have

$$d\,\theta \sin.\,\theta = \frac{f\,d\,f}{\varrho\,u}\,;$$

and consequently

$$2\,\pi \,.\, u^2 \, d\,u \int d\,\theta \sin.\,\theta \,.\, \varphi_{,}\,(f) = 2\,\pi \,.\, \frac{u\,d\,u}{\varrho} \int f\,d\,f \,.\, \varphi_{,}\,(f).$$

The integral relative to θ ought to be taken from $\theta = 0$ to $\theta = \pi$, and at these two limits we have $f = \varrho - u$, and $f = \varrho + u$; thus the integral relative to f must be taken from $f = \varrho - u$ to $f = \varrho + u$; let therefore $\int f\,d\,f \,.\, \varphi_{,}\,(f) = \psi\,(f)$, we shall have

$$\frac{2\,\pi \,.\, u\,d\,u}{\varrho} \int f\,d\,f\,\varphi_{,}\,(f) = \frac{2\,\pi \,.\, u\,d\,u}{\varrho} \{\psi\,(\varrho + u) - \psi\,(\varrho - u)\}.$$

The coefficient of $d\,\varrho$, in the differential of the second member of this equation, taken relatively to ϱ, will give the attraction of the spherical shell upon the point attracted; and it is easy thence to conclude that in nature where $\varphi\,(f) = \frac{1}{f^2}$ this attraction is equal to

$$\frac{4\,\pi \,.\, u^2\,d\,u}{\varrho^2}.$$

That is to say, that it is the same as if all the mass of the spherical shell were collected at its center. This furnishes a new demonstration of the property already established of the attraction of spheres.

Let us determine $\varphi\,(f)$ on the condition that the attraction of the shell is the same as if its mass were condensed into its center. This mass is equal to $4\,\pi \,.\, u^2\,d\,u$, and if it were condensed into its center, its action

upon the point attracted would be $4\pi . u^2 d u . \varphi(\varrho)$; we shall therefore have

$$2\pi . u d u \left(\frac{d . \left\{ \frac{1}{\varrho} . (\psi[\varrho+u] - \psi[\varrho-u]) \right\}}{d\varrho} \right) = 4\pi u^2 d u \varphi(\varrho); \quad (D)$$

integrating relatively to ϱ, we shall get

$$\psi(\varrho + u) - \psi(\varrho - u) = 2\varrho u \int d\varrho . \varphi(\varrho) + \varrho U,$$

U being a function of u and constants, added to the integral $2u \int d\varrho \varphi(\varrho)$. If we represent $\psi(\varrho + u) - \psi(\varrho - u)$ by R, we shall have by differentiating the preceding equation

$$\left(\frac{d^2 R}{d\varrho^2}\right) = 4 u \varphi(\varrho) + 2\varrho u . \frac{d\varphi(\varrho)}{d\varrho}$$

$$\left(\frac{d^2 R}{d u^2}\right) = \varrho \left(\frac{d^2 U}{d u^2}\right);$$

But we have, by the nature of the function R,

$$\left(\frac{d^2 R}{d\varrho^2}\right) = \frac{d^2 R}{d u^2};$$

wherefore

$$2 u \left\{ 2\varphi(\varrho) + \varrho \frac{d\varphi(\varrho)}{d\varrho} \right\} = \varrho \left(\frac{d^2 U}{d u^2}\right);$$

or

$$\frac{2\varphi(\varrho)}{\varrho} + \frac{d . \varphi(\varrho)}{d\varrho} = \frac{1}{2u} \left(\frac{d^2 U}{d u^2}\right).$$

Thus the first member of this equation being independent of u and the functions of ϱ, each of its members must be equal to an arbitrary which we shall designate by 3 A; we therefore have

$$\frac{2\varphi\varrho}{\varrho} + \frac{d\varphi\varrho}{d\varrho} = 3 A;$$

whence in integrating we derive

$$\varphi\varrho = A\varrho + \frac{B}{\varrho^2}$$

B being a new arbitrary constant. All the laws of attraction in which a sphere acts upon an exterior point placed at the distance ϱ from its center, as if all the mass were condensed into its center, are therefore comprised in the general formula

$$A\varrho + \frac{B}{\varrho^2};$$

it is easy to see in fact that this value satisfies equation (D) whatever may be A and B.

If we suppose $A = 0$, we shall have the law of nature, and we see that

in the infinity of laws which render attraction very small at great distances, that of nature is the only one in which spheres have the property of acting as if their masses were condensed into their centers.

This law is also the only one in which a body placed within a spherical shell, every where of an equal thickness, is equally attracted on all sides. It results from the preceding analysis that the attraction of the spherical shell, whose thickness is d u, upon a point placed in its interior, has the expression

$$2\pi u^2 d u \left(\frac{d . \frac{1}{\varrho} \{\psi (u + \varrho) - \psi (u - \varrho)\}}{d \varrho} \right).$$

To make this function nothing, we must have

$$\psi (u + \varrho) - \psi (u - \varrho) = \varrho . U,$$

U being a function of u independent of ϱ, and it is easy to see that this obtains in the law of nature, where $\varphi (f) = \frac{B}{\varrho^2}$. But to show that it takes place only in this law, we shall denote by $\psi' (f)$ the difference of $\psi (f)$ divided by d f, we shall also denote by $\psi'' (f)$ the difference of $\psi' (f)$ divided by d f, and so on; thus we shall get, by differentiating twice successively, the preceding equation relatively to ϱ,

$$\psi'' (u + \varrho) - \psi'' (u - \varrho) = 0.$$

This equation obtaining whatever may be u and ϱ, it thence results that $\psi'' (f)$ ought to be equal to a constant whatever f may be, and that therefore $\psi''' (f) = 0$. But, by what precedes,

$$\psi (f) = f . \varphi, (f),$$

whence we get

$$\psi''' (f) = 2 \varphi (f) + f \varphi' (f);$$

we therefore have

$$0 = 2 \varphi (f) + f \varphi' (f);$$

which gives by integration

$$\varphi (f) = \frac{B}{f^2},$$

and consequently the law of nature.

554. Let us resume the equation (C) of No. 541. If this equation could generally be integrated, we should have an expression of V, which would contain two arbitrary functions, which we should determine by finding the attraction of a spheroid, upon a point situated so as to facilitate this research, and by comparing this attraction with its general expression. But the integration of the equation (C) is possible only in some particular cases, such as that where the attracting spheroid is a sphere, which reduces this equation to ordinary differences; it is also possible in

the case where the attracting body is a cylinder whose base is an oval or curve returning into itself, and whose length is infinite. This particular case contains the theory of Saturn's ring.

Fix the origin of ϱ upon the same axis of the cylinder, which we shall suppose of an infinite length on each side of the origin. Naming ϱ' the distance of the point attracted from the axis; we shall have

$$\varrho' = \varrho \sqrt{1 - m^2}.$$

It is evident that V only depends on ϱ' and ϖ, since it is the same for all the points relatively to which these two variations are the same; it contains therefore only m inasmuch as ϱ' is a function of this variable. This gives

$$\left(\frac{dV}{dm}\right) = \left(\frac{dV}{d\varrho'}\right).\left(\frac{d\varrho'}{dm}\right) = -\frac{\varrho m}{\sqrt{1-m^2}}\left(\frac{dV}{d\varrho'}\right);$$

$$\left(\frac{d^2V}{dm^2}\right) = \frac{\varrho^2 m^2}{1-m^2}.\left(\frac{d^2V}{d\varrho'^2}\right) - \frac{\varrho}{(1-m^2)^{\frac{3}{2}}}.\left(\frac{dV}{d\varrho'}\right);$$

the equation (C) hence becomes

$$0 = \varrho'^2\left(\frac{d^2V}{d\varrho^2}\right) + \frac{d^2V}{d\varpi^2} + \varrho'\left(\frac{dV}{d\varrho'}\right);$$

whence by integrating we get

$$V = \varphi\{\varrho' \cos.\varpi + \varrho'\sqrt{-1}\sin.\varpi\} + \psi\{\varrho'\cos.\varpi - \varrho'\sqrt{-1}\sin.\varpi\};$$

$\varphi(\varrho')$ and $\psi(\varrho')$ being arbitrary functions of ϱ', which we can determine by seeking the attraction of the cylinder when ϖ is nothing and when it is a right angle.

If the base of the cylinder is a circle, V will be evidently a function of ϱ' independent of ϖ; the preceding equation of partial differences will thus become

$$0 = \varrho'^2\left(\frac{d^2V}{d\varrho'^2}\right) + \varrho'\left(\frac{dV}{d\varrho'}\right);$$

which gives by integrating,

$$-\left(\frac{dV}{d\varrho'}\right) = \frac{H}{\varrho'},$$

H being a constant. To determine it, we shall suppose ϱ' relatively to the radius of the base of the cylinder extremely great, which supposition permits us to consider the cylinder as an infinite straight line. Let A be this base, and z the distance of any point whatever of the axis of the cylinder, to the point where this axis is met by ϱ'; the action of the cylinder considered as concentrated or condensed upon its axis, will be, parallel to ϱ', equal to

$$\int \frac{A\varrho'.dz}{(\varrho'^2 + z^2)^{\frac{3}{2}}},$$

the integral being taken from $z = -\infty$ to $z = \infty$; this reduces the integral to $\frac{2A}{\varrho'}$; which is the expression of $-\left(\frac{dV}{d\varrho'}\right)$ when ϱ' is very considerable. Comparing this with the preceding one we have $H = 2A$, and we see that whatever is ϱ', the action of the cylinder upon an exterior point, is $\frac{2A}{\varrho'}$.

If the attracted point is within a circular cylindrical shell, of a constant thickness, and infinite length, we shall have $-\left(\frac{dV}{d\varrho}\right) = \frac{H}{\varrho'}$; and since the attraction is nothing when the point attracted is upon the axis of the shell, we have $H = 0$, and consequently, a point placed in the interior of the shell is equally attracted on all sides.

545. We have thus determined the attraction of a sphere and of a spherical shell: let us now consider the attraction of spheroids terminated by surfaces of the second order.

Let x, y, z be the three rectangular coordinates of an element of the spheroid; designating d M this element, and taking for unity the density of the spheroid which we shall suppose homogeneous, we shall have

$$dM = dx.dy.dz.$$

Let a, b, c be the rectangular coordinates of the point attracted by the spheroid, and denote by A, B, C the attractions of the spheroid upon this point resolved parallel to the axes of x, y, z and directed to the origin of the coordinates.

It is easy to show that we have

$$A = \iiint \frac{(a-x)\,dx.dy.dz}{\{(a-x)^2 + (b-y)^2 + (c-z)^2\}^{\frac{3}{2}}};$$

$$B = \iiint \frac{(b-y)\,dx.dy.dz}{\{(a-x)^2 + (b-y)^2 + (c-z)^2\}^{\frac{3}{2}}};$$

$$C = \iiint \frac{(c-z)\,dx.dy.dz}{\{(a-x)^2 + (b-y)^2 + (c-z)^2\}^{\frac{3}{2}}}.$$

All these triple integrals ought to be extended to the entire mass of the spheroid. The integrations under this form present great difficulties, which we can often in part remove by transforming the differentials into others more convenient. This is the general principle of such transformations.

Let us consider the differential function $P\,dx.dy.dz$, P being any function whatever of x, y, z. We may suppose x a function of y and z and of a new variable p: let $\varphi(y, z, p)$ denote this function; in this case,

we shall have, making y and z constant, $dx = \beta . dp$, β being a function of y, z and p. The preceding differential will thus become $\beta . P . dp . dy . dz$; and to integrate it, we must substitute in P, for x, its value $\varphi (y, z, p)$.

In like manner we may suppose in this new differential, $y = \varphi' (z, p, q)$, q being a new variable, and $\varphi' (z, p, q)$ being any function of the three variables z, p and q. We shall have, considering z and p constant, $dy = \beta' dq$, β' being a function of z, p, q; the preceding differential will thus take this new form $\beta \beta' P . dp . dq . dz$, and to integrate it, we must substitute in β P for y its value $\varphi' (z, p, q)$.

Lastly we may suppose z equal to $\varphi'' (p, q, r)$, r being a new variable, and $\varphi'' (p, q, r)$ being any function whatever of p, q, r. We shall have, considering p and q constant, $dz = \beta'' dr$, β'' being a function of p, q, r; the preceding differential will thus become $\beta . \beta' . \beta'' . P . dp . dq . dr$ and to integrate it, we must substitute in $\beta . \beta' . P$ for z its value $\varphi'' (p, q, r)$. The proposed differential function is thence transformed to another relative to the three new variables p, q, r, which are connected with the preceding by the equations

$$x = \varphi (y, z, p); \quad y = \varphi' (z, p, q); \quad z = \varphi'' (p, q, r).$$

It only remains to derive from these equations the values of β, β', β''. For that purpose we shall observe that they give x, y, z, in functions of the variables p, q and r; let us consider therefore the three first variables as functions of the three last. Since β'' is the coefficient of dr in the differential of z, taken by considering p and q constant, we have

$$\beta'' = \left(\frac{dz}{dp}\right).$$

β' is the coefficient of dq, in the differential of y taken on the supposition that p and z are constant; we shall therefore have β', by differentiating y on the supposition that p is constant, and by eliminating dr by means of the differential of z taken on the supposition that p is constant, and equating it to zero. Thus we shall have the two equations

$$dy = \left(\frac{dy}{dq}\right) dq + \left(\frac{dy}{dr}\right) dr$$

$$0 = \left(\frac{dz}{dq}\right) dq + \left(\frac{dz}{dr}\right) dr;$$

which give

$$dy = dq \times \frac{\left(\frac{dy}{dq}\right)\left(\frac{dz}{dr}\right) - \left(\frac{dy}{dr}\right)\left(\frac{dz}{dq}\right)}{\left(\frac{dz}{dr}\right)};$$

wherefore

$$\beta' = \frac{\left(\frac{dy}{dq}\right)\left(\frac{dz}{dr}\right) - \left(\frac{dy}{dr}\right)\left(\frac{dz}{dq}\right)}{\left(\frac{dz}{dr}\right)}.$$

Finally, β is the coefficient of d p, in the differential of x taken on the supposition that y and z are constant. This gives the three following equations

$$dx = \left(\frac{dx}{dp}\right) dp + \left(\frac{dx}{dq}\right) dq + \left(\frac{dx}{dr}\right) dr;$$

$$0 = \left(\frac{dy}{dp}\right) dp + \left(\frac{dy}{dq}\right) dq + \left(\frac{dy}{dr}\right) dr;$$

$$0 = \left(\frac{dz}{dp}\right) dp + \left(\frac{dz}{dq}\right) dq + \left(\frac{dz}{dr}\right) dr.$$

If we make

$$\begin{aligned} \varepsilon = &\left(\frac{dx}{dp}\right)\left(\frac{dy}{dq}\right)\left(\frac{dz}{dr}\right) - \left(\frac{dx}{dp}\right)\left(\frac{dy}{dr}\right)\left(\frac{dz}{dq}\right) \\ &+ \left(\frac{dx}{dq}\right)\left(\frac{dy}{dr}\right)\left(\frac{dz}{dp}\right) - \left(\frac{dx}{dq}\right)\left(\frac{dy}{dp}\right)\left(\frac{dz}{dr}\right) \\ &+ \left(\frac{dx}{dr}\right)\left(\frac{dy}{dp}\right)\left(\frac{dz}{dq}\right) - \left(\frac{dx}{dr}\right)\left(\frac{dy}{dq}\right)\left(\frac{dz}{dp}\right); \end{aligned}$$

we shall have

$$dx = \frac{\varepsilon\, dp}{\left(\frac{dy}{dq}\right)\left(\frac{dz}{dr}\right) - \left(\frac{dy}{dr}\right)\left(\frac{dz}{dq}\right)}.$$

which gives

$$\beta = \frac{\varepsilon}{\left(\frac{dy}{dq}\right)\left(\frac{dz}{dr}\right) - \left(\frac{dy}{dr}\right)\left(\frac{dz}{dq}\right)};$$

wherefore $\beta . \beta' . \beta'' = \varepsilon$ and the differential P. d x. d y. d z is transformed into ε. P. d p. d q. d r; P being here what P becomes when we substitute for x, y, z their values in p, q, r. The whole is therefore reduced to finding the variables p, q, r such that the integrations may become possible.

Let us transform the coordinates x, y, z into the radius drawn from the point attracted to the molecule, and into the angles which this radius makes with given straight lines or with given planes. Let r be this radius, p the angle which it forms with a straight line drawn through the attracted point parallel to the axis of x, and let q be the angle which

its projection makes on the plane of y, z with the axis of y; we shall have

$$x = a - r\cos.\,p;\quad y = b - r\sin.\,p\cos.\,q;\quad z = c - r\sin.\,p\sin.\,q.$$

We shall then find $\varepsilon = -r^2\sin.\,p$, and the differential $dx.dy.dz$ will thus be transformed into $-r^2\sin.\,p.dp.dq.dr$: this is the expression of the element dM, and since this expression ought to be positive in considering sin. p, dp, dq, dr as positive, we must change its sign, which amounts to changing that of ε, and to making $\varepsilon = r^2\sin.\,p$.

The expressions of A, B, C will thus become

$$A = \iiint dr\,dp\,dq.\sin.\,p\cos.\,p;$$
$$B = \iiint dr\,dp\,dq.\sin.^2 p\cos.\,p;$$
$$C = \iiint dr\,dp\,dq.\sin.^2 p\sin.\,q.$$

It is easy to arrive by another way at these expressions, by observing that the element dM may be supposed equal to a rectangular parallelopiped, whose dimensions are dr, r dp and r dq sin. p, and by then observing that the attraction of the element, parallel to the three axes of x, y, z is

$$\frac{dM}{r^2}\cos.\,p;\quad \frac{dM}{r^2}\sin.\,p\cos.\,q;\quad \frac{dM}{r^2}\sin.\,p\sin.\,q.$$

The triple integrals of the expressions of A, B, C must extend to the entire mass of the spheroid: the integrations relative to r are easy, but they are different according as the point attracted is within or without the spheroid; in the first case, the straight line which passing through the point attracted, traverses the spheroid, is divided into two parts by this point; and if we call r and r′ these parts, we shall have

$$A = \iint(r + r')\,dp\,dq.\sin.\,p\cos.\,p;$$
$$B = \iint(r + r')\,dp\,dq.\sin.^2 p\cos.\,p;$$
$$C = \iint(r + r')\,dp\,dq.\sin.^2 p\sin.\,q;$$

the integrals relative to p and q ought to be taken from p and q equal to zero, to p and q equal to two right angles.

In the second case, if we call r, the radius at its entering the spheroid, and r′ the radius at its farther surface, we shall have

$$A = \iint(r' - r)\,dp\,dq.\sin.\,p\cos.\,p;$$
$$B = \iint(r' - r)\,dp\,dq.\sin.^2 p\cos.\,q;$$
$$C = \iint(r' - r)\,dp\,dq.\sin.^2 p\sin.\,q.$$

The limits of the integrals relative to p and to q, must be fixed at the points where $r' - r = 0$, that is to say, where the radius r is a tangent to the surface of the spheroid.

546. Let us apply these results to spheroids bounded by surfaces of the

second order. The general equation of these surfaces, referred to the three orthogonal coordinates x, y, z is

$$0 = A + B.x + C.y + E.z + F.x^2 + H.xy + L.y^2 + M.xz + N.yz + O.z^2.$$

The change of the origin of coordinates introduces three arbitraries, since the position of this new origin relating to the first depends upon three arbitrary coordinates. The changing the position of the coordinates around their origin introduces three arbitrary angles; supposing, therefore, the coordinates of the origin and position in the preceding equation to change at the same time, we shall have a new equation of the second degree whose coefficients will be functions of the preceding coefficients and of the six arbitraries. If we then equate to zero the first powers of the coordinates, and their products two and two, we shall determine these arbitraries, and the general equation of the surfaces of the second order, will take this very simple form

$$x^2 + m y^2 + n z^2 = k^2;$$

it is under this form that we shall discuss it.

In these researches we shall only consider solids terminated by finite surfaces, which supposes m and n positive. In this case, the solid is an ellipsoid whose three semi-axes are what the variables x, y, z become when we suppose two of them equal to zero; we shall thus have k, $\frac{k}{\sqrt{m}}$, $\frac{k}{\sqrt{n}}$ for the three semi-axes respectively parallel to x, to y and to z. The solid content of the ellipsoid will be $\frac{4\pi . k^3}{3\sqrt{m\,n}}$.

If, however, in the preceding equation we substitute for x, y, z their values in p, q, r given by the preceding No., we shall have

$$r^2 (\cos.^2 p + m \sin.^2 p \cos.^2 q + n \sin.^2 p \sin.^2 q)$$
$$- 2r (a \cos. p + m b \sin. p \cos. q + n c \sin. p \sin. q) = k^2 - a^2 - m b^2 - n c^2;$$

so that if we suppose

$$I = a \cos. p + m b \sin. p \cos. q + n c \sin. p \sin. q;$$
$$L = \cos.^2 p + m \sin.^2 p \cos.^2 q + n \sin.^2 p \sin.^2 q;$$
$$R = I^2 + (k^2 - a^2 - m b^2 - n c^2). L$$

we shall have

$$r = \frac{I \pm \sqrt{R}}{L};$$

whence we obtain r′ by taking +, and r by taking —; we shall therefore have

$$r + r' = \frac{2I}{L}; \quad r' - r = \frac{2\sqrt{R}}{L}.$$

Hence relatively to the interior points of the spheroid, we get

$$A = 2\iint \frac{dp.dq.I.\sin.p.\cos.p}{L};$$

$$B = 2\iint \frac{dp.dq.I.\sin.^2 p.\cos.q}{L};$$

$$C = 2\iint \frac{dp.dq.I.\sin.^2 p.\sin.q}{L};$$

and relatively to the exterior points

$$A = 2\iint \frac{dp.dq.\sin.p.\cos.p\sqrt{R}}{L};$$

$$B = 2\iint \frac{dp.dq.\sin.^2 p\cos.q\sqrt{R}}{L};$$

$$C = 2\iint \frac{dp.dq.\sin.^2 p\sin.q\sqrt{R}}{L};$$

the three last integrals being to be taken between the two limits which correspond to $R = 0$.

547. The expressions relative to the interior points being the most simple, we shall begin with them. First, we shall observe that the semi-axis k of the spheroid does not enter the values of I and L; the values of A, B, C are consequently independent; whence it follows that we may augment at pleasure, the shells of the spheroid which are above the point attracted, without changing the attraction of the spheroid upon this point, provided the values of m and n are constant. Thence results the following theorem.

A point placed within an elliptic shell whose interior and exterior surfaces are similar and similarly situated, is equally attracted on all sides.

This theorem is an extension of that which we have demonstrated in 542, relative to a spherical shell.

Let us resume the value of A. If we substitute for I and L their values, it will become

$$A = 2\iint \frac{dp.dq.\sin.p.\cos.p.(a\cos.p + mb\sin.p\cos.q + nc\sin.p\sin.q)}{\cos.^2 p + m\sin.^2 p\cos.^2 q + n\sin.^2 p\sin.^2 q}.$$

Since the integrals relative to p and q, must be taken from p and q equal to zero, to p and q equal to two right angles, it is clear we have generally $\int P\,dp.\cos.p = 0$, P being a rational function of sin. p and of cos.^{2} p; because the value of p being taken at equal distances greater and less than the right angle, the corresponding values of P. cos. p are equal and have contrary signs; thus we have

$$A = 2a\iint \frac{dp.dq.\sin.p\cos.^2 p}{\cos.^2 p + m\sin.^2 p\cos^2 q + n\sin^2 p\sin.^2 q}.$$

If we integrate relatively to q from q = 0 to q = two right angles, we shall find

$$A = \frac{2a\pi}{\sqrt{mn}} \int \frac{dp \cdot \sin. p \cos.^2 p}{\sqrt{\left(1 + \frac{1-m}{m} \cos.^2 p\right) \cdot \left(1 + \frac{1-n}{n} \cos.^2 p\right)}};$$

an integral which must be taken from cos. p = 1 to cos. p = — 1. Let cos. p = x, and call M the entire mass of the spheroid; we shall have by 545, $M = \frac{4\pi \cdot k^3}{\sqrt{mn}}$ and consequently $\frac{4\pi}{\sqrt{mn}} = \frac{3M}{k^3}$; we shall therefore have

$$A = \frac{3aM}{k^3} \int \frac{x^2 dx}{\sqrt{\left(1 + \frac{1-m}{m} \cdot x^2\right) \cdot \left(1 + \frac{1-n}{n} \cdot x^2\right)}};$$

which must be taken from x = 0, to x = 1.

Integrating in the same manner the expressions of B, C we shall reduce them to simple integrals; but it is easier to get these integrals from the preceding expression of A. For that purpose, we shall observe that this expression may be considered as a function of a and of the squares k^2, $\frac{k^2}{m}$, $\frac{k^2}{n}$ of the semi-axes of the spheroid, parallel to the coordinates a, b, c of the point attracted; calling therefore k'^2 the square of the semi-axis parallel to b, and consequently $k'^2 \cdot m$, and $k'^2 n$ the squares of the two other semi-axes, B will be a similar function of b, k'^2, $k'^2 m$, $k'^2 \frac{m}{n}$; thus to get B we must change in the expression of A, a into b, k into k' or $\frac{k}{\sqrt{m}}$, m into $\frac{1}{m}$, and n into $\frac{n}{m}$, which gives

$$B = \frac{3bM}{k^3} \int \frac{m^{\frac{3}{2}} \cdot x^2 dx}{\sqrt{\{1 + (m-1)x^2\} \cdot \left\{1 + \frac{m-n}{n} \cdot x^2\right\}}}.$$

Let

$$x = \frac{t}{\sqrt{m + (1-m) \cdot t^2}};$$

we shall have

$$B = \frac{3bM}{k^3} \int \frac{t^2 dt}{\left(1 + \frac{1-m}{m} \cdot t^2\right)^{\frac{5}{2}} \left(1 + \frac{1-n}{n} \cdot t^2\right)^{\frac{1}{2}}};$$

an integral relative to t which must be taken, like the integral relative to x

from $t = 0$ to $t = 1$, because $x = 0$ gives $t = 0$ and $x = 1$, gives $t = 1$

Hence it follows that if we suppose

$$\frac{1-m}{m} = \lambda^2; \quad \frac{1-n}{n} = \lambda'^2; \quad F = \int \frac{x^2 dx}{\sqrt{(1+\lambda^2 x^2).(1+\lambda'^2 x^2)}};$$

we shall have

$$B = \frac{3\,b\,M}{k^3}\left(\frac{d.\lambda F}{d\lambda}\right).$$

If we change in this expression, b into c, λ into λ' and reciprocally, we shall have the value of C. The attractions A, B, C of the spheroid, parallel to its three axes are thus given by the following formulas

$$A = \frac{3\,a\,M}{k^3}.F; \quad B = \frac{3\,b\,M}{k^3}\left(\frac{d.\lambda F}{d\lambda}\right); \quad C = \frac{3\,c\,M}{k^3}\left(\frac{d.\lambda' F}{d\lambda'}\right).$$

We may observe that these expressions obtaining for all the interior points, and consequently for those infinitely near to the surface, they also hold good for the points of the surface.

The determination of the attractions of a spheroid thus depends only on the value of F; but although this value is only a definite integral, it has, however, all the difficulty of indefinite integrals when λ and λ' are indeterminate, for if we represent this definite integral, taken from $x = 0$ to $x = 1$, by $\varphi(\lambda^2, \lambda'^2)$, it is easy to see that the indefinite integral will be $x^3 \varphi(\lambda x^2, \lambda'^2 x^2)$, so that the first being given, the second is likewise given. The indefinite integral is only possible in itself when one of the quantities λ, λ' is nothing, or when they are equal: in these two cases, the spheroid is an ellipsoid of revolution, and k will be its semi-axis of revolution if λ and λ' are equal. In this last case we have

$$F = \int \frac{x^2 dx}{1+\lambda^2 x^2} = \frac{1}{\lambda^3}\{\lambda - \text{tan.}^{-1}\lambda\}.$$

To get the partial differences $\left(\frac{d.\lambda F}{d\lambda}\right)$, $\left(\frac{d.\lambda' F}{d\lambda'}\right)$, which enter the expressions of B, C, we shall observe that

$$dF = \frac{d\lambda}{\lambda}\left(\frac{d.\lambda F}{d\lambda}\right) + \frac{d\lambda'}{\lambda'}\left(\frac{d.\lambda' F}{d\lambda'}\right) - F\left(\frac{d\lambda}{\lambda} + \frac{d\lambda'}{\lambda'}\right).$$

but when $\lambda = \lambda'$, we have

$$\left(\frac{d.\lambda F}{d\lambda}\right) = \left(\frac{d.\lambda' F}{d\lambda'}\right); \quad \frac{d\lambda}{\lambda} = \frac{d\lambda'}{\lambda'};$$

wherefore

$$\left(\frac{d.\lambda F}{d\lambda}\right).d\lambda = \tfrac{1}{2}.\lambda\, dF + F\,d\lambda = \frac{1}{2\lambda}\,d.\lambda^2 F.$$

Substituting for F its value, we shall have

$$\left(\frac{d.\lambda F}{d\lambda}\right) = \frac{1}{2\lambda^3}\left(\text{tan.}^{-1}\lambda - \frac{\lambda}{1+\lambda^2}\right).$$

we shall therefore have relatively to ellipsoids of revolution, whose semi-axis of revolution is k,

$$A = \frac{3\,a.\,M}{k^3.\,\lambda^3}(\lambda - \text{tan.}^{-1}\lambda);$$

$$B = \frac{3\,b.\,M}{2\,k^3.\,\lambda^3}\left(\text{tan.}^{-1}\lambda - \frac{\lambda}{1+\lambda^2}\right);$$

$$C = \frac{3\,c.\,M}{2\,k^3.\,\lambda^3}\left(\text{tan.}^{-1}\lambda - \frac{\lambda}{1+\lambda^2}\right).$$

548. Now let us consider the attraction of spheroids upon an exterior point. This research presents greater difficulties than the preceding because of the radical $\surd R$ which enters the differential expressions, and which under this form renders the integrations impossible. We may render them possible by a suitable transformation of the variables of which they are functions; but instead of that method, let us use the following one, founded solely upon the differentiation of functions.

If we designate by V the sum of all the elements of the spheroid divided by their respective distances from the point attracted, and x, y, z the coordinates of the element d M of the spheroid, and a, b, c those of the point attracted, we shall have

$$V = \int \frac{d\,M}{\surd\left((a-x)^2 + (b-y)^2 + (c-z)^2\right)}.$$

Then designating, as above, by A, B, C the attractions of the spheroid parallel to the axes of x, y, z, and directed towards their origin, we shall have

$$A = \int \frac{(a-x).\,d\,M}{\{(a-x)^2 + (b-y)^2 + (c-z)^2\}^{\frac{3}{2}}} = -\left(\frac{d\,V}{d\,a}\right).$$

In like manner we get

$$B = -\left(\frac{d\,V}{d\,b}\right),\ C = -\left(\frac{d\,V}{d\,c}\right);$$

whence it follows that if we know V, it will be easy thence to obtain by differentiation alone, the attraction of a spheroid parallel to any straight line whatever, by considering this straight line as one of the rectangular coordinates of the point attracted; a remark we have already made in 541.

The preceding value of V, reduced into a series, becomes

$$V = \int \frac{d\,M}{\surd(a^2+b^2+c^2)}\left\{1 + \tfrac{1}{2}.\frac{2\,a\,x+2\,b\,y+2\,c\,z-x^2-y^2-z^2}{a^2+b^2+c^2} + \tfrac{3}{8}.\frac{(2\,a\,x+2\,b\,y+2\,c\,z-x^2-y^2-z^2)^2}{(a^2+b^2+c^2)^2} + \&c.\right\}.$$

This series is ascending relatively to the dimensions of the spheroid

and descending relatively to the coordinates of the point attracted. If we only retain the first term, which is sufficient when the attracted point is at a very great distance, we shall have

$$V = \frac{M}{\sqrt{a^2 + b^2 + c^2}};$$

M being the entire mass of the spheroid. This expression will be still more exact, if we place the origin of coordinates at the center of gravity of the sphere; for by the property of this center we have

$$\int x.\, dM = 0;\ \int y.\, dM = 0;\ \int z.\, dM = 0;$$

so that if we consider a very small quantity of the first order, the ratio of the dimensions of the spheroid to its distance from the point attracted, the equation

$$V = \frac{M}{\sqrt{a^2 + b^2 + c^2}}$$

will be exact to quantities nearly of the third order.

We shall now investigate a rigorous expression of V relatively to elliptic spheroids.

549. If we adopt the denominations of 544, we shall have

$$V = \int \frac{dM}{r} = \iiint r\, dr\, dp\, dq \sin. p = \tfrac{1}{2} \iint (r'^2 - r^2)\, dp\, dq . \sin. p.$$

Substituting for r and r' their values found in 544, we shall have

$$V = 2 \iint \frac{dp . dq \sin. p . I . \sqrt{R}}{L^2}.$$

Let us resume the values of A B, C relative to the exterior points, and given in 546,

$$A = 2 \iint \frac{dp . dq \sin. p \cos. p \sqrt{R}}{L};$$

$$B = 2 \iint \frac{dp . dq \sin.^2 p \cos. q \sqrt{R}}{L};$$

$$C = 2 \iint \frac{dp . dq \sin.^2 p \sin. q \sqrt{R}}{L}.$$

Since at the limits of the integrals, we have $\sqrt{R} = 0$, it is easy to see that by taking the first differences of V, A, B, C relatively to any of the six quantities a, b, c, k, m, n, we may dispense with regarding the variations of the limits; so that we have, for example,

$$\left(\frac{dV}{da}\right) = 2 \iint dp\, dq \sin. p \left(\frac{d. \frac{I \sqrt{R}}{L^2}}{da} \right);$$

for the integral

$$\int \frac{dp \sin. p\, I \sqrt{R}}{L^2}$$

is towards these limits, very nearly proportional to $R^{\frac{2}{2}}$, which renders equal to zero, its differential at these limits. Hence it is easy to see by differentiation that if for brevity we make

$$a\,A + b\,B + c\,C = F;$$

we shall have between the four quantities B, C, F, and V the following equation of partial differences,

$$\begin{aligned} 0 = {} & \frac{a^2 + b^2 + c^2 - k^2}{2} . k . \left\{ \left(\frac{d\,V}{d\,k}\right) - \left(\frac{d\,F}{d\,k}\right) \right\} + k^2 (V - F) \\ & + k^2 \frac{m - 1}{m} . b . \left\{ \left(\frac{d\,F}{d\,b}\right) - \tfrac{1}{2} \left(\frac{d\,V}{d\,b}\right) - B \right\} \\ & + k^2 \frac{n - 1}{n} . c \left\{ \left(\frac{d\,F}{d\,c}\right) - \tfrac{1}{2} \left(\frac{d\,V}{d\,c}\right) - C \right\} \\ & - k^2 (m - 1) . \left(\frac{d\,F}{d\,m}\right) - k^2 (n - 1) . \left(\frac{d\,F}{d\,n}\right) . \quad (1) \end{aligned}$$

We may eliminate from this equation, the quantities B, C, F by means of their values

$$-\left(\frac{d\,V}{d\,b}\right), -\left(\frac{d\,V}{d\,c}\right) \text{ and } -a\left(\frac{d\,V}{d\,a}\right) - b\left(\frac{d\,V}{d\,a}\right) - c\left(\frac{d\,V}{d\,c}\right).$$

We shall thus get an equation of partial differences in V alone. Let therefore

$$V = \frac{4\,\pi . k^3}{3\,\sqrt{m\,n}} . v = M . v,$$

M being by 545, the mass of the elliptic spheroid; and for the variables m and n let us here introduce θ and ϖ which shall be such that we have

$$\theta = \frac{1 - m}{m} . k^2; \quad \varpi = \frac{1 - n}{n} . k^2;$$

θ will be the difference of the square of the axis of the spheroid parallel to y and the square of the axis parallel to x; ϖ will be the difference of the square of the axis of z and the square of the axis of x; so that if we take for the axis of x, the smallest of the three axes of the spheroid, $\sqrt{\theta}$ and $\sqrt{\varpi}$ will be its two excentricities. Thus we shall have

$$k\left(\frac{d\,V}{d\,k}\right) = M . \left\{ 2\,\theta \left(\frac{d\,v}{d\,\theta}\right) + 2\,\varpi \left(\frac{d\,v}{d\,\varpi}\right) + k\left(\frac{d\,v}{d\,k}\right) + 3\,v \right\};$$

$$\left(\frac{d\,V}{d\,m}\right) = -M \left\{ \frac{k^2}{m^2} . \left(\frac{d\,v}{d\,\theta}\right) + \frac{v}{2\,m} \right\};$$

$$\left(\frac{d\,V}{d\,m}\right) = -M \left\{ \frac{k^2}{n^2} \left(\frac{d\,v}{d\,\varpi}\right) + \frac{v}{2\,n} \right\};$$

V being considered in the first members of those equations as a function of a, b, c, k, m, n; and v being considered in their second members as a function of a, b, c, θ, ϖ, k.

If we make

$$Q = a\left(\frac{d\,v}{d\,a}\right) + b\left(\frac{d\,v}{d\,b}\right) + c\left(\frac{d\,v}{d\,c}\right);$$

we shall have $F = -M\,Q$, and we shall get the values of $k\left(\frac{d\,F}{d\,k}\right)$, $\left(\frac{d\,F}{d\,m}\right)$, $\left(\frac{d\,F}{d\,n}\right)$ by changing in the preceding values of $k\left(\frac{d\,V}{d\,k}\right)$, $\left(\frac{d\,V}{d\,m}\right)$, $\left(\frac{d\,V}{d\,n}\right)$, v into $-Q$. Moreover V and F are homogeneous functions in a, b, c, k, $\sqrt{\theta}$, $\sqrt{\varpi}$ of the second dimension, for V being the sum of the elements of the spheroid, divided by their distances from the point attracted, and each element being of three dimensions, V is necessarily of two dimensions, as also F which has the same number of dimensions as V; v and Q are therefore homogeneous functions of the same quantities and of the dimension -1; thus we shall have by the nature of homogeneous functions,

$$a.\left(\frac{d\,v}{d\,a}\right) + b.\left(\frac{d\,v}{d\,b}\right) + c.\left(\frac{d\,v}{d\,c}\right) + 2\theta.\left(\frac{d\,v}{d\,\theta}\right) + 2\varpi.\left(\frac{d\,v}{d\,\varpi}\right) + k\left(\frac{d\,v}{d\,k}\right) = -v;$$

an equation which may be put under this form

$$2\theta\left(\frac{d\,v}{d\,\theta}\right) + 2\varpi\left(\frac{d\,v}{d\,\varpi}\right) + k\left(\frac{d\,v}{d\,k}\right) = -v - Q.$$

We shall have in like manner

$$a.\left(\frac{d\,Q}{d\,a}\right) + b\left(\frac{d\,Q}{d\,b}\right) + c\left(\frac{d\,Q}{d\,c}\right) + 2\theta\left(\frac{d\,Q}{d\,\theta}\right) + 2\varpi\left(\frac{d\,Q}{d\,\varpi}\right) + k\left(\frac{d\,Q}{d\,k}\right) = -Q;$$

then, if in equation (1) we substitute for V, F and their partial differences; if moreover we substitute $\frac{k^2}{k^2+\theta}$ for m and $\frac{k^2}{k^2+\varpi}$ for n, we shall have

$$0 = (a^2+b^2+c^2)\left[v + \tfrac{1}{2}Q - \tfrac{1}{2}\left\{a\left(\frac{d\,Q}{d\,a}\right) + b\left(\frac{d\,Q}{d\,b}\right) + c\left(\frac{d\,Q}{d\,c}\right)\right\}\right]$$
$$+ \theta^2\left(\frac{d\,Q}{d\,\theta}\right) + \varpi^2\left(\frac{d\,Q}{d\,\varpi}\right) - \frac{k^3}{2}\left(\frac{d\,Q}{d\,k}\right) + \tfrac{1}{2}(\theta+\varpi).\,Q \qquad (2)$$
$$b\theta.\left(\frac{d\,Q}{d\,b}\right) + c\varpi\left(\frac{d\,Q}{d\,c}\right) - \tfrac{1}{2}b\theta\left(\frac{d\,v}{d\,b}\right) - \tfrac{1}{2}c\varpi\left(\frac{d\,v}{d\,c}\right).$$

550. Conceive the function v expanded into a series ascending relatively to the dimensions k, $\sqrt{\theta}$, $\sqrt{\varpi}$ of the spheroid, and consequently descending relatively to the quantities a, b, c: this series will be of the following form:

$$v = U^{(0)} + U^{(1)} + U^{(2)} + U^{(3)} + \&c.;$$

$U^{(0)}$, $U^{(1)}$, &c. being homogeneous functions of a, b, c, k, $\sqrt{\theta}$, $\sqrt{\varpi}$, and separately homogeneous relatively to the three first and to the three last

of these six quantities; the dimensions relative to the three first always decreasing, and the dimensions relative to the three last increasing continually. These functions being of the same dimension as v, are all of the dimension — 1.

If we substitute in equation (2) for v its preceding expanded value; if we call s the dimension of $U^{(i)}$ in k, $\sqrt{\theta}$, $\sqrt{\varpi}$, and consequently $-s-1$ its dimension in a, b, c; if in like manner we name s′ the dimension of $U^{(i+1)}$ in k, $\sqrt{\theta}$, $\sqrt{\varpi}$, and consequently $-s'-1$ its dimension in a, b, c; if we then consider that by the nature of homogeneous functions we have

$$a\left(\frac{dU^{(i)}}{da}\right)+b\left(\frac{dU^{(i)}}{db}\right)+c\left(\frac{dU^{(i)}}{dc}\right)=-(s+1)U^{(i)};$$

$$a\left(\frac{dU^{(i+1)}}{da}\right)+b\left(\frac{dU^{(i+1)}}{db}\right)+c\left(\frac{dU^{(i+1)}}{dc}\right)=-(s'+1)U^{(i+1)};$$

we shall have, by rejecting the terms of a dimension superior in k, $\sqrt{\theta}$, $\sqrt{\varpi}$ to that of the terms which we retain,

$$U^{(i+1)}=\frac{\left\{\begin{array}{l}\frac{1}{2}(s+1).k^2.\left(\frac{dU^{(i)}}{dk}\right)-(s+1).\theta^2\left(\frac{dU^{(i)}}{d\theta}\right)\\ -(s+1).\varpi^2.\left(\frac{dU^{(i)}}{d\varpi}\right)-\frac{s+1}{2}.(\theta+\varpi).U^{(i)}\\ -(s+\frac{3}{2})b\,\theta.\left(\frac{dU^{(i)}}{db}\right)-(s+\frac{3}{2}).c\varpi.\left(\frac{dU^{(i)}}{dc}\right)\end{array}\right\}}{s'.\frac{s'+3}{2}.(a^2+b^2+c^2)} \quad \ldots \quad (3)$$

This equation gives the value of $U^{(i+1)}$, by means of $U^{(i)}$ and of its partial differences; but we have

$$U^{(0)}=\frac{1}{(a^2+b^2+c^2)^{\frac{1}{2}}};$$

since, retaining only the first term of the series, we have found in 548, that

$$V=\frac{M}{(a^2+b^2+c^2)^{\frac{1}{2}}}.$$

Substituting therefore this value of $U^{(0)}$ in the preceding formula, we shall get that of $U^{(1)}$; by means of that of $U^{(1)}$ we shall have that of $U^{(2)}$ and so on. But it is remarkable that none of these quantities contains k: for it is evident by the formula (3) that $U^{(0)}$, not containing $U^{(1)}$, does not contain it; that $U^{(1)}$ not containing it, $U^{(2)}$ will not contain it, and so on; so that the entire series $U^{(0)}+U^{(1)}+$ &c. is independent of k, or which is the same thing $\left(\frac{dv}{dk}\right)=0$. The values of v, $-\left(\frac{dv}{da}\right)$, $-\left(\frac{dv}{db}\right)$,

$-\left(\frac{dv}{dc}\right)$, are therefore the same for all elliptic spheroids similarly situated, and which have the same excentricities $\sqrt{\theta}$, $\sqrt{\varpi}$; but $-M\left(\frac{dv}{da}\right)$, $-M\left(\frac{dv}{db}\right)$, $-M\left(\frac{dv}{dc}\right)$, express by 548, the attractions of the spheroid parallel to its three axes; therefore the attractions of different elliptic spheroids which have the same center, the same position of the axes and the same excentricities, upon an exterior point, are to one another as their masses.

It is easy to see by formula (3) that the dimensions of $U^{(0)}$, $U^{(1)}$, $U^{(2)}$, &c. in $\sqrt{\theta}$ and $\sqrt{\varpi}$, increase two units at a time, so that $s=2i$, $s'=2i+2$; moreover we have by the nature of homogeneous functions

$$\varpi\left(\frac{dU^{(i)}}{d\varpi}\right) = i.U^{(i)} - \theta\left(\frac{dU^{(i)}}{d\theta}\right);$$

this formula will therefore become

$$U^{(i+1)} = \frac{\left\{\begin{array}{l}(2i+1).\theta.(\varpi-\theta)\left(\frac{dU^{(i)}}{d\theta}\right) - (2i+\frac{3}{2}).b\theta\left(\frac{dU^{(i)}}{db}\right)\\ -(2i+\frac{3}{2})c\varpi\left(\frac{dU^{(i)}}{dc}\right) - \frac{1}{2}(2i+1).\{\theta+(2i+1).\varpi\}U^{(i)}\end{array}\right\}}{(i+1)(2i+5)(a^2+b^2+c^2)}. \quad (4)$$

By means of this equation, we shall have the value of v in a series very convergent, whenever the excentricities $\sqrt{\theta}$, $\sqrt{\varpi}$ are very small, or when the distance $\sqrt{a^2+b^2+c^2}$ of the point attracted from the center of the spheroid is very great relatively to the dimensions of the spheroid.

If the spheroid is a sphere, we shall have $\theta = 0$, and $\varpi = 0$, which give $U^{(1)} = 0$, $U^{(2)} = 0$, &c.; wherefore

$$v = U^{(0)} = \frac{1}{\sqrt{a^2+b^2+c^2}};$$

and

$$V = \frac{M}{\sqrt{a^2+b^2+c^2}};$$

whence it follows that the value of V is the same as if all the mass of the sphere were condensed into its center, and that thus, a sphere attracts any exterior point, as if all its whole mass were condensed into its center; a result already obtained in 542.

551. The property of the function of v being independent of k, furnishes the means of reducing its value to the most simple form of which it is susceptible; for since we can make k vary at pleasure without changing this value, provided the spheroid retain the same excentricities, $\sqrt{\theta}$ and

$\sqrt{\varpi}$, we may suppose k such that the spheroid shall be infinitely flattened, or so contrived that its surface pass through the point attracted. In these two cases, the research of the attractions of the spheroid is rendered more simple; but since we have already determined the attractions of elliptic spheroids, upon points at the surface, we shall now suppose k such that the surface of the spheroid passes through the point of attraction.

If we call k', m', n' relatively to this new spheroid what in 545, we named k, m, n relatively to the spheroid we there considered; the condition that the point attracted is at the surface, and that also a, b, c are the coordinates of a point of the surface, will give

$$a^2 + m' b^2 + n' c^2 = k^2;$$

and since we suppose the excentricities $\sqrt{\theta}$ and $\sqrt{\varpi}$ to remain the same, we shall have

$$\frac{1 - m'}{m'} . k'^2 = \theta; \quad \frac{1 - n'}{n'} . k'^2 = \varpi;$$

whence we obtain

$$m' = \frac{k'^2}{k'^2 + \theta}; \quad n' = \frac{k'^2}{k'^2 + \varpi};$$

we shall therefore have to determine k', the equation

$$a^2 + \frac{k'^2}{k'^2 + \theta} b^2 + \frac{k'^2}{k'^2 + \varpi} . c^2 = k'^2. \quad . \quad . \quad . \quad (5)$$

It is easy hence to conclude that there is only one spheroid whose surface passes through the point attracted, θ and ϖ remaining the same. For if we suppose, which we always may do, that θ and ϖ are positive, it is clear that augmenting in the preceding equation, k'^2 by any quantity which we may consider an aliquot part of k'^2, each of the terms of the first member of this equation, will increase in a less ratio than k'^2; therefore if in the first state of k'^2, there subsist an equality between the two members of this equation, this equality will no longer obtain in the second state; whence it follows that k'^2 is only susceptible of one real and positive value.

Let M' be the mass of the new spheroid, and A', B', C' its attractions parallel to the axes of a, b, c; if we make

$$\frac{1 - m'}{m'} = \lambda^2; \quad \frac{1 - n'}{n} = \lambda'^2$$

$$F = \int \frac{x^2 dx}{\sqrt{(1 + \lambda^2 . x^2) . (1 + \lambda'^2 . x^2)}};$$

by 547, we shall have

$$A' = \frac{3 a M' F}{k'^3}; \quad B' = \frac{3 b M'}{k'^3} \left(\frac{d . \lambda F}{d \lambda}\right); \quad C' = \frac{3 c M'}{k'^3} \left(\frac{d . \lambda' F}{d \lambda'}\right).$$

Changing in these values of A', B', C', M' into M, we shall have by the preceding No., the values of A, B, C relatively to the first spheroid but the equations

$$\frac{1-m'}{m'}\cdot k'^2=\theta;\quad \frac{1-n'}{n'}\cdot k'^2=\varpi,$$

give

$$\lambda^2=\frac{\theta}{k'^2};\quad \lambda'^2=\frac{\varpi}{k'^2};$$

k'^2 being given by equation (5) which we may put under this form

$$0=k'^6-(a^2+b^2+c^2-\theta-\varpi)\,k'^4-\{(a^2+c^2)\,\theta+(a^2+b^2)\,\varpi-\theta\varpi\}\,k'^2-a^2.\theta\varpi;$$

we shall therefore have

$$A=\frac{3\,a\,M}{k'^3}\cdot F;\quad B=\frac{3\,b\,M}{k'^3}\left(\frac{d.\lambda F}{d\lambda}\right);\quad C=\frac{3\,c\,M}{k'^3}\left(\frac{d.\lambda' F}{d\lambda'}\right).$$

These values obtain relatively to all points exterior to the spheroid, and to extend them to those of the surface, and even to the interior points we have only to change k' to k.

If the spheroid is one of revolution, so that $\theta=\varpi$, the formula (5) will give

$$2\,k'^2=a^2+b^2+c^2-\theta+\sqrt{(a^2+b^2+c^2-\theta)^2+4\,a^2.\theta};$$

and by 547, we shall have

$$A=\frac{3\,a\,M}{k'^3.\lambda^3}(\lambda-\tan.^{-1}\lambda)$$

$$B=\frac{3\,b\,M}{2\,k'^3.\lambda^3}\left(\tan.^{-1}\lambda-\frac{\lambda}{1+\lambda^2}\right)$$

$$C=\frac{3\,c\,M}{2\,k'^3.\lambda^3}\left(\tan.^{-1}\lambda-\frac{\lambda}{1+\lambda^2}\right).$$

Thus we have terminated the complete theory of the attractions of elliptic spheroids; for all that remains to be done is the integration of the differential expression of F, and this integration in the general sense is impossible, not only by known methods, but also in itself. The value of F cannot be expressed in finite terms by algebraic, logarithmic or circular quantities; or which is tantamount, by any algebraic function of quantities whose exponents are constant, nothing or variable. Functions of this kind being the only ones which can be expressed independently of the symbol $\int$, all the integrals which cannot be reduced to such functions, are impossible in finite terms.

If the elliptic spheroid is not homogeneous, and if it is composed of elliptic shells varying in position, excentricity and density according to any law whatever, we shall have the attraction of one of its shells, by de-

termining as above the difference of the attractions of two homogeneous elliptic spheroids, having the same density as the shell, one of which shall have for its surface the exterior surface of the shell, and the other the interior surface of the shell. Then summing this differential attraction, we shall have the attraction of the whole spheroid.

THE DEVELOPEMENT INTO SERIES, OF THE ATTRACTIONS OF ANY SPHEROIDS WHATEVER.

552. Let us consider generally the attractions of any spheroids whatever. We have seen in No. 547, that the expression V of the sum of the elements of the spheroid, divided by their distances from the attracted points, possesses the advantage of giving by its differentiation, the attraction of this spheroid parallel to any straight line whatever. We shall see moreover, when treating of the figure of the planets, that the attraction of their elements presents itself under this form in the equation of their equilibrium; thus we proceed particularly to investigate V.

Let us resume the equation of No. 548,

$$V = \int \frac{d\,M}{\sqrt{(a-x)^2 + (b-y)^2 + (c-z)^2}};$$

a, b, c being the coordinates of the point attracted; x, y, z those of the element d M of the spheroid; the origin of coordinates being in the interior of the spheroid. This integral must be taken relatively to the variables x, y, z, and its limits are independent of a, b, c; hence we shall find by differentiation,

$$0 = \left(\frac{d^2 V}{d a^2}\right) + \left(\frac{d^2 V}{d b^2}\right) + \left(\frac{d^2 V}{d c^2}\right); \quad \ldots\ldots\ldots \quad (1)$$

an equation already obtained in 541,

Let us transform the coordinates to others more commodious. For that purpose, let r be the distance of the point attracted from the origin of coordinates; θ the angle which the radius r makes with the axis of a; ϖ the angle which the plane formed by the radius and this axis, makes with the plane of the axis of a, and of b; we shall have

$$a = r \cos.\theta;\ b = r \sin.\theta \cos.\theta;\ c = r \sin.\theta \sin.\varpi.$$

If in like manner we name R, θ', ϖ' what r, θ, ϖ become relatively to the element d M of the spheroid; we shall have

$$x = R \cos.\theta';\ y = R \sin.\theta' \cos.\varpi';\ z = R \sin.\theta'. \sin.\varpi'.$$

Moreover, the element d M of the spheroid is equal to a rectangular parallelopiped whose dimensions are d R, R d θ', R d ϖ' sin. θ', and con-

sequently it is equal to $\varrho . R^2 . dR . d\theta' . d\varpi' . \sin. \theta'$, ϱ being its density; we shall thus have

$$V = \iiint \frac{\varrho R^2 . dR . d\theta' . d\varpi' \sin. \theta'}{\sqrt{r^2 - 2rR\{\cos. \theta . \cos. \theta' + \sin. \theta \sin. \theta' \cos. (\varpi' - \varpi)\} + R^2}};$$

the integral relative to R must be taken from $R = 0$ to the value of R at the surface of the spheroid; the integral relative to ϖ' must be taken from $\varpi' = 0$ to ϖ' equal to the circumference; and the integral relative to θ' must be taken from $\theta' = 0$ to θ' equal to the semi-circumference. Differentiating this expression of V, we shall find

$$0 = \left(\frac{d^2 V}{d\theta^2}\right) + \frac{\cos. \theta}{\sin. \theta} \cdot \left(\frac{d V}{d\theta}\right) + \frac{\left(\frac{d^2 V}{d\varpi^2}\right)}{\sin.^2 \theta} + r\left(\frac{d^2 . r V}{d r^2}\right); \quad \ldots \quad (2)$$

an equation which is only equation (1) in another form.

If we make $\cos. \theta = m$, we may give it this form

$$0 = \left(\frac{d . \left\{(1 - m^2)\left(\frac{d V}{d m}\right)\right\}}{d m}\right) + \frac{\left(\frac{d^2 V}{d\varpi^2}\right)}{1 - m^2} + r\left(\frac{d^2 . r V}{d r^2}\right) \ldots (3)$$

We have already arrived at these several equations in 541.

553. First, let us suppose the point attracted to be exterior to the spheroid. If we wish to expand V into a series, it ought in this case, to descend relatively to powers of r, and consequently to be of this form

$$V = \frac{U^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \frac{U^{(2)}}{r^3} + \&c.$$

Substituting this value of V in equation (3) of the preceding No., the comparison of the same powers of r will give, whatever i may be

$$0 = \left(\frac{d . \left\{(1 - m^2)\left(\frac{d U^{(i)}}{d m}\right)\right\}}{d m}\right) + \frac{\left(\frac{d^2 U^{(i)}}{d\varpi^2}\right)}{1 - m^2} + i(i + 1) U^{(i)}.$$

It is evident from the integral expression alone of V that $U^{(i)}$ is a rational and entire function of m, $\sqrt{1 - m^2} . \sin. \varpi$, and $\sqrt{1 - m^2} . \cos. \varpi$, depending upon the nature of the spheroid. When $i = 0$, this function becomes a constant; and in the case of $i = 1$, it assumes the form

$$H m + H' \sqrt{1 - m^2} . \sin. \varpi + H'' \sqrt{1 - m^2} . \cos. \varpi;$$

H, H', H'' being constants.

To determine generally $U^{(i)}$ call T the radical

$$\frac{1}{\sqrt{r^2 - 2Rr\{\cos. \theta \cos. \theta' + \sin. \theta \sin. \theta' \cos. (\varpi' - \varpi)\} + R^2}},$$

we shall have

$$0=\left(\frac{d\left\{(1-m^2)\left(\frac{dT}{dm}\right)\right\}}{dm}\right)+\frac{\left(\frac{d^2T}{d\varpi^2}\right)}{1-m^2}+r\left(\frac{d^2.rT}{dr^2}\right).$$

This equation will still subsist if we change θ into θ', ϖ into ϖ', and reciprocally; because T is a similar function of θ', ϖ' and of θ, ϖ.

If we expand T, in a series descending relatively to r, we shall have

$$T=\frac{Q^{(0)}}{r}+Q^{(1)}.\frac{R}{r^2}+Q^{(2)}.\frac{R^2}{r^3}+\&c.$$

$Q^{(i)}$ being, whatever i may be, subject to the condition that

$$0=\left(\frac{d\left\{(1-m^2)\left(\frac{dQ^{(i)}}{dm}\right)\right\}}{dm}\right)+\frac{\left(\frac{d^2Q^{(i)}}{d\varpi^2}\right)}{1-m^2}+i\,(i+1).Q^{(i)};$$

and moreover it is evident, that $Q^{(i)}$ is a rational and entire function of m, and $\sqrt{1-m^2}.\cos.(\varpi'-\varpi)$: $Q^{(i)}$ being known, we shall have $U^{(i)}$ by means of the equation

$$U^{(i)}=\int\varrho\,R^{(i+2)}.dR.d\varpi'.d\theta'.\sin.\theta'.Q^{(i)}.$$

Now suppose the point attracted in the interior of the spheroid: we must then develope the integral expression of V, in a series ascending relatively to r, which gives for V a series of the form

$$V=v^{(0)}+r.v^{(1)}+r^2.v^{(2)}+r^3.v^{(3)}+\&c.$$

$v^{(i)}$ being a rational and whole function of m, $\sqrt{1-m^2}.\sin.\varpi$ and $\sqrt{1-m^2}\cos.\varpi$, which satisfies the same equation of partial differences that $U^{(i)}$ does; so that we have

$$0=\left(\frac{d\left\{(1-m^2)\left(\frac{dv^{(i)}}{dm}\right)\right\}}{dm}\right)+\frac{\left(\frac{d^2v^{(i)}}{d\varpi^2}\right)}{1-m^2}+i\,(i+1).v^{(i)}.$$

To determine $v^{(i)}$, we shall expand the radical T into a series ascending according to r, and we shall have

$$T=\frac{Q^{(0)}}{R}+Q^{(1)}.\frac{r}{R^2}+Q^{(2)}.\frac{r^2}{R^3}+\&c.$$

the quantities $Q^{(0)}$, $Q^{(1)}$, $Q^{(2)}$, &c. being the same as above; we shall therefore get

$$v^{(i)}=\int\frac{\varrho.dR.d\varpi'.d\theta'.\sin.\theta'.Q^{(i)}}{R^{i-1}}.$$

But since the preceding expression of T is only convergent so long as R is equal to or greater than r, the preceding value of V only relates to the shells of the spheroid, which envelope the point attracted. This point being exterior, relatively to the other shells, we shall determine that part of V which is relative to them by the first series of V.

554. First let us consider those spheroids which differ but very little from the sphere, and determine the functions $U^{(0)}$, $U^{(1)}$, $U^{(2)}$, &c. $v^{(0)}$, $v^{(1)}$, $v^{(2)}$, &c. relatively to these spheroids. There exists a differential equation in V, which holds good at their surface, and which is remarkable because it gives the means of determining those functions without any integration.

Let us suppose generally, that gravity is proportional to a power n of the distance; let d M be an element of the spheroid, and f its distance from the point attracted; call V the integral $\int f^{n+1} d M$, which shall extend to the entire mass of the spheroid. In nature we have $n = -2$, it becomes $\int \frac{d M}{f}$, and we have expressed it in like manner by V in the preceding Nos. The function V possesses the advantage of giving, by its differentiation, the attraction of the spheroid, parallel to any straight line whatever; for considering f as a function of the three coordinates of the point attracted perpendicular to one another, and one of which is parallel to this straight line. Call r this coordinate, the attraction of the spheroid along r and directed towards its origin, will be $\int . f^n . \left(\frac{d f}{d r}\right) . d M$. Consequently it will be equal to $\frac{1}{n+1} \left(\frac{d V}{d r}\right)$, which, in the case of nature, becomes $- \left(\frac{d V}{d r}\right)$, conformably with what has been already shown.

Suppose, however, that the spheroid differs very little from a sphere of the radius a, whose center is upon the radius r perpendicular to the surface of the spheroid, the origin of the radius being supposed to be arbitrary, but very near to the center of gravity of the spheroid; suppose, moreover, that the sphere touches the spheroid, and that the point attracted is at the point of contact of the two surfaces. The spheroid is equal to the sphere plus the excess of the spheroid above the sphere; but we may conceive this excess as being formed of an infinite number of molecules spread over the surface of the sphere, these molecules being supposed negative wherever the sphere exceeds the spheroid; we shall therefore have the value of V by determining this value, 1st, relatively to the sphere; 2dly, relatively to the different molecules.

Relatively to the sphere, V is a function of a, which we denote by A; if we name d m one of the molecules of the excess of the spheroid above the sphere, and f its distance from the point attracted; the value of V rela-

tive to this excess will be $\int . f^{n+1} . d m$; we shall therefore have, for the entire value of V, relative to the spheroid,

$$V = A + \int . f^{n+1} . d m.$$

Conceive that the point attracted is elevated by an infinitely small quantity d r, above the surface of the spheroid and the sphere upon r or a produced; the value of V, relative to this new position of the attracted point, will become

$$V + \left(\frac{d V}{d r}\right) d r;$$

A will increase by a quantity proportional to d r, and which we shall represent by $A' . d r$. Moreover, if we name γ the angle formed by the two radii drawn from the center of the sphere to the point attracted, and to the molecule d m, the distance f of this element or molecule from the point attracted, will be in the first position of the point, equal to

$$\sqrt{2 a^2 (1 - \cos. \gamma)};$$

in the second position it will be

$$\sqrt{(a + d r)^2 - 2 a (a + d r) \cos. \gamma + a^2},$$

or

$$f \left(1 + \frac{d r}{2 a}\right);$$

the integral $\int . f^{n+1} d m$, will thus become

$$\left\{1 + \frac{n + 1}{2} . \frac{d r}{a}\right\} \int . f^{n+1} d m;$$

we shall therefore have

$$\left(\frac{d V}{d r}\right) . d r = A' d r + \frac{n + 1}{2} . \frac{d r}{a} \int . f^{n+1} . d m;$$

substituting for $\int . f^{n+1} . d m$, its value V — A, we shall have

$$\left(\frac{d V}{d r}\right) = A' - \frac{(n + 1) A}{2 a} + \frac{n + 1}{2 a} . V; \quad . . . \quad (1)$$

In the case of nature, the equation (1) becomes

$$- a \left(\frac{d A}{d r}\right) = - a A' - \tfrac{1}{2} A + \tfrac{1}{2} V.$$

The value of V relative to the sphere of radius a, is, by 550, equal to $\frac{4 \pi a^3}{3 r}$, which gives $A = \frac{4 \pi a^3}{3}$; $A' = - \frac{4 \pi a}{3}$; we shall therefore get

$$- a \left(\frac{d V}{d r}\right) = \frac{2 \pi . a^2}{3} + \tfrac{1}{2} V; \quad . . . . . \quad (2)$$

We must here observe that this equation obtains, whatever may be the position of the straight line r, and even in the case where it is not perpen-

dicular to the surface of the spheroid, provided that it passes very near its center of gravity, for it is easy to see that the attraction of the spheroid, resolved parallel to these straight lines, and which, as we have seen, is equal to $-\left(\frac{dV}{dr}\right)$, is, whatever may be their position, always the same, to quantities nearly of the order of the square of the excentricity of the spheroid.

555. Let us resume the general expression of V of 553, relative to a point attracted exterior to the spheroid,

$$V = \frac{U^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \frac{U^{(2)}}{r^3} + \&c.$$

the function $U^{(i)}$ being, whatever i may be, subject to the equation of partial differences

$$0 = \left(\frac{d\left\{(1-m^2).\left(\frac{dU^{(i)}}{dm}\right)\right\}}{dm}\right) + \frac{\left(\frac{d^2U^{(i)}}{d\varpi^2}\right)}{1-m^2} + i(i+1).U^{(i)}.$$

By differentiating the value of V relatively to r, we have

$$-\left(\frac{dV}{dr}\right) = \frac{U^{(0)}}{r^2} + \frac{2U^{(1)}}{r^3} + \frac{3U^{(2)}}{r^4} + \&c.$$

Let us represent by a $(1 + \alpha y)$ the radius drawn from the origin of r to the surface of the spheroid, α being a very small constant coefficient, whose square and higher powers we shall neglect, and y being a function of m and ϖ depending on the nature of the spheroid. We shall have to quantities nearly of the order α, $V = \frac{4\pi a^3}{3r}$; whence it follows that in the preceding expression of V, 1st, the quantity $U^{(0)}$ is equal to $\frac{4\pi a^3}{3}$ *plus* a very small quantity of the order α, and which we shall denote by $U'^{(0)}$; 2dly, that the quantities $U^{(1)}$, $U^{(2)}$, &c. are small quantities of the order α. Substituting $a(1 + \alpha.y)$ for r in the preceding expressions of V and of $-\left(\frac{dV}{dr}\right)$, and neglecting quantities of the order α^2, we shall have relatively to an attracted point placed at the surface

$$\tfrac{1}{2}V = \tfrac{2}{3}\pi.a^2(1-\alpha y) + \frac{U'^{(0)}}{2a} + \frac{U^{(1)}}{2a^2} + \frac{U^{(2)}}{2a^3} + \&c.;$$

$$-a\left(\frac{dV}{dr}\right) = \tfrac{4}{3}\pi a^2(1-2\alpha y) + \frac{U'^{(0)}}{a} + \frac{2.U^{(1)}}{a^2} + \frac{3U^{(2)}}{a^3} + \&c.$$

If we substitute these values in equation (2) of the preceding No. we shall have

$$4\alpha\pi a^2.y = \frac{U'^{(0)}}{a} + \frac{3.U^{(1)}}{a^2} + \frac{5U^{(2)}}{a^3} + \frac{7U^{(3)}}{a^4} + \&c.$$

It thence follows that the function y is of this form

$$y = Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$$

the quantities $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}$, &c. as well as $U^{(0)}$, $U^{(1)}$, &c. being subject to the equation of partial differences

$$0 = \left(\frac{d \left\{ (1 - m^2) . \left(\frac{d Y^{(i)}}{d m} \right) \right\}}{d m} \right) + \frac{\left(\frac{d^2 Y^{(i)}}{d \varpi^2} \right)}{1 - m^2} + i (i+1) . Y^{(i)};$$

this expression of y is not therefore arbitrary, but it is derived from the developement of the attractions of spheroids. We shall see in the following No. that y cannot be thus developed except in one manner only; we shall therefore have generally, by comparing similar functions,

$$U^{(i)} = \frac{4 \alpha \pi}{2 i + 1} a^{i+3} . Y^{(i)};$$

whence, whatever r may be, we derive

$$V = \frac{4 \pi a^3}{3 r} + \frac{4 \alpha \pi . a^3}{r} \left\{ Y^{(0)} + \frac{a}{3 r} . Y^{(1)} + \frac{a^2}{5 r^2} Y^{(2)} + \&c. \right\}; \quad . . \quad (3)$$

To get V, therefore, it remains only to reduce y to the form above described; for which object we shall give, in what follows, a very simple method.

If we had $y = Y^{(i)}$, the part of V relative to the excess of the spheroid above the sphere whose radius is a, or which is the same thing, relative to a spherical shell whose radius is a, and thickness α a y, would be $\frac{4 \alpha \pi a^{i+3} . Y^{(i)}}{(2 i + 1) r^{i+1}}$; this value would consequently be proportional to y, and it is evident that it is only in this case that the proportionality can subsist.

556. We may simplify the expression $Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$ of y, and cause to disappear the two first terms, by taking for a, the radius of a sphere equal in solidity to the spheroid, and by fixing the arbitrary origin of r at the center of gravity of the spheroid. To show this, we shall observe that the mass M of the spheroid supposed homogeneous, and of a density represented by unity, is by 552, equal to $\int R^2 \, d R \, d m \, d \varpi$, or to $\frac{1}{3} \int R'^3 \, d m \, d \varpi$, R′ being the radius R produced to the surface of the spheroid. Substituting for R′ its value a (1 + α y) we shall have

$$M = \frac{4 \pi a^3}{3} + \alpha a^3 \int y \, d m \, d \varpi.$$

All that remains to be done, therefore, is to substitute for y its value $Y^{(0)} + Y^{(1)} + \&c.$ and then to make the integrations. For this purpose here is a general theorem, highly useful also in this analysis.

" If $Y^{(i)}$ and $Z^{(i)}$ be rational and entire functions of m, $\sqrt{1-m^2}.\sin.\varpi$ " and $\sqrt{1-m^2}.\cos.\varpi$, which satisfy the following equations:

$$\text{"}\ 0=\left(\frac{d.\left\{(1-m^2).\left(\frac{dY^{(i)}}{dm}\right)\right\}}{dm}\right)+\frac{\left(\frac{d^2Y^{(i)}}{d\varpi^2}\right)}{1-m^2}+i(i+1).Y^{(i)};$$

$$\text{"}\ 0=\left(\frac{d\left\{(1-m^2).\left(\frac{dZ^{(i)}}{dm}\right)\right\}}{dm}\right)+\frac{\left(\frac{d^2Z^{(i)}}{d\varpi^2}\right)}{1-m^2}+i'(i'+1).Z^{(i')};$$

" we shall have generally

$$\text{"}\int Y^{(i)}.Z^{(i')}.dm\,d\varpi=0,$$

" whilst i and i' are whole positive numbers differing from one another; " the integrals being taken from $m=-1$ to $m=1$, and from $\varpi=0$ " to $\varpi=2\pi$."

To demonstrate this theorem, we shall observe that in virtue of the first of the two preceding equations of partial differences, we have

$$\int Y^{(i)}.Z^{(i')}.dm.d\varpi=-\frac{1}{i(i+1)}.\int Z^{(i')}\left(\frac{d.\left\{(1-m^2).\left(\frac{dY^{(i)}}{dm}\right)\right\}}{dm}\right).dm.d\varpi$$

$$-\frac{1}{i(i+1)}.\int\frac{Z^{(i')}.\left(\frac{d^2Y^{(i)}}{d\varpi^2}\right)}{1-m^2}.dm.d\varpi;$$

But integrating by parts relatively to m we have

$$\int Z^{(i')}.\left(\frac{d\left\{(1-m^2)\left(\frac{dY^{(i)}}{dm}\right)\right\}}{dm}\right).dm=(1-m^2)\left(\frac{dY^{(i)}}{dm}\right).Z^{(i')}$$

$$-(1-m^2)\,Y^{(i)}.\left(\frac{dZ^{(i')}}{dm}\right)$$

$$+\int Y^{(i)}\left(\frac{d\left\{(1-m^2)\left(\frac{dZ^{(i')}}{dm}\right)\right\}}{dm}\right)dm;$$

and it is clear that if we take the integral from $m=-1$ to $m=1$, the second member of this equation will be reduced to its last term. In like manner, integrating by parts relatively to ϖ, we get

$$\int Z^{(i')}.\left(\frac{d^2Y^{(i)}}{d\varpi^2}\right).d\varpi=\text{const.}+Z^{(i')}.\left(\frac{dY^{(i)}}{d\varpi}\right)$$

$$-Y^{(i)}\left(\frac{dZ^{(i')}}{d\varpi}\right)+\int Y^{(i)}\left(\frac{d^2Z^{(i')}}{d\varpi^2}\right).d\varpi;$$

and this second member also reduces to its last term, when the integral

is taken from $\varpi = 0$ to $\varpi = 2\pi$, because the values of $Y^{(i)}$, $\left(\frac{d\,Y^{(i)}}{d\,\varpi}\right)$, $Z^{(i')}$, $\left(\frac{d\,Z^{(i')}}{d\,\varpi}\right)$ are the same at these two limits; thus we shall have

$$\int Y^{(i)} . Z^{(i')} . d\,m . d\,\varpi =$$

$$-\frac{1}{i\,(i+1)} . \int Y^{(i)} . d\,m . d\,\varpi \left\{ \left(\frac{d . \left\{ (1-m^2) . \left(\frac{d\,Z^{(i')}}{d\,m} \right) \right\}}{d\,m} \right) + \frac{\frac{d^2 Z^{(i')}}{d\,\varpi^2}}{1-m^2} \right\};$$

whence we derive, in virtue of the second of the two preceding equations of partial differences,

$$\int Y^{(i)} . Z^{(i')} . d\,m . d\,\varpi = \frac{i\,(i'+1)}{i\,(i+1)} . \int Y^{(i)} . Z^{(i')} . d\,m . d\,\varpi,$$

we therefore have

$$0 = \int Y^{(i)} . Z^{(i')}\, d\,m . d\,\varpi,$$

when i is different from i'.

Hence it is easy to conclude that y can be developed into a series of the form $Y^{(0)} + Y^{(1)} + Y^{(2)} +$ &c. in one way only; for we have generally

$$\int y . Z^{(i)}\, d\,m\, d\,\varpi = \int Y^{(i)} . Z^{(i)}\, d\,m . d\,\varpi;$$

If we could develope y into another series of the same form, $Y_{,}^{(0)} + Y_{,}^{(1)} + Y_{,}^{(2)} +$ &c. we should have

$$\int y . Z^{(i)} = \int Y_{,}^{(i)} . Z^{(i)}\, d\,m . d\,\varpi;$$

wherefore

$$\int Y_{,}^{(i)} . Z^{(i)} . d\,m\, d\,\varpi = \int Y^{(i)} . Z^{(i)}\, d\,m . d\,\varpi.$$

But it is easy to perceive that if we take for $Z^{(i)}$ the most general function of its kind, the preceding equation can only subsist in the case wherein $Y_{,}^{(i)} = Y^{(i)}$; the function y can therefore be developed thus in only one manner.

If in the integral $\int y\, d\,m . d\,\varpi$, we substitute for y its value $Y^{(0)} + Y^{(1)} + Y^{(2)} +$ &c., we shall have generally $0 = \int Y^{(i)}\, d\,m . d\,\varpi$, i being equal to or greater than unity; for the unity which multiplies $d\,m . d\,\varpi$ is comprised in the form $Z^{(0)}$, which extends to every constant and quantity independent of m and ϖ. The integral $\int y\, d\,m . d\,\varpi$ reduces therefore to $\int Y^{(0)}\, d\,m . d\,\varpi$, and consequently to $4\,\pi\, Y^{(0)}$; we have therefore

$$M = \tfrac{4}{3}\,\pi\, a^3 + 4\,\alpha\,\pi\, a^3 . Y^{(0)};$$

thus, by taking for α, the radius of the sphere equal in solidity to the spheroid, we shall have $Y^{(0)} = 0$, and the term $Y^{(0)}$ will disappear from the expression of y.

The distance of the element d M, or $R^2 . d R d m . d \varpi$, from the plane of the meridian from whence we measure the angle ϖ, is equal to $R \sqrt{1 - m^2} . \sin. \varpi$; the distance of the center of gravity of the spheroid from this plane, will be therefore $\int R^3 d R d m . d \varpi \sqrt{1 - m^2} . \sin. \varpi$, and integrating relatively to R, it will be $\frac{1}{4} \int R'^4 d m . d \varpi \sqrt{1 - m^2} \sin. \varpi$, R′ being the radius R produced to the surface of the spheroid. In like manner the distance of the element d M from the plane of the meridian perpendicular to the preceding, being $R \sqrt{1 - m^2} . \cos. \varpi$, the distance of the center of gravity of the spheroid from this plane will be $\frac{1}{4} \int R'^4 d m . d \varpi . \sqrt{1 - m^2} . \cos. \varpi$. Finally, the distance of the element d M from the plane of the equator being m, the distance of the center of gravity of the spheroid from this plane will be $\frac{1}{4} \int R'^4 m . d m . d \varpi$. These functions m, $\sqrt{1 - m^2} . \sin. \varpi$, $\sqrt{1 - m^2} . \cos. \varpi$, are of the form $Z^{(i)}$, $Z^{(i)}$ being subject to the equation of partial differences

$$0 = \left(\frac{d \left\{ (1 - m^2) . \left(\frac{d Z^{(i)}}{d m} \right) \right\}}{d m} \right) + \frac{\left(\frac{d^2 Z^{(i)}}{d \varpi^2} \right)}{1 - m^2} + 2 Z^{(i)}.$$

If we conceive R'^4 developed into the series $N^{(0)} + N^{(1)} + N^{(2)} + \&c.$ $N^{(i)}$ being a rational and entire function of m, $\sqrt{1 - m^2} . \sin. \varpi$, $\sqrt{1 - m^2} . \cos. \varpi$, subject to the equation of partial differences.

$$0 = \left(\frac{d \left\{ (1 - m^2) . \left(\frac{d N^{(i)}}{d m} \right) \right\}}{d m} \right) + \frac{\left(\frac{d^2 N^{(i)}}{d \varpi^2} \right)}{1 - m^2} + i (i+1) . N^{(i)};$$

the distances of the center of gravity of the spheroid, from the three preceding planes, will be, in virtue of the general theorem above demonstrated,

$$\frac{1}{4} \int N^{(1)} . d m . d \varpi . \sqrt{1 - m^2} . \sin. \varpi,$$
$$\frac{1}{4} \int N^{(1)} . d m . d \varpi . \sqrt{1 - m^2} . \cos. \varpi;$$
$$\frac{1}{4} \int N^{(1)} m . d m . d \varpi.$$

$N^{(1)}$ is, by No. 553, of the form $A m + B \sqrt{1 - m^2} . \sin. \varpi + C \sqrt{1 - m^2} . \cos. \varpi$, A, B, C being constants; the preceding distances will thus become $\frac{\pi}{3} . B$, $\frac{\pi}{3} . C$, $\frac{\pi}{3} . A$. The position of the center of gravity of the spheroid, thus depends only on the function $N^{(1)}$. This gives a very simple way of determining it. If the origin of the radius R′ is at the center; this origin being upon the three preceding planes, the distances of the center of gravity from these planes will be nothing. This gives $A = 0$, $B = 0$, $C = 0$; therefore $N^{(1)} = 0$.

These results obtain whatever may be the spheroid: when it is very little different from a sphere, we have $R' = a(1 + \alpha y)$, and $R'^4 = a^4(1 + 4\alpha y)$; thus, y being equal to $Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$, we have $N^{(1)} = 4\alpha a^4 Y^{(1)}$, the function $Y^{(1)}$ disappears, therefore, from the expression of y, when we fix the origin of R' at the center of gravity of the spheroid.

557. Now let the point attracted be in the interior of the spheroid, we shall have by 553

$$V = v^{(0)} + r \cdot v^{(1)} + r^2 \cdot v^{(2)} + r^3 v^{(3)} + \&c.$$

$$v^{(i)} = \int \frac{dR \cdot d\varpi' \cdot d\theta' \cdot \sin.\theta' \cdot Q^{(i)}}{R^{i-1}}.$$

Suppose that this value of V is relative to a shell whose interior surface is spherical and of the radius a, and the radius of whose exterior surface is $a(1 - \alpha y)$; the thickness of the shell is $\alpha a y$. If we denote by y' what y becomes when we change θ, ϖ into θ', ϖ', we may, neglecting quantities of the order α^2, change r into a, and d R into $\alpha a y'$, in the integral expression of $v^{(i)}$; thus we shall have

$$v^{(i)} = \frac{\alpha}{a^{i-2}} \int y' \, d\varpi' \cdot d\theta' \cdot \sin.\theta' \cdot Q^{(i)}.$$

Relatively to a point placed without the spheroid, we have, by 553,

$$V = \frac{U^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \&c.;$$

$$U^{(i)} = \int R^{i+2} \cdot dR \cdot d\varpi' \cdot d\theta' \cdot \sin.\theta' \cdot Q^{(i)}.$$

If we suppose this value of V relative to a shell, whose interior and exterior radii are respectively a, $a(1 + \alpha y)$, we shall have

$$U^{(i)} = \alpha \cdot a^{i+3} \cdot \int y' \cdot d\varpi' \cdot d\theta' \cdot \sin.\theta' \cdot Q^{(i)};$$

wherefore

$$v^{(i)} = \frac{U^{(i)}}{a^{2i+1}}.$$

We have by 555

$$U^{(i)} = \frac{4\alpha\pi a^{i+3} \cdot Y^{(i)}}{2i + 1};$$

therefore

$$v^{(i)} = \frac{4a\pi Y^{(i)}}{(2i + 1) a^{i-2}};$$

which gives

$$V = 4\alpha\pi a^2 \left\{ Y^{(0)} + \frac{r}{3a} \cdot Y^{(1)} + \frac{r^2}{5a^2} \cdot Y^{(2)} + \&c. \right\}.$$

To this value of V we must add that which is relative to the spherical shell of the thickness a — r which envelopes the attracted point *plus* that which is relative to the sphere of radius r, and which is below the same

point. If we make cos. $\theta' = m'$, we shall have, relatively to the first of the two parts of V,

$$v^{(i)} = \int \frac{d\,R \,.\, d\,\varpi' \,.\, d\,m' \,.\, Q^{(i)}}{R^{i-1}};$$

an integral which, relative to m', must be taken from $m' = -1$ to $m = 1$ Integrating relative to R, from $R = r$ to $R = a$, we shall have

$$v^{(i)} = \frac{1}{2-i}\left(\frac{1}{a^{i-2}} - \frac{1}{r^{i-2}}\right) \int d\,\varpi' \,.\, d\,m' \,.\, Q^{(i)};$$

But we have generally, by the theorem of the preceding No., $\int d\,\varpi' \,.\, d\,m' \,.\, Q^{(i)} = 0$ when i is equal to or greater than unity; when $i = 0$, we have, by 553, $Q^{(i)} = 1$; moreover the integration relative to ϖ' must be taken from $\varpi' = 0$ to $\varpi' = 2\pi$; we shall therefore have

$$v^{(0)} = 2\pi(a^2 - r^2).$$

This value of $v^{(0)}$ is that part of V which is relative to the spherical shell whose thickness is $a - r$.

The part of V which is relative to the sphere whose radius is r is equal to the mass of this sphere, divided by the distance of the attracted point from its center: it is consequently equal to $\frac{4\pi r^2}{3}$. Collecting the different parts of V, we shall have its whole value

$$V = 2\pi a^2 - \tfrac{2}{3}\pi r^2 + 4\alpha\pi a^2\left\{Y^{(0)} + \frac{r}{3a}Y^{(1)} + \frac{r^2}{5a^2}Y^{(2)} + \&c.\right\}; \quad (4)$$

Suppose the point attracted, placed within a shell very nearly spherical, whose interior radius is

$$a + \alpha a\,\{Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.\}$$

and whose exterior radius is

$$a' + \alpha a'\,\{Y'^{(0)} + Y'^{(1)} + Y'^{(2)} + \&c.\}$$

The quantities $\alpha a\, Y^{(0)}$ and $\alpha a'\, Y'^{(0)}$ may be comprised in the quantities a, a'. Moreover, by fixing the origin of coordinates at the center of gravity of the spheroid whose radius is

$$a + \alpha a\,\{Y^{(0)} + Y^{(1)} + \&c.\},$$

we may cause $Y^{(1)}$ to disappear from the expression of this radius; and then the interior radius of the shell will be of this form,

$$a + \alpha a\,\{Y^{(2)} + Y^{(3)} + \&c.\},$$

and the exterior radius will be of the form,

$$a' + \alpha a'\,\{Y'^{(1)} + Y'^{(2)} + \&c.\}.$$

We shall have the value of V relative to this shell, by taking the difference of the values of V relative to two spheroids, the smaller of which shall have for the radius of its surface the first quantity, and the greater

the second quantity for the radius of its surface; calling therefore $\Delta . V$, what V becomes relatively to this shell, we shall have

$$\Delta V = 2\pi(a'^2 - a^2) + 4\alpha\pi\left\{\frac{r\,a'}{3}Y'^{(1)} + \frac{r^2}{5}\{Y'^{(2)} - Y^{(2)}\} + \frac{r^3}{7}\left(\frac{Y'^{(3)}}{a'} - \frac{Y^{(3)}}{a}\right) + \&c.\right\}$$

If we wish that the point placed in the interior of the shell, should be equally attracted on all sides, $\Delta . V$ must be reduced to a constant independent of r, θ, ϖ; for we have seen that the partial differences of $\Delta . V$, taken relatively to these variables, express the partial attractions of the shell upon the point attracted; we therefore, in this case have $Y'^{(1)} = 0$, and generally

$$Y'^{(i)} = \left(\frac{a'}{a}\right)^{i-2} . Y^{(i)};$$

so that the radius of the interior surface being given, that of the exterior surface will be found.

When the interior surface is elliptic, we have $Y^{(3)} = 0$, $Y^{(4)} = 0$, &c. and consequently $Y'^{(3)} = 0$, $Y'^{(4)} = 0$; the radii of the two surfaces, interior and exterior, are therefore

$$a\{1 + \alpha Y^{(2)}\}; \quad a'\{1 + \alpha Y^{(2)}\};$$

thus we see that these two surfaces are similar and similarly situated, which agrees with what we found in 547.

558. The formulas (3), (4) of Nos. 555, and 557, comprehend all the theory of the attractions of *homogeneous* spheroids, differing but little from the sphere; whence it is easy to obtain that of heterogeneous spheroids, whatever may be the law of the variation of the figure and density of their shells. For that purpose let a $(1 + \alpha y)$ be the radius of one of the shells of a heterogeneous spheroid, and suppose y to be of this form

$$Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.$$

the coefficients which enter the quantities $Y^{(0)}$, $Y^{(1)}$, &c. being functions of a, and consequently variable from one shell to another. If we differentiate relatively to a, the value of V given by the form (3) of No. 555; and call ϱ the density of the shell whose radius is a $(1 + \alpha y)$, ϱ being a function of a only; the value of V corresponding to this shell will be, for an exterior attracted point,

$$\frac{4\pi}{3r}\varrho\,d\,a^3 + \frac{4\alpha\pi . \varrho}{r}d\left\{a^3 Y^{(0)} + \frac{a^4}{3r} . Y^{(1)} + \frac{a^5}{5r^2} . Y^{(2)} + \&c.\right\};$$

this value will be, therefore, relatively to the whole spheroid,

$$V = \frac{4\pi}{3r}\int\varrho\,d\,a^3 + \frac{4\alpha\pi}{r}\int\varrho\,d\left\{a^3 Y^{(0)} + \frac{a^4}{3r}Y^{(1)} + \frac{a^5}{5r^2}Y^{(2)} + \&c.\right\}; \quad . \quad . \quad (5)$$

the integrals being taken from $a = 0$ to that value of a which subsists at the surface of the spheroid, and which we denote by a.

To get the part of V relative to an attracted point in the interior of the spheroid, we shall determine first the part of this value relative to all the shells to which this point is exterior. This first part is given by formula (5) by taking the integral from $a = 0$ to $a = a$, a being relative to the shell in which is the point attracted. We shall find the second part of V relative to all the shells in the interior of which is placed the point attracted, by differentiating the formula (4) of the preceding No. relatively to a; then multiplying this differential by ϱ, and taking the integral from $a = a$, to $a = \alpha$, the sum of the two parts of V will be its entire value relative to an interior point, which sum will be

$$V = \frac{4\pi}{3}\int \varrho \, d\, a^3 + \frac{4\alpha\pi}{r}\int \varrho \, d\, . \left\{ a^3 Y^{(0)} + \frac{a^4}{3r} Y^{(1)} + \frac{a^5}{5r^2} Y^{(2)} + \&c. \right\}$$

$$+ 2\pi \int \varrho \, d\, a^2 + 4\alpha\pi \int \varrho \, d\, . \left\{ a^2 Y^{(0)} + \frac{a\, r}{3} Y^{(1)} + \frac{r^2}{5} Y^{(2)} + \&c. \right\} . \quad (6)$$

the two first integrals being taken from $a = 0$ to $a = a$, and the two last being taken from $a = a$ to $a = \alpha$; after the integrations, moreover, we must substitute a for r in the terms multiplied by α, and $\frac{1 - \alpha y}{\alpha}$ for $\frac{1}{r}$ in the term $\frac{4\pi}{3r}\int \varrho \, d\, . \, a^3$.

559. Now let us consider any spheroids whatever. The research of their attraction is reduced, by 553, to forming the quantities $U^{(i)}$ and $v^{(i)}$, by that No. we have

$$U^{(i)} = \int \varrho \, R^{i+2} \, . \, d\, R \, d\, m' \, d\, \varpi' \, . \, Q^{(i)};$$

in which the integrals must be taken from $R = 0$ to its value at the surface, from $m' = -1$ to $m' = 1$, and from $\varpi' = 0$ to $\varpi' = 2\pi$.

To determine this integral, $Q^{(i)}$ must be known. This quantity may be developed into a finite function of cosines of the angle $\varpi - \varpi'$, and of its multiples. Let $\beta \cos . n (\varpi - \varpi')$ be the term of $Q^{(i)}$ depending on $\cos . n (\varpi - \varpi')$, β being a function m, m'. If we substitute for $Q^{(i)}$ its value in the equation of partial differences in $Q^{(i)}$ of No. 553, we shall have, by comparing the terms multiplied by $\cos . n (\varpi - \varpi')$, this equation of ordinary differences,

$$0 = \frac{d \left\{ (1 - m^2) . \left(\frac{d\beta}{d m} \right) \right\}}{d m} - \frac{n^2 \beta}{1 - m^2} + i (i + 1) . \beta;$$

$Q^{(i)}$ being the coefficient of $\frac{R^{(i)}}{r^{i+1}}$, in the developement of the radical

$$\frac{1}{\sqrt{r^2 - 2Rr\{m m' + \sqrt{1 - m'^2} . \sqrt{1 - m^2} . \cos . (\varpi - \varpi') + R^2\}}}.$$

The term depending on cos. n $(\varpi - \varpi')$, in the developement of this radical, can only result from the powers of cos. $(\varpi - \varpi')$, equal to n, n+2, n + 4, &c.; thus cos. $(\varpi - \varpi')$ having the factor $\sqrt{1 - m^2}$, β must have the factor $(1 - m^2)^{\frac{n}{2}}$. It is easy to see, by the consideration of the developement of the radical, that β is of this form

$$(1 - m^2)^{\frac{n}{2}} . \{A . m^{i-n} + A^{(1)} . m^{i-n-2} + A^{(2)} . m^{i-n-4} + \&c.\}.$$

If we substitute this value in the differential equation in β, the comparison of like powers of m will give

$$A^{(s)} = - \frac{(i - n - 2s + 2) . (i - n - 2s + 1)}{2s(2i - 2s + 1)} . A^{(s-1)};$$

whence we derive, by successively putting s = 1, s = 2, &c. the values of $A^{(1)}$, $A^{(2)}$, and consequently,

$$\beta = A(1-m^2)^{\frac{n}{2}} \left\{ m^{i-n} - \frac{(i-n)(i-n-1)}{2(2i-1)} m^{i-n-2} + \frac{(i-n)(i-n-1)(i-n-2)(i-n-3)}{2.4.(2i-1)(2i-3)} m^{i-n-4} \right.$$
$$\left. - \frac{(i-n)(i-n-1)(i-n-2)(i-n-3)(i-n-4)(i-n-5)}{2.4.6(2i-1)(2i-3)(2i-5)} m^{i-n-6} + \&c. \right\}.$$

A is a function of m' independent of m; but m and m' entering alike into the preceding radical, they ought to enter similarly into the expression of β; we have therefore

$$A = \gamma (1 - m'^2)^{\frac{n}{2}} \left\{ m'^{i-n} - \frac{(i-n)(i-n-1)}{2(2i-1)} . m'^{i-n-2} + \&c. \right\};$$

γ being a coefficient independent of m and m'; therefore

$$\beta = \gamma (1 - m'^2)^{\frac{n}{2}} \left\{ m'^{i-n} - \frac{(i-n)(i-n-1)}{2(2i-1)} m'^{i-n-2} + \&c. \right\} . (1 - m^2)^{\frac{n}{2}} \times$$
$$\left\{ m^{i-n} - \frac{(i-n)(i-n-1)}{2(2i-1)} m^{i-n-2} + \&c. \right\}.$$

Thus we see that β is split into three factors, the first independent of m and m'; the second a function of m' alone; and the third a like function of m. We have only now to determine γ.

For that purpose, we shall observe, that if i − n be even; we have, supposing m = 0, and m' = 0,

$$\theta = \frac{\gamma . \{1.2. \ldots i - n\}^2}{\{2.4. \ldots (i-n).(2i-1).(2i-3). \ldots (i+n+1)\}^2}$$
$$= \frac{\gamma . \{1.3.5. \ldots (i-n-1).1.3.5. \ldots (i+n-1)\}^2}{\{1.3.5. \ldots (2i-1)\}^2}$$

If i — n is odd, we shall have, in retaining only the first power of m, and m′,

$$\beta = \frac{\gamma . m . m' \{1. 2. \ldots . (i - n)\}^2}{\{2. 4. \ldots . (i - n - 1)\ (2i - 1)\ (2i - 3) \ldots (i + n + 2)\}^2}$$
$$= \frac{\gamma . m . m' \{1. 3. 5. \ldots . (i - n) . 1. 3. 5. \ldots . (i + n)\}^2}{\{1. 3. 5. \ldots . (2i - 1)\}^2}.$$

The preceding radical becomes, neglecting the squares of m, m′,

$$\{r^2 - 2Rr\cos.(\varpi - \varpi') + R^2\}^{-\frac{1}{2}} + Rr . mm' \{r^2 - 2rR\cos.(\varpi - \varpi') + R^2\}^{-\frac{3}{2}}; \quad . \quad (f)$$

If we substitute for cos. $(\varpi - \varpi')$, its value in imaginary exponentials, and if we call c the number whose hyperbolic logarithm is unity, the part independent of m m′, becomes

$$\{r - R . c^{(\varpi - \varpi')\sqrt{-1}}\}^{-\frac{1}{2}} . \{r - R . c^{-(\varpi - \varpi')\sqrt{-1}}\}^{-\frac{1}{2}}.$$

The coefficient of

$$\frac{R^i}{r^{i+1}} . \frac{c^{n(\varpi - \varpi')\sqrt{-1}} + c^{-n(\varpi - \varpi')\sqrt{-1}}}{2}, \text{ or of } \frac{R^i}{r^{i+1}} . \cos. n(\varpi - \varpi')$$

in the developement of this function is

$$\frac{2. 1. 3. 5. \ldots . (i + n - 1) . 1. 3. 5. \ldots . (i - n - 1)}{2. 4. 6. \ldots . (i + n)\ 2. 4. 6. \ldots . (i - n)}.$$

This is the value of β when i — n is even. Comparing it with that which in the same case we have already found, we shall have

$$\gamma = 2\left(\frac{1. 3. 5. \ldots . (2i - 1)}{1. 2. 3. \ldots . i}\right)^2 \times \frac{i\,(i - 1) \ldots . (i - n + 1)}{(i + 1)\ (i + 2) \ldots . (i + n)}$$

When n = 0, we must take only half this coefficient, and then we have

$$\gamma = \left(\frac{1. 3. 5. \ldots . 2i - 1}{1. 2. 3. \ldots . i}\right)^2.$$

In like manner, the coefficient of $\frac{R^i}{r^{i+1}} m . m' \cos. n(\varpi - \varpi')$ in the function (f) is

$$\frac{2. 1. 3. 5. \ldots . (i + n) . 1. 3. 5. \ldots . (i - n)}{2. 4. 6. (i + n - 1) . 2. 4. 6. \ldots . (i - n - 1)};$$

this is the coefficient of m m′ in the value of β, when we neglect the squares of m, m′, and when i — n is odd. Comparing this with the value already found, we shall have

$$\gamma = 2\left(\frac{1. 3. 5. \ldots . (2i - 1)}{1. 2. 3. \ldots . i}\right)^2 . \frac{i\,(i - 1) \ldots . (i - n + 1}{(i + 1)\ (i + 2) \ldots . (i + n)};$$

an expression which is the same as in the case of i — n being even.

If n = 0, we also have

$$\gamma = \left(\frac{1. 3. 5. \ldots . (2i - 1)}{1. 2. 3. \ldots . i}\right)^2.$$

560. From what precedes, we may obtain the general form of functions $Y^{(i)}$ of m, $\sqrt{1-m^2}.\sin.\varpi$, and $\sqrt{1-m^2}.\cos.\varpi$, which satisfy the equation of partial differences

$$0=\left(\frac{d\left\{(1-m^2).\left(\frac{dY}{dm}\right)\right\}}{dm}\right)+\frac{\left(\frac{d^2Y^{(i)}}{d\varpi^2}\right)}{1-m^2}+i(i+1)\,Y^{(i)}.$$

Designating by β, the coefficient of sin. $n\varpi$, or of cos. $n\varpi$, in the function $Y^{(i)}$, we shall have

$$0=\frac{d\left\{(1-m^2)\left(\frac{d\beta}{dm}\right)\right\}}{dm}-\frac{n^2\beta}{1-m^2}+i(i+1).\beta.$$

β is equal to $(1-m^2)^{\frac{n}{2}}$ multiplied by a rational and entire function of m, and in this case, by the preceding No., we have

$$\beta=A^{(n)}(1-m^2)^{\frac{n}{2}}\left\{m^{i-n}-\frac{(i-n)(i-n-1)}{2(2i-1)}m^{i-n-2}+\&c.\right\},$$

$A^{(n)}$ being an arbitrary constant; thus the part of $Y^{(i)}$ depending on the angle $n\varpi$, is

$$(1-m^2)^{\frac{n}{2}}\left\{m^{i-n}-\frac{(i-n)(i-n-1)}{2(2i-1)}.m^{i-n-2}+\&c.\right\}.\{A^{(n)}\sin.n\varpi + B^{(n)}\cos.n\varpi\};$$

$A^{(n)}$ and $B^{(n)}$ being two arbitraries. If we make successively in this function, $n=0$, $n=1$, $n=2\ldots n=i$; the sum of all the functions which thence result, will be the general expression of $Y^{(i)}$, and this expression will contain $2i+1$ arbitraries $B^{(0)}$, $A^{(1)}$, $B^{(1)}$, $A^{(2)}$, $B^{(2)}$, &c.

Let us now consider a rational and entire function S of the order s, of the three rectangular coordinates x, y, z. If we represent by R the distance of the point determined by these coordinates from their origin; by θ the angle formed by R and the axis of x; and by ϖ the angle which the plane of x, y forms with the plane passing through R and the axis of x; we shall have

$$x=Rm;\quad y=R.\sqrt{1-m^2}.\cos.\varpi;\quad z=R\sqrt{1-m^2}.\sin.\varpi.$$

Substituting these values in S, and developing this function into sines and cosines of the angle ϖ and its multiples, if S is the most general function of the order s, then sin. $n\varpi$, and cos. $n\varpi$, will be multiplied by functions of the form

$$(1-m^2)^{\frac{n}{2}}\{A.m^{s-n}+B.m^{s-n-1}+C.m^{s-n-2}+\&c.\};$$

thus the part of S, depending on the angle $n\varpi$, will contain $2(s-n+1)$ indeterminate constants. The part of S depending on the angle ϖ and its multiples will contain therefore $s(s+1)$ indeterminates; the part inde-

pendent of ϖ will contain $s + 1$, and S will therefore contain $(s + 1)^2$ indeterminate constants.

The function $Y^{(0)} + Y^{(1)} +$ &c. $Y^{(s)}$ contains in like manner $(s + 1)^2$ indeterminate constants, since the function $Y^{(i)}$ contains $2i + 1$; we may therefore put S into a function of this form, and this may be effected as follows:

From what precedes we shall learn the most general expression of $Y^{(s)}$, we shall take it from S and determine the arbitraries of $Y^{(s)}$ so that the powers and products of m and $\sqrt{1 - m^2}$ of the order s shall disappear from the difference $S - Y^{(s)}$; this difference will thus become a function of the order $s - 1$ which we shall denote by S'. We shall take the most general expression of $Y^{(s-1)}$; we shall subtract it from S', and determine the arbitraries of $Y^{(i-1)}$ so that the powers and products of m and $\sqrt{1 - m^2}$ of the order $s - 1$ may disappear from the difference $S' - Y^{(s-1)}$. Thus proceeding we shall determine the functions $Y^{(s)}$, $Y^{(s-1)}$, $Y^{(s-2)}$, &c. of which the sum is S.

561. Resume now, the equation of No. 559,

$$U^{(i)} = \int \varrho . R^{i+2} \, dR . dm' . d\varpi' . Q^{(i)}.$$

Suppose R a function of m', ϖ' and of a parameter a, constant for all shells of the same density, and variable from one shell to another. The difference d R being taken on the supposition that m', ϖ' are constant we shall have

$$dR = \left(\frac{dR}{da}\right) da;$$

therefore

$$U^{(i)} = \frac{1}{i + 3} . \int \varrho \left(\frac{d . R^{i+3}}{da}\right) da . dm' . d\varpi' . Q^{(i)}.$$

Let R^{i+3} be developed into a series of the form

$$Z'^{(0)} + Z'^{(1)} + Z'^{(2)} + \text{&c.},$$

$Z'^{(i)}$ being whatever i may be, a rational and entire function of m', $\sqrt{1 - m'^2}$. sin. ϖ', and $\sqrt{1 - m'^2}$. cos. ϖ', which satisfies the equation of partial differences

$$0 = \left(\frac{d\left\{(1 - m^2).\left(\frac{dZ'^{(i)}}{dm'}\right)\right\}}{dm'} \right) + \frac{\left(\frac{d^2 Z'^{(i)}}{d\varpi'^2}\right)}{1 - m'^2} + i(i + 1) Z'^{(i)}.$$

The difference of $Z'^{(i)}$ taken relatively to a, satisfies also this equation, and consequently it is of the same form; by the general theorem of 556, we ought therefore only to consider the term $Z'^{(i)}$ in the developement of R^{i+3}, and then we have

$$U^{(i)} = \frac{1}{i + 3} . \int \varrho \left(\frac{dZ'^{(i)}}{da}\right) . da . dm' . d\varpi' . Q^{(i)}.$$

When the spheroid is homogeneous and differing but little from a sphere, we may suppose $\varrho = 1$, and $R = a(1 + \alpha y')$; then we have, by integrating relatively to a

$$U^{(i)} = \frac{1}{i+3} \int Z'^{(i)}.\, dm'.\, d\varpi'.\, Q^{(i)}.$$

Moreover, if we suppose y' developed into a series of the form

$$Y'^{(0)} + Y'^{(1)} + Y'^{(2)} + \&c.,$$

$Y^{(i)}$ satisfying the same equation of partial difference as $Z'^{(i)}$; we shall have, neglecting quantities of the order α^2, $Z'^{(i)} = (i+3).\alpha.a^{i+3} Y'^{(i)}$; we shall therefore have

$$U^{(i)} = \alpha.a^{i+3}.\int Y'^{(i)}.\, dm'.\, d\varpi'.\, Q^{(i)}.$$

If we denote by $Y^{(i)}$ what $Y'^{(i)}$ becomes when we change m' and ϖ' into m and ϖ; we shall have by No. 554,

$$U^{(i)} = \frac{4\alpha\pi.a^{i+3}}{2i+1}.\, Y^{(i)};$$

we therefore have this remarkable result,

$$\int Y'^{(i)}.\, dm'.\, d\varpi'.\, Q^{(i)} = \frac{4\pi Y^{(i)}}{2i+1} \quad \ldots\ldots \quad (1)$$

This equation subsisting whatever may be $Y'^{(i)}$ we may conclude generally that the double integration of the function $\int Z'^{(i)}\, dm'.\, d\varpi'.\, Q^{(i)}$ taken from $m' = -1$ to $m' = 1$, and from $\varpi' = 0$ to $\varpi' = 2\pi$, only transforms $Z'^{(i)}$ into $\frac{4\pi Z^{(i)}}{2i+1}$; $Z^{(i)}$ being what $Z'^{(i)}$ becomes when we change m' and ϖ' into m and ϖ; we therefore have

$$U^{(i)} = \frac{4\pi}{(i+3)(2i+1)} \int \varrho \left(\frac{d\, Z^{(i)}}{da}\right).\, da;$$

and the triple integration upon which $U^{(i)}$ depends, reduces to one integration only taken relatively to a, from $a = 0$ to its value at the surface of the spheroid.

The equation (1) presents a very simply method of integrating the function $\int Y^{(i)}.\, Z^{(i)}.\, dm.\, d\varpi$, from $m = -1$ to $m = 1$, and from $\varpi = 0$ to $\varpi = 2\pi$. In fact, the part of $Y^{(i)}$ depending on the angle $n\varpi$, is by what precedes, of the form $\lambda\{A^{(n)} \text{ sin. } n\varpi + B^{(n)} \text{ cos. } n\varpi\}$, λ being equal to

$$(1 - m^2)^{\frac{n}{2}}.\left\{m^{i-n} - \frac{(i-n)(i-n-1)}{3(2i-1)}.\, m^{i-n-2} + \&c.\right\};$$

we shall have therefore

$$Y'^{(i)} = \lambda'\{A^{(n)} \text{ sin. } n\varpi' + B^{(n)} \text{ cos. } n\varpi'\};$$

λ' being what λ becomes when m is changed into m'. The part of $Q^{(i)}$ depending on the angle $n\varpi$, is by the preceding No., $\gamma\lambda\lambda'$ cos. $n(\varpi - \varpi')$,

or $\gamma\,\lambda'.\,\lambda.\{\cos.\,n\,\varpi.\,\cos.\,n\,\varpi' + \sin.\,n\,\varpi.\,\sin.\,n\,\varpi'\}$; thus that part of the integral $\int Y^{(i)}.\,d\,m.\,d\,\varpi'.\,Q^{(i)}$ which depends on the angle $n\,\varpi$, will be

$\gamma\,\lambda.\,\sin.\,n\,\varpi.\int\lambda'^{2}.\,d\,m'.\,d\,\varpi'.\,\sin.\,n\,\varpi'\,\{A^{(n)}\sin.\,n\,\varpi' + B^{(n)}\cos.\,n\,\varpi'\}$

$\gamma\,\lambda.\,\cos.\,n\,\varpi\int\lambda'^{2}.\,d\,m'.\,d\,\varpi'.\,\cos.\,n\,\varpi'\,\{A^{(n)}\sin.\,n\,\varpi' + B^{(n)}\cos.\,n\,\varpi'\}$.

Executing the integrations relative to ϖ', that part becomes

$$\gamma\,\lambda\,\varpi\,\{A^{(n)}\sin.\,n\,\varpi + B^{(n)}\cos.\,n\,\varpi\}.\int\lambda'^{2}.\,d\,m';$$

but in virtue of equation (1), the same part is equal to

$$\frac{4\,\pi}{2\,i+1}.\,\lambda.\,\{A^{(n)}\sin.\,n\,\varpi + B^{(n)}\cos.\,n\,\varpi\}$$

we therefore have

$$\int\lambda'^{2}.\,d\,m' = \frac{4}{(2\,i+1)\,\gamma}.$$

Now represent by $\lambda\,\{A'^{(n)}\sin.\,n\,\varpi + B'^{(n)}\cos.\,n\,\varpi\}$ that part of $Z^{(i)}$ which depends on the angle $n\,\varpi$. This part ought to be combined with the corresponding part of $Y^{(i)}$; because the terms depending on the sines and cosines of the angle ϖ and its multiples, disappear by integration, in the function $\int Y^{(i)}\,Z^{(i)}\,d\,m.\,d\,\varpi$, integrated from $\varpi = 0$ to $\varpi = 2\,\pi$; we shall thus obtain, in regarding only that part of $Y^{(i)}$ which depends on the angle $n\,\varpi$,

$$\int Y^{(i)}.\,Z^{(i)}\,d\,m\,d\,\varpi =$$

$$\int\lambda^{2}.\,d\,m.\,d\,\varpi\{A^{(n)}\sin.\,n\,\varpi + B^{(n)}\cos.\,n\,\varpi\}\{A'^{(n)}\sin.\,n\,\varpi + B'^{(n)}\cos.\,n\,\varpi\}$$

$$= \varpi\{A^{(n)}.\,A'^{(n)} + B^{(n)}.\,B'^{(n)}\}.\int\lambda^{2}\,d\,m = \frac{4\,\pi}{(2\,i+1)\,\gamma}.\{A^{(n)}.\,A'^{(n)} + B^{(n)}.\,B'^{(n)}\}.$$

Supposing therefore successively in the last member $n = 0$, $n = 1$, $n = 2 \ldots n = i$; the sum of all the terms, will be the value of the integral $\int Y^{(i)}\,Z^{(i)}\,d\,m.\,d\,\varpi$.

If the spheroid is one of revolution, so that the axis with which the radius R forms the angle ϖ, may be the axis of revolution; the angle ϖ will disappear from the expression of $Z^{(i)}$, which then takes the following form:

$$\frac{1.3.5\ldots(2\,i-1)}{1.2.3\ldots i}\,A^{(i)}\left\{m^{(i)} - \frac{i.(i-1)}{2.(2\,i-1)}\,m^{i-2} + \frac{i.(i-1)\,(i-2)\,(i-3)}{2.4.(2\,i-1)(2\,i-3)}\,m^{i-4} - \&c.\right\};$$

$A^{(i)}$ being a function of a. Call $\lambda^{(i)}$ the coefficient of $A^{(i)}$, in this function: the product

$$\left(\frac{1.3.5\ldots(2\,i-1)}{1.2.3\ldots i}\right)^{2}.\left\{1 - \frac{i.(i-1)}{2.(2\,i-1)} + \&c.\right\}^{2},$$

is by the preceding No., the coefficient of $\frac{R^{i}}{r^{i+1}}$ in the developement of the radical

$$\{r^{2} - 2\,R\,r\,\{m\,m' + \sqrt{1-m^{2}}.\sqrt{1-m'^{2}}\cos.(\varpi - \varpi')\} + R^{2}\}^{-\frac{1}{2}};$$

when we therein suppose m and m′ equal to unity. This coefficient is then equal to 1; we have therefore

$$\frac{1.3.5\ldots(2i-1)}{1.2.3\ldots i}\left\{1-\frac{i(i-1)}{2(2i-1)}+\&c.\right\}=1,$$

that is to say, $\lambda^{(i)}$ reduces to unity, when m = 1. We have then

$$U^{(i)}=\frac{4\pi\lambda^{(i)}}{(i+3).(2i+1)}\cdot\int\varrho\left(\frac{dA^{(i)}}{da}\right)da.$$

Relatively to the axis of revolution, m = 1, and consequently,

$$U^{(i)}=\frac{4\pi}{(1+3)(2i+1)}\cdot\int\varrho\left(\frac{dA^{(i)}}{da}\right)da;$$

therefore if we suppose that relatively to a point placed upon this axis produced, we have

$$V=\frac{B^{(0)}}{r}+\frac{B^{(1)}}{r^2}+\frac{B^{(2)}}{r^3}+\&c.;$$

we shall have the value of V relative to another point placed at the mean distance from the origin of coordinates, but upon a radius which makes with the axis of revolution, an angle whose cosine is m; by multiplying the terms of this value respectively by $\lambda^{(0)}$, $\lambda^{(1)}$, $\lambda^{(2)}$, &c.

In the case when the spheroid is not of revolution, this method will give the part of V independent of the angle ϖ: we shall determine the other part in this manner. Suppose for the sake of simplicity, the spheroid such that it is divided into two equal and similar parts by the equator, whether by the meridian where we fix the origin of the angle ϖ, or by the meridian which is perpendicular to the former. Then V will be a function of m^2, $\sin.^2\varpi$, and $\cos.^2\varpi$, or which comes to the same, it will be a function of m^2, and of the cosine of the angle 2ϖ and its multiples; $U^{(i)}$ will therefore be nothing, when i is odd, and in the case when it is even, the term which depends on the angle $2n\varpi$, will be of the form

$$C^{(i)}.(1-m^2)^n\left\{m^{i-2n}-\frac{(i-2n)(i-2n-1)}{2(2i-1)}m^{i-2n-2}+\&c.\right\}\cos.2n\varpi.$$

Relatively to an attracted point placed in the plane of the equator, where m = 0, that part of V which depends on this term becomes

$$\pm\frac{C^{(i)}}{r^{i+1}}\cdot\frac{1.3.5\ldots(i-2n-1)}{2(i+2n+1)(i+2n+2)\ldots(2i-1)}\cdot\cos.2n\varpi;$$

whence it follows that having developed V into a series ordered according to the cosines of the angle 2ϖ and its multiples, when the point attracted is situated in the plane of the equator; to extend this value to any attracted point whatever, it will be sufficient to multiply the terms which depend on $\frac{\cos.2n\varpi}{r^{i+1}}$ by the function

$$\pm\frac{2(i+2n+1)\ldots(2i-1)}{1.3.5\ldots(i-2n-1)}\cdot(1-m^2)^n\left\{m^{i-2n}-\frac{(i-2n)(i-2n-1)}{2(2i-1)}\right.$$
$$\left.m^{i-2n-2}+\&c.\right\};$$

we shall hence obtain, therefore, the entire value of V, when this value shall be determined in a series, for the two cases where the part attracted is situated upon the polar axis produced, and where it is situated in the plane of the equator; this greatly simplifies the research of this value.

The spheroid which we are considering comprehends the ellipsoid. Relatively to an attracted point situated upon the polar axis, which we shall suppose to be the axis of x, by 546, we have $b = 0$, $c = 0$, and then the expression of V of No. 549, is integrable relatively to p. Relatively to a point situated in the plane of the equator, we have $a = 0$, and the same expression of V still becomes, by known methods, integrable relatively to q, by making tan. $q = t$. In the two cases, the integral being taken relatively to one of these variables in its limits, it then becomes possible relatively to the other, and we find that M being the mass of the spheroid, the value of $\frac{V}{M}$ is independent of the semi-axis k of the spheroid perpendicular to the equator, and depends only on the excentricities of the ellipsoid. Multiplying therefore the different terms of the values of $\frac{V}{M}$ relative to these two cases, and reduced into series proceeding according to the powers of $\frac{1}{r}$, by the factors above mentioned, to get the value of $\frac{V}{M}$ relative to any attracted point whatever; the function which thence results will be independent of k, and only depend on the excentricities; this furnishes a new demonstration of the theorem already proved in 550.

If the point attracted is placed in the interior of the spheroid, the attraction which it undergoes, depends, as we have seen in No. 553, on the function $v^{(i)}$, and by the No. cited, we have

$$v^{(i)} = \int\frac{\varrho\, d\,R\, d\,m'\, d\,\varpi'.\, Q^{(i)}}{R^{i-1}};$$

an equation which we can put under this form

$$v^{(i)} = \frac{1}{2-i}\cdot\int\varrho\left(\frac{d\,R^{2-i}}{d\,a}\right) d\,a.\,d\,m'.\,d\,\varpi'.\,Q^{(i)}.$$

Suppose R^{2-i} developed into a series of the form

$$z'^{(0)} + z'^{(1)} + z'^{(2)} + \&c.$$

$z'^{(i)}$ satisfying the equation of partial differences,

$$0 = \left(\frac{d\left\{(1-m'^2)\left(\frac{d\,z'^{(i)}}{d\,m'}\right)\right\}}{d\,m'} \right) + \frac{\left(\frac{d^2 z'^{(i)}}{d\,\varpi'^2}\right)}{1-m'^2} + i(i+1)\,z'^{(i)};$$

if moreover we call $z^{(i)}$ what $z'^{(i)}$ becomes when we change m' into m, and ϖ' into ϖ, we shall have by what precedes,

$$v^{(i)} = \frac{4\pi}{(2i+1)(2-i)} \int \varrho \left(\frac{d\,z^{(i)}}{d\,a}\right) d\,a;$$

thus therefore we shall get the expression of V relative to all the shells of the spheroid which envelope the point attracted. The value of V relative to shells to which it is interior, we have already shown how to determine.

ON THE FIGURE OF A FLUID HOMOGENEOUS MASS IN EQUILIBRIUM, AND ENDOWED WITH A ROTATORY MOTION.

562. Having exposed in the preceding Nos. the theory of the attractions of spheroids, we now proceed to consider the figure which they must assume in virtue of the mutual action of their parts, and the other forces which act upon them. We shall first seek the figure which satisfies the equilibrium of a fluid homogeneous mass endowed with a rotatory motion, and of that problem we shall give a rigorous solution.

Let a, b, c be the rectangular coordinates of any point of the surface of the mass, and P, Q, R the forces which solicit it parallel to the coordinates, the forces being supposed as tending to diminish them. We know that when the mass is in equilibrium, we have

$$0 = P.da + Q.db + R.dc;$$

provided that in estimating the forces P, Q, R, we reckon the centrifugal force due to the motion of rotation.

To estimate these forces, we shall suppose that the figure of the fluid mass, is that of the ellipsoid of revolution, whose axis of rotation, is the axis itself of revolution. If the forces P, Q, R which result from this hypothesis, substituted in the preceding equation of equilibrium give the differential equation of the surface of the ellipsoid; the preceding hypothesis is legitimate, and the elliptic figure satisfies the equilibrium of the fluid mass.

Suppose that the axis of a is that also of revolution; the equation of the surface of the ellipsoid will be of this form

$$a^2 + m(b^2 + c^2) = k^2;$$

the origin of the coordinates a, b, c being at the center of the ellipsoid, k will be the semi-axis of revolution, and if we call M the mass of the ellipsoid, by 546, we shall have

$$M = \frac{4\pi\varrho k^3}{3m}$$

ϱ being the density of the fluid. If we make as in 547, $\frac{1-m}{m} = \lambda^2$, we shall have $m = \frac{1}{1+\lambda^2}$, and consequently

$$M = \frac{4\pi}{3} k^3 . (1+\lambda^2);$$

an equation which will give the semi-axis k, when λ is known.

Let

$$A' = \frac{4\pi\varrho(1+\lambda^2)}{\lambda^3}(\lambda - \tan.^{-1}\lambda)$$

$$B' = \frac{4\pi\varrho}{2\lambda^3}\{(1+\lambda^2)\tan.^{-1}\lambda - \lambda)\};$$

we shall have by 547, regarding only the attraction of the fluid mass

$$P = A'a;\ Q = B'b;\ R = B'c.$$

If we call g, the centrifugal force at the distance 1, from the axis of rotation; this force at the distance $\sqrt{b^2+c^2}$ from the same axis, will be $g\sqrt{b^2+c^2}$: resolving this parallel to the coordinates b, c there will result in Q the term $-gb$, and in R the term $-gc$; thus we shall have, reckoning all the forces which animate the molecules of the surface,

$$P = A'a;\ Q = (B'-g)b;\ R = (B'-g).c;$$

the preceding equation of equilibrium, will therefore become

$$0 = a\,da + \frac{B'-g}{A'}(b\,db + c\,dc).$$

The differential equation of the surface of the ellipsoid is by substituting for m its value $\frac{1}{1+\lambda^2}$,

$$0 = a\,da + \frac{b\,db + c\,dc}{1+\lambda^2};$$

by comparing this with the preceding one, we shall have

$$(1+\lambda^2)(B'-g) = A'; \quad \ldots\ldots\ldots (1)$$

if we substitute for A', B' their values, and if we make $\frac{g}{\frac{4}{3}\pi\varrho} = q$; we shall have

$$0 = \frac{9.\lambda + 2q.\lambda^3}{9+3.\lambda^2} - \tan.^{-1}\lambda; \quad \ldots\ldots\ldots (2)$$

determining therefore λ by this equation which is independent of the coordinates a, b, c, the equation of equilibrium will coincide with the equation of the surface of the ellipsoid; whence it follows, that the elliptic figure satisfies the equilibrium, at least, when the motion of rotation is such that the value of λ^2 is not imaginary, or when being negative, it is neither equal to nor greater than unity. The case where λ^2 is imaginary would give an imaginary solid; that where $\lambda^2 = -1$, would give a paraboloid, and that where λ^2 is negative and greater than unity, would give a hyperboloid.

563. If we call p the gravity at the surface of the ellipsoid, we shall have

$$p = \sqrt{P^2 + Q^2 + R^2}.$$

In the interior of the ellipsoid, the forces P, Q, R, are proportional to the coordinates a, b, c; for we have seen in No. 547, that the attractions of the ellipsoid, parallel to these coordinates, are respectively proportional to them, which equally takes place for the centrifugal force resolved parallel to the same coordinates. Hence it follows, that the gravities at different points of a radius drawn from the center of the ellipsoid to its surface, have parallel directions, and are proportional to the distances from the center; so that if we know the gravity at its surface, we shall have the gravity in the interior of the spheroid.

If in the expression of p, we substitute for P, Q, R, their values given in the preceding No., we shall have

$$p = \sqrt{A'^2 a^2 + (B' - g)^2 . (b^2 + c^2)};$$

whence we derive, in virtue of equation (1) of the preceding No.

$$p = A' \sqrt{a^2 + \frac{b^2 + c^2}{(1 + \lambda^2)^2}};$$

but the equation of the surface of the ellipsoid gives $\frac{b^2 + c^2}{1 + \lambda^2} = k^2 - a^2$; we shall therefore have

$$p = A' \sqrt{\frac{k^2 + \lambda^2 a^2}{1 + \lambda^2}},$$

a is equal to k at the pole, and it is nothing at the equator; whence it follows, that the gravity at the pole is to the gravity at the equator, as $\sqrt{1 + \lambda^2}$ is to unity, and consequently, as the diameter of the equator is to the polar axis.

Call t the perpendicular at the surface of the ellipsoid, produced to meet the axis of revolution, we shall have

$$t = \sqrt{(1 + \lambda^2)(k^2 + \lambda^2 a^2)};$$

wherefore

$$p = \frac{A' t}{1 + \lambda^2};$$

thus gravity is proportional to t.

Let ψ be the complement of the angle which t makes with the axis of revolution; ψ will be the latitude of the point of the surface, which we are considering, and by the nature of the ellipse, we shall have

$$t = \frac{(1 + \lambda^2)\, k}{\sqrt{1 + \lambda^2 \cos.^2 \psi}};$$

we therefore shall have

$$p = \frac{A' k}{\sqrt{1 + \lambda^2 . \cos.^2 \psi}};$$

and substituting for A' its value, we shall get

$$p = \frac{4 \pi \varrho . k . (1 + \lambda^2) . (\lambda - \tan.^{-1} \lambda)}{\lambda^3 \sqrt{1 + \lambda^2 . \cos.^2 \psi}}; \quad . \quad . \quad . \quad . \quad (3)$$

this equation gives the relation between gravity and the latitude; but we must determine the constants which it contains.

Let T be the number of seconds in which the fluid mass will effect a revolution; the centrifugal force at the distance 1 from the axis of revolution, will be equal to $\frac{4 \pi^2}{T^2}$; we therefore have

$$q = \frac{g}{\frac{4}{3} \pi . \varrho} = \frac{12 \pi^2}{4 \pi \varrho T^2};$$

which gives

$$4 \pi \varrho = \frac{12 . \pi^2}{q . T^2}.$$

The radius of curvature of the elliptic meridian is

$$\frac{(1 + \lambda^2)\, k}{(1 + \lambda^2 \cos.^2 \psi)^{\frac{3}{2}}};$$

calling therefore c the length of a degree at the latitude ψ, we shall have

$$\frac{(1 + \lambda^2)\, \pi k}{(1 + \lambda^2 \cos.^2 \psi)^{\frac{3}{2}}} = 200\ c.$$

This equation combined with the preceding one, gives

$$\frac{4 \pi \varrho (1 + \lambda^2) . k}{\sqrt{1 + \lambda^2 \cos.^2 \psi}} = 200\ c\ (1 + \lambda^2 \cos.^2 \psi) . \frac{12 \pi}{q T^2};$$

thus we shall have

$$p = 200\ c\ (1 + \lambda^2 \cos.^2 \psi) \frac{\lambda - \tan.^{-1} \lambda}{\lambda^3} . \frac{12 \pi}{q T^2}.$$

Let l be the length of the simple pendulum which oscillates seconds;

from dynamics it results that $p = \pi^2 l$ (see § X.); comparing these two expressions of p, we get

$$q = \frac{2400\, c\, (\lambda - \tan.^{-1}\lambda)\, (1 + \lambda^2 \cos.^2 \psi)}{\pi l T^2 \lambda^3}; \quad \ldots \quad (4)$$

this equation and equation (2) of the preceding No. will give the values of q and λ by means of the length l of the seconds' pendulum, and the length c of the degree of the meridian, both being observed at the latitude ψ.

Suppose $\psi = 50°$, these equations will give

$$q = \frac{800\, c}{\pi l T^2} - \tfrac{1}{4}\left(\frac{800\, c}{\pi l T^2}\right)^2 + \&c.;$$

$$\lambda^2 = \frac{5}{2} q + \frac{75}{14} q^2 + \&c.;$$

observations give, as we shall see hereafter,

$$c = 100000; \; l = 0.741608;$$

moreover we have $T = 99727$; we shall thus obtain

$$q = 0.00344957; \; \lambda^2 = 0.00868767.$$

The ratio of the axis of the equator to the polar axis, being $\sqrt{1 + \lambda^2}$, it becomes in this case 1.00433441; these two axes are very nearly in the ratio of 231.7 to 230.7, and by what precedes, the gravities at the pole and at the equator are in the same ratio.

We shall have the semi polar axis k, by means of the equation

$$k = \frac{200\, c\, (1 + \frac{1}{2}\lambda^2)^{\frac{5}{2}}}{\pi (1 + \lambda^2)} = \frac{200\, c}{\pi} \{1 - \tfrac{1}{4}\lambda^2 + \&c.\};$$

which gives

$$k = 6352534.$$

To get the attraction of a sphere, whose radius is k, and density any whatever; we shall observe that a sphere, having the radius k and density ϱ, acts upon a point placed at its surface, with a force equal to $\frac{4}{3}\pi\varrho . k$, and consequently, in virtue of equation (3) equal to $\frac{\lambda^3 p \sqrt{1 + \frac{1}{2}\lambda^2}}{3 (1+\lambda^2) (\lambda - \tan.^{-1}\lambda)}$, or to $p\left(1 - \frac{3}{20}\lambda^2 + \&c.\right)$, or finally to 0.998697. p, p being the gravity upon the parallel of 50°. Hence it is easy to obtain the attractive force of a sphere of any radius and density whatever, upon a point placed within or without it.

564. If the equation (2) of No. 562, were susceptible of many real roots, many figures of equilibrium might result from the same motion of rotation; let us examine therefore whether this equation has several real

roots. For that purpose, call φ the function $\frac{9\lambda + 2q\lambda^3}{9 + 3\lambda^2} - \text{tan.}^{-1}\lambda$, which being equated to zero, produces the equation (2). It is easy to see, that by making λ increase from zero to infinity, the expression of φ begins and ends by being positive; thus, by imagining a curve whose abscissa is λ and ordinate φ, this curve will cut its axis when $\lambda = 0$; the ordinates will afterwards be positive and increasing; when arrived at their maximum, they will decrease; the curve will cut the axis a second time at a point which will determine the value of λ corresponding to the state of equilibrium of the fluid mass; the ordinates will then be negative, and since they are positive when $\lambda = \infty$; the curve necessarily cuts the axis a third time, which determines a second value of λ which satisfies the equilibrium. Thus we see, that for one and the same value of q, or for one given motion of rotation, there are several figures for which the equilibrium may subsist.

To determine the number of these figures, we shall observe, that we have

$$d\varphi = \frac{6\lambda^2 d\lambda \{q\lambda^4 + (10q - 6)\lambda^2 + 9q\}}{(3\lambda^2 + 9)^2 . (1 + \lambda^2)}.$$

The supposition of $d\varphi = 0$, gives

$$0 = q\lambda^4 + (10q - 6)\lambda^2 + 9q;$$

whence we derive, considering only the positive values of λ

$$\lambda = \sqrt{\frac{3}{q} - 5 \pm \sqrt{\left(\frac{3}{q} - 5\right)^2 - 9}}.$$

These values of λ determine the *maxima* and *minima* of the ordinate φ; there being only two similar ordinates on the side of positive abscissas, on that side the curve cuts its axis in three points, one of them being the origin; thus, the number of figures which satisfy the equilibrium is reduced to two.

The curve on the side of negative abscissas being exactly the same as on the side of positive abscissas; it cuts its axis on each side in corresponding points equidistant from the origin of coordinates; the negative values of λ which satisfy the equilibrium, are therefore, as to the sign taken, the same as the positive values; which gives the same elliptic figures, since the square of λ only enters the determination of these figures; it is useless therefore to consider the curve on the side of negative abscissas.

If we suppose q very small, which takes place for the earth, we may satisfy equation (2) of 562, in the two hypotheses of λ^2 being very small,

and of λ^2 being very great. In the first, by the preceding No., we have

$$\lambda^2 = \frac{5}{2}\,q + \frac{75}{14}\,q^2 + \&c.$$

To get the value of λ^2 in the second hypothesis, we shall observe that then tan.$^{-1}\lambda$ differs very little from $\frac{1}{2}\pi$, so that if we suppose $\lambda = \frac{\pi}{2} - \alpha$, α will be a very small angle of which the tangent is $\frac{1}{\lambda}$; we shall therefore have, p. 27. Vol. I.

$$\alpha = \frac{1}{\lambda} - \frac{1}{3\lambda^3} + \frac{1}{5\lambda^5} - \&c.$$

and consequently

$$\text{tan.}^{-1}\lambda = \frac{\pi}{2} - \frac{1}{\lambda} + \frac{1}{3\lambda^3} - \frac{1}{5\lambda^5} + \&c.;$$

equation (2) of No. 562, will thus become

$$\frac{9\lambda + 2q.\lambda^3}{9 + 3\lambda^2} = \frac{\pi}{2} - \frac{1}{\lambda} + \frac{1}{3\lambda^3} - \&c.;$$

whence by the reversion of series we get

$$\lambda = \frac{3\pi}{4q} - \frac{8}{\pi} + \frac{4q}{\pi}\left(1 - \frac{64}{3\pi^2}\right) + \&c.$$

$$= 2.356195.\frac{1}{q} - 2.546479 - 1.478885\,q + \&c.$$

We have seen in the preceding No., that relatively to the earth, $q = 0.00344957$; this value of q substituted in the preceding expression, gives $\lambda = 680.49$. Thus the ratio of the two axes equatorial and polar, a ratio which is equal to $\sqrt{1 + \lambda^2}$, is in the case of a very thin spheroid, equal to 680.49.

The value of q has a limit beyond which the equilibrium is impossible, the figure being elliptic. Suppose, in fact, that the curve cuts its axis only at its origin, and that in the other points it only touches; at the point of contact we shall have $\varphi = 0$, and $d\varphi = 0$; the value of φ will never therefore be negative on the side of positive abscissas, which are the only ones we shall here consider. The value of q determined by the two equations $\varphi = 0$, $d\varphi = 0$, will therefore be the limit of those with which the equilibrium can take place, so that a greater value will render the equilibrium impossible; for q being supposed to increase by f, the function φ increases by the term $\frac{2f\lambda^3}{9 + 3\lambda^2}$; thus, the value of φ corresponding to q, being never negative, whatever λ may be, the same function corresponding to $q + f$, is constantly positive, and can never become no-

thing; the equilibrium is then therefore impossible. It results also from this analysis, that there is only one real and positive value of q, which would satisfy the two equations $\varphi = 0$, and $d\varphi = 0$. These equations give

$$q = \frac{6\lambda^2}{(1+\lambda^2)(9+\lambda^2)}$$

$$0 = \frac{7\lambda^5 + 30\lambda^3 + 27\lambda}{(1+\lambda^2)(3+\lambda^2)(9+\lambda^2)} - \text{tan.}^{-1}\lambda.$$

The value of λ which satisfies this last equation is $\lambda = 2.5292$; whence we get $q = 0.337007$; the quantity $\sqrt{1+\lambda^2}$, which expresses the ratio of the equatorial axis to the polar axis, is in this case equal to 2.7197.

The value of q relatively to the earth is equal to 0.00344957. This value corresponds to a time of rotation of 0.99727 days; but we have generally $q = \frac{g}{\frac{4}{3}\pi\varrho}$, so that relatively to masses of the same density, q is proportional to the centrifugal force g of the rotatory motion, and consequently in the inverse ratio of the square of the time of rotation; whence it follows, that relatively to a mass of the same density as the earth, the time of rotation which answers to $q = 0.337007$, is 0.10090 days. Whence result these two theorems.

"Every homogeneous fluid mass of a density equal to the mean density of the earth, cannot be in equilibrium having an elliptic figure, if the time of its rotation is less than 0.10090 days. If this time be greater, there will be always two elliptic figures and no more which satisfy the equilibrium."

"If the density of the fluid mass is different from that of the earth; we shall have the time of rotation in which the equilibrium ceases to be possible under an elliptic figure, by multiplying 0.10090 days by the square root of the ratio of the mean density of the earth to that of the fluid mass."

This relatively to a fluid mass, whose density is only a fourth part of that of the earth, which nearly is the case with the sun, this time will be 0.20184 days; and if the density of the earth supposed fluid and homogeneous were about 98 times less than its actual density, the figure which it ought to take to satisfy its actual motion of rotation, would be the limit of all the elliptic figures with which the equilibrium can subsist. The density of Jupiter being about five times less than that of the earth, and the time of its rotation being 0.41377 days; we see that this duration is in the limits of those of equilibrium.

It may be thought that the limit of q, is that where the fluid would begin to fly off by reason of a too rapid motion of rotation; but it is easy to be convinced of the contrary, by observing that by 563, the gravity at the equator of the ellipsoid is to that at the pole in the ratio of the polar axis to that of the equator, a ratio which in this case, is that of 1 to 2.7197; the equilibrium ceases therefore to be possible, because with a motion of rotation more rapid, it is impossible to give to the fluid mass, an elliptic figure such that the resultant of its attraction and of the centrifugal force, may be perpendicular to the surface.

Hitherto we have supposed λ^2 positive, which gives the spheroids flattened towards the poles; let us now examine whether the equilibrium can subsist with a figure lengthened towards the poles, or with a prolate spheroid. Let $\lambda^2 = -\lambda'^2$; λ'^2 must be positive and less than unity, otherwise, the ellipsoid will be changed into a hyperboloid. The preceding value of $d\varphi$ gives

$$\varphi = \int \frac{\lambda^2 d\lambda \{q\lambda^4 + (10q - 6)\lambda^2 + 9q\}}{(1+\lambda^2)(9+3\lambda^2)^2};$$

the integral being taken from $\lambda = 0$. Substituting for λ its value $\pm\lambda'\sqrt{-1}$, we shall have

$$\varphi = \pm\sqrt{-1}.\int \frac{\lambda'^2 d\lambda' \{q.(1-\lambda'^2).(9-\lambda'^2)+6\lambda'^2\}}{(1-\lambda'^2)(9-3\lambda'^2)};$$

but it is evident that the elements of this last integral are all of the same sign from $\lambda'^2 = 0$, to $\lambda'^2 = 1$; the function φ can therefore never become nothing in this interval. Thus then the equilibrium cannot subsist in the case of the prolate spheroid.

565. If the motion of rotation primitively impressed upon the fluid mass, is more rapid than that which belongs to the limit of q, we must not thence infer that it cannot be in equilibrium with an elliptic figure; for we may conceive, that by flattening it more and more, it will take a rotatory motion less and less rapid; supposing therefore that there exists, as in the case of all known fluids, a force of tenacity between its molecules, this mass, after a great number of oscillations, may at length arrive at a rotatory motion, comprised within the limits of equilibrium, and may continue in that state. But this possibility it would also be interesting to verify; and it would be equally interesting to know whether there would not be many possible states of equilibrium; for what we have already demonstrated upon the possibility of two states of equilibrium, corresponding to one motion of rotation, does not infer the possibility of two states of equilibrium corresponding to one primitive force; because the two

states of equilibrium relative to one motion of rotation, require two primitive forces, either different in quantity or differently applied.

Consider therefore a fluid mass agitated primitively by any forces whatever, and then left to itself, and to the mutual attractions of all its parts. If through the center of gravity of this mass supposed immoveable, we conceive a plane relatively to which the sum of the areas described upon this plane, by each molecule, multiplied respectively by the corresponding molecules, is a *maximum* at the origin of motion; this plane will always have this property, whatever may be the manner in which the molecules act upon one another, whether by their tenacity, by their attraction, and their mutual collision, even in the very case where there is finite loss of motion in an instant of time; thus, when after a great number of oscillations, the fluid mass shall take a uniform rotatory motion about a fixed axis, this axis shall be perpendicular to the plane above-mentioned, which will be that of the equator, and the motion of rotation will be such that the sum of the areas described during the instant d t, by the molecules projected upon this plane, will be the same as at the origin of motion; we shall denote by E d t this last sum.

We shall here observe, that the axis in question, is that relatively to which the sum of the moments of the primitive forces of the system was a *maximum*. It retains this property during the motion of the system, and finally becomes the axis of rotation; for what is above asserted as to the *plane of the maximum of projected areas*, equally applies to the *axis of the greatest moment of forces*; since the elementary area described by the projection of the radius-vector of a body upon a plane, and multiplied by its mass, is evidently proportional to the moment of the finite force of this body relatively to the axis perpendicular to this plane.

Let, as above, g be the centrifugal force due to the rotatory motion at the distance 1 from the axis; $\sqrt{g}$ will be the angular velocity of rotation (p. 166. Vol. I.); then call k the semi-axis of rotation of the fluid mass, and $k\sqrt{1+\lambda^2}$ the semi-axis of its equator. It is easy to show that the sum of the areas described during the instant d t, by all the molecules projected upon the plane of the equator and multiplied respectively by the corresponding molecules, is

$$\frac{4\pi\varrho}{15}(1+\lambda^2)^2 . k^5 \, dt \sqrt{g}$$

we shall therefore have

$$\frac{4\pi\varrho}{15}(1+\lambda^2)^2 . k^5 \sqrt{g} = E.$$

Then calling M, the fluid mass, we shall have

$$\tfrac{4}{3}\pi k^3 \varrho (1 + \lambda^2) = M;$$

the quantity $\frac{g}{\frac{4}{3}\pi . \varrho}$, which we have called q, in No. 562, thus becomes $q' (1 + \lambda^2)^{-\frac{2}{3}}$, denoting by q' the function $\frac{25 E^2 (\frac{4}{3}\pi\varrho)^{\frac{1}{3}}}{M^{\frac{1}{3}}}$ The equation of the same No. becomes

$$0 = \frac{9\lambda + 2q'\lambda^3 (1 + \lambda^2)^{-\frac{2}{3}}}{9 + 3\lambda^2} - \text{tan.}^{-1}\lambda.$$

This equation will determine λ; we shall then have k by means of the preceding expression of M.

Call φ the function

$$\frac{9\lambda + 2q'\lambda^3 (1 + \lambda^2)^{-\frac{2}{3}}}{9 + 3\lambda^2} - \text{tan.}^{-1}\lambda,$$

which, by the condition of equilibrium, ought to be equal to zero: this equation begins by being positive, when λ is very small, and ends by being negative, when λ is infinite; there exists therefore between $\lambda = 0$, and $\lambda =$ infinity, a value of λ which renders this function nothing, and consequently, there is always, whatever q' may be, an elliptic figure, with which the fluid mass may be in equilibrium.

The value of φ may be put under this integral form

$$\varphi = 2\int \frac{\lambda^4 d\lambda \left\{\frac{27q'}{\lambda^2} + 18q' - \{q'\lambda^2 + 18(1 + \lambda^2)^{\frac{2}{3}}\}\right\}}{(9 + 3\lambda^2)^2 (1 + \lambda^2)^{\frac{5}{3}}}.$$

When it becomes nothing the function

$$\frac{27q'}{\lambda^2} + 18q' - \{q'\lambda^2 + 18(1 + \lambda^2)^{\frac{2}{3}}\},$$

has already passed through zero to become negative; but from the instant when this function begins to be negative, it continues to be so as λ increases, because the positive part $\frac{27q'}{\lambda^2} + 18q'$ decreases whilst the negative part $-\{q'\lambda^2 + 18(1 + \lambda^2)^{\frac{2}{3}}\}$ increases; the function φ cannot therefore twice become nothing; whence it follows, that there is but one real and positive value of λ which satisfies the equation of equilibrium, and consequently, the fluid can be in equilibrium with one elliptic figure only.

ON THE FIGURE OF A SPHEROID DIFFERING VERY LITTLE FROM A SPHERE, AND COVERED WITH A SHELL OF FLUID IN EQUILIBRIUM.

566. We have already discussed the equilibrium of a homogeneous fluid mass, and we have found that the elliptic figure satisfies this equilibrium; but in order to get a complete solution of the problem, we must determine *a priori* all the figures of equilibrium, or we must be certified that the elliptic is the only figure which will fulfil these conditions; besides, it is very probable that the celestial bodies have not homogeneous masses, and that they are denser towards the center than at the surface. In the research, therefore, of their figure, we must not rest satisfied with the case of homogeneity; but then this research presents great difficulties. Happily it is simplified by the consideration of the little difference which exists between the spherical figure and those of the planets and satellites; by which we are permitted to neglect the square of this difference, and of the quantities depending on it. Notwithstanding, the research of the figure of the planets is still very complex. To treat it with the greatest generality, we proceed to consider the equilibrium of a fluid mass which covers a body formed of shells of variable density, endowed with a rotatory motion, and sollicited by the attraction of other bodies. For that purpose, we proceed to recapitulate the laws of equilibrium of fluids, as laid down in works upon hydrostatics.

If we name ϱ the density of a fluid molecule, Π the pressure it sustains, F, F′, F″, &c. the forces which act upon it, and d f, d f′, d f″ the elements of the directions of these forces; then the general equation of the equilibrium of the fluid mass will be

$$\frac{d\Pi}{\varrho} = F\,d\,f + F'\,d\,f' + F''\,d\,f'' + \&c.$$

Suppose that the second member of this equation is an exact difference; designating by $d\varphi$ this difference, ϱ will necessarily be a function of Π and of φ: the integral of this equation will give φ in a function of Π; we may therefore reduce to a function of Π only, from which we can obtain Π in a function of ϱ; thus, relatively to shells of a given constant density, we shall have $d\,\Pi = 0$, and consequently

$$0 = F\,d\,f + F'\,d\,f' + F''\,d\,f'' + \&c.;$$

an equation which indicates the tangential force at the surface of those shells is nothing, and consequently, that the resultant of all the forces F, F′, F″, &c. is perpendicular to this surface; so that the shells are spherical.

The pressure Π being nothing at the exterior surface, ϱ must there be constant, and the resultant of all the forces which animate each molecule of the surface is perpendicular to it. This resultant is what we call gravity. The conditions of equilibrium of a fluid mass, are therefore 1st, that the direction of gravity be perpendicular to each point of the exterior surface: 2dly, that in the interior of the mass the directions of the gravity of each molecule be perpendicular to the surface of the shells of a constant density. Since we may take, in the interior of a homogeneous mass, such shells as we wish for shells of a constant density, the second of two preceding conditions of equilibrium, is always satisfied, and it is sufficient for the equilibrium that the first should be fulfilled; that is to say, that the resultant of all the forces which animate each molecule of the exterior surface should be perpendicular to the surface.

567. In the theory of the figure of the celestial bodies, the forces F, F', F'', &c. are produced by the attraction of their molecules, by the centrifugal force due to their motion of rotation, and by the attraction of distant bodies. It is easy to be certified that the difference $F\,d\,f + F'\,d\,f' + \&c.$ is there exact; but we shall clearly perceive that, by the analysis which we are about to make of these different forces, in determining that part of the integral $\int(F\,d\,f + F'\,d\,f' + \&c.)$ which is relative to each of them.

If we call d M any molecule of the spheroid, and f its distance from the point attracted, its action upon this latter will be $\frac{d\,M}{f^2}$. Multiplying this action by the element of its direction, which is $-\,d\,f$, since it tends to diminish f, we shall have, relatively to the action of the molecule d M, $\int F\,d\,f = \frac{d\,M}{f}$; whence it follows that that part of the integral $\int(F\,d\,f + F'\,d\,f' + \&c.)$, which depends on the attraction of the molecules of the spheroid, is equal to the sum of all these molecules divided by their respective distances from the molecule attracted. We shall represent this sum by V, as we have already done.

We propose, in the theory of the figure of the planets, to determine the laws of the equilibrium of all their parts, about their common center of gravity; we must, therefore, transfer into a contrary direction to the molecule attracted, all the forces by which this center is animated in virtue of the reciprocal action of all the parts of the spheroid; but we know that, by the property of this center, the resultant of all the actions upon this point is nothing. To get, therefore, the total effect of the attraction

of the spheroid upon the molecules attracted, we have nothing to add to V.

To determine the effect of the centrifugal force, we shall suppose the position of the molecule determined by the three rectangular coordinates x', y', z', whose origin we fix at the center of gravity of the spheroid. We shall then suppose that the axis of x' is the axis of rotation, and that g expresses the centrifugal force due to the velocity of rotation at the distance l from the axis. This force will be nothing in the direction of x' and equal to $g\,y'$ and $g\,z'$ in the direction of y' and of z'; multiplying, therefore, these two last forces respectively by the elements $d\,y'$, $d\,z'$ of their directions, we shall have $\frac{1}{2}\,g\,(y'^2 + z'^2)$ for that part of the integral $\int(F\,d\,f + F'\,d\,f' + \&c.)$, which is due to the centrifugal force of the rotatory motion.

If we call, as above, r the distance of the molecule attracted from the center of gravity of the spheroid, θ the angle which the radius r forms with the axis of x', and ϖ the angle the plane which passes through the axis of x', and through the molecule, forms with the plane of x', y'; finally, if we make $\cos.\,\theta = m$, we shall have

$$x' = r\,m\,;\quad y' = r\sqrt{1 - m^2}.\cos.\varpi\,;\quad z' = r\sqrt{1 - m^2}.\sin.\varpi\,;$$

whence we get

$$\tfrac{1}{2}\,g\,(y'^2 + z'^2) = \tfrac{1}{2}\,g\,r^2\,(1 - m^2).$$

We shall put this last quantity under the following form:

$$\tfrac{1}{3}\,g\,r^2 - \tfrac{1}{2}\,g\,r^2\,(m^2 - \tfrac{1}{3})$$

to assimilate its terms to those of the expression V which are given in No. 559; that is to say, to give them the property of satisfying the equation of partial differences

$$0 = \left(\frac{d\,(1 - m^2).\left(\frac{d\,Y^{(i)}}{d\,m}\right)}{d\,m}\right) + \frac{\left(\frac{d^2\,Y^{(i)}}{d\,\varpi^2}\right)}{1 - m^2} + i\,(i + 1)\,Y^{(i)};$$

in which $Y^{(i)}$ is a rational and entire function of m, $\sqrt{1 - m^2}.\cos.\varpi$ and $\sqrt{1 - m^2}\sin.\varpi$ of the degree i; for it is clear that each of the two terms $\frac{1}{3}\,g\,r^2$ and $-\frac{1}{2}\,g\,r^2\,(m^2 - \frac{1}{3})$ satisfies for $Y^{(i)}$, the preceding equation.

It remains now for us to determine that part of the integral $\int(F\,d\,f + F'\,d\,f' + \&c.)$ which results from the action of distant bodies. Let S be the mass of one of these bodies, f its distance from the molecule attracted, and s its distance from the center of gravity of the spheroid. Multiplying its action by the element $-\,d\,f$ of its direction, and then inte-

grating we shall have $\frac{S}{f}$. This is not the entire part of the integral $\int(F\,df + F'\,df' + \&c.)$ which is due to the action of S; we have still to transfer, in a contrary direction to the molecule, the action of this body upon the center of gravity of the spheroid. For that purpose, call v the angle which s forms with the axis of x′, and ψ the angle which the plane passing through this star and through the body S, makes with the plane of x′, y′. The action of $\frac{S}{s^2}$ of this body upon the center of gravity of the spheroid, resolved parallel to the axes of x′, y′, z′, will produce the three following forms:

$$\frac{S}{s^2}\cos.\,v;\quad \frac{S}{s^2}\sin.\,v\cos.\,\psi;\quad \frac{S}{s^2}\sin.\,v\sin.\,\psi.$$

Transferring them in a contrary direction to the molecule attracted, which amounts to prefixing to them the sign —, then multiplying them by the elements dx′, dy′, dz′, of their directions, and integrating them, the sum of the integrals will be

$$-\frac{S}{s^2}.\{x'\cos.\,v + y'\sin.\,v.\cos.\,\psi + z'\sin.\,v\sin.\,\psi\} + \text{const.};$$

the entire part of the integral $\int(F\,df + F'\,df' + \&c.)$, due to the action of the body S, will therefore be

$$\frac{S}{f} - \frac{S}{s^2}\{x'\cos.\,v + y'\sin.\,v\cos.\,\psi + z'\sin.\,v\sin.\,\psi\} + \text{const.};$$

and since this quantity ought to be nothing relatively to the center of gravity of the spheroid, which we suppose immoveable, and that relatively to this point, f becomes s, and x′, y′, z′, are nothing, we shall have

$$\text{const.} = -\frac{S}{s}.$$

However, f is equal to

$$\{(s\cos.\,v - x')^2 + (s.\sin.\,v\cos.\,\psi - y')^2 + (s\sin.\,v\sin.\,\psi - z')^2\}^{\frac{1}{2}};$$

which gives, by substituting for x′, y′, z′, their preceding values

$$\frac{S}{f} = \frac{S}{\sqrt{s^2 - 2sr\{\cos.\,v\cos.\,\theta + \sin.\,v\sin.\,\theta\cos.\,(\varpi - \psi)\} + r^2\}}}.$$

If we reduce this function into a series descending relatively to powers of s, and if we thus represent the series,

$$\frac{S}{s}\left\{P^{(0)} + \frac{r}{s}P^{(1)} + \frac{r^2}{s^2}P^{(2)} + \&c.\right\};$$

we shall have generally by 561 and 562,

$$P^{(i)} = \frac{1.3.5..(2i-1)}{1.2.3.....i}\left\{\partial^i - \frac{i(i-1)}{2(2i-1)}\partial^{i-2} + \frac{i(i-1)(i-2)(1-3)}{2.4(2i-1)(2i-3)}\partial^{i-4} - \&c.\right\};$$

∂ being equal to cos. v cos. θ + sin. v sin. θ . cos. (ϖ—ψ); it is evident that by 553, we have

$$0 = \left(\frac{d\left\{(1-m^2)\left(\frac{d\,P^{(i)}}{d\,m}\right)\right\}}{d\,m} \right) + \frac{\left(\frac{d^2 P^{(i)}}{d\,\varpi^2}\right)}{1-m^2} + i\,(i+1)\,P^{(i)};$$

so that the terms of the preceding have this property, common with those of V. This being shown, we have

$$\frac{S}{f} - \frac{S}{s} - \frac{S}{s^2}(x' \cos.\, v + y' \sin.\, v \cos.\, \psi + z' \sin.\, v \sin.\, \psi)$$
$$= \frac{S\,r^2}{s^3}\left\{ P^{(2)} + \frac{r}{s} P^{(3)} + \frac{r^2}{s^2} P^{(4)} + \&c. \right\}.$$

If there were other bodies S′, S″, &c.; denoting by s′, v′, ψ', $P'^{(i)}$; s″, v″, ψ'', $P''^{(i)}$, &c. what we have called s, v, ψ, $P^{(i)}$, relatively to the body S, we shall have the parts of the integral $\int(F\,df + F'\,df' + \&c.)$ due to their action, by marking with one, two, &c. dashes, the letters s, v, ψ, and P in the preceding expression of that part of this integral, which is due to the action of S.

If we collect all the parts of this integral, and make

$$\frac{g}{3} = \alpha\, Z^{(0)};$$

$$\frac{S}{s^3} P^{(2)} + \frac{S'}{s'^3} P'^{(2)} + \&c. - \frac{g}{2}\left(m^2 - \frac{1}{3}\right) = \alpha\, Z^{(2)};$$

$$\frac{S}{s^4} P^{(3)} + \frac{S'}{s'^4} P'^{(3)} + \&c. = \alpha\, Z^{(3)}$$

&c.

α being a very small coefficient, because the condition that the spheroid is very little different from a sphere, requires that the forces which produce this difference should themselves be very small; we shall have

$\int(F\,df + F'\,df' + \&c.) = V + \alpha\, r^2\{Z^{(0)} + Z^{(2)} + r\,Z^{(3)} + r^2 Z^{(4)} + \&c.\}$

$Z^{(i)}$ satisfying, whatever i may be, in the equation of partial differences

$$0 = \left(\frac{d\left\{(1-m^2)\,\frac{d\,Z^{(i)}}{d\,m}\right\}}{d\,m} \right) + \frac{\left(\frac{d^2 Z^{(i)}}{d\,\varpi^2}\right)}{1-m^2} + i\,(i+1)\,Z^{(i)}.$$

The general equation of equilibrium will therefore be

$$\int \frac{d\,\Pi}{\varrho} = V + \alpha\, r^2\{Z^{(0)} + Z^{(2)} + r\,Z^{(3)}\, r^2\, Z^{(4)} + \&c.\} \quad . \quad (1)$$

If the extraneous bodies are very distant from the spheroid, we may neglect the quantities $r^3 Z^{(3)}$, $r^4 Z^{(4)}$, &c., because the different terms of these quantities being divided respectively by s^4, s^3, &c. s'^4, s'^3, &c. these terms become very small when s, s′, &c. are very great compared with r. This

case subsists for the planets and satellites with the exception of Saturn, whose ring is too near his surface for us to neglect the preceding terms. In the theory of the figure of that planet, we must therefore prolong the second member of equation (1), which possesses the advantage of forming a series always convergent; and since then the number of corpuscles exterior to the spheroid is infinite, the values of $Z^{(0)}$, $Z^{(2)}$, &c. are given in definite integrals, depending on the figure and interior constitution of the ring of Saturn.

568. The spheroid may be entirely fluid; it may be formed of a solid nucleus covered by a fluid. In both cases the equation (1) of the preceding No. will determine the figure of the shells of the fluid part, by considering, that since Π must be a function of ϱ, the second member of this equation must be constant for the exterior surface, and for that of the shells in equilibrium, and can only vary from one shell to another.

The two preceding cases reduce to one when the spheroid is homogeneous; for it is indifferent as to the equilibrium whether it is entirely fluid, or contains an interior solid nucleus. It is sufficient by No. 556, that at the exterior surface we have

$$\text{constant} = V + \alpha r^2 \{Z^{(0)} + Z^{(2)} + r Z^{(3)} + \&c.\}.$$

If we substitute in this equation for V its value given by formula (3) of No. 555, and if we observe that by No. 556, $Y^{(0)}$ disappears by taking for a the radius of a sphere of the same volume as the spheroid, and that $Y^{(1)}$ is nothing when we fix the origin of coordinates at the center of the spheroid; we shall have

$$\text{constant} = \frac{4\pi a^3}{3r} + \frac{4\pi\alpha a^5}{r^3}\left\{\frac{1}{5}Y^{(2)} + \frac{a}{7r}Y^{(3)} + \frac{a^2}{9r^2}Y^{(4)} + \&c.\right\}$$
$$+ \alpha r^2 \{Z^{(0)} + Z^{(2)} + r Z^{(3)} + r^2 Z^{(4)} + \&c.\}$$

Substituting in the equation of the surface of the spheroid for r its value at the surface $1 + \alpha y$, or

$$a + \alpha a \{Y^{(2)} + Y^{(3)} + Y^{(4)} + \&c.\}$$

which gives

$$\text{const.} = \frac{4\pi}{3}a^2 - \frac{8\alpha\pi a^2}{3}\left\{\frac{1}{5}Y^{(2)} + \frac{2}{7}Y^{(3)} + \frac{3}{9}Y^{(4)} + \&c.\right\}$$
$$+ \alpha a^2 \{Z^{(0)} + Z^{(2)} + a Z^{(3)} + a^2 Z^{(4)} + \&c.\}$$

We shall determine the arbitrary constant of the first member of this equation, by means of this equation,

$$\text{const.} = \frac{4}{3}\pi a^2 + \alpha a^2 Z^{(0)};$$

we shall then have by comparing like functions, that is to say, such as are subject to the same equation of partial differences,

$$Y^{(i)} = \frac{3\,(2\,i+1)}{8\,(i-1)\,\pi}\,.\,a^{i-2}\,Z^{(i)};$$

i being greater than unity. The preceding equation may be put under the form

$$Y^{(i)} = \frac{3}{4\,\pi}\,.\,a^{i-2}\,.\,Z^{(i)} + \frac{9}{8\,a\,\pi}\int r^{i-2}\,d\,r\,Z^{(i)}$$

the integral being taken from $r = 0$ to $r = a$. The radius $a\,(1 - \alpha\,y)$ of the surface of the spheroid will hence become

$$a\,(1+\alpha\,y) = a\left\{\begin{array}{l} 1 + \frac{3\,\alpha}{4\,\pi}\{Z^{(2)} + a\,Z^{(3)} + a^2\,Z^{(4)} + \&c.\} \\ \quad + \frac{9\,\alpha}{8\,a\,\pi}\int d\,r\,\{Z^{(2)} + r\,Z^{(3)} + r^2\,Z^{(4)} + \&c.\} \end{array}\right\}, \quad (2)$$

We may put this equation under a finite form, by considering that we have by the preceding No.

$$\alpha\,\{Z^{(2)} + r\,Z^{(3)} + r^2\,Z^{(4)} + \&c.\} = -\frac{g}{2}(m^2 - \tfrac{1}{3}) - \frac{S}{s\,r^2} - \frac{S\,.\,\delta}{s^2\,r} + \frac{S}{r^2\,\sqrt{s^2 - 2\,s\,r\,\delta + r^2}} - \frac{S'}{s'\,r^2} - \&c.,$$

so that the integral $\int d\,r\,\{Z^{(2)} + r\,Z^{(3)} + \&c.\}$ is easily found by known methods.

569. The equation (1) of 567 not only has the advantage of showing the figure of the spheroid, but also that of giving by differentiation the law of gravity at its surface; for it is evident that the second member of this equation being the integral of the sum of all the forces with which each molecule is animated, multiplied by the elements of their respective directions, we shall have that part of the resultant which acts along the radius r, by differentiating the second member relatively to r; thus calling p the force by which a molecule of the surface is sollicited towards the center of gravity of the spheroid, we shall have

$$p = -\left(\frac{d\,V}{d\,r}\right) - \frac{\alpha}{d\,r}\,d\,\{r^2\,Z^{(0)} + r^2\,Z^{(2)} + r^3\,Z^{(3)} + r^4\,Z^{(4)} + \&c.\}.$$

If we substitute in this equation for $-\left(\frac{d\,V}{d\,r}\right)$, its value at the surface $\frac{2}{3}\pi\,a + \frac{V}{2\,a}$, given by equation (2) of No. 554, and for V, its value given by equation (1) of No. 567; we shall have

$$p = \frac{4}{3}\pi\,a - \frac{1}{2}\,\alpha\,a\,\{Z^{(2)} + a\,Z^{(3)}\,a^2\,Z^{(4)} + \&c.\} - \frac{\alpha}{d\,r}\,.\,d\,.\,\{r^2\,Z^{(0)} + r^2\,Z^{(2)} + r^3\,Z^{(3)} + r^4\,Z^{(4)} + \&c.\} \quad (3)$$

r must be changed into a after the differentiations in the second member of this equation, which by the preceding No. may always be reduced to a finite function.

p does not represent exactly gravity, but only that part of it which is directed towards the center of gravity of the spheroid, by supposing it resolved into two forces, one of which is perpendicular to the radius r, and the other p is directed along this radius. The first of these two forces is evidently a small quantity of the order α; denoting it therefore by $\alpha\gamma$, gravity will be equal to $\sqrt{p^2 + \alpha^2\gamma^2}$, a quantity which, neglecting the terms of the order α^2, reduces to p. We may thus consider p as expressing gravity at the surface of the spheroid, so that the equations (2) and (3) of the preceding No. and of this, determine both the figure of homogeneous spheroids in equilibrium, and the law of gravity at their surfaces; they contain the complete theory of the equilibrium of these spheroids, on the supposition that they differ very little from the sphere.

If the extraneous bodies S, S', &c. are nothing, and therefore the spheroid is only sollicited by the attraction of its molecules, and the centrifugal force of its rotatory motion, which is the case of the Earth and primary planets with the exception of Saturn, when we only regard the permanent state of their figures; then designating by $\alpha\varphi$, the ratio of the centrifugal force to gravity at the equator, a ratio which is very nearly equal to $\frac{g}{\frac{4}{3}\pi}$, the density of the spheroid being taken for unity; we shall find,

$$a(1 + \alpha y) = a\{1 - \frac{5\alpha\varphi}{4}(m^2 - \tfrac{1}{3})\};$$

$$p = \tfrac{4}{3}\pi a\{1 - \tfrac{2}{5}\alpha\varphi + \frac{5\alpha\varphi}{4}(m^2 - \tfrac{1}{3})\};$$

the spheroid is then therefore an ellipsoid of revolution, upon which increments of gravity, and decrements of the radii, from the equator to the poles, are very nearly proportional to the square of the sine of the latitude, m being to quantities of the order α, equal to this sine.

a, by what precedes, is the radius of a sphere, equal in solidity to the spheroid; gravity at the surface of this sphere will be $\frac{4}{3}\pi a$; thus we shall have the point of the surface of the spheroid, where gravity is the same as at the surface of the sphere, by determining m by the equation

$$0 = -\tfrac{2}{5} + \tfrac{5}{4}(m^2 - \tfrac{1}{3});$$

which gives

$$m = \sqrt{\frac{13}{15}}.$$

570. The preceding analysis conducts us to the figure of a homogeneous fluid mass in equilibrium, without employing other hypotheses than that of a figure differing very little from the sphere: it also shows that the elliptic figure which satisfies this equilibrium, is the only figure which does it. But as the expansion of the radius of the spheroid into a series of the form $a\{1 + \alpha Y^{(0)} + \alpha Y^{(1)} + \&c.\}$ may give rise to some difficulties, we proceed to demonstrate directly, and independently of this expansion, that the elliptic figure is the only figure of the equilibrium of a homogeneous fluid mass endowed with a rotatory motion; which by confirming the results of the preceding analysis, will at the same time serve to remove any doubts we may entertain against the generality of this analysis.

First suppose the spheroid one of revolution, and that its radius is $a(1 + \alpha y)$, y being a function of m, or of the cosine of the angle θ which this radius makes with the axis of revolution. If we call f any straight line drawn from the extremity of this radius in the interior of the spheroid; p the complement of the angle which this straight line makes with the plane which passes through the radius $a(1 + \alpha y)$ and through the axis of revolution; q the angle made by the projection of f upon this plane and by the radius; finally, if we call V the sum of all the molecules of the spheroid, divided by their distances from the molecules placed at the extremity of the radius $a(1 + \alpha y)$; each molecule being equal to $f^2 \, df . dp . dq . \sin. p$, we shall have

$$V = \tfrac{1}{2} \int f'^2 \, dp . dq . \sin. p,$$

f' being what f becomes at its quitting the spheroid. We must now determine f' in terms of p and q.

For that purpose, we shall observe that if we call θ', the value of θ relative to this point of exit, and $a(1 + \alpha y')$, the corresponding radius of the spheroid, y' being a similar function of cos. θ' or of m' that y is of m; it is easily seen that the cosine of the angle formed by the two straight lines f' and $a(1 + \alpha y)$ is equal to sin. p . cos. q; and therefore that in the triangle formed by the three straight lines f', $a(1 + \alpha y)$ and $a(1 + \alpha y')$ we have

$$a^2 (1 + \alpha y')^2 = f'^2 - 2 a f' (1 + \alpha y) \sin. p . \cos. q + a^2 (1 + \alpha y)^2.$$

This equation gives for f'^2 two values; but one of them being of the order α^2 is nothing when we neglect the quantities of that order; the other becomes

$$f'^2 = 4 a^2 \sin.^2 p \cos.^2 q (1 + 2 \alpha y) + 4 \alpha a^2 (y' - y);$$

which gives

$$V = 2a^2 \int dp\, dq \sin. p \{(1 + 2\alpha y) \sin.^2 p \cos.^2 q + \alpha (y' - y)\}.$$

It is evident that the integrals must be taken from $p = 0$, to $p = \pi$, and from $q = -\frac{1}{2}\pi$ to $q = \frac{1}{2}\pi$; we shall therefore have

$$V = \tfrac{4}{3}\pi a^2 - \tfrac{4}{3}\alpha\pi a^2 y + 2\alpha a^2 \int dp . dq . y' \sin. p .$$

y' being a function of cos. θ', we must determine this cosine in a function of p and q; we may therefore in this determination neglect the quantities of the order α, since y' is already multiplied by α; hence we easily find

$$a \cos. \theta' = (a - f' \sin. p \cos. q) \cos. \theta + f' \sin. p . \sin. q . \sin. \theta;$$

whence we derive, substituting for f' its value $2a \sin. p \cos. q$,

$$m' = m \cos.^2 p - \sin.^2 p \cos. (2q + \theta).$$

Here we must observe, relatively to the integral $\int y' dp . dq . \sin. p$, taken relatively to q from $2q = -\pi$ to $2q = \pi$, that the result would be the same, if this integral were taken from $2q = -\theta$ to $2q = 2\pi - \theta$, because the values of m', and consequently of y' are the same from $2q = -\pi$ to $2q = -\theta$ as from $2q = \pi$ to $2q = 2\pi - \theta$; supposing therefore $2q + \theta = q'$, which gives

$$m' = m \cos.^2 p - \sin.^2 p \cos. q';$$

we shall have

$$V = \tfrac{4}{3}\pi a^2 - \tfrac{4}{3}\alpha\pi a^2 y + \alpha a^2 \int y' dp\, dq' \sin. p;$$

the integrals being taken from $p = 0$ to $p = \pi$ and from $q' = 0$ to $q' = 2\pi$.

Now if we denote by $a^2 N$ the integral of all the forces extrinsic to the attraction of the spheroid, and multiplied by the elements of their directions; by 568 we shall have in the case of equilibrium

$$\text{constant} = V + a^2 N,$$

and substituting for V its value, we shall have

$$\text{const.} = \tfrac{4}{3}\alpha\pi . y - \alpha \int y' dp . dq' \sin. p - N;$$

an equation which is evidently but the equation of equilibrium of No. 568, presented under another form. This equation being linear, it thence results that if any number i of radii $a(1 + \alpha y)$, $a(1 + \alpha v)$, and satisfy it; the radius $a\{+\frac{\alpha}{i}(y + v + \&c.)\}$ will also satisfy it.

Suppose that the extraneous forces are reduced to the centrifugal force due to the rotation, and call g this force at the distance 1 from the axis of rotation; we shall have, by 567, $N = \frac{1}{2} g (1 - m^2)$; the equation of equilibrium will therefore be

$$\text{const.} = \tfrac{4}{3}\alpha\pi y - \alpha \int y' dp\, dq' \sin. p - \tfrac{1}{2} g (1 - m^2).$$

Differentiating three times successively, relatively to m, and observing that $\left(\frac{d\, m'}{d\, m}\right) = \cos.^2 p$, in virtue of the equation

$$m' = m \cos.^2 p - \sin.^2 p \cos. q';$$

we shall have

$$0 = \tfrac{4}{5}\pi \left(\frac{d^3 y}{d m^3}\right) - \int d p\, d q' \sin. p \cos.^6 p \left(\frac{d^3 y'}{d m'^3}\right),$$

but we have $\int d p\, d q' \sin. p \cos.^6 p = \frac{4\pi}{7}$; we may therefore put the preceding equation under this form,

$$0 = \int d p\, d q' \sin. p \cos.^6 p \left\{\tfrac{7}{5}\left(\frac{d^3 y}{d m^3}\right) - \left(\frac{d^3 y'}{d m'^3}\right)\right\}.$$

This equation subsists, whatever m may be; but it is evident, that amongst all the values between $m = -1$ and $m = 1$, there is one which we shall designate by h, and which is such that, abstraction being made of the sign, each of the values of $\left(\frac{d^3 y}{d m^3}\right)$ will not exceed that which is relative to h; denoting therefore by H, this latter value, we shall have

$$0 = \int d p\, d q' \sin. p \cos.^6 p \left\{\tfrac{7}{5} H - \left(\frac{d^3 y'}{d m'^3}\right)\right\}.$$

The quantity $\frac{7}{5} H - \left(\frac{d^3 y'}{d m'^3}\right)$ has evidently the same sign as H, and the factor sin. p . cos.^{6} p, is constantly positive in the whole extent of the integral; the elements of this integral have, therefore, all of them the same sign as H; whence it follows that the entire integral cannot be nothing, at least H cannot be so, which requires that we have generally $0 = \left(\frac{d^3 y}{d m^3}\right)$, whence by integrating we get

$$y = l + m.m + n.m^2;$$

l, *m*, n, being arbitrary constants.

If we fix the origin of the radii in the middle of the axis of revolution, and take for a the half of this axis, y will be nothing when $m = 1$ and when $m = -1$, which gives $m = 0$ and $n = -1$; the value of y thus becomes, $l(1 - m^2)$; substituting in the equation of equilibrium,

$$\text{const.} = \tfrac{4}{5}\alpha\pi y - \alpha \int y'\, d p\, d q' \sin. p - \tfrac{1}{2} g (1 - m^2);$$

we shall find $\alpha l = \frac{15 g}{16 \pi} = \frac{5}{4}\alpha\varphi$, $\alpha\varphi$ being the ratio of the centrifugal force to the equatorial gravity, a ratio which is very nearly equal to $\frac{3 g}{4\pi}$; the radius of the spheroid will therefore be

$$a\left\{1 + \frac{5\alpha\varphi}{4}(1 - m^2)\right\};$$

whence it follows that the spheroid is an ellipsoid of revolution, which is conformable to what precedes.

Thus we have determined directly and independently of series, the figure of a homogeneous spheroid of revolution, which turns round its axis, and we have shown that it can only be that of an ellipsoid which becomes a sphere when $\varphi = 0$; so that the sphere is the only figure of revolution which would satisfy the equilibrium of an immoveable homogeneous fluid mass.

Hence we may conclude generally, that if the fluid mass is sollicited by any very small forces, there is only one possible figure of equilibrium; or, which comes to the same, there is only one radius $a(1 + \alpha y)$ which can satisfy the equation of equilibrium,

$$\text{const.} = \tfrac{4}{3}\alpha\pi \cdot y - \alpha \int y' \, dp \cdot dq' \sin. p - N;$$

y being a function of θ and of the longitude ϖ, and y' being what y becomes when we change θ and ϖ into θ' and ϖ'. Suppose, in fact, that there are two different rays $a(1 + \alpha y)$ and $a(1 + \alpha y + \alpha v)$ which satisfy this equation; we shall have

$$\text{const.} = \tfrac{4}{3}\alpha\pi (y + v) - \alpha \int (y' + v') \, dp \, dq' \sin. p - N.$$

Taking the preceding equation from this, we shall have

$$\text{const.} = \tfrac{4}{3}\pi v - \int v' \, dp \, dq' \sin. p.$$

This equation is evidently that of a homogeneous spheroid in equilibrium, whose radius is $a(1 + \alpha v)$, and which is not sollicited by any force extraneous to the attraction of its molecules. The angle ϖ disappearing in this equation, the radius $a(1 + \alpha v)$ will still satisfy it if ϖ be successively changed to $\varpi + d\varpi$, $\varpi + 2\,d\varpi$, &c., whence it follows, that if we call v_1, v_2, &c. what v becomes in virtue of these changes; the radius

$$a\{1 + \alpha v \, d\varpi + \alpha v_1 \, d\varpi + \alpha v_2 \, d\varpi + \&c.\},$$

or

$$a(1 + \alpha \int v \, d\varpi),$$

will satisfy the preceding equation. If we take the integral $\int v \, d\varpi$ from $\varpi = 0$ to $\varpi = 2\pi$, the radius $a(1 + \alpha \int v \, d\varpi)$ becomes that of a spheroid of revolution, which, by what precedes, can only be a sphere: see the condition which results for v.

Suppose that a is the shortest distance of the center of gravity of the spheroid whose radius is $a(1 + \alpha v)$, to the surface, and fix the pole or origin of the angle θ at the extremity of a; v will be nothing at the pole, and positive every where else; it will be the same for the integral $\int v \, d\varpi$. But, since the center of gravity of the spheroid whose radius is $a(1 + \alpha v)$, is at the center of the sphere whose radius is a, this point will, in like manner, be the center of gravity of the spheroid whose radius is

$a\,(1 + \alpha \int v\, d\varpi)$; the different radii drawn from this center to the surface of this last spheroid are therefore unequal to one another, if v is not nothing; there can only therefore be a sphere in the case of $v = 0$; thus we learn for a certainty, that a homogeneous spheroid, sollicited by any small forces whatever, can only be in equilibrium in one manner.

571. We have supposed that N is independent of the figure of the spheroid; which is what very nearly takes place when the forces, extraneous to the action of the fluid molecules, are due to the centrifugal force of rotatory motion, and to the attraction of bodies exterior to the spheroid. But if we conceive at the center of the spheroid a finite force depending on the distance r, its action upon the molecules placed at the surface of the fluid, will depend on the nature of this surface, and consequently N will depend upon y. This is the case of a homogeneous fluid mass which covers a sphere of a density different from that of the fluid; for we may consider this sphere as of the same density as the fluid, and may place at its center a force reciprocal to the square of the distances; so that, if we call c the radius of the sphere, and ϱ its density, that of the fluid being taken for unity, this force at the distance r will be equal to $\frac{4}{3}\pi . \frac{c^3 (\varrho - 1)}{\varrho^2}$. Multiplying by the element $-\,d\,r$ of its direction the integral of the product will be $\frac{4}{3}\pi . \frac{c^3 (\varrho - 1)}{\varrho}$, a quantity which we must add to $a^2 N$; and since at the surface we have $r = a\,(1 + \alpha\, y)$, in the equation of equilibrium of the preceding No., we must add to N,

$$\frac{4}{3}\pi . \frac{(\varrho - 1)\, c^3}{a^3} . (1 - \alpha\, y).$$

This equation will become

$$\text{const.} = \frac{4\,\alpha\,\pi}{3}\left\{1 + (\varrho - 1) . \frac{c^3}{a^3}\right\} y - \alpha \int y'\, dp . dq \sin. p - N.$$

If we denote by $a\,(1 + \alpha\, y + \alpha\, v)$, a new expression of the radius of the spheroid in equilibrium, we shall have to determine v, the equation

$$\text{const.} = \frac{4}{3}\pi \left\{1 + (\varrho - 1)\frac{c^3}{a^3}\right\} - \int v'\, dp\, dq'\, \sin. p;$$

an equation which is that of the equilibrium of the spheroid, supposing it immoveable, and abstracting every external force.

If the spheroid is of revolution, v will be a function of cos. θ or m only; but in this case we may determine it by the analysis of the preceding No.; for if we differentiate this equation $i + 1$ times successively relatively to m, we shall have

$$0 = \frac{4}{3}\pi \left\{1 + (\varrho - 1)\frac{c^3}{a^3}\right\} \left(\frac{d^{i+1} v}{d\, m^{i+1}}\right) - \int \left(\frac{d^{i+1} v'}{d\, m'^{i+1}}\right) dp\, dq'\, \sin. p \cos.^{2i+2} p$$

but we have

$$\int d\,p\,d\,q'\sin.p\cos.^{2i+2}p = \frac{4\pi}{2i+3};$$

the preceding equation may therefore be put under this form,

$$0 = \int d\,p\,d\,q'\sin.p\cos.^{2i+2}p\left\{\frac{2i+3}{3}\left(1+\frac{\varrho-1}{2}\cdot\frac{c^3}{a^3}\right)\left(\frac{d^{i+1}v}{d\,m^{i+1}}\right)-\left(\frac{d^{i+1}v'}{dm'^{i+1}}\right)\right\}$$

We may take i such that, abstraction being made of the sign, we have

$$\frac{2i+3}{3}\left\{1+(\varrho-1)\frac{c^3}{a^3}\right\} > 1;$$

Supposing, therefore, that i is the smallest positive whole number which renders this quantity greater than unity, we may see, as in the preceding No., that this equation cannot be satisfied unless we suppose $\left(\frac{d^{i+1}v}{d\,m^{i+1}}\right)=0$, which gives

$$v = m^i + A\,m^{i-1} + B\,m^{i-2} + \&c.$$

Substituting in the preceding equation of equilibrium for v, this value, and for v'

$$m'^i + A\,m'^{i-1} + B\,m'^{i-2} + \&c.$$

m' being by the preceding No. equal to m cos.^{2} p — sin.^{2} p cos. q', first we shall find

$$1 + (\varrho - 1)\frac{c^3}{a^3} = \frac{3}{2i+1};$$

which supposes ϱ equal to or less than unity; thus, whenever a, c, and ϱ are not such as to satisfy this equation, i being a positive whole number, the fluid can be in equilibrium only in one manner. Then we shall have

$$A = 0;\quad B = -\frac{i(i-1)}{2(2i-1)};\ \&c.$$

so that

$$v = m^i - \frac{i(i-1)}{2(2i-1)}\cdot m^{i-2} + \frac{i(i-1)(i-2)(i-3)}{2.4.(2i-1)(2i-3)}m^{i-4} - \&c.;$$

there are, therefore, generally two figures of equilibrium, since α v is susceptible of two values, one of which is given by the supposition of $\alpha = 0$, and the other is given by the supposition of v being equal to the preceding function of m.

If the spheroid has no rotatory motion, and is not sollicited by any extraneous force, the first of these two figures is a sphere, and the second has for its meridian a curve of the order i. These two curves coincide in the case of i = 1, because the radius a (1 + α m) is that of a sphere in which the origin of the radii is at the distance α from its center; but then it is easy to see that ϱ = 1, that is, the spheroid is homogeneous, a result agreeing with that of the preceding No.

572. When we have figures of revolution which satisfy the equilibrium, it is easy to obtain those which are not of revolution by the following method. Instead of fixing the origin of the angle θ at the extremity of the axis of revolution, suppose it at the distance γ from this extremity, and call θ' the distance from this same extremity of the point of the surface whose distance from the new origin of the angle θ is θ. Call, moreover, $\varpi - \beta$ the angle comprised between the two arcs θ and γ; we shall have

$$\cos.\theta' = \cos.\gamma \cos.\theta + \sin.\gamma \sin.\theta . \cos.(\varpi - \beta);$$

designating therefore by $\Gamma.(\cos.\theta')$ the function

$$\cos.^{i}\theta' - \frac{i(i-1)}{2.(2i+1)} . \cos.^{i-2}\theta' + \&c.;$$

the radius of the immoveable spheroid in equilibrium, which we have seen is equal to $a\{1 + \alpha\,\Gamma.(\cos.\theta')\}$, will be

$$a + \alpha a \Gamma.\{\cos.\gamma . \cos.\theta + \sin.\gamma . \sin.\theta \cos.(\varpi - \beta)\};$$

and although it is a function of the angle ϖ, it belongs to a solid of revolution, in which the angle θ is not at the extremity of the axis of revolution.

Since this radius satisfies the equation of equilibrium, whatever may be α, β, and γ, it will also satisfy in changing these quantities into α', β', γ', α'', β'', γ'', &c. whence it follows that this equation being linear, the radius

$$\begin{aligned} &a + \alpha\ a\,\Gamma.\{\cos.\gamma\ \cos.\theta + \sin.\gamma\ \sin.\theta \cos.(\varpi - \beta)\} \\ &\quad + \alpha' a\,\Gamma.\{\cos.\gamma' \cos.\theta + \sin.\gamma' \sin.\theta \cos.(\varpi - \beta')\} \\ &\quad + \&c. \end{aligned}$$

will likewise satisfy it. The spheroid to which this radius belongs is no longer one of revolution; it is formed of a sphere of the radius a, and of any number of shells similar to the excess of the spheroid of revolution whose radius is $a + \alpha a \Gamma.(m)$ above the sphere whose radius is a, these shells being placed arbitrarily one over another.

If we compare the expression of $\Gamma.(\cos.\theta')$ with that of $P^{(i)}$ of No. 567, we shall see that these two functions are similar, and that they differ only by the quantities γ and β, which in $P^{(i)}$ are ν and ψ, and by a factor independent of m and ϖ; we have, therefore,

$$0 = \left(\frac{d.(1-m^2).\left(\frac{d.\Gamma(\cos.\theta')}{dm}\right)}{dm}\right) + \frac{\left(\frac{d^2\Gamma.(\cos.\theta')}{d\varpi^2}\right)}{1-m^2} + i(i+1)\Gamma.(\cos.\theta')$$

It is easy hence to conclude, that if we represent by $\alpha\,Y^{(i)}$ the function

$$\begin{aligned} &\alpha\ .\ \Gamma.\{\cos.\gamma\ \cos.\theta + \sin.\gamma\ \sin.\theta .\ \cos.(\varpi - \beta)\} \\ &+ \alpha'.\ \Gamma.\{\cos.\gamma' \cos.\theta + \sin.\gamma' \sin.\theta .\ \cos.(\varpi - \beta')\} \\ &+ \&c. \end{aligned}$$

$Y^{(i)}$ will be a rational and entire function of m, $\sqrt{1-m^2}\cos.\varpi$, $\sqrt{1-m^2}\sin.\varpi$, which will satisfy the equation of partial differences,

$$0=\left(\frac{d\left\{(1-m^2).\left(\frac{dY^{(i)}}{dm}\right)\right\}}{dm}\right)+\frac{\left(\frac{d^2Y^{(i)}}{d\varpi^2}\right)}{1-m^2}+i(i+1)Y^{(i)};$$

choosing for $Y^{(i)}$, therefore, the most general function of that nature, the function $a(1+\alpha Y^{(i)})$ will be the most general expression of the equilibrium of an immoveable spheroid.

We may arrive at the same result by means of the series for V in 555; for the equation of equilibrium being, by the preceding No.,

$$\text{const.} = V + a^2 N;$$

if we suppose that all the forces extraneous to the reciprocal action of the fluid molecules, are reducible to a single attractive force equal to $\frac{4}{3}\pi.\frac{(\varrho-1)c^3}{r^2}$, placed at the center of the spheroid, by multiplying this force by the element $-dr$ of its direction, and then integrating, we shall have

$$\frac{4}{3}\pi.\frac{(\varrho-1)c^3}{r}=a^2N;$$

and since at the surface $r=a(1+\alpha y)$ the preceding equation of equilibrium will become

$$\text{const.} = V+\frac{4}{3}\alpha\pi.\frac{c^3}{a}(1-\varrho)y.$$

Substituting in this equation for V its value given by formula (3) of No. 555, in which we shall put for r its value $a(1+\alpha y)$, and by substituting for y its value

$$Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.;$$

we shall have

$$0=\left\{(1-\varrho)\frac{c^3}{a^3}+2\right\}Y^{(0)}+(1-\varrho)\frac{c^3}{a^3}Y^{(1)}+\left\{(1-\varrho)\frac{c^3}{a^3}-\frac{2}{5}\right\}Y^{(2)}$$
$$\ldots\ldots+\left\{(1-\varrho)\frac{c^3}{a^3}-\frac{2i-2}{2i+1}\right\}Y^{(i)}+\&c.;$$

the constant a being supposed such, that $\text{const.}=\frac{4}{3}\pi a^2$. This equation gives $Y^{(0)}=0$, $Y^{(1)}=0$, $Y^{(2)}=0$, &c. unless the coefficient of one of these quantities, of $Y^{(i)}$ for example, is nothing, which gives

$$(1-\varrho)\frac{c^3}{a^3}=\frac{2i-2}{2i+1},$$

i being a positive whole number, and in this case all these quantities except $Y^{(i)}$ are nothing; we shall therefore have $y=Y^{(i)}$, which agrees with what is found above.

Thus we see, that the results obtained by the expansion of V into a se-

ries, have all possible generality, and that no figure of equilibrium has escaped the analysis founded upon this expansion; which confirms what we have seen *à priori*, by the analysis of 555, in which we have proved that the form which we have given to the radius of spheroids, is not arbitrary but depends upon the nature itself of their attractions.

573. Let us now resume equation (1) of No. 567. If we therein substitute for V its value given by formula (6) of No. 558, we shall have relatively to the different fluid shells

$$\int\frac{d\,\Pi}{\varrho}=2\pi\int\varrho\,d\,a^2+4\alpha\pi\int\varrho\,d\left\{a^2\,Y^{(0)}+\frac{ar}{3}Y^{(1)}+\frac{r^2}{5}Y^{(2)}+\frac{r^3}{7a}Y^{(3)}+\&c.\right\}$$
$$+\frac{4\pi}{3r}\int\varrho d\,a^3+\frac{4\alpha\pi}{r}\int\varrho\,d\left\{a^3\,Y^{(0)}+\frac{a^4}{3r}Y^{(1)}+\frac{a^5}{5r^2}Y^{(2)}+\frac{a^6}{7r^3}Y^{(3)}+\&c.\right\}$$
$$+\alpha\,r^2\{Z^{(0)}+Z^{(2)}+r\,Z^{(3)}+r^2\,Z^{(4)}+\&c.\};\quad\ldots\quad(1)$$

the differentials and integrals being relative to the variable a; the two first integrals of the second member of this equation must be taken from $a=a$ to $a=1$, a being the value of a, relative to the *leveled* fluid shell, which we are considering, and this value at the surface being taken for unity: the two last integrals ought to be taken from $a=0$ to $a=a$: finally, the radius r ought to be changed into $a\,(1+\alpha y)$ after all the differentiations and integrations. In the terms multiplied by α it will suffice to change r into a; but in the term $\frac{4\pi}{3r}\int\varrho\,d\,.\,a^3$ we must substitute $a\,(1+\alpha y)$ for r; which changes it into this

$$\frac{4\pi}{3a}(1-\alpha y)\int\varrho\,d\,a^3,$$

and consequently, into the following

$$\frac{4\pi}{3a}\{1-\alpha\,Y^{(0)}-\alpha\,Y^{(1)}-\alpha\,Y^{(3)}-\&c.\}.\int\varrho\,d\,a^3.$$

Hence if in equation (1) we compare like functions, we shall have

$$\int\frac{d\,\Pi}{\varrho}=2\pi\int\varrho\,d\,a^2+4\alpha\pi\int\varrho\,d\,(a^2\,Y^{(0)})+\frac{4\pi}{3a}\int\varrho\,d\,a^3$$
$$-\frac{4\alpha\pi}{3a}Y^{(0)}\int\varrho\,d\,a^3+\frac{4\alpha\pi}{a}\int\varrho\,d\,(a^3\,Y^{(0)})+\alpha\,a^2\,Z^{(0)};$$

the two first integrals of the second member of this equation being taken from $a=a$ to $a=1$, the three other integrals must be taken from $a=0$ to $a=a$. This equation determining neither a nor $Y^{(0)}$, but only a relation between them, we see that the value of $Y^{(0)}$ is arbitrary, and may be determined at pleasure. We shall have then, i being equal to, or greater than unity,

$$0 = \frac{4\pi a^i}{2i+1}\int \varrho\, d.\left(\frac{Y^{(i)}}{a^{i-2}}\right) - \frac{4\pi}{3a} Y^{(i)} \int \varrho\, d\, a^3$$
$$+ \frac{4\pi}{(2i+1)\, a^{i+1}} \int \varrho\, d\, (a^{i+3} Y^{(i)}) + a^i Z^{(i)}; \quad . \quad . \quad (2)$$

the first integral being taken from $a = a$, to $a = 1$, and the two others being taken from $a = 0$ to $a = a$. This equation will give the value of $Y^{(i)}$ relative to each fluid shell, when the law of the densities ϱ shall be known.

To reduce these different integrals within the same limits, let

$$\frac{4\pi}{2i+1}\int \varrho\, d\left(\frac{Y^{(i)}}{a^{i-2}}\right) + Z^{(i)} = \frac{4\pi}{2i+1} Z'^{(i)},$$

the integral being taken from $a = 0$ to $a = 1$; $Z'^{(i)}$ will be a quantity independent of a, and the equation (2) will become

$$0 = (2i+1)\, a^i Y^{(i)} \int \varrho\, d\, a^3 + 3\, a^{2i+1} . \int \varrho\, d\left(\frac{Y^{(i)}}{a^{i-2}}\right)$$
$$- 3 \int \varrho\, d\, (a^{i+3} Y^{(i)}) - 3\, a^{2i+1} Z'^{(i)};$$

all the integrals being taken from $a = 0$ to $a = a$.

We may make the signs of integration disappear by differentiating relatively to a, and we shall have the differential equation of the second order,

$$\left(\frac{d^2 Y^{(i)}}{d\, a^2}\right) = \left\{\frac{i\,(i+1)}{a^2} - \frac{6\varrho a}{\int \varrho\, d\, a^3}\right\} Y^{(i)} - \frac{6\varrho a^2}{\int \varrho\, d . a^3}\left(\frac{d\, Y^{(i)}}{d\, a}\right).$$

The integral of this equation will give the value of $Y^{(i)}$ with two arbitrary constants; these constants are rational and entire functions of the order i, of m, $\sqrt{1-m^2}$. sin. ϖ, and $\sqrt{1-m^2}$. cos. ϖ, such, that representing them by $U^{(i)}$, they satisfy the equation of partial differences,

$$0 = \left(\frac{d\left\{(1-m^2).\left(\frac{d\, U^{(i)}}{d\, m}\right)\right\}}{d\, m}\right) + \frac{\left(\frac{d^2 U^{(i)}}{d\, \varpi^2}\right)}{1-m^2} + i\,(i+1).\, U^{(i)}.$$

One of these functions will be determined by means of the function $Z'^{(i)}$ which disappears by differentiation, and it is evident that it will be a multiple of this function. As to the other function, if we suppose that the fluid covers a solid nucleus, it will be determined by means of the equation of the surface of the nucleus, by observing that the value of $Y^{(i)}$ relative to the fluid shell contiguous to this surface, is the same as that of the surface. Thus the figure of the spheroid depends upon the figure of the internal nucleus, and upon the forces which sollicit the fluid.

574. If the mass is entirely fluid, nothing then determining one of the arbitrary constants, it would seem that there ought to be an infinity of

figures of equilibrium. Let us examine this case particularly, which is the more interesting inasmuch as it appears to have subsisted primitively for the celestial bodies.

First, we shall observe that the shells of the spheroid ought to decrease in density from the center to the surface; for it is clear that if a denser shell were placed above a shell of less density, its molecules would penetrate into the other in the same manner that a ponderous body sinks into a fluid of less density; the spheroid will not therefore be in equilibrium. But whatever may be its density at the center, it can only be finite; reducing therefore the expression of ϱ into a series ascending relatively to the powers of a, this series will be of the form $\beta - \gamma . a^n$ — &c. β, γ and n being positive; we shall thus have

$$\frac{a^3 . \varrho}{\int \varrho d . a^3} = 1 - \frac{n \gamma . a^n}{(n+3)\beta} + \&c.;$$

and the differential equation in $Y^{(i)}$ will become

$$\left(\frac{d^2 Y^{(i)}}{d a^2}\right) = \left\{(i-2)(i+3) + \frac{6 n \gamma . a^n}{(n+3).\beta} - \&c.\right\} . \frac{Y^{(i)}}{a^2}$$
$$- \frac{6}{a}\left\{1 - \frac{n \gamma . a^n}{(n+3)\beta} + \&c.\right\} . \left(\frac{d Y^{(i)}}{d a}\right).$$

To integrate this equation, suppose that $Y^{(i)}$ is developed into a series ascending according to the powers of a, of this form

$$Y^{(i)} = a^s . U^{(i)} + a^{s'} . U'^{(i)} + \&c.;$$

the preceding differential equation will give

$$(s+i+3)(s-i+2) a^{s-2} U^{(i)} + (s'+i+3)(s'-i+2) a^{s'-2} U'^{(i)} + \&c.$$
$$= \frac{6 n \gamma a^n}{(n+3)\beta} \{(s+1) a^{s-2} . U^{(i)} + (s'+1) a^{s'-2} U'^{(i)} + \&c.\} \quad . \quad (e)$$

Comparing like powers of a, we have $(s + i + 3)(s - i + 2) = 0$, which gives $s = i - 2$, and $s = -i - 3$. To each of these values of s, belongs a particular series, which, being multiplied by an arbitrary, will be an integral of the differential equation in $Y^{(i)}$; the sum of these two integrals will be its complete integral. In the present case, the series which answers to $s = -i - 3$ must be rejected; for there thence results for a $Y^{(i)}$, an infinite value, when a shall be infinitely small, which would render infinite the radii of the shells which are infinitely near to the center. Thus of the two particular integrals of the expression of $Y^{(i)}$, that which answers to $s = i - 2$ ought alone to be admitted. This expression then, contains no more than one arbitrary which will be determined by the function $Z^{(i)}$.

$Z^{(1)}$ being nothing by No. 567, $Y^{(1)}$ is likewise nothing, so that the center of gravity of each shell, is at the center of gravity of the entire

spheroid. In fact the differential equation in $Y^{(i)}$ of the preceding No. gives

$$\left(\frac{d^2 Y^{(i)}}{d a^2}\right) = \left(\frac{2}{a^2}\right) - \frac{6 \varrho a^2}{\int \varrho d . a^3} . Y^{(1)} - \frac{6 \varrho a^2}{\int \varrho d . a^3} . \left(\frac{d Y^{(1)}}{d a}\right).$$

We satisfy this equation by making $Y^{(1)} = \frac{U^{(1)}}{a}$, $U^{(1)}$ being independent of a. This value of $Y^{(1)}$ is that which answers to the equation $s = i - 2$; it is, consequently, the only one which we ought to admit. Substituting it in the equation (2) of the preceding No., and supposing $Z^{(1)} = 0$, the function $U^{(1)}$ disappears, and consequently remains arbitrary; but the condition that the origin of the radius r is at the center of gravity of the terrestrial spheroid, renders it nothing; for we shall see in the following No. that then $Y^{(1)}$ is nothing at the surface of every spheroid covered over with a shell of fluid in equilibrium; we shall have, therefore, in the present case $U^{(1)} = 0$; thus, $Y^{(1)}$ is nothing relatively to all the fluid shells which form the spheroid.

Now consider the general equation,

$$Y^{(i)} = a^s . U^{(i)} + a^{s'} . U'^{(i)} + \&c.;$$

s being, as we have seen, equal to $i - 2$, s is nothing or positive, when i is equal to or greater than 2; moreover, the functions $U'^{(i)}$, $U''^{(i)}$, &c. are given in $U^{(i)}$, by the equation (e) of this No.; so that we have

$$Y^{(i)} = h . U^{(i)};$$

h being a function of a, and $U^{(i)}$ being independent of it. If we substitute this value of $Y^{(i)}$ in the differential equation in $Y^{(i)}$, we shall have

$$\frac{d^2 h}{d a^2} = \left\{ i (i + 1) - \frac{6 \varrho a^3}{\int \varrho d . a^3} \right\} . \frac{h}{a^2} - \frac{6 \varrho a^2}{\int \varrho d . a^3} . \frac{d h}{d a}.$$

The product $i (i + 1)$ is greater than $\frac{6 \varrho a^3}{\int \varrho d . a^3}$, when i is equal to or greater than 2, for the fraction $\frac{\varrho a^3}{\int \varrho d . a^3}$ is less than unity; in fact its denominator $\int \varrho d . a^3$ is equal to $\varrho a^3 - \int a^3 d \varrho$, and the quantity $- \int a^3 d \varrho$ is positive, since ϱ decreases from the center to the surface.

Hence it follows that h and $\frac{d h}{d a}$ are constantly positive, from the center to the surface. To show this, suppose that both these quantities are positive in going from the center; d h ought to become negative before h, and it is clear that in order to do this it must pass through zero; but from the instant it is nothing, $d^2 h$ becomes positive in virtue of the preceding equation, and consequently d h begins to increase; it can never therefore become negative. Whence it follows that h and d h always pre-

serve the same sign from the center to the surface. Now both of these quantities are positive in going from the center; for we have in virtue of equation (e), $s' - 2 = s + n - 2$, which gives $s' = i + n - 2$; hence we have

$$(s' + i + 3)\ (s' - i + 2)\ U'^{(i)} = \frac{6\,n\,(s+1)\,\gamma\,U^{(i)}}{(n+3)\,\beta};$$

whence we derive

$$U^{(i)} = \frac{6\,(i-1)\,\gamma\,.\,U^{(i)}}{(n+3)\,(2\,i+n+1)\,.\,\beta};$$

we shall therefore get

$$h = a^{i-2} + \frac{6\,(i-1)\,\gamma\,.\,a^{i+n-2}}{(n+3)\,(2\,i+n+1)\,\beta} + \&c.;$$

$$\frac{d\,h}{d\,a} = (i-2)\,a^{i-3} + \frac{6\,(i-1)\,(i+n-2)\,\gamma\,.\,a^{i+n-3}}{(n+3)\,(2\,i+n+1)\,\beta} + \&c.$$

γ, β, n, being positive, we see that at the center h and d h are positive, when i is equal to or greater than 2; they are therefore constantly positive from the center to the surface.

Relatively to the Earth, to the Moon, to Jupiter, &c. $Z^{(i)}$ is nothing or insensible, when i is equal to or greater than 3; the equation (2) of the preceding No. then becomes

$$0 = \{3\,a^{2i+1} \int \varrho\,d\left(\frac{h}{a^{i-2}}\right) - (2\,i+1)\,a^i\,h \int \varrho\,d\,a^3 + 3 \int \varrho\,d\,(a^{i+3}h)\}.\ U^{(i)};$$

the first integral being taken from $a = a$, to $a = 1$, and the two others being taken from $a = 0$, to $a = a$. At the surface where $a = 1$, this equation becomes

$$0 = \{-(2\,i+1)\,h \int \varrho\,d\,.\,a^3 + 3 \int \varrho\,d\,(a^{i+3}h)\}.\ U^{(i)};$$

an equation which we can put under this form

$$0 = \{-(2\,i-2)\,\varrho\,h + (2\,i+1)\,h \int a^3\,d\varrho - 3 \int a^{i+3}\,h\,.\,d\varrho\}\ U^{(i)}.$$

$d\varrho$ is negative from the center to the surface, and h increases in the same interval; the function $(2\,i+1)\,h \int a^3\,d\varrho - 3 \int a^{i+3}\,h\,d\varrho$ is therefore negative in the same interval; thus in the preceding equation the coefficient of $U^{(i)}$ is negative and cannot be nothing at the surface; $U^{(i)}$ ought therefore to be nothing, which gives $Y^{(i)} = 0$; the expression of the radius of the spheroid thus reduces to $a + \alpha\,a\,\{Y^{(0)} + Y^{(2)}\}$; that is to say, that the surface of each *leveled* shell of the spheroid is elliptic, and consequently its exterior surface is elliptic.

$Z^{(2)}$, relatively to the Earth is, by No. 567, equal to $-\frac{g}{2\alpha}\,(m^2 - \frac{1}{3})$; the equation (2) of the preceding No. gives therefore

$$0 = \{\tfrac{4}{3}\pi a^5 \int \varrho\, dh - \tfrac{4}{3}\pi a^2 h \int \varrho\, da^3 + \tfrac{4}{3}\pi \int \varrho\, d(a^5 h)\}\, U^{(1)} - \frac{g}{2\alpha} a^5 (m^2 - \tfrac{1}{3}).$$

At the surface, the first integral $\int \varrho\, dh$ is nothing; we have therefore at this surface where $a = 1$,

$$U^{(2)} = \frac{-\frac{1}{2\alpha} \cdot g \cdot (m^2 - \frac{1}{3})}{\frac{4}{3}\pi \cdot h \cdot \int \varrho\, d \cdot a^3 - \frac{4}{3}\pi \int \varrho\, d(a^5 h)}.$$

Let $\alpha \varphi$ be the ratio of the centrifugal force to the equatorial gravity; the expression of gravity to quantities of the order α, being equal to $\frac{4}{3}\pi \int \varrho\, d \cdot a^3$; we shall have $g = \frac{4}{3}\pi \alpha \varphi \int \varrho\, d \cdot a^3$; wherefore

$$U^{(2)} = \frac{-\varphi (m^2 - \frac{1}{3})}{2h - \frac{2}{5} \cdot \frac{\int \varrho \cdot d(a^5 h)}{\int \varrho \cdot a^2 da}};$$

comprising therefore in the arbitrary constant a, what we have taken for unity, the function

$$\alpha Y^{(0)} - \frac{\alpha h \varphi}{3h - \frac{3}{5} \cdot \frac{\int \varrho\, d(a^5 h)}{\int \varrho \cdot a^2 da}},$$

the radius of the terrestrial spheroid at the surface will be

$$1 + \frac{\alpha h \varphi (1 - m^2)}{2h - \frac{2}{5} \cdot \frac{\int \varrho\, d(a^5 h)}{\int \varrho \cdot a^2 da}}.$$

The figure of the earth supposed fluid, can therefore only be that of an ellipsoid of revolution; all of whose shells of constant density are elliptic, and of revolution, and in which the ellipticities increase, and the densities decrease from the center to the surface. The relation between the ellipticities and densities is given by the differential equation of the second order,

$$\frac{d^2 h}{da^2} = \frac{6h}{a^2}\left(1 - \frac{\varrho a^3}{3\int \varrho\, a^2 da}\right) - \frac{2\varrho a^2}{\int \varrho \cdot a^2 da} \cdot \frac{dh}{da}.$$

This equation is not integrable by known methods except in some particular suppositions of the densities ϱ; but if the law of the ellipticities were given, we should easily obtain that of the corresponding densities. We have seen that the expression of h given by the integral of this equation contains, in the present question, only one arbitrary, which disappears from the preceding value of the radius of the spheroid; there is therefore only one figure of equilibrium differing but little from a sphere, which is possible, and it is easy to see that the limits of the flattening of this figure are $\frac{\alpha \varphi}{2}$ and $\frac{5}{4}\alpha \varphi$, the former of which corresponds to the case where all

the mass of the spheroid is collected at its center, and the second to the case where this mass is homogeneous.

The directions of gravity from any point of the surface to the center do not form a straight line, but a curve whose elements are perpendicular to the *leveled* shells which they traverse: this curve is the orthogonal trajectory of all the ellipses which by their revolution form these shells. To determine its nature, take for the axis, the radius drawn from the center to a point of the surface, θ being the angle which this radius forms with the axis of revolution. We have just seen that the general expression of any shell of the spheroid is $a + \alpha k . a h . (1 - m^2)$, k being independent of a: whence it is easy to conclude that if we call $\alpha y'$, the ordinate let fall from any point of the curve upon its axis, we shall have

$$\alpha y' = \alpha a k . \sin. 2\theta \left\{ c - \int \frac{h \, d a}{a} \right\},$$

c being the entire value of the integral $\int \frac{h \, d a}{a}$, taken from the center to the surface.

575. Now consider the general case in which the spheroid always fluid at its surface, may contain a solid nucleus of any figure whatever, but differing but little from the sphere. The radius drawn from the center of gravity of the spheroid to its surface, and the law of gravity at this surface have some general properties, which it is the more essential to consider, inasmuch as these properties are independent of every hypothesis.

The first of these properties is, that in the state of equilibrium the fluid part of the spheroid must always be disposed so, that the function $Y^{(1)}$ may disappear from the expression of the radius drawn from the center of gravity of the whole spheroid to its surface; so that the center of gravity of this surface coincides with that of the spheroid.

To show this, we shall observe that R being supposed to represent the radius drawn from the center of gravity of the spheroid to any one of its molecules, the expression of this molecule will be $\varrho R^2 . d R . d m . d \varpi$, and we shall have by 556, in virtue of the properties of the center of gravity,

$$0 = \int \varrho R^3 . d R . d m . d \varpi . m;$$

$$0 = \int \varrho R^3 . d R . d m . d \varpi . \sqrt{1 - m^2} . \sin. \varpi;$$

$$0 = \int \varrho R^3 . d R . d m . d \varpi . \sqrt{1 - m^2} . \cos. \varpi.$$

Conceive the integral $\int \varrho R^3 . d R$ taken relatively to R from the origin of R to the surface of the spheroid, and then developed into a series of the form

$$N^{(0)} + N^{(1)} + N^{(2)} + \&c.;$$

$N^{(i)}$ being whatever i may be, subject to the equation of partial differences,

$$0 = \left(\frac{d\left\{(1-m^2)\left(\frac{dN^{(i)}}{dm}\right)\right\}}{dm} \right) + \frac{\left(\frac{d^2 N^{(i)}}{d\varpi^2}\right)}{1-m^2} + i(i+1)N^{(i)};$$

we shall have by No. 556, when i is different from unity,

$$0 = \int N^{(i)}.\, m\, dm\, .\, d\varpi;\quad 0 = \int N^{(i)}.\, dm\, .\, d\varpi\, .\, \sqrt{1-m^2}.\sin.\varpi;$$

and

$$0 = \int N^{(i)}.\, dm\, .\, d\varpi\, .\, \sqrt{1-m^2}.\cos.\varpi.$$

The three preceding equations given by the nature of the center of gravity, will become

$$0 = \int N^{(1)}\, m\, dm\, .\, d\varpi;\quad 0 = \int N^{(1)}\, dm\, .\, d\varpi\, .\, \sqrt{1-m^2}.\sin.\varpi;$$

$$0 = \int N^{(1)}\, dm\, .\, d\varpi\, .\, \sqrt{1-m^2}.\cos.\varpi.$$

$N^{(1)}$ is of the form

$$H\, m + H'.\sqrt{1-m^2}.\sin.\varpi + H''.\sqrt{1-m^2}.\cos.\varpi.$$

Substituting this value, in these three equations, we shall have

$$H = 0;\quad H' = 0;\quad H'' = 0;$$

where $N^{(1)} = 0$; this is the condition necessary that the origin of R is at the center of gravity of the spheroid.

Now let us see, what $N^{(1)}$ becomes relatively to the spheroids differing little from the sphere, and covered over with a fluid in equilibrium. In this case we have $R = a(1 + \alpha y)$, and the integral $\int \varrho . R^3 . dR$, becomes $\frac{1}{4}\int \varrho\, d\, .\, \{a^4(1 + 4\alpha y)\}$, the differential and integral being relative to the variable a, of which ϱ is a function. Substituting for y its value $Y^{(0)} + Y^{(1)} + Y^{(2)} +$ &c., we shall have

$$N^{(1)} = \alpha \int \varrho\, d\,(a^4\, Y^{(1)}).$$

The equation (2) of No. 573 gives, at the surface where $a = 1$, and observing that $Z^{(1)}$ is nothing

$$\int \varrho\, d\,(a^4\, Y^{(1)}) = Y^{(1)} \int \varrho\, d\, .\, a^3,$$

the value of $Y^{(1)}$ in the second member of this equation, being relative to the surface; thus, $N^{(1)}$ being nothing, when the origin of R is at the center of gravity of the spheroid, we have in like manner $Y^{(1)} = 0$.

576. The permanent state of equilibrium of the celestial bodies, makes known also some properties of their radii. If the planets did not turn exactly, or at least if they turned not nearly, round one of their three principal axes of rotation, there would result in the position of their axes of rotation, changes which for the earth above all would be sensible; and since the most exact observations have not led to the discovery of any, we may conclude that long since, all the parts of the celestial bodies, and princi-

pally the fluid parts of their surfaces, are so disposed as to render stable their state of equilibrium, and consequently their axes of rotation. It is in fact very natural to suppose that after a great number of oscillations, they must settle in this state, in virtue of the resistances which they suffer. Let us see, however, the conditions which thence result in the expression of the radii of the celestial bodies.

If we name x, y, z the rectangular coordinates of a molecule d M of the spheroid, referred to three principal axes, the axis of x being the axis of rotation of the spheroid; by the properties of these axes as shown in dynamics, we have

$$0 = \int x y \,.\, d M;\ 0 = \int x z \,.\, d M;\ 0 = \int y z \,.\, d M;$$

the integrals ought to be extended to the entire mass of the spheroid, R being the radius drawn from the origin of coordinates to the molecule d M; θ being the angle formed by R and by the axis of rotation; and ϖ being the angle which the plane formed by this axis and by R, makes with the plane formed by this axis and by that of the principal axes, which is the axis of y; we shall have

$$x = R m;\ y = R \sqrt{1 - m^2}.\cos.\varpi;\ z = R \sqrt{1 - m^2}.\sin.\varpi;$$
$$d M = \varrho R^2 d R d m \,.\, d \varpi.$$

The three equations given by the nature of the principal axes of rotation, will thus become

$$0 = \int \varrho \,.\, R^4.\, d R \,.\, d m \,.\, d \varpi \,.\, m \sqrt{1 - m^2}.\cos.\varpi;$$
$$0 = \int \varrho \,.\, R^4.\, d R \,.\, d m \,.\, d \varpi \,.\, m \sqrt{1 - m^2}.\sin.\varpi;$$
$$0 = \int \varrho \,.\, R^4.\, d R \,.\, d m \,.\, d \varpi \,.\, (1 - m^2)\sin. 2\varpi.$$

Conceive the integral $\int \varrho R^4 d R$ taken relatively to R, from $R = 0$, to the value of R at the surface of the spheroid, and developed into a series of the form $U^{(0)} + U^{(1)} + U^{(2)} + \&c.$; $U^{(i)}$ being, whatever i may be, subject to the equation of partial differences,

$$0 = \left(\frac{d \left\{ (1 - m^2) . \left(\frac{d U^{(i)}}{d m} \right) \right\}}{d m} \right) + \frac{\left(\frac{d^2 U^{(i)}}{d \varpi^2} \right)}{1 - m^2} + i(i + 1) . U^{(i)}.$$

We shall have by the theorem of No. 556, where i is different from 2, and by observing that the functions $m \sqrt{1 - m^2}.\cos.\varpi$, $m \sqrt{1 - m^2}.\sin.\varpi$, and $(1 - m^2)\sin. 2\varpi$, are comprised in the form $U^{(2)}$;

$$0 = \int U^{(i)} . d m \,.\, d \varpi \,.\, m \,.\, \sqrt{1 - m^2}.\cos.\varpi;$$
$$0 = \int U^{(i)} . d m \,.\, d \varpi \,.\, m \,.\, \sqrt{1 - m^2}.\sin.\varpi;$$
$$0 = \int U^{(i)} . d m \,.\, d \varpi \,.\, (1 - m^2)\sin. 2\varpi.$$

The three equations relative to the nature of the axes of rotation, will thus become

$$0 = \int U^{(2)}.\, d\,m.d\,\varpi.m.\sqrt{1-m^2}.\cos.\varpi;$$
$$0 = \int U^{(2)}.\, d\,m.d\,\varpi.m.\sqrt{1-m^2}.\sin.\varpi;$$
$$0 = \int U^{(2)}.\, d\,m.d\,\varpi.(1-m^2)\sin.2\varpi.$$

These equations therefore depend only on the value of $U^{(2)}$: this value is of the form

$$H\,(m^2-\tfrac{1}{3}) + H'\,m\sqrt{1-m^2}.\sin.\varpi + H''\,m\sqrt{1-m^2}.\cos.\varpi + H'''\,(1-m^2)\sin.2\varpi + H''''\,(1-m^2)\cos.2\varpi:$$

substituting it in the three preceding equations, we shall have

$$H' = 0;\quad H'' = 0;\quad H''' = 0.$$

It is to these three conditions that the conditions necessary to make the three axes of x, y, z the true axes of rotation are reduced, and then $U^{(2)}$ will be of the form

$$H\,(m^2-\tfrac{1}{3}) + H''''\,(1-m^2)\cos.2\varpi.$$

When the spheroid is a solid differing but little from the sphere, and covered with a fluid in equilibrium, we have $R = a\,(1+\alpha\,y)$, and consequently

$$\int \varrho\, R^4.\, d\,R = \tfrac{1}{5}\int \varrho\, d.\{a^5.\,(1+5\,\alpha\,y)\}.$$

If we substitute for y, its value $Y^{(0)} + Y^{(1)} + Y^{(2)} +$ &c.; we shall have

$$U^{(2)} = \alpha \int \varrho\, d\,(a^5\,Y^{(2)}).$$

The equation (2) of No. 573, gives for the surface of the spheroid,

$$\frac{4\,\pi}{5}\int \varrho\, d\,(a^5\,Y^{(2)}) = \tfrac{4}{3}\,\pi\,Y^{(2)}\int \varrho\, d.\,a^3 - Z^{(2)};$$

$Y^{(2)}$ and $Z^{(2)}$ in the second member of this equation being relative to the surface; we have therefore,

$$U^{(2)} = \tfrac{5}{3}\,\alpha\,Y^{(2)}\int \varrho\, d.\,a^3 - \frac{5\,\alpha\,Z^{(2)}}{4\,\pi}.$$

The value of $Z^{(2)}$ is of the form

$$-\frac{g}{2}(m^2-\tfrac{1}{3}) + g'\,m\sqrt{1-m^2}.\sin.\varpi + g''\,m\sqrt{1-m^2}.\cos.\varpi;$$
$$+ g'''\,(1-m^2)\sin.2\varpi + g''''\,(1-m^2)\cos.2\varpi;$$

and that of $Y^{(2)}$ is of the form

$$-h\,(m^2-\tfrac{1}{3}) + h'\,m\sqrt{1-m^2}.\sin.\varpi + h''\,m\sqrt{1-m^2}.\cos.\varpi$$
$$+ h'''.\,(1-m^2)\sin.2\varpi + h''''\,(1-m^2)\cos.2\varpi.$$

Substituting in the preceding equation, these values, and $H\,(m^2-\frac{1}{3}) + H''''\,(1-m^2)\cos 2\varpi$, for $U^{(2)}$; we shall have

$$h' = \frac{g'}{4\,\pi\int \varrho.\,a^2\,d\,a};\quad h'' = \frac{g''}{4\,\pi\int \varrho.\,a^2\,d\,a};\quad h''' = \frac{g'''}{4\,\pi\int \varrho.\,a^2\,d\,a}.$$

Such are the conditions which result from the supposition that the spheroid turns round one of its principal axes of rotation. This supposition determines the constants h', h'', h''' by means of the values g', g'', g'''; but it leaves indeterminate the quantities h and h'''' as also the functions $Y^{(3)}$, $Y^{(4)}$, &c.

If the forces extraneous to the attraction of the molecules of the spheroid are reduced to the centrifugal force due to its rotatory motion; we shall have $g' = 0$, $g'' = 0$, $g''' = 0$; wherefore $h' = 0$, $h'' = 0$, $h''' = 0$, and the expression of $Y^{(2)}$, will be of the form

$$-h(m^2 - \tfrac{1}{3}) + h''''(1 - m^2)\cos. 2\varpi.$$

577. Let us consider the expression of gravity at the surface of the spheroid. Call p this force; it is easy to see by No. 569, that we shall have its value by differentiating the second member of the equation (1) of 573 relatively to r, and by dividing its differential by $-dr$; which gives at the surface

$$p = \frac{4\pi}{3r^2}\int \varrho\, d.a^3 + \frac{4\alpha\pi}{r^2}\int \varrho\, d.\left\{a^3 Y^{(0)} + \frac{2a^4}{3r}Y^{(1)} + \frac{3a^5}{5r^2}Y^{(2)} + \frac{4a^6}{7r^3}Y^{(3)} + \&c\right\}$$
$$- \alpha r\{2Z^{(0)} + 2Z^{(2)} + 3r.Z^{(3)} + 4r^2.Z^{(4)} + \&c.\};$$

these integrals being taken from $a = 0$, to $a = 1$. The radius r at the surface is equal to $1 + \alpha y$, or equal to

$$1 + \alpha\{Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.\};$$

we shall hence obtain

$$p = \frac{4\pi}{3}\int \varrho\, d.a^3 - \frac{8\alpha\pi}{3}\{Y^{(0)} + Y^{(1)} + Y^{(2)} + \&c.\}\int \varrho\, d.a^3$$
$$+ 4\alpha\pi\int \varrho\, d.\{a^3 Y^{(0)} + \frac{2a^4}{3}Y^{(1)} + \frac{3a^5}{5}Y^{(2)} + \&c.\}$$
$$- \alpha\{2Z^{(0)} + 2Z^{(2)} + 3Z^{(3)} + 4Z^{(4)} + \&c.\}.$$

The integrals of this expression may be made to disappear by means of equation (2) of No. 573, which becomes at the surface,

$$\frac{4\pi}{2i+1}\cdot\int \varrho\, d.(a^{i+3}Y^{(i)}) = \tfrac{4}{3}\pi Y^{(i)}.\int \varrho\, d.a^3 - Z^{(i)};$$

supposing therefore

$$P = \tfrac{4}{3}\pi\int \varrho\, d.a^3 - \frac{8\alpha\pi}{3}Y^{(0)} + 4\alpha\pi\int \varrho\, d.(a^3 Y^{(0)}) - 2\alpha Z^{(0)};$$

we shall have

$$p = P + \alpha P.\{Y^{(2)} + 2Y^{(3)} + 3Y^{(4)} + \ldots + (i-1)Y^{(i)} + \&c.\}$$
$$- \alpha\{5Z^{(2)} + 7Z^{(3)} + 9Z^{(4)} + \ldots + (2i+1)Z^{(i)} + \&c.\}.$$

By observations of the lengths of the seconds' pendulum, has been recognised the variation of gravity at the surface of the earth. By dynamics it appears that these lengths are proportional to gravity; let

therefore l, L be the lengths of the pendulum corresponding to the gravities p, P; the preceding equation will give

$$l = L + \alpha L \{Y^{(2)} + 2Y^{(3)} + 3Y^{(4)} + \ldots + (i-1)Y^{(i)}\}$$
$$-\frac{\alpha L}{P}.\{5\,Z^{(2)} + 7Z^{(3)} + \ldots\ldots\ldots + (2i+1)Z^{(i)}\}.$$

Relatively to the earth $\alpha Z^{(2)}$ reduces by 567, to $-\frac{g}{2}(m^2 - \frac{1}{3})$, or, which comes to the same, to $-\frac{\alpha\varphi}{2}.\,P.\,(m^2 - \frac{1}{3})$, $\alpha\varphi$ being the ratio of the centrifugal force to the equatorial gravity; moreover, $Z^{(3)}$, $Z^{(4)}$, &c. are nothing; we have therefore

$$l = L + \alpha L.\{Y^{(2)} + 2Y^{(3)} + 3Y^{(4)} + \ldots + (i-1)Y^{(i)}\}$$
$$+ \tfrac{5}{2}\alpha\varphi.L.(m^2 - \tfrac{1}{3}).$$

The radius of curvature of the meridian of a spheroid which has for its radius $1 + \alpha y$, is

$$1 + \alpha\left(\frac{d.my}{dm}\right) + \alpha\left(\frac{d.\left\{(1-m^2)\left(\frac{dy}{dm}\right)\right\}}{dm}\right)$$

designating therefore by c, the magnitude of the degree of a circle whose radius is what we have taken for unity; the expression of the degree of the spheroid's meridian, will be

$$c\left\{1 + \alpha\left(\frac{d.my}{dm}\right) + \alpha\left(\frac{d.\left\{(1-m^2)\left(\frac{dy}{dm}\right)\right\}}{dm}\right)\right\}.$$

y is equal to $Y^{(0)} + Y^{(1)} + Y^{(2)} +$ &c. We may cause $Y^{(0)}$ to disappear, by comprising it in the arbitrary constant which we have taken for the unit; and $Y^{(1)}$ by fixing the origin of the radius at the center of gravity of the entire spheroid. This radius thus becomes,

$$1 + \alpha\{Y^{(2)} + Y^{(3)} + Y^{(4)} + \text{&c.}\}.$$

If we then observe that

$$\left(\frac{d\left\{(1-m^2).\left(\frac{dY^{(i)}}{dm}\right)\right\}}{dm}\right) = -i(i+1)Y^{(i)} - \frac{\left(\frac{d^2Y^{(i)}}{d\varpi^2}\right)}{1-m^2};$$

the expression of the degree of the meridian will become

$$c - \alpha c\{5Y^{(2)} + 11Y^{(3)} + \ldots + (i^2+i-1)Y^{(i)} + \text{&c.}\}$$
$$+ \alpha c m\left\{\left(\frac{dY^{(2)}}{dm}\right) + \left(\frac{dY^{(3)}}{dm}\right) + \text{&c.}\right\}$$
$$- \alpha c.\frac{\left(\frac{d^2Y^{(2)}}{d\varpi^2}\right) + \left(\frac{d^2Y^{(3)}}{d\varpi^2}\right) + \text{&c.}}{1-m^2}.$$

If we compare these expressions of the terrestrial radius with the length of the pendulum, and the magnitude of the degree of the meridian, we see that the term $\alpha\, Y^{(i)}$ of the expression of the radius is multiplied by $i - 1$, in the expression of the length of the pendulum, and by $i^2 + i - 1$ in that of the degree; whence it follows, that whilst $i - 1$ is considerable, this term will be more sensible in the observations of the length of the pendulum than in that of the horizontal parallax of the moon which is proportional to the terrestrial radius; it will be still more sensible in the measures of degrees than in the lengths of the pendulum. The reason of it is, that the terms of the expression of the terrestrial radius undergo two variations in the expression of the degree of the meridian; and each differentiation multiplies these terms by the corresponding exponent of m, and this renders them the more considerable. In the expression of the variation of two consecutive degrees of the meridian, the terms of the terrestrial radius undergo three consecutive differentiations; those which disturb the figure of the earth from that of an ellipsoid, may thence become very sensible, and the ellipticity obtained by this variation may be very different from that which the observed lengths of the pendulum give. These three expressions have the advantage of being independent of the interior constitution of the earth, that is to say, of the figure and density of its shells; so that if we are going to determine the functions $Y^{(2)}$, $Y^{(3)}$, &c. by measures of degrees of meridians and parallaxes, we shall have immediately the length of the pendulum; we may therefore thus ascertain whether the law of universal gravity accords with the figure of the earth, and with the observed variations of gravity at its surface. These remarkable relations between the expressions of the degrees of the meridian and of the lengths of the pendulum may also serve to verify the hypotheses proper to represent the measures of degrees of this meridian: this will be perceptible from the application we now proceed to make to the hypothesis proposed by Bouguer, to represent the degrees measured northward in France and at the equator.

Suppose that the expression of the terrestrial radius is $1 + \alpha\, Y^{(2)} + \alpha\, Y^{(4)}$, and that we have

$$Y^{(2)} = -A\,(m^2 - \tfrac{1}{3}); \quad Y^{(4)} = -B\left(m^4 - \tfrac{6}{7}\,m^2 + \frac{3}{35}\right);$$

it is easy to see that these functions of m satisfy the equations of partial differences which $Y^{(2)}$ and $Y^{(4)}$ ought to satisfy. The variation of the degrees of the meridian will be, by what precedes,

$$\alpha\, c \left\{3\,A - \frac{102}{7}\,B\right\}^{2} + 15\,\alpha\, c\, B\,.\,m^4.$$

Bouguer supposes this variation proportional to the fourth power of the sine of the latitude, or, which nearly comes to the same, to m^4; the term multiplied by m^2, therefore, being made to disappear from the preceding function, we shall have

$$B = \frac{7}{34} . A;$$

thus in this case the radius drawn from the center of gravity of the earth at its surface, will be in taking that of the equator for unity,

$$1 - \frac{7 \alpha A}{34} (4 m^2 + m^4).$$

The expression of the length l of the pendulum, will become, denoting by L, its value at the equator,

$$L + \tfrac{5}{2} \alpha \varphi . L m^2 - \frac{\alpha . A L}{34} (16 m^2 + 21 m^4).$$

Finally, the expression of the degree of the meridian, will be, calling c its length at the equator,

$$c + \frac{105}{34} . \alpha A . c . m^4.$$

We shall observe here, that agreeably to what we have just said, the term multiplied by m^4 is three times more sensible in the expression of the length of the pendulum than in that of the terrestrial radius, and five times more sensible in the expression of the length of a degree, than in that of the length of the pendulum; finally, upon the mean parallel it would be four times more sensible in the expression of the variation of consecutive degrees, than in that of the same degree. According to Bouguer, the difference of the degrees at the pole and equator is $\frac{959}{56753}$; it is the ratio which, on his hypothesis, the measures of degrees at Pello, Paris and the equator, require. This ratio is equal to $\frac{105}{34} . \alpha A$; we have therefore

$$\alpha A = 0.0054717.$$

Taking for unity the length of the pendulum at the equator, the variation of this length, in any place whatever, will be

$$- \frac{0.0054717}{34} . \{16 m^2 + 21 m^4\} + \tfrac{5}{2} \alpha \varphi . m^2.$$

By No. 563, we have $\alpha \varphi = 0.00345113$, which gives $\frac{5}{2} \alpha \varphi = 0.0086278$, and the preceding formula becomes

$$0.0060529 . m^2 - 0.0033796 . m^4.$$

At Pello, where m = sin. 74°. 22′, this formula gives 0.0027016 for the variation of the length of the pendulum. According to the observations, this variation is 0.0044625, and consequently much greater; thus, since the hypothesis of Bouguer cannot be reconciled with the observations made on the length of the pendulum, it is inadmissible.

578. Let us apply the general results which we have just found, to the case where the spheroid is not sollicited by any extraneous forces, and where it is composed of elliptic shells, whose center is at the center of gravity of the spheroid. We have seen that this case is that of the earth supposed to be originally fluid: it is also that of the earth in the hypothesis where the figures of the shells are similar. In fact, the equation (2) of No. 573 becomes at the surface where a = 1,

$$0 = Y^{(i)} \int \varrho\, a^2\, da - \frac{1}{2i+1} \int \varrho\, d\,(a^{i+3} Y^{(i)}) - \frac{Z^{(i)}}{4\pi}.$$

The shells being supposed similar, the value of $Y^{(i)}$ is, for each of them, the same as at the surface; it is consequently independent of a, and we have

$$Y^{(i)} \int \varrho\, a^2\, da \left\{ 1 - \frac{i+3}{2i+1} . a^i \right\} = \frac{Z^{(i)}}{4\pi}.$$

When i is equal to or greater than 3, $Z^{(i)}$ is nothing relatively to the earth; besides the factor $1 - \frac{i+3}{2i+1} . a^i$ is always positive; therefore $Y^{(i)}$ is then nothing. $Y^{(1)}$ is also nothing by No. 575, when we fix the origin of the radii at the center of gravity of the spheroid. Finally, by No. 577, we have $Z^{(2)}$ equal to

$$-\frac{\varphi}{2} (m^2 - \tfrac{1}{3})\, 4\pi \int \varrho . a^2\, da\,; \quad Y^{(0)} = -\frac{\varphi}{2} . (\mu^2 - \tfrac{1}{3}) \int \varrho\, a^2\, da\,;$$

we have therefore

$$Y^{(2)} = -\frac{\frac{\varphi}{2} (m^2 - \frac{1}{3}) \int \varrho\, a^2\, da}{\int \varrho\, a^2\, da\, (1 - a^2)}.$$

Thus the earth is then an ellipsoid of revolution. Let us consider therefore generally the case where the figure of the earth is elliptic and of revolution.

In this case, by fixing the origin of terrestrial radii at the center of gravity of the earth, we have

$$Y^{(1)} = 0\,; \quad Y^{(3)} = 0\,; \quad Y^{(4)} = 0\,; \text{ \&c.}$$
$$Y^{(2)} = -h\,(m^2 - \tfrac{1}{3}),$$

h being a function of a; moreover we have

$$Z^{(1)} = 0; \quad Z^{(3)} = 0; \quad Z^{(4)} = 0; \text{ \&c.}$$

$$\alpha Z^{(2)} = -\frac{\alpha\varphi}{2}(m^2 - \tfrac{1}{3}).\frac{4\pi}{3}\int \varrho\, d.\, a^3;$$

the equation (2) of No. 573 will therefore give at the surface

$$0 = 6.\int \varrho\, d\,(a^5 h) + 5.(\varphi - 2h)\int \varrho\, d.\, a^3 \quad . \quad . \quad . \quad (1)$$

This equation contains the law which ought to exist to sustain the equilibrium between the densities of the shells of the spheroid and their ellipticities; for the radius of a shell being $a\{1+\alpha Y^{(0)} - \alpha h(\mu^2 - \frac{1}{3})\}$; if we suppose, as we may, that $Y^{(0)} = -\frac{1}{3}h$, this radius becomes $a(1 - \alpha h.\mu^2)$, and αh is the ellipticity of the shell.

At the surface, the radius is $1 - \alpha h.\mu^2$; whence we see that the decrements of the radii, from the equator to the poles, are proportional to μ^2, and consequently to the square of the sines of the latitude.

The increment of the degrees of the meridian from the equator to the poles is, by the preceeding No., equal to $3\alpha h c.\mu^2$, c being the degree of the equator; it is therefore also proportional to the square of the sine of the latitude.

The equation (1) shows us that the densities being supposed to decrease from the center to the surface, the ellipticity of the spheroid is less than in the case of homogeneity, at least whilst the ellipticities do not increase from the surface to the center in a greater ratio than the inverse ratio of the square of the distances to this center. In fact, if we suppose $h = \frac{u}{a^2}$, we shall have

$$\int \varrho\, d\,(a^5 h) = \int \varrho\, d\,(a^3 u) = u\int \varrho\, d.\, a^3 + \int (d\,u.\int a^3 d\,\varrho).$$

If the ellipticities increase in a less ratio than $\frac{1}{a^2}$, u increases from the center to the surface, and consequently d u is positive; besides, d ϱ is negative by the supposition that the densities decrease from the center to the surface; thus, $\int(d\,u\int a^3\, d\,\varrho)$ is a negative quantity, and making at the surface

$$\int \varrho\, d\,(a^5 h) = (h - f)\int \varrho\, d.\, a^3,$$

f will be a positive quantity. Hence equation (1) will give

$$h = \frac{5\varphi - 6f}{4};$$

αh will therefore be less than $\frac{5\alpha\varphi}{4}$, and consequently it will be less than

in the case of homogeneity, where $d\varrho$ being equal to nothing f is also equal to zero.

Hence it follows, that in the most probable hypotheses, the flattening of the spheroid is less than $\frac{5\alpha\varphi}{4}$; for it is natural to suppose that the shells of the spheroid are denser towards the center, and that the ellipticities increase from the surface to the center in a less ratio than $\frac{1}{a^2}$, this ratio giving an infinite radius for shells infinitely near to the center, which is absurd. These suppositions are the more probable, inasmuch as they become necessary in the case where the fluid is originally fluid; then the denser shells are, as we have seen, the nearer to the center, and the ellipticities so far from increasing from the surface to the center, on the contrary, decrease.

If we suppose that the spheroid is an ellipsoid of revolution, covered with a homogeneous fluid mass of any depth whatever, by calling a' the semi-minor axis of the solid ellipsoid, and $\alpha h'$ its ellipticity, we shall have at the surface of the fluid,

$$\int \varrho\, d\,(a^5 h) = h - a'^5 h' + \int \varrho\, d\,(a^5 h);$$

the integral of the second member of this equation being taken relatively to the interior ellipsoid, from its center to its surface, and the density of the fluid which covers it being taken for unity. The equation (1) will give for the expression of the ellipticity αh, of the terrestrial spheroid,

$$\alpha h = \frac{5\alpha\varphi\{1 - a'^3 + \int \varrho\, d\, a^3\} - 6\alpha h'.a'^5 + 6\alpha \int \varrho\, d\,(a^5 h)}{4 - 10\,a'^3 + 10.\int \varrho\, d.\, a^3};$$

the integrals being taken from $a = 0$ to $a = a'$.

Let us now consider the law of gravity, or which comes to the same, that of the length of the pendulum at the elliptic surface in equilibrium. The value of l, found in the preceding No., becomes in this case

$$l = L + \alpha L\{\tfrac{5}{2}\varphi - h\}(m^2 - \tfrac{1}{3});$$

making, therefore, $L' = L - \frac{1}{3}\alpha L(\frac{5}{2}\varphi - h)$, we shall have, in neglecting quantities of the order α^2,

$$l = L' + \alpha L'(\tfrac{5}{2}\varphi - h)\mu^2;$$

an equation from which it results that L' is the length of the seconds' pendulum at the equator, and that this length increases from the equator to the poles, proportionally to the square of the sine of the latitude.

If we call $\alpha\varepsilon$ the excess of the length of the pendulum at the pole above its length at the equator, divided by the latter, we shall have

$$\alpha\varepsilon = \alpha(\tfrac{5}{2}\varphi - h);$$

and consequently

$$\alpha\,\varepsilon + \alpha\,h = \tfrac{5}{2}\,\alpha\,\varphi;$$

a remarkable equation between the ellipticity of the earth and the variation of the length of the pendulum from the equator to the poles. In the case of homogeneity $\alpha\,h = \frac{5}{4}\,\alpha\,\varphi$; hence in this case $\alpha\,\varepsilon = \alpha\,h$; but *if the spheroid is heterogeneous, as much as* $\alpha\,h$ *is above or below* $\frac{5}{4}\,\alpha\,\varphi$, *so much is* $\alpha\,\varepsilon$ *above or below the same quantity.*

579. The planets being supposed covered with a fluid in equilibrium, it is necessary, in the estimate of their attractions, to know the attraction of spheroids whose surface is fluid and in equilibrium: we may express it very simply in this way. Resume the equation (5) of No. 558; the signs of integration may be made to disappear by means of equation (2) of No. 573, which gives at the surface of the spheroid,

$$\frac{4\pi}{2i+1}\int\varrho\,d\,(a^{i+3}\,.\,Y^{(i)}) = \frac{4\pi}{3}\,Y^{(i)}\int\varrho\,d\,.\,a^3 - Z^{(i)};$$

thus fixing the origin of the radii r at the center of gravity of the spheroid which makes $Y^{(1)}$ disappear; then observing that $Z^{(1)}$ is nothing, and that $Y^{(0)}$ being arbitrary, we may suppose $\frac{4\pi}{3}\,Y^{(0)} - Z^{(0)} = 0$, the equation (5) of 558, will give

$$V = \frac{4\pi}{3r}\int\varrho\,d\,.\,a^3 + \frac{4\alpha\pi}{3r^3}\left\{Y^{(2)} + \frac{Y^{(3)}}{r} + \frac{Y^{(4)}}{r^2} + \&c.\right\}\int\varrho\,d\,.\,a^3$$
$$- \frac{\alpha}{r^3}\left\{Z^{(2)} + \frac{Z^{(3)}}{r} + \frac{Z^{(4)}}{r^2} + \&c.\right\};$$

an expression in which we ought to observe that $\frac{4\pi}{3}\int\varrho\,d\,.\,a^3$ expresses the mass of the spheroid, since, in the case of r being infinite, the value of V is equal to the mass of the spheroid divided by r. Hence the attraction of the spheroid parallel to r will be $-\left(\frac{dV}{dr}\right)$; the attraction perpendicular to this radius, in the plane of the meridian will be $-\frac{\sqrt{1-m^2}}{r}\left(\frac{dV}{dm}\right)$; finally, the attraction perpendicular to this same radius in the direction of the parallel will be

$$-\frac{\left(\frac{dV}{d\varpi}\right)}{r\sqrt{1-m^2}}.$$

The expression of V, relatively to the earth supposed elliptic, becomes

$$V = \frac{M}{r} + \frac{\frac{1}{2}\alpha\varphi - \alpha h}{r^3}\,M\left(m^2 - \tfrac{1}{3}\right);$$

M being the mass of the earth.

580. Although the law of attraction in the inverse ratio of the square of the distance is the only one that interests us, yet equation (1) of 554 affords a determination so simple of the gravity at the surface of homogeneous spheroids in equilibrium, whatever is the exponent of the power of the distance to which the attraction is proportional, that we cannot here omit it. The attraction being as any power n of the distance, if we denote by d m a molecule of the spheroid, and by f its distance from the point attracted, the action of d m upon this point multiplied by the element — d f of its direction, will be $-d\,\mu\,f^n.\,df$. The integral of this quantity, taken relatively to f, is $-\frac{d\,\mu\,f^{n+1}}{n+1}$, and the sum of these integrals extended to the entire spheroid is $-\frac{V}{n+1}$; supposing, as in 554, that $V = \int f^{n+1}\,d\,\mu$.

If the spheroid be fluid, homogeneous, and endowed with rotatory motion, and not sollicited by any extraneous force, we shall have at the surface, in the case of equilibrium, by No. 567,

$$\text{const.} = -\frac{V}{n+1} + \tfrac{1}{2}\,g\,r^2\,(1 - m^2),$$

r being the radius drawn from the center of gravity of the spheroid at its surface, and g the centrifugal force at the distance 1 from the axis of rotation.

The gravity p at the surface of the spheroid is equal to the differential of the second member of this equation taken relatively to r, and divided by — d r, which gives

$$p = \frac{1}{n+1}\cdot\left(\frac{d\,V}{d\,r}\right) - g\,r\,(1 - m^2).$$

Let us now resume equation (1) of 554, which is relative to the surface,

$$\left(\frac{d\,V}{d\,r}\right) = A' - \frac{(n+1)\,A}{2\,a} + \frac{(n+1)\,V}{2\,a};$$

this equation, combined with the preceding ones, gives

$$p = \text{const.} + \left\{\frac{(n+1)\,r}{4\,a} - 1\right\} g\,r\,(1 - m^2).$$

At the surface, r is very nearly equal to a; by making them entirely so, for the sake of simplicity, we shall have

$$p = \text{const.} + \frac{n-3}{4}\,g\,(1 - m^2)$$

Let P be the gravity at the equator of the spheroid, and $a\ \varphi$

the ratio of the centrifugal force to gravity at the equator; we shall have

$$p = P\left\{1 + \frac{3 - n}{4}\alpha\varphi.m^2\right\};$$

whence it follows that, from the equator to the poles, gravity varies as the square of the sine of the latitude. In the case of nature, where $n = -2$, we have

$$p = P\{1 + \tfrac{5}{4}\alpha\varphi.m^2\};$$

which agrees with what we have before found.

But it is remarkable that if $n = 3$, we have $p = P$, that is to say, that if the attraction varies as the cube of the distance, the gravity at the surface of homogeneous spheroids is every where the same, whatever may be the motion of rotation.

581. We have only retained, in the research of the figure of the celestial bodies, quantities of the order α; but it is easy, by the preceding analysis, to extend the approximations to quantities of the order α^2, and to superior orders. For that purpose, consider the figure of a homogeneous fluid mass in equilibrium, covering a spheroid differing but little from a sphere, and endowed with a rotatory motion; which is the case of the earth and planets. The condition of equilibrium at the surface gives, by No. 557, the equation

$$\text{const.} = V - \frac{g}{2}r^2(m^2 - \tfrac{1}{3}).$$

The value of V is composed, 1st, of the attraction of the spheroid covered by the fluid upon the molecule of the surface, determined by the coordinates r, θ, and ϖ; 2dly, of the attraction of the fluid mass upon this molecule. But the sum of these two attractions is the same as the sum of the attractions, 1st, of a spheroid supposing the density of each of its shells diminished by the density of the fluid; 2dly, of a spheroid of the same density as the fluid, and whose exterior surface is the same as that of the fluid. Let V′ be the first of these attractions and V″ the second, so that $V = V' + V''$; we shall have, supposing g of the order α and equal to $\alpha g'$,

$$\text{const.} = V' + V'' - \frac{\alpha g'}{2}.r^2.(m^2 - \tfrac{1}{3}).$$

We have seen in 553 that V′ may be developed into a series of the form

$$\frac{U^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \frac{U^{(2)}}{r^3} + \&c.$$

$U^{(i)}$ being subject to the equation of partial differences,

$$0 = \frac{d\left\{(1 - m^2)\left(\frac{dU^{(i)}}{dm}\right)\right\}}{dm} + \frac{\left(\frac{d^2U^{(i)}}{d\varpi^2}\right)}{1 - m^2} + i(i+1)U^{(i)}.$$

and by the analysis of 561, we may determine $U^{(i)}$, with all the accuracy that may be wished for, when the figure of the spheroid is known.

In like manner V″ may be developed into a series of the form

$$\frac{U_{,}^{(0)}}{r} + \frac{U_{,}^{(1)}}{r^2} + \frac{U_{,}^{(2)}}{r^3} + \&c.$$

$U_{,}^{(i)}$ being subject to the same equation of partial differences as $U^{(i)}$. If we take for the unit of density that of the fluid, we have, by 561,

$$U_{,}^{(i)} = \frac{4\pi}{(i+3)(2i+1)} \cdot Z^{(i)};$$

r^{i+3} being supposed developed into the series

$$Z^{(0)} + Z^{(1)} + Z^{(2)} + \&c.$$

in which $Z^{(i)}$ is subject to the same equation of partial differences, as $U^{(i)}$. The equation of equilibrium will therefore become

$$\text{const.} = \frac{U^{(0)}}{r} + \frac{U_{,}^{(0)}}{r} + \Sigma \cdot \frac{1}{r^{i+1}} \left\{ U^{(i)} + \frac{4\pi}{(i+3)(2i+1)} Z^{(i)} \right\} - \alpha g' r^2 (m^2 - \tfrac{1}{3});$$

i being equal to greater than unity.

If the distance r from the molecule attracted to the center of the spheroid were infinite, V would be equal to the sum of the masses of the spheroid and fluid divided by r; calling, therefore, m this mass, we have $U^{(0)} + U_{,}^{(0)} = m$. Carrying the approximation only to quantities of the order α^2, we may suppose

$$r = 1 + \alpha y + \alpha^2 y';$$

which gives

$$r^{i+3} = 1 + (i+3)\alpha y + \frac{(i+2)(i+3)}{1.2} \alpha^2 . y^2 + (i+3)\alpha^2 . y'.$$

Suppose

$$y = Y^{(1)} + Y^{(2)} + Y^{(3)} + \&c.$$
$$y' = Y'^{(1)} + Y'^{(2)} + Y'^{(3)} + \&c.$$
$$y'' = M^{(0)} + M^{(1)} + M^{(2)} + \&c.$$

$Y^{(i)}$, $Y'^{(i)}$, and $M^{(i)}$ being subject to the same equation of partial differences as $U^{(i)}$; we shall have

$$Z^{(i)} = (i+3)\alpha Y^{(i)} + \frac{(i+2)(i+3)}{1.2} \alpha^2 M^{(i)} + (i+3)\alpha^2 Y'^{(i)}.$$

Then observe that $U^{(i)}$ is a quantity of the order α, since it would be nothing if the spheroid were a sphere; thus carrying the approximation only to terms of the order α^2, $U^{(i)}$ will be of this form $\alpha U'^{(i)} + \alpha^2 U''^{(i)}$. Substituting therefore these values in the preceding equation of equilibrium, and there changing r into $1 + \alpha y + \alpha^2 y'$, we shall have to quantities of the order α^3,

$$\text{const.} = \mu\{1 - \alpha y + \alpha^2 y^2 - \alpha^2 y'\}$$
$$+ \Sigma \left\{ \begin{array}{l} \alpha U'^{(i)} + \alpha^2 U''^{(i)} - (i+1)\alpha^2 y U'^{(i)} \\ + \frac{4\alpha\pi}{2i+1} Y^{(i)} - \frac{4\alpha^2\pi(i+1)}{2i+1} y Y^{(i)} + \frac{4\alpha^2\pi}{2i+1} Y^{(i)} \\ + \frac{4\alpha^2\pi(i+2)}{2(2i+1)} M^{(i)} \end{array} \right\}$$
$$- \frac{\alpha g'(1+2\alpha y)}{2}(m^2 - \tfrac{1}{3}).$$

Equating separately to zero the terms of the order α, and those of the order α^2, we shall have the two equations,

$$\Sigma\left(\mu - \frac{4\pi}{2i+1}\right) Y^{(i)} = \Sigma U'^{(i)} - \frac{g'}{2}(m^2 - \tfrac{1}{3});$$

$$\Sigma\left(\mu - \frac{4\pi}{2i+1}\right) Y'^{(i)} = C' + \Sigma \left\{ \begin{array}{l} U''^{(i)} - (i+1) y U'^{(i)} - \frac{4\pi(i+1)}{2i+1} y Y^{(i)} \\ + \left\{\mu + \frac{4\pi(i+2)}{2(2i+1)}\right\} M^{(i)} \end{array} \right\}$$
$$- g' y (m^2 - \tfrac{1}{3});$$

C' being an arbitrary constant. The first of these equations detects $Y^{(i)}$ and consequently the value of y. Substituting in the second member of the second equation, we shall develope by the method of No. 560. in a series of the form

$$N^{(0)} + N^{(1)} + N^{(2)} + \&c.$$

$N^{(i)}$ being subject to the same equation of partial differences as $U^{(i)}$, and we shall determine the constant C' in such a manner that $N^{(0)}$ is nothing; thus we shall have

$$Y'^{(i)} = \frac{N^{(i)}}{\mu - \frac{4\pi}{2i+1}}$$

and consequently

$$Y' = \frac{N^{(1)}}{\mu - \frac{4}{3}\pi} + \frac{N^{(2)}}{\mu - \frac{4}{5}\pi} + \frac{N^{(3)}}{\mu - \frac{4}{7}\pi} + \&c.$$

The expression of the radius r of the surface of the fluid will thus be determined to quantities of the order α^3, and we may, by the same process, carry the approximation as far as we wish. We shall not dwell any longer upon this object, which has no other difficulty than the length of calculations; but we shall derive from the preceding analysis this important conclusion, namely, that we may affirm that the equilibrium is rigorously possible, although we cannot assign the rigorous figure which satisfies it; for we may find a series of figures, which, being substituted in the equation of equilibrium, leave remainders successively smaller and smaller, and which become less than any given quantity.

COMPARISON OF THE PRECEDING THEORY WITH OBSERVATIONS.

582. To compare with observations the theory we have above laid down, we must know the curve of the terrestrial meridians, and those which we trace by a series of geodesic operations. If through the axis of rotation of the earth, and through the zenith of a plane at its surface we imagine a plane to pass produced to the heavens; this plane will trace a great circle which will be the meridian of the plane: all points of the surface of the earth which have their zenith upon this circumference, will lie under the same celestial meridian, and they will form, upon this surface, a curve which will be the corresponding terrestrial meridian.

To determine this curve, represent by $u = 0$ the equation of the surface of the earth; u being a function of three rectangular coordinates x, y, z. Let x′, y′, z′, be the three coordinates of the vertical which passes through the place on the earth's surface determined by the coordinates x, y, z; we shall have by the theory of curved surfaces, the two following equations,

$$0 = \left(\frac{d\,u}{d\,x}\right) d\,y' - \left(\frac{d\,u}{d\,y}\right) d\,x';$$

$$0 = \left(\frac{d\,u}{d\,x}\right) d\,z' - \left(\frac{d\,u}{d\,z}\right) d\,x'.$$

Adding the first multiplied by the indeterminate λ to the second, we get

$$d\,z' = \frac{\left(\frac{d\,u}{d\,z}\right) + \lambda \left(\frac{d\,u}{d\,y}\right)}{\left(\frac{d\,u}{d\,x}\right)} . d\,x' - \lambda\, d\,y'.$$

This equation is that of any plane parallel to the said vertical: this vertical produced to infinity coinciding with the celestial meridian, whilst its foot is only distant by a finite quantity from the plane of this meridian, may be deemed parallel to that plane. The differential equation of this plane may therefore be made to coincide with the preceding one by suitably determining the indeterminate λ.

Let

$$d\,z' = a\,d\,x' + b\,d\,y',$$

be the equation of the plane of the celestial meridian; comparing it with the preceding one, we shall get

$$\left(\frac{d\,u}{d\,z}\right) - a \left(\frac{d\,u}{d\,x}\right) - b \left(\frac{d\,u}{d\,y}\right) = 0; \quad \ldots\ldots \quad (a)$$

To get the constants a, b, we shall suppose known the coordinates of

the foot of the vertical parallel to the axes of rotation of the earth and that of a given place on its surface. Substituting successively these coordinates in the preceding equation, we shall have two equations, by means of which we shall determine a and b. The preceding equation combined with that of the surface $u = 0$, will give the curve of the terrestrial meridian which passes through the given plane.

If the earth were any ellipsoid whatever, u would be a rational and entire function of the second degree in x, y, z; the equation (a) would therefore then be that of a plane whose intersection with the surface of the earth, would form the terrestrial meridian: in the general case, this meridian is a curve of double curvature.

In this case the line determined by geodesic measures, is not that of the terrestrial meridian. To trace this line, we form a first horizontal triangle of which one of the angles has its summit at the origin of this curve, and whose two other summits are any visible objects. We determine the direction of the first side of the curve, relatively to two sides of the triangle, and to its length from the point where it meets the side which joins the two objects. We then form a second horizontal triangle with these objects, and a third one still farther from the origin of the curve. This second triangle is not in the plane of the first; it has nothing in common with the former, but the side formed by the two first objects; thus the first side of the curve being produced, lies above the plane of this second triangle; but we bend it down upon the plane so as always to form the same angles with the side common to the two triangles, and it is easy to see that for this purpose it must be bent along a vertical to this plane. Such is therefore the characteristic property of the curve traced by geodesic operations. Its first side, of which the direction may be supposed any whatever, touches the earth's surface; its second side is this tangent produced and bent vertically; its third is the tangent of the second side bent vertically, and so on.

If through the point where the two sides meet, we draw in the tangent plane at the surface of the spheroid, a line perpendicular to one of the sides, it is clear that it will be perpendicular to the other; whence it follows, that the sum of the sides is the shortest line which can be drawn upon the surface between their extreme points. Thus the lines traced by geodesic operations, have the property of being the shortest we can draw upon the surface of the spheroid between any two of their points; and p.294, Vol.I. they would be described by a body moving uniformly in this surface.

Let x, y, z be the rectangular coordinates of any part whatever of the curve; $x + dx$, $y + dy$, $z + dz$ will be those of points infinitely near to it. Call ds the element of the curve, and suppose this element produced by a quantity equal to ds; $x + 2dx$, $y + 2dy$, $z + 2dz$ will be the coordinates of extremity of the curve thus produced. By bending it vertically, the coordinates of this extremity will become $x + 2dx + d^2x$, $y + 2dy + d^2y$, $z + 2dz + d^2z$; thus $-d^2x$, $-d^2y$, $-d^2z$ will be the coordinates of the vertical, taken from its foot; we shall therefore have by the nature of the vertical, and by supposing that $u = 0$ is the equation of the earth's surface,

$$0 = \left(\frac{du}{dx}\right) d^2y - \left(\frac{du}{dy}\right) d^2x;$$

$$0 = \left(\frac{du}{dx}\right) d^2z - \left(\frac{du}{dz}\right) d^2x;$$

equations which are different from those of the terrestrial meridian. In these equations ds must be constant; for it is clear that the production of ds meets the foot of the vertical at an infinitely small quantity of the fourth order nearly.

Let us see what light is thrown upon the subject of the figure of the earth by geodesic measures, whether made in the directions of the meridians, or in directions perpendicular to the meridians. We may always conceive an ellipsoid touching the terrestrial surface at every point of it, and upon which, the geodesic measures of the longitudes and latitudes from the point of contact, for a small extent, would be the same as at the surface itself. If the entire surface were that of an ellipsoid, the tangent ellipsoid would every where be the same; but if, as it is reasonable to suppose, the figure of the meridians is not elliptic, then the tangent ellipsoid varies from one country to another, and can only be determined by geodesic measures, made in different directions. It would be very interesting to know the osculating ellipsoids at a great number of places on the earth's surface.

Let $u = x^2 + y^2 + z^2 - 1 - 2\alpha u'$, be the equation to the surface of the spheroid, which we shall suppose very little different from a sphere whose radius is unity, so that α is a very small quantity whose square may be neglected. We may always consider u' as a function of two variables x, y; for by supposing it a function of x, y, z, we may eliminate z by means of the equation $z = \sqrt{1 - x^2 - y^2}$. Hence, the three equations found above, relatively to the shortest line upon the earth's surface, become

$$\left.\begin{aligned} x\,d^2y - y\,d^2x &= \alpha\left(\frac{d\,u'}{d\,x}\right)d^2y - \alpha\left(\frac{d\,u'}{d\,y}\right)d^2x; \\ x\,d^2z - z\,d^2x &= \alpha\left(\frac{d\,u'}{d\,x}\right)d^2z; \\ y\,d^2z - z\,d^2y &= \alpha\left(\frac{d\,u'}{d\,y}\right).\,d^2z. \end{aligned}\right\} \quad . \quad . \quad (O)$$

This line we shall call the *Geodesic line.*

Call r the radius drawn from the center of the earth to its surface, θ the angle which this radius makes with the axis of rotation, which we shall suppose to be that of z, and φ the angle which the plane formed by this axis and by r makes with the plane of x, y; we shall have

$$x = r\sin.\theta.\cos.\varphi;\ y = r\sin.\theta\sin.\varphi;\ z = r\cos.\theta;$$

whence we derive

$$r^2\sin.^2\theta.\,d\varphi = x\,dy - y\,dx;$$

$$-r^2\,d\theta = (x\,dz - z\,dx)\cos.\varphi + (y\,dz - z\,dy)\sin.\varphi$$

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2\,d\theta^2 + r^2\,d\varphi^2\sin.^2\theta.$$

Considering then u', as a function of x, y, and designating by ψ the latitude; we may suppose in this function $r = 1$, and $\psi = 100^\circ - \theta$, which gives

$$x = \cos.\psi\cos.\varphi;\ y = \cos.\psi\sin.\varphi;$$

thus we shall have

$$\left(\frac{d\,u'}{d\,x}\right)dx + \left(\frac{d\,u'}{d\,y}\right)dy = \left(\frac{d\,u'}{d\,\psi}\right)d\psi + \left(\frac{d\,u'}{d\,\varphi}\right)d\varphi;$$

but we have

$$x^2 + y^2 = \cos.^2\psi;\ \frac{y}{x} = \tan.\varphi;$$

whence we derive

$$d\psi = -\frac{x\,dx + y\,dy}{\sin.\psi\cos.\psi};\ d\varphi = \frac{x\,dy - y\,dx}{x^2}.\cos.^2\varphi.$$

Substituting these values of $d\psi$ and of $d\varphi$ in the preceding differential equation in u', and comparing separately the coefficients of dx and dy; we shall have

$$\left(\frac{d\,u'}{d\,x}\right) = -\frac{\cos.\varphi}{\sin.\psi}\cdot\left(\frac{d\,u'}{d\,\psi}\right) - \frac{\sin.\varphi}{\cos.\psi}\cdot\left(\frac{d\,u'}{d}\right);$$

$$\left(\frac{d\,u'}{d\,y}\right) = -\frac{\sin.\varphi}{\sin.\psi}\cdot\left(\frac{d\,u'}{d\,\psi}\right) + \frac{\cos.\varphi}{\cos.\psi}\cdot\left(\frac{d\,u'}{d\,\varphi}\right);$$

which give

$$\left(\frac{d\,u'}{d\,x}\right)d^2y - \left(\frac{d\,u'}{d\,y}\right)d^2x = -\frac{\left(\frac{d\,u'}{d\,\psi}\right)}{\sin.\psi\cos.\psi}\cdot(x\,d^2y - y\,d^2x)$$

$$-\frac{\left(\frac{d\,u'}{d\,\varphi}\right)}{\cos.^2\psi}(x\,d^2x + y\,d^2y).$$

But neglecting quantities of the order α, we have $x\,d^2y - y\,d^2x = 0$; and the two equations

$$x\,d^2z - z\,d^2x = 0,\quad y\,d^2z - z\,d^2y = 0,$$

give

$$z\,d^2z = \frac{z^2(x\,d^2x + y\,d^2y)}{x^2 + y^2}$$

and

$$x^2 + y^2 + z^2 = 1$$

gives

$$x\,d^2x + y\,d^2y + z\,d^2z + ds^2 = 0;$$

substituting for $z\,d^2z$ its preceding value, we shall have

$$x\,d^2x + y\,d^2y = -(x^2 + y^2)\,ds^2 = -ds^2\cos.^2\psi;$$

wherefore

$$\left(\frac{du'}{dx}\right)d^2y - \left(\frac{du'}{dy}\right).d^2x = \left(\frac{du'}{d\varphi}\right).ds^2.$$

The first of equations (O), will thus give by integration,

$$r^2\,d\varphi\sin.^2\theta = c\,ds + \alpha\,ds\int ds\left(\frac{du'}{d\varphi}\right);\quad .\ .\ .\ .\ .\ (p)$$

c being the arbitrary constant.

The second of equations (O) gives

$$d.(x\,dz - z\,dx) = \alpha\left(\frac{du'}{dx}\right)d^2z;$$

but it is easy to see by what precedes, that we have

$$d^2z = -ds^2.\sin.\psi;$$

we have therefore

$$d\,(x\,dz - z\,dx) = -\alpha\,ds^2\left(\frac{du'}{dx}\right)\sin.\psi;$$

in like manner we have

$$d\,(y\,dz - z\,dy) = -\alpha\,ds^2\left(\frac{du'}{dy}\right)\sin.\psi;$$

we shall therefore have

$$r^2\,d\theta = c'\,ds\sin.\varphi + c''\,ds\cos.\varphi$$
$$-\alpha\,ds\cos.\varphi\int ds\left\{\left(\frac{du'}{d\psi}\right)\cos.\varphi + \left(\frac{du'}{d\varphi}\right)\sin.\varphi\tan.\psi\right\}$$
$$-\alpha\,ds\sin.\varphi\int ds\left\{\left(\frac{du'}{d\psi}\right)\sin.\varphi - \left(\frac{du'}{d\varphi}\right)\cos.\varphi\tan.\psi\right\};\ .\ (q)$$

First consider the case in which the first side of the Geodesic line is parallel to the corresponding plane of the celestial meridian. In this case $d\varphi$ is of the order α, as also dr; we have, therefore, neglecting quantities of the order α^2, $ds = -r\,d\theta$, the arc s being supposed to increase from

the equator to the poles. ψ expressing the latitude, it is easy to see that we have $\theta = 100^{\circ} - \psi - \left(\frac{d\,r}{d\,\psi}\right)$, which gives

$$d\,\theta = -d\,\psi - \alpha\, d\,\psi \left(\frac{d^2 u'}{d\,\psi^2}\right);$$

we have therefore

$$d\,s = d\,\psi . \left\{1 + \alpha\, u' + \alpha \left(\frac{d^2 u'}{d\,\psi^2}\right)\right\}.$$

Thus naming ε the difference in latitude of the two extreme points of the arc s, we shall have

$$s = \varepsilon + \alpha\,\varepsilon \left\{u_{/}' + \left(\frac{d^2 u_{/}'}{d\,\psi^2}\right)\right\} + \frac{\alpha\,\varepsilon^2}{1.2} \cdot \left\{\left(\frac{d\,u_{/}'}{d\,\psi}\right) + \left(\frac{d^3 u_{/}'}{d\,\psi^3}\right)\right\} + \&c.;$$

$u_{/}'$ being here the value of u' at the origin of s.

If the earth were a solid of revolution, the geodesic line would be always in the plane of the same meridian; it departs from it if the parallels are not circles; the observations of this deflection may therefore clear up this important point of the theory of the earth. Resume the equation (p) and observe that in the present case, $d\,\varphi$ and the constant c of this equation are of the order α, and that we may there suppose $r = 1$, $d\,s = d\,\psi$, $\theta = 100^{\circ} - \psi$; we shall thus get

$$d\,\varphi \cos.^2 \psi = c\,d\,\psi + \alpha\, d\,\psi \int d\,\psi \left(\frac{d\,u'}{d\,\varphi}\right).$$

However, if we call V the angle which the plane of the celestial meridian makes with that of x, y, whence we compute the origin of the angle φ; we shall have $d\,x' = \text{tan. } V = d\,y'$; x', y', z' being the coordinates of that meridian whose differential equation, as we have seen in the preceding No., is

$$d\,z' = a\,d\,x' + b\,d\,y'.$$

Comparing it with the preceding one, we see that a, b are infinite and such that $-\frac{a}{b} = \text{tan. } V$, the equation (a) of the preceding No. thus gives

$$0 = \left(\frac{d\,u}{d\,x}\right) . \text{tan. } V - \left(\frac{d\,u}{d\,y}\right),$$

whence we derive

$$0 = x \text{ tan. } V - y - \alpha \left(\frac{d\,u'}{d\,x}\right) \text{tan. } V + \alpha \left(\frac{d\,u'}{d\,y}\right).$$

We may suppose $V = \varphi$, in the terms multiplied by α; moreover $\frac{y}{x} = \text{tan. } \varphi$; w have therefore

$$\cos.\psi \cos.\varphi \{\tan.\varphi - \tan.V\} = \frac{\alpha\left(\frac{d\,u'}{d\,\varphi}\right)}{\cos.\psi \cos.\varphi},$$

which gives

$$\varphi - V = \frac{\alpha\left(\frac{d\,u'}{d\,\varphi}\right)}{\cos.^2\psi}.$$

The first side of the *Geodesic line*, being supposed parallel to the plane of the celestial meridian, the differentials of the angle V, and of the distance $(\varphi - V)\cos.\psi$ from the origin of the curve to the plane of the celestial meridian ought to be nothing at this origin; we have therefore at this point

$$\frac{d\,\varphi}{d\,\psi} = (\varphi - V)\tan.\psi = \frac{\alpha\left(\frac{d\,u'}{d\,\varphi}\right)\tan.\psi}{\cos.^2\psi};$$

and consequently, the equation (p) gives

$$c = \alpha\left(\frac{d\,u_{,}'}{d\,\varphi}\right)\tan.\psi_{,},$$

$u_{,}$ and $\psi_{,}$ being referred to the origin of the arc s.

At the extremity of the measured arc, the side of the curve makes with the plane of the corresponding celestial meridian an angle very nearly equal to the differential of $(\varphi - V)\cos.\psi$, divided by $d\,\psi$, V being supposed constant in the differentiation; by denoting therefore this angle by ϖ, we shall have

$$\varpi = \frac{d\,\varphi}{d\,\psi}\cos.\psi - (\varphi - V)\sin.\psi.$$

If we substitute for $\frac{d\,\varphi}{d\,\psi}$ its value obtained from the equation (p), and for $\varphi - V$, its preceding value, we shall have

$$\varpi = \frac{\alpha}{\cos.\varphi}\cdot\left\{\left(\frac{d\,u_{,}'}{d\,\varphi}\right)\tan.\psi_{,} - \left(\frac{d\,u'}{d\,\varphi}\right)\tan.\psi + \int d\,\psi\left(\frac{d\,u'}{d\,\varphi}\right)\right\};$$

the integral being taken from the origin of the measured arc, to its extremity. Call ε the difference in latitude of its two extreme points; ε being supposed sufficiently small for ε^2 to be rejected, we shall have

$$\varpi = -\frac{\alpha\,\varepsilon\tan.\psi}{\cos.\psi}\left\{\left(\frac{d\,u'}{d\,\varphi}\right)\tan.\psi + \left(\frac{d^2\,u'}{d\,\varphi\,d\,\psi}\right)\right\};$$

in which the values of ψ, $\left(\frac{d\,u'}{d\,\varphi}\right)$, and $\left(\frac{d^2\,u'}{d\,\varphi\,d\,\psi}\right)$ must be referred, for the greater exactness, to the middle of the measured arc. The angle ϖ must be

supposed positive, when it quits the meridian, in the direction of the increments of φ.

To obtain the difference in longitude of the two meridians corresponding to the extremities of the arc, we shall observe, that $u_{,}'$, $V_{,}$, $\psi_{,}$, and $\varphi_{,}$, being the values of u', V, ψ, and φ, at the first extremity, we have

$$\varphi_{,} - V_{,} = \frac{\alpha\left(\frac{d\, u_{,}'}{d\,\varphi}\right)}{\cos.^{2}\psi_{,}}; \quad \varphi - V = \frac{\alpha\left(\frac{d\, u'}{d\,\varphi}\right)}{\cos.^{2}\psi};$$

but we have very nearly, neglecting the square of ε,

$$\varphi - \varphi_{,} = \frac{c\,\varepsilon}{\cos.^{2}\psi_{,}}; \quad c = \alpha\left(\frac{d\, u_{,}'}{d\,\varphi}\right) \tan.\psi_{,};$$

we shall have, therefore,

$$V - V = -\frac{\alpha\,\varepsilon}{\cos.^{2}\psi_{,}} \cdot \left\{\left(\frac{d\, u_{,}'}{d\,\varphi}\right)\tan.\psi_{,} + \left(\frac{d^{2} u_{,}'}{d\,\varphi\, d\,\psi}\right)\right\};$$

whence results this very simple equation,

$$(V - V_{,}) \sin.\psi_{,} = \varpi;$$

thus we may, by observation alone, and independently of the knowledge of the figure, determine the difference in longitude of the meridians corresponding to the extremities of the measured arc; and if the value of the angle ϖ is such that we cannot attribute it to errors of observations, we shall be certain that the earth is not a spheroid of revolution.

Let us now consider the case where the first side of the *Geodesic line* is perpendicular to the corresponding plane of the celestial meridian. If we take this plane for that of x, y, the cosine of the angle formed by this side upon the plane, will be $\frac{\sqrt{d\,x^{2} + d\,z^{2}}}{d\,s}$; thus this cosine being nothing at the origin, we have $d\,x = 0$, $d\,z = 0$, which gives

$$d.\, r \sin.\theta \cos.\varphi = 0; \quad d.\, r \cos.\theta = 0;$$

and consequently

$$r\, d\,\theta = r\, d\,\varphi \sin.\theta.\cos.\theta.\tan.\varphi;$$

but we have, to quantities of the order α^{2}, $d\,s = r\, d\,\varphi \sin.\theta$; we shall have, therefore, at the origin,

$$\frac{d\,\theta}{d\,s} = \frac{\tan.\varphi.\cos.\theta}{r}.$$

The constant c'', of the equation (q), is equal to the value of $x\, d\, z - z\, d\, x$, at the origin; it is therefore nothing, and the equation (q) gives at the origin,

$$\frac{d\,\theta}{d\,s} = \frac{c'}{r^{2}} \sin.\varphi;$$

we have, therefore, observing that φ is here of the order α, and that thus neglecting quantities of the order α^2, we have sin. φ = tan. φ,

$$c' = r_{,} \cos. \theta_{,},$$

the quantities $r_{,}$ and $\theta_{,}$ being relative to the origin; therefore, if we consider that at this origin the angle φ is what we have before called it, $\varphi_{,} - V_{,}$, and whose value we have found equal to $\frac{\alpha\left(\frac{d u_{,}'}{d \varphi}\right)}{\cos.^2 \psi_{,}}$; we shall have at this point

$$\frac{d\theta_{,}}{ds} = \alpha\left(\frac{d u_{,}'}{d \varphi}\right)\frac{\sin. \psi_{,}}{\cos.^2 \psi_{,}}.$$

The equation (q) then gives

$$\frac{d^2 \theta_{,}}{ds^2} = \frac{\cos. \theta_{,}}{r_{,}} \cdot \frac{d \varphi_{,}}{ds} - \alpha . \left(\frac{d u_{,}'}{d \psi}\right);$$

but we have

$$\frac{d \varphi_{,}}{ds} = \frac{1}{r_{,} \sin. \theta_{,}}; \quad r_{,} = 1 + \alpha u_{,}'; \quad \theta_{,} = 100^{\circ} - \psi_{,} - \alpha\left(\frac{d u_{,}'}{d \psi}\right);$$

we shall get therefore

$$\frac{d^2 \theta_{,}}{ds^2} = (1 - 2 \alpha u_{,}') \tan. \psi_{,} + \alpha\left(\frac{d u_{,}'}{d \psi}\right) \tan.^2 \psi_{,}.$$

Observing that at the origin,

$$\frac{d \varphi_{,}}{ds} = \frac{1}{r_{,} \sin. \theta_{,}} = \frac{1}{\cos. \psi_{,}}\left\{1 - \alpha u_{,}' + \alpha\left(\frac{d u_{,}'}{d \psi}\right) \tan. \psi_{,}\right\},$$

the equation (p) gives

$$c = r_{,} \sin. \theta_{,};$$

whence we get

$$\frac{d^2 \varphi_{,}}{ds^2} = -\frac{2 \alpha . \frac{d u_{,}'}{ds}}{r_{,} \sin. \theta_{,}} - \frac{2 . \frac{d \theta_{,}}{ds} \cos. \theta_{,}}{r_{,} \sin.^2 \theta_{,}} + \frac{\alpha\left(\frac{d u_{,}'}{d \varphi_{,}}\right)}{\cos.^2 \psi_{,}},$$

and consequently

$$\frac{d^2 \varphi_{,}}{ds^2} = -\alpha\left(\frac{d u_{,}'}{d \varphi}\right) . \frac{2 - \cos.^2 \psi_{,}}{\cos.^4 \psi_{,}}.$$

The equation

$$\theta = 100^{\circ} - \psi - \alpha\left(\frac{d u'}{d \psi}\right),$$

gives, by retaining amongst the terms of the order s^2, only those which are independent of α,

$$\psi - \psi_{,} = -s . \frac{d \theta_{,}}{ds} - \tfrac{1}{2} s^2 . \frac{d^2 \theta_{,}}{ds^2} - \frac{\alpha s}{\cos. \psi_{,}} . \left(\frac{d^2 u_{,}'}{d \varphi \, d \psi}\right);$$

wherefore

$$\psi - \psi_{,} = -\frac{\alpha s}{\cos.\psi_{,}} \cdot \left\{ \left(\frac{d u_{,}'}{d \varphi}\right) \tan.\psi_{,} + \alpha \left(\frac{d^2 u_{,}'}{d \varphi\, d \psi}\right) - \tfrac{1}{2} s^2 \tan.\psi_{,} \right\}.$$

The difference of latitudes at the two extremities of the measured arc, will therefore give

$$-\frac{\alpha s}{\cos.\psi_{,}} \cdot \left\{ \left(\frac{d u_{,}'}{d \varphi}\right) \tan.\psi_{,} + \alpha \left(\frac{d^2 u_{,}'}{d \varphi\, d \psi}\right) \right\};$$

It is remarkable, that for the same arc, measured in the direction of the meridian, this function, by what precedes, is equal to $\frac{\varpi}{\tan.\psi_{,}}$; it may thus be determined in two ways, and we shall be able to judge whether the values thus found of the difference of latitudes, or of the azimuthal angle ϖ, are due to the errors of observations, or to the excentricity of the terrestrial parallels.

Retaining only the first power of s, we have

$$\varphi - \varphi_{,} = s \cdot \frac{d \varphi_{,}}{d s} = \frac{s}{\cos.\psi_{,}} \cdot \left\{ 1 - \alpha u_{,}' + \alpha \left(\frac{d u_{,}'}{d \psi}\right) \tan.\psi_{,} \right\}.$$

$\varphi - \varphi_{,}$ is not the difference in longitude of the two extremities of the arc s; this difference is equal to $V - V_{,}$; but we have, by what precedes,

$$\varphi - V = \frac{\alpha \left(\frac{d u'}{d \varphi}\right)}{\cos.^2 \psi};$$

which gives

$$\varphi - V - (\varphi_{,} - V_{,}) = \frac{\alpha s \left(\frac{d^2 u_{,}'}{d \varphi \,.\, d s}\right)}{\cos.^2 \psi_{,}} = \frac{\alpha s \left(\frac{d^2 u_{,}'}{d \varphi^2}\right)}{\cos.^3 \psi_{,}};$$

wherefore

$$V - V_{,} = \frac{s}{\cos.\psi_{,}} \cdot \left\{ 1 - \alpha u_{,}' + \alpha \left(\frac{d u_{,}'}{d \psi}\right) \tan.\psi_{,} - \frac{\alpha \left(\frac{d^2 u_{,}'}{d \varphi^2}\right)}{\cos.^2 \psi_{,}} \right\}.$$

For greater exactness, we must add to this value of $V - V_{,}$ the term depending on s^3, and independent of α, which we obtain in the hypothesis of the earth being a sphere. This term is equal to $-\frac{1}{3} s^3 \cdot \frac{\tan.^2 \psi_{,}}{\cos.\psi_{,}}$; thus we have

$$V - V_{,} = \frac{s}{\cos.\psi_{,}} \cdot \left\{ 1 - \alpha . u_{,}' + \alpha . \left(\frac{d u_{,}'}{d \psi}\right) \tan.\psi_{,} - \frac{\alpha \left(\frac{d^2 u_{,}'}{d \varphi^2}\right)}{\cos.^2 \psi_{,}} - \tfrac{1}{3} s^2 \tan.^2 \psi' \right\}.$$

It remains to determine the azimuthal angle at the extremity of the arc s. For that purpose, call x', and y', the coordinates x, y, referred to

the meridian of the last extremity of the arc s; it is easy to see that the cosine of the azimuthal angle is equal to $\frac{\sqrt{dx'^2 + dz^2}}{ds}$. If we refer the coordinates x, y, to the plane of the meridian corresponding to the first extremity of the arc; its first side being supposed perpendicular to the plane of this meridian, we shall have

$$\frac{dx_{,}}{ds} = 0; \frac{dz_{,}}{ds} = 0; \frac{dy_{,}}{ds} = 1;$$

wherefore, retaining only the first power of s,

$$\frac{dx}{ds} = s.\frac{d^2x_{,}}{ds^2}; \quad \frac{dz}{ds} = s.\frac{d^2z_{,}}{ds^2};$$

but we have

$$x' = x\cos.(V - V_{,}) + y\sin.(V - V_{,});$$

thus $V - V_{,}$ being, by what precedes, of the order α, we shall have

$$\frac{dx'}{ds} = s.\frac{d^2x_{,}}{ds^2} + (V - V_{,})\frac{dy_{,}}{ds}.$$

Again, we have

$$x = r\sin.\theta\cos.\varphi; \quad z = r\cos.\theta;$$

we therefore shall obtain, rejecting quantities of the order α^2, and observing that $\varphi_{,}$, $\frac{d^2\varphi_{,}}{ds^2}$, and $\frac{d\theta_{,}}{ds}$ are quantities of the order α,

$$\frac{d^2x_{,}}{ds^2} = \alpha.\frac{d^2u_{,}'}{ds^2}\sin.\theta_{,} + r_{,}.\frac{d^2\theta_{,}}{ds^2}\cos.\theta_{,} - r_{,}\sin.\theta_{,}.\frac{d\varphi_{,}^2}{ds^2}.$$

Thence we have

$$\alpha.\frac{d^2u_{,}'}{ds^2} = \alpha\left(\frac{d^2u_{,}'}{d\varphi^2}\right)\frac{d\varphi_{,}^2}{ds^2} - \alpha\left(\frac{du_{,}'}{d\psi}\right).\frac{d^2\theta_{,}}{ds^2} = \frac{\alpha\left(\frac{d^2u_{,}'}{d\varphi^2}\right)}{\cos.^2\psi_{,}} - \alpha\left(\frac{du_{,}'}{d\psi}\right)\tan.\psi_{,};$$

moreover, $ds = r_{,}\sin.\theta_{,}.d\varphi_{,}$; we shall, therefore, have by substituting for $r_{,}$, $\theta_{,}$, $\frac{d\varphi_{,}}{ds}$, and $\frac{d^2\theta_{,}}{ds^2}$, their preceding values,

$$\frac{d^2x_{,}}{ds^2} = (1 - \alpha u_{,}').\frac{\sin.^2\psi_{,}}{\cos.\psi_{,}} + \alpha\left(\frac{du_{,}'}{d\psi}\right)\tan.^2\psi_{,}\sin.\psi_{,}$$

$$- \frac{1}{\cos.\psi_{,}}.\left\{1 - \alpha u_{,} + \alpha\left(\frac{du_{,}'}{d\psi}\right)\tan.\psi_{,}\right\} + \frac{\alpha\left(\frac{d^2u_{,}'}{d\varphi^2}\right)}{\cos.\psi_{,}}$$

Neglecting the superior powers of s, we have, as we have seen,

$$V - V_{,} = \frac{s}{\cos.\psi_{,}}.\left\{1 - \alpha u_{,}' + \alpha\left(\frac{du_{,}'}{d\psi}\right)\tan.\psi_{,} - \frac{\alpha\left(\frac{d^2u_{,}'}{d\varphi^2}\right)}{\cos.^2\psi_{,}}\right\},$$

and $\frac{dy_{,}}{ds} = 1$; we therefore have

$$\frac{dx_{,}}{ds}=s(1-\alpha u_{,}')\frac{\sin.^{2}\psi_{,}}{\cos.\psi_{,}}+\alpha s\left(\frac{du_{,}'}{d\psi}\right)\tan.^{2}\psi_{,}.\sin.\psi_{,}-\alpha s\left(\frac{d^{2}u_{,}'}{d\varphi^{2}}\right)\frac{\sin.^{2}\psi_{,}}{\cos.^{3}\psi_{,}};$$

in like manner we shall find

$$\frac{dz}{ds}=-s(1-\alpha u_{,}')\sin.\psi_{,}-\alpha s\left(\frac{du_{,}'}{d\psi}\right)\tan.^{2}\psi_{,}\cos.\psi_{,}+\alpha s\left(\frac{d^{2}u_{,}'}{d\varphi^{2}}\right)\frac{\sin.\psi_{,}}{\cos.^{2}\psi_{,}};$$

the cosine of the azimuthal angle, at the extremity of the arc s, will thus be

$$s\tan.\psi_{,}\left\{1-\alpha u_{,}'+\alpha\left(\frac{du_{,}'}{d\psi}\right)\tan.\psi_{,}-\frac{\alpha\left(\frac{d^{2}u_{,}'}{d\varphi^{2}}\right)}{\cos.^{2}\psi_{,}}\right\}.$$

This cosine being very small, it may be taken for the complement of the azimuthal angle, which consequently is equal to

$$100^{\circ}-s\tan.\psi_{,}\left\{1-\alpha u_{,}'+\alpha\left(\frac{du_{,}'}{d\psi}\right)\tan.\psi_{,}-\frac{\alpha\left(\frac{d^{2}u_{,}'}{d\varphi^{2}}\right)}{\cos.^{2}\psi_{,}}\right\}.$$

For the greater exactness, we must add to this angle that part depending on s^{3}, and independent of α, which we obtain in the hypothesis of the earth's sphericity. This part is equal to $\frac{1}{3}s^{3}(\frac{1}{2}+\tan.^{2}\psi_{,})\tan.\psi_{,}$. Thus the azimuthal angle at the extremity of the arc s is equal to

$$100^{\circ}-s\tan.\psi_{,}\left\{1-\alpha u_{,}'+\alpha\left(\frac{du_{,}'}{d\psi}\right)\tan.\psi_{,}-\frac{\alpha\left(\frac{d^{2}u_{,}'}{d\varphi^{2}}\right)}{\cos.^{2}\psi_{,}}-\tfrac{1}{3}s^{2}(\tfrac{1}{2}\tan.^{2}\psi_{,})\right\}.$$

The radius of curvature of the Geodesic line, forming any angle whatever with the plane of the meridian, is equal to

$$\frac{ds^{2}}{\sqrt{(d^{2}x)^{2}+(d^{2}y)^{2}+(d^{2}z)^{2}}},$$

d s being supposed constant; let R be this radius. The equation $x^{2}+y^{2}+z^{2}=1+2\alpha u'$ gives

$$x\,d^{2}x+y\,d^{2}y+z\,d^{2}z=-ds^{2}+\alpha d^{2}u';$$

if we add the square of this equation to the squares of equations (O), we shall have, rejecting terms of the order α^{2},

$$(x^{2}+y^{2}+z^{2})\{(d^{2}x)^{2}+(d^{2}y)^{2}+(d^{2}z)^{2}\}=ds^{4}-2\alpha ds^{2}d^{2}u'$$

whence we derive

$$R=1+\alpha u'+\alpha\frac{d^{2}u'}{ds^{2}}.$$

In the direction of the meridian, we have

$$\alpha.\frac{d^{2}u'}{ds^{2}}=\alpha\left(\frac{d^{2}u'}{d\psi^{2}}\right);$$

wherefore

$$R=1+\alpha u'+\alpha\left(\frac{d^{2}u'}{d\psi^{2}}\right).$$

In the direction perpendicular to the meridian, we have by what precedes,

$$\alpha \cdot \frac{d^2 u_{,}'}{d s^2} = \frac{\alpha \left(\frac{d^2 u_{,}'}{d \varphi^2}\right)}{\cos.^2 \psi_{,}} - \alpha \left(\frac{d u_{,}'}{d \varphi}\right) \tan. \psi_{,};$$

wherefore

$$R = 1 + \alpha u_{,}' - \alpha \left(\frac{d u_{,}'}{d \psi}\right) \tan. \psi_{,} + \frac{\alpha \left(\frac{d^2 u_{,}'}{d \varphi^2}\right)}{\cos.^2 \psi_{,}}.$$

If in the preceding expression of $V - V_{,}$, we make $\frac{s}{R} = s'$, it takes this very simple form relative to a sphere of the radius R,

$$V - V_{,} = \frac{s'}{\cos. \psi_{,}} \cdot \left\{ 1 - \frac{1}{3} s'^2 \cdot \tan.^2 \psi_{,} \right\}.$$

The expression of the azimuthal angle becomes

$$100^{\circ} - s' \tan. \psi_{,} \{1 - \tfrac{1}{3} s'^2 (\tfrac{1}{2} + \tan.^2 \psi_{,})\}.$$

Call λ, the angle which the first side of the *Geodesic line* forms with the plane corresponding to the celestial meridian, we shall have

$$\frac{d^2 u'}{d s^2} = \left(\frac{d u'}{d \varphi}\right)\frac{d^2 \varphi}{d s^2} + \left(\frac{d u'}{d \psi}\right)\frac{d^2 \psi}{d s^2} + \left(\frac{d^2 u'}{d \varphi^2}\right)\frac{d \varphi^2}{d s^2} + 2\left(\frac{d^2 u'}{d \varphi d \psi}\right)\frac{d \varphi}{d s}\frac{d \psi}{d s} + \left(\frac{d^2 u'}{d \psi^2}\right)\frac{d \psi^2}{d s^2}.$$

But supposing the earth a sphere, we have

$$\frac{d \varphi_{,}}{d s} = \frac{\sin. \lambda}{\cos. \psi_{,}}; \quad \frac{d^2 \varphi_{,}}{d s^2} = \frac{2 \sin. \lambda \cos. \lambda}{\cos \psi_{,}} \tan. \psi_{,};$$

$$\frac{d \psi_{,}}{d s} = \cos. \lambda; \quad \frac{d^2 \psi_{,}}{d s^2} = - \sin.^2 \lambda \tan. \psi_{,};$$

wherefore,

$$\frac{d^2 u_{,}'}{d s^2} = 2 \frac{\sin. \lambda \cos. \lambda}{\cos. \psi_{,}} \left\{ \left(\frac{d u_{,}'}{d \varphi}\right) \tan. \psi_{,} + \left(\frac{d^2 u_{,}'}{d \varphi d \psi}\right) \right\} - \sin.^2 \lambda \tan. \psi_{,} \left(\frac{d u_{,}'}{d \psi}\right)$$
$$+ \left(\frac{d^2 u_{,}'}{d \varphi^2}\right) \frac{\sin.^2 \lambda}{\cos.^2 \psi_{,}} + \left(\frac{d^2 u_{,}'}{d \psi^2}\right) \cos.^2 \lambda;$$

the radius of curvature R, in the direction of this *Geodesic line*, is therefore

$$1 + \alpha u_{,}' + 2\alpha \frac{\sin. \lambda \cos. \lambda}{\cos. \psi_{,}} \left\{ \left(\frac{d u_{,}'}{d \varphi}\right) \tan. \psi_{,} + \left(\frac{d^2 u_{,}'}{d \varphi d \psi}\right) \right\} - \alpha \sin.^2 \lambda \tan. \psi_{,} \left(\frac{d u_{,}'}{d \psi}\right)$$
$$+ \alpha \left(\frac{d^2 u_{,}'}{d \varphi^2}\right) \cdot \frac{\sin.^2 \lambda}{\cos.^2 \psi_{,}} + \alpha \left(\frac{d^2 u_{,}'}{d \psi^2}\right) \cos.^2 \lambda.$$

To abridge this, let

$$K = 1 + \alpha u_{,}' - \tfrac{1}{2} \alpha \tan. \psi_{,} \left(\frac{d u_{,}'}{d \psi}\right) + \tfrac{1}{2} \cdot \frac{\alpha \left(\frac{d^2 u_{,}'}{d \varphi^2}\right)}{\cos.^2 \psi_{,}} + \tfrac{1}{2} \alpha \left(\frac{d^2 u_{,}'}{d \psi^2}\right);$$

$$A = \frac{\alpha}{\cos. \psi_{,}} \cdot \left\{ \left(\frac{d\, u_{,}'}{d\, \varphi} \right) \tan. \psi_{,} + \left(\frac{d^2 u_{,}'}{d\, \varphi\, d\, \psi} \right) \right\};$$

$$B = \frac{\alpha}{2} \tan. \psi_{,} \left(\frac{d\, u_{,}'}{d\, \psi} \right) - \frac{\frac{\alpha}{2} \left(\frac{d^2 u_{,}'}{d\, \varphi^2} \right)}{\cos.^2 \psi_{,}} + \frac{\alpha}{2} \left(\frac{d^2 u_{,}'}{d\, \psi^2} \right);$$

we shall have

$$R = K + A \sin. 2\lambda + B \cos. 2\lambda.$$

The observations of azimuthal angles, and of the difference of the latitudes at the extremities of the two *geodesic lines*, one measured in the direction of the meridian, and the other in the direction perpendicular to the meridian, will give, by what precedes, the values of A, B and K; for the observations give the radii of curvature in these two directions. Let R, and R′ be these radii; we shall have

$$K = \frac{R' + R''}{2};$$

$$B = \frac{R' - R''}{2};$$

and the value of A will be determined, either by the azimuth of the extremity of the arc measured in the direction of the meridian, or by the difference in latitude of the two extremities of the arc measured in a direction perpendicular to the meridian. We shall thus get the radius of curvature of the *geodesic line*, whose first side forms any angle whatever with the meridian.

If we call 2 E, an angle whose tangent is $\frac{A}{B}$, we shall have

$$R = K + \sqrt{A^2 + B^2}. \cos. (2\lambda - 2E);$$

the greatest radius of curvature corresponds with $\lambda = E$; the corresponding geodesic line forms therefore the angle E, with the plane of the meridian. The least radius of curvature corresponds with $\lambda = 100^{\circ} + E$; let r be the least radius, and r′ the greatest, we shall have

$$R = r + (r' - r) \cos.^2 (\lambda - E),$$

$\lambda - E$ being the angle which the *geodesic line* corresponding to R, forms with that which corresponds with r′.

We have already observed, that at each point of the earth's surface, we may conceive an osculatory ellipsoid upon which the degrees, in all directions, are sensibly the same to a small extent around the point of osculation. Express the radius of this ellipsoid by the function

$$1 - \alpha \sin.^2 \psi \{1 + h \cos. 2 (\varphi + \beta)\},$$

the longitudes φ being reckoned from a given meridian. The expression

of the terrestrial meridian measured in the direction of the meridian, will be, by what precedes,

$$\varepsilon - \frac{\alpha\,\varepsilon}{2}.\,\{1 + h\cos.\,2\,(\varphi + \beta)\}.\,\{1 + 3\cos.\,2\psi - 3\,\varepsilon\sin.\,2\psi\}.$$

If the measured arc is considerable, and if we have observed, as in France, the latitudes of some points intermediate between the extremity; we shall have by these measures, both the length of the radius taken for unity, and the value of $\alpha\,\{1 + h\cos.\,2\,(\varphi + \beta)\}$. We then have, by what precedes,

$$\varpi = -2\,\alpha\,h.\,\varepsilon.\,\frac{\tan.^2\psi\,(1 + \cos.^2\psi)}{\cos.\,\psi}.\,\sin.\,2\,(\varphi + \beta);$$

the observation of the azimuthal angles at the two extremities of the arc will give $\alpha\,h\sin.\,2\,(\varphi + \beta)$. Finally, the degree measured in the direction perpendicular to the meridian, is

$1^\circ + 1^\circ.\,\alpha\,\{1 + h\cos.\,2\,(\varphi + \beta)\}\sin.^2\psi + 4^\circ.\,\alpha\,h\tan.^2\psi\cos.\,2\,(\varphi + \beta)$; the measure of this degree will therefore give the value of $\alpha\,h\sin.\,2\,(\varphi+\beta)$. Thus the osculatory ellipsoid will be determined by these several measures: it would be necessary for an arc so great, to retain the square of ε in the expression of the angle ϖ; and the more so, if, as it has been observed in France, the azimuthal angle does not vary proportionally to the measured arc: at the same time we must add a term of the form $\alpha\,k\sin.\,\psi\cos.\,\psi\sin.\,(\varphi + \beta')$, to get the most general expression of this radius.

583. The elliptic figure is the most simple after that of the sphere: we have seen above that this ought to be the figure of the earth and planets, on the supposition of their being originally fluid, if besides they have retained their primitive figure. It was natural therefore to compare with this figure the measured degrees of the meridian; but this comparison has given for the figure of the meridians different ellipses, and which disagree too much with observations to be admissible. However, before we renounce entirely the elliptic, we must determine that in which the greatest *defect* of the measured degrees, is smaller than in every other elliptic figure, and see whether it be within the limits of the errors of observations. We arrive at this by the following method.

Let $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, &c. be the measured degrees of the meridians; $p^{(1)}$, $p^{(2)}$, $p^{(3)}$, &c. the squares of the sines of the corresponding latitudes: suppose that in the ellipse required, the degree of the meridian is expressed by the formula $z + p\,y$; calling $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, &c. the errors of observation, we shall have the following equations, in which we shall suppose that $p^{(1)}$, $p^{(2)}$, $p^{(3)}$, &c. form an increasing progression,

$$\begin{aligned} a^{(1)} - z - p^{(1)} y &= x^{(1)} \\ a^{(2)} - z - p^{(2)} y &= x^{(2)} \quad \ldots\ldots\ldots \text{(A)} \\ &\text{- - - - - - - - - - - -} \\ a^{(n)} - z - p^{(n)} y &= x^{(n)} \end{aligned}$$

n being the number of measured degrees.

We shall eliminate from these equations the unknown quantities z and y, and we shall have n — 2 equations of condition, between the n errors $x^{(1)}$, $x^{(2)}$, ... $x^{(n)}$. We must, however, determine that system of errors, in which the greatest, abstraction being made of the signs, is less than in every other system.

First suppose that we have only one equation of condition, which may be represented by

$$a = m x^{(1)} + n x^{(2)} + p x^{(3)} + \&c.$$

a being positive. We shall have the system of the values of $x^{(1)}$, $x^{(2)}$, &c. which gives, not regarding signs, the least value to the greatest of them; supposing them all nearly equal, and to the quotient of *a* divided by the sum of the coefficients, m, n, p, &c. taken positively. As to the sign which each quantity ought to have, it must be the same as that of its coefficient in the proposed equation.

If we have two equations of condition between the errors, the system which will give the smallest value possible to the greatest of them will be such that, signs being abstracted, all the errors will be equal to one another, with the exception of one only which will be smaller than the rest, or at least not greater. Supposing therefore that $x^{(1)}$ is this error, we shall determine it in function $x^{(2)}$, $x^{(3)}$, &c. by means of one of the proposed equations of condition; then substituting this value of $x^{(1)}$ in the other equation of condition, we shall form one between $x^{(2)}$, $x^{(3)}$, &c.; which represent by the following

$$a = m x^{(2)} + n x^{(3)} + \&c.$$

a being positive; we shall have, as above, the values of $x^{(2)}$, $x^{(3)}$, &c. by dividing a by the sum of the coefficients m, n, &c. taken positively, and by giving successively to the quotient the signs of m, n, &c. These values substituted in the expression of $x^{(1)}$ in terms of $x^{(2)}$, $x^{(3)}$, &c. will give the value of $x^{(1)}$; and if this value, abstracting signs, is not greater than that of $x^{(2)}$, this system of values will be that which we must adopt; but if greater, then the supposition that $x^{(1)}$ is the least error, is not legitimate, and we must successively make the same supposition as to $x^{(2)}$, $x^{(3)}$, &c. until we arrive at an error which is in this respect satisfactory.

If we have three equations of condition between the errors; the system which will give the smallest value possible to the greatest of them, will be

such, that, abstracting signs, all the errors will be equal, with exception of two, which will be less than the others.

Supposing therefore that $x^{(1)}$, $x^{(2)}$ are these two errors, we shall eliminate them from the third of the equations of condition by means of the other two, and we shall have an equation between the errors $x^{(3)}$, $x^{(4)}$, &c.: represent it by

$$a = m\,x^{(3)} + n\,x^{(4)} + \&c.$$

a being positive. We shall have the values of $x^{(3)}$, $x^{(4)}$, &c. by dividing a by the sum of the coefficients m, n, &c. taken positively, and by giving successively to the quotient, the signs of m, n, &c. These values substituted in the expressions of $x^{(1)}$, and of $x^{(2)}$ in terms of $x^{(3)}$, $x^{(4)}$, &c. will give the values of $x^{(1)}$, and of $x^{(2)}$, and if these last values, abstracting signs, do not surpass $x^{(3)}$, we shall have the system of errors, which we ought to adopt; but if one of these values exceed $x^{(3)}$, the supposition that $x^{(1)}$, and $x^{(2)}$ are the smallest errors is not legitimate, and we must use the same supposition upon another combination of errors $x^{(1)}$, $x^{(2)}$, &c. taken two and two, until we arrive at a combination in which this supposition is legitimate. It is easy to extend this method to the case where we should have four or more equations of condition, between the errors $x^{(1)}$, $x^{(2)}$, &c. These errors being thus known, it will be easy to obtain the values of z and y.

The method just exposed, applies to all questions of the same nature; thus, having the number n of observations upon a comet, we may by this means determine that parabolic orbit, in which the greatest error is, abstracting signs, less than in any other parabolic orbit, and thence recognise whether the parabolic hypothesis can represent these observations. But when the number of observations is considerable, this method becomes too tedious, and we may in the present problem, easily arrive at the required system of errors, by the following method.

Conceive that $x^{(i)}$, abstracting signs, is the greatest of the errors $x^{(1)}$, $x^{(2)}$, &c.; we shall first observe, that therein must exist another error $x^{(i')}$, equal, and having a contrary sign to $x^{(i)}$; otherwise we might, by making z to vary properly in the equation

$$a^{(i)} - z - p^{(i)}.y = x^{(i)},$$

diminish the error $x^{(i)}$, retaining to it the property of being the extreme error, which is against the hypothesis. Next we shall observe that $x^{(i)}$ and $x^{(i')}$ being the two extreme errors, one positive, and the others negative, and equal to one another, there ought to exist a third error $x^{(i'')}$, equal, abstracting signs, to $x^{(i)}$. In fact, if we take the equation corre-

sponding to $x^{(i)}$, from the equation corresponding to $x^{(i')}$, we shall have

$$a^{(i')} - a^{(i)} - \{p^{(i')} - p^{(i)}\}. y = x^{(i')} - x^{(i)}.$$

The second member of this equation is, abstracting signs, the sum of the extreme errors, and it is clear, that in varying y suitably, we may diminish it, preserving to it the property of being the greatest of the sums which we can obtain by adding or subtracting the errors $x^{(1)}$, $x^{(2)}$, &c. taken two and two; provided there is no third error $x^{(i'')}$ equal, abstracting signs, to $x^{(i)}$; but the sum of the extreme errors being diminished, and these errors being made equal, by means of the value of z, each of these errors will be diminished, which is contrary to the hypothesis. There exists therefore three errors $x^{(i)}$, $x^{(i')}$, $x^{(i'')}$ equal to one another, abstracting signs, and of different signs the one from the other two.

Suppose that this one is $x^{(i')}$; then the number i' will fall between the two numbers i and i''. To show this, let us imagine that it is not the case, and that i' is below or above both the numbers i, i''. Taking the equation corresponding to i', successively from the two equations corresponding to i and to i'', we shall have

$$a^{(i)} - a^{(i')} - (p^{(i)} - p^{(i')})\, y = x^{(i)} - x^{(i')};$$
$$a^{(i'')} - a^{(i')} - (p^{(i'')} - p^{(i')})\, y = x^{(i'')} - x^{(i')}.$$

The second members are equal and have the same sign; these are also, abstracting signs, the sum of the extreme errors; but it is evident, that varying y suitably, we may diminish each of these sums, since the coefficient of y, has the same sign in the two first members: moreover, we may, by varying z properly, preserve to $x^{(i')}$ the same value; $x^{(i)}$ and $x^{(i'')}$ will therefore then be, abstracting signs, less than $x^{(i')}$ which will become the greatest of the errors without having an equal; and in this case, we may, as we have seen, diminish the extreme error; which is contrary to the hypothesis. Thus the number i' ought to fall between i and i''.

Let us now determine which of the errors $x^{(1)}$, $x^{(2)}$, &c. are the extreme errors. For that purpose, take the first of the equations (A) successively from the following ones, and we shall have this series of equations,

$$a^{(2)} - a^{(1)} - (p^{(2)} - p^{(1)})\, y = x^{(2)} - x^{(1)},$$
$$a^{(3)} - a^{(1)} - (p^{(3)} - p^{(1)})\, y = x^{(3)} - x^{(1)}; \quad \ldots\ldots (B)$$
&c.

Suppose y infinite; the first members of these equations will be negative, and then the value of $x^{(1)}$ will be greater than $x^{(2)}$, $x^{(3)}$, &c.: diminishing y continually, we shall at length arrive at a value that will render positive one of the first members, which, before arriving at this state, will

be nothing. To know which of these members first becomes equal to zero, we shall form the quantities,

$$\frac{a^{(2)} - a^{(1)}}{p^{(2)} - p^{(1)}}; \quad \frac{a^{(3)} - a^{(1)}}{p^{(3)} - p^{(1)}}; \quad \frac{a^{(4)} - a^{(1)}}{p^{(4)} - p^{(1)}}; \text{ &c.}$$

Call $\beta^{(1)}$ the greatest of these quantities, and suppose it to be $\frac{a^{(r)} - a^{(1)}}{p^{(r)} - p^{(1)}}$: if there are many values equal to $\beta^{(1)}$, we shall consider that which corresponds to the number r the greatest, substituting $\beta^{(1)}$ for y, in the $(r - 1)^{th}$ of the equations (B), $x^{(r)}$ will be equal to $x^{(1)}$, and diminishing y, it will be equal to $x^{(1)}$, the first member of this equation then becoming positive. By the successive diminutions of y, this member will increase more rapidly than the first members of the equations which precede it; thus, since it becomes nothing when the preceding ones are still negative, it is clear that, in the successive diminutions of y, it will always be the greatest which proves that $x^{(r)}$ will be constantly greater than $x^{(1)}$, $x^{(2)}, \ldots x^{(r-1)}$, when y is less than $\beta^{(1)}$.

The first members of the equations (B) which follow the $(r - 1)^{th}$ will be at first negative, and whilst that is the case, $x^{(r+1)}$, $x^{(r+2)}$, &c. will be less than $x^{(1)}$, and consequently less than $x^{(r)}$, which becomes the greatest of all the errors $x^{(1)}$, $x^{(2)}, \ldots x^{(n)}$, when y begins to be less than $\beta^{(1)}$. But continuing to diminish y, we shall get a value of it, such that some of the errors $x^{(r+1)}$, $x^{(r+2)}$, &c. begin to *exceed* $x^{(r)}$.

To determine this value of y, we shall take the rth of equations (A) successively from the following ones, and we shall have

$$a^{(r+1)} - a^{(r)} - \{p^{(r+1)} - p^{(r)}\} y = x^{(r+1)} - x^{(r)};$$
$$a^{(r+2)} - a^{(r)} - \{p^{(r+2)} - p^{(r)}\} y = x^{(r+2)} - x^{(r)}.$$

Then we shall form the quantities

$$\frac{a^{(r+1)} - a^{(r)}}{p^{(r+1)} - p^{(r)}}; \quad \frac{a^{(r+2)} - a^{(r)}}{p^{(r+2)} - p^{(r)}}; \text{ &c.}$$

Call $\beta^{(2)}$, the greatest of these quantities, and suppose that it is $\frac{a^{(r')} - a^{(r)}}{p^{(r')} - p^{(r)}}$: if many of these quantities are equal to $\beta^{(2)}$, we shall suppose that r' is the greatest of the numbers to which they correspond. Then $x^{(r)}$ will be the greatest of the errors $x^{(1)}$, $x^{(2)}$, &c. $\ldots x^{(n)}$ so long as y is comprised between $\beta^{(1)}$, and $\beta^{(2)}$; but when by diminishing y, we shall arrive at $\beta^{(2)}$; then $x^{(r')}$ will begin to exceed $x^{(r)}$, and to become the greatest of the errors.

To determine within what limits we shall form the quantities

$$\frac{a^{(r'+1)} - a^{(r')}}{p^{(r'+1)} - p^{(r')}}; \quad \frac{a^{(r'+2)} - a^{(r')}}{p^{(r'+2)} - p^{(r')}}; \text{ &c.}$$

Let $\beta^{(3)}$ be the greatest of these quantities, and suppose that it is

$\frac{a^{(r''+1)} - a^{(r')}}{p^{(r''+1)} - p^{(r')}}$: if several of the quantities are equal to $\beta^{(3)}$, we shall suppose that r'' is the greatest of the numbers to which they correspond, $x^{(r')}$ will be the greatest of all the errors from $y = \beta^{(2)}$, to $y = \beta^{(3)}$. When $y = \beta^{(3)}$, then $x^{(r'')}$ begins to be this greatest error. Thus proceding, we shall form the two series,

$$\begin{array}{l} x^{(1)};\ x^{(r)};\ x^{(r')};\ x^{(r'')};\ \ldots\ x^{(n)} \\ \infty;\ \beta^{(1)};\ \beta^{(2)};\ \beta^{(3)};\ \ldots\ \beta^{(q)};\ -\infty;\ \ldots\ldots \end{array} \quad (C)$$

The first indicates the errors $x^{(1)}$, $x^{(r)}$, $x^{(r')}$, &c. which become successively the greatest: the second series formed of decreasing quantities, indicates the limits of y, between which these errors are the greatest; thus, $x^{(1)}$ is the greatest error from $y = \infty$, to $y = \beta^{(1)}$; $x^{(r)}$ is the greatest error from $y = \beta^{(1)}$, to $y = \beta^{(2)}$; $x^{(r')}$ is the greatest error from $y = \beta^{(2)}$, to $y = \beta^{(3)}$, and so on.

Resume now the equations (B) and suppose y negative and infinite. The first members of these equations will be positive, $x^{(1)}$ will therefore then be the least of the errors $x^{(1)}$, $x^{(2)}$, &c.: augmenting y continually, some of these members will become negative, and then $x^{(1)}$ will cease to be the least of the errors. If we apply here the reasoning just used in the case of the greatest errors, we shall see that if we call $\lambda^{(1)}$ the least of the quantities

$$\frac{a^{(2)} - a^{(1)}}{p^{(2)} - p^{(1)}};\ \frac{a^{(3)} - a^{(1)}}{p^{(3)} - p^{(1)}};\ \frac{a^{(4)} - a^{(1)}}{p^{(4)} - p^{(1)}};\ \&c.$$

and if we suppose that it is $\frac{a^{(s)} - a^{(1)}}{p^{(s)} - p^{(1)}}$, s being the greatest of the numbers to which $\lambda^{(1)}$ corresponds, if several of these quantities are equal to $\lambda^{(1)}$, $x^{(1)}$ will be the least of the errors from $y = -\infty$, to $y = \lambda^{(1)}$. In like manner if we call $\lambda^{(2)}$ the least of the quantities

$$\frac{a^{(s+1)} - a^{(s)}}{p^{(s+1)} - p^{(s)}};\ \frac{a^{(s+2)} - a^{(s)}}{p^{(s+2)} - p^{(s)}};\ \&c.$$

and suppose it to be $\frac{a^{(s')} - a^{(s)}}{p^{(s')} - p^{(s)}}$, s' being the greatest of the numbers to which $\lambda^{(2)}$ corresponds, if several of these quantities are equal to $\lambda^{(2)}$; $x^{(s')}$ will be the smallest of the errors from $y = \lambda^{(1)}$, to $y = \lambda^{(2)}$; and so forth. In this manner we shall form the two series

$$\begin{array}{l} x^{(1)};\ x^{(s)};\ x^{(s')};\ x^{(s'')};\ \ldots\ x^{(p)} \\ -\infty;\ \lambda^{(1)};\ \lambda^{(2)};\ \lambda^{(3)};\ \ldots\ \lambda^{(q')};\ \infty;\ \ldots\ldots \end{array} \quad (D)$$

The first indicates the errors $x^{(1)}$, $x^{(s)}$, $x^{(s')}$, &c. which are successively the least as we augment y: the second series formed of increasing terms, indicates the limits of the values of y between which each of these errors

is the least; thus $x^{(1)}$ is the least of the errors from $y = -\infty$, to $y = \lambda^{(1)}$ $x^{(s)}$ is the least of the errors, from $y = \lambda^{(1)}$, to $y = \lambda^{(2)}$, and thus of the rest.

Hence the value of y which, to the required ellipse, will be one of the quantities $\beta^{(1)}$, $\beta^{(2)}$, $\beta^{(3)}$; &c. $\lambda^{(1)}$, $\lambda^{(2)}$, &c.; it will be in the first series, if the two extreme errors of the same sign are positive. In fact, these two errors being then the greatest, they are in the series $x^{(1)}$, $x^{(r)}$, $x^{(r')}$, &c.; and since one and the same value of y renders them equal they ought to be consecutive, and the value of y which suits them, can only be one of the quantities $\beta^{(1)}$, $\beta^{(2)}$, &c.; because two of these errors cannot at the same time be made equal and the greatest, except by one only of these quantities. Here, however, is a method of determining that of the quantities $\beta^{(1)}$, $\beta^{(2)}$, &c. which ought to be taken for y.

Conceive, for example, that $\beta^{(3)}$ is this value; then there ought to be found by what precedes between $x^{(r')}$, and $x^{(r'')}$, an error which will be the *minimum* of all the errors, since $x^{(r')}$, and $x^{(r'')}$ will be the *maxima* of these errors; thus in the series $x^{(1)}$, $x^{(s)}$, $x^{(s')}$, &c. some one of the numbers s, s', &c. will be comprised between r and r'. Suppose it to be s. That $x^{(s)}$ may be the last of the value of y, it ought to be comprised between $\lambda^{(1)}$ and $\lambda^{(2)}$; therefore if $\beta^{(3)}$ is comprised by these limits, it will be the value sought of y, and it will be useless to seek others. In fact, suppose we take that of the equations (A), which answers to $x^{(s)}$ successively from the two equations which respond to $x^{(r')}$ and to $x^{(r'')}$; we shall have

$$a^{(r')} - a^{(s)} - \{p^{(r')} - p^{(s)}\}\, y = x^{(r')} - x^{(s)};$$

$$a^{(r'')} - a^{(s)} - \{p^{(r'')} - p^{(s)}\}\, y = x^{(r'')} - x^{(s)}.$$

All the members of these equations being positive, by supposing $y = \beta^{(3)}$, it is clear, that if we augment y, the quantity $x^{(r')} - x^{(s)}$ will increase; the sum of the extreme errors, taken positively, will be therefore augmented. If we diminish y, the quantity $x^{(r'')} - x^{(s)}$ will be augmented, and consequently also the sum of their extremes; $\beta^{(3)}$ is therefore the value of y, which gives the least of these sums; whence it follows that it is the only one which satisfies the problem.

We shall try in this way the values of $\beta^{(1)}$, $\beta^{(2)}$, $\beta^{(3)}$, &c., which is easily done by inspection; and if we arrive at a value which fulfils the preceding conditions, we shall be assured of the value required of y.

If any of these values of β does not fulfil these conditions, then this value of y will be some one of the terms of the series $\lambda^{(1)}$, $\lambda^{(2)}$, &c. Conceive, for example, that it is $\lambda^{(2)}$, the two extreme errors $x^{(s)}$ and $x^{(s')}$ will then be negative, and it will have, by what precedes, an intermediate error,

which will be a *maximum*, and which will fall consequently in the series $x^{(1)}$, $x^{(r)}$, $x^{(r')}$, &c. Suppose that this is $x^{(r)}$, r being then necessarily comprised between s and s'; $\lambda^{(2)}$ ought, therefore, to be comprised between $\beta^{(1)}$ and $\beta^{(2)}$. If that is the case, this will be a proof that $\lambda^{(2)}$ is the value required of y. We shall try thus all the terms of the series $\lambda^{(2)}$, $\lambda^{(3)}$, $\lambda^{(4)}$, &c. up to that which fulfils the preceding conditions.

When we shall have thus determined the value of y, we shall easily obtain that of z. For this, suppose that $\beta^{(2)}$ is the value of y, and that the three extreme errors are $x^{(r)}$, $x^{(r')}$, $x^{(s)}$; we shall have $x^{(s)} = -x^{(r)}$, and consequently

$$a^{(r)} - z - p^{(r)}.\, y = x^{(r)};$$

$$a^{(s)} - z - p^{(s)}.\, y = -x^{(r)};$$

whence we get

$$z = \frac{a^{(r)} + a^{(s)}}{2} - \frac{p^{(r)} + p^{(s)}}{2}.\, y;$$

then we shall have the greatest error $x^{(r)}$, by means of the equation

$$x^{(r)} = \frac{a^{(r)} - a^{(s)}}{2} + \frac{p^{(s)} - p^{(r)}}{2}\, y.$$

584. The ellipse determined in the preceding No. serves to recognise whether the hypothesis of an elliptic figure is in the limits of the errors of observations; but it is not that which the measured degrees indicate with the greatest probability. This last ellipse, it seems, should fulfil the following conditions, viz. 1st, that the sum of the errors committed in the measures of the entire measured arcs be nothing: 2dly, that the sum of these errors, all taken positively, may be a *minimum*. Thus considering the entire ones instead of the degrees which have thence been deduced, we give to each of the degrees by so much the more influence upon the ellipticity which thence results for the earth, as the corresponding arc is considerable, as it ought to be. The following is a very simple method of determining the ellipse which satisfies these two conditions.

Resume the equations (A) of 589, and multiply them respectively by the numbers which express how many degrees the measured arcs contain, and which we will denote by $i^{(1)}$, $i^{(2)}$, $i^{(3)}$, &c. Let A be the sum of the quantities $i^{(1)}.\, a^{(1)}$, $i^{(2)}.\, a^{(2)}$, &c. divided by the sum of the numbers $i^{(1)}$, $i^{(2)}$, &c.; let, in like manner, P denote the sum of the quantities $i^{(1)}.\, p^{(1)}$, $i^{(2)}.\, p^{(2)}$, &c. divided by the sum of the numbers $i^{(1)}$, $i^{(2)}$, &c.; the condition that the sum of the errors $i^{(1)}.\, x^{(1)}$, $i^{(2)}.\, x^{(2)}$, &c. is nothing, gives

$$0 = A - z - P.\, y.$$

If we take this equation from each of the equations A of the preceding No., we shall have equations of the following form:

$$\left.\begin{array}{l} b^{(1)} - q^{(1)}.\, y = x^{(1)} \\ b^{(2)} - q^{(2)}.\, y = x^{(2)} \\ b^{(3)} - q^{(3)}.\, y = x^{(3)} \\ \quad \&c. \end{array}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (O)$$

Form the series of quotients $\frac{b^{(1)}}{q^{(1)}}$, $\frac{b^{(2)}}{q^{(2)}}$, &c. and dispose them according to their order of magnitude, beginning with the greatest; then multiply the equations O, to which they respond, by the corresponding numbers $i^{(1)}$, $i^{(2)}$, &c.; finally, dispose these thus multiplied in the same order as the quotients.

The first members of the equations disposed in this way, will form a series of terms of the form

$$h^{(1)} y - c^{(1)}; \quad h^{(2)} y - c^{(2)}; \quad h^{(3)} y - c^{(3)}; \text{ &c. } . \; . \; . \quad (P)$$

in which we shall suppose $h^{(1)}$, $h^{(2)}$ positive, by changing the sign of the terms when y has a negative coefficient. These terms are the errors of the measured arcs, taken positively or negatively.

Then it is evident, that in making y infinite, each term of this series becomes infinite; but they decrease as we diminish y, and end by being negative—at first, the first, then the second, and so on. Diminishing y continually, the terms once become negative continue to be so, and decrease without ceasing. To get the value y, which renders the sum of these terms all taken positively a *minimum*, we shall add the quantities $h^{(1)}$, $h^{(2)}$, &c. as far as when their sum begins to surpass the semi-sum of all these quantities; thus calling F this sum, we shall determine r such that

$$h^{(1)} + h^{(2)} + h^{(3)} + \ldots . + h^{(r)} > \tfrac{1}{2} F;$$
$$h^{(1)} + h^{(2)} + h^{(3)} + \ldots . + h^{(r-1)} < \tfrac{1}{2} F.$$

We shall then have $y = \frac{c^{(r)}}{h^{(r)}}$, so that the error will be nothing relatively to the same degree which corresponds to that of the equations (O), of which the first member equated to zero, gives this value of y.

To show this, suppose that we augment y by the quantity δy, so that $\frac{c^{(r)}}{h^{(r)}} + \delta y$ may be comprised between $\frac{c^{(r-1)}}{h^{(r-1)}}$ and $\frac{c^{(r)}}{h^{(r)}}$. The $(r - 1)$ first terms of the series (P) will be negative, as in the case of $y = \frac{c^{(r)}}{h^{(r)}}$; but in taking them with the sign +, their sum will decrease by the quantity

$$\{h^{(1)} + h^{(2)} \ldots . h^{(r-1)}\}\, \delta y.$$

The first term of this series, which is nothing when $y = \frac{c^{(r)}}{h^{(r)}}$, will become positive and equal to $h^{(r)}\,\delta y$; the sum of this term and the following, which are positive, will increase by the quantity

$$\{h^{(r)} + h^{(r+1)} + \&c.\}\,\delta y;$$

but by supposition we have

$$h^{(1)} + h^{(2)} \ldots . h^{(r-1)} < h^{(r)} + h^{(r+1)} + \&c.;$$

the entire sum of the terms of the series (P), all taken positively, will therefore be augmented, and as it is equal to the sum of the errors $i^{(1)}.x^{(1)} + i^{(2)}.x^{(2)}$, &c. of the entire measured arcs, all taken with the sign +, this last sum will be augmented by the supposition of $y = \frac{c^{(r)}}{h^{(r)}} + \delta y$. It is easy to prove, in the same way, that by augmenting y, so as to be comprised between $\frac{c^{(r-1)}}{h^{(r-1)}}$ and $\frac{c^{(r-2)}}{h^{(r-2)}}$, or between $\frac{c^{(r-2)}}{h^{(r-2)}}$ and $\frac{c^{(r-3)}}{h^{(r-3)}}$, &c. the sum of the errors taken with the sign + will be greater than when $y = \frac{c^{(r)}}{h^{(r)}}$.

Now diminish y by the quantity δy so that $\frac{c^{(r)}}{h^{(r)}} - \delta y$ may be comprised between $\frac{c^{(r)}}{h^{(r)}}$ and $\frac{c^{(r+1)}}{h^{(r+1)}}$, the sum of the negative terms of the series (P) will increase, in changing their sign, by the quantity

$$\{h^{(1)} + h^{(2)} + \ldots . h^{(r)}\}\,\delta y;$$

and the sum of the positive terms of the same series will decrease by the quantity

$$\{h^{(r+1)} + h^{(r+2)} + \&c.\}\,\delta y;$$

and since we have

$$h^{(1)} + h^{(2)} + \ldots . h^{(r)} > h^{(r+1)} + h^{(r+2)} + \&c.,$$

it is clear that the entire sum of the errors, taken with the sign +, will be augmented. In the same manner we shall see that, by diminishing y, so that it should be between $\frac{c^{(r+1)}}{h^{(r+1)}}$ and $\frac{c^{(r+2)}}{h^{(r+2)}}$, or between $\frac{c^{(r+2)}}{h^{(r+2)}}$ and $\frac{c^{(r+3)}}{h^{(r+3)}}$, &c. the sum of the errors taken with the sign + is greater than when $y = \frac{c^{(r)}}{h^{(r)}}$; this value of y is therefore that which renders this sum a *minimum.*

The value of y gives that of z by means of the equation

$$z = A - P \cdot y.$$

The preceding analysis being founded on the variation of the degrees from the equator to the poles, being proportional to the square of the sine of the latitude, and this law of variation subsisting equally for gravity, it is clear that it applies also to observations upon the length of the seconds' pendulum.

The practical application of the preceding theory will fully establish its soundness and utility. For this purpose, ample scope is afforded by the actual admeasurements of arcs on the earth's surface, which have been made at different times and in different countries. Tabulated below you have such results as are most to be depended on for care in the observations, and for accuracy in the calculations.

Latitudes.	Lengths of Degrees.	Where made.	By whom made.
0°.0000	25538ᵗ.85	Peru.	Bouguer.
37 .0093	25666 .65	Cape of Good Hope	La Caille.
43 .5556	25599 .60	Pennsylvania.	Mason & Dixon.
47 .7963	25640 .55	Italy.	Boscovich & le Maire.
51 .3327	25658 .28	France.	Delambre & Mechain.
53 .0926	25683 .30	Austria.	Liesganig.
73 .7037	25832 .25	Laponia.	Clairaut, &c.

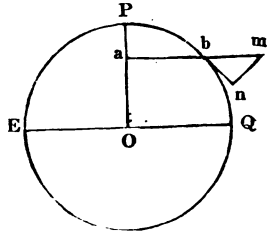
SUPPLEMENT

TO

BOOK III.

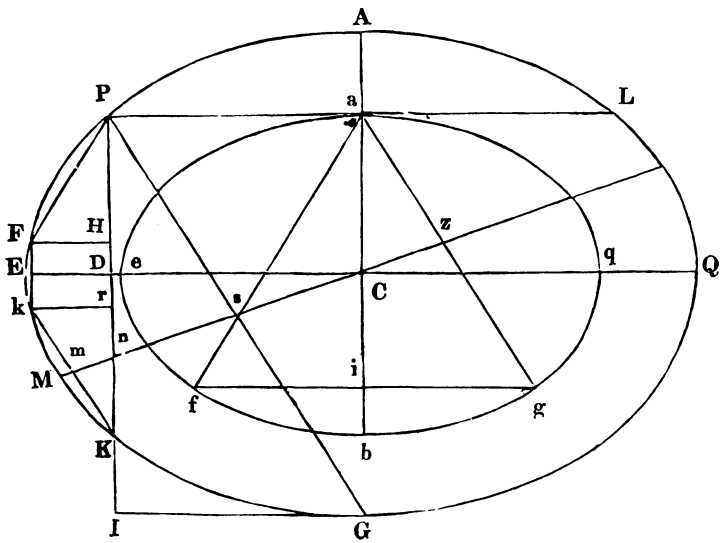
FIGURE OF THE EARTH.

585. If a fluid body had no motion about its axis, and all its parts were at rest, it would put on the form of a sphere; for the pressures on all the columns of fluid upon the central particle would not be equal unless they were of the same length. If the earth be supposed to be a fluid body, and to revolve round its axis, each particle, besides its gravity, will be urged by a centrifugal force, by which it will have a tendency to recede from the axis. On this account, Sir Isaac Newton concluded that the earth must put on a spheroidical form, the polar diameter being the



shortest. Let P E Q represent a section of the earth, P p the axis, E Q the equator, (b m) the centrifugal force of a part revolving at (b). This force is resolved into (b n), (n m), of which (b n) draws fluid from (b) to Q, and therefore tends to diminish P O, and increases E Q.

It must first be considered what will be the form of the curve P E p, and then the ratio of P O : G O may be obtained.



586. Lemma, Let E A Q, e a q, be similar and concentric ellipses, of which the interior is touched at the extremity of the minor axis by P a L; draw a f, a g, making any equal angle with a C; draw P F and P G respectively parallel to a f, a g; then will P F + P G = a f + a g. For draw P K, F k perpendicular to E Q, and F H, k r perpendicular to P K, ∴ F E = E K, ∴ H D = D r and P D = D K, ∴ P H = K r; also F H = K r, ∴ if K k be joined, K k = P F; draw the diameter M C z bisecting K k, G P, a g, in (m), (s), (z).

Then

K m : K n : : P s : P n : : a z : a C : : a g : a b.

∴ K m + P s : K n + P n : : a g : a b

but

K n + n P = K P = 2 P D = 2 a C = a b ∴ K m + P s = a g.

∴ 2 K m + 2 P s = 2 a g, or P F + P G = a g + a f.

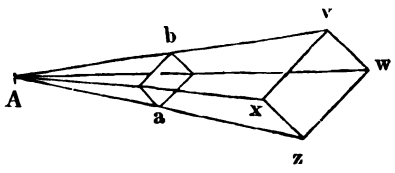
Cor. P H + P I = 2 a i. For

P F : P H : : P G : P I : : a g : a i.

∴ P F + P G : P H + P I : : a g : a i : : 2 a g : 2 a i.

but

P F + P G = 2 a g, ∴ P H + P I = 2 a i.



587. The attraction of a particle A towards any pyramid, the area of whose base is indefinitely small, $\propto$ length, the angle A being given, and the attraction to each particle varying as $\frac{1}{\text{distance}^2}$.

For let

$$\begin{aligned} a &= \text{area } (v\ x\ z\ w) \\ m &= (A\ z) \\ x &= (A\ a) \end{aligned}$$

Then section a b $= \dfrac{\text{section } v\ x\ z\ w \,.\, (A\ a)^2}{(A\ z)^2} = \dfrac{a\ x^2}{m^2}$

$\therefore \varphi'$. attraction $= \dfrac{a\ x^2\ x'}{m^2\ x^2} = \dfrac{a\ x'}{m^2}$

$\therefore$ attraction $= \dfrac{a\ x}{m^2}$.

$\therefore$ attractions of particles at vertices of similar pyramids $\propto$ lengths.

588. If two particles be similarly situated in respect to two similar solids, the attraction to the solids $\propto$ lengths of solids.

For if the two solids be divided into similar pyramids, having the particles in the vertices, the attractions to all the corresponding pyramids $\propto$ their lengths $\propto$ lengths of solids, since the pyramids being similarly situated in the two similar solids, their lengths must be as the lengths of the solids: $\therefore$ whole attractions $\propto$ lengths of the solids, or as any two lines similarly situated in them.

Cor. 1. Attraction of (a) to the spheroid a q f: attraction of A to A Q F : : a C : A C.

Cor. 2. The gravitation of two particles P and p in one diameter P C are proportional to their distances from the center. For the gravitation of (p) is the same as if all the matter between the surfaces A Q E, a q e, were taken away (Sect. XIII. Prop. XCI. Cor. 3.) $\therefore$ P and p are similarly situated in similar solids, $\therefore$ attractions on P and p are proportional to P C and p C, lines similarly situated in similar solids.

589. All particles equally distant from E Q gravitate towards E Q with equal forces.

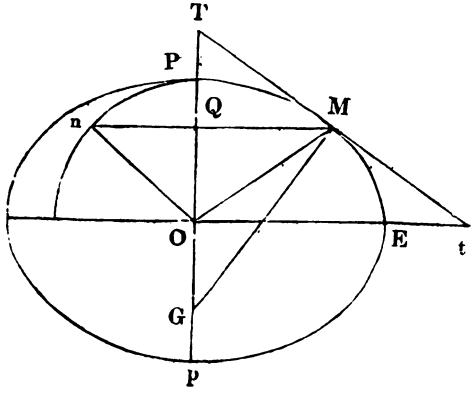
For P G and P F may be considered as the axes of two very slender pyramids, contained between the plane of the figure and another plane, making a very small angle with it. In the same manner we may conceive of (a f) and (a g). Now the gravity of P to these pyramids is as P F + P G; and in the direction P d is as P H + P I. Again, the gravity of (a) to the pyramids (a f), (a g) is as (a f + a g), or in the direction (a i) as 2 a i; but P H + P I = 2 a i : ∴ gravity of P in the direction P d = gravity of (a) in the same direction.

It is evident, by carrying the ordinate (f g) along the diameter from (b) to (a); the lines (a f), (a g) will diverge from (a b), and the pyramids of which these lines are the axes, will compose the whole surface of the interior ellipse. The pyramids, of which P F, P G are the axes, will, in like manner, compose the surface of the exterior ellipse, and this is true for every section of the spheroid passing through P m. Hence the attraction of P to the spheroid P A Q in the direction P d equals the attraction of (a) to the spheroid (p a q) in the same direction.

590. Attraction of P in the direction P D : attraction of A in the same direction : : P D : A C.

For the attraction of (a) in the direction P D : attraction of A in the same direction : : P D : A C, and the attraction of (a) = attraction of P. ∴ attraction of P : attraction of A : : P D : A C.

Similarly, the attraction of P in the direction E C : attraction of A in the direction E C : : P a : E C.



591. Draw M G perpendicular to the ellipse at M, and with the radius O P describe the arc P n.

Then Q G : Q M : : Q M : Q T

$$\therefore Q G = \frac{Q M^2}{Q T}.$$

And

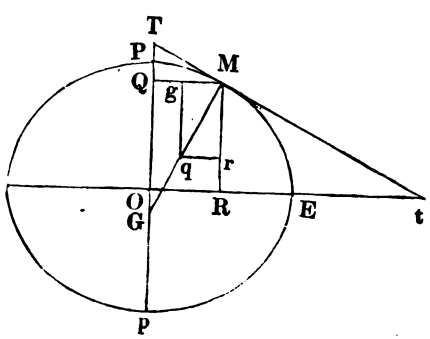
$$OQ : OP :: OP : OT$$
$$\therefore OQ = \frac{OP^2}{OT}.$$
$$\therefore QG : QO :: \frac{QM^2}{QT} : \frac{OP^2}{OT} :: \frac{QM^2 . OT}{QT} : OP$$

but

$$OT : OQ :: OP^2 : QO^2$$
$$\therefore OT : TQ :: OP^2 : OP^2 - OQ^2$$
$$:: OP^2 : nQ^2$$
$$:: OP^2 : PQ . Qp :: OE^2 : QM^2$$
$$\therefore \frac{OT}{TQ} = \frac{OE^2}{QM^2}.$$
$$\therefore QG : QO :: OE^2 : OP^2$$
$$\therefore QG = \frac{OE^2}{OP^2} . QO.$$

592. A fluid body will preserve its figure if the direction of its gravity, at every point, be perpendicular to its surface; for then gravity cannot put its surface in motion.

593. If the particles of a homogeneous fluid attract each other with forces varying as $\frac{1}{\text{distance}^2}$, and it revolve round an axis, it will put on the form of a spheroid.



For if P E p P be a fluid, P p the axis round which it revolves, then may the spheroid revolve in such a time that the centrifugal force of any particle M combined with its gravity, may make this whole force act perpendicularly to the surface. For let E = attraction at the equator, P = attraction at the pole, F = centrifugal force at the equator.

Then (590),

attraction of M in the direction M R : P : : Q O : P O

$\therefore$ attraction of M in the direction $M R = \frac{P \cdot Q O}{P O}$.

Similarly, the attraction of M in the direction $M Q = \frac{E \cdot O R}{O E}$.

But the centrifugal force of bodies revolving in equal times $\propto$ radii.

$$F \propto \frac{V^2}{r} \propto \frac{c^2}{r \cdot P^2}$$

$$\propto \frac{4 \pi^2 r}{P^2}$$

(and P being given) $\propto r$

$\therefore$ centrifugal force of $M = \frac{F \cdot O R}{O E}$

$\therefore$ whole force of M in the direction $M O = \frac{(E - F) \cdot O R}{O E}$.

Take $M r = \frac{P \cdot Q O}{P O}$, $M g = \frac{(E - F) \cdot O R}{O E}$, complete the parallelogram, and M q will be the compound force; O E and O P $\therefore$ must have such a ratio to each other that M q may be always perpendicular to the curve. Suppose M q perpendicular to the curve, then, by similar triangles, q g or M r : M g : : Q G : Q M.

$$:: \frac{P \cdot Q O}{P O} : \frac{(E - F) O R}{O E} :: \frac{O E^2}{O P^2} \cdot Q O : O R$$

$$\therefore \frac{P \cdot Q O \cdot O R}{P O} = (E - F) \cdot \frac{O R}{O E} \cdot \frac{O E^2}{O P^2} \cdot Q O$$

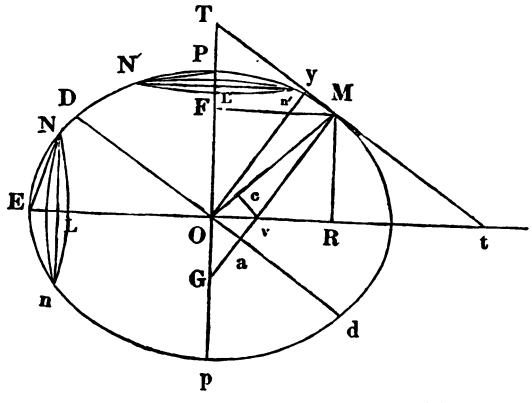
$$\therefore P = (E - F) \cdot \frac{O E}{O P}$$

$$\therefore P : E - F : : O E : O P,$$

in which no lines are concerned except the two axes; $\therefore$ to a spheroid having two axes in such a ratio, the whole force will, at every point, be perpendicular to the surface, and $\therefore$ the fluid will be at rest.

594. The attraction of any point M in the direction $M R = \frac{P \cdot M R}{P O}$;

$\therefore$ if P be represented by P O, M R will represent the attraction of M in the direction M R, and M v will represent the whole attraction acting perpendicularly to the surface.



Draw (v c) perpendicular to M O.

Then

M O : M a : : M v : M c : : attraction in the direction M v : M O.

$\therefore$ attraction in the direction $MO = \frac{Mv \cdot Ma}{MO} = \frac{OP^2}{MO} \propto \frac{1}{MO}$.

By similar triangles T O y, M v R, (the angle T O y being equal to the angle v M R.)

$$TO : Oy :: vM : MR$$

$$\therefore TO \cdot MR = Oy \cdot vM = Ma \cdot Mv = TO \cdot OF = OP^2.$$

595. Required the attraction of an oblong spheroid on a particle placed at the extremity of the major axis, the excentricity being very small.

Let axis major : axis minor : : 1 : 1 — n. Attraction of the circle N n (Prop XC.)

$$\propto 1 - \frac{EL}{EN} \propto 1 - \frac{x}{\sqrt{n^2 + (1-n)^2 (2n - n^2)}}$$

$$\propto 1 - x \{2x - n \cdot (4x - 2n^2)\}^{\frac{1}{2}}$$

$$\propto 1 - x \{(2x)^{-\frac{1}{2}} + \frac{1}{2}(2x)^{-\frac{3}{2}} n \cdot (4n - 2n^2)\}$$

$$\propto 1 - \frac{\sqrt{x}}{\sqrt{2}} - \frac{n}{4\sqrt{2}} \cdot (4\sqrt{x} - 2x^{\frac{1}{2}})$$

$$\therefore A' \propto x' - \frac{x^{\frac{1}{2}} x'}{\sqrt{2}} - \frac{n}{4\sqrt{2}} \cdot (4x^{\frac{1}{2}} x' - 2x^{\frac{3}{2}} x')$$

$$\therefore A \propto x - \frac{\sqrt{2}}{3} \cdot x^{\frac{3}{2}} - \frac{n}{4\sqrt{2}} \cdot \left(\frac{8x^{\frac{3}{2}}}{3} - \frac{4x^{\frac{5}{2}}}{5}\right)$$

Let $x = 2\,E\,O = 2$,

$$\therefore A \propto 2 - \frac{4}{3} - \frac{n}{4\sqrt{2}} \cdot \left(\frac{16\sqrt{2}}{3} - \frac{16\sqrt{2}}{5}\right)$$

$$\propto \frac{2}{3} - \frac{8n}{15} \propto 1 - \frac{4n}{5}.$$

$\therefore$ attraction of the oblong spheroid on E : attraction of a circumscribed sphere on E : : (since in the sphere $n = 0$.)

$$:: 1 - \frac{4n}{5} : 1.$$

596. Required the attraction of an oblate spheroid on a particle placed at the extremity of the minor axis.

Let axis minor : axis major : : 1 : 1 + n.

$$\therefore A' \propto x'\left\{1 - \frac{x}{\sqrt{x^2 + (1+n)^2 \cdot (2x - x^2)}}\right\}$$

$$\propto x'\left\{1 - \frac{x}{\sqrt{2x + 4nx - 2nx^2}}\right\}$$

$$\propto x'\left\{1 - x\left((2x)^{-\frac{1}{2}} - \frac{1}{2}(2x)^{-\frac{3}{2}}\sqrt{4nx - 2nx^2}\right)\right\}$$

$$\propto x' - \frac{x^{\frac{1}{2}}x'}{\sqrt{2}} + \frac{nx^{\frac{1}{2}}x'}{\sqrt{2}} - \frac{nx^{\frac{3}{2}}x'}{2\sqrt{2}}$$

$$\therefore A \propto x - \frac{\sqrt{2} \cdot x^{\frac{3}{2}}}{3} + \frac{\sqrt{2} \cdot nx^{\frac{3}{2}}}{3} - \frac{nx^{\frac{5}{2}}}{5\sqrt{2}}$$

$\therefore$ whole attraction

$$\propto 2 - \frac{4}{3} + \frac{4n}{3} - \frac{4n}{5} \propto \frac{2}{3} + \frac{8n}{15} \propto 1 + \frac{4n}{5}$$

$\therefore$ attraction of the oblate sphere on P : attraction of the sphere inscribed on P : : $1 + \frac{4n}{5}$: 1.

Since these spheroids, by hypothesis, approximate to spheres, they may, without sensible error, be assumed for spheres, and their attractions will be nearly proportional to their quantities of matter. But oblong sphere : oblate : : oblate : circumscribed sphere. $\therefore$ A of oblong sphere on E : A′ of oblate on E : : A′ : A″ of circumscribed sphere on E.

$$\therefore A' : A'' :: A : A' :: \sqrt{A} : \sqrt{A''} :: \sqrt{1 - \frac{4n}{5}} : 1 :: 1 - \frac{2n}{5} : 1$$

Also

att^{n}. of oblate sph. on P : att^{n}. of insc^{d}. sph. on P :: $1 + \frac{4n}{5} : 1$

att^{n}. of insc^{d}. sph. on P : att^{n}. of circumsc^{d}. sph. on E :: $1 : 1 + n$

att^{n}. of circumsc^{d}. sph. on E : attr^{n}. of oblate sph. on E :: $1 : 1 - \frac{2n}{5}$

$\therefore$ attraction of the oblate sphere on P : attraction of the oblate sphere on E :: $1 + \frac{4n}{5} : \overline{1+n} \cdot \overline{1 - \frac{2n}{5}}$

$:: 1 + \frac{4n}{5} : 1 + \frac{3n}{5} :: 1 + \frac{n}{5} : 1$ nearly.

$$1 + \tfrac{3n}{5}\,\big)\,1 + \tfrac{4n}{5}\,\big(\,1 + \tfrac{n}{5}$$

$$1 + \tfrac{3n}{5}$$

$$\tfrac{n}{5}$$

$$\tfrac{n}{5} + \tfrac{3n^2}{25}$$

$$-\tfrac{3n^2}{25}$$

$\therefore P : E :: 1 + \frac{n}{5} : 1$;

but (593), $P : E - F :: O E : O P$

$:: 1 + n : 1 :: P + F : E$ nearly

$\overline{1+n} \cdot \overline{E - F} = P$

$\therefore \overline{1+n} \cdot E - F - n F = P$

$\therefore \overline{1+n} \cdot E - n F = P + F$

and since (n) is very small, as also F compared with E,

$\therefore 1\overline{+n} \cdot E = P + F$

$\therefore 1 + n : 1 :: P + F : E$

$\therefore P = E + \frac{nE}{5}$

$\therefore E + \frac{nE}{5} + F : E :: 1 + n : 1$

$\therefore E + \frac{nE}{5} + F = E + nE$

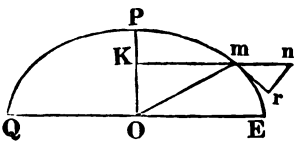
$\therefore F = \frac{4nE}{5}$

$\therefore n = \frac{5F}{4E}$

$\therefore 4E : 5F :: 1 : n$

or "four times the primitive gravity at the equator : five times the centrifugal force at the equator : : one half polar axis : excentricity."

597. The centrifugal force opposed to gravity $\propto$ cos.^{2} latitude.



Let (m n) = centrifugal force at (m), F = centrifugal force at E.

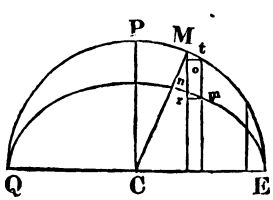
$\therefore$ (n r) is that part of the centrifugal force at (m) which is opposed to gravity.

Now

$$\left.\begin{array}{l} F : m n :: O E : K m \\ \text{and} \\ m n : n r :: O m : K m \end{array}\right\} \therefore F : n r :: O m^2 : K m^2 \\ :: r^2 : \text{cos.}^2 \text{lat.}$$

$$\therefore m r \propto \text{cos.}^2 \text{lat.}$$

598. From the equator to the pole, the increase of the length of a degree of the meridian $\propto$ sin.^{2} lat.



$$n r : M s :: n G : M G :: C P : C R :: 1 - n : 1.$$

$$\therefore n r = \overline{1 - n}.\ M S = \overline{1 - n}.\ \varphi' \text{ sin. } \theta = 1 - n.\ \text{cos. } \theta.\ \theta'$$

$$m r = s t = \varphi'.\ \text{cos. } \theta = -\ \text{sin. } \theta.\ \theta'$$

$$\therefore m r^2 = \text{sin.}^2 \theta.\ \theta^2$$

$$\therefore m n^2 = n r^2 + m r^2 = \theta'^2.\ \text{sin.}^2 \theta + (1 - n)^2.\ \text{cos.}^2 \theta.\ \theta'^2$$

$$= \theta'^2.\ (\text{sin.}^2 \theta + \overline{1 - 2n}.\ \text{cos.}^2 \theta)$$

$$= \theta'^2\ (\text{sin.}^2 \theta + \text{cos.}^2 \theta - 2 n.\ \text{cos.}^2 \theta)$$

$$= \theta'^2.\ (1 - 2 n.\ \text{cos.}^2 \theta)$$

$$\therefore m n = \theta'.\ (1 - n.\ \text{cos.}^2 \theta)$$

$\therefore$ at the equator, since

$$\theta = 0;\ m' n' = \theta'\ (1 - n)$$

$$\therefore \text{increase} = \theta' (1 - n . \cos.^2 \theta - 1 + n) = \theta' . n (1 - \cos.^2 \theta)$$
$$= \theta' . n \sin.^2 \theta,$$
$$\therefore \text{increase} \propto n \theta' . \sin.^2 \theta$$
$$\propto \sin.^2 \theta, \propto \sin.^2 \text{latitude}.$$

599. Given the lengths of a degree at two given latitudes, required the ratio between the polar and equatorial diameters.

Let P and p be the lengths of a degree at the pole and equator, m and n the lengths in latitudes whose sines are S and s, and cosines C and c. Then as length of a degree $\propto$ radius of curvature, (for the arc of the meridian intercepted between an angle of one degree, which is called the length of a degree, may be supposed to coincide with the circle of curvature for that degree, and will $\therefore \propto$ radius of curvature.)

$$\propto \frac{C D^2}{P F}.$$

Now at the pole C D^2 becomes A C^2, and P F becomes B C

$$\therefore \text{length of a degree} \propto \frac{A C^2}{B C}, \propto \frac{a^2}{b};$$

similarly the length of a degree at the equator

$$\propto \frac{B C^2}{A C}, \propto \frac{b^2}{a},$$

$$P : p :: \frac{a^2}{b} : \frac{b^2}{a} :: a^3 : b^3 :: 1 : (1 - n')^3.$$

Now

$$m - p : n - p . (598) :: S^2 : s^2,$$
$$\therefore m - n : n - p :: S^2 - s^2 : S^2,$$
$$\therefore n - p = \frac{\overline{m - n} . S^2}{S^2 - s^2},$$

but

$$P - p : n - p :: 1^2 : s^2 :: P - p : \frac{\overline{m - n} . S^2}{S^2 - s^2},$$
$$\therefore P - p = \frac{m - n}{S^2 - s^2},$$
$$\therefore P = p + \frac{m - n}{S^2 - s^2},$$
$$\frac{\overline{m - n} . s^2}{S^2 - s^2} = n - p,$$
$$\therefore P = n - \frac{\overline{m - n} . S^2}{S^2 - s^2},$$

$$= \frac{n S^2 - n s^2 - m s^2 + n s^2}{S^2 - s^2} = \frac{n S^2 - m s^2}{S^2 - s^2},$$

$$\therefore P = p + \frac{m - n}{S^2 - s^2} = \frac{n S^2 - m s^2 + m - n}{S^2 - s^2},$$

$$= \frac{m.(1 - s^2) - n.(1 - S^2)}{S^2 - s^2},$$

$$= \frac{m c^2 - n C^2}{S^2 - s^2},$$

$$\therefore P : p :: \frac{m c^2 - n C^2}{S^2 - s^2} : \frac{n S^2 - m s^2}{S^2 - s^2},$$

$$:: m c^2 - n C^2 : n S^2 - m s^2 :: 1 : (1 - n')^3$$

$$\therefore (m c^2 - n C^2)^{\frac{1}{3}} : (n S^2 - m s^2)^{\frac{1}{3}} :: 1 : 1 - n'.$$

600. The variation in the length of a pendulum $\propto$ sin.^{2} latitude.

Let l = length of a pendulum vibrating seconds at the equator.

L = length of one vibrating seconds at latitude θ.

The force of gravity at the pole = 1, $\therefore$ the force of gravity at the equator = 1 — F, and the force of gravity in latitude θ (603) = 1 — F. cos.$^2 \theta$,

$$\therefore L : l :: 1 - F. \cos.^2 \theta : 1 - F \text{ (since } \delta \propto \propto \delta F)$$

$$\therefore L - l : l :: F. (1 - \cos.^2 \theta) : 1 - F :: F. \sin.^2 \theta : 1 - F,$$

$$\therefore L - l = \frac{l F. \sin.^2 \theta}{1 - F} \propto \sin.^2 \theta.$$

From the poles to the equator, the decrease of the length of a pendulum always vibrating in the same time, $\propto$ cos.^{2} latitude.

Let L' = length of a pendulum vibrating seconds at the pole,

$$\therefore L' : L :: 1 : 1 - F. \cos^2 \theta,$$

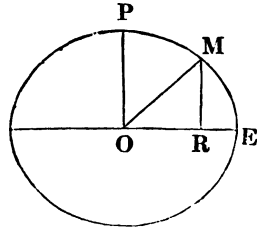
$$\therefore L' : L' - L :: 1 : F. \cos^2 \theta,$$

$$\therefore L' - L \propto \cos.^2 \theta.$$

601. The increase of attraction from the equator to the pole $\propto$ sin.^{2} lat.

Let

$$O E : O P :: 1 : 1 - n.$$



Let

$M O = a$, the angle $M O E = \theta$,

$$\therefore M R^2 = \frac{P O^2}{O E^2}. \{O E^2 - O R^2\},$$

or

$$a^2. \sin.^2 \theta = (1 - n)^2. (1 - a^2 \cos.^2 \theta)$$

$$= \overline{1 - 2n}. (1 - a^2 \cos.^2 \theta)$$

$$\therefore a^2. \{\sin.^2 \theta + \overline{1 - 2n}. \cos.^2 \theta\} = 1 - 2n$$

$$\therefore a^2 = \frac{1-2n}{\sin.^2\theta + \cos.^2\theta - 2n.\cos.^2\theta} = \frac{1-2n}{1-2n.\cos.^2\theta},$$

$$\therefore a = \frac{1-n}{1-n.\cos.^2\theta} = \overline{1-n}.\frac{1+n.\cos.^2\theta}{1^2-n^2.\cos.^4\theta} = \overline{1-n}.(1+n\cos.^2\theta),$$

$$= 1-n\,(1-\cos.^2\theta) = 1-n.\sin.^2\theta,$$

$$\therefore \frac{1}{a} = \frac{1}{1-n.\sin.^2\theta} = 1+n.\sin.^2\theta = \frac{1}{MO},$$

but (594) the attraction in the direction M O $\propto \frac{1}{MO}$,

$\therefore$ attraction in the direction M O (A) : attraction at E (A')

$$:: 1+n.\sin.^2\theta : 1,$$

$$\therefore A - A' : A' :: n.\sin.^2\theta : 1,$$

$$\therefore A - A' = A'.n.\sin.^2\theta,$$

$\therefore$ increase of attraction $\propto \sin.^2\theta \propto \sin.^2$ latitude.

602. Given the lengths of two pendulums vibrating seconds in two known latitudes; find the lengths of pendulums that will vibrate seconds at the equator and pole.

Let L, l be the lengths of pendulums vibrating seconds at the equator and pole.

L', l' be the lengths in given latitudes whose sines are S, s, cosines C, c.

$$\therefore L' - L : l' - L :: S^2 : s^2$$

$$\therefore L's^2 - Ls^2 = l'S^2 - Ls^2$$

$$\therefore L.(S^2 - s^2) = l'S^2 - L's^2,$$

$$\therefore L = \frac{l'S^2 - L's^2}{S^2 - s^2}.$$

Again

$$L' - L : l - L :: S^2 : 1,$$

$$\therefore L' - L = lS^2 - LS^2,$$

$$\therefore l = \frac{L' - L.(1-S^2)}{S^2},$$

$$= \frac{L'}{S^2} - \frac{(l'S^2 - L's^2)(1-S^2)}{S^2.(S^2-s^2)},$$

$$= \frac{L'S^2 - L's^2 - l'S^2 + l'S^4 + L's^2 - L'S^2s^2}{S^2.(S^2-s^2)},$$

$$= \frac{L'S^2 - l'.S^2 + l'.S^4 - L'.S^2s^2}{S^2.(S^2-s^2)},$$

$$= \frac{L'.(1-s^2) - l'.(1-S^2)}{S^2-s^2} = \frac{L'c^2 - l'C^2}{S^2-s^2}.$$

603. Given the lengths of two pendulums vibrating seconds in two known latitudes; required the ratio between the equatorial and polar diameters.

Since the lengths $\propto$ forces, the times being the same,

$\therefore$ L : l :: force at the equator : force at the pole

$$:: (10)\ \frac{1}{l} : \frac{1}{l - n} :: l - n : l :: OP : OE,$$

$\therefore$ OP : OE :: polar diameter : equatorial diameter

$$:: L : l :: l' S^2 - L' s^2 : L' c^2 - l' C^2.$$

604. To compare the space described in one second by the force of gravity in any given latitude, with that which would be described in the same time, if the earth did not revolve round its axis.

The space which would be described by a body, if the rotatory motion of the earth were to cease, equals the space actually described by a body at the pole in the same time; and if the force at the pole equal 1, the force at the latitude θ (597) equal $1 - F . \cos.^2 \theta$, and since $S = m F T^2$, and T is the same, $\therefore S \propto F$.

$\therefore$ space actually described when the earth revolves : space which would be described if the earth were at rest :: $1 - F . \cos.^2 \theta$: 1.

605. Let the earth be supposed a sphere of a given magnitude, and to revolve round its axis in a given time; to compare the weight of a body at the equator, with its weight in a given latitude.

The centrifugal force $= \frac{V^2}{r} = \frac{4 \pi^2 . r}{P^2} = F$ equal a given quantity, since (r) and P are known. Now the force at the equator $= 1 - F$, and the force at latitude $\theta = 1 - F . \cos.^2 \theta$, and the weight $\propto$ attractive force

$$\therefore W : W' :: 1 - F : 1 - F . \cos.^2 \theta.$$

606. Find the ratio of the times of oscillation of a pendulum at the equator and at the pole, supposing the earth to be a sphere, and to revolve round its axis in a given time.

$L \propto F T^2$ but L is constant, $\therefore T^2 \propto \frac{1}{F}$,

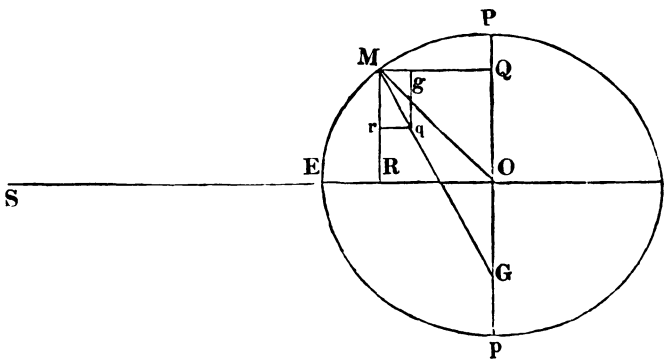
$\therefore$ T. oscillation at the pole : T. oscillation at the equator

$$:: \sqrt{\text{force at the equator}} : \sqrt{\text{force at the pole}}$$

$$:: \sqrt{1 - F} : 1.$$

ON THE THEORY OF THE TIDES.

607. If a spherical body at rest be acted upon by some other body, it may put on the form of a spheroid.



Let P E p be the earth at rest; (S) a body acting upon it; (O) its center; (M) a particle on its surface.

Let P = polar, E = equatorial, } attraction on the earth.

Then the attraction on M is parallel to $M\,Q = \frac{E \cdot O\,R}{O\,E}$.

Similarly the attraction on M is parallel to $M\,R = \frac{P \cdot O\,Q}{O\,P}$.

Let (m) = mean addititious force of S on P.

(n) = mean addititious force of S on E.

Now since the addititious force (Sect. XI.) $\propto$ distance,

$\therefore$ the whole addititious force of S on $M = \frac{m \cdot M\,O}{P\,O}$,

and

$\frac{m \cdot M\,O}{P\,O}$: addititious force in the direction M R :: M O : M R,

$\therefore$ addititious force in the direction $M\,R = \frac{m \cdot M\,R}{P\,O} = \frac{m \cdot O\,Q}{P\,O}$.

Again, since

$m : n :: P\,O : E\,O$,

$\therefore \frac{m}{P\,O} = \frac{n}{E\,O}$,

$\therefore$ whole addititious force of S on $M = \frac{n \cdot M\,O}{E\,O}$,

$\therefore$ addititious force in the direction $MQ = \frac{n.MQ}{EO} = \frac{n.OR}{EO}$,

$\therefore$ whole disturbing force of S on M in the direction MQ = twice the addititious force in that direction, and is negative $= -\frac{2n.OR}{OE}$.

$\therefore$ whole attraction of M in the direction $MQ = \{E - 2n\}.\frac{OR}{OE}$,

and the whole attraction of M in the direction $MR = \{P + m\}.\frac{OQ}{OP}$.

$$\left.\begin{aligned} \text{Take } Mg &= \{E - 2n\}.\frac{OR}{OE} \\ Mr &= \{P + m\}.\frac{OQ}{OP} \end{aligned}\right\}$$

complete the parallelogram (m q), and produce M q to meet P p in G.

Now if the surface be at rest, M G will be perpendicular to the surface.

$\therefore Mr : Mg :: gq : gM :: GQ : QM$,

or

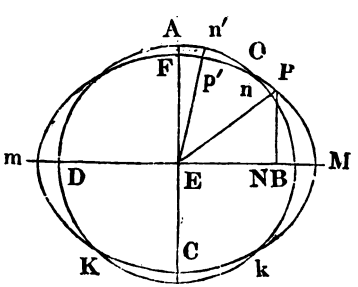
$$\{P+m\}.\frac{OQ}{OP} : \{E-2n\}\frac{OR}{OE} :: \frac{OE^2}{OP^2}OQ \cdot OR.$$

$\therefore P + m : E - 2n :: OE : OP$,

$\therefore$ figure may be an ellipse.

608. Suppose the Moon to move in the equator; to find the greatest elevation of tide.

Let A B C D be the undisturbed sphere; M P m K a spheroid formed by the attraction of the Moon; M the place to which the Moon is vertical.



Let

$$\left.\begin{aligned} AE &= 1 \\ EM &= 1 + \alpha \\ EF &= 1 - \beta \end{aligned}\right\}$$

Then since the sphere and spheroid have the same solid content,

$$\therefore \frac{4\pi.(AE)^3}{3} = \frac{4\pi.EM.(FE)^2}{3},$$

$$\therefore \frac{4\pi}{3} = \frac{4\pi.(1+\alpha).(1-\beta)^2}{3} = \frac{4\pi.(1+\alpha)(1-2\beta+\beta^2)}{3},$$

$$\therefore 1 = 1 + \alpha - 2\beta - 2\alpha\beta + \beta^2 + \alpha\beta^2$$
$$= 1 + \alpha - 2\beta \text{ nearly, } (\alpha) \text{ and } (\beta) \text{ being very small,}$$
$$\therefore \alpha = 2\beta \text{ or greatest elevation} = 2 \times \text{greatest depression.}$$

614. To find the greatest height of the tide at any place, as (n).

Let

$$E P = \theta - \angle P E M = \theta - \alpha + \beta = \frac{3\alpha}{2} = E M - E F = M,$$

$$\therefore P N^2 = \varrho^2 . \sin.^2\theta = \frac{E F^2}{E M^2} . \{E M^2 - E N^2\}$$
$$= \left(\frac{1-\beta}{1+\alpha}\right)^2 . \{(1+\alpha)^2 - \varrho^2 \cos.^2\theta\}.$$

Now $\frac{(1-\beta)^2}{(1+\alpha)^2}$ by actual division (all the terms of two or more dimensions being neglected) $= 1 - 2 . (\alpha + \beta) = 1 - 2M,$

$$\therefore P N^2 = \varrho^2 . \sin.^2\theta = (1 - 2M) . \{1 + 2\alpha - \varrho^2 . \cos.^2\theta\}$$
$$\left(\text{since } 2\alpha = \frac{3\alpha}{2} . \frac{4}{3} = \frac{4}{3}M\right) = (1 - 2M)\left\{1 + \frac{4M}{3} - \varrho^2 . \cos.^2\theta\right\}.$$

$$\therefore \varrho^2 . \{\sin.^2\theta + (1 - 2M)\cos.^2\theta\} = (1 - 2M) . \left(1 + \frac{4M}{3}\right)$$

$$\therefore \varrho^2 = \frac{1 - \frac{2M}{3}}{\sin.^2\theta + \cos.^2\theta - 2M . \cos.^2\theta} = \frac{1 - \frac{2M}{3}}{1 - 2M . \cos.^2\theta},$$

$$\therefore \varrho = \frac{1 - \frac{M}{3}}{1 - M . \cos.^2\theta} = (1 + M . \cos.^2\theta) . \left(1 - \frac{M}{3}\right) \text{ nearly,}$$
$$= 1 + M . \cos.^2\theta - \frac{M}{3},$$

$\therefore \varrho - 1 = M . \cos.^2\theta - \frac{M}{3} = E P - E n = P n =$ elevation required.

615. Similarly if the angle $M E p = \theta'$, $\therefore E p = 1 + M \cos.^2\theta' - \frac{M}{3}$,

$$\therefore 1 - E p = p n' = \text{depression} = \frac{M}{3} - M . \cos.^2\theta$$
$$= M - M . \cos.^2\theta' - \frac{2M}{3} = M \sin.^2\theta' - \frac{2M}{3}.$$

616. $B M = a = \frac{2M}{3}$,

$$\therefore B M - P n = \frac{2M}{3} + M . \sin.^2\theta - \frac{2M}{3}$$
$$= M . \sin.^2\theta \propto \sin.^2\theta,$$

$\therefore$ greatest elevation $\propto$ sin.^{2} horizontal angle from the time of high tide.

617. At (O) P n = 0,

$$\therefore \text{M} \cdot \cos.^2 \theta - \frac{\text{M}}{3} = 0,$$

$$\therefore \text{M} \cdot \cos.^2 \theta = \frac{\text{M}}{3},$$

$$\therefore \cos. \theta = \frac{1}{\sqrt{3}},$$

$$\therefore \theta = 54^\circ\ ,\ 44'.$$

Hitherto we have considered the moon only as acting on the spheroid. Now let the sun also act, and let the elevation be considered as that produced by the joint action of the sun and moon in their different positions.

Let us suppose a spheroid to be formed by the action of the sun, whose semi-axis major = (1 + a), axis minor = (1 — b).

618. Let (a + b) = S, (φ) = the angular distance of any place from the point to which the sun is vertical. It may be shown in the same manner as was proved in the case of the moon, that

$$\text{S} \cdot \cos.^2 \varphi - \frac{\text{S}}{3} = \text{elevation due to the sun,}$$

and

$$\text{S} \cdot \sin.^2 \varphi' - \frac{2\,\text{S}}{3} = \text{depression due to the sun,}$$

(φ') being the angular distance of the place of low water from the point to which the sun is vertical,

$$\therefore \text{M} \cdot \cos.^2 \theta + \text{S} \cdot \cos.^2 \varphi - \frac{\text{M} + \text{S}}{3} = \text{compound elevation.}$$

Similarly $\text{M} \cdot \sin.^2 \theta' + \text{S} \cdot \sin.^2 \varphi' - \frac{2}{3}\sqrt{\text{M} + \text{S}}$ = compound depression.

619. Let the sun and moon be both vertical to the same place,

$$\therefore \theta = \varphi = 0,$$

$$\therefore \text{M} + \text{S} - \frac{\text{M} + \text{S}}{3} = \frac{2}{3}\sqrt{\text{M} + \text{S}} = \text{compound elevation,}$$

and

$$\theta' = \varphi' = 90^\circ,$$

$$\therefore \text{M} + \text{S} - \tfrac{2}{3} \cdot \sqrt{\text{M} + \text{S}} = \tfrac{1}{3}\sqrt{\text{M} + \text{S}} = \text{compound depression,}$$

$\therefore$ compound elevation + compound depression = $\sqrt{\text{M} + \text{S}}$ = height of spring tide.

620. Let the moon be in the quadratures with the sun, then at a place under the moon,

$$(\theta) = 0, \text{ and } (\varphi) = 90^\circ,$$

$\therefore$ compound elevation $= M - \frac{M + S}{3}$,

also $(\theta') = 90$, and $(\varphi') = 0$,

$\therefore$ compound depression $= M - \frac{2}{3}\sqrt{M + S}$,

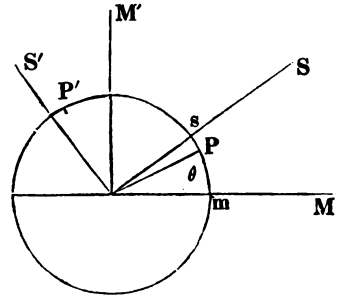
$\therefore$ height of the tide at the place under the moon $= 2M - \sqrt{M + S}$

$= M + S =$ height of neap tide.

Similarly at a place under the sun, height of tide $= S - M$.

621. Given the elongation of the sun and moon, to find the place of compound high tide.

Compound elevation $= M \cos.^2 \theta + S \cos.^2 \varphi - \frac{M + S}{3} =$ maximum at high water.



$\therefore -2M \cos. \theta \sin. \theta\, \theta' - 2S \cos. \varphi \sin. \varphi\, \varphi' = 0$,

but

$(\theta' + \varphi) =$ elongation $= E$

$=$ constant quantity,

$\therefore \theta' + \varphi' = 0$

$\therefore \theta' = -\varphi'$,

$\therefore 2M \cos. \theta \sin. \theta = 2S \cos. \varphi \sin. \varphi'$,

$\therefore M \sin. 2\theta = S \sin. 2\varphi$,

$\therefore M : S :: \sin. 2\varphi : \sin. 2\theta$,

$\therefore M + S : M - S :: \sin. 2\varphi + \sin. 2\theta : \sin. 2\varphi - \sin. 2\theta$,

$:: \tan. (\varphi + \theta) : \tan. (\varphi - \theta)$,

and since $(\varphi + \theta)$ is known, $\therefore (\varphi - \theta)$ is obtained, and $\therefore (\varphi)$ and (θ) are found, i. e. the distance of the sun and moon from the place of compound high tide is determined.

622. Let P be the place of high tide,

P' the place of low water, 90° distant from P,

$P m = \theta - P m 1 = 90 + \theta = \theta' - P s = \varphi - P' s$

$= 90 - \varphi = \varphi'$.

Now the greatest depression $= M \sin.^2 \theta' + S \sin.^2 \varphi' - \frac{2}{3}\sqrt{M + S}$,

but

$\sin.^2 \theta' = \sin.^2 (90 + \theta) = \sin.^2$ supplemental angle $(90 - \theta) = \cos.^2 \theta$,

and

$\sin.^2 \varphi' = \sin.^2 (90 - \varphi) = \cos.^2 \varphi$,

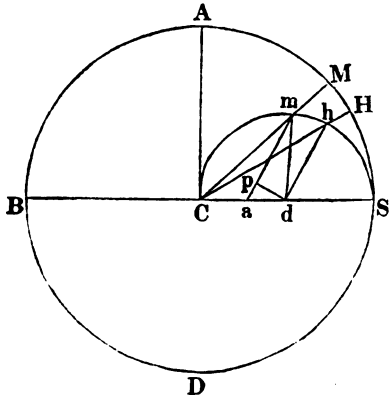
$\therefore$ the greatest depression $= M \cos.^2 \theta + S \cos.^2 \varphi - \frac{2}{3}\sqrt{M + S}$,

and the greatest elevation $= M \cos.^2 \theta + S \cos.^2 \varphi - \frac{1}{3}\sqrt{M + S}$,

$\therefore$ the greatest whole tide $=$ the greatest elevation $+$ greatest depression

$$= 2\,M\cos.^2\theta + 2\,S\cos.^2\varphi - \overline{M+S},$$
$$= M\,\{2\cos.^2\theta - 1\} + S\,(2\cos.^2\varphi - 1)$$
$$= M\cos.2\theta + S\cos.2\varphi.$$

623. Hence Robison's construction.



Let A B D S be a great circle, S and M the places to which the sun and moon are vertical; on S C, as diameter, describe a circle, bisect S C in (d); and take S d : d a :: M : S. Take the angle S C M = $(\varphi+\theta)$, and let C M cut the inner circle in (m), join (m a) and draw (h d) parallel to it; through (h) draw C h H meeting the outer circle in H; then will H be the place of high water.

For draw (d p) perpendicular to (m a) and join (m d).

Let the angle S C H = φ, and the angle M C H = θ.

Since M : S :: S d : d a

$$\therefore M+S : M-S :: Sd + da : Sd - da$$
$$:: dm + da : dm - da$$
$$:: \tan.\frac{dam + dma}{2} : \tan.\frac{dam - dma}{2}$$
$$:: \tan.\frac{Sdm}{2} : \tan.\frac{dam - dma}{2}$$
$$:: \tan.SCM : \tan.\frac{Sdh - mdh}{2}$$
$$:: \tan.SCM : \tan.(SCH - HCM)$$
$$:: \tan.(\varphi+\theta) : \tan.(\varphi-\theta)$$

$\therefore$ H is the place of high water 621.

Also (m a) equals the height of the whole tide. For (a p) = a d. cos. p a d
$$= S.\cos.Sdh = S.\cos.2\varphi$$
and
$$(p\,m) = m\,d.\cos.p\,m\,d = M.\cos.mdh = M.\cos.2\theta$$

$\therefore$ a m = a p + p m = M. cos. 2θ + S. cos. 2φ = height of the tide.

At new moon, $\theta = \varphi = 0$
At full moon, $\theta = 0$, $\varphi = 180°$ } $\therefore$ tide = M + S = spring tide.

When the moon is in quadratures, (m a) coincides with C A,

$\therefore \theta = 0$, $\varphi = 90°$,

$\therefore$ tide = M — S = neap tide.

624. The fluxion of the tide, i. e. the increase or decrease in the height of the tide $\propto \varphi'$. (m a) $\propto \varphi'$. {M. cos. 2θ + S. cos. 2φ}. But the sun for any place is considered as constant,

$$\therefore \varphi'. (\text{m a}) \propto -\text{M. sin. } 2\theta. 2\theta',$$

$\therefore \varphi'$. (m a) is a maximum at the octants of the tide with the moon

$$\propto -\text{M. sin. } 2\theta$$

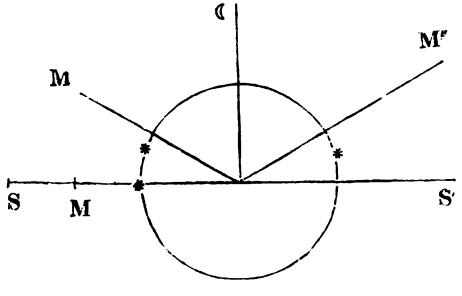
since at the octants, $2\theta = 90°$.

The fluxion of the tide is represented in the figure by (d p).

For let (m u) be a given arc of the moon's synodical motion, draw (n v) perpendicular on (m a), $\therefore$ (m v) is the difference of the tides.

Now m u : m v : : m d : d p and m u and m d are constant, $\therefore$ m v $\propto$ d p and d p is a maximum, when it coincides with (d a), i. e. when the tide is in octants; for then 2 (m a d) = 90°.

625. At the new and full moon, it is high water when the sun and



moon are on the meridian; i. e. at noon and midnight. At the quadratures of the moon, it is high water when the moon is on the meridian, because then (m) coincides with C.

For let M. cos.$^2\theta$ + S. cos.$^2\varphi$ — $\frac{\text{M} + \text{S}}{3}$ = maximum; then since in quadratures $(\varphi + \theta) = 90°$, $\therefore \varphi = 90° - \theta$,

$\therefore$ M. cos.$^2\theta$ + S. sin.$^2\theta$ — $\frac{1}{3}\overline{\text{M} + \text{S}}$ = maximum,

$\therefore$ 2 M. cos. θ. sin. θ. θ' = 2 S. sin. θ. cos. θ. θ',

$\therefore \overline{\text{M} - \text{S}}$. 2. sin. θ. cos. $\theta = \overline{\text{M} - \text{S}}$. sin. $2\theta = 0$, $\therefore$ sin. $2\theta = 0$,

$\therefore \theta = 0$, that is, the moon is on the meridian.

626. From the new moon to the quadratures, the place of high tide follows the moon, i. e. is westward of it; since the moon moves from west to east, from the quadratures to the full moon, the place of high tide is before the moon. There is therefore some place at which its distance from the moon (θ) equals a maximum.

Now (621) $\quad M : S :: \sin. 2\varphi : \sin. 2\theta$

$$\therefore M. \sin. 2\theta = S. \sin. 2\varphi$$

$$\therefore M. 2\theta'. \cos. 2\theta = S. 2\varphi'. \cos. 2\varphi = 0,$$

$$\therefore \cos. 2\varphi = 0, \therefore \varphi = 45°.$$

627. By (621) $M. \sin. 2\theta = S. \sin. 2\varphi$

$$\therefore \theta'. M. \cos. 2\theta = \varphi'. S. \cos. 2\varphi$$

but

$$\varphi + \theta = e, \therefore \varphi' + \theta' = e', \therefore \theta' = e' - \varphi'$$

$$\therefore (e' - \varphi') M. \cos. 2\theta = \varphi'. S. \cos. 2\varphi$$

$$\therefore e'. M. \cos. 2\theta = \varphi'. \{S. \cos. 2\varphi + M. \cos. 2\theta\}$$

$$\therefore \varphi = \frac{e'. M. \cos. 2\theta}{M. \cos. 2\theta + S. \cos. 2\varphi}.$$

Next, considering the moon to be out of the equator, its action on the tides will be affected by its declination, and the action of the sun will not be considered.

By Art. (614) the elevation $= M \cos.^2\theta - \frac{M}{3}$

$$\therefore \text{elevation above low water mark} = M. \cos.^2\theta - \frac{M}{3} + b$$

now

$$b = \frac{a}{2} = \frac{M}{3}$$

$$\therefore \text{elevation above low water} = M. \cos.^2\theta$$

$$= \text{magnitude of the tide.}$$

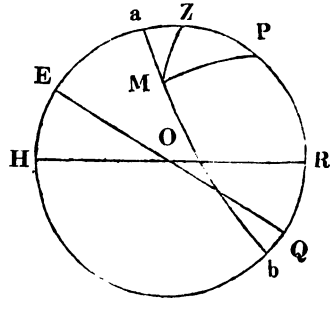
Let the angle Z P M which measures the time from the moon's passing the meridian equal t.

Let the latitude of the place

$= 90° - P Z = l$

Let the declination

$= 90° - P M = d$



$$\cos. ZPM = \frac{\cos. ZM - \cos. Z P \cos. P M}{\sin. Z P \sin. Z M}$$

or

$$\cos. t = \frac{\cos. \theta - \sin. l \sin. d}{\cos. l \cos. d}$$

$$\therefore \cos. \theta = \cos. t \cos. l \cos. d + \sin. l \sin. d$$

$\therefore$ magnitude of the tide $= M. \{\cos. t \cos. l \cos d + \sin. l \sin. d\}^2$

$\therefore$ for the same place and the same declination of the moon, the magnitude of the tide depends upon the value of (cos. t). Now the greatest and least values of (cos. t) are (+ 1) and (— 1), and since the moon only acts, it is high water when the moon is on the meridian, and the mean tide $= \frac{\text{greatest} + \text{least}}{2}$,

$$\text{greatest} = M. \{\sin. l \sin. d + \cos. l \cos. d\}^2$$
$$\text{least} = M. \{\sin. l \sin. d - \cos. l \cos. d\}^2$$
$$\therefore \frac{\text{greatest} + \text{least}}{2} = M. \{\sin.^2 l \sin.^2 d + \cos.^2 l \cos.^2 d\}$$
$$2 \sin.^2 l = 1 - \cos. 2 l$$
$$2 \sin.^2 d = 1 - \cos. 2 d$$
$$\therefore 4. \sin.^2 l \sin.^2 d = 1 - \{\cos. 2 l + \cos. 2 d\} + \cos. 2 l \cos. 2 d$$
$$2. \cos.^2 l = \cos. 2 l + 1$$
$$2. \cos.^2 d = \cos. 2 d + 1$$
$$\therefore 4. \cos.^2 l \cos.^2 d = 1 + (\cos. 2 l + \cos. 2 d) + \cos. 2 l \cos. 2 d$$
$$\therefore 4. \{\sin.^2 l \sin.^2 d + \cos.^2 l \cos.^2 d\} = 2 + 2. \cos. 2 l \cos. 2 d$$
$$\therefore \text{mean tide} = M. \sin.^2 l \sin.^2 d + \cos.^2 l \cos.^2 d$$
$$= M. \frac{1 + \cos. 2 l \cos. 2 d}{2}$$

It is low water at that place from whose meridian the moon is distant 90°, $\therefore \cos. \theta = 0$, $\therefore$ for low water

$$\cos. t = - \frac{\sin. l \sin. d}{\cos. l \cos. d} = - \tan. l \tan. d.$$

When $(l + d) = 90°$, $\therefore \tan. l \tan. d = \tan. l \tan. (90° - l)$

$$= \tan. l \cot. l = \frac{\tan. l}{\tan. l} = 1$$

$\therefore \cos. t = - 1$, $\therefore t = 180°$, $\therefore$ time from the moon's passing the meridian in this case equals twelve hours, $\therefore$ under these circumstances there is but one tide in twenty-four hours.

When $l = d$, $\therefore \cos. t = - \tan.^2 l$

and the greatest elevation $= M \{\cos. t \cos. l \cos. d + \sin. l \sin. d\}^2$

(since $\cos. t = 1$) $= M. \{\cos.^2 l + \sin.^2 l\} = M.$

When $d = 0$, $\therefore$ greatest elevation $= M \cos.^2 l$.

When $l = 0$, $\therefore$ greatest elevation $= M \cos.^2 d$.

At high water $t = 0$, $\therefore$ greatest elevation when the moon is in the meridian above the horizon, or, the superior tide $= M \{\cos. l \cos. d + \sin. l \sin. d\}^2 = M \cos.^2 (l - d) = T$.

For the inferior tide $t = 180°$, $\therefore \cos. t = - 1$,

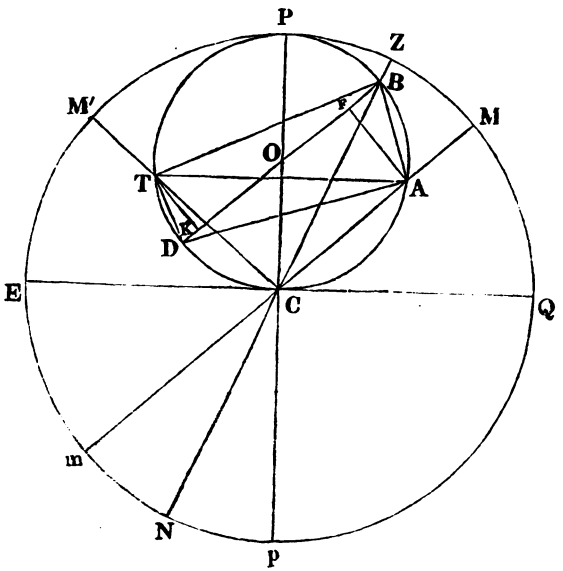
$\therefore$ inferior tide $= M\{\sin. l \sin. d - \cos. l \cos. d\}^2$

$= M\{-1(\cos. l \cos. d - \sin. l \sin. d)\}^2$

$= M \cos.^2(l + d) = T'$.

Hence Robison's construction.

With C P = M, as a radius, describe a circle P Q p E representing



a terrestrial meridian; P, p, the poles of the earth; E Q the equator; (Z) the zenith; (N) the nadir of a place on this meridian; M the place of the moon. Then

Z Q latitude of the place = l
M Q declination = d
$\}$ $\therefore$ Z M the zenith distance = l — d.

Join C M, cutting the inner circle in A; draw A T parallel to E Q. Join C T and produce it to M'; then M' is the place of the moon after half a revolution, $\therefore$ M' N = nadir distance

$= M E + E N = M Q + Z Q = l + d.$

Join C Z cutting the inner circle in B; join B with the center O and produce it to D; join A D, B T, A B, D T; and draw T K, A F perpendiculars on B D.

$\angle A D B = \angle B C A = Z Q - M Q = l - d$
$\angle T D B = 180^\circ - \angle T C B = \angle M C N = l + d$
$\}$ and the angles B A D, B T Z are right angles

$$B D : D A :: D A : D F = \frac{D A^2}{B D} = \frac{B D^2 . \cos. B D A}{B D} = BD . \cos.^2(l - d)$$

$= M \cos.^2(l - d)$ = height of the sup^r. tide.

Again

$$BD : DT :: DT : DK = \frac{DT^2}{BD} = \frac{BD^2 . \cos. BDT}{BD} = BD \cos. \overline{l + d}$$

$$= M \cos. \overline{l + d} = \text{point of the inferior tide.}$$

If the moon be in the zenith, the superior tide equals the maximum.

For then $l - d = 0$, $\therefore$ cos. $\overline{l - d}$ = maximum, and $BD = DF$.

If the moon be in the equator, $d = 0$, $\therefore DF = DK$.

The superior tide $= M \cos.^2 (l - d) = T$

The inferior tide $= M \cos.^2 (l + d) = T$.

Now $T > T'$, if (d) be positive, i. e. if the moon and place be both on *the same* side of the equator.

$T < T'$ if (d) be negative, i. e. if the moon and place be on *different* sides of the equator.

If $(d) = 90^\circ - l$, $\therefore DK = M \cos.^2 (l + 90^\circ - l) = M \cos.^2 90^\circ = 0$.

If $(d) = 90^\circ + l$, and in this case (d) be positive, and (l) negative,

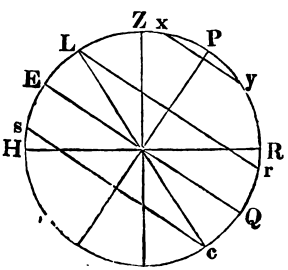
$\therefore DF = \cos.^2 (d - l) . M = M \cos.^2 (90^\circ + l - l) = M \cos.^2 90^\circ = 0$.

PROBLEMS

FOR

VOLUME III.

PROB. I. *The altitude* P R *of the pole is equal to the latitude of the place.*



For Z E measures the latitude.

= P R by taking Z P from E P and Z R.

PROB. 2. *One half the sum of the greatest and least altitudes of a circumpolar star is equal to the altitude of the pole.*

The greatest and least altitudes are at x, y on the meridian.

Also

$$x\,R + y\,R = x\,y + 2\,y\,R = 2\,(P\,y + R\,y) = 2 \text{ . altitude of the pole.}$$

PROB. 3. *One half the difference of the sun's greatest and least meridian altitudes is equal to the inclination of the ecliptic to the equator.*

The sun's declination is greatest at L, at which time it describes the parallel L r.

$\therefore$ L H is the greatest altitude,

The sun's declination is least at C, when it describes the parallel s C.

$\therefore$ s H is the least altitude,

and

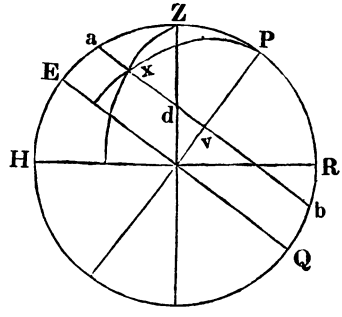
$$\tfrac{1}{2}\,(L\,H - s\,H) = \tfrac{1}{2}\,L\,s = L\,E.$$

PROB. 4. *One half the sum of the sun's greatest and least meridian altitudes is equal to the colatitude of the place.*

$$\tfrac{1}{2}\,(L\,H + s\,H) = \tfrac{1}{2}\,(H\,E + E\,L + H\,E - E\,s)$$
$$= \tfrac{1}{2}\,(2\,H\,E) = H\,E.$$

PROB. 5. *The angle which the equator makes with the horizon is equal to the colatitude* = E H.

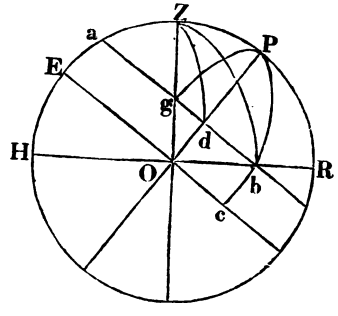
PROB. 6. When the sun describes b a in twelve hours, he will describe c a in six; if on the meridian at a it be noon, at c it will be six o'clock. Also at d he will be due east. He travels 15° in one hour. The angle a P x, measured by the number of degrees contained in a x (supposing x equals the sun's place), converted into the time at the rate of 15° for one hour, *gives the time from apparent noon*, or from the sun's arrival at a.



PROB. 7. *Given the sun's declination, and latitude of the place; find the time of rising, and azimuth at that time.*

Given Z E, ∴ Z P = colat. given.
Given b c, ∴ P b = codec. given.
Given b Z = 90°.

Required the angle Z P b, measuring a b, which measures the time from sun rise to noon.



Take the angles adjacent to the side 90°, and complements of the other three parts, for the circular parts.

∴ r. cos. Z P b = cot. Z P cot. P b

or

r. cos. hour ∠ = tan. lat. tan. dec.

∴ log. tan. lat. + log. tan. dec. — 10 = log. cos. hour ∠ required.

Also the angle P Z b measures b R, the azimuth referred to the north, and

$$\text{r. cos. P b} = \text{cos. P Z. cos. Z}$$

$$\therefore \text{cos. Z} = \frac{\text{r. cos. p}}{\text{sin. L}}.$$

PROB. 7. (a) r. cos. hour ∠ = tan. latitude tan. declination, *for sun rise.*

Hence *the length of the day* = 2. cos. hour ∠ = $\frac{2\text{. tan. lat. tan. dec.}}{}$

h may be found thus, from $\triangle$ Z P b cos. $h = \frac{\text{cos. Z b} - \text{Z cos. P. cos. P b}}{\text{sin. Z P. sin. P b}}$

$= (\text{since Z b} = 90^0,) \frac{\text{sin. L}}{\text{cos. L}} \cdot \frac{\text{cos. p}}{\text{sin. p}}$, or since $h > 90^0$,

— cos. h = — tan. L. cot. p, or cos. h = tan. L. cot. p.

and the angle P Z b may be similarly found,

$$\text{r. cos. Z} = \frac{\text{cos. P b} - \text{cos. Z P. cos. Z b}}{\text{sin. Z P. sin. Z b}}$$

$$= \frac{\text{cos. p}}{\text{cos. L}}.$$

PROB. 8. *Find the sun's altitude at six o'clock in terms of the latitude and declination.*

The sun is at d at six o'clock. The angle Z P d = right angle.

Z p = colat. P d = codec. Required Z d (= coalt.)

r. cos. Z d = cos. Z P. cos. d P

or

r. sin. altitude = sin. latitude × sin. declination.

PROB. 9. *Find the time when the sun comes to the prime vertical (that vertical whose plane is perpendicular to the meridian as well as to the horizon), and his altitude at that time, in terms of the latitude and declination.*

Z P = colatitude. P g = codeclination. The angle P Z g = right angle. Required the angle Z P g.

$$\therefore \text{r. cos. Z P g} = \text{tan. Z P. cot. P g.}$$

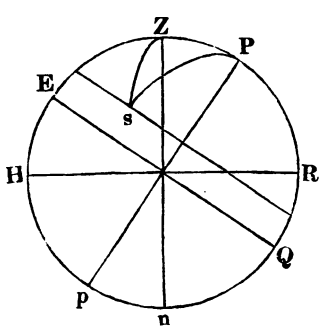
$$= \text{cot. latitude tan. declination.}$$

Also required Z g equal to the coaltitude,

$$\text{r. cos. P g} = \text{cos. P Z. cos. Z g.}$$

$$\therefore \frac{\text{r. sin. declination}}{\text{sin. latitude}} = \text{sin. altitude.}$$

PROB. 10. *Given the latitude, declination, and altitude of the sun; find the hour and azimuth.*



Let s be the place.

Given Z P, Z s, P s. Find the angle Z P s.

Let Z P, Z s, P s = a, b, c, be given, to find B.

$$\text{sin. B} = \frac{2\text{r}}{\text{sin. a . sin. c}} \times \sqrt{\text{s . (s — a) . (s — b) . (s — c)}}$$

where $\text{s} = \frac{\text{a} + \text{b} + \text{c}}{2}$.

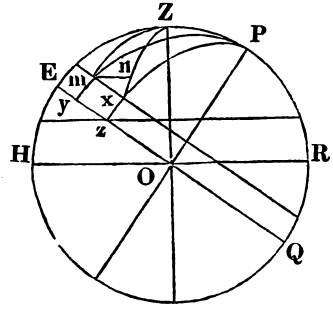
Also find C. $\sqrt{\qquad\qquad}$. (Or by Nap. 1st and 2d Anal.)

$$\text{sin. C} = \frac{2\,r}{\text{sin. a . sin. b}}.$$

Similarly, sin. A = sin. $\angle$ of position $= \frac{2\,r}{\text{sin. b . sin. c}} \sqrt{\qquad\qquad}$.

PROB. 11. *Given the error in the altitude. Find the error in the time in terms of latitude and azimuth.*

Let m n be parallel to H, and n x be the error in the altitude.



$\therefore \angle$ m P x = error in the time = y z.

y z : m x :: rad. : cos. m y

m x : x n :: rad. : sin. n m x

$\therefore$ y z : x n :: r^2 : cos. my. sin. n m x

or

$$y\,z = \frac{r^2 . n\,x}{\text{cos. m y . sin. n m x}}$$

$$= \frac{r^2 . n\,x}{\text{cos. m y . sin. Z x P}};$$

but

$$\frac{\text{sin. Z x P}}{\text{sin. x Z P}} = \frac{\text{sin. Z P}}{\text{sin. P x}}$$

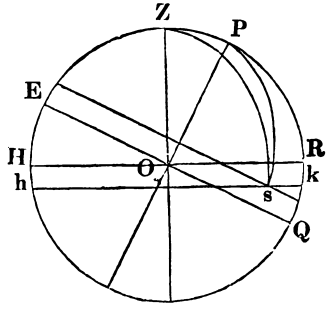
$$\therefore \text{sin. Z x P} = \frac{\text{sin. P Z . sin. x Z P}}{\text{cos. m y}}$$

$$\therefore y\,z = \frac{r^2 . n\,x}{\text{cos. L. sin. azimuth}}.$$

COR. Sin. of the azimuth is greatest when a z = 90°, or when the sun is on the prime vertical, $\therefore$ y z is then least.

Also, the perpendicular ascent of a body is quickest on the prime vertical, for if y z and the latitude be given, n x $\propto$ azimuth, which is the greatest.

PROB. 12. *Given the latitude and declination. Find the time when twilight begins.*



(Twilight begins when the sun is 18° below the horizon.)

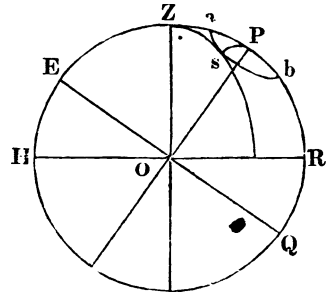
h k is parallel to H R and 18° below H R.

$\therefore$ Twilight begins when the sun is in h k.

$\therefore$ Z s = 90° + 18°, P s = D, Z P = colat.

Find the angle Z P s.

PROB. 13. *Find the time when the apparent diurnal motion of a fixed star is perpendicular to the horizon in terms of the latitude and declination.*



Let a b be the parallel described by the star.

Draw a vertical circle touching it at s.

$\therefore$ s is the place where the motion appears perpendicular to H R.

$\therefore$ Z P, P s, and $\angle$ Z S P $= 90^0$ is given. Find Z P s.

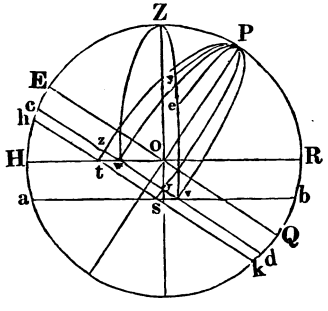
PROB. 14. *Find the time of the shortest twilight, in terms of the latitude and declination.*

a b is parallel to H R 18^0 below H R.

The parallels of declination c d, h k, are indefinitely near each other.

The angles v P w, s P t, measure the durations of twilight for c d, h k.

Since twilight is shortest, the increment of duration is nothing.



$$\therefore \text{v P w} = \text{s P t}$$
$$\therefore \text{v r} = \text{w z}$$
$$\text{and } \text{r s} = \text{t z}$$

and the angle v r s = right angle
= w z t.

$\therefore \angle$ r v s = z w t, and $\angle$ Z w c $= 90^0 -$ z w t $= 90^0 -$ Z w P.

$\therefore \angle$ z w t = Z w P.

Similarly,

$$\angle \text{r v s} = \text{Z v P}$$
$$\therefore \text{Z w P} = \text{Z a P}.$$

Take v e $= 90^0$. Join P e. Draw P y perpendicular to Z c.

In the triangles Z P w, P v e, Z w = e v, P w = P v, and the angles contained are equal, $\therefore$ Z P = P e.

$\therefore$ In the triangles Z P y, P e y, Z P = P e, P y com ; and the angles at y are right angles.

$\therefore$ Z e is bisected in y.

$$\text{r . cos. P v} = \text{cos. P y . cos. v y}$$
$$\text{r . cos. P e} = \text{cos. P y . cos. y e.}$$

$\therefore$ cos. P v : cos. P e :: cos. v y : cos. y e

(but v y is greater than 90^0, $\therefore$ therefore cos. v y is negative.)

:: — cos. (— compl. y e) : cos. y e

:: sin. y e : cos. y e

:: tan. y e : r.

$$\therefore \text{cos. p} = \frac{\text{sin. L. tan. y e}}{\text{r}} = \frac{\text{sin. L. tan. } \frac{18^0}{2}}{\text{r}} = \frac{\text{sin. L. tan. } 9^\circ}{\text{r}}$$

P Z is never greater than 90^0, Z y is equal to 9, $\therefore$ P y is never greater than 90^0, $\therefore$ cos. P y is always positive; v y is always greater than 90^0, $\therefore$ cos. v y is always negative, $\therefore$ cos. P v is negative, $\therefore$ the sun's declination is south.

Also, if instead of R b = 18^0, we take it equal to 2 s equal the sun's diameter, we get from the expression sin. $D = \frac{\text{sin. L. tan. s}}{\text{r}}$ the time when the sun is the shortest time in bringing his body over the horizon.

PROB. 15. *Find the duration of the shortest twilight.*

$\angle$ w P Z = v P e, $\therefore \angle$ Z P e = v P w.

$\therefore$ 2 Z P e is equal to the duration of the shortest twilight.

$$\text{r} \cdot \text{sin. Z y} = \text{sin. Z P} \cdot \text{sin. Z P y}$$

or

$$\text{sin. Z P y} = \frac{\text{sin. } 90^0 \cdot \text{r}}{\text{cos. L.}},$$

which doubled is equal to the duration required.

PROB. (A). Given the sun's azimuth at six, and also the time when due east. Find the latitude.

From the triangle Z P c,

r . cos. L = tan. P c . cot. P Z c.

From the triangle Z P d,

r . cos. h = cot. L . cot. P d.

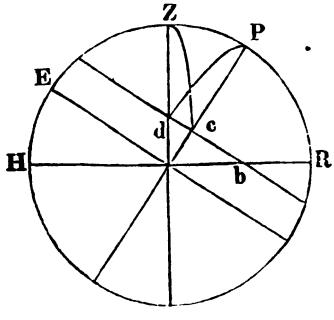
$$\therefore \text{tan. P c} = \frac{\text{cos. L}}{\text{cot. Z}}$$

$$\text{cot. P d} = \frac{\text{cos. h}}{\text{cot. L}}$$

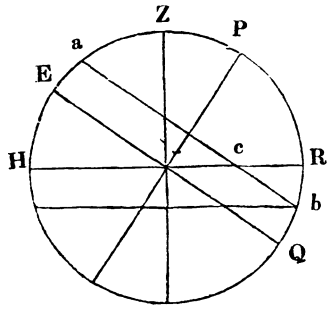
$$\therefore \text{tan. P d} = \frac{\text{cot. L}}{\text{cos. h}}$$

$$\therefore \frac{\text{cos. L}}{\text{cot. Z}} = \frac{\text{cot. L}}{\text{cos. h}}$$

$$\therefore \text{sin. L} = \frac{\text{cot. Z}}{\text{cos. h}}.$$



PROB. 16. *Find the declination when it is just twilight all night.*



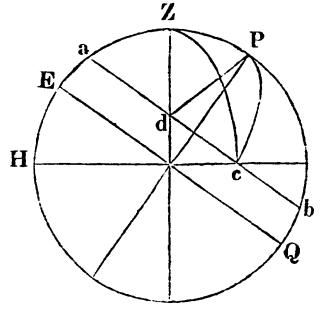
$$\text{Dec. } b\,Q = Q\,R - b\,R$$
$$= \text{colat.} - 18^0$$
$$= 90^0 - L - 18^0$$
$$= 72^0 - L$$

PROB. 17. Given the declination, find the latitude, the sun being due east, when one half the time has elapsed between his rising and noon.

Given $\angle\, Z\,P\,c$, and $Z\,P\,d = \frac{1}{2}\,Z\,P\,c$.

Given also $P\,d = p$,

and $\angle\, P\,Z\,d$ right angle.



$\therefore$ by Nap.

$$r \cdot \cos. h = \tan. Z\,P \cdot \cot. p$$

$$\therefore \cot. L = \frac{r \cdot \cos. h}{\cot. p}.$$

If the angle $Z\,P\,c$ be not given.

From the triangle $Z\,P\,d$,

$$\cos. Z\,P\,d = \tan. Z\,P \cdot \cot. p.$$

From the triangle $Z\,P\,c$,

$$r \cdot \cos. Z\,P\,c = \cot. Z\,P \cdot \cot. p,$$

or
$$\left.\begin{aligned} \cos. h &= \cot. \lambda \cdot \cot. p \\ \cos. 2\,h &= \tan. \lambda \cdot \cot. p \end{aligned}\right\}$$

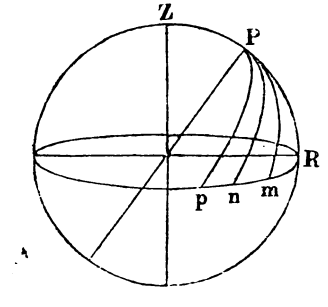
$$= 2 \cos.^2 h - 1 = 2 \cot.^2 \lambda \cdot \cot.^2 p - 1 = \frac{2 \cot.^2 p - 1}{\tan.^2 \lambda}.$$

$$\therefore \tan.^3 \lambda \cdot \cot. p = 2 \cot.^2 p - \tan.^2 \lambda$$

$$\therefore \tan.^3 \lambda + \frac{\tan.^2 \lambda}{\cot. p} - 2 \cot. p = 0,$$

from the solution of which cubic equation, $\tan. \lambda$ is found.

PROB. 18. Given the angle between two and three o'clock in the horizontal dial equal to a. Find the longitude.



From the triangle $P\,R\,n$,

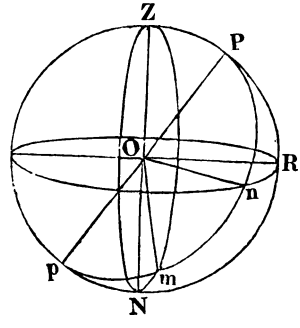
$$r \cdot \sin. P\,R = \tan. R\,n \cdot \cot. 30$$
$$= \tan. R\,n \cdot \sqrt{3}.$$

From the triangle $P\,R\,p$,

$$r \cdot \sin. P\,R = \tan. R\,p \cdot \cot. 45$$
$$= \tan\, R\,p.$$

$$\therefore \text{tan. n p} = \text{tan. a} = \text{tan. R p} - \text{R n}$$
$$= \frac{\text{tan. R p} - \text{tan. R n}}{1 + \text{tan. R p} \cdot \text{tan. R n}}$$
$$= \frac{\text{sin. } \lambda \left(1 - \frac{1}{\sqrt{3}}\right)}{1 + \frac{\text{sin.}^2 \lambda}{\sqrt{3}}} = \frac{\text{sin.} \lambda \cdot (\sqrt{3} - 1)}{\sqrt{3} + \text{sin.}^2 \lambda}.$$

PROB. 19. In what longitude is the angle between the hour lines of twelve and one on the horizontal dial equal to twice the angle between the same hour lines of the vertical sun dial?



From the triangle P R n,

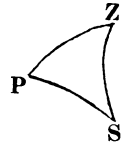
$$\text{sin. } \lambda = \text{cot. } 15 \cdot \text{tan. R n}$$

From the triangle p N m,

$$\text{sin. p M} = \text{cot. } 15 \cdot \text{tan. N m}$$
$$= \text{cos. } \lambda = \text{cot. } 15 \cdot \text{tan.} \frac{\text{R n}}{2}.$$
$$\therefore \frac{\text{sin. } \lambda}{\text{cos. } \lambda} = \frac{\text{tan. R n}}{\text{tan.} \frac{\text{R n}}{2}} = \text{tan. } \lambda$$
$$= \frac{\text{tan.} \frac{\text{R n}}{2} + \text{tan.} \frac{\text{R n}}{2}}{\frac{1 - \text{tan.}^2 \frac{\text{R n}}{2}}{\text{tan.} \frac{\text{R n}}{2}}} = \frac{2}{1 - \text{tan.}^2 \frac{\text{R n}}{2}}.$$

PROB. 20. *Given the altitude, latitude, and declination of the sun, find the time.*

$$\text{cos. h} = \frac{\text{cos. Z S} - \text{cos. Z P} \cdot \text{cos. P S}}{\text{sin. Z P} \cdot \text{sin. P S}}$$
$$= \frac{\text{sin. A} - \text{sin. L} \cdot \text{cos. p}}{\text{cos. L} \cdot \text{sin. p}}$$
$$\therefore 1 + \text{cos. h} = \frac{\text{cos. L} \cdot \text{sin. p} + \text{sin. A} - \text{sin. L} \cdot \text{cos. p}}{\text{cos. L} \cdot \text{sin. p}}$$
$$= \frac{\text{sin. (p} - \text{L)} + \text{sin. A}}{\text{cos. L} \cdot \text{sin. p}}$$

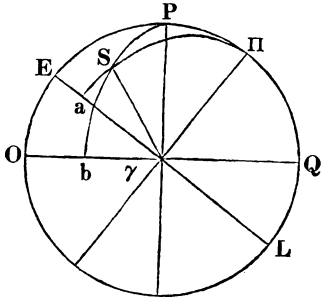


or

$$2 \text{ cos.}^2 \frac{\text{h}}{2} = \frac{2 \cdot \text{cos} \left(\frac{\text{A} + \text{p} - \text{L}}{2}\right) \cdot \text{sin.} \left(\frac{\text{A} + \text{L} - \text{p}}{2}\right)}{\text{cos. L} \cdot \text{sin. p}}$$

$\therefore \cos.^2 \frac{h}{2} = \left(\frac{\cos.(\text{——}).\ \sin.(\text{——})}{\text{——}}\right)$ the form adapted to the Logarithmic computation, or, see Prob. (18).

Prob. 21. Given a star's right ascension and declination. Find the latitude and longitude of the star.



Given

γ b, b S, $\angle$ S b γ right angle

$\therefore$ find $\angle$ S γ b and S γ.

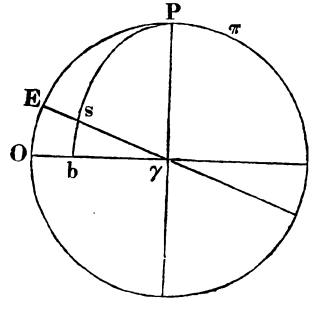
$\therefore \angle$ S γ a = S γ b — Obl.

$\therefore$ S γ is known, $\angle$ S γ a is known and S a γ is a right angle,

$\therefore$ find S a = latitude

γ a = longitude.

Given the sun's right ascension and declination. Find the obliquity of the ecliptic.



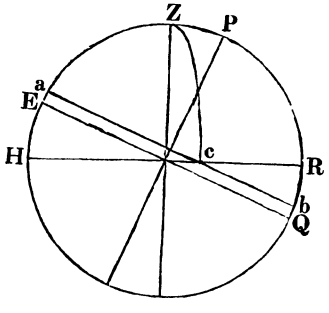
P S being known P γ = 90°, $\angle$ S P γ = R A,

$\therefore$ in the $\triangle$ S P γ, $\angle$ S γ P is known.

$\therefore$ obliquity = 90° — S γ P is known.

Prob. 22. In what latitude does the twilight last all night? Declination given.

(Twilight begins when the sun is 18° below the horizon in his ascent, and ends when he is there in his descent, lasting in each case as long as he is in travelling 18°.)



R Q = H E = colat. = b Q + b R

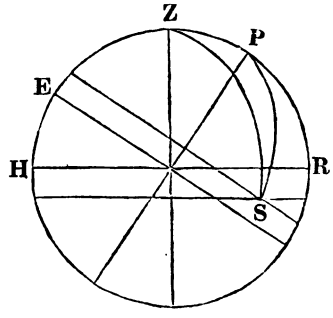
= D + 18°.

$\therefore$ 90° — 18 — D = L

= 72° — D.

(See Prob. 16.)

Find the general equation for the hour at which the twilight begins.



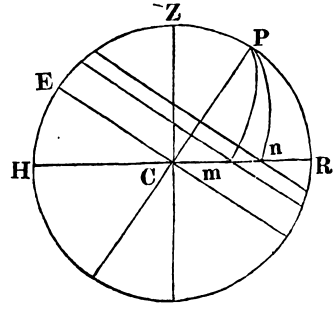
Let the sides P Z, P S, Z S, be a b c.

$$\text{Then } \sin.^2\frac{C}{2} = \frac{\sin.\left(\frac{a+b+c}{2} - a\right)\sin.\left(\frac{a+b+c}{2} - b\right)}{\sin. a . \sin. b}.$$

or

$$\sin.^2\frac{h}{2} = \left.\begin{array}{l}\sin.\left(\frac{\text{colat.} + p + 108^0}{2} - \text{colat.}\right) \\ \times \left(\frac{\text{sin. cotan.} + p + 108^0}{2} - p\right)\end{array}\right\} \cdot$$
$$\cos. L . \sin. p$$

PROB. 24. Given the difference between the times of rising of the stars, and their declinations: required the latitude of the place.



Given P m, P n, and the $\angle$ m P n included.

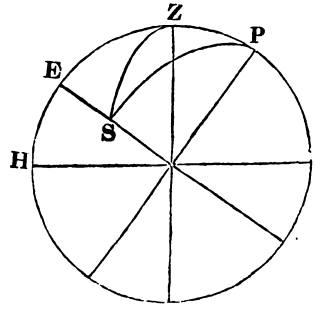
From Napier's first and second analogies, the $\angle$ P m n is known,

$\therefore$ P m C = complement of P m n is known,

$\therefore$ P C = 90^0, P m is given, and the $\angle$ P m C is found,

$\therefore$ P R = latitude is known.

PROB. 25. Given the sun in the equator, also latitude and altitude: find the time.



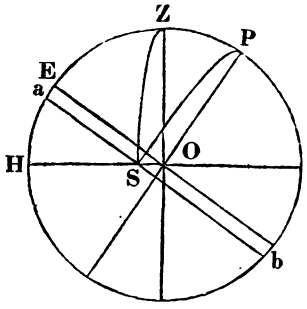
Given

Z P, Z S, P S = 90^0 find the $\angle$ Z P S.

PROB. 26. The sun's declination = 8^0 south, required the latitude, when he rises in the south-east point of the horizon, and also the time of rising.

$P S = 90^0 + 8^0$, $Z S = 90^0$, $\angle S Z P = 45^0 + 90^0$.

Find Z P, and the $\angle$ Z P S.

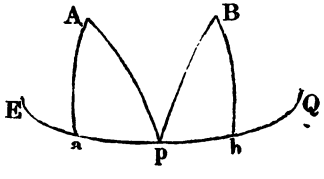


PROB. 27. Determine a point in E Q, that the sum of the arcs drawn from it to two given places on the earth's surface shall be minimum.

Let A, B, be the spectator's situations, whereof the latitude and longitude are known.

Let E Q be the equator, p the point required; a b = difference of the longitudes is known. Let a p = x.

$\therefore$ p b = a — x. Let L, L′ be the latitudes.



In $\triangle$ A a p, r. cos. A p = cos. L′. cos. x.

In $\triangle$ B b p, r. cos. B p = cos. L′. cos. $\overline{a - x}$,

$\therefore$ cos. L . cos. x + cos. L′. cos. (a—x) = max.

$\therefore$ cos. L . (— sin. x) . d x + cos. L′. $\times$ sin. (a — x). (— d x) = 0,

$\therefore$ — cos. L . sin. x = cos. L′. sin. a. cos. x — cos. L′. cos. a. sin. x.

Let sin. x = y

$\therefore$ — cos. L . y = cos. L′. sin. a. $\sqrt{1 - y^2}$ — cos. L′. cos. a. y

$\therefore$ transposing and squaring

$\cos.^2 L.\ y^2 - 2.\ \cos.\ L.\ \cos.\ L'.\ \cos.^2 y^2 + \cos.^2 L'.\ \cos.^2 a.\ y^2$
$= \cos.^2 L'.\ \sin.^2 a - \cos.^2 L'.\ \sin.^2 a\ y^2$,

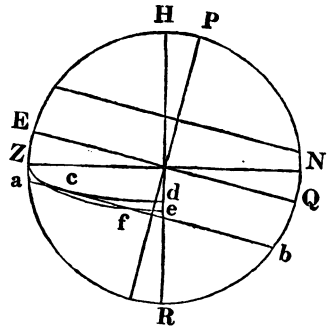
$\therefore y^2 = \&c. = n.$ and $y = \sqrt{n}$.

PROB. 28. To a spectator situated within the tropics, the sun's azimuth will admit of a maximum twice every day, from the time of his leaving the solstice till his declination equal the latitude of the place. Required proof.

a b the parallel of declination passing through capricorn.

From Z a circle may be drawn touching the parallel of the declination till this parallel coincides with Z. ∴ every day till that time the sun will have a maximum azimuth twice a day, and at that time he will have it only once at Z.

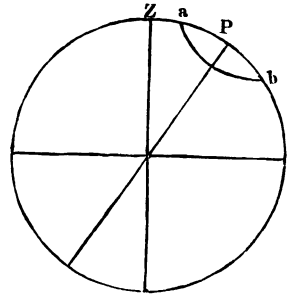
(Also the sun will have the same azimuth twice a day, i. e. he will be twice at f.)



PROB. 29. The true zenith distance of the polar star when it first passes the meridian is equal to m, and at the second passage is equal to n. Required the latitude.

Given b Z = m, a Z = n,

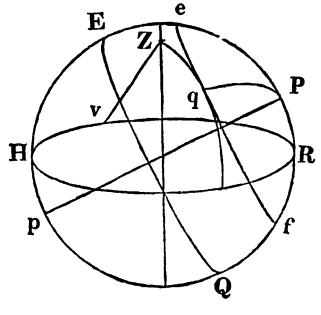
$$Z\,P = \text{colat.} = \tfrac{1}{2}.\,\overline{m + n}.$$



PROB. 30. If the sun's declination E e, is greater than E Z, draw the circle Z m touching the parallel of the declination,

∴ R m is the greatest azimuth that day

If Z v be a straight line drawn perpendicular to the horizon, the shadow of this line being always opposite the sun, will, in the morning as the sun rises from f, recede from the south point H, till the sun reaches his greatest azimuth, and then will approach H; also twice in the day the shadow will be upon every particular point, because the sun has the same azimuth twice a day, in this situation. ∴ shadow will go backwards upon the horizon.



But if we consider P p a straight line or the earth's axis produced, the sun will revolve about it, ∴ the shadow will not go backwards.

$$r.\ \text{cot.}\ Z\,P\,q = \text{tan.}\ P\,q.\ \text{cot.}\ P\,Z,$$

or

cot. (time of the greatest azimuth) = tan. p. tan. L.

All the bodies in our system are elevated by refraction 33′, and depressed by parallax.

$\therefore$ at their rise they will be distant from Z, $90^{\circ} + 33' -$ horizontal parallax.

A fix d star has no parallax, $\therefore$ distance from Z $= 90^{\circ} + 33'$.

PROB. 31. Given two altitudes and the time between them, and the declination. Find the latitude of the place.

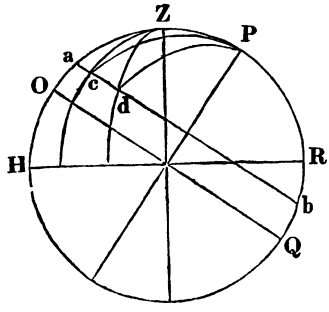
Given Z c, Z d, P c, P d, $\angle$ c P d.

From $\triangle$ c P d, find c d, and $\angle$ P d c.

From $\triangle$ Z c d, find $\angle$ Z d c,

$\therefore$ Z d p $=$ c d P $-$ c d Z,

$\therefore$ From $\triangle$ Z P d, find Z P $=$ colat.



PROB. 32. To find the time in which the sun passes the meridian or the horizontal wire of a telescope.

Let m n equal the diameter of the sun equal d″ in space.

V v : m n :: r : cosine declination,

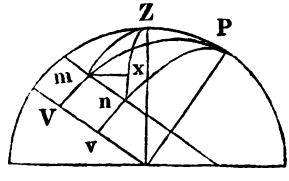
$$\therefore \text{V v} = \frac{\text{m n}}{\text{cosine declination}} \text{ radius 1,}$$

$=$ d″. second declination in seconds of space,

$\therefore$ 15″ in space : 1″ in time

:: d″ second dec. : $\frac{\text{d″ second dec.}}{15''}$

$=$ time in seconds of passing the merid.



Hence the sun's diameter in R A $=$ V v $=$ d″. second declination.

(n x $=$ d″ $=$ sun's diameter)

V v : m n : : r : sin. P n

m n : n x : : r : sin. x n P

V v : n x : : r^2 : sin. P n . sin. Z n P,

$$\therefore \text{V v} = \frac{r^2 . \text{n x}}{\text{sin. P n . sin. Z n P}} = \frac{r^2 \text{ n x}}{\text{sin. Z P . sin. P Z n}}$$

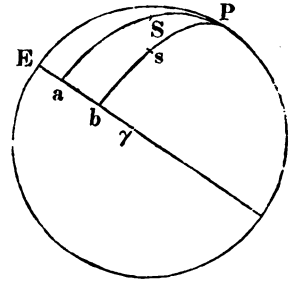
$$= \frac{r^2 . d''}{\text{cos. } \lambda . \text{ sin. azimuth}}$$

$$\therefore \text{time of describing V v} = \frac{r^2 . d''}{15'' . \text{ cos. } \lambda . \text{ sin. azimuth}}$$

which also gives the time of the sun's rising above the horizon.

PROB. 33. *Flamstead's method of determining the right ascension of a star.*

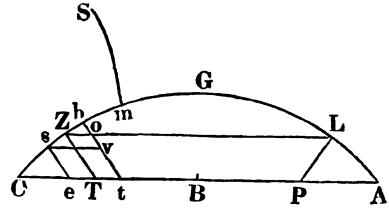
LEMMA. The right ascension of stars passing the meridian at different times, differs as the difference of the times of their passing.



For the angle a P b measures the difference of the times of passing, which is measured by a b = a γ — b γ.

Hence, as the interval of the times of the succeeding passages of any fixed star : 360 (the difference of its right ascensions between those times) : : the interval between the passages of any two fixed stars : to the difference of their right ascensions.

Let A G c be the equator, A B C the ecliptic, S the place of a star, S m a secondary to the equator. Let the sun be near the equinox at P, when on the meridian.



Take C T = P A, ∴ the sun's declination at T = that at P. Draw P L, T Z, perpendicular to A G c.

∴ Z L parallel to A C.

Observe the meridian altitude of the sun at P, and the time of the passage of his center over the meridian.

Observe what time the star passes over the meridian, thence find the apparent difference of their right ascensions.

When the sun approaches T, observe his meridian altitude on one day, when he is close to T, and the next day when he has passed through T, so that at t it may be greater, and at e less than the meridian altitude at P. Draw t b, and e s, perpendiculars.

Observe on the two days before mentioned, the differences b m, s m, of the sun's right ascension, and that of the star.

Draw s v parallel to A C.

Considering the variation of the right ascension and declination to be uniform for a short time, v b (change of the meridian altitudes in one day) : o b difference of the declinations) : : s b (= s m — b m) : Z b. Whence Z b. Add or substract Z b to or from T m. Whence Z m. Add, or take the

difference of, (according to circumstances), Z m, L m, whence Z L, $\therefore \frac{180 - Z L}{2}$ gives A L, the sun's right ascension at the time of the first observation.

$\therefore$ A L + L m = the star's right ascension. Whence the right ascension of all the stars.

PROB. 34. Given the altitudes of two known stars. Find λ.

Right ascensions being known, $\therefore$ a b = the difference of right ascensions, is known,

$\therefore \angle$ a P b is known.

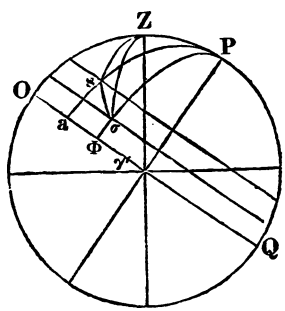
$\therefore$ From $\triangle$ s P σ, $\angle$ s σ P is known,

and σ s,

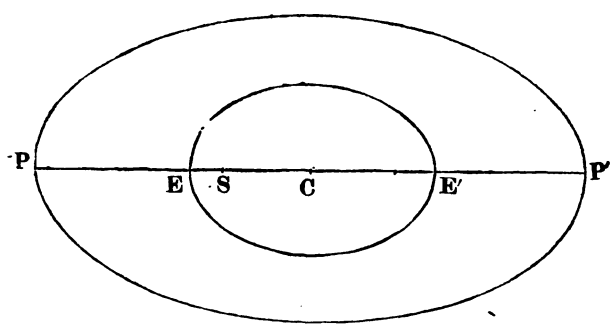
From $\triangle$ Z s σ, $\angle$ s σ Z is known,

$\therefore \angle$ Z σ P is known,

$\therefore$ from $\triangle$ Z σ P, Z P is known.



PROB. 35. Given the apparent diameter of a planet, at the nearest and most distant points of the earth's orbit. Required the radius of the planet's orbit.



D $\propto \frac{1}{\text{distance}}$; D greatest, D' nearest diameter.

$\therefore$ D : D' :: E P : E' P

:: E P — E C : C P + E C,

$\therefore$ D C P + D E C = D' C P — D' E C,

$\therefore$ C P = E C $\frac{D + D'}{D' - D}$.

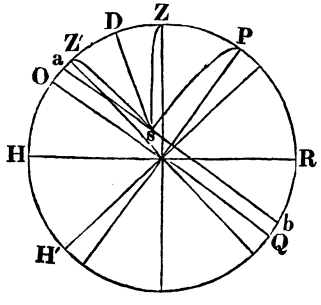
PROB. 36. Given the sun's greatest apparent diameter, and least, as 101 and 100. Find the excentricity of the earth's orbit.

D $\propto \frac{1}{\text{rad.}}$; the sun at S, P P′ the earth's orbit.

$$100 : 101 :: S P : S P' :: C P - C S : C P + C S$$
$$\therefore 100\ C P + 100\ C S = 101\ C P - 101\ C S$$
$$\therefore 201\ C S = C P$$
$$\therefore C S = \frac{C P}{201}, \text{ on the same scale of notation.}$$

PROB. 37. Two places are on the same meridian.

Find the hour on a given day, when the sun will have the same altitude at each place.



Z Z′, two zeniths of places, ∴ Z Z′ is known, S the place of the sun in the parallel a b, Z S = S Z′.

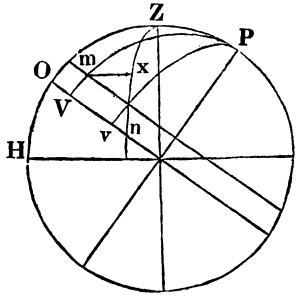
From S draw perpendicular S D,

$$\therefore Z D = Z' D,$$
$$\therefore P Z + \frac{Z Z'}{2} = P D, \text{ is known,}$$

P S is known, ∠ S D P right ∠,

∴ ∠ D P S = hour is known.

PROB. 38. Find the time in which the sun passes the vertical wire of a telescope.



Meridian = the vertical wire,

∴ the time of passing the meridian = the time of passing the vertical wire.

Take m n = the sun's diameter = d.

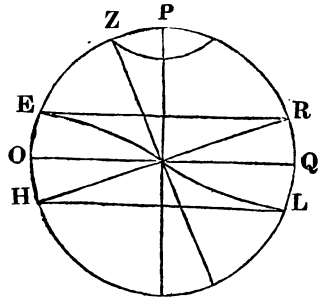
V v : m n : r : cos. declination,

$$\therefore V v = \frac{d\ r}{\text{cos. dec.}},$$

∴ V v converted into the time at the rate of 15′ for 1° = the time required.

PROB. 40. If a man be in the arctic circle, the longest day = 24 hours, the shortest = 0.

P Z = obliquity = Q R,
$\therefore$ Z R = P Q = 90
Z H = P Q = 90^0
$\therefore$ H R is the horizon, and the nearest parallel touches at R,
$\therefore$ the day = 24 hours, and the farthest parallel touches at H,
$\therefore$ the day = 0 hours.

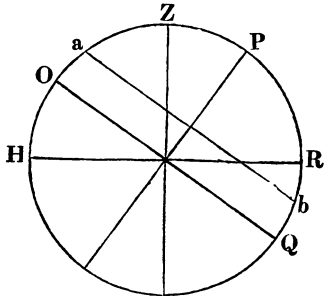


PROB. 41. Given the sun's meridian altitude = 62^0, midnight depression = 22^0. Find the longitude and declination.

Q a = b Q
or H a — H Q = R Q — R b
= H Q — R b,

$$\therefore \frac{H\,a + R\,b}{2} = H\,Q = \cos.\,\lambda$$

$= 42^0$, $\therefore \lambda = 48^0$,
$\therefore$ D = 62 — 42 = 20.



PROB. 42. Given the sun's declination, apparent diameter, altitude, and longitude. Find the time of passing the horizontal wire of a telescope.

s = the place of the sun.

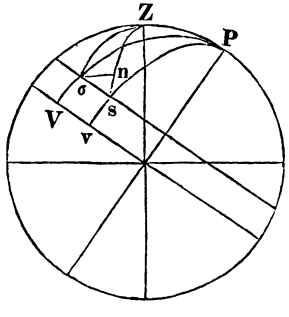
Take s n in a vertical circle = the sun's diameter = d.

Draw n σ parallel to the horizon,

V v : σ s : : r : cos. dec.
σ s : n s : : r : sin. n s P,
$\therefore$ V v : d : : r^2 : sin. P s sin. n s P
: : r^2 : sin. Z P sin. P Z s,

$$\therefore V\,v = \frac{d\,r^2}{\text{sin. cos. } \lambda \text{ sin. azimuth,}}$$

converted into the time, gives the time required



PROB. 43. Given the longitude, right ascension, and declination of two stars; find the time when both are on the same azimuthal circle, and also of the azimuth.

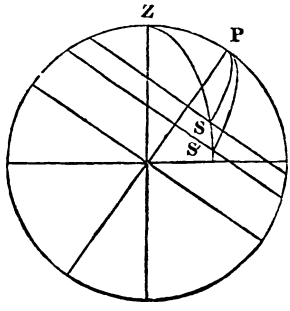
Given P S, P S′, and ∠ S P S′ = difference of right ascension.

∴ ∠ P S S′ is known,

∴ ∠ P S Z is known,

and Z P given, and P S given,

∴ ∠ P Z S, is known = azimuth, and Z P S = time for the first star, or (Z P S + S P S′) = time for the second star.



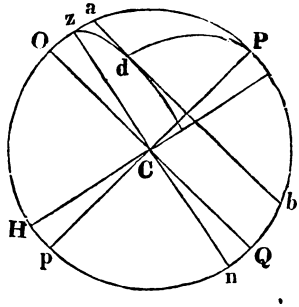
PROB. 44. Given the longitude and declination. Find the time when the sun ascends perpendicular to H R.

D must be greater than λ, or a Q greater than Z Q.

Draw the vertical circle tangent to the parallel of declination, at d.

P Z given, P d given, ∠ P d Z is a right ∠,

∴ ∠ Z P d is known.



PROB. 45. Find the length of the longest day in longitude = 45°.

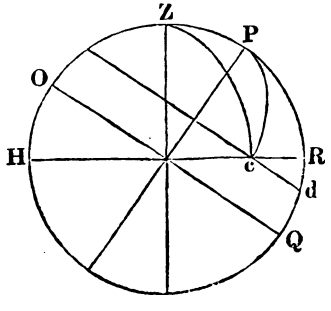
Q d = obliquity,

∴ P d = 90 — obliquity = P c,

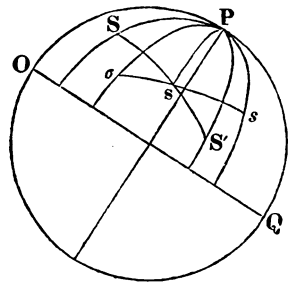
Z P = 45,

Z c = 90,

∴ 2 hours is known.



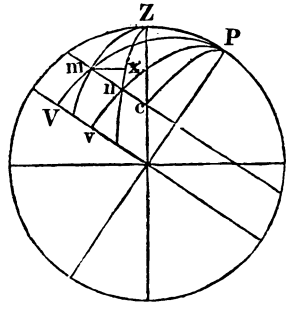
PROB. 46. Find the right ascension and declination of a star, when in a line with two known stars, and also in another line with two other known stars.



The star is in the same line with S, S', and in the same line with s, σ,

$\therefore$ in the intersection s

PROB. 47. The least error in the time due to the given error in altitude $= b''$. Find the longitude.



n x is the given error in altitude,

V v : m n : : r : cos. declination,

m n : n x : : r : sin. x n P.

V v : n x : : r^2 : sin. P n sin. Z n P,

$$V\,v = \frac{n\,x\,r^2}{\sin.\,P\,n\,\sin.\,Z\,n\,P}$$

$$= \frac{n\,x\,r^2}{\sin.\,Z\,P\,\sin.\,P\,Z\,n}$$

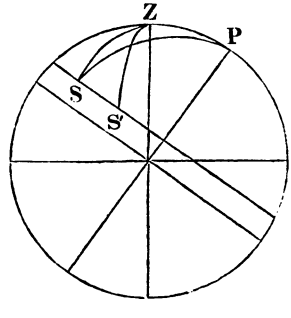
$$= \frac{n\,x\,r^2}{\cos.\,\lambda\,\sin.\,\text{azimuth}},$$

$\therefore$ V v is least when the sin. azimuth is greatest, or the azimuth $= 90^0$, i. e. the prime vertical.

$$\therefore b = \frac{n\,x\,r^2}{\cos.\,\lambda},$$

$$\therefore \cos.\,\lambda = \frac{n\,x\,r^2}{b}.$$

PROB. 48. Given two altitudes, and two azimuths of the sun. Find the longitude.



Z S is known, Z S' is known, $\angle$ S Z S' = difference of the azimuth,

$\therefore \angle$ Z S S' is known,

$\therefore \angle$ Z S P = Z S S' $- 90^0$ is known,

$\therefore$ Z S P, Z S, S Z P, known,

find Z P.

PROB. 49. Near the solstice, the declination $\propto$ longitude, nearly.

$$r \sin. D = \sin. L \sin. \gamma,$$

$$\therefore r\, d(D) \cos. D = \sin. \gamma\, d(L) \cos. L$$

$$\therefore r \frac{d(D)}{d(L)} = \frac{\sin. \gamma \cos. L}{\cos. D}$$

$$= \frac{\sin. \gamma \cos. 90 - d(L)}{\cos. \gamma}, \text{ since D}$$

may be considered the measure of γ,

$$= \tan. \gamma \sin. d(L)$$

$$= \tan. \gamma\, d(L), \text{ since d (L) small,}$$

$$\therefore \frac{d(D)}{d(L)^2} = -\frac{\tan. \gamma}{r} = \text{constant quantity,}$$

$$\therefore d(D) \propto d(L) \text{ nearly.}$$

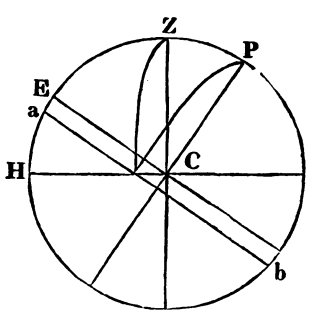
PROB. 50. Given the apparent time T of the revolution of a spot on the sun's surface, find the real time.

Considering the spot as the inferior planet in inferior conjunction, $T = \frac{P\,p}{P - p}$ where P equals the earth's periodic time, p equals the planet's,

$$\therefore T P - T p = P p,$$

$$\therefore p = \frac{T P}{T + P}.$$

PROB. 51. The sun's declination equal 8 south, find the latitude of the place where he rises in the south east point, and also the time of his rising.



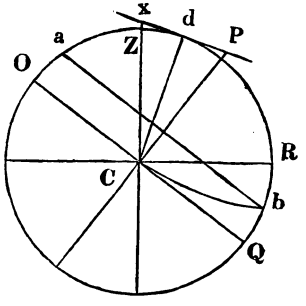
$Z c = 90^0$, $P c = 98^0$, $\angle c Z S = 135^0$,

whence Z P, and $\angle$ h

PROB. 52. How high must a man be raised to see the sun at midnight?

Z P = R Q. Take P d = Q b

$\therefore$ b d = 90°.



Draw x d to the tangent at d,

$\therefore$ if the person be raised to Z x, he will see the sun at b,

$\angle$ d C b = 90° = x C R,

$\therefore$ x C d = R C b measured by R b given.

$\therefore$ in the rectilinear $\triangle$ x d C, $\angle$ x d C = right angle,

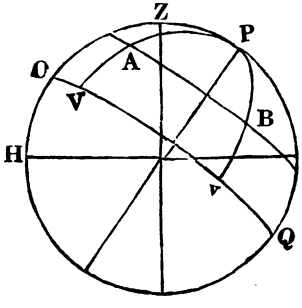
$\angle$ x C d being known from the dec.

C d = radius of the earth.

$\therefore$ C x being known,

$\therefore$ C x — 90°, or Z x is known.

PROB. 53. Given the latitudes and longitudes of two places, find the straight line which joins them. They lie in the same declination of the circle.

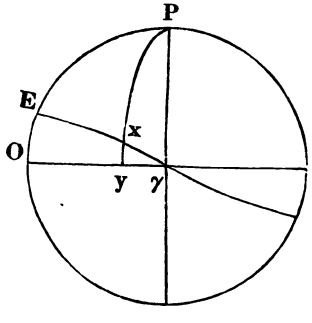


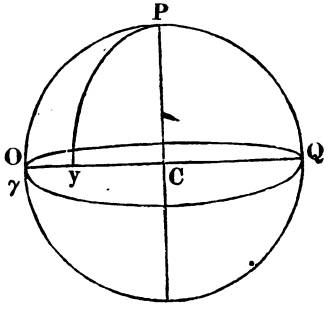
V v : A B : : 1 : cosine declination,

$\therefore$ A B is known,

and the straight line joining A, B, is the chord of A, B, = 2 sin. $\frac{A\,B}{2}$.

PROB. 54. A clock being properly adjusted to keep the sidereal time, required to find when γ is on the meridian.





Observe the sun's center on the meridian, when the declination = x y, is known,

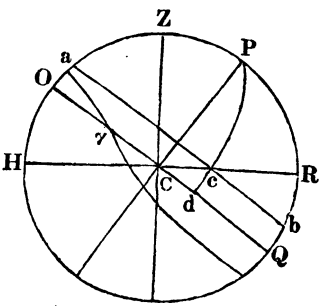
$\angle$ x y γ = right angle
x γ y = I, being known,
x y is known.

Whence y γ = time from noon to γ being on the meridian, or from γ being on the meridian to noon, whence two values of γ y are found.

If the declination north and before solstice the $>$ value gives the time,
——— after——— $<$ ———
If the declination south and before———12+$<$ ———
——— after———12+$>$ ———

(Thelwall.)

Prob. 55. Given the sun's declination and longitude, find his right ascension, his oblique ascension, his azimuth and amplitude, and the time of his rising, and the length of the day.



γ C is given, from $\triangle$ c C d, c d is given; I. and right angle, find c d.

$\therefore$ C γ = R A, C d = oblique asc^{n}.
and C d measures $\angle$ C P c,
$\therefore$ 90^0 + C P c = time of rising,
2 (90^0 + C P c) = length of the day.

NOTES.

To show that (see p. 16.)

$$\Sigma.\mu \times \Sigma.\mu \frac{x\,dy - y\,dx}{dt} + \Sigma.\mu y \times \Sigma.\mu \frac{dx}{dt} - \Sigma.\mu x \times \Sigma \mu \frac{dy}{dt}$$
$$= \Sigma.\mu\mu' \left\{ \frac{(x' - x)(dy' - dy) - (y' - y)(dx' - dx)}{dt} \right\}.$$

Not considering the common factor $\frac{1}{dt}$, we have

$$\Sigma.\mu \times \Sigma.\mu\,(x\,dy - y\,dx)$$
$$= (\mu + \mu' + \mu'' + \ldots)\,\{\mu\,(x\,dy - y\,dx) + \mu'\,(x'\,dy' - y'\,dx') + \mu''\,(x''\,dy'' - y''\,dx'') + \&c.\}$$
$$= \mu^2\,(x\,dy - y\,dx) + \mu'^2\,(x'\,dy' - y'\,dx') + \mu''^2\,(x''\,dy'' - y''\,dx'') + \&c.$$
$$+ \mu\mu'\,(x\,dy - y\,dx + x'\,dy' - y'\,dx') + \mu\mu''\,(x\,dy - y\,dx + x''\,dy'' - y''\,dx'') + \&c.$$
$$+ \mu'\mu''\,(x'\,dy' - y'\,dx') + (x''\,dy'' - y''\,dx'') + \mu'\mu'''\,(x'\,dy' - y'\,dx' + x'''\,dy''' - y'''\,dx''') + \&c$$
$$+ \mu''\mu'''\,(x''\,dy'' - y''\,dx'' + x'''\,dy''' - y'''\,dx''') + \&c.$$
$$\&c.$$

the law of which is evident

Again,

$$\Sigma.\mu y \times \Sigma.\mu\,dx - \Sigma.\mu x \times \Sigma.\mu\,dy$$
$$= (\mu y + \mu' y' + \mu'' y'' + \ldots.)\,(\mu\,dx + \mu'\,dx' + \mu''\,dx'' + \ldots. \&c.)$$
$$- (\mu x + \mu' x' + \mu'' x'' + \ldots.)\,(\mu\,dy + \mu'\,dy' + \mu''\,dy'' + \ldots..)$$
$$= -\mu^2\,(x\,dy - y\,dx) - \mu'^2\,(x'\,dy' - y'\,dx') - \&c.$$
$$+ \mu\mu'\,(y\,dx' - x\,dy' + y'\,dx - x'\,dy) + \mu\mu''\,(y\,dx'' - x\,dy'' + y''\,dx - x''\,dy) + \&c.$$
$$+ \mu'\mu''\,(y'\,dx'' - x'\,dy'' + y''\,dx' - x''\,dy') + \mu'\mu'''\,(y'\,dx''' - x'\,dy''' + y'''\,dx' - x'''\,dy') + \&c.$$
$$+ \&c.$$

Hence by adding together these results the aggregate is

$$\mu\mu'(x\,dy - y\,dx + x'\,dy' - y'\,dx' + y\,dx' - x\,dy' + y'\,dx - x'\,dy) + \mu\mu''(x\,dy - y\,dx + \&c.) + \&c.$$
$$\mu'\mu''(x'\,dy' - y'\,dx' + x''\,dy'' - y''\,dx'' + y'\,dx'' - x'\,dy'' + y''\,dx' - x''\,dy') + \&c.$$
$$\&c.$$

But

$$x\,dy - y\,dx + x'\,dy' - y'\,dx' + y\,dx' - x\,dy' + y'\,dx - x'\,dy$$
$$= dy\,(x - x') + dx\,(y' - y) + dy'\,(x' - x) + dx'\,(y - y')$$
$$= (x' - x)\,dy' - dy) - (y' - y)\,(dx' - dx);$$

and in like manner the coefficients of $\mu\mu''$, $\mu\mu'''$ ..... $\mu'\mu''$, $\mu'\mu'''$, ... &c. are found to be respectively

$$(x'' - x)\,(dy'' - dy) - (y'' - y)\,(dx'' - dx),$$
$$(x''' - x)\,(dy''' - dy) - (y''' - y)\,(dx''' - dx),$$

$$(x'' - x')\,dy'' - dy') - (y'' - y')\,(dx'' - dx'),$$
$$(x''' - x')\,(dy''' - dy') - (y''' - y')\,(dx''' - dx')$$
$$\&c.$$

Hence then the sum of all the terms in $\mu\mu'$, $\mu\mu''$ ..... $\mu'\mu''$, $\mu'\mu''$ ..... $\mu''\mu'''$, $\mu'\mu''''$ ..... is briefly expressed by

$$\Sigma\,.\,\mu\mu'\{(x' - x)\,(dy' - dy) - (y' - y)\,(dx' - dx)\}$$

and the suppressed coefficient $\frac{1}{dt}$ being restored, the only difficulty of p. 16 will be fully explained.

That $\Sigma\,.\left(\frac{d\lambda}{dx}\right) = 0$, &c. has been shown.

2. To show that $\int(2\,\Sigma\,.\,\mu\,dx \times \Sigma\,.\,\mu\,d^2x) = (\Sigma\,.\,\mu\,dx)^2$..... page 17.

$$\Sigma\,.\,\mu\,d^2x = \mu\,d^2x + \mu'\,d^2x' + \&c.$$
$$= d\,.\,\mu\,dx + d\,.\,\mu'\,dx' + \&c.$$
$$= d\,(\mu\,dx + \mu'\,dx' + \&c.)$$
$$= d\,.\,\Sigma\,.\,\mu\,dx.$$

Hence

$$\int(2\,\Sigma\,.\,\mu\,dx \times \Sigma\,.\,\mu\,d^2x) = \int 2\,.\,\Sigma\mu\,dx \times d.\Sigma\,.\,\mu\,dx$$
$$= (\Sigma\,.\,\mu\,dx)^2$$

being of the form $\int 2\,n\,du = u^2$.

3. To show that (page 17).

$$\Sigma.\mu \times \Sigma.\mu\,(dx^2 + dy^2 + dz^2) - \{(\Sigma.\mu\,dx)^2 + (\Sigma.\mu\,dy)^2 + (\Sigma.\mu\,dz)^2\} = \Sigma.\mu\mu'\,\{(dx' - dx)^2 + (dy' - dy)^2 + (dz' - dz)^2\}.$$

Since the quantities are similarly involved, for brevity, let us find the value of $\Sigma.\mu \times \Sigma.\mu\,dx^2 - (\Sigma.\mu\,dx)^2$.

$$\text{It} = (\mu + \mu' + \mu'' + \ldots.)(\mu\,dx^2 + \mu'\,dx'^2 + \mu''\,dx''^2 + \ldots.) - (\mu\,dx + \mu'\,dx' + \mu''\,dx'' + \ldots.)^2;$$

Consequently when the expression is developed, the terms $\mu^2\,dx^2$, $\mu'^2\,dx'^2$, $\mu''^2\,dx''^2$, &c. will be destroyed, and the remaining ones will be

$$\mu\mu'\,(dx^2 + dx'^2 - 2\,dx\,dx') = \mu\mu'\,(dx' - dx)^2$$
$$\mu\mu''\,(dx^2 + dx''^2 - 2\,dx\,dx'') = \mu\mu''\,(dx'' - dx)^2$$

$$\mu'\mu''\,(dx'^2 + dx''^2 - 2\,dx'\,dx'') = \mu'\mu''\,(dx'' - dx')^2$$
$$\mu'\mu'''\,(dx'^2 + dx'''^2 - 2\,dx'\,dx''') = \mu'\mu'''\,(dx''' - dx')^2$$

$$\mu''\mu'''\,(dx''^2 + dx'''^2 - 2\,dx''\,dx''') = \mu''\mu'''\,(dx''' - dx'')^2$$

&c.

Hence, of the partial expression

$$\Sigma.\mu \times \Sigma.\mu\,dx^2 - (\Sigma.\mu\,dx)^2 = \Sigma.\mu\mu'\,(dx' - dx)^2.$$

In like manner

$$\Sigma.\mu \times \Sigma.\mu\,dy^2 - (\Sigma.\mu\,dy)^2 = \Sigma.\mu\mu'\,(dy' - dy)^2$$
$$\Sigma.\mu \times \Sigma.\mu\,dz^2 - (\Sigma.\mu\,dz)^2 = \Sigma.\mu\mu'\,(dz' - dz)^2$$

and the aggregate of these three, whose first members amount to the proposed form, is

$$\Sigma.\mu\mu'\,\{(dx' - dx)^2 + (dy' - dy)^2 + (dz' - dz)^2\}$$

4. To show that (p. 19.)

$$\frac{\Sigma.\frac{\mu x}{\rho^3}}{\Sigma.\mu} = \frac{3x'}{(\rho')^3}$$

nearly.

It is shown already in page 19 that

$$\frac{x}{\varrho^3} = \frac{x}{(\varrho')^3} - \frac{3x'}{(\varrho')^5}(x'x_, + y'y_, + z'z_,).$$

But since $x_, = x - x'$, $y_, = y - y'$, $z_, = z - z'$, by substitution and multiplying both members by μ, we get

$$\frac{\mu x}{\varrho^3} = \frac{\mu x}{(\varrho')^3} - \frac{3x'}{(\varrho')^5}(x' \cdot \mu x + y' \cdot \mu y + z' \cdot \mu z) + \frac{3x'}{(\varrho')^5} \cdot \mu$$

nearly.

Similarly

$$\frac{\mu' x'}{\varrho'^3} = \frac{\mu' x'}{(\varrho')^3} - \frac{3x'}{(\varrho')^5}(x' \cdot \mu' x' + y' \cdot \mu' y' + z' \cdot \mu' z') + \frac{3x'}{(\varrho')^5} \cdot \mu'$$

nearly.

&c.

Hence

$$\Sigma \cdot \frac{\mu x}{\varrho^3} = \frac{\Sigma \cdot \mu x}{(\varrho')^3} - \frac{3x'}{(\varrho')^5}(x' \Sigma \cdot \mu x + y' \Sigma \cdot \mu y + z' \Sigma \cdot \mu z) + \frac{3x'}{(\varrho')^5} \Sigma \mu.$$

But by the property of the center of gravity,

$$\Sigma \cdot \mu x = 0, \quad \Sigma \cdot \mu y = 0, \quad \Sigma \cdot \mu z = 0.$$

Hence

$$\Sigma \cdot \frac{\mu x}{\varrho^3} = \frac{3x'}{(\varrho')^5} \times \Sigma \mu \text{ nearly.}$$

5. To show that (p. 22.)

$$\frac{x}{\varrho} d^2 x + \frac{y}{\varrho} d^2 y + \frac{z}{\varrho} d^2 z = d^2 \varrho - \varrho \, d v^2 \cos.^2 \theta - \varrho \, d \theta^2$$

and that

$$\frac{x}{\varrho}\left(\frac{dQ}{dx}\right) + \frac{y}{\varrho}\left(\frac{dQ}{dy}\right) + \frac{z}{\varrho}\left(\frac{dQ}{dz}\right) = \left(\frac{dQ}{d\varrho}\right).$$

First, we have

$$x\,d^2x + y\,d^2y + z\,d^2z$$
$$= d(x\,dx + y\,dy + z\,dz) - (dx^2 + dy^2 + dz^2).$$

But

$$x^2 + y^2 + z^2 = \varrho^2,$$
$$x\,dx + y\,dy + z\,dz = \varrho\,d\varrho$$

and because

$$x = \varrho \cos. \theta \times \cos. v$$
$$y = \varrho \cos. \theta \times \sin. v$$
$$z = \varrho \sin. \theta.$$

$$\therefore d x^2 + d y^2 = \{d (\varrho \cos. \theta) . \cos. v - \varrho \cos. \theta \times d v \sin. v\}^2 + \{(\varrho \cos. \theta) \sin. v + \dot{\varrho} \cos. \theta \, d v \cos. v\}^2 = (d . \varrho \cos. \theta)^2 + \varrho^2 d v^2 \cos.^2 \theta,$$

$$\therefore d x^2 + d y^2 + d z^2 = (d . \varrho \sin. \theta)^2 + (d . \varrho \cos. \theta)^2 + \varrho^2 d v^2 \cos.^2 \theta = d \varrho^2 + \varrho^2 d \theta^2 + \varrho^2 d v^2 \cos.^2 \theta.$$

Hence, since also

$$d . \varrho \, d \varrho = d \varrho^2 + \varrho \, d^2 \varrho,$$

$$\frac{x d^2 x + y d^2 y + z d^2 z}{\varrho} = d^2 \varrho - \varrho \, d v^2 \cos.^2 \theta - \varrho \, d \theta^2.$$

Secondly, since ϱ is evidently independent of the angles θ and v, the three equations (1), give us

$$\left(\frac{d x}{d \varrho}\right) = \cos. \theta \cos. v = \frac{x}{\varrho},$$

$$\left(\frac{d y}{d \varrho}\right) = \cos. \theta \sin. v = \frac{y}{\varrho},$$

$$\left(\frac{d z}{d \varrho}\right) = \sin. \theta = \frac{z}{\varrho}$$

Hence

$$\frac{x}{\varrho}\left(\frac{d Q}{d x}\right) + \frac{y}{\varrho}\left(\frac{d Q}{d y}\right) + \frac{z}{\varrho}\left(\frac{d Q}{d z}\right) = \left(\frac{d Q}{d x}\right)\left(\frac{d x}{d \varrho}\right) + \left(\frac{d Q}{d y}\right)\left(\frac{d y}{d \varrho}\right) + \left(\frac{d Q}{d z}\right)\left(\frac{d z}{d \varrho}\right).$$

But since Q is a function of ϱ (observe the equations 1), and ϱ is a function of x, y, z, viz. $\sqrt{x^2 + y^2 + z^2}$,

$$d Q = \left(\frac{d Q}{d \varrho}\right) d \varrho = \left(\frac{d Q}{d \varrho}\right) \times \left\{ d x \left(\frac{d \varrho}{d x}\right) + d y \left(\frac{d \varrho}{d y}\right) + d z \left(\frac{d \varrho}{d z}\right) \right\}$$

$$= d x . \left(\frac{d Q}{d \varrho}\right)\left(\frac{d \varrho}{d x}\right) + d y \left(\frac{d Q}{d \varrho}\right)\left(\frac{d \varrho}{d y}\right) + d z \left(\frac{d Q}{d \varrho}\right)\left(\frac{d \varrho}{d z}\right).$$

But

$$d x \left(\frac{d Q}{d \varrho}\right)\left(\frac{d \varrho}{d x}\right) = d x . \left(\frac{d Q}{d x}\right) = d \varrho \left(\frac{d x}{d \varrho}\right)\left(\frac{d Q}{d x}\right)$$

and like transformations may be effected in the other two terms. Consequently we have

$$d Q = d \varrho \left(\frac{d x}{d \varrho}\right)\left(\frac{d Q}{d x}\right) + d \varrho \left(\frac{d y}{d \varrho}\right)\left(\frac{d Q}{d y}\right) + d \varrho \left(\frac{d z}{d \varrho}\right)\left(\frac{d Q}{d z}\right).$$

Hence and from what was before proved, we get

$$\frac{d^2\varrho}{dt^2} - \frac{\varrho\, dv^2}{dt^2}\cos.^2\theta - \frac{\varrho\, d\theta^2}{dt^2} = \frac{x}{\varrho}\left(\frac{dQ}{dx}\right) + \frac{y}{\varrho}\left(\frac{dQ}{dy}\right) + \frac{y}{\varrho}\left(\frac{dQ}{dz}\right) = \left(\frac{dQ}{d\varrho}\right).$$

6. To show that $x\, d^2 y - y\, d^2 x = d\,(\varrho^2\, d\, v \cos.^2\theta)$, and that $x\left(\frac{dQ}{dy}\right) - y\left(\frac{dQ}{dx}\right) = \left(\frac{dQ}{dv}\right)$, see p. 22.

First, since

$$x\, d^2 y = d.x\, dy - dx\, dy$$
$$y\, d^2 x = d.y\, dx - dx\, dy,$$
$$\therefore x\, d^2 y - y\, d^2 x = d.(x\, dy - y\, dx).$$

But from equations (1), p. 22,

$$dy = \sin. v\, .d\,(\varrho \cos.\theta) + \varrho \cos.\theta\, .\cos. v\, dv$$
$$dx = \sin. v\, .d\,(\varrho \cos.\theta) - \varrho \cos.\theta\, .\sin. v\, dv,$$

$$\left.\begin{aligned} \therefore x\, dy &= \sin. v \cos. v\, .\frac{d\,(\varrho^2 \cos.^2\theta)}{2} + \varrho^2 \cos.^2\theta \cos.^2 v\, dv \\ y\, dx &= \sin. v \cos. v\, .\frac{d\,(\varrho^2 \cos.^2\theta)}{2} - \varrho^2 \cos.^2\theta \sin.^2 v\, dv \end{aligned}\right\}$$

the difference of which is

$$\varrho^2 \cos.^2\theta \times dv.$$

Consequently

$$x\, d^2 y - y\, d^2 x = d.(\varrho^2\, d\, v \cos.^2\theta).$$

Secondly by equations (1) p. 22, we have

$$\left(\frac{dy}{dv}\right) = \varrho \cos.\theta \cos. v = x$$

$$\left(\frac{dx}{dv}\right) = -\varrho \cos.\theta \sin. v = -y,$$

$$\therefore x\left(\frac{dQ}{dy}\right) - y\left(\frac{dQ}{dx}\right) = \left(\frac{dy}{dv}\right)\left(\frac{dQ}{dy}\right) + \left(\frac{dx}{dv}\right)\left(\frac{dQ}{dx}\right).$$

But since dividing the two first of the equations (1) p. 22, we have $\frac{y}{x}$ = tan. v, v is a function of x, y only. Consequently, as in the note preceding this it may be shown that

$$\left(\frac{d\,Q}{d\,v}\right) = \left(\frac{d\,y}{d\,v}\right)\left(\frac{d\,Q}{d\,y}\right) + \left(\frac{d\,x}{d\,v}\right)\left(\frac{d\,Q}{d\,x}\right)$$
$$= x\left(\frac{d\,Q}{d\,y}\right) - y\left(\frac{d\,Q}{d\,x}\right).$$

Hence

$$\frac{d.(\varrho^2 \frac{d\,v}{d\,t}\cos.^2\theta)}{d\,t} = \left(\frac{d\,Q}{d\,v}\right).$$

7. To find the value of $\left(\frac{d\,Q}{d\,\theta}\right)$ in terms of ϱ, v, θ, (see the last line but two of p. 22)

Since θ is a function of x, y, z, we have

$$\left(\frac{d\,Q}{d\,\theta}\right) = \left(\frac{d\,Q}{d\,x}\right)\left(\frac{d\,x}{d\,\theta}\right) + \left(\frac{d\,Q}{d\,y}\right)\left(\frac{d\,y}{d\,\theta}\right) + \left(\frac{d\,Q}{d\,z}\right)\left(\frac{d\,z}{d\,\theta}\right).$$

But from equations (1) p. 22, we get

$$\left(\frac{d\,x}{d\,\theta}\right) = -\varrho \sin.\theta \cos.\,v$$
$$\left(\frac{d\,y}{d\,\theta}\right) = -\varrho \sin.\theta \sin.\,v$$
$$\left(\frac{d\,z}{d\,\theta}\right) = \varrho \cos.\theta.$$

Hence multiplying the values of

$$\left(\frac{d\,Q}{d\,x}\right), \quad \left(\frac{d\,Q}{d\,y}\right), \quad \left(\frac{d\,Q}{d\,z}\right),$$

viz.

$$\frac{d^2 x}{d\,t^2}, \quad \frac{d^2 y}{d\,t^2}, \quad \frac{d^2 z}{d\,t^2} \text{ (see p. 22)},$$

by the partial differences we get

$$\left(\frac{d\,Q}{d\,\theta}\right) = \frac{1}{d\,t^2}\{d^2 z\,\varrho \cos.\theta - d^2 y.\varrho \sin.\theta \sin.\,v - d^2 x\,\varrho \sin.\theta \cos.\,v\}$$

Now the first term gives

$$\varrho \cos.\theta . d^2 z = \varrho\{d^2\varrho \sin.\theta \cos.\theta + 2\,d\varrho\,d\theta \cos.^2\theta + \varrho \cos.^2\theta\, d^2\theta - \varrho\, d\theta^2 \sin.\theta \cos.\theta\},$$

and the two other terms gives when added, by means of the equations (1) p. 22,

$$-\frac{\sin.\theta}{\cos.\theta}(v\,d^2 y + x\,d^2 x) = -\frac{\sin.\theta}{\cos.\theta}\{d\,(y\,d\,y + x\,d\,x) - (d\,x^2 + d\,y^2)\}$$

But

$$\begin{aligned} d\,(y\,dy + x\,dx) &= \tfrac{1}{2}\,d\,.\,(d\,.\,\overline{x^2 + y^2}) = \tfrac{1}{2}\,d^2\,(\varrho^2\cos.^2\theta) \\ &= d\,\{\varrho\cos.\theta\,d\,(\varrho\cos.\theta)\} \\ &= (d\,.\,\varrho\cos.\theta)^2 + \varrho\cos.\theta\,d^2\,(\varrho\cos.\theta) \end{aligned}$$

and

$$dx^2 + dy^2 = (d\,.\,\varrho\cos.\theta)^2 + \varrho^2\cos.^2\theta\,.\,dv^2.$$

Hence

$$\begin{aligned} &-\frac{\sin.\theta}{\cos.\theta}\,(y\,d^2y + x\,d^2x) \\ &= -\frac{\sin.\theta}{\cos.\theta}\,\{\varrho\cos.\theta\,d^2\,(\varrho\cos.\theta) - \varrho^2\cos.^2\theta\,dv^2\} \\ &= -\varrho\sin.\theta\,\{d^2\,(\varrho\cos.\theta) - \varrho\cos.\theta\,dv^2\} \\ &= -\varrho\sin.\theta\,\{d^2\varrho\cos.\theta - 2\,d\varrho\,d\theta\sin.\theta - d^2\theta\,\varrho\sin.\theta \\ &\qquad\qquad - \overline{d\theta^2 + dv^2}\,\varrho\cos.\theta\}. \end{aligned}$$

Adding this value to the preceding one of the first term, we have

$$\begin{aligned} \left(\frac{dQ}{d\theta}\right) &= \frac{\varrho}{dt^2} \times \{\varrho\,d^2\theta + 2\,d\varrho\,d\theta + \varrho\,dv^2\sin.\theta\cos.\theta\} \\ &= \varrho^2\cdot\frac{d^2\theta}{dt^2} + \varrho^2\frac{dv^2}{dt^2}\sin.\theta\cos.\theta + \frac{2\varrho\,d\varrho\,d\theta}{dt^2}, \end{aligned}$$

the value required.

8. To develope $\frac{1}{1 + e\cos.\theta}$ in terms of the cosines of θ and of its multiples, see p. 25.

If c be the member whose hyperbolic logarithm is unity, we know that

$$\cos.\theta = \frac{c^{\theta\sqrt{-1}} + c^{-\theta\sqrt{-1}}}{2}$$

which value of $\cos.\theta$ being substituted in the proposed expression, we have

$$\begin{aligned} \frac{1}{1 + e\cos.\theta} &= \frac{2\,c^{\theta\sqrt{-1}}}{e\,c^{2\theta\sqrt{-1}} + 2\,c^{\theta\sqrt{-1}} + e} \\ &= \frac{2\,c^{\theta\sqrt{-1}}}{e} \times \frac{1}{c^{2\theta\sqrt{-1}} + \frac{2}{e}\,c^{\theta\sqrt{-1}} + 1}. \end{aligned}$$

But since

$$c^{2\theta\sqrt{-1}} + \frac{2}{e}\,c^{\theta\sqrt{-1}} + 1 = 0$$

gives

$$c^{\theta\sqrt{-1}} = -\frac{1+\sqrt{\frac{1}{e^2}-1}}{e} = -\left(\frac{1-\sqrt{1-e^2}}{e+\quad e}\right)$$

and since, if we make

$$\frac{1}{e}(1-\sqrt{1-e^2}) = \lambda \text{ which also } = \frac{e}{1+\sqrt{1-e^2}},$$

we also have

$$\frac{1}{e}(1+\sqrt{1-e^2}) = \frac{1}{\lambda};$$

the expression proposed becomes

$$\frac{1}{1+e\cos.\theta} = \frac{2c^{\theta\sqrt{-1}}}{e} \times \frac{1}{(c^{\theta\sqrt{-1}}+\lambda)\left(c^{\theta\sqrt{-1}}+\frac{1}{\lambda}\right)}$$

$$= \frac{2\lambda}{e} \times \frac{1}{(1+\lambda c^{\theta\sqrt{-1}})(1+\lambda c^{-\theta\sqrt{-1}})}$$

$$= \frac{2\lambda}{e(1-\lambda^2)} \times \left(\frac{1}{1+\lambda c^{\theta\sqrt{-1}}} - \frac{\lambda c^{-\theta\sqrt{-1}}}{1+\lambda c^{-\theta\sqrt{-1}}}\right)$$

But

$$\frac{\lambda}{e} = \frac{1}{1+\sqrt{1-e^2}},$$

and

$$1-\lambda^2 = \frac{2\sqrt{1-e^2}}{1+\sqrt{1-e^2}},$$

$$\therefore \frac{1}{1+e\cos.\theta} = \frac{1}{\sqrt{(1-e^2)}} \times \left(\frac{1}{1+\lambda c^{\theta\sqrt{-1}}} - \frac{\lambda c^{-\theta\sqrt{-1}}}{1+\lambda c^{-\theta\sqrt{-1}}}\right),$$

which when $\theta = v - \varpi$ is the same expression as that in page 25.

Again by division

$$\frac{1}{1+\lambda c^{\theta\sqrt{-1}}} = 1 - \lambda c^{\theta\sqrt{-1}} + \lambda^2 c^{2\theta\sqrt{-1}} - \&c.$$

and

$$\frac{\lambda c^{-\theta\sqrt{-1}}}{1+\lambda c^{-\theta\sqrt{-1}}} = \lambda c^{-\theta\sqrt{-1}} - \lambda^2 c^{-2\theta\sqrt{-1}} + \&c.$$

Taking the latter from the former, we get

$$\frac{\sqrt{1-e^2}}{1+e\cos.\theta} = 1 - \lambda(c^{\theta\sqrt{-1}}+c^{-\theta\sqrt{-1}}) + \lambda^2(c^{2\theta\sqrt{-1}}+c^{-2\theta\sqrt{-1}})$$

$$= 1 - 2\lambda\cos.\theta + 2\lambda^2\cos.2\theta - 2\lambda^3\cos.3\theta + \&c.$$

and substituting for θ the value $v - \varpi$, we get the expression in page 25.

9. To demonstrate the Theorem of page 28.

Let us take the case of three variables x, y, z. Then our system of differential equations is

$$0 = H x + G \frac{d x}{d t} + F \frac{d^2 x}{d t^2};$$

$$0 = H y + G \frac{d y}{d t} + F \frac{d^2 y}{d t^2};$$

$$0 = H z + G \frac{d z}{d t} + F \frac{d^2 z}{d t^2};$$

in which F, G, H, are symmetrical functions of x, y, z; that is such as would not be altered by substituting x for y, and y for x; and so on for the other variables taken in pairs; for instance, functions of this kind

$$\sqrt{x^2 + y^2 + z^2 + t^2}, (x y + x z + x t + y z + y t + z t) \frac{p}{q},$$

$$(x y z + x z t + y z t),$$

log. (x y z t) and so on.

Multiply the first of the equations by the arbitrary α, the second by β, and the third by γ and add them together; the result is

$$0 = H (\alpha x + \beta y + \gamma z) + G \left(\alpha \frac{d x}{d t} + \beta \frac{d y}{d t} + \gamma \frac{d z}{d t}\right) + F \left(\alpha \frac{d^2 x}{d t^2} + \beta \frac{d^2 y}{d t^2} + \gamma \frac{d^2 z}{d t^2}\right).$$

Now since α, β, γ, are arbitrary, we may assume

$$\alpha x + \beta y + \gamma z = 0,$$

which gives

$$\alpha \frac{d x}{d t} + \beta \frac{d y}{d t} + \gamma \frac{d z}{d t} = 0,$$

$$\alpha \frac{d^2 x}{d t^2} + \beta \frac{d^2 y}{d t^2} + \gamma \frac{d^2 z}{d t^2} = 0.$$

and substituting for x, $\frac{d^2 x}{d t}$, $\frac{d^2 x}{d t^2}$, their values hence derived in the first of the proposed equations, we have

$$-\frac{H}{\alpha}(\beta y+\gamma z)-\frac{G}{\alpha}\left(\beta\frac{dy}{dt}+\gamma\frac{dz}{dt}\right)-\frac{F}{\alpha}\left(\beta\frac{dy^2}{dt^2}+\gamma\frac{d^2z}{dt^2}\right)=$$

$$-\frac{\beta}{\alpha}\left(Hy+G\frac{dy}{dt}+F\frac{d^2y}{dt^2}\right)-\frac{\gamma}{\alpha}\left(Hz+G\frac{dz}{dt}+F\frac{d^2z}{dt^2}\right)=$$

$$-\frac{\beta}{\alpha}\times 0-\frac{\gamma}{\alpha}\times 0=0.$$

In the same it will appear that

$$\alpha x+\beta y+\gamma z=0$$

verifies each of the other two equations. It is therefore the integral of each of them, and may be put under the form

$$z=ax+by$$

in valuing only two arbitraries a and b, which are sufficient, two arbitraries only being required to complete the integral of an equation of the second order.

In the equations (0) p. 27.

$$\frac{m}{\varrho^3}=H,\quad G=0\quad\text{and}\quad F=1$$

and ϱ^3 being $=(x^2+y^2+z^2)^{\frac{3}{2}}$ is symmetrical with regard to x, y, z. Hence the theorem here applies and gives for the integral of any of the equations 0

$$z=ax+by,$$

see page 28.

Again, let us now take four variables x, y, z, u; then the theorem proposes the integration of

$$0=Hx+G\frac{dx}{dt}+F\frac{d^2x}{dt^2}$$

$$0=Hy+G\frac{dy}{dt}+F\frac{d^2y}{dt^2}$$

$$0=Hz+G\frac{dz}{dt}+F\frac{d^2z}{dt^2}$$

$$0=Hu+G\frac{du}{dt}+F\frac{d^2u}{dt^2}.$$

Multiplying these by the arbitraries α, β, γ, δ and adding them we get, as before

$$0=H\left(\alpha x+\beta y+\gamma z+\delta u\right)$$
$$+G\left(\alpha\frac{dx}{dt}+\beta\frac{dy}{dt}+\gamma\frac{dz}{dt}+\delta\frac{du}{dt}\right)$$
$$+F\left(\alpha\frac{d^2x}{dt^2}+\beta\frac{d^2y}{dt^2}+\gamma\frac{d^2z}{dt^2}+\delta\frac{d.\,u}{dt^2}\right)$$

Assume

$$\alpha x + \beta y + \gamma z + \delta u = 0.$$

and upon trial it will be found as before, that this equation satisfies each of the four proposed equations, or it is their integrals supposing them to subsist simultaneously. As before, however, there are more arbitraries than are necessary for the integral of each, two only being required.

Hence the integral of each will be of the form

$$x + y + \gamma z + \delta u = 0.$$

This form might have been obtained at once, by adding the two last of the proposed equations multiplied by γ and δ to the two first of them, and assuming the coefficient of $H = 0$, as before.

In the same manner if we have (n) differential equations of the i-th order, the order involving the n variables $x^{(1)}, x^{(2)} \ldots . x^{(n)}$, and of the general form

$$0 = H x^{(s)} + G \frac{d x^{(s)}}{l t} + F \frac{d^2 x^{(s)}}{d t^2} + \ldots . A \frac{d^{i-1} x^{(s)}}{d t^{i-1}} + \frac{d^i x^{(s)}}{d t^i}$$

we shall find by multiplying i of them (for instance the i wherein first $s = 1, 2 \ldots .$ i) by the arbitraries $\alpha^{(1)}, \alpha^{(}, \ldots . \alpha^{(i)}$; adding these results together and their aggregate to the sum of the other equations; and assuming the coefficient of $H = 0$, that

$$\alpha^{(1)} x^{(1)} + \alpha^{(2)} x^{(2)} + \ldots . \alpha^{(i)} x^{(i)} + x^{i+1} + x^{i+2} + \ldots . x^n = 0$$

will satisfy each of the proposed differential equations subsisting simultaneously; and since it has an arbitrary for every integration, it must be the complete integral of any one of them.

This result is the same in substance as that enunciated in the theorem of p. 28, inasmuch as it is obtained by adding together the equations whose first members are $x^{(1)}$, $x^{(2)}$, &c. and making such arrangements as are permitted by a change of the arbitraries. In short if we had multiplied the i last equations instead of the i first by the arbitraries, and added the results to the n — i first equations, our assumption would have been

$$x^{(1)} + x^{(2)} + \ldots . x^{(n-i)} + \alpha^{(1)} x^{(n-i+1)} + \alpha^{(2)} x^{(n-i+2)} + \ldots . \alpha^i x^n = 0 \ldots . (\alpha)$$

which is derived at once by adding together the n — i equations in page 28.

If we wish to obtain these n — i equations from the equation (a), it may be effected by making assumptions of the required form, provided by so doing we do not destroy the arbitrary nature $\alpha^{(1)}, \alpha^{(2)}, \ldots\ldots \alpha^{(i)}$. The necessary assumptions do, however, evidently still leave them arbitrary. Those assumptions are therefore legitimate, and will give the forms of Laplace.

END OF VOLUME SECOND.

A HISTORY OF THE CONCEPTIONS OF LIMITS AND FLUXIONS IN GREAT BRITAIN

FROM

NEWTON TO WOODHOUSE

Copyright in Great Britain under the Act of 1911

The Open Court Series of Classics of Science and Philosophy, No. 5

A HISTORY OF THE CONCEPTIONS OF LIMITS AND FLUXIONS IN GREAT BRITAIN

FROM

NEWTON TO WOODHOUSE

BY

FLORIAN CAJORI, PH.D.

PROFESSOR OF HISTORY OF MATHEMATICS IN THE UNIVERSITY OF CALIFORNIA

With portraits of Berkeley and Maclaurin

CHICAGO AND LONDON

THE OPEN COURT PUBLISHING COMPANY

1919

TABLE OF CONTENTS

PAGE

INTRODUCTION 1

CHAPTER I

NEWTON

Newton's Publications printed before 1734 2
Principia 3
Wallis's *De Algebra Tractatus* 14
Quadratura Curvarum (1704) 17
An Account of the *Commercium Epistolicum* 26
Newton's Correspondence and Manuscripts not in print in 1734 . 29
Remarks 32

CHAPTER II

PRINTED BOOKS AND ARTICLES ON FLUXIONS BEFORE 1734

John Craig, De Moivre, David Gregory, Fatio de Duillier, Cotes, Ditton. Cheyne 37
John Harris, 1702, 1705, 1710 40
Charles Hayes, 1704 41
William Jones, 1706 43
Humphry Ditton, 1706 43
Commercium Epistolicum D. Johannis Collins, 1712 47
Joseph Raphson, 1715 49
Brook Taylor, 1715 50
James Stirling, 1717, 1730 50
Edmund Stone, 1730 50
Remarks 55

CHAPTER III

BERKELEY'S ANALYST (1734); CONTROVERSY WITH JURIN AND WALTON

PAGE

The *Analyst* 57
Jurin's first reply to Berkeley 64
Walton's first reply to Berkeley 69
Berkeley's reply to Jurin and Walton 72
Walton's second reply to Berkeley 78
Jurin's second reply to Berkeley 80
Berkeley's second reply to Walton 85
The second edition of Walton's second reply 87
Remarks 89

CHAPTER IV

JURIN'S CONTROVERSY WITH ROBINS AND PEMBERTON

Robins's *Discourse on Fluxions* 96
Jurin's review of his own letters to Berkeley 101
Robins's rejoinder 106
The debate continued 109
Pemberton enters the debate 129
Debate over Robins's review of treatises by Leonhard Euler, Robert Smith, and James Jurin 139
Remarks 146

CHAPTER V

TEXT-BOOKS IMMEDIATELY FOLLOWING BERKELEY'S ATTACK

John Colson, 1736 149
James Hodgson, 1736 155
Thomas Bayes, 1736 157
John Muller, 1736 162
Anonymous translation of Newton's *Method of Fluxions*, 1737 . 164
James Smith, 1737 165
Thomas Simpson, 1737 169
Benjamin Martin, 1739, 1759 171
An anonymous text, 1741 172
John Rowe, 1741, 1757, 1767 175
Berkeley ten years after 178
Remarks 179

CHAPTER VI

MACLAURIN'S "TREATISE OF FLUXIONS, 1742"

PAGE
Remarks 188

CHAPTER VII

TEXT-BOOKS OF THE MIDDLE OF THE CENTURY

John Stewart, 1745 190
William Emerson, 1743 (?), 1757, 1768 192
Thomas Simpson, 1750 194
Nicholas Saunderson, 1756 197
John Rowning, 1756 198
Israel Lyons, 1758 201
William West, 1762 202
James Wilson, 1761 202
Remarks 206

CHAPTER VIII

ROBERT HEATH AND FRIENDS OF EMERSON IN CONTROVERSY WITH JOHN TURNER AND FRIENDS OF SIMPSON

Robert Heath 207
Main articles in the controversy 209
Ladies' Diary, 1751, 1752 219
Popular impression of the nature of fluxions 222
Remarks 223

CHAPTER IX

ABORTIVE ATTEMPTS AT ARITHMETISATION

John Kirkby, 1748 225
John Petvin, 1750 230
John Landen, 1758 231
James Glenie, 1793 235
Remarks 238

CHAPTER X

LATER BOOKS AND ARTICLES ON FLUXIONS

PAGE

Encyclopædia Britannica, 1771, 1779, 1797 240
Robert Thorp, 1777 241
F. Holliday, 1777 243
Charles Hutton, 1796, 1798 244
S. Vince, 1795, 1805 245
Agnesi—Colson—Hellins, 1801 247
T. Newton, 1805 250
William Dealtry, 1810, 1816 252
New editions 253
Remarks 253

CHAPTER XI

CRITICISMS OF FLUXIONS BY BRITISH WRITERS UNDER THE INFLUENCE OF D'ALEMBERT, LAGRANGE, AND LACROIX

Review of Lagrange's *Fonctions analytiques*, 1799 . . . 255
Review of a memoir of Stockler, 1799 259
Review of Lacroix's *Calcul différentiel*, 1800 260
Review of Carnot's *Réflexions*, 1801 262
Robert Woodhouse, 1803 263
William Hales, 1804 267
Encyclopædia Britannica, 1810 269
Lacroix's *Elementary Treatise*, 1816 271
Remarks 274

CHAPTER XII

MERITS AND DEFECTS OF THE EIGHTEENTH-CENTURY BRITISH FLUXIONAL CONCEPTIONS

Merits 277
Defects 279
Addenda 289
INDEX 294

LIMITS AND FLUXIONS

INTRODUCTION

1. EVERY great epoch in the progress of science is preceded by a period of preparation and prevision. The invention of the differential and integral calculus is said to mark a "crisis" in the history of mathematics. The conceptions brought into action at that great time had been long in preparation. The fluxional idea occurs among the schoolmen—among Galileo, Roberval, Napier, Barrow, and others. The differences or differentials of Leibniz are found in crude form among Cavalieri, Barrow, and others. The undeveloped notion of limits is contained in the ancient method of exhaustion; limits are found in the writings of Gregory St. Vincent and many others. The history of the conceptions which led up to the invention of the calculus is so extensive that a good-sized volume could be written thereon. We shall not yield to the temptation of lingering on this pre-history at this time, but shall proceed at once to the subject-matter of the present monograph.

CHAPTER I

NEWTON

2. IT was in the year 1734 that Bishop Berkeley made his famous attack upon the doctrine of fluxions, which was the starting-point of all philosophical discussion of the new mathematics in England during the eighteenth century. In what follows we quote from the writings of Newton that were printed before 1734 such parts as bear on his conceptions of fluxions, so that the reader may judge for himself what grounds there were for Berkeley's great assault. To assist us in the interpretation of some of these printed passages, we quote also from manuscripts and letters of Newton which at that time were still unprinted. In the next chapter we give an account of the foundations of fluxions as displayed by other writers in books and articles printed in Great Britain before 1734. It is hoped that the material contained in these first two chapters will enable the student to follow closely and critically the debates on fluxions.

From Newton's Publications printed before 1734

I. PRINCIPIA

3. Three editions of the *Principia* were brought out in Newton's lifetime; the first in 1687, the

second in 1713, the third in 1726. We give extracts which bear on the theory of limits and fluxions and indicate the changes in phraseology introduced in the second and third editions. We give also translations into English based on the text of the 1726, or third, edition.

Principia, Book I, Section I, Lemma I

First edition :

4. "Quantitates, ut & quantitatum rationes, quæ ad æqualitatem dato tempore constanter tendunt & eo pacto propius ad invicem accedere possunt quam pro data quavis differentia; fiunt ultimo æquales.

5. "Si negas, sit earum ultima differentia D. Ergo nequeunt propius ad æqualitatem accedere quam pro data differentia D : contra hypothesin."

Second and third editions :

6. "Quantitates, ut & quantitatum rationes, quæ ad æqualitatem tempore quovis finito constanter tendunt, & ante finem temporis illius propius ad invicem accedunt quam pro data quavis differentia, fiunt ultimo æquales.

7. "Si negas, fiant ultimò inæquales, & sit earum, etc." [As in the first edition.]

Translation by Robert Thorp:[1]

8. "Quantities, and the ratios of quantities, which, in any finite time, tend continually to equality; and

[1] *Mathematical Principles of Natural Philosophy, by Sir Isaac Newton, Knight. Translated into English, and illustrated with a Commentary, by* ROBERT THORP, M.A., vol. i, London, 1777.

before the end of that time, approach nearer to each other than by any given difference, become ultimately equal.

"If you deny it, let them be ultimately unequal; and let their ultimate difference be D. Therefore, they cannot approach nearer to equality than by that given difference D. Which is against the supposition."

Principia, Book I, Section I, Lemma II

Translation by Motte:[1]

9. "If in any figure A*ac*E, terminated by the right lines A*a*, AE, and the curve *ac*E, there be inscribed any number of parallelograms A*b*, B*c*, C*d*, etc., comprehended under equal bases AB, BC, CD, etc., and the sides B*b*, C*c*, D*d*, etc., parallel to one side A*a* of the figure; and the parallelograms *a*K*bl*, *b*L*cm*, *c*M*dn*, etc., are completed. Then if the breadth of those parallelograms be supposed to be diminished, and their number be augmented *in infinitum*; I say, that the ultimate ratios which the inscribed figure AK*b*L*c*M*d*D, the circumscribed figure A*albmcndo*E, and curvilinear figure A*abcd*E, will have to one another, are ratios of equality.

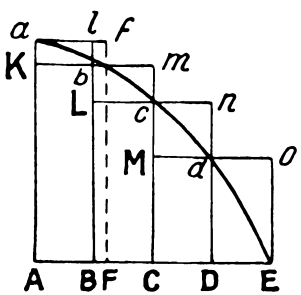


FIG. 1.

[1] *The Mathematical Principles of Natural Philosophy, by Sir Isaac Newton; translated into English by* ANDREW MOTTE, London, 1729. (Two volumes.)

"For the difference of the inscribed and circumscribed figures is the sum of the parallelograms K*l*, L*m*, M*n*, D*o*, that is (from the equality of all their bases), the rectangle under one of their bases K*b* and the sum of their altitudes A*a*, that is, the rectangle AB*la*. But this rectangle, because its breadth AB is supposed diminished *in infinitum*, becomes less than any given space. And therefore (by Lem. I) the figures inscribed and circumscribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. Q.E.D."

Principia, Book I, Section I, Lemma XI, Scholium (first part omitted)

10. ". . . Quæ de curvis lineis deque superficiebus comprehensis demonstrata sunt, facile applicantur ad solidorum superficies curvas & contenta. Præmisi vero hæc Lemmata, ut effugerem tædium deducendi perplexas[1] demonstrationes, more veterum Geometrarum, ad absurdum. Contractiores enim redduntur demonstrationes per methodum Indivisibilium. Sed quoniam durior est Indivisibilium hypothesis, & propterea methodus illa minus Geometrica censetur; malui demonstrationes rerum sequentium ad ultimas quantitatum evanescentium summas & rationes, primasque nascentium, id est, ad limites summarum & rationum deducere; & propterea limitum illorum demonstrationes qua potui brevitate præmittere. His enim idem præstatur

[1] In the third edition "longas" takes the place of "perplexas."

quod per methodum Indivisibilium; & principiis demonstratis jam tutius utemur. Proinde in sequentibus, siquando quantitates tanquam ex particulis constantes consideravero, vel si pro rectis usurpavero lineolas curvas; nolim indivisibilia, sed evanescentia divisibilia, non summas & rationes partium determinatarum, sed summarum & rationum limites semper intelligi; vimque talium demonstrationum ad methodum præcedentium Lemmatum semper revocari.

11. "Objectio est, quod quantitatum evanescentium nulla sit ultima proportio; quippe quæ, antequam evanuerunt, non est ultima, ubi evanuerunt, nulla est. Sed & eodem argumento æque contendi posset nullam esse corporis ad certum locum pergentis [1] velocitatem ultimam. Hanc enim, antequam corpus attingit locum, non esse ultimam, ubi attingit, nullam esse. Et responsio facilis est. Per velocitatem ultimam intelligieam,[2] qua corpus movetur; neque antequam attingit locum ultimum & motus cessat, neque postea, sed tunc cum attingit; id est, illam ipsam velocitatem quacum corpus attingit locum ultimum & quacum motus cessat. Et similiter per ultimam rationem quantitatum evanescentium intelligendam esse rationem quantitatum non antequam evanescunt, non postea, sed quacum evanescunt. Pariter & ratio prima

[1] In the second and third editions "pervenientis" takes the place of "pergentis." In the third edition the sentence reads, ". . . ad certum locum, ubi motus finiatur, pervenientis velocitatem ultimam."

[2] In the second and third editions "intelligi eam" takes the place of "intelligieam."

nascentium est ratio quacum nascuntur. Et summa prima & ultima est quacum esse (vel augeri et[1] minui) incipiunt & cessant. Extat limes quem velocitas in fine motus attingere potest, non autem transgredi. Hæc est velocitas ultima Et par est ratio limitis quantitatum & proportionum omnium incipientium & cessantium. Cumque hic limes sit certus & definitus, problema est vere Geometricum eundem determinare. Geometrica vero omnia in aliis geometricis determinandis ac demonstrandis legitime usurpantur.

12. "Contendi etiam potest, quod si dentur ultimæ quantitatum evanescentium rationes, dabuntur & ultimæ magnitudines; & sic quantitas omnis constabit ex indivisibilibus, contra quam Euclides de incommensurabilibus, in libro decimo Elementorum, demonstravit. Verum hæc objectio falsæ innititur hypothesi. Ultimæ rationes illæ quibuscum quantitates evanescunt, revera non sunt rationes quantitatum ultimarum, sed limites ad quos quantitatum sine limite decrescentium rationes semper appropinquant, & quas propius assequi possunt quam pro data quavis differentia, nunquam vero transgredi, neque prius attingere quam quantitates diminuuntur in infinitum. Res clarius intelligetur in infinite magnis. Si quantitates duæ quarum data est differentia augeantur in infinitum, dabitur harum ultima ratio, nimirum ratio æqualitates nec tamen ideo dabuntur quantitates ultimæ seu maximæ quarum ista est ratio. Igitur in sequentibus,

[1] In the third edition "aut" takes the place of "et."

siquando facili rerum imaginationi[1] consulens, dixero quantitates quam minimas, vel evanescentes vel ultimas; cave intelligas quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite."

Translation by Robert Thorp:

13. "Those things which have been demonstrated of curve lines, and the surfaces which they comprehend, are easily applied to the curve surfaces and contents of solids. But I premised these lemmas to avoid the tediousness of deducing long demonstrations to an absurdity, according to the method of the ancient geometers. For demonstrations are rendered more concise by the method of indivisibles. But, because the hypothesis of indivisibles is somewhat harsh, and therefore that method is esteemed less geometrical, I chose rather to reduce the demonstrations of the following propositions to the prime and ultimate sums and ratios of nascent and evanescent quantities; that is, to the limits of those sums and ratios: and so to premise the demonstrations of those limits, as briefly as I could. For hereby the same thing is performed, as by the method of indivisibles; and those principles being demonstrated, we may now use them with more safety. Therefore, if hereafter I shall happen to consider quantities, as made up of particles, or shall use little curve lines for right ones, I would not be understood to mean indivisible, but evanescent

[1] In the third edition "conceptui" takes the place of "imaginationi."

divisible quantities; not the sums and ratios of determinate parts, but always the limits of sums and ratios: and, that the force of such demonstrations always depends on the method laid down in the preceding lemmas.

14. "It is objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not ultimate; and, when they have vanished, is none. But, by the same argument, it might as well be maintained, that there is no ultimate velocity of a body arriving at a certain place, when its motion is ended: because the velocity, before the body arrives at the place, is not its ultimate velocity; when it has arrived, is none. But the answer is easy: for by the ultimate velocity is meant that, with which the body is moved, neither before it arrives at its last place, when the motion ceases, nor after; but at the very instant when it arrives; that is, that very velocity with which the body arrives at its last place, when the motion ceases. And, in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities, not before they vanish, nor after, but that with which they vanish. In like manner, the first ratio of nascent quantities is that with which they begin to be: and the first or last sum is that, with which they begin and cease to be, or to be augmented or diminished. There is a limit, which the velocity at the end of the motion may attain, but cannot exceed. This is the

ultimate velocity. And there is a like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. But whatever is geometrical we may be allowed to use in determining and demonstrating any other thing that is likewise geometrical.

15. "It may be also argued, that if the ultimate ratios of evanescent quantities are given, their ultimate magnitudes will be also given; and so all quantities will consist of indivisibles, which is contrary to what Euclid has demonstrated concerning incommensurables, in the tenth book of his *Elements*. But this objection is founded on a false supposition, for those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but the limits to which the ratios of quantities, decreasing without end, always converge; and to which they may approach nearer than by any difference, but can never go beyond, nor attain to, unless the quantities are diminished indefinitely. This will appear more evident in quantities indefinitely great. If two quantities, whose difference is given, are augmented continually, their ultimate ratio will be given, to wit, the ratio of equality; but the ultimate or greatest quantities themselves, of which that is the ratio, will not therefore be given. If then in what follows, for the more easy apprehension of things, I shall ever mention quantities *the least possible*, or *evanescent*, or *ultimate*, beware lest you understand quan-

tities of a determinate magnitude; but conceive them to be continually diminished without limit."

Principia, Book II, Section II, Lemma II.

16. " . . . Has quantitates, ut indeterminatas & instabiles, & quasi motu fluxuve perpetuo crescentes vel decrescentes, hic considero; & earum[1] incrementa vel decrementa momentanea sub nomine momentorum intelligo: ita ut incrementa pro momentis addititiis seu affirmativis, ac decrementa pro subductitiis seu negativis habeantur. Cave tamen intellexeris particulas finitas. Momenta, quam primum finitæ sunt magnitudinis, desinunt esse momenta. Finiri enim repugnat aliquatenus perpetuo eorum incremento vel decremento.[2] Intelligenda sunt principia jamjam nascentia finitarum magnitudinum. Neque enim spectatur in hoc lemmate magnitudo momentorum, sed prima nascentium proportio. Eodem recidit si loco momentorum usurpentur vel velocitates incrementorum ac decrementorum (quas etiam motus, mutationes & fluxiones quantitatum nominare licet) vel finitæ quævis quantitates velocitatibus hisce proportionales. Termini[3] autem cujusque generantis coefficiens est quantitas, quæ oritur applicando genitam

[1] The first edition gives "eorum" instead of "earum."

[2] In the place of "Momenta, quam primum finitæ sunt magnitudinis, desinunt esse momenta. Finiri enim repugnat aliquatenus perpetuo eorum incremento vel decremento," the second and third editions have this: "Particulæ finitæ non sunt momenta, sed quantitates ipsæ ex momentis genitæ."

[3] In the second and third editions "Lateris" takes the place of "Termini."

ad hunc terminum. Igitur sensus lemmatis est, ut, si quantitatum quarumcunque perpertuo motu crescentium vel decrescentium A, B, C, etc., momenta, vel mutationum velocitates[2] dicantur a, b, c, etc., momentum vel mutatio geniti[3] rectanguli AB fuerit $a\mathrm{B}+b\mathrm{A}$, et geniti[3] contenti ABC momentum fuerit $a\mathrm{BC}+b\mathrm{AC}+c\mathrm{AB}$: . . .

17. "*Cas.* 1. Rectangulum quodvis motu perpetuo auctum AB, ubi de lateribus A & B deerant momentorum dimidia[4] $\frac{1}{2}a$ & $\frac{1}{2}b$, fuit $\mathrm{A}-\frac{1}{2}a$ in $\mathrm{B}-\frac{1}{2}b$, seu $\mathrm{AB}-\frac{1}{2}a\mathrm{B}-\frac{1}{2}b\mathrm{A}+\frac{1}{4}ab$; & quam primum latera A & B alteris momentorum dimidiis aucta sunt, evadit $\mathrm{A}+\frac{1}{2}a$ in $\mathrm{B}+\frac{1}{2}b$, seu $\mathrm{AB}+\frac{1}{2}a\mathrm{B}+\frac{1}{2}b\mathrm{A}+\frac{1}{4}ab$. De hoc rectangulo subducatur rectangulum prius, et manebit excessus $a\mathrm{B}+b\mathrm{A}$. Igitur laterum incrementis totis a et b generatur rectanguli incrementum $a\mathrm{B}+b\mathrm{A}$. Q.E.D."

English Translation by Andrew Motte:

18. " . . . These quantities I here consider as variable and indetermined and increasing or decreasing as it were by perpetual motion or flux; and I understand their momentaneous increments or decrements by the name of Moments; so that the

[1] In the second and third editions "hoc latus" takes the place of "hunc terminum."

[2] In the third edition "vel his proportionales mutationum velocitates" takes the places of "vel mutationum velocitates."

[3] "geniti" is left out in the first edition.

[4] In this history, the solidus (/) will be used sometimes in printing fractions which come in the line of the text. The reader must remember that this notation is modern; it occurs in none of the passages which we quote from seventeenth- and eighteenth-century books. In some cases the use of the solidus has made it necessary to insert parentheses which do not occur in the original.

increments may be esteemed as added, or affirmative moments; and the decrements as subducted, or negative ones. But take care not to look upon finite particles as such. Finite particles are not moments, but the very quantities generated by the moments. We are to conceive them as the just nascent principles of finite magnitudes. Nor do we in this Lemma regard the magnitude of the moments, but their first proportion as nascent. It will be the same thing, if, instead of moments, we use either the Velocities of the increments and decrements (which may also be called the motions, mutations, and fluxions of quantities) or any finite quantities proportional to those velocities. The coefficient of any generating side is the quantity which arises by applying the Genitum to that side. Wherefore the sense of the Lemma is, that if the moments of any quantities A, B, C, etc., increasing or decreasing by a perpetual flux, or the velocities of the mutations which are proportional to them, be called a, b, c, etc., the moment or mutation of the generated rectangle AB will be $a\mathrm{B}+b\mathrm{A}$; the moment of the generated content ABC will be $a\mathrm{BC}+b\mathrm{AC}+c\mathrm{AB}$: . . .

19. "*Case* 1. Any rectangle as AB augmented by a perpetual flux, when, as yet, there wanted of the sides A and B half their moments $\frac{1}{2}a$ and $\frac{1}{2}b$, was $\mathrm{A}-\frac{1}{2}a$ into $\mathrm{B}-\frac{1}{2}b$, or $\mathrm{AB}-\frac{1}{2}a\mathrm{B}-\frac{1}{2}b\mathrm{A}+\frac{1}{4}ab$; but as soon as the sides A and B are augmented by the other half moments; the rectangle becomes $\mathrm{A}+\frac{1}{2}a$ into $\mathrm{B}+\frac{1}{2}b$, or $\mathrm{AB}+\frac{1}{2}a\mathrm{B}+\frac{1}{2}b\mathrm{A}+\frac{1}{4}ab$. From this rectangle subduct the former rectangle, and there

will remain the excess $a\mathrm{B}+b\mathrm{A}$. Therefore with the whole increments a and b of the sides, the increment $a\mathrm{B}+b\mathrm{A}$ of the rectangle is generated. Q.E.D."

II. Wallis's De Algebra Tractatus

20. The Latin edition of John Wallis's *Algebra*, which appeared in 1693, contains on pages 390–396 a treatise on the "Quadrature of Curves" which Newton had prepared many years before, and from which he cited many things in his letter of October 24, 1676. In revised phraseology and with a new Introduction, the "Quadrature of Curves" was republished in 1704, as we shall see presently. Through the researches of Rigaud[1] we know now that what is given in Wallis's *Algebra*, p. 390, line 18, to p. 396, line 19, are Newton's own words, except, no doubt, the word "clarissimus," as applied to himself. From this part we quote as follows:[2]—

21. Page 391: "Per *fluentes quantitates* intelligit indeterminatas, id est quæ in generatione Cuvarum per motum localem perpetuo augentur vel diminuuntur, & per earum *fluxionem* intelligit celeritatem incrementi vel decrementi. Nam quamvis *fluentes quantitates* & earum *fluxiones* prima fronte conceptu difficiles videantur, (solent enim nova difficilius concipi), earundem tamen notionem cito faciliorem evasuram putat, quam sit notio momen-

[1] S. P. Rigaud, *Historical Essay on Sir Isaac Newton's Principia*, Oxford, 1838, p. 22.

[2] Johannis Wallis, S.T.D., *De Algebra Tractatus; Historicus & Practicus.* Oxoniæ, MDCXCIII.

torum aut *partium minimarum* vel *differentiarum infinite parvarum* : propterea quod figurarum & quantitatum generatio per motum continuum magis naturalis est & facilius concipitur, & Schemata in hac methodo solent esse simpliciora, quam in illa partium. . . ."

22. Page 392 : "Sint v, x, y, z fluentes quantitates, & earum fluxiones his notis $\dot{v}$, $\dot{x}$, $\dot{y}$, $\dot{z}$ designabuntur respective. Et quoniam hæ fluxiones sunt etiam indeterminatæ quantitates, & perpetua mutatione redduntur majores vel minores, considerat velocitates quibus augentur vel diminuuntur tanquam earum fluxiones, & punctis binis notat in hunc modum $\ddot{v}$, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, & perpetuum incrementum vel decrementum harum fluxionum considerat ut ipsarum fluxiones, . . ."

23. Page 392 : "Sit enim o quantitas infinite parva, & sint $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ Synchrona momenta seu incrementa momentanea quantitatum fluentium z, y, & x : & hæ quantitates proximo temporis momento per accessum incrementorum momentaneorum evadent $z+o\dot{z}$, $y+o\dot{y}$, $x+o\dot{x}$: . . . " After substituting these in $x^3-xyy+aaz=0$, then subtracting the original expression and dividing the remainder by o, he remarks (page 393) : "Terminos multiplicatos per o tanquam infinite parvos dele, & manebit æquatio $3\dot{x}x^2-\dot{x}yy-2xy\dot{y}+aa\dot{z}=0$."

Translation :

24. Page 391 : "By *flowing quantities* he understands indeterminates, that is, those which, in the

generation of curves by local motion are always increased or diminished, and by their *fluxions* he understands the velocity of increase or decrease. For, however difficult of comprehension *flowing quantities* and their fluxions appear at first sight (for new things are usually perceived with difficulty), yet he thinks a notion of them will be obtained more easily than the notion of *moments* either of *least parts* or of *infinitely small differences*; because the generation of figures and quantities is more naturally and easily conceived, and the drawings in this method are usually more simple than in that of parts."

25. Page 392: "Let the flowing quantities be designated v, x, y, z, and their fluxions by the marks $\dot{v}$, $\dot{x}$, $\dot{y}$, $\dot{z}$, respectively. And since these fluxions are likewise indeterminate quantities, and by perpetual motion become greater or lesser, he considers the velocities by which they are increased or diminished as their fluxions, and marks them with double dots in this way $\ddot{v}$, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, and he considers the perpetual increase or decrease of these fluxions as fluxions of themselves. . . ."

26. Page 392: "Let o be an infinitely small quantity, and $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ the synchronous moments or momentaneous increments of the flowing quantities z, y, x: and these quantities at the next moment of time, by the accession of the momentaneous increments become $z+o\dot{z}$, $y+o\dot{y}$, $x+o\dot{x}$: . . ." After substituting these in $x^3-xyy+aaz=0$, then subtracting the original expression and divid-

ing the remainder by o, he remarks (page 393): "Destroy the terms multiplied by o as infinitely small, and there will remain the equation $3\dot{x}x^2 - \dot{x}yy - 2xy\dot{y} + aa\dot{z} = 0$."

III. Quadratura Curvarum,[1] 1704

"Introductio

27. "Quantitates Mathematicas non ut ex partibus quam minimis constantes, sed ut motu continuo descriptas hic considero. Lineæ describuntur ac describendo generantur non per appositionem partium sed per motum continuum punctorum, superficies per motum linearum, solida per motum superficierum, anguli per rotationem laterum, tempora per fluxum continuum, et sic in cæteris. Hæ Geneses in rerum natura locum vere habent et in motu corporum quotidie cernuntur. Et ad hunc modum Veteres ducendo rectas mobiles in longitudinem rectarum immobilium genesin docuerunt rectangulorum.

28. "Considerando igitur quod quantitates æqualibus temporibus crescentes et crescendo genitæ, pro velocitate majori vel minori qua crescunt ac

[1] *Tractatus de Quadratura Curvarum*, published in 1704 in London, as an appendix to Newton's *Opticks*. It was reprinted under the editorship of William Jones in London in the year 1711, in a volume containing also three other papers of Newton, viz., the *De analysi per æquationes infinitas*, *Enumeratio linearum tertii ordinis*, and *Methodus differentialis*. An English translation of the *Quadratura Curvarum*, made by John Stewart, was brought out in 1745 at London, in a volume containing also Newton's *Analysis by Equations of an Infinite Number of Terms*. A German translation of the *Quadratura Curvarum* by Gerhard Kowalewski appeared at Leipzig in 1908 in *Ostwald's Klassiker der exakten Wissenschaften*, Nr. 164.

generantur, evadunt majores vel minores; methodum quærebam determinandi quantitates ex velocitatibus motuum vel incrementorum quibus generantur; et has motuum vel incrementorum velocitates nominando *Fluxiones* et quantitates genitas nominando *Fluentes*, incidi paulatim Annis 1665 et 1666 in Methodum Fluxionum qua hic usus sum in Quadratura Curvarum.

29. "Fluxiones sunt quam proxime ut Fluentium

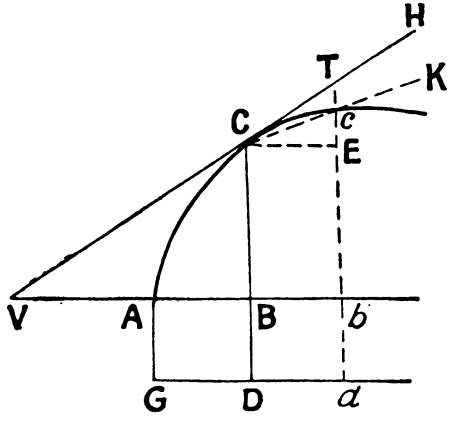


FIG. 2.

augmenta æqualibus temporis particulis quam minimis genita, et ut accurate loquar, sunt in prima ratione augmentorum nascentium; exponi autem possunt per lineas quascunque quæ sunt ipsis proportionales. Ut si areæ ABC, ABDG ordinatis BC, BD super basi AB uniformi cum motu progredientibus describantur, harum arearum fluxiones erunt inter se ut ordinatæ describentes BC et BD, et per ordinatas illas exponi possunt, propterea quod ordinatæ illæ sunt ut arearum augmenta nascentia.

Progrediatur ordinata BC de loco suo BC in locum quemvis novum *bc*. Compleatur parallelogrammum BCE*b*, ac ducatur recta VTH quæ curvam tangat in C ibsisque *bc* et BA productis occurrat in T et V : et abscissæ AB, ordinatæ BC, et lineæ curvæ AC*c* augmenta modo genita erunt B*b*, E*c*, et C*c* ; et in horum augmentorum nascentium ratione prima sunt latera trianguli CET, ideoque fluxiones ipsarum AB, BC et AC sunt ut trianguli illius CET latera CE, ET et CT et per eadem latera exponi possunt, vel quod perinde est per latera trianguli consimilis VBC.

30. "Eodem recidit si sumantur fluxiones in ultima ratione partium evanescentium. Agatur recta C*c* et producatur eadem ad K. Redeat ordinata *bc* in locum suum priorem BC, et cœuntibus punctis C et *c*, recta CK coincidet cum tangente CH, et triangulum evanescens CE*c* in ultima sua forma evadet simile triangulo CET, et ejus latera evanescentia CE, E*c* et C*c* erunt ultimo inter se ut sunt trianguli alterius CET latera CE, ET et CT, et propterea in hac ratione sunt fluxiones linearum AB, BC et AC. Si puncta C et *c* parvo quovis intervallo ab invicem distant recta CK parvo intervallo a tangente CH distabit. Ut recta CK cum tangente CH coincidat et rationes ultimæ linearum CE, E*c* et C*c* inveniantur, debent puncta C et *c* coire et omnino coincidere. Errores quam minimi in rebus mathematicis non sunt contemnendi.

31. "Simili argumento si circulus centro B radio BC descriptus in longitudinem abscissæ AB ad

angulos rectos uniformi cum motu ducatur, fluxio solidi geniti ABC erit ut circulus ille generans, et fluxio superficiei ejus erit ut perimeter circuli illius et fluxio lineæ curvæ AC conjunctim. Nam quo tempore solidum ABC generatur ducendo circulum illum in longitudinem abscissæ AB, eodem superficies ejus generatur ducendo perimetrum circuli illius in longitudinem curvæ AC. . . .

32. "*Fluat quantitas x uniformiter et invenienda sit fluxio quantitatis* x^n. Quo tempore quantitas x fluendo evadit $x+o$, quantitas x^n evadet $\overline{x+o}|^n$, id est per methodum serierum infinitarum, $x^n + nox^{n-1} + (nn-n)/2\, oox^{n-2}$ + etc. Et augmenta o et $nox^{n-1} + (nn-n)/2\, oox^{n-2}$ + etc. sunt ad invicem ut 1 et $nx^{n-1} + (nn-n)/2\, ox^{n-2}$ + etc. Evanescant jam augmenta illa, et eorum ratio ultima erit 1 ad nx^{n-1}: ideoque fluxio quantitatis x est ad fluxionem quantitatis x^n ut 1 ad nx^{n-1}.

33. "Similibus argumentis per methodum rationum primarum et ultimarum colligi possunt fluxiones linearum seu rectarum seu curvarum in casibus quibuscunque, ut et fluxiones superficierum, angulorum et aliarum quantitatum. In finitis autem quantitatibus Analysin sic instituere, et finitarum nascentium vel evanescentium rationes primas vel ultimas investigare, consonum est geometriæ veterum: et volui ostendere quod in Methodo Fluxionum non opus sit figuras infinite parvas in geometriam introducere. Peragi tamen potest Analysis in figuris quibuscunque seu finitis seu infinite parvis quae figuris evanescentibus finguntur

similes, ut et in figuris quæ pro infinite parvis haberi solent, modo caute procedas."

Translation by John Stewart:

"*Introduction*

34. "I consider mathematical quantities in this place not as consisting of very small parts; but as describ'd by a continued motion. Lines are describ'd, and thereby generated not by the apposition of parts, but by the continued motion of points; superficies's by the motion of lines; solids by the motion of superficies's; angles by the rotation of the sides; portions of time by a continual flux: and so in other quantities. These geneses really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner the ancients, by drawing moveable right lines along immoveable right lines, taught the genesis of rectangles.

35. "Therefore considering that quantities, which increase in equal times, and by increasing are generated, become greater or less according to the greater or less velocity with which they increase and are generated; I sought a method of determining quantities from the velocities of the motions or increments, with which they are generated; and calling these velocities of the motions or increments *Fluxions*, and the generated quantities *Fluents*, I fell by degrees upon the Method of Fluxions, which I have made use of here in the Quadrature of Curves, in the years 1665 and 1666.

36. "Fluxions are very nearly as the augments of the fluents generated in equal but very small particles of time, and, to speak accurately, they are in the *first ratio* of the nascent augments; but they may be expounded by any lines which are proportional to them.

37. "Thus if the area's ABC, ABDG be described by the ordinates BC, BD moving along the base AB with an uniform motion, the fluxions of these area's shall be to one another as the describing ordinates BC and BD, and may be expounded by these ordinates, because that these ordinates are as the nascent augments of the area's.

38. "Let the ordinate BC advance from it's place into any new place *bc*. Complete the parallelogram BCE*b*, and draw the right line VTH touching the curve in C, and meeting the two lines *bc* and BA produc'd in T and V: and B*b*, E*c* and C*c* will be the augments now generated of the absciss AB, the ordinate BC and the curve line AC*c*; and the sides of the triangle CET are in the *first ratio* of these augments considered as nascent, therefore the fluxions of AB, BC and AC are as the sides CE, ET and CT of that triangle CET, and may be expounded by these same sides, or, which is the same thing, by the sides of the triangle VBC, which is similar to the triangle CET.

39. "It comes to the same purpose to take the fluxions in the *ultimate ratio* of the evanescent parts. Draw the right line C*c*, and produce it to K. Let the ordinate *bc* return into it's former

place BC, and when the points C and *c* coalesce, the right line CK will coincide with the tangent CH, and the evanescent triangle C*Ec* in it's ultimate form will become similar to the triangle CET, and it's evanescent sides CE, E*c* and C*c* will be *ultimately* among themselves as the sides CE, ET and CT of the other triangle CET, are, and therefore the fluxions of the lines AB, BC and AC are in the same ratio. If the points C and *c* are distant from one another by any small distance, the right line CK will likewise be distant from the tangent CH by a small distance. That the right line CK may coincide with the tangent CH, and the ultimate ratios of the lines CE, E*c* and C*c* may be found, the points C and *c* ought to coalesce and exactly coincide. The very smallest errors in mathematical matters are not to be neglected.

40. "By the like way of reasoning, if a circle describ'd with the center B and radius BC be drawn at right angles along the absciss AB, with an uniform motion, the fluxion of the generated solid ABC will be as that generating circle, and the fluxion of it's superficies will be as the perimeter of that circle and the fluxion of the curve line AC jointly. For in whatever time the solid ABC is generated by drawing that circle along the length of the absciss, in the same time it's superficies is generated by drawing the perimeter of that circle along the length of the curve AC. . . ."

41. "*Let the quantity x flow uniformly, and let it be proposed to find the fluxion of* x^n.

"In the same time that the quantity x, by flowing, becomes $x+o$, the quantity x^n will become $\overline{x+o}|^n$, that is, by the method of infinite series's, $x^n+nox^{n-1}+(n^2-n)/2\ oox^{n-2}+$etc. And the augments o and $nox^{n-1}+(n^2-n)/2\ oox^{n-2}+$etc. are to one another as 1 and $nx^{n-1}+(n^2-n)/2\ ox^{n-2}+$etc. Now let these augments vanish, and their ultimate ratio will be 1 to nx^{n-1}.

42. "By like ways of reasoning, the fluxions of lines, whether right or curve in all cases, as likewise the fluxions of superficies's angles and other quantities, may be collected by the method of *prime* and *ultimate* ratios. Now to institute an analysis after this manner in finite quantities and investigate the *prime* or *ultimate* ratios of these finite quantities when in their nascent or evanescent state, is consonant to the geometry of the ancients: and I was willing to show that, in the Method of Fluxions, there is no necessity of introducing figures infinitely small into geometry. Yet the analysis may be performed in any kind of figures, whether finite or infinitely small, which are imagin'd similar to the evanescent figures; as likewise in these figures, which, by the Method of Indivisibles, used to be reckoned as infinitely small, provided you proceed with due caution."

43. In the Quadrature of Curves proper, under "Proposition I" the proof of the rule for finding the fluxion of expressions like $x^3-xy^2+a^2z-b^3=0$ contains the following passages which indicate the use made of the symbol "o" and of the term

"moment," and the mode of passing to the limit. We quote :—

"*Demonstratio*

44. "Nam sit o quantitas admodum parva et sunto $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, quantitatum z, y, x, momenta id est incrementa momentanea synchrona. Et si quantitates fluentes jam sunt z, y et x, hæ post momentum temporis incrementis suis $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ auctæ, evadent $z+o\dot{z}$, $y+o\dot{y}$, $x+o\dot{x}$, quæ in æquatione prima pro z, y et x scriptæ dant æquationem . . .

$$3\dot{x}x^2+3\dot{x}\dot{x}ox+\dot{x}^3oo-\dot{x}yy-2x\dot{y}y \\ -2\dot{x}o\dot{y}y-xo\dot{y}\dot{y}-\dot{x}oo\dot{y}\dot{y}+aa\dot{z}=0.$$

Minuatur quantitas o in infinitum, et neglectis terminis evanescentibus restabit $3\dot{x}x^2-\dot{x}yy-2x\dot{y}y+aa\dot{z}=0$. Q.E.D."

Translation by John Stewart :

"Demonstration

45. "For let o be a very small quantity, and let $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ be the moments, that is the momentaneous synchronal increments of the quantities z, y, x. And if the flowing quantities are just now z, y, x, then after a moment of time, being increased by their increments $o\dot{z}$, $o\dot{y}$, $o\dot{x}$: these quantities shall become $z+o\dot{z}$, $y+o\dot{y}$, $x+o\dot{x}$: which being wrote in the first equation for z, y and x, give this equation . . .

$$3\dot{x}x^2+3\dot{x}\dot{x}ox+\dot{x}^3oo-\dot{x}yy-2x\dot{y}y \\ -2\dot{x}o\dot{y}y-xo\dot{y}\dot{y}-\dot{x}oo\dot{y}\dot{y}+aa\dot{z}=0.$$

Let the quantity o be diminished infinitely, and neglecting the terms which vanish, there will remain $3\dot{x}x^2 - \dot{x}yy - 2x\dot{y}y + aa\dot{z} = 0$. Q.E.D."

IV. An Account of the "Commercium Epistolicum"

46. It is now generally accepted that the account [1] of the *Commercium Epistolicum*, published in the *Philosophical Transactions*, London, 1717, was written by Newton. The reasons for attributing it to him are stated by De Morgan[2] and by Brewster.[3] In abstract the account is as follows:—

47. (Pp. 177–178.) In a letter of October 24, 1676, to Oldenburgh, Newton explained that in deducing areas he considered the area as growing "by continual Flux"; "from the Moments of Time he gave the Name of Moments to the momentaneous Increases, or infinitely small Parts of the Abscissa and Area generated in Moments of Time. The Moment of a Line he called a Point, in the Sense of Cavalerius, tho' it be not a geometrical Point, but a Line infinitely short, and the Moment of an Area or Superficies he called a Line, in the sense of Cavalerius, tho' it be not a geometrical Line,

[1] *Philosophical Transactions*, vol. xxix, for the years 1714, 1715, 1716. London, 1717. "An Account of the Book entituled *Commercium Epistolicum Collinii et aliorum, De Analysi promota . . .*," pp. 173–224. This account was translated into Latin and inserted in the edition of the *Commercium Epistolicum* of 1725.

[2] See De Morgan's articles in the *Philosophical Magazine*, S. 4, vol. iii, June, 1852, pp. 440–444; vol. iv, November 1852, p. 323.

[3] Sir David Brewster, *Memoirs of the Life, Writings, and Discoveries of Sir Isaac Newton*, 2nd ed., vol. ii, Edinburgh, 1860, pp. 35, 36.

but a Superficies infinitely narrow. And when he consider'd the Ordinate as the Moment of the Area, he understood by it the Rectangles under the geometrical Ordinate and a Moment of the Abscissa, tho' that Moment be not always expressed." Again, p. 179: "And this is the Foundation of the Method of Fluxions and Moments, which Mr. Newton in his Letter dated Octob. 24, 1676, comprehended in this Sentence. *Data æquatione quotcunque fluentes quantitates involvente, invenire Fluxiones; et vice versa.* In this Compendium Mr. Newton represents the uniform Fluxion of Time, or of any Exponent of Time by an Unit; the Moment of Time or its Exponent by the Letter *o*; the Fluxions of other Quantities by any other Symbols; the Moments of those Quantities by the Rectangles under those Symbols and the Letter *o*; and the Area of the Curve by the Ordinate inclosed in a Square, the Area being put for a Fluent and the Ordinate for its Fluxion. When he is demonstrating a Proposition he uses the Letter *o* for a finite Moment of Time, or of its Exponent, or of any Quantity flowing uniformly, and performs the whole Calculation by the Geometry of the Ancients in finite Figures or Schemes without any Approximation: and so soon as the Calculation is at an End, and the Equation is reduced, he supposes that the moment *o* decreases *in infinitum* and vanishes. But when he is not demonstrating but only investigating a Proposition, for making Dispatch he supposes the Moment *o* to be infinitely little, and

forbears to write it down, and uses all manner of Approximations which he conceives will produce no Error in the Conclusion." In Newton's *Principia* "he frequently considers Lines as Fluents described by Points, whose Velocities increase or decrease, the Velocities are the first Fluxions, and their Increase the second." The Compendium of his Analysis was written "in or before the year 1669" (p. 180). "And the same Way of working he used in his Book of Quadratures, and still uses to this day" (p. 182). On p. 204 we read: "Mr. Newton used the letter *o* in his *Analysis* written in or before the Years 1669, and in his Book of *Quadratures*, and in his *Principia Philosophiæ*, and still uses it in the very same Sense as at first. . . . These Symbols *o* and $\dot{x}$ are put for things of a different kind. The one is a Moment, the other a Fluxion or Velocity as has been explained above. . . . Prickt Letters never signify Moments, unless when they are multiplied by the Moment *o* either exprest or understood to make them infinitely little, and then the Rectangles are put for the Moments" (p. 204). Further on we read: "It [the method of fluxions] is more elegant [than the Differential Method of Leibniz], because in his Calculus there is but one infinitely little Quantity represented by a symbol, the symbol *o*. We have no Ideas of infinitely little Quantities, and therefore Mr. Newton introduced Fluxions into his Method, that it might proceed by finite Quantities as much as possible. It is more Natural and Geometrical, because founded upon the

primæ quantitatum nascentium rationes, which have a Being in Geometry, whilst *Indivisibles*, upon which the Differential Method is founded, have no Being either in Geometry or in Nature. There are *rationes primæ quantitatum nascentium*, but not *quantitates primæ nascentes*. Nature generates Quantities by continual Flux or Increase; and the ancient Geometers admitted such a Generation of Areas and Solids" (p. 205).

From Newton's Correspondence and Manuscripts not in print in 1734

48. Manuscripts of Newton, some of them still unpublished, show that he first thought of fluents and fluxions in 1665 and 1666, when he was in his twenty-third and twenty-fourth years.[1] The notation by dots occurs as early as 1665. As pointed out by De Morgan,[2] these early papers are infinitesimal in character. They were first published in 1838.[3] A manuscript, dated Nov. 13, 1665, gives rules for finding the velocities p, q, r, etc., of two or more lines x, y, z, etc., described by bodies A, B, C, etc., the lines being related to each other

[1] See a list of Newton's manuscripts and publications on fluxional calculus prepared by Philip E. B. Jourdain, in his edition of *Augustus De Morgan's Essays on the Life and Work of Newton*, The Open Court Publishing Co., 1914, pp. 107–112.

[2] Augustus De Morgan, "On the Early History of Infinitesimals in England," *The London, Edinburgh, and Dublin Philosophical Magazine*, 4th S., vol. iv, 1852, pp. 321–330. This article is an important historical contribution, of which extensive use is made in the present history.

[3] See S. P. Rigaud, *Historical Essay on the first Publication of Sir Isaac Newton's Principia*, Oxford, 1838, Appendix, pp. 20–24.

by an equation, such as $x^3 - 2a^2y + zzx - yyx + zyy - z^3 = 0$. "If the body A, with the velocity p, describe the infinitely little line o in one moment, in the same moment B, with the velocity q, will describe the line oq / p," and the body C, with the velocity r, will describe the line or / p. So that, if the described lines be x, y and z "in one moment," they will be $x + o$, $\dot{y} + oq / p$, $z + or / p$ "in the next." He finds that the relation of the velocities p, q, r, in the above example, is $3pxx + pzz - pyy - 2aaq - 2yxq + 2zyq + 2zxr + yyr - 3zzr = 0$. In proving his rules for differentiation, Newton divides by o, and in the resulting expression observes that "those terms in which o is, are infinitely less than those in which it is not. Therefore, blotting them out, there rests" the relation sought. The notation by dots, "pricked letters," occurs on a leaf, dated May 20, 1665, which has never been printed.[1]

49. It is evident that Newton permitted twenty-eight years to pass between the time of his first researches on fluxions and 1693, the date when the earliest printed account of his notation of fluxions appeared from his pen in the Latin edition of Wallis's *Algebra*. Moments and fluxions are mentioned in his *Principia*, as has been shown by our extracts.

50. Of importance in the interpretation of the meanings of "moment" in the second edition of

[1] S. P. Rigaud, *op. cit.*, Appendix, p. 23. Consult also the remarks on this passage made by G. Enestrom in *Bibliotheca mathematica*, 3. F., Bd. 11, Leipzig, 1910-1911, p. 276, and Bd. 12, 1911-1912, p. 268, and by A. Witting in Bd. 12, pp. 56-60. See also *A catalogue of the Portsmouth collection of books and papers, written by or belonging to Isaac Newton*, Cambridge, 1889.

the *Principia* (1713) is a letter of May 15, 1714, from Newton to Keill,[1] from which we quote the following :—

51. ". . . altho I use prickt Letters in the first Proposition of the book of Quadratures, yet I do not there make them necessary to the method. For in the Introduction to that book I describe the method at large & illustrate it wth various examples without making any use of such letters. And it cannot be said that when I wrote that Preface I did not understand the method of fluxions because I did not there make use of prickt letters in solving of Problems.[2] The book of Quadratures is ancient, many things being cited out of it by me in my Letter of 24 Octob. 1676. . . .

52. "ffluxions & moments are quantities of a different kind. ffluxions are finite motions, moments are infinitely little parts. I put letters with pricks for fluxions, & multiply fluxions by the letter *o* to make them become infinitely little and the rectangles I put for moments. And wherever prickt letters represent moments & are without the letter *o* this letter is always understood. Wherever $\dot{x}$, $\dot{y}$, $\ddot{y}$, $\dot{\ddot{y}}$, etc., are put for moments they are put for $\dot{x}o$, $\dot{y}o$, $\ddot{y}oo$, $\dot{\ddot{y}}o^3$. In demonstrating Propositions I always write down the letter *o* & proceed by the Geometry of Euclide and Apollonius without any

[1] J. Edleston, *Correspondence of Sir Isaac Newton and Professor Cotes*, London, 1850, pp. 176, 177.

[2] John Bernoulli, in the *Acta Eruditorum* for February and March, 1713, had criticised a passage in the *Principia*, and claimed that Newton did not understand the second fluxions when writing that passage.

approximation. In resolving Questions or investigating truths I use all sorts of approximations wch I think will create no error in the conclusion and neglect to write down the letter *o*, and this do for making dispatch. But where $\dot{x}$, $\dot{y}$, $\ddot{y}$, $\dot{\ddot{y}}$ are put for fluxions without the letter *o* understood to make them infinitely little quantities they never signify differences. The great Mathematician[1] therefore acts unskilfully in comparing prickt letters with the marks *dx* and *dy*, those being quantities of a different kind."

Remarks

53. The extracts from Newton's writings demonstrate the following :—

(1) At first Newton used infinitesimals (infinitely small quantities), as did Leibniz and other mathematicians of that age. As early as 1665, when Newton was a young man of twenty-three, he used them and speaks of "blotting them out."[2] He uses infinitesimals in the *Principia* of 1687[3] and in his account of the quadrature of curves in Wallis's *Algebra* of 1693, where Newton speaks of himself in the third person.[4] It is worthy of emphasis, in contrast to Leibniz, that Newton uses only infinitesimals of the *first* order. Moreover, as De Morgan remarked long ago,[5] "the early distinction between the systems of the two is this, that Newton, holding to the conception of the *velocity* or *fluxion*,

[1] John Bernoulli. See Edleston, *op. cit.*, p. 171. [2] See our § 48.
[3] See our §§ 10, 13, 16, 18. [4] See our §§ 21, 26.
[5] De Morgan, *Philosophical Magazine*, 4 S., vol. iv, 1852, p. 324.

used the infinitely small increment as a means of determining it; while, with Leibnitz, the relation of the infinitely small increments is itself the object of determination."

(2) As early as 1665, Newton speaks of describing an "infinitely little line" in "one moment," and then uses the expression "in the next" moment.[1] Here "moment" cannot mean a point of time, destitute of duration; it means an infinitely small duration, an infinitesimal of time. Doubtless this use of "moment" with reference to time suggested the more extended and general use of the term "momentum" or "momenta" as found in the *Principia*[2] and later publications.

(3) The use of dots, "prickt letters," to indicate velocities or fluxions goes back to 1665,[3] but they are not used by Newton in print until 1693 in Wallis's *Algebra*; they are used extensively in Newton's *Quadrature of Curves* of 1704.[4]

(4) Newton first used the word "fluxion" in print in 1687 in the *Principia*.[5]

(5) The first refinement of the doctrine of fluxions is found in Newton's *Principia*, where he speaks of "prime and ultimate ratios"[6] and of "limits."[7]

(6) The high-water mark of Newton's efforts to place the doctrine of fluxions upon a thoroughly logical basis is found in his *Quadrature of Curves*, 1704. It indicates the almost complete exclusion

[1] See our § 48.
[2] See our §§ 16, 18, 21, 24.
[3] See our § 48.
[4] See our §§ 22, 25, 44, 45.
[5] See our §§ 16, 18.
[6] See our §§ 10, 13.
[7] See our §§ 4, 6, 8, 10, 13.

of quantities infinitely little. "I consider mathematical quantities in this place not as consisting of very small parts," says Newton.[1] Also "the very smallest errors in mathematical matters are not to be neglected,"[2] and "in the method of fluxions there is no necessity of introducing figures infinitely small into geometry."[3] In view of these statements the symbol *o* used in the *Quadrature of Curves*, a "quantitas ad modum parva,"[4] must be interpreted as a small *finite* quantity. In this connection De Morgan's remarks are of interest:[5] "In 1704, Newton in the *Quadratura Curvarum* renounced and abjured the infinitely small quantity; but he did it in a manner which would lead any one to suppose that he had never held it. . . . And yet, there is something like a recognition of *some one* having used infinitely small quantities in *Fluxions*, contained in the following words: *volui ostendere quod in Methodo Fluxionum non opus sit figuras infinite parvas in Geometriam introducere*: nothing is wanted except an avowal that the *some one* was Newton himself. The want of this avowal was afterwards a rock of offence. Berkeley, in the *Analyst*, could not or would not see that Newton of 1687 and Newton of 1704 were of two different modes of thought."

We do not interpret Newton's expressions of 1704 as declarations that a logical exposition of

[1] See our §§ 27, 34. [2] See our §§ 30, 39.
[3] See our §§ 33, 41. [4] See our §§ 44, 45.
[5] De Morgan, *Philosophical Magazine*, 4 S., vol. iv, 1852, p. 328.

fluxions cannot be given on the basis of infinitesimals or that infinitely small quantities are impossible; for he says,[1] "the analysis may be performed in any kind of figures whether finite or infinitely small, which are imagined similar to the evanescent figures."

In fact, not even in 1704 did Newton succeed in completely banishing from his doctrine of fluxions the infinitely little. If what he used in 1704 is not the infinitely little, it is so closely related thereto, that it cannot be called either a finite magnitude or an absolute zero.

In 1704, fluxions are "in the *first ratio* of the nascent augments," or "in the *ultimate ratio* of the evanescent parts."[2] Unless the fully developed theory of limits is read into these phrases, they will involve either infinitely little parts or other quantities no less mysterious. At any rate, the history of fluxions shows that these expressions did not meet the demands for clearness and freedom from mysticism. Newton himself knew full well the logical difficulty involved in the words "prime and ultimate ratios"; for in 1687 he said,[3] "it is objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not ultimate; and, when they have vanished, is none." How does Newton meet this, his own unanswerable argument? He does so simply by stating the difficulty in another

[1] See our §§ 33, 42. [2] See our §§ 29, 30, 33, 36, 38, 39, 42.
[3] See our §§ 11, 14.

form: "But, by the same argument, it might as well be maintained, that there is no ultimate velocity of a body arriving at a certain place, when its motion is ended: because the velocity, before the body arrives at the place, is not its ultimate velocity; when it has arrived, is none. But the answer is easy: for by the ultimate velocity is meant that . . . at the very instant when it arrives." If "instant," as used here, is not an infinitesimal, the passage would seem to be difficult or impossible of interpretation.

(7) A return to the open use of the infinitely small quantities is seen in writings of Newton after the year 1704. It might be argued that such a return was necessary in the second edition of the *Principia*, 1713, unless the work were largely rewritten. Newton's *Analysis per æquationes numero terminorum infinitas* was first printed in 1711, and might have been rewritten so as to exclude infinitesimals as fully as was done in the *Quadrature of Curves* of 1704. But the infinitely little is permitted to remain.[1] There is no disavowal of such quantities either in the *Commercium Epistolicum*, with the editors of which Newton was in touch, or in Newton's own account of this publication, contributed to the *Philosophical Transactions*.[2]

(8) The theory of limits is involved in the first lemma of the *Principia*,[3] and in the explanation of prime and ultimate ratios as given in that work.

[1] See our § 66.
[2] See our § 47.
[3] See our §§ 4, 6, 8, 9, 10, 12, 13, 15.

CHAPTER II

PRINTED BOOKS AND ARTICLES ON FLUXIONS BEFORE 1734

54. THE earliest printed publication in Great Britain on the new calculus was from the pen of John Craig, a Scotsman by birth, who settled in Cambridge and became a friend of Newton. Later he was rector of Gillingham in Dorsetshire. He was "an inoffensive, virtuous man," fond of mathematics. In 1685 he published at London a book entitled, *Methodus figurarum . . . quadraturas determinandi.* At that time nothing could be known about fluxions except through private communication. In 1684 Leibniz published his first ideas of Differential Calculus in the Leipzig *Acts.* Craig used in 1685 the calculus of Leibniz and also the notation of Leibniz. Continental writers call Craig the introducer of the theory of Leibniz into England. On p. 28 of his book, Craig derives $dp = 48nr^4y^2dy$ from $p = 16nr^4y^3$, and arrives at a differential equation (æquationem differentialem). The meanings of dp, dy, dx, etc., are not explained but taken for granted, reference being made to Leibniz. In 1693 Craig published another book in which the notation of Leibniz is used. He con-

tributed also several papers to the *Philosophical Transactions* (London), but never, before 1718, did he use fluxional symbols. In preparing the book of 1685 he had received from Newton the binomial theorem which he used before it had appeared in print, but he had no communication about fluxions. "We have here the singular indifference," says De Morgan, "which Newton at that time, and long afterwards, showed toward his own calculus."[1] Craig wrote a tract in 1693, and articles for the *Philosophical Transactions* in 1701, 1703, 1704, 1708, using the differential calculus all this time. In the issue No. 284, 1703, he employs the Leibnizian sign of integration $\int$. Craig submitted to Newton one of his early manuscripts (probably the one printed in 1693). With regard to this event De Morgan wrote to Hamilton, the inventor of quaternions: "Few of us know that Leibniz was perfectly well known in England before the dispute, and that Newton's first provocative to an imperfect publication was *ds* and infinitely small quantities paraded under his own eyes by an English writer (Craig), who lent him his MSS. to read."[2] Craig's publication of 1718 followed the great controversy on the invention of the calculus; now he uses fluxions exclusively and says not a word on the differential calculus. The book does not discuss fundamentals, and no explanation of $\dot{x}$ is given. As conjectured

[1] De Morgan, *Philosophical Magazine*, 4 S., vol. iv, 1852, p. 326.

[2] *Life of Sir William Rowan Hamilton*, by Robert P. Graves, vol. iii, 1889, p. 415.

by De Morgan, it may have been Craig's manuscript that suggested to Newton the need of making his own fluxions accessible to the public. At any rate, in 1693 there appeared the account of fluxions in Wallis's *Algebra*. [See *Addenda*, p. 289.]

55. Abraham De Moivre, a French mathematician who in 1688, after the revocation of the Edict of Nantes, came to London, contributed in 1695 to No. 216 of the *Philosophical Transactions* (London) an article in which he uses $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, and lets both "fluxion" and "moment" stand for things infinitely small. In the same number of the *Transactions*, the astronomer Edmund Halley has an article on logarithms in which he uses infinitely small *ratiunculæ* and *differentiolæ*, but neither the notation of Leibniz nor that of Newton. In 1697, David Gregory used in No. 231 of the *Transactions* $\dot{x}$ and speaks of "fluxio fluxionis" without, however, explaining his terms.

56. Fatio de Duillier, a Swiss by birth, who had settled in London and become member of the Royal Society, wrote in 1699 a treatise, *Lineæ brevissimi descensus investigatio geometrica*, uses fluxions as infinitely small quantities. This publication is noted as containing a statement which started the Newton-Leibniz controversy on the invention of the Calculus.

57. It is remarkable that Roger Cotes, in 1701, when an undergraduate at Trinity College, Cambridge [Newton's own College], wrote a letter on mathematical subjects, in which $\dot{x}$ is used as

"infinitely little."[1] In 1702–3 Humphry Ditton, in vol. xxiii of the *Transactions*, used the fluxional notation, without explanation.

58. Other writings that do not define their terms are the *Fluxionum methodus inversa*, 1704, by the London physician, George Cheyne, and De Moivre's *Animadversiones in D. Georgii Cheynai Tractatum*, London, 1704. However, Cheyne lets $\dot{x} = 1$, from which we infer that, with him, $\dot{x}$ was finite. [See *Addenda*, p. 289.]

59. The next writer on fluxions was John Harris, a voluminous author of books on various subjects. He was at one time Secretary of the Royal Society. In 1702 he published at London *A New Short Treatise of Algebra*, which devotes the last 22 pages, out of a total of 136 pages, to fluxions. It is the first book in the English language in which this subject is treated. The doctrine of fluxions is the "Arithmetick of the *Infinitely small* Increments or Decrements of *Indeterminate* or *Variable Quantities*, or as some call them the *Moments* or *Infinitely small Differences* of such Variable Quantities. These Infinitely small Increments or Decrements, our incomparable Mr. Isaac Newton calls very properly by this name of Fluxions" (p. 115). A few lines further on it says that Newton "calls the *celerity* or *Velocity* of the Augmentation of Diminution of these *Flowing Quantities*, by the name of *Fluxions*." A second edition of this book appeared in 1705. As authors on fluxions, Harris in 1705

[1] J. Edleston, *Correspondence of Sir Isaac Newton and Professor Cotes*, London, 1850, p. 196.

mentions Newton, Wallis, Nieuwentiit, Carré, Leibniz, l'Hospital, de Moivre, and Hayes.

60. John Harris also published a *Lexicon Technicum*, of which the second volume, London, 1710, contains an article, "Fluxions."

"This general Method of finding the Fluxions of all Powers and Roots, I had from the Hon. Fr. Robartes, Esq. If a Quantity gradually increases or decreases, its immediate Increment or Decrement is called its *Fluxion*. Or the Fluxion of a Quantity is its Increase or Decrease indefinitely small. . . . Since $\dot{x}\dot{x}$ . . . is infinitely smaller than $2\dot{x}\dot{x}$, whereby it can make no sensible Change in that Quantity, it may be laid aside as of no Value. . . . Authors' Names who have written of Fluxions: *D. Bernoulli Tractatus de Principiis Calculi Exponentialis*; *Nieuwentiit's Analysis Infinitorum*, Amster., 1695; *Dr. Cheyne's Fluxions*, with *Moivre's Animadversions* on them, and the Doctor's reply; *Hays's Fluxions*, Lond., 1704; *Analyse des Infiniment Petits. Par l'Hospital*, Fr., Paris, 1696; *Le Calcule Integrale, par M. Carré*, Paris, 1700; *Mr. Abraham de Moivre's* Use of *Fluxions*, in the Solution of Geometrick Problems. See *Philos. Trans.*, N. 216; *Mr. Humphry Ditton's Institution of Fluxions.*"

61. In the above list of writers are Charles Hayes and Humphry Ditton, authors of English texts now demanding our attention. Hayes starts his elucidation of fundamentals (p. 1) as follows:[1]

[1] *A Treatise of Fluxions: or, An Introduction to Mathematical Philosophy*, Charles Hayes, London, 1704.

"Magnitude is divisible in *infinitum*, and the Parts after this infinite Division, being infinitely little, are what Analysts call *Moments* or *Differences*; And if we consider Magnitude as Indeterminate and perpetually Increasing or Decreasing, then the infinitely little Increment or Decrement is call'd the *Fluxion* of that Magnitude or Quantity: And whether they be called *Moments*, *Differences* or *Fluxions*, they are still suppos'd to have the same Proportion to their Whole's, as a Finite Number has to an Infinite; or as a finite Space has to an infinite Space. Now those infinitely little Parts being extended, are again infinitely Divisible; and these infinitely little Parts of an infinitely little Part of a given Quantity, are by Geometers call'd *Infinitesimæ Infinitesimarum* or *Fluxions* of *Fluxions*. Again, one of those infinitely little Parts may be conceiv'd to be Divided into an infinite Number of Parts which are call'd *Third Fluxions*, etc."

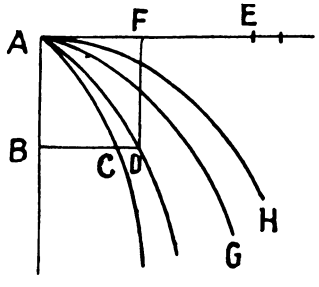


FIG. 3.

He endeavours to justify this doctrine by illustrations. The angle of contact FAC formed by the line AE and the ordinary parabola AC, is less than any rectilineal angle; the angle FAD, formed by AE with the cubical parabola AD, is infinitely less than the angle FAC, and so on. Hayes defines the doctrine of *Fluxions* as the "Arithmetick of infinitely small Increments or Decrements

of Indeterminate or variable Quantities." He cautions the reader: "But we must take great heed, not to consider the Fluxions, or Increments, or Decrements as finite Quantities" (p. 4). He rejects $\dot{x}\dot{z}y$ and $\dot{x}\dot{z}\dot{y}$ "as being incomparably less" than $\dot{x}zy$.

The same year in which Hayes wrote this first English book on fluxions which could make any claim to attention, saw the appearance of Newton's *Quadratura Curvarum*. The contrast in the definition of "fluxion" was sharp. Hayes called it "an infinitely small increment"; Newton called it a "velocity," a finite quantity.

62. William Jones, in his *Synopsis Palmariorum Matheseos*, London, 1706, devotes a few pages to fluxions and fluents, using the Newtonian notation. On p. 225 he gives, in substance, Newton's lemma, in these words: "Quantities, as also their Ratio's, that continually tend to an Equality, and therefore that approach nearer the one to the other, than any Difference that can possibly be assign'd, do at last become equal." Then he says: "Hence all Curved Lines may be considered as composed of an Infinite Number of Infinitely little right Lines." He uses "infinitely small" quantities, but defines a fluxion as "the Celerity of the Motion," fluxions being "in the first Ratio of their Nascent Augments." Jones represents here the Newton of the *Principia*, and of the *Quadrature of Curves* as given in 1793.

63. The earliest book exhibiting a careful study

of Newton's tract of 1704 was Humphry Ditton's *Institution of Fluxions*, 1706.[1] Ditton was prominent as a divine as well as a mathematician. Like so many other English writers on fluxions during the eighteenth century, he had not been at either of the great universities. He states in his preface that he has also consulted and drawn from the writings of John Bernoulli and some other Continental writers.

64. The reader of Ditton's book is impressed by the fact that he labours strenuously to make everything plain. He takes the reader fully into his confidence. This is evident in the extracts which follow (pp. 12–21):—

"Suppose any flowing Quantities, . . . as also their Increments . . . which Increment imagine to be generated in equal very small Particles of Time. I conceive we may say without Scruple, that the *Fluxions are the velocities of those Increments, consider'd not as actually generated, but quatenus Nascentia, as arising and beginning to be generated.* As there is a vast difference between the Increments consider'd as Finite, or really and actually generated; and the same considered only as *Nascentia* or in the first Moment of their Generation: So there is as great a difference also between the Velocities of the Increments, consider'd in this two fold respect. . . .

[1] *An Institution of Fluxions: Containing the First Principles, The Operations, with some of the Uses and Applications of that Admirable Method; According to the Scheme prefix'd to his Tract of Quadratures, by (its First Inventor) the Incomparable Sir Isaac Newton.* By Humphry Ditton, London, 1706.

The Reason of that [difference], is this. Because there is (speaking strictly and accurately) an Infinity of Velocities to be consider'd, in the Generation and Production of a *Real Increment*; . . . So that if we conceiv'd the Fluxion, to be the Velocity of the Increment, *as actually Generated*; we must conceive it to be an Infinite Variety or Series of Velocities. Whereas the Velocity, with which any sort of Increment *arises*, or *begins to be generated*; is a thing that one may form a very clear and distinct Idea of, and leaves the Mind in no Ambiguity or Confusion at all. . . . However, if we take those Particles of time exceeding small indeed, and Neglect the Acceleration of the Velocity as inconsiderable, we may say the Fluxions are proportional to those Increments; remembering at the same time, that they are *but nearly*, and not *accurately* so. . . . If in the *Differential Calculus*, some Terms are reiected and thrown out of an Equation, because they are nothing *Comparatively*, or with respect to other Terms in the same Equation; that is, because they are infinitely small in proportion to those other Terms, and so may be neglected upon that Score: On the other hand, in the Method of Fluxions, those same Terms go out of the Equation, because they are multiplied into a Quantity, which . . . does at last really vanish. . . . N. B. Speaking here of Infinitely small Quantities, or Infinitesimals as some Authors (and particularly Mr. Neiwentiit) chuse to term them, I cannot but take notice of a notion, which that Excellent and Ingenious Person advances in his *Analysis Infinitorum*.

It is this; That a *Quantity Infinitely Great, a Finite or any given Quantity, an Infinitesimal, and Nihilum Geometricum*, are in Geometrical Proportion. I confess I cannot discover the truth of this. . . . Let m denote an infinite Quantity, d any finite one; then is d / m the Infinitesimal of d, according to Mr. Neiwentiit. Now his Assertion is, that $m : d :: d / m : 0$; therefore since from the nature of Geometrical Proportion, 'tis also $m : d :: d / m : dd / mm$; it follows that dd / mm is $= 0$ . . . then $d / m = 0$. Now Mr. Neiwentiit will hardly allow his Infinitesimal to be nothing; and yet . . . I think it must follow, that $d = 0$." Proceeding geometrically, Ditton explains the fluxions of lines, areas, solids, and surfaces. Next he takes up algebraical expressions. To find the fluxion of x^n, he lets x flow uniformly and represents the augment of x in a given particle of time by the symbol o. While x becomes $x+o$, x^n becomes $(x+o)^n$. Expanding the binomial, he finds that the two augments are as 1 to $nx^{n-1}+(n^2-n)ox^{n-2} / 2+$ etc. "And the Ratio of them (making o to vanish) will be that of 1 to nx^{n-1}." According to his notation $\dot{x}$ is a fluxion of x, and x is a fluxion of $\dot{x}$. Taking o as a very small quantity, he lets the expressions $o\dot{z}$, $o\dot{y}$ represent the *moments*, or increments of the flowing quantity z, y generated in a very small part of time. "If therefore *now, at the present Moment*, the flowing Quantities are z, y, x; the next *Moment* (when augmented by these Increments) they will become $x+o\dot{z}$, $y+o\dot{y}$, $x+o\dot{x}$." He expresses the general mode of procedure for finding

the fluxion, which coincides with the modern mode of finding a derivative. Ditton considers the increments as *finite* (p. 53). "These *Momenta* are in proportion to one another as the Fluxions of the flowing Quantities respectively, for $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, are as $\dot{z}$, $\dot{y}$, $\dot{x}$; and Mr. Newton had before expresly told us; that the Increments generated in a very small Particle of time were *very nearly*, as the Fluxions." Evidently Ditton does not here overlook that $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ represent the increments only "very nearly." He observes (p. 98) that we may "go on with ease to the second, third, and any other Fluxions; neither are there any new Difficulties to be met with."

A second edition of Ditton's book was brought out in 1726 by John Clarke.

65. Ditton's first edition appeared at a time when the Newton-Leibniz controversy was under way. Leibniz had appealed to the Royal Society for justice. That Society appointed a committee which published a report containing letters and other material bearing on the case, in a book called the *Commercium Epistolicum*,[1] which figures prominently in the lamentable controversy. From this book the early use of infinitely small quantities on the part of Newton is conspicuously evident. The book makes it clear also that some of Newton's warmest supporters were guilty of gross inaccuracy in the use of the word "fluxion."

[1] *Commercium Epistolicum D. Johannis Collins, et aliorum de analysi promota: jussu societatis regiæ in lucem editum.* Londini, MDCCXII.

66. Newton's *Analysis per æquationes numero terminorum infinitas*, which was sent on July 31, 1669, through Barrow to Collins, and which was first published at London in 1711, was reprinted in the *Commercium Epistolicum*. In this *Analysis* infinitely small quantities are used repeatedly, but the word "fluxion" and the fluxional notation do not occur. In a letter to H. Sloane, who was then Secretary of the Royal Society of London, written in answer to a letter of Leibniz dated March 4, 1711, John Keill, professor of astronomy at Oxford, recounts the achievements of Isaac Barrow and James Gregory, and says: "If in place of the letter *o*, which represents an infinitely small quantity in James Gregory's *Geometriæ pars universalis* (1667), or in place of the letters *a* or *e* which Barrow employs for the same thing, we take the $\dot{x}$ or $\dot{y}$ of Newton or the dx or dy of Leibniz, we arrive at the formulas of fluxions or of the differential calculus."[1] Thus Keill, the would-be great champion of Newton, instead of warning the reader against confusing differentials and fluxions, himself comes dangerously close to conveying the erroneous idea that $\dot{x}$ and $\dot{y}$ are infinitely small, the same as dx and dy. He comes so near to this as to be guilty of lack of caution, if not of inaccuracy.

More serious is a statement further on. The en-

[1] "Nam si pro Litera *o*, quæ in Jacobi Gregorii Parte Matheseos Universali quantitatem infinite parvam repræsentat; aut pro Literis *a* vel *e* quas ad eandem designandam adhibet Barrovius; ponamus $\dot{x}$ vel $\dot{y}$ Newtoni, vel dx seu dy Leibnitii, in Formulas Fluxionum vel Calculi Differentialis incidemus" (p. 112).

listment of the services of a clever lawyer would be needed to acquit the editors of the *Commercium Epistolicum* of gross error when, in the final summary of their case against Leibniz, they declare (p. 121), "that the Differential Method is one and the same with the Method of Fluxions, excepting the name and the notation; Mr. Leibniz calling those Quantities Differences, which Mr. Newton calls Moments or Fluxions; and marking them with the letter *d*, a mark not used by Newton."

67. Joseph Raphson, in his *History of Fluxions* (which appeared as a posthumous work at London, in 1715, printed in English, and in the same year also in Latin, the Latin edition containing new correspondence bearing on the Newton-Leibniz controversy), says on p. 5 that Newton "makes use of Points, and denotes those first *Differences* (which by a Name congruous to their Generation, being consider'd as the first Increments or Decrements of a continued Motion, he calls *Fluxions*) thus, viz. $\dot{x}$, $\dot{y}$, $\dot{z}$." This misrepresentation of Newton is the more astonishing when we recollect that Raphson was very partial to Newton, and also meant his History "to open a plain and easy way for Beginners to understand these Matters." Newton never looked upon a *fluxion* as anything different from velocity; with him it was always a *finite* quantity. To make matters worse, Raphson continues: "To these Quantities he adds others of another Gender, and which in relation to Finite ones may be conceiv'd as infinitely great, and denotes them thus $'x$, $'y$, $'z$,

whereof the first or finite Quantities themselves, viz. x, y, z, may be conceiv'd as *Fluxions*." And again, "a Point . . . may be consider'd as the Fluxion of a Line, a Line as the Fluxion of a Plane, and a Plane as the Fluxion of a Solid, and a finite Solid as the Fluxion of a (partially) infinite one, and that again as the Fluxion of one of an higher Gender of Infinity, and so on *ad inf.* which we shall further illustrate in some Dissertations at the end of this Treatise."

68. Brook Taylor brought out at London in 1715 his *Methodus incrementorum directa et inversa*, in which he looks upon fluxions strictly from the standpoint of the Newtonian exposition in the *Quadrature of Curves*, 1704.

69. James Stirling uses $\dot{x}$ and $\dot{y}$ as infinitesimals in his *Lineæ tertii ordinis*, Oxford, 1717. He draws the infinitely small right triangle at the contact of a curve with its asymptote, the horizontal side being "quam minima" and equal to $\dot{x}$, the vertical side being $\dot{y}$. In the appendix to this booklet of 1717, $\dot{x}$ and $\dot{y}$ are again infinitely small. In his *Methodus differentialis*, London, 1730, there is no direct attempt to explain fundamentals, any more than there was in 1717, but on p. 80 he puts the fluxion of an independent variable equal to unity, from which we infer that a fluxion is with him now a finite velocity.

70. For twenty-four years after Ditton no new text appeared. In 1730 Edmund Stone, a self-taught mathematician who had studied De l'Hospital,

sent forth a new book, the first part of which was a translation.[1]

The following extract is from Stone's translation of De l'Hospital's Preface, the words in the square brackets [ ] being interpolated by Stone :—

"By means of this Analysis we compare the infinitely small (Differences or) Parts of finite Magnitudes, and find their Ratio's to each other; and hereby likewise learn the Ratio's of finite Magnitudes, those being in reality so many infinitely great Magnitudes, in respect of the other infinitely small ones. This Analysis may ever be said to go beyond the Bounds of Infinity itself; as not being confined to infinitely small (Differences or) Parts, but discovering the Ratio's of Differences of Differences, or of infinitely small Parts of infinitely small Parts, and even the Ratio's of infinitely small Parts of these again, without End. So that it not only contains the Doctrine of Infinites, but that of an Infinity of Infinites. It is an Analysis of this kind that can alone lead us to the Knowledge of the true Nature and Principles of Curves: For Curves being no other than Polygons, having an Infinite Number of Sides, and their Differences arising altogether from the different Angles which their infinitely small Sides make with each other, it is the Doctrine of Infinites alone that must enable us to determine the Position of these Sides, in order to get the

[1] *The Method of Fluxions, both Direct and Inverse. The former being a Translation from the Celebrated Marquis De l'Hospital's Analyse des Infinements Petits: And the Latter Supply'd by the Translator,* E. Stone, F.R.S. London, MDCCXXX.

Curvature formed by them; and thence the Tangents, Perpendiculars, Points of Inflexion and Retrogression, reflected and refracted Rays, etc., of Curves.

"Polygons circumscribed about or inscribed in Curves, whose Number of Sides infinitely augmented till at last they coincide with the Curves, have always been taken for Curves themselves. . . . It was the Discovery of the Analysis of Infinites that first pointed out the vast Extent and Fecundity of this Principle. . . . Yet this itself is not so simple as Dr. Barrow afterwards made it, from a close Consideration of the Nature of Polygons, which naturally represent to the Mind a little Triangle consisting of a Particle of a Curve (contained between two infinitely near Ordinates), the Difference of the correspondent Absciss's; and this Triangle is similar to that formed by the Ordinate, Tangent, and Subtangent. . . . Dr. Barrow . . . also invented a kind of Calculus suitable to the Method (*Lect. Geom.*, p. 80), tho' deficient. . . . The Defect of this Method was supplied by that of Mr. Leibnitz'z,[1] [*or rather the great Sir Isaac Newton*].[2] He began where Dr. Barrow and others left off: His *Calculus* has carried him into Countries hitherto unknown. . . . I must here in justice own (as Mr Leibnitz himself has done, in *Journal des Sçavans* for August 1694) that the learned Sir Isaac Newton likewise discover'd something like the *Calculus Differentialis*, as appears by his excellent

[1] *Acta Eruditi. Lips.*, ann. 1684, p. 467.
[2] See *Commercium Epistolicum.*

Principia, published first in the Year 1687, which almost wholly depends upon the Use of the said *Calculus*. But the Method of Mr. Leibnitz'z is much more easy and expeditious, on account of the Notation he uses. . . ."

In the preface of "The Translator to the Reader" Stone points out that the work he is bringing out "becomes the more necessary, because there are but two English Treatises on the Subject . . . the one being Hay's *Introduction to Mathematical Philosophy*, and the other, Ditton's *Institution of Fluxions*"; the former "too prolix," the latter "much too sparing in Examples" and "too redundant" in the explanation of fluxions, so that "it is next to impossible for one who has not been conversant about Infinites to apprehend it. That of our Author is much easier, tho less Geometrical, who calls a Differential (or Fluxion) the infinitely small Part of a Magnitude." "But," continues Stone, "I would not here be thought in any wise to lessen the Value of Sir Isaac Newton's Definition: When the Learner has made some Progress, I would have him then make himself Master of it." Stone then proceeds to explain the nature of fluxions, following closely Newton's language in his *Quadrature of Curves*.

71. In De l'Hospital's treatise, as translated by Stone, we read:

"The infinitely small Part whereby a variable Quantity is continually increased or decreas'd, is called the *Fluxion* of that Quantity."

Here Stone simply writes "fluxion" where De l'Hospital writes "différence," which is a mischievous procedure, seeing that the two words stand for things totally different. De l'Hospital's wording is "La portion infiniment petite dont une quantité variable augmente ou diminuë continuellement, en est appellée la Différence." Stone also changes from the Leibnizian to the Newtonian notation, by writing $\dot{x}$ instead of dx. Then follow two postulates:

"Grant that two Quantities, whose Difference is an infinitely small Quantity, may be taken (or used) indifferently for each other: or (which is the same thing) that a Quantity, which is increased or decreas'd only by an infinitely small Quantity, may be consider'd as remaining the same.

"Grant that a Curve Line may be consider'd as the Assemblage of an infinite Number of infinitely small right Lines: or (which is the same thing) as a Polygon of an infinite Number of Sides, each of an infinitely small Length, which determine the Curvature of the Line by the Angles they make with each other."

De l'Hospital's "prendre la différence" is rendered by Stone "to find the fluxions." The fluxion of xy is found by taking the product of $x+\dot{x}$ and $y+\dot{y}$, and neglecting $\dot{x}\dot{y}$, "because $\dot{x}\dot{y}$ is a Quantity infinitely small, in respect of the other Terms $y\dot{x}$ and $x\dot{y}$."

72. Further on in Stone's translation (p. 73) we read:

"The infinitely small Part generated by the continual increasing or decreasing of the Fluxion of a variable Quantity, is called the *Fluxion* of the *Fluxion* of that Quantity, or *second Fluxion.*" In like manner he defines *third Fluxion*; "fluxion of the second fluxion" taking the place of "différence de la différence seconde."

In the appendix, containing Stone's *Inverse Method of Fluxions*, a *fluent* is defined thus:

"The *fluent* or *flowing Quantity* of a given fluxionary Expression, is that Quantity whereof the given fluxionary Expression is the Fluxion."

Remarks

73. The earliest treatment of the new analysis which became current in England was that of Leibniz. The Scotsman Craig used it for over a quarter of a century before rejecting it in favour of fluxions. Harris, Hayes, and Stone drew their inspiration from French writers who followed Leibniz. A hopeless confusion arose in the use of the term "fluxion." Newton always took it to be a velocity, but many writers, including Newton's friends who prepared the *Commercium Epistolicum*, simply said "fluxion" instead of "differential," thus putting a home label upon goods of foreign manufacture. A strict follower of the Newton of 1704 was Ditton; fluxions are taken as infinitesimals by Fatio de Duillier, Cotes (in 1701), Harris, Hayes, Raphson, Stirling (in 1717), and Stone.

Stone comes out strongly with the view that a

circle is a polygon of an infinite number of sides. He also uses the infinitesimal triangle. Hayes and Stone have no hesitation in speaking of "fluxions of fluxions," and "infinitely little parts of an infinitely little part." No writers, unless we except Newton (1704) and Ditton, dispense with the use of infinitely small quantities. The dropping of such quantities from an equation was usually permitted without scruple.

What an opportunity did this medley of untenable philosophical doctrine present to a close reasoner and skilful debater like Berkeley! [See *Addenda*, p. 289.]

CHAPTER III

BERKELEY'S ANALYST (1734); CONTROVERSY WITH JURIN AND WALTON

74. BISHOP BERKELEY'S publication of the *Analyst*[1] is the most spectacular event of the century in the history of British mathematics. The arguments in the *Analyst* were so many bombs thrown into the mathematical camp.

The views expressed in the *Analyst* are foreshadowed in Berkeley's *Principles of Human Knowledge* (§§ 123–134), published nearly a quarter of a century earlier. The "Infidel mathematician," it is generally supposed, was Dr. Halley. Mathematicians complain of the incomprehensibility of religion, argues Berkeley, but they do so unreasonably, since their own science is incomprehensible. "Our Sense is strained and puzzled with the perception of objects extremely minute, even so the Imagination, . . . is very much strained and puzzled to frame clear ideas of the least particles of time, or the least increments generated therein:

[1] *The Analyst: or, a Discourse addressed to an Infidel Mathematician. Wherein it is examined whether the Object, Principles, and Inferences of the Modern Analysis are more distinctly conceived, or more evidently deduced, than religious Mysteries and Points of Faith.* London, 1734.

and much more so to comprehend the moments, or those increments of the flowing quantities in *statu nascenti*, in their very first origin or beginning to exist, before they become finite particles. And it seems still more difficult to conceive the abstracted velocities of such nascent imperfect entities. But the velocities of the velocities—the second, third, fourth, and fifth velocities, etc.—exceed, if I mistake not, all human understanding" (*Analyst*, § 4). . . .

75. "In the *calculus differentialis* . . . our modern analysts are not content to consider only the differences of finite quantities: they also consider the differences of those differences, and the differences of the differences of the first differences: and so on *ad infinitum*. That is, they consider quantities infinitely less than the least discernible quantity; and others infinitely less than those infinitely small ones; and still others infinitely less than the preceding infinitesimals, and so on without end or limit" (§ 6).

76. "I proceed to consider the principles of this new analysis. . . . Suppose the product or rectangle AB increased by continual motion: and that the momentaneous increments of the sides A and B are a and b. When the sides A and B are deficient, or lesser by one-half of their moments, the rectangle was $\overline{A-\frac{1}{2}a} \times \overline{B-\frac{1}{2}b}$, *i.e.* $AB-\frac{1}{2}aB-\frac{1}{2}bA+\frac{1}{4}ab$. And as soon as the sides A and B are increased by the other two halves of their moments, the rectangle becomes $\overline{A+\frac{1}{2}a} \times \overline{B+\frac{1}{2}b}$ or $AB+\frac{1}{2}aB+\frac{1}{2}bA+\frac{1}{4}ab$. From the latter rectangle subduct the former, and

the remaining difference will be $a\mathrm{B}+b\mathrm{A}$. Therefore the increment of the rectangle generated by the entire increments a and b is $a\mathrm{B}+b\mathrm{A}$. Q.E.D. But it is plain that the direct and true method to obtain the moment or increment of the rectangle AB, is to take the sides as increased by their whole increments, and so multiply them together, $\mathrm{A}+a$ by $\mathrm{B}+b$, the product whereof $\mathrm{AB}+a\mathrm{B}+b\mathrm{A}+ab$ is the augmented rectangle; whence, if we subduct AB the remainder $a\mathrm{B}+b\mathrm{A}+ab$ will be the true increment of the rectangle, . . . and this holds universally by the quantities a and b be what they will, big or little, finite or infinitesimal, increments, moments, or velocities" (§ 9). . . . The point of getting rid of ab cannot be obtained by legitimate reasoning." . . .

77. "The points or mere limits of nascent lines are undoubtedly equal, as having no more magnitude one than another, a limit as such being no quantity. If by a momentum you mean more than the very initial limit, it must be either a finite quantity or an infinitesimal. But all finite quantities are expressly excluded from the notion of a momentum. Therefore the momentum must be an infinitesimal. . . . For aught I see, you can admit no quantity as a medium between a finite quantity and nothing, without admitting infinitesimals" (§ 11).

78. Berkeley next premises the following *lemma*, which figures prominently in the debates about fluxions:

"'If, with a view to demonstrate any proposition,

a certain point is supposed, by virtue of which certain other points are attained; and such supposed point be itself afterwards destroyed or rejected by a contrary supposition; in that case, all the other points attained thereby, and consequent thereupon, must also be destroyed and rejected, so as from thenceforward to be no more supposed or applied in the demonstration.' This is so plain as to need no proof" (§ 12).

79. Berkeley examines now the method of obtaining the fluxion of x^n by writing $x+o$ in the place of x, expanding by the binomial formula, writing down the increments of x and x^n, which are in the ratio of

$$1 \text{ to } nx^{n-1} + \frac{(nn-n)}{2} ox^{n-2} + \text{ etc.},$$

or, when the increment o is made to vanish, in the ratio of 1 to nx^{n-1}. Berkeley argues:

"But it should seem that this reasoning is not fair or conclusive. For when it is said, let the increments vanish, *i.e.* let the increments be nothing, or let there be no increments, the former supposition that the increments were something, or that there were increments, is destroyed, and yet a consequence of that supposition, *i.e.* an expression got by virtue thereof, is retained. Which, by the foregoing lemma, is a false way of reasoning. Certainly when we suppose the increments to vanish, we must suppose their proportions, their expressions, and everything else derived from the supposition of their existence, to vanish with them (§ 13).

. . . All which seems a most inconsistent way of arguing, and such as would not be allowed of in Divinity (§ 14). . . . Nothing is plainer than that no just conclusion can be directly drawn from two inconsistent suppositions (§ 15). . . . It may perhaps be said that [in the *calculus differentialis*] the quantity being infinitely diminished becomes nothing, and so nothing is rejected. But, according to the received principles, it is evident that no geometrical quantity can by any division or subdivision whatsoever be exhausted, or reduced to nothing. Considering the various arts and devices used by the great author of the fluxionary method; in how many lights he placeth his fluxions; and in what different ways he attempts to demonstrate the same point; one would be inclined to think, he was himself suspicious of the justness of his own demonstrations, and that he was not enough pleased with any notion steadily to adhere to it" (§ 17). . . .

80. "And yet it should seem that, whatever errors [in the *calculus differentialis*] are admitted in the premises, proportional errors ought to be apprehended in the conclusion, be they finite or infinitesimal: and that therefore the ἀκρίβεια of geometry requires nothing should be neglected or rejected. In answer to this you will perhaps say, that the conclusions are accurately true, and that therefore the principles and methods from whence they are derived must be so too. But . . . the truth of the conclusion will not prove either the form or the matter of a syllogism to be true" (§ 19).

81. Berkeley proceeds to show that correct results are derived from false principles by a *compensation of errors*, a view advanced again later by others, particularly by the French critic L. N. M. Carnot. Taking $y^2=px$, Berkeley says that the subtangent is not $ydx \mid dy$ if dy is the true increment of y corresponding to dx; the accurate subtangent, obtained by similar triangles, is $ydx \mid (dy+z)$, where $z=dydy \mid (2y)$. That is, if dy is the true increment, then in $ydx \mid dy$ there is an "error of defect." But in $ydx \mid dy$, as used in the differential calculus, the dy is not its true value, viz. $dy=pdx \mid (2y)-dydy \mid (2y)$ (obtained by writing $x+dx$ for x and $y+dy$ for y, in the equation $y^2=px$), but its erroneous value, $pdx \mid (2y)$. There is here an "error of excess." "Therefore the two errors being equal and contrary destroy each other (§ 21); . . . by virtue of a twofold mistake you arrive, though not at science, yet at truth." Berkeley gives other illustrations of cases where "one error is redressed by another."

82. "A point may be the limit of a line: a line may be the limit of a surface: a moment may terminate time. But how can we conceive a velocity by help of such limits? It necessarily implies both time and space, and cannot be conceived without them. And if the velocities of nascent and evanescent quantities, *i.e.* abstracted from time and space, may not be comprehended, how can we comprehend and demonstrate their proportions; or consider their *rationes primæ* and *ultimæ*? For, to

consider the proportion or *ratio* of things implies that such things have magnitude; that such their magnitudes may be measured" (§ 31). . . .

83. "If it be said that fluxions may be expounded or expressed by finite lines proportional to them; which finite lines, as they may be distinctly conceived and known and reasoned upon, so they may be substituted for the fluxions, . . . I answer that if, in order to arrive at these finite lines proportional to the fluxions, there be certain steps made use of which are obscure and inconceivable, be those finite lines themselves ever so clearly conceived, it must nevertheless be acknowledged that your proceeding is not clear nor your method scientific" (§ 34).

Berkeley discusses this matter with reference to a geometric figure, and argues that "a point therefore is considered as a triangle, or a triangle is supposed to be formed in a point. Which to conceive seems quite impossible" (§ 34). . . .

84. "And what are these fluxions? The Velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?" (§ 35). . . .

"And if the first [fluxions] are incomprehensible, what shall we say of the second and third fluxions, etc.?" (§ 44).

"To the end that you may more clearly comprehend the force and design of the foregoing

remarks . . ., I shall subjoin the following Queries" (§ 50).

Then follow sixty-seven queries, of which the sixteenth is a good specimen: "*Qu.* 16. Whether certain maxims do not pass current among analysts which are shocking to good sense? And whether the common assumption, that a finite quantity divided by nothing is infinite, be not of this number?"

Jurin's First Reply to Berkeley

85. A reply to Berkeley's *Analyst* was made by the noted physician, James Jurin, at one time a student in Trinity College, Cambridge, who had imbibed Newtonian teachings from Newton himself. Jurin wrote under the pseudonym of "Philalethes Cantabrigiensis." The letter[1] is dated April 10, 1734.

86. Philalethes says that the charge in the *Analyst* "consists of three principal points: (1) Of Infidelity with regard to the Christian Religion. (2) Of endeavouring to make others Infidels, and succeeding in those endeavours by means of *the deference* which is paid *to their judgment, as being*

[1] *Geometry No Friend to Infidelity: or, a Defence of Sir Isaac Newton and the British Mathematicians, In a Letter to the Author of the Analyst. Wherein it is examined, How far the Conduct of such Divines as intermix the Interest of Religion with their private Disputes and Passions, and allow neither Learning nor Reason to those they differ from, is of Honour or Service to Christianity, or agreeable to the Example of our Blessed Saviour and his Apostles.* By Philalethes Cantabrigiensis. Ne Deus intersit, nisi dignus vindice nodus Inciderit. London: Printed for T. Cooper at the Globe in Ivy-Lane. MDCCXXXIV. Price 1s.

presumed to be of all men the greatest masters of reason. (3) Of error and false reasoning in their own science."

87. The early part of Jurin's reply is given to a discussion of the religious side. If there is no more certainty in modern analysis, argues Jurin, than in the Christian religion, this comparison brings no honour to Christianity; it is not true that mathematicians are infidels, leading others to infidelity. If it were true, this fact ought not in prudence to be published. Even if it be shown that the method of fluxions is built upon false principles, will it follow that all other parts of mathematics rest on inaccurate and false reasoning? Your attack, I surmise, is really, not so much in the interest of Christianity, as to demonstrate your superiority as a reasoner, by showing Newton and Barrow, two of the greatest mathematicians, less clear and just than you are. But because a mathematician "is thought to reason well in Geometry," his "decisions against the Christian Religion" will not "pass even upon weak and vulgar minds." "Sir Isaac Newton was a greater Mathematician than any of his contemporaries in France, . . . yet I have not heard that the French Mathematicians are converted to the Protestant Religion by his authority." Your objections against Newton's Fluxions may be "reduced under three heads: (1) Obscurity of this doctrine; (2) False reasoning in it by Sir Isaac Newton, and implicitly received by his followers; (3) Artifices and fallacies used by

Sir Isaac Newton, to make this false reasoning pass upon his followers." Jurin continues: "It must be owned that this doctrine . . . is not without difficulties," but "have you not altered his expressions in such a manner, as to mislead and confound your readers, instead of informing them," thereby increasing the difficulties? "Where do you find Sir Isaac Newton using such expressions as *the velocities of the velocities, the second, third and fourth velocities, the incipient celerity of an incipient celerity, the nascent augment of a nascent augment?*" As to the "moment or increment of the rectangle AB," the mathematicians take it to be $a\mathrm{B}+b\mathrm{A}$; you say that the rigorous value is $a\mathrm{B}+b\mathrm{A}+ab$. "Do not they know that in estimating any finite quantity how great soever . . ., a globe, suppose, as big as the earth, . . . or even the orb of the fixed stars . . ., this omission shall not cause them to deviate from the truth so much as a single pin's head, nay not the millionth part of a pin's head?" The operations by fluxions are no more objectionable than those by decimal fractions, where we take ·33333, etc., instead of $\frac{1}{3}$. You say that the Marquis de l'Hospital, in his *Analyse des infiniment petits*, Prop. 2, having found the fluxion of xy to be $xdy+ydx+dxdy$, drops the $dxdy$ "without the least ceremony." But does he not especially require in a postulate, "that a quantity, which is augmented or diminished by another quantity infinitely less than the first, may be considered as if it continued the same, *i.e.* had received no such augmentation or

diminution?" As to Newton, he takes (*Principia*, lib. ii, lemma 2, cas. 1; our § 17) initially $(A-\frac{1}{2}a)(B-\frac{1}{2}b)$ and finally $(A+\frac{1}{2}a)(B+\frac{1}{2}b)$, thereby deriving $aB+bA$, not as the increment of AB, but as the increment of $(A-\frac{1}{2}a)(B-\frac{1}{2}b)$. ". . . Rigorously speaking, the moment of the rectangle AB is not, as you suppose, the increment of the rectangle AB; but it is the increment of the rectangle $\overline{A-\frac{1}{2}a} \times \overline{B-\frac{1}{2}b}$." A moment may be either an increment or a decrement; you obtain the increment $aB+bA+ab$, the decrement of AB is $aB+bA-ab$. Which of those two will you call the moment of AB? "I apprehend the case will stand thus: $aB+bA+ab+aB+bA-ab$ making twice the moment of the rectangle AB; it follows that $aB+bA$ will make the single moment of the same rectangle";[1] the velocity which the flowing rectangle has, is its velocity "neither before nor after it becomes AB, but at the very instant of time that it is AB." In like manner with the moment of the rectangle. Let me advise you hereafter to "first examine and weigh every word he [Newton] uses." Lastly, I must observe that the moment of AB, namely $aB+bA$, and the increment of the same rectangle, $aB+bA+ab$, "are perfectly and exactly equal, supposing a and b to be diminished *ad infinitum*."

88. As to your second instance of false reasoning, in Newton's book on *Quadratures*, apparently that is "so truly Bœotian a blunder" that I know not how "a Newton could be guilty of it." You

[1] Jurin, *op. cit.*, p. 46.

interpret "*Evanescant jam augmenta illa*" (our § 32), as "let now the increments vanish, *i.e.* let the increments be nothing, or let there be no increments." But "do not the words *ratio ultima* stare us in the face, and plainly tell us that though there is a last proportion of evanescent increments, yet there can be no proportion of increments which are nothing, of increments which do not exist?" You grossly misinterpreted Newton.

89. As to the third head of your objections, since Newton did not reason falsely, "he had no occasion to make use of arts and fallacies to impose upon his followers." "Having now . . . driven you entirely out of your intrenchments . . . I should sally out and attack you in your own." "But as they seem rather designed for shew, than use, . . . to dazzle the imagination . . . [they] will likewise immediately disappear like *the Ghost of a departed quantity*," if you exorcise them "with a few words out of the first section of the *Principia*." You say that the paradox, "that Mathematicians should deduce true Propositions from false Principles" is accounted for by the fact that one error "is compensated by another contrary and equal error." But the two are no errors at all, as is evident from the fact that true results follow when only the first operation is carried out, so that no compensation is possible. Jurin argues that the first supposed fallacy, without the second, gives as the subtangent of $y^2 = ax$, the value $2x(2y + dy) \div (2y)$; the second supposed fallacy,

without the first, gives $2x(2y) \div (2y + dy)$. Both these expressions are equal to $2x$, "which is the result either of two errors, or of none at all." If you claim that $2x(2y + dy) \div (2y) > 2x$, how much greater is it, supposing $2x = 1000$ miles? Not as much as the thousand-millionth part of an inch. Jurin ends with a discussion of Lock on abstract ideas.

Walton's First Reply to Berkeley

90. Little is known about John Walton. He was Professor of Mathematics in Dublin, and participated in this controversy. Otherwise, practically nothing about him has been handed down.

His reply to Berkeley was published in 1735 at Dublin.[1] Berkeley attacked the method of fluxions more particularly as given in Newton's earlier exposition; Walton defended the theory on the basis of the later treatment as given by Newton in his *Quadratura Curvarum* (1704), and in the *Principia*, Book II.

91. Walton begins by stating that inasmuch as the credulous may "become infected" by Berkeley's attack on fluxions, it seems necessary to give a short account of the nature of fluxions. "The momentaneous Increments or Decrements of flow-

[1] *A Vindication of Sir Isaac Newton's Principles of Fluxions, against the Objections contained in the Analyst.* By J. Walton.—Siquid novisti rectius istis, candidus imperti: Si non, his utere mecum. *Hor.* In the fulness of his Sufficiency he shall be in Straits: Every Hand of the *Wicked* shall come upon him. *Job.*—Dublin, Printed; and reprinted at London, and sold by J. Roberts in Warwick-Lane. 1735. [Price Six Pence.]

ing Quantities, he [Newton] elsewhere calls by the name of *Moments*, . . : By Moments we may understand the nascent or evanescent Elements or Principles of finite Magnitudes, but not Particles of any determinate Size, or Increments actually generated; for all such are Quantities, themselves generated of Moments."

92. "The magnitudes of the momentaneous Increments or Decrements of Quantities are not regarded in the Method of Fluxions, but their first or last Proportions only; that is, the Proportions with which they begin or cease to exist." . . . "The ultimate Ratios with which synchronal Increments of Quantities vanish, are not the Ratios of finite Increments, but Limits which the Ratios of the Increments attain, by having their magnitudes infinitely diminish'd. . . . There are certain determinate Limits to which all such Proportions perpetually tend, and approach nearer than by any assignable Difference, but never attain before the Quantities themselves are infinitely diminish'd; or 'till the Instant they evanesce and become nothing." "The Fluxions of Quantities are very nearly as the Increments of their Fluents generated in the least equal Particles of Time," and they "are accurately in the first or last Proportions of their nascent or evanescent Increments." "The Fluxions of Quantities are only velocities. . . ." Again, ". . . to obtain the Ratios of Fluxions, the corresponding synchronal or isochronal Increments must be lessened *in infinitum*. For the

Magnitudes of synchronal or isochronal Increments must be infinitely diminished and become evanescent, in order to obtain their first or last Ratios, to which Ratios the Ratios of their corresponding Fluxions are equal." The moment of the rectangle AB is $A\dot{b}+B\dot{a}$, for consider $A\dot{b}+B\dot{a}+\dot{a}\dot{b}$ and $A\dot{b}+B\dot{a}$, "under a constant Diminution of the Increments $\dot{a}$ and $\dot{b}$ . . . [they] constantly tend to an Equality . . . [and] they become equal, and their Ratio becomes a Ratio of Equality. . . ." Hence $A\dot{b}+B\dot{a}+\dot{a}\dot{b}$ "is not the Moment or Fluxion of the Rectangle AB, except in the very Instant when it begins or ceases to exist." Here fluxions appear to be no longer velocities (finite magnitudes) but moments. Walton next quotes a Latin passage from the *Quadratura Curvarum*. He says that Berkeley seems "to have been deceived by an Opinion that there can be no first or last Ratios of mathematical Quantities," but Walton insists that if quantities are generated together, or if they vanish together, they will do so "under certain Ratios, which are their first or last Ratios." Walton claims that Berkeley's lemma "is in no Way pertinent to the Case for which it was intended"; he explains the Newtonian process of finding the fluxion of x^n, supposing x to increase uniformly, and points out that this is done without rejecting quantities "on account of their exceeding smallness." Commenting on Berkeley's contention that "no geometric Quantity, by being infinitely diminished, can ever be exhausted or become

nothing," Walton states that the fluxional calculus assumes that "Quantities can be generated by Motion . . . and consequently they may also by Motion be destroy'd."

93. Walton's *Vindication* follows Newton's exposition closely; Berkeley's claim that Walton followed in Jurin's track and borrowed from him, is, I believe, incorrect. Take the vital question of rejecting infinitesimals: Jurin claims that, being so very small, they do not appreciably affect the result; Walton takes the stand that there is no rejection whatever of infinitesimals. The main criticism to be passed on Walton's first essay consists, in our judgment, in a failure to meet Berkeley's objections squarely and convincingly.

Berkeley's Reply to Jurin and Walton

94. Jurin's and Walton's articles were answered by Berkeley in a publication entitled, *A Defence of Free-Thinking in Mathematics.*[1]

Berkeley restates the purpose he had in writing the *Analyst*: "Now, if it be shewn that fluxions are really most incomprehensible mysteries, and that those who believe them to be clear and scientific do entertain an implicit faith in the author of that method: will not this furnish a fair *argumentum ad hominem* against men who reject that very thing in religion which they admit in human learn-

[1] *A Defence of Free-Thinking in Mathematics. In Answer to a Pamphlet of Philalethes Cantabrigiensis. . . . Also an Appendix concerning Mr. Walton's Vindication. . . . By the Author of "The Minute Philosopher,"* Dublin, 1735.

ing?[1] (§ 3) . . . I say that an infidel, who believes the doctrine of fluxions, acts a very inconsistent part in pretending to reject the Christian religion —because he cannot believe what he doth not comprehend" (§ 7). . . .

[1] Berkeley is not the only one who invoked the aid of the Doctrine of Fluxions in theological discussion. In a criticism (*A Review of the Fiery Eruption*, etc., London, 1752, p. 128) of Bishop William Warburton's *Julian*, concerning earthquakes and fiery eruptions, which, Warburton argued, defeated Julian's attempt to rebuild the temple at Jerusalem, it is stated that a connection (needed in the argument) was established between the preservation of Christianity and the destruction of Judaism by the following clever procedure:—

"The great modern Father of the mathematics had invented a new and curious way of improving that science by a fiction; according to which quantities are supposed to be generated by the continual flux or motion of others. In the application of this method it became necessary to consider these quantities, sometimes in a nascent, and at other times in an evanescent state, by which ingenious contrivance they could be made either continually to tend to and at last absolutely to become nothing, or *vice versa*, according to the intention and occasions of the Artist. Now by extending this noble invention to the two religions, it evidently appeared, that, from the time of the first coming of Christ, Judaism entered into its *evanescent* state, as on the other hand Christianity did into a *nascent* state, by which means both being put into a proper flux, one was seen continually decaying, and the other continually improving, till at last by the destruction of the Temple Judaism actually vanished and became nothing, and the Christian religion then bursted out a perfectly generated Entity. . . . As the great author of the mathematical method of fluxions had for very good reasons studiously avoided giving any definition of the precise magnitude of those moments, by whose help he discovers the exact magnitude of the generated quantities, so our Author [Warburton] by the same rule of application, and under the influence of the same authority, was fairly excused from defining that precise degree of perfection and imperfection in which the two religions subsisted, during the respective *evanescent* and *nascent* state of each, by the help of which he discovered the precise time when Judaism was perfectly abolished, and Christianity perfectly established. But we may well suppose, that the most alluring charm in this extraordinary piece of ingenuity, was the creating of a new character by it: For questionless he may now be justly stiled the great founder and inventor of the *fluxionary method* of theology. . . . This fancy of a necessary connexion between the Temple-edifice, and the being of Christianity, . . . this pretended Christianity which is of such an unsubstantial nature, that it must necessarily vanish at the restoration of the Temple, can be nothing else but a mere *Ghost*, . . . evidently the Ghost of departed Judaism."

95. "I have said (and I venture still to say) that a fluxion is incomprehensible: that second, third, and fourth fluxions are yet more incomprehensible: that it is not possible to conceive a simple infinitesimal: that it is yet less possible to conceive an infinitesimal of an infinitesimal, and so onward. What have you to say in answer to this? Do you attempt to clear up the notion of a fluxion or a difference? Nothing like it" (§ 17).

96. Berkeley quotes from Newton's *Principia* and *Quadrature of Curves*, and then asks, "Is it not plain that if a fluxion be a velocity, then the fluxion of a fluxion may, agreeably thereunto, be called the velocity of a velocity? In like manner, if by a fluxion is meant a nascent augment, will it not then follow that the fluxion of a fluxion or second fluxion is the nascent augment of a nascent augment?" (§ 23).

97. "I had observed that the great author had proceeded illegitimately, in obtaining the fluxion or moment of the rectangle of two flowing quantities. . . . In answer to this you allege that the error arising from the omission . . . is so small that it is insignificant (§ 24). . . . If you mean to defend the reasonableness and use of approximations . . . I have nothing to say. . . . That the method of fluxions is supposed accurate in geometrical rigour is manifest to whoever considers what the great author writes about it . . . *In rebus mathematicis errores quam minimi non sunt contemnendi*" (§ 25; our § 30).

98. Berkeley justifies his use of the expression "increment of a rectangle" by quoting from Newton (our § 17), "*rectanguli incrementum* $a\mathrm{B}+b\mathrm{A}$."

"You say 'you do not consider AB as lying at either extremity of the moment, but as extended to the middle of it; as having acquired the one half of the moment, and as being about to acquire the other; or, as having lost one half of it, and being about to lose the other.' Now, in the name of truth, I entreat you to tell what this moment is, . . . Is it a finite quantity, or an infinitesimal, or a mere limit, or nothing at all? . . . If you take it in either of the two former senses, you contradict Sir Isaac Newton. And, if you take it in either of the latter, you contradict common sense; it being plain that what hath no magnitude, or is no quantity, cannot be divided" (§ 30).

". . . You observe that the moment of the rectangle determined by Sir Isaac Newton, and the increment of the rectangle determined by me are perfectly and exactly equal, supposing a and b to be diminished *ad infinitum*: and, for proof of this, you refer to the first lemma of the first section of the first book of Sir Isaac's Principles. I answer that if a and b are real quantities, then ab is something, and consequently makes a real difference: but if they are nothing, then the rectangles whereof they are coefficients become nothing likewise: and consequently the *momentum* or *incrementum*, whether Sir Isaac's or mine, are in that case nothing at all. As for the above-mentioned

lemma, . . . however that way of reasoning may do in the method of *exhaustions*, where quantities less than assignable are regarded as nothing; yet, for a fluxionist writing about momentums, to argue that quantities must be equal because they have no assignable difference, seems the most injudicious step that could be taken; . . . for, it will thence follow that all homogeneous momentums are equal, and consequently the velocities, mutations, or fluxions, proportional thereto, are all likewise equal" (§ 32).

99. As regards Newton's *evanescant jam augmenta illa* (our § 32), Berkeley argues that it means either "let the increments vanish," or else "let them become infinitely small," but the latter "is not Sir Isaac's sense," since on the very same page in the Introduction to the Quadrature of Curves he says that there is no need of considering infinitely small figures. Taking advantage of the fact that the Newton of the *Principia* (1687) differed from the Newton of the *Quadratura Curvarum* (1704), Berkeley broke out into the following philippic: "You Sir, with the bright eyes, be pleased to tell me, whether Sir Isaac's momentum be a finite quantity, or an infinitesimal, or a mere limit? If you say a finite quantity; be pleased to reconcile this with what he saith in the scholium of the second lemma of the first section of the first book of his Principles (our § 12): *Cave intelligas quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite.* If you say, an infinitesimal; reconcile this

with what is said in his Introduction to the Quadratures (our § 33): *Volui ostendere quod in methodo fluxionum non opus sit figuras infinite parvas in geometriam introducere.* If you should say, it is a mere limit; be pleased to reconcile this with what we find in the first case of the second lemma in the second book of his Principles (our § 17): *Ubi de lateribus A et B deerant momentorum dimidia*, etc.—where the moments are supposed to be divided. I should be very glad a person of such a luminous intellect would be so good as to explain whether by fluxions we are to understand the nascent or evanescent quantities themselves, or their motions, or their velocities, or simply their proportions . . . that you would then condescend to explain the doctrine of the second, third, and fourth fluxions, and show it to be consistent with common sense if you can" (§ 36).

100. In an appendix to the *Defence of Free-Thinking in Mathematics*, Berkeley replies to Walton, stating that the issues raised by him had been previously raised by "the other," that he delivered a technical discourse without elucidating anything, that his scholars have a right to be informed as to the meaning of fluxions and should therefore ask him "the following questions." Then follow many questions, of which we give a few:

"Let them ask him—Whether he can conceive velocity without motion, or motion without extension, or extension without magnitude? . . . Whether nothing be not the product of nothing

multiplied by something; and, if so, . . . when *ab* is nothing, whether $A\dot{b}+B\dot{a}$ be not also nothing? *i.e.* whether the momentum of AB be not nothing? Let him then be asked, what his momentums are good for, when they are thus brought to nothing? . . . I wish he were asked to explain the difference between a magnitude infinitely small and a magnitude infinitely diminished. . . . Let him be farther asked, how he dares to explain the method of Fluxions, by the *Ratio* of magnitudes infinitely diminished, when Sir Issac Newton hath expressly excluded all consideration of quantities infinitely small? If this able vindicator should say that quantities infinitely diminished are nothing at all, and consequently that, according to him, the first and last *Ratio's* are proportions between nothings, let him be desired to make sense of this. . . . If he should say the ultimate proportions are the *Ratio's* of mere limits, then let him be asked how the limits of lines can be proportioned or divided?"

Walton's Second Reply to Berkeley

101. In a second reply[1] to Berkeley, Walton states that in the Appendix to the *Defence*, Berkeley "has composed a Catechism which he recommends to my Scholars" and which Walton quotes. I am first to be asked, "Whether I can conceive Velocity without Motion, or Motion without Extension. . . .

[1] J. Walton, *Catechism of the Author of the Minute Philosopher Fully answer'd.* Printed at Dublin. Reprinted at London, and sold by J. Roberts, 1735. It is a pamphlet of 30 pages.

I answer, I can conceive Velocity and Motion in a Point of Space; that is, without any assignable Length or Extension described by it . . . for . . . if a cause acts continually upon a given Thing . . . there must be a continual Increase of its Velocity: the Velocity cannot be the same in any two different Points," as in the case of falling bodies. Referring to $A\dot{b}+B\dot{a}$, Walton continues: "I agree with him that nothing is the Product of nothing multipl'd by something; but must know what he means by the vanishing of the Gnomon[1] and Sum of the two Rectangles . . . before I give him a direct Answer. If by vanishing he means that they vanish and become nothing as Areas, I grant they do; but absolutely deny, upon such an Evanescence of the Gnomon and Sum of the two Rectangles by the moving back of the Sides of the Gnomon till they come to coincide with those of the Rectangle, that nothing remains. For there still remain the moving Sides, which are now become the Sides of the Rectangle, . . . the Motion of the Gnomon is the same with the Sum of the Motions of the Two Rectangles, when they evanesce, and are converted into the two Sides of the Rectangle AB. If a point moves forward to generate a Line, and afterwards the same Point moves back again to destroy the Line with the very same Degrees of Velocity, in all Parts of the Line

[1] If a parallelogram is extended in length and breadth and if the original parallelogram be removed, the remaining figure is called the gnomon.

which it had in those Parts when moving forward to generate it; in the Instant the Line vanishes as a Length . . . the generating point will remain, together with the Velocity it had at the very Beginning of its Motion. And the Case is the very same with respect to the Rectangle increasing by the Motion of its Sides." This point is elaborated with great fullness. After some illustrations, Walton exclaims: "This is a full and clear Answer to this part of the catechism, and shows that its Author has been greatly mistaken in supposing that I explained the Doctrine of Fluxions by the *Ratio of Magnitudes infinitely diminish'd*, or by *Proportions between nothings*. . . . I do not wonder that this Author should have no clear Ideas or Conceptions of second, third or fourth Fluxions, when he has no clear Conceptions of the common Principles of Motion, nor of the first and last Ratios of the isochronal Increments of Quantities generated and destroyed by Motion. . . . In order to prevent my being Catechised any more by this Author," Walton makes a confession "of some Part of my Faith in Religion."

Jurin's Second Reply to Berkeley

102. Jurin brought out a second publication,[1] of 112 pages, which was in reply to Berkeley's *Defence of Free-Thinking*. Passing by unimportant preliminaries, we come to Jurin's definitions of "flow-

[1] *The Minute Mathematician: or, The Free-Thinker no Just-Thinker*. By Philalethes Cantabrigiensis. London, 1735.

ing quantity," "fluxion" ("the velocity with which a flowing quantity increases or decreases"), "increment," "nascent increment" ("an increment just beginning to exist from nothing . . . but not yet arrived at any assignable magnitude how small soever"), "evanescent increment" (similarly defined). He then endeavours to prove the proposition: "The Fluxions, or Velocities of flowing quantities . . . are exactly in the first proportion of the nascent increments, or in the last proportion of the evanescent increments." He insists that "the first ratio of the nascent increments must be the same, whether the velocities be uniform or variable"; hence, "the nascent increments must be exactly as the velocities with which they begin to be generated." In further explanation, Jurin says that, according to Newton, nascent increments are "less than any finite magnitude," "their magnitude cannot be assigned or determined," "the proportion between them . . . being all that is requisite in his Method." In further explanation of the proportion of evanescent increments he says, it "is not their proportion before they vanish," "nor is it their proportion after they have vanished," "but it is their proportion at the instant that they vanish." Jurin then states that Berkeley has "taken as much pains as . . . any man living, except a late Philosopher of our University, to make nonsense of Sir Isaac Newton's principles." There is no occurrence in Newton's writings of "velocity without motion," "motion without extension," which

Berkeley pretends to derive from them. Jurin succeeds, we think, in establishing the contention that there is no greater difficulty in explaining the second or third fluxion, than there is in explaining the first. "The second fluxion is the velocity with which the first fluxion increases." Jurin confesses that his statement in his first reply to Berkeley, to the effect that certain errors were of "no significance in practice," was intended for popular consumption, for men such as one meets in London.

103. "One of them, indeed, could make nothing of what I had said about the length of a subtangent, or the magnitude of the orb of the fixed stars; but was fully satisfied by the information given him by one of his acquaintance to the following effect. The Author of the Minute Philosopher has found out that, if Sir Isaac Newton were to measure the height of St. Paul's Church by Fluxions, he would be out about three quarters of a hair's breadth: But yonder is one *Philalethes* at Cambridge, who pretends that Sir Isaac would not be out above the tenth part of hair's breadth. Hearing this, and that two books had been written in this controversy, the honest gentleman flew into a great passion, and after muttering something to himself about some body's being overpaid, he went on making reflections, which I don't care to repeat, as not being much for your honour or mine."

104. Jurin thereupon takes up the rectangle AB. The terms "moment" and "increment" are involved in the discussion of it. Jurin de-

clares: "I absolutely and fully agree with you that the *incrementum* in the conclusion is the *momentum* in the Lemma," that "the *momentum* in the Lemma" is "the *momentum* of the rectangle AB." Further, Jurin says, "the *incrementum* in the conclusion is manifestly the excess of the rectangle $\overline{A+\frac{1}{2}a} \times \overline{B+\frac{1}{2}b}$, above the rectangle $\overline{A-\frac{1}{2}a} \times \overline{B-\frac{1}{2}b}$, i.e. the increment of the rectangle $\overline{A-\frac{1}{2}a} \times \overline{B-\frac{1}{2}b}$. Therefore we are agreed that the moment of the rectangle AB is the increment of the rectangle $\overline{A-\frac{1}{2}a} \times \overline{B-\frac{1}{2}b}$. Consequently you were mistaken in supposing that the moment of the rectangle AB was the increment of the same rectangle AB. . . . The moment AB is neither the increment nor the decrement of AB," for if it really was the increment of AB, and also its decrement, we would have $Ab+Ba+ab = Ab+Ba-ab$, i.e. $2ab=0$. Hence the rectangle *ab* "is by his own confession equal to nothing." Jurin concludes that the fluxion of AB is not the velocity with which the increment or decrement of AB is generated, but the "middle arithmetical proportional between these two velocities," this being "in like manner as I had before supposed an arithmetical mean between the increment and decrement of AB, which mean is the moment of AB." Berkeley had considered four definitions of a moment, that of a finite quantity, or an infinitesimal or a mere limit, or nothing at all; and he had found each either to contradict Newton or to contradict common sense. Jurin does not accept "any one of those senses." A moment,

says Jurin, is defined by Newton as "nascent increment," its magnitude is "utterly unassignable." Jurin continues:

105. "You seem much at a loss to conceive how a nascent increment, a quantity just beginning to exist, but not yet arrived to any assignable magnitude, can be divided or distinguished into two equal parts. Now to me there appears no more difficulty in conceiving this, than in apprehending how any finite quantity is divided or distinguished into halves. For nascent quantities may bear all imaginable proportions to one another, as well as finite quantities."

106. Near the close Jurin enters upon the discussion of Berkeley's *Lemma*, given in the *Analyst*: "If one supposition be made, and be afterwards destroy'd by a contrary supposition, then everything that followed from the first supposition, is destroyed with it." Not so, says Jurin, when the supposition and its contradiction are made at different times. "Let us imagine yourself and me to be debating this matter, in an open field, . . . a sudden violent rain falls . . . we are all wet to the skin . . . it clears up . . . you endeavour to persuade me I am not wet. The shower, you say, is vanished and gone, and consequently your . . . wetness . . . must have vanished with it." You say that your explanation of the correctness of results as due to a compensation of errors, was intended by you to apply, not to Newton, but to Marquis de l'Hospital; your statements were such that not I

alone, but Mr. Walton as well, inferred that you were charging Newton with committing double errors. The rest of Jurin's ill-arranged article is given either to a renewed and fuller elucidation of his previous contentions or to poetical outbursts. Sure of the soundness of his exposition, he exclaims, "I meet with nothing in my way but *the Ghosts of departed* difficulties and objections."

Berkeley's Second Reply to Walton

107. Walton's *Catechism . . . fully Answered* was followed by Berkeley's *Reasons for not replying to Mr. Walton's Full Answer*, 1735. This last reply has been called "a combination of reasoning and sarcasm," in which "he affects to treat his opponent as a disguised convert." Says Berkeley: "He seems at bottom a facetious man, who, under the colour of an opponent, writes on my side of the question, and really believes no more than I do of Sir Isaac Newton's doctrine about fluxions, which he exposes, contradicts, and confutes, with great skill and humour, under the mask of a grave vindication." Berkeley objects to Walton's motion and velocity "in a point" of space; "consider the reasoning: The same velocity cannot be in two points of space; therefore velocity can be in a point of space. . . . I can as easily conceive Mr. Walton should walk without stirring, as I can his idea of motion without space."[1] Newton calls absolute

[1] Walton is not consistent in his use of the term "motion." In some passages it means translation; in others it means velocity, or else both

motion "a translation from absolute place to absolute place,"[1] and relative motion, "from one relative place to another. Mr. Walton's is plainly neither of these sorts of motion"; hence, he argues against Newton. "When ab is nothing, that is, when a and b are nothing, he denies that $Ab+Ba$ is nothing. This is one of the inconsistencies which I leave the reader to reconcile." In his *Vindication* he holds that, "to obtain the last ratio of synchronal increments, the magnitude of those increments must be infinitely diminished"; in his *Catechism . . . fully Answered* "he chargeth me as greatly mistaken in supposing that he explained the doctrine of fluxions by the ratio of magnitudes infinitely diminished."[2] In his *Catechism . . . fully Answered* "he tells us that 'fluxions are measured by the first and last proportion of isochronal increments generated or destroyed by motion.' A little later he says, these ratios subsist when the isochronal increments have no magnitude." Can "isochronal increments subsist when they have no magnitude"? Berkeley

translation and velocity, as when he says, ". . . isochronal increments must be made to vanish by a Retroversion of the Motion before we can obtain the Motions with which they vanish, or begin to be generated; that is, before we can obtain the *Fluxions* of the Quantities, the Name given by Sir Isaac Newton to those Motions." J. Walton, *Catechism . . . fully Answered*, pp. 18, 19.

[1] I. Newton, *Principia*, Definitions, Scholium, def. viii.

[2] What Walton actually wrote was, that Berkeley had been mistaken in supposing that he explained fluxions "by the *Ratios of Magnitudes infinitely diminish'd*, or by *Proportions between nothings*." Three pages earlier Walton had denied that Newton and he measured fluxions "by the Proportions of Magnitudes infinitely small." Evidently Walton meant to exclude the "infinitely small," but used "magnitudes infinitely diminished" at one time as magnitudes "infinitely small," and at another time as signifying something else, namely, "increments" that "vanish."

then quotes from his own *Analyst*: "As it is impossible to conceive velocity without time or space, without either finite length or finite duration, it must seem above the power of man to comprehend even the first fluxions." In the endeavour to explain this matter, Walton's skill has been "vain and impertinent."

The Second Edition of Walton's Second Reply

108. Walton begins[1] by explaining what Newton means by Velocity. It is "the ratio of the Quantity of Motion to the Quantity of Matter in the body"; that is, if V is the velocity, M the quantity of motion, F the force generating the motion, D the density, B the bulk or magnitude, W the weight, then "V is M / Q, and is as F / W, or as F / DB," for, "the Quantity of Motion is the Quantity of Matter and Velocity taken together; that is, M is QV" (p. 35). "The Author [Berkeley] therefore has been grossly mistaken in asserting that *Velocity* necessarily implies both Time and Space, and cannot be conceived without them.—And that there is *no* Measure of Velocity except Time and Space." It appears that "a body in Motion, will have a Velocity inherent in itself during the Whole Time of its Motion: and consequently there must be a Velocity where-ever the Body is, exclusive of Time and Space . . . its

[1] *The Catechism of the Author of the Minute Philosopher fully answer'd. The Second Edition. With an Appendix, in Answer to the Reasons for not replying to Mr. Walton's Full Answer.* By J. Walton . . . Dublin: Printed by S. Powell, for William Smith at the Hercules, Bookseller: in Dame-Street, 1735.

[a point's] Velocity will exist in a Point, and successively will exist in every Point of Space through which the Point moves" (p. 37). Berkeley thinks that "from the generated Velocity not being the same in any two different Points of the described Space it will not follow that Velocity can exist in a Point of Space. But in this he is mistaken. For the continual Action of a Moving Force necessarily preserves a *continual* Velocity; and if the generated Velocity be not the same in any two different Points of the described Space, a Velocity must of Consequence exist in every Point of that Space" (p. 38). This account of velocity "is agreeable to Sir Isaac Newton's Notion of Velocity; who constantly excludes described Space from his Idea of that Term." Motion being measured by QV, "the continual translation of a Body therefore into a new Place is, . . . an *Effect* of this Tendency forward in the Body, and not the Tendency itself; consequently Space described is an *Effect* of Velocity, and not Velocity itself" (p. 47). On the question of first and last ratios it cannot be said that Walton here throws new light. He insists that he explained fluxions not "by the Ratio of Magnitudes infinitely diminish'd, but by the first and last Ratios of Increments generated or destroyed in equal times: that is, by the Ratios of the Velocities with which those Increments begin or cease to exist" (p. 53). To Berkeley's charge that Walton "supposed two Points to exist at the same Time in one Point, and to be moved different

Ways without stirring from that Point," Walton replies that there is no difficulty in supposing two points existing in a given place each having its own velocity, but he never said that they can go in different directions "without stirring from the Point." Berkeley, in his remarks about the fourth fluxion of a cube, did not observe all the conditions which he [Walton] had imposed. "He [Berkeley] intreats me to explain whether Sir Isaac's Momentum be a finite Quantity, or an Infinitesimal, or a mere Limit. I tell him, that Sir Isaac's Momentum is a *finite* quantity; it is a Product contained under the moving Quantity and its Velocity, or under the moving Quantity and first Ratio of that Space described by it in a given Particle of Time." Since both these factors are finite, the product is finite (p. 62). "By Moments therefore he is not to understand generated Increments of Fluents, but certain finite Products or Quantities of very different Nature from generated Increments, expressing only the Motions with which those Increments begin or cease to exist" (p. 63).

Remarks

109. Berkeley's *Analyst* must be acknowledged to be a very able production, which marks a turning-point in the history of mathematical thought in Great Britain.

His contention that no geometrical quantity can be exhausted by division[1] is in consonance with

[1] See our § 79.

the claim made by Zeno in his "dichotomy," and the claim that the actual infinite cannot be realised. The modern reader may not agree with Berkeley on this point, nor in the claim that second or third fluxions are more mysterious than the first fluxion. Nevertheless, a reader of Berkeley feels that he spoke in the *Analyst* with perfect sincerity. Interesting is De Morgan's comment :[1] "Dishonesty must never be insinuated of Berkeley. But the *Analyst* was intentionally a publication involving the principle of Dr. Whately's argument against the existence of Buonaparte ; and Berkeley was strictly to take what he found. The *Analyst* is a tract which could not have been written except by a person who knew how to answer it. But it is singular that Berkeley, though he makes his fictitious character nearly as clear as afterwards did Whately, has generally been treated as a real opponent of fluxions. Let us hope that the arch Archbishop will fare better than the arch Bishop."

110. Sir William Rowan Hamilton once wrote De Morgan : "On the whole, I think that Berkeley persuaded *himself* that he was in earnest against Fluxions, especially of orders higher than the first, as well as against matter." To this De Morgan replied : "I have no doubt Berkeley knew that the fluxions were sound enough."[2]

[1] A. De Morgan, *Philosophical Magazine*, 4 S., vol. iv, 1852, p. 329, note.

[2] *Life of Sir William Rowan Hamilton*, by R. P. Graves, vol. iii, 1889, p. 581.

111. One is not so easily convinced of the ability and sincerity of Jurin. That at first he should argue that quantities may be dropped because small, and afterwards admit that this argument was intended for popular consumption, is not reassuring.[1] That he should fail to see the soundness of Berkeley's criticism of Newton's proof $(A+\frac{1}{2}a)(B+\frac{1}{2}b)-(A-\frac{1}{2}a)(B-\frac{1}{2}b)$ for the increment of AB is somewhat surprising, even if it must be admitted that neither Walton nor any other eighteenth-century mathematician appears to have seen and admitted the defect. In this connection we quote from a letter which Hamilton wrote De Morgan in 1862 when Hamilton was seeing his *Elements of Quaternions* through the press:[2]

"When your letter arrived this morning, I was deep in Berkeley's 'Defence of Freethinking in Mathematics'; . . . I think there is more than mere plausibility in the Bishop's criticisms on the remarks attached to the Second Lemma of the Second Book of the Principia; and that it is very difficult to understand the *logic* by which Newton proposes to prove, that the *momentum* (as he calls it) of the *rectangle* (or product) AB is equal to $aB+bA$, if the *momenta* of the sides (or factors) A and B be denoted by a and b. His mode of getting rid of ab appeared to me long ago (I must confess it) to involve so much of *artifice*, as to

[1] See our §§ 97, 102, 103.

[2] *Life of Sir William Rowan Hamilton*, by R. P. Graves, vol. iii, p. 569.

deserve to be called *sophistical*; although I should not like to say so publicly. He subtracts, you know, $(A-\frac{1}{2}a)(B-\frac{1}{2}b)$ from $(A+\frac{1}{2}a)(B+\frac{1}{2}b)$; whereby, of course, ab disappears in the result. But by *what right*, or *what reason* other than to give an unreal air of *simplicity* to the calculation, does he *prepare* the *products* thus? Might it not be argued similarly that the difference,

$$(A+\tfrac{1}{2}a)^3-(A-\tfrac{1}{2}a)^3=3aA^2+\tfrac{1}{4}a^3,$$

was the moment of A^3; and is it not a sufficient *indication* that the mode of procedure adopted is not the fit one for the subject, that it quite *masks* the notion of a *limit*; or rather has the appearance of treating that notion as foreign and irrelevant, notwithstanding all that had been said so well before, in the First Section of the First Book? Newton does not seem to have cared for being very consistent in his *philosophy*, if he could anyway get hold of *truth*, or what he considered to be such. . . ."

We give also Hermann Weissenborn's objection[1] to Newton's procedure of taking half of the increments a and b; with equal justice one might take, says he, $(A+\frac{2}{3}a)(B+\frac{2}{3}b)-(A-\frac{1}{3}a)(B-\frac{1}{3}b)$, and the result would then be $Ab+Ba+\frac{1}{3}ab$.

112. Walton's two (or three) articles do not seem to have been read much. They are seldom mentioned. The pamphlets are now rare. Pro-

[1] H. Weissenborn, *Principien der höheren Analysis in ihrer Entwickelung von Leibniz bis auf Lagrange*, Halle, 1856, p. 42.

fessor G. A. Gibson had not seen them when he wrote on the *Analyst* controversy.[1] Walton seemed to have a good intuitive grasp of fluxions, but lacked deep philosophic insight. He showed himself inexperienced in the conduct of controversies, and did not know how to protect himself against attack from a skilful adversary.

113. It is worthy of notice that Walton[2] expressed himself on the nature of limits, by claiming that the limit was reached. As to the nature of "variable velocity," it is interesting to see that Berkeley realised the difficulty of the concept, and probably saw that there was no variable velocity as a physical fact, while Walton was compelled to take refuge in less primitive mechanical concepts in order to uphold his side of the argument.[3] Unjustifiable is Walton's identification of Newton's "moment" with "momentum" of mechanics.

114. Berkeley's Lemma[4] was rejected by Jurin and Walton. We shall see that it found no recognition from mathematicians in England during the eighteenth century, but was openly and repeatedly accepted as valid in its application to limits, by Woodhouse at the beginning of the nineteenth century. The Newtonian derivation of the fluxion of x^n (see our §§ 32, 41), accomplished by dividing both o and $(x+o)^n - x^n$ by the finite increment o, and then putting o equal to zero in the quotient, is

[1] G. A. Gibson, "The Analyst Controversy," in *Proceedings of the Edinburgh Math. Soc.*, vol. xvii, 1899, p. 18.
[2] See our § 92. [3] See our § 108. [4] See our §§ 78, 92, 106.

certainly open to the logical objection raised by Berkeley. Eighteenth-century mathematicians did not attach due importance to this point.

115. The existence of infinitesimals (infinitely small quantities) was denied by Berkeley, but, it would seem, not denied by Jurin and Walton. All three finally abjured the philosophy which permits their being dropped because so small. It is well known that many mathematicians of prominence have believed in the reality of such quantities. From Leibniz to Lagrange all Continental writers of note used them. Lagrange headed a small school that was opposed to them, when he published his *Fonctions analytiques*. There followed a reaction against Lagrange. De Morgan once remarked: "Duhamel, Navier, Cournot, are pure infinitesimalists. Some of them say an infinitely small quantity is one which may be made as small *as you please*. This is an evasion; but they do not mean that dx is *finite*. . . . By-the-way, Poisson was a believer in the *reality* of infinitely small quantities—as I am."[1]

". . . For myself, I am now fixed in the faith of the *subjective reality* of *infinitesimal quantity*. But *what* an infinitely small quantity is, I know no more than I know what a *straight line* is; but I know it *is*; and there I stop short. But I do not believe in *objectively realised* infinitesimals."

[1] *Life of Sir William Rowan Hamilton*, by Robert P. Graves, vol. iii, pp. 572, 580. Consult also De Morgan's article, "On Infinity; and on the Sign of Equality," in *Trans. of the Cambridge Phil. Society*, vol. xi, Cambridge, 1871 [read May 16, 1864].

116. We must not neglect to express our appreciation of the fact that Berkeley withdrew from the controversy after he had said all that he had to say on his subject. Some of the debates that came later were almost interminable, because the participants continued writing even after they had nothing more to say.

CHAPTER IV

JURIN'S CONTROVERSY WITH ROBINS AND PEMBERTON

Robins's "Discourse," and Review of it

117. Benjamin Robins was a native of Bath and a self-educated mathematician of considerable reputation.

The debate carried on by Bishop Berkeley with Jurin and Walton induced Benjamin Robins to issue a publication, entitled, *A Discourse Concerning the Nature and Certainty of Sir Isaac Newton's Methods of Fluxions, and of Prime and Ultimate Ratios*, 1735.[1] Evidently Robins felt that Berkeley's attacks should be met, and that Jurin was not the man to defend Newton satisfactorily. Robins was a man of mathematical power; his exposition is regarded by Professor G. A. Gibson as very able, and far superior to that of Jurin.[2] Without naming either Berkeley or Jurin, and without referring to their articles, Robins proceeds to his task. The whole foundation of the doctrine of fluxions is

[1] This paper is republished, along with subsequent articles on the same subject, in the *Mathematical Tracts of the late Benjamin Robins, Esq. . . .* in two volumes, edited by James Wilson, M.D. London, 1761, vol. ii, pp. 1–77.

[2] G. A. Gibson, *loc. cit.*, pp. 22–25.

based by Robins upon the following two definitions and certain general propositions annexed to them :

I. Definition : " . . . we . . . define an ultimate magnitude to be the limit, to which a varying magnitude can approach within any degree of nearness whatever, though it can never be made absolutely equal to it."

Here for the first time is the stand taken openly, clearly, explicitly, that a variable (say the perimeter of a polygon inscribed in a circle) can never *reach its limit* (the circumference). The gain from the standpoint of debating is very great ; a regular inscribed polygon whose sides are steadily doubling at set intervals of time, say, every half second, presents a picture to the imagination which is comparatively simple. The hopeless attempt of imagining the limit as reached need not be made. But this great gain is made at the expense of generality. Robins descends to a very special type of variation which is not the variation encountered in ordinary mechanics ; it is an exceedingly artificial variation. According to Robins's definition, Achilles never caught the tortoise. It would not be difficult to assume a time rate in the doubling of the sides of a polygon inscribed in a circle, so that the circumference *is reached.* Thus, let the first doubling of the number of sides take place in 1 second, the second doubling in $\frac{1}{2}$ a second, the third in $\frac{1}{4}$ a second, and so on. It is easy to see that under this mode of variation the polygons *do reach the*

limit, the circumference. The process here transcends our power of imagination, but lies within the limits of reason. We are dwelling upon this point because of its extreme importance in the history of the theory of limits.

118. Robins constructs upon his first definition the theorem, "that, when varying magnitudes keep constantly the same proportion to each other, their ultimate magnitudes are in the same proportion." As a corollary of this he states "that the ultimate magnitudes of the same or equal varying magnitudes are equal."

II. Definition: "If there be two quantities, that are (one or both) continually varying, either by being continually augmented, or continually diminished; though the proportion, they bear to each other, should by this means perpetually vary, but in such a manner, that it constantly approaches nearer and nearer to some determined proportion, and can also be brought at length in its approach nearer to this determined proportion than to any other, that can be assigned, but can never pass it: this determined proportion is then called the ultimate proportion, or the ultimate ratio of those varying quantities."

Theorem: "To this definition of the sense, in which the term ultimate ratio, or ultimate proportion is to be understood, we must subjoin the following proposition: That all the ultimate ratios of the same varying ratio are the same with each other."

119. Robins remarks thereupon that attempts at the exposition of this method, so far as it depends upon his first definition, were made by Lucas Valerius in a treatise on the centre of gravity, and by Andrew Tacquet in a treatise on the cylindrical and annular solids; but the development involving his second definition was first made by Newton. There are a number of writers, not mentioned by Robins, who might be cited as forerunners in the theory of limits; such, for instance, as Gregory St. Vincent and Stevin.

Newton's definition of momenta as the momentaneous increments or decrements of varying quantities, "may possibly be thought obscure." Robins elucidates thus: "In determining the ultimate ratios between the contemporaneous differences of quantities, it is often previously required to consider each of these differences apart, in order to discover, how much of those differences is necessary for expressing that ultimate ratio" (§ 154). For instance, $Ab+Ba$ only, and not the whole increment $Ab+Ba+ab$, is called the momentum of the rectangle under A, B.

120. Of this *Discourse*, a long account of twenty-six pages, written by Robins himself, although his name does not appear,[1] was given in *The Present State of the Republick of Letters*, London, October, 1735, in which it is stated that Robins wrote his *Discourse* with the view of removing the doubts which had lately arisen concerning fluxions and

[1] This account is republished in the *Mathematical Tracts of the late Benjamin Robins*, edited by James Wilson, London, 1761, vol. ii, p. 78.

prime and ultimate ratios; that Robins carefully distinguished both these methods from the method of indivisibles and also from each other. After an historical excursion viewing the works of the ancients, of Cavalieri and Wallis, the introduction by Newton of the concept of motion is taken up. "If the proportion between the celerity of increase of two magnitudes produced together is in all parts known," then "the relation between the magnitudes themselves must from thence be discoverable." This is the basis for fluxions. The "method of prime and ultimate ratios proceeds entirely upon the consideration of the increments produced." By it Newton avoids "the length of the ancient demonstrations by exhaustions," on which, according to Robins, the method of fluctions rests. "Newton did not mean, that any point of time was assignable, wherein these varying magnitudes would become actually equal, or the ratios really the same; but only that no difference whatever could be named, which they should not pass." Newton's term momentum is used simply for greater brevity, hence need not be considered. Newton's description is capable of an interpretation too much resembling the language of indivisibles—in fact, he sometimes did use indivisibles at first; Robins has freed the doctrine from this imputation in a manner that "shall agree to the general sense of his [Newton's] description."

Jurin's Review of his own Letters to Berkeley

121. In the November, 1735, number of the *Republick of Letters*, Philalethes Cantabrigiensis (Jurin) appears with an article, *Considerations upon some passages contained in two Letters to the Author of the Analyst.* The two letters in question are the two replies Jurin himself had made to Berkeley. The article is really a reply to Robins, though Robins's name is not mentioned. Jurin claims to have adhered strictly to Newton's language; some other defenders of Newton, says he, are guilty of departing from it. Their objections to his own defence are threefold:

"I. My explication of *Lemma* 1, *Lib.* I, *Princip.*" See our §§ 4, 6, 8.

"II. The sense of the *Scholium ad Sect.* 1, *Libr.* I, *Princ.*, particularly as to,

"1. The doctrine of Limits, 2. The meaning of the term evanescent, or vanishing." See our §§ 10–15.

"III. The sense of *Lemma* 2, *Lib.* II, *Princip.*" See our §§ 16–19.

122. As to the first objection, Jurin insists that Newton's words *fiunt ultimo æquales* mean that the quantities "do at last become actually, perfectly, and absolutely equal." He takes the hands of a clock between 11 and 12. The arcs traced by the hands "1. Constantly tend to equality, 2. During an hour, 3. And will come nearer to one another than to have any given difference, 4. Before the

end of the hour; . . . at the end of the hour, the two quantities must become equal." Further, "by taking the consideration of a finite time, Sir Isaac Newton is able to assign a point of time, at which he can demonstrate the quantities to be actually equal." Consider, says Jurin, the ordinate to a point of a hyperbola and that ordinate continued to the asymptote: they do not become equal in a finite time; Newton's *Lemma* is, "with great judgment, so worded on purpose, as necessarily to exclude this and such like cases." Thus Newton's inscribed and circumscribed rectangles of *Principia*, *Lib.* I, *Sec.* 1, *Lemma* 2 (fig. 1 in our § 9), were thought by Nieuwentiit and others never to be capable of coincidence with the curve (say the quadrant of a circle); but Jurin *assumes* the variation to be of such a nature that the limit is actually reached, as demanded by Newton's *Lemma*. For, suppose a point to move on the horizontal radius from the circumference to the centre A *in one hour*; suppose also that, when this moving point is at B on that radius, there be two rectangles described upon AB (one inscribed, the other circumscribed), and that upon every other part of the horizontal radius that is equal to AB, namely the parts BC, CD, DE, taken in order, rectangles be similarly erected "at the same point of time," then as the moving point nears the centre, the rectangles diminish in size and increase in number, and they must together become equal to the quadrant at the end of the

hour. Jurin points out that he has introduced here all the suppositions of Newton's first *Lemma*, namely that, (1) the two figures tend constantly to equality, (2) during one hour, *i.e.* a finite time, (3) and come nearer to one another than to any given difference, (4) before the end of the hour, *i.e.* before the end of a finite time. Jurin continues:

"If any man shall say, that a right-angled figure, inscribed in a curvilineal one, can never be equal to that curvilineal figure; much less to another right-lined figure, circumscribed about the curve; I agree with him. I am ready to own that, during the hour, these figures are one inscribed, and the other circumscribed; that neither of them is equal to the curvilineal figure, much less one to another. But then, on the other hand, it must be granted me, that, at the instant the hour expires, there is no longer any inscribed or circumscribed figure; but each of them coincides with the curvilineal figure, which is the limit, the *limes curvilineus*, at which they then arrive."

123. Jurin thereupon proceeds to Lemma 7 of Book I, Section 1 in Newton's *Principia*, which, he says, requires additional consideration. It relates to fig. 4, where ACB is any arc and "the points A and B approach one another and meet." Lemma 7, in Andrew Motte's translation, reads as follows:—

"The same things being supposed; I say, that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality."

Jurin says that here the chord AB, the arch ACB,

and the tangent AD come to vanish when B reaches A, and their last ratio is unity. Newton "directs our imagination, not to these vanishing quantities themselves, but to others which are proportional to them, and always preserve a finite magnitude," such as A*b*, the arch A*cb*, A*d*. Since at the instant when A and B coincide, "the angle BAD, or *b*A*d*, will vanish; it is easy to conceive that, . . . the chord A*b* must coincide with the tangent A*d*, . . . consequently, AB, AD must likewise, at the same instant of time, arrive at the same proportion of a perfect equality."

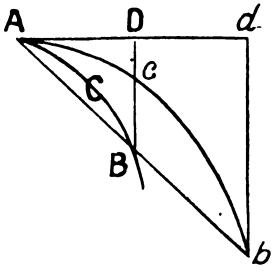


FIG. 4.

124. Proceeding to the last *Scholium* in Book I, Section 1 of the *Principia*, Jurin starts by defining the word *limit*. "I apprehend therefore that, by the limit of a variable quantity, is meant some determinate quantity, to which the variable quantity is supposed continually to approach, and to come nearer to it than to have any given difference, but never to go beyond it. And by the limit of a variable ratio, is meant some determinate ratio, to which the variable ratio is supposed continually to approach, and to come nearer to it than to have any given difference, but never to go beyond it. By arriving at a limit I understand Sir Isaac Newton to mean, that the variable quantity, or ratio, becomes absolutely equal to the determinate quantity, or ratio, to which it is supposed to tend."

With unusual lucidity, for that period, Jurin says on the subject of limits: "Now whether a quantity, or ratio, shall arrive at its limit, or shall not arrive at it, depends entirely upon the supposition we make of the time, during which the quantity, or ratio, is conceived constantly to tend or approach towards its limit." If we assume the approach to be made in a finite time, the limit is reached, otherwise it is not reached. Of a variable which "can never *attingere limitem*" Newton gives one illustration at the end of the *Scholium*: that of two quantities having at first a common difference and increasing together by equal additions, *ad infinitum*. Since they can never be really and in fact increased *ad infinitum*, says Jurin, their ratio cannot arrive at its limit. What Newton wanted to meet was the objection, "that if the last ratio's of evanescent quantities could be assigned, the last magnitudes of those quantities might likewise be assigned." Newton says No, "for those last ratio's, with which the quantities vanish, strictly speaking, are not the ratio's of the last quantities ... . but they are the limits" which those ratios can never "arrive at," "before the quantities are diminished *ad infinitum*." As to the sense in which Newton uses the word evanescent or vanishing, in the Scholium under consideration, Jurin inclines to the view that "both imply one single instant, or point of time."

125. In the *Principia*, Book II, Section 2, Lemma 2 (our §§ 16–19), Newton defines *moment* as "a momentaneous increment, or decrement, of a

flowing quantity, proportional to the velocity of the flowing quantities." According to Jurin, Newton puts *a*, *b*, *c* to signify either the moments, or the velocities, of the flowing quantities A, B, C. Leibniz looks upon them as differences. Newton, says Jurin, never used indivisibles, and his method to find the differences of variable quantities is not "rigorously geometrical," but is more rigorous than the treatment given by Leibniz.

Robins's Rejoinder

126. Robins replied in the *Republick of Letters* for December, 1735, in a *Review of some of the Principal Objections that have been made to the Doctrine of Fluxions and Ultimate Proportions; with some Remarks on the different Methods that have been taken to obviate them.* Robins does not here mention Philalethes any more than the latter referred directly to Robins. The objections to fluxions, says Robins, are levelled at Newton's expression, *fluxiones sunt in ultima ratione decrementorum evanescentium vel prima nascentium.* "Which being usually thus translated, that fluxions are in the ultimate ratio of the evanescent decrements, or in the first ratio of the nascent augments, it has from hence been ask'd, what these nascent or evanescent augments are?" There are difficulties of interpretation, whether the augments have quantity or have not. One way out of this difficulty which has been pointed out, is to say: "the limit of the proportion that the decrements bear to each other

as they diminish, is the true proportion of the fluxions" (p. 438). Here a new difficulty arises: Does the varying ratio reach its limit "actually, perfectly, and absolutely," or does it not? On the understanding that it does not reach the limit, "all that has at any time been demonstrated by the ancient method of exhaustions may be most easily and elegantly deduced." Rigour of demonstration does not require ultimate coincidence. "Coincidence of the variable quantity and its limit, could it be always prov'd, would yet bring no addition to the accuracy of these demonstrations" (p. 441). Hence, "why to the natural difficulty of these subjects should the obscurity of so strained a conception be added?" Is this view a correct interpretation of Newton? A literal translation of his Lemma 1, Section 1, Book I, *Principia* (see our §§ 4, 6, 8), is: "Quantities, and the ratios of quantities, that during any finite time constantly approach each other, and before the end of that time approach nearer than any given difference, are ultimately equal." What is the meaning of "given difference"? If it be a "difference first assigned" according to which the degree of approach of these quantities may be afterwards regulated; then . . . ratios, and their limits, tho' they do never actually coincide, will come within the description of this *Lemma*; since the difference being once assign'd, the approach of these quantities may be so accelerated, that in less than any given time the variable quantity, and its limit, shall differ by

less than the assign'd difference." Here Robins expresses the idea that the rapidity of approach toward the limit can be arbitrarily altered, yet he does not apparently perceive—certainly he does not admit—that this rapidity may be altered in such a way that the variable actually does reach its limit. On the contrary, he maintains that "where the approach is determin'd by a subdivision into parts," "it is obvious, that no coincidence can ever be obtain'd." A coincidence such as Philalethes explains in the case of rectangles circumscribed and inscribed in a curve, if it could take place, is not a coincidence such as Newton intended, for Newton did not in this instance use motion, but continual subdivision. Robins tries to establish his view of the matter by giving an instance of erroneous results being deduced by letting the variable reach its limit. He takes a hyperbola and revolves its principal axis in the plane of the curve, around the point of intersection of this axis and an asymptote, until the two lines coincide. At the end "the hyperbola coincides with the asymptote," which is "absurd." As a matter of fact there is no absurdity. In $b^2x^2 - a^2y^2 = a^2b^2$, the slope of the asymptote is $m = b / a$. Robins's process amounts to making $m = o$, which gives a real locus when $b = o$, namely the locus $y^2 = o$. The only objection lies in still calling the final curve a "hyperbola."

The Debate Continued

127. Robins's article was followed in the January, 1736, number of the *Republick of Letters* by Philalethes's *Considerations occasioned by a Paper in the last Republick of Letters, concerning some late Objections against the Doctrine of Fluxions, and the different Methods that have been taken to obviate them.* Jurin denies having said that there was an "intermediate state" between augments being "any real quantity" and being "actually vanished"; he says he gave Newton's declaration that "their magnitude *cannot be assigned or determined.*" Such intermediate magnitudes, in Jurin's opinion, cannot be "represented to the mind," but their ratio can be represented to the mind, by contemplating the ratio, "not in the vanishing quantities themselves, but in other quantities permanent and stable, which are always proportional to them" (p. 76). As to Newton's Lemma 1 in Section 1, Book I of the *Principia*, if the great author meant to *conclude*, that the quantities "approach nearer than any given difference," then he first supposed what he would prove, and proved only what he had before supposed. Of this he could not be guilty. Besides, Newton's words,[1] "fiunt æquales," do absolutely subvert such an interpretation. Jurin says that he does not claim that coincidence is *necessary* for rigorous proof; he admits that in Robins's treatment of prime and ultimate ratios, coincidence is

[1] Newton's words are "fiunt ultimo æquales." See our § 4.

not necessary; only, Robins's method is not that of Newton. To establish this last point, Philalethes quotes Newton's *lemma* in Latin, then prints Robins's and his own translation of it. In case of variation, the upper line is Robins's translation, the lower is Jurin's:—

Quantities, and { *the* / *also* } *ratio's of quantities, that during any finite time constantly* { *approach each other,* / *tend to equality,* } *and before the end of that time approach nearer* { *than any given difference, are ultimately equal.* / *to one another than to have any given difference, do at last become equal.* }

It is not clear to Jurin what Robins means by "are ultimately equal," nor can Jurin conceive "how quantities, which do never *become actually equal*, . . . can come within the description of a Lemma, which Lemma expressly affirms, that they *become equal.*" *Fiunt ultimo æquales* means "become at last equal." Jurin quotes different places in the *Principia* which support his point. He denies that Newton proceeds, in the case of inscribed and circumscribed rectangles, by continual divisions of the base of the figure, and gives references in support of his contention. Of interest are the following admissions made by Jurin (p. 87): "This equality therefore we are obliged to acknowledge, although we should not be able, by stretch of imagination, to pursue these figures, and, as it were, to keep them in sight all the way, till the

very point of time that they arrive at this equality. For a demonstrated truth must be owned, though we do not perfectly see every step by which the thing is brought about."

"We have therefore no occasion for *the delineation of a line less than any line that can be assigned.* We acknowledge such delineation to be utterly impossible; as likewise the *conception* of such a line, as an actually existing, fixed, invariable, determinate quantity." Jurin here begins to disavow infinitesimals. "I am very free to own that Sir Isaac Newton does not *always* consider this coincidence, or rather equality, of the variable quantity, or ratio and its ultimate, as necessary in his method."

128. The debate between Jurin and Robins had reduced itself by this time, not so much to the discussion of the logical foundations of fluxions, as to the discussion of what Newton's own views on the subject had been. Robins prepared a long paper on the subject for the April, 1736, issue of the *Republick of Letters*, under the title: *A Dissertation shewing, that the Account of the doctrine of Fluxions, and of prime and ultimate ratios, delivered in a treatise entitled, 'A discourse concerning the nature and certainty of Sir Isaac Newton's methods of fluxions, and of prime and ultimate ratios,' is agreeable to the real sense and meaning of their great inventor.* The paper covers 45 pages. Robins repeats the fundamental definitions and historical statements given in his earlier papers, and directs

some attacks against Berkeley. To set forth the views of Newton, quotations are made from his works. He quotes from the Introduction to the *Quadratura Curvarum* (see our §§ 27–42). From the *Quadratura Curvarum* itself he quotes:

"Quantitates indeterminatas, ut motu perpetuo crescentes vel decrescentes, id est, ut fluentes vel defluentes, in sequentibus considero, designoque literis z, y, x, v, et earum fluxiones, seu celeritates crescendi noto iisdem literis punctatis. Sunt et harum fluxionum fluxiones, sive mutationes magis aut minus celeres, quas ipsarum z, y, x, v fluxiones secundas nominare licet," etc.

Robins quotes also from the anonymous account of John Collins's *Commercium Epistolicum*, which figures so prominently in the controversy between the followers of Newton and of Leibniz. This account was published in the *Philosophical Transactions*, vol. xxix, for the years 1714, 1715, 1716, of the Royal Society of London, of which Robins was a member. Robins goes on the assumption that the anonymous article was written by Newton himself, an assumption denied by no one at that time or since, though Jurin in a reply wants to know on what authority Newton's authorship is asserted. Robins quotes as follows (see our § 47):

"When he [Newton] considers lines as fluents described by points, whose velocities increase or decrease, the velocities are the first fluxions, and their increase the second."

129. Robins says that Berkeley, "for the support

of his objections against this doctrine [of fluxions], found it necessary to represent the idea of fluxions as inseparably connected with the doctrine of prime and ultimate ratios, intermixing this plain and simple description of fluxions with the terms used in that other doctrine, to which the idea of fluxions has no relation: and at the same time by confounding this latter doctrine with the method of Leibniz and the foreigners, has proved himself totally unskill'd in both. These two methods of Sir Isaac Newton are so absolutely distinct, that their author had formed his idea of fluxions before his other method was invented, and that method is no otherwise made use of in the doctrine of fluxions, than for demonstrating the proportion between different fluxions. For, in Sir Isaac Newton's words [see our §§ 29, 36], as the fluxions of quantities are nearly proportional to the contemporaneous increments generated in very small portions of time, so they are exactly in the first ratio of the *augmenta nascentia* of their fluents. With regard to this passage the writer of the *Analyst* has made a two-fold mistake. First, he charges Sir Isaac Newton, as saying these fluxions are very nearly as the increments of the flowing quantity generated in the least equal particles of time. Again, he always represents these *augmenta nascentia*, not as finite indeterminate quantities, the nearest limit of whose continually varying proportions are here called their first ratio, but as quantities just starting out from non-existence, and

yet not arrived at any magnitude, like the infinitesimals of differential calculus. But this is contrary to the express words of Sir Isaac Newton, who after he had shewn how to assign by his method of prime and ultimate ratios the proportion, that different fluxions have to one another, he thus concludes. *In finitis autem quantitatibus Analysin sic instituere et finitarum nascentium vel evanescentium rationes primas vel ultimas investigare consonum est geometriæ veterum: et volui ostendere, quod in methodo fluxionum non opus sit figuras infinite parvas in geometriam introducere.*" (See our §§ 33, 41.)

130. Robins proceeds to explain that the method of prime and ultimate ratios is "no other than the abbreviation and improvement of the form of demonstrating used by the ancients on the like occasions." It has nothing to do with infinitely small quantities, which have led into error not only Leibniz in studying the resistance of fluids and the motion of heavenly bodies, but also Bernoulli likewise in the resistance of fluids and in the study of isoperimetrical curves. Such infinitely small quantities led Parent to make wrong deductions. It was argued that because a heavy body descends through the chord of a circle terminating at its lowest point in the same time as along a vertical diameter, "the time of the fall through the smallest arches must be equal to the time of the fall through the diameter." To relieve Newton of the suspicion of not being free from the obscure methods of indivisibles, Robins says he [Robins] defined an "ultimate

magnitude" and "ultimate ratio" as limits. This exposition Robins had given in full in his *Discourse.* The difference of interpretation of Newton's Lemma 1 in the *Principia* (Book I, Section 1), given by himself and by Jurin, arises from Jurin's misinterpretation of Newton's word *given.* He "supposes it to stand for *assignable*; whereas it properly signifies only what is actually assigned." Jurin claims that by our interpretation, "Newton is rendred obnoxious to the charge of first supposing what he would prove" (p. 307). Robins says in reply that the statement, quantities which "are perpetually approaching each other in such a manner, that any difference how minute soever being given, a finite time may be assigned, before the end of which the difference of those quantities or ratios shall become less than that given difference," is an obvious but not an identical proposition. Robins argues, "that Sir Isaac Newton had neither demonstrated the actual equality of all quantities capable of being brought under this lemma, nor that he intended so to do" (p. 309); when quantities "are incapable of such equality, the phrase of ultimately equal must of necessity be interpreted in a somewhat laxer sense," as in *Principia*, Book I, Prop. 71, "pro æqualibus habeantur, are to be esteemed equal." When Newton says that the number of inscribed parallelograms should be augmented *in infinitum*, he does not mean that the number becomes infinitely great, but that they are augmented *endlessly.* The nature of the motion

assumed by Jurin to explain how the limit may be reached is excessively complex. Moreover, "to assert that any collection of these inscribed and circumscribed parallelograms can ever become actually equal to the curve, is certainly an impropriety of speech, . . . the essence of indivisibles consists in endeavouring to represent to the mind such inscribed or circumscribed figure, as actually subsisting, equal to the curve" (p. 312). Our interpretation "thus removes this doctrine quite beyond the reach of every objection" (p. 315). Robins argues that Newton's *ultimæ rationes, quibuscum quantitates evanescunt* are not *rationes quantitatum ultimarum*; but only limits, to which the ratios of these quantities, which themselves decrease without limit, continually approach; and to which these ratios can come within any difference, that may be given, but never pass, nor even reach those limits" (p. 316). "Newton has expressly told us, that the quantities, he calls *nascentes* and *evanescentes*, are by him always considered as finite quantities" (p. 321).

131. The *momenta* of quantities occur in Newton's *De analysi per æquationes numero terminorum infinitas*, drawn up in 1666. Newton says "that he there called the moment of a line a point in the sense of Cavalerius, and the moment of an area a line in the same sense" (see our § 47), that "from the moments of time he gave the name of moments to the momentaneous increases, or infinitely small parts of the abscissa and area generated in moments of time . . .

because we have no ideas of infinitely little quantities, he introduced fluxions into his method, that it might proceed by finite quantities as much as possible." Prime and ultimate ratios he introduced later. Newton says in that place that in his proofs he uses *o* for a finite moment of time, though sometimes, for brevity, he supposes *o* infinitely little. Thus Newton used *o* in two senses; in the fluxions published in 1693 in Wallis's algebra, *o* is used in the sense of indivisibles; in 1704 he gave it a second signification in the *Quadratura Curvarum*. Robins sums up his dissertation thus: "Hence it is very manifest, that as Sir Isaac Newton used at first indivisibles, so he soon corrected those faulty notions by his doctrine of fluxions, and afterwards by that of prime and ultimate ratios. And all the absurdity of expression, and all the inconsistency with himself charged on him by the author of the *Analyst*, arises wholly from mis-representation." This paper was badly arranged and below the level of Robins's earlier contributions.

132. Robins's long paper in the *Republick of Letters* was followed in the July and August (vol. xxviii, 1736) numbers by *Considerations upon some passages of a Dissertation concerning the Doctrine of Fluxions, published by Mr. Robins in the Republick of Letters for April last*, by Philalethes Cantabrigiensis. The paper extends over 136 pages, and could not be easily accommodated in a single number. From now on the disputants, particularly Jurin, are no longer in a calm frame of mind. The

debate is one over words and ceases to be illuminating. Their judgments were perverted by the heat of controversy. Even theological or political controversies could not easily surpass the verbosity and haze exhibited here.

Jurin's first objection to Robins's last analysis is the statement that the method of fluxions has no relation to the method of first and last ratios; Jurin quotes from Newton in support of his contention. The charge that he (Jurin) represents *augmentia nascentia* not as finite, but as just starting out of non-existence, "like infinitesimals of the differential calculus," Jurin denies, saying: Leibniz's differentials "are fixed, determinate, invariable"; he himself has represented the nascent augments as "quantities just starting out from non-existence, and yet not arrived at any magnitude, and not as finite quantities" (p. 52), and quotes Newton in support of this view. According to the article in the *Philosophical Transactions*, No. 342, attributed to Newton, moments are represented "by the rectangles under the fluxions and the moment *o*"; "in his calculus there is but one infinitely little quantity represented by a symbol, the symbol *o*: it is also said, Prick'd letters never signify moments, unless when they are multiplied by the moment *o* either exprest or understood to make them infinitely little, and then the rectangles are put for moments." Jurin charges that Robins has now published four different interpretations of Newton's much-discussed lemma. Newton's phrase, *fiunt ultimo æquales*, the

use of the words "perpetually" and "endlessly," "the last difference," are again discussed at length. Jurin quotes from Robins a passage which appears to show that "Mr. Robins is now of opinion, that Sir Isaac's demonstration is applicable to such quantities, as at last become actually equal, as well as to quantities, which only approach without limit to the ratio of equality" (p. 67); therefore, the lemma, "by Mr. Robins's own confession, may be taken in the sense I have always understood it in" (p. 68). However, this is in direct conflict with Robins's earlier assertions. In the discussion about the inscribed rectangles, both Robins and Jurin agree that if the "base of the curve" (our abscissa) be continually subdivided as in Euclid I 10 or V 10, it is manifest "that such subdivision can never be actually finished" (p. 78); but Newton proceeds differently—he supposes a line to be described by a moving point. Jurin thereupon repeats exactly the argument in Zeno's "Dichotomy," though he does not mention Zeno, to show that a point moving across the page in, say, one hour passes over 1 / 2 of the distance, then over 1 / 4 of it, then over 1 / 8, 1 / 16, etc., and insists that "all the possible subdivisions of the line" will be "actually finished" and "brought to a period at the end of the hour." This is given in support of his previous argument that the rectangles inscribed in a curve may reach the limit. "If Mr. Robins will tell me, that the imagination cannot pursue these parallelograms to the very end of the hour, I may ask him,

whether the imagination can any better pursue the subdivision of the line, or even of the hour itself, to the end of the hour, which subdivisions he must own to *be brought to a period* by the end of the hour. But there is no need to *strain our imagination, to labour in every case*, or indeed in any case, *after some idea of motion however intricate; no subtle inquiry is at all necessary*, since we are obliged to own the conclusion to be true and certain. . . ."

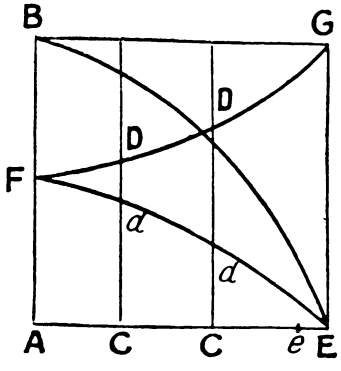


FIG. 5.

"However, since Mr. Robins is pleased to talk so much about straining our imagination, . . . let us see, if we cannot find some plain and easy way of representing to the imagination, that actual equality, at which the inscribed and circumscribed figures will arrive with each other, and with the curvilineal figure, at the expiration of the finite time" (p. 111).

Let the curvilineal figure ABE equal in area the rectangle with sides EA and AF. When the moving point describing the base EA in a finite time is at C, let the rectangle with the base EA and height C*d* be equal to the sum of the parallelograms inscribed in ABE (not drawn in the figure) which stand on CA and upon as many other adjoining parts of EA as can be taken equal to CA. Let E*dd* be the curve traced by the moving point *d*.

Let the area of the rectangle with EA as base and CD as height = sum of the circumscribed parallelograms (not drawn in the figure) standing on CA and upon as many other parts of EA as can be taken equal to CA and adjoining to it; also let GDD be the curve traced by the movable point D. Then as the curvilineal, the inscribed, and the circumscribed figures are respectively equal to EA × AF, EA × Cd, EA × CD, these figures must be proportional to AF, Cd, and CD. These three lines will "be equal to one another at the end of the finite time." Now since Cd and CD approach each other, during a finite time, within less than any given distance before the end of that time, these three lines will, by that *Lemma*, be equal to one another at the end of the finite time. The limit is reached (p. 114).

133. As a further illustration, Jurin takes a rectilinear figure, the right triangle ABE, where EA = AB = a, AF = $\frac{1}{2}a$, EC = x, the point C moving from E to A as before. Upon AC as a base, imagine an inscribed rectangle (height CH), and a circumscribed rectangle (height CK). As in the previous figure, imagine other inscribed and circumscribed rectangles, standing upon as many other parts of EA as can be taken equal to CA, and adjoining to it in order. When CA is an aliquot part of AE, then $a \times C d$ is the sum of the inscribed rectangles and $a \times$ CD is the sum of the circumscribed rectangles, where $Cd = x / 2$, and $CD = a - x / 2$. Let Kd = CD. The ordinate Kd,

drawn to the base BG, will be terminate by EF. When CA is not an aliquot part of AE, if we divide the base into as many parts as may be, there will be left a portion E*e*, which, let us call *r*. Then $Cd = \overline{x+r} \times \overline{a-r} \div (2a)$ and all these ordinates will be bounded by E*dd*F. In the same way, $Kd = a - \overline{x+r} \times \overline{a-r} \div (2a)$, and the ordinate will be bounded by E*dd*F. When $x = a$, r vanishes, $Cd = \frac{1}{2}a$ and $Kd = \frac{1}{2}a$. Hence the inscribed and circumscribed figures do then become equal to each other, and to the triangle ABE; again, the limit is reached.

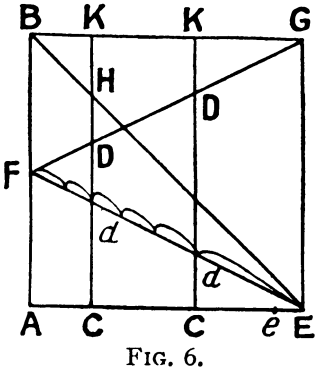


FIG. 6.

Jurin takes Robins to task for asserting that "equality can properly subsist only between figures distinct from each other." To Robins's query, "Does Philalethes here suppose the truth of Sir Isaac Newton's demonstrations to depend on this actual equality of the variable quantity and its limit?" Jurin answers, "I do . . . In the manner Mr. Robins defines, and treats of prime and ultimate ratios, I allow his demonstrations to be just without this actual equality. But Sir Isaac Newton does not define and treat of prime and ultimate ratios, in the same manner with Mr. Robins; nor are Mr. Robins's demonstrations at all like Sir Isaac Newton's demonstration" (p. 128). The inability of our imagination to pursue the rectangles in reach-

ing the limit is no valid argument against the contention that the limit is reached; even in the ancient geometry there are demonstrated truths that lie beyond the reach of the imagination, as for instance, that three cones may equal a cylinder, all of the same base and height (p. 130). The meaning of *moment*, a truly difficult concept, is discussed again, Jurin holding that Newton took it as "a momentaneous increment, . . . less than any finite quantity whatsoever, and proportional to the velocity of the flowing quantity," while Robins seemingly claimed that Newton meant them to be finite quantities (p. 151). With respect to Newton's early use of the infinitely little, Jurin and Robins were in disagreement, and Robins was in our opinion nearer the truth. Robins claimed that Newton at first used infinitely little quantities; that afterwards he improved his method by discarding them; Jurin claimed that Newton's alleged absurdity of expression and inconsistency with himself, as charged by Berkeley and others, "arises wholly from misinterpretation, or misunderstanding him" (p. 179).

134. Jurin's article appeared in the July and August numbers, 1736, of the *Republick of Letters*. Robins could not wait in patience until the entire article of Jurin had been printed. In the August number he replies to the part of Jurin's article that had appeared in the July number. The August number was given up to Jurin and Robins, to the entire exclusion of all other articles and of the usual

book reviews. On the last page of the August number, the editor apologises to the readers and assures them "they shall hereafter have no occasion to complain upon this head." In Robins's reply,[1] both "Robins" and "Philalethes" appear in the third person, as if the writer were some outsider. Robins says: "Newton does not intermix his simple and plain description of fluxions with the terms used in the doctrine of prime and ultimate ratios; for his description of fluxions is contained in the two first paragraphs of his Introduction to the *Quadratures*, in which no terms of the other doctrine occur" (p. 89). The *Lemma* is, of course, taken up again, Robins claiming his interpretation legitimate, "for two quantities may constantly tend to equality during some finite space of time, and before the end of that time come nearer together than to have any difference, which shall be given; and yet at the end of that time have still a real difference," while Jurin's interpretation was not "any difference that shall be given," but "any assignable difference," which would mean that the limit must be reached. Mr. Robins says (p. 97): "It is not difficult to assign a very probable reason, which led Sir Isaac Newton to the use of this expression [fiunt ultimo æquales], for before him it had not been unusual for geometers to speak of the last sums of infinite progressions, which is an ex-

[1] "Remarks on the *Considerations relating to Fluxions*, etc., that were published by Philalethes Cantabrigiensis in the *Republick of Letters* for the last month," *Republick of Letters*, August, 1736, pp. 87–110.

pression something similar to this. Surely here no one will pretend, that an infinite number of terms can in strict propriety of speech, and without a figure, be said to be capable of being actually summed up and added together." Robins makes the only direct reference that was made in this debate to Zeno's paradoxes. He mentions Achilles and the Tortoise, but in a manner devoid of interest. Referring to the line which Jurin supposes traced in one hour, Robins says: "Perhaps it may be easiest understood by comparing the present point with the old argument against motion from Achilles and the Tortoise. It is impossible to pursue in the imagination their motion by the means proposed in that argument to the point of their meeting, because the motion of each is described by the terms of an infinite progression." Robins does not seem fully to realise that Achilles and the Tortoise present a case in which a variable reaches its limit.

135. The editor of the *Republick of Letters* permitted the two disputants to continue their wranglings in an *Appendix* to the September issue.[1]

Philalethes's attempt to represent to the imagination the actual equality at which the inscribed and circumscribed figures will arrive with each other, and with the curvilinear figure, is criticised by Robins

[1] *An Appendix to the Present State of the Republick of Letters for the Month of September*, 1736. *Being Remarks on the Remainder of the Considerations relating to Fluxions, etc., that was published by Philalethes Cantabrigiensis in the Republick of Letters for the last Month. To which is added by Dr. Pemberton a Postscript occasioned by a passage in the said Considerations.* London, 1736.

on the ground that the continued curve "is not to be described, but by an endless number of parabola's" (of which the curve is the envelope); thus, Philalethes gave "as an equation expressing the nature of a single curve, one which in reality includes an infinite series." "Philalethes supposed a last form of the inscribed figures, that was equal to the curve." Robins observed "that equality implies the things, which have that property, to be distinct from each other. For to say a thing is equal to itself is certainly no proper expression." But "there is no such last form distinct from the curve," as Philalethes admits; hence Philalethes "gives up the point."

136. In the *Principia*, Newton does not deliver the doctrine of fluxions, but the doctrine of prime and ultimate ratios. "The understanding of this book is attended with difficulty." The expression *ultima summa* is defective: "Can any sum of a set of quantities, whose number is supposed infinite, in strict propriety of speech be called their last sum?" Later, Robins says: "Let Philalethes reconcile the actual arrival of these quantities to the ratio supposed, and at the same instant vanishing away. Is not this saying, that the two quantities become nothing, and bear proportion at the same instant of time?" (p. (14)). Philalethes "has thought himself unjustly accused by Mr. Robins of supposing a nascent increment to be some intermediate state of that increment between its finite magnitude, and its being absolutely nothing. To have proved this

assertion groundless he ought to have shown, that this definition does not attempt at describing such an intermediate state" (p. (15)). Robins asserts: "Whoever has read Sir Isaac Newton's *Lectiones Opticæ*, and will deny, that he has at any time made use of indivisibles, must be very much a stranger to that doctrine, and to the style of those writers who follow it" (p. (19)). "What reflexion is it upon Sir Isaac Newton to suppose, that he made use of the methods he had learned from others before he had invented better of his own: or that in an analysis of a problem for dispatch he still continued to make use of such methods, when he conceived it would create no error in the conclusion? Has not Sir Isaac Newton said this of himself, and has Mr. Robins said anything more?" (p. (15)). "Does Philalethes here mean, that a quantity can become less than any finite quantity whatsoever, before it vanishes into nothing? If not, then the point is given up to Mr. Robins, who only contends, that vanishing quantities can never by their diminution be brought at last into any state or condition, wherein to bear the proportion called their ultimate: if otherwise, since Philalethes supposes . . . that it is nonsense, that it implies a contradiction to imagine a quantity actually existing fixed, determinate, invariable, indivisible, less than any finite quantity whatsoever; because this imports as much as the conception of a quantity less than any quantity, that can be conceived: how can a quantity supposed to be less than any finite quantity whatso-

ever be rendered more the object of the conception by being understood to be brought into this condition by a constant diminution from a variable divisible quantity?" (p. (20)). "Sir Isaac Newton has introduced into use the term moment throughout the whole second book of the *Principia*, and for no other purpose than for the sake of brevity; for his doctrine of prime and ultimate ratios had been before fully explained, and every proposition of the second book might have been treated on without the use of this term, though perhaps with a somewhat greater compass of words" (p. (23)). "Mr. Robins has endeavored to defend Sir Isaac Newton both against the accusation of the author of the *Analyst*, and the misrepresentation of Philalethes. He has shown, that Sir Isaac Newton's doctrine of prime and ultimate ratios has no connexion with indivisibles, and that, if he ever allowed himself in the use of indivisibles, he knew that he did so, and did not confound both the methods together, as the author of the *Analyst* accuses him, and Philalethes without knowing it has owned" (p. (27)). "Had Philalethes been versed in the ancients, and in the later writers who have imitated them, he could have been at no loss about the true sense of *data quavis differentia* used by Sir Isaac Newton in his first *Lemma*. For this expression is borrowed from the writers, that made use of exhaustions" (p. (29)). "What separates the doctrine of prime and ultimate ratios from indivisibles is the declaration made in the *Scholium* to the first Section of the *Principia*,

that Sir Isaac Newton understood by the ultimate sums and ratios of magnitudes no more than the limits of varying magnitudes and ratio's; and he puts the defence of his method upon this, that the determining any of these limits is the subject of a problem truly geometrical. To insist, that the variable magnitudes and ratio's do actually attain, and exist under these limits, is the very essence of indivisibles" (p. (34)).

Robins's reply in the August and September, 1736, numbers of the *Republick of Letters* is condensed in form, yet covers 61 pages. It is impossible for us to convey an adequate idea of the amount of detail entering in the discussion. Altogether Robins shows greater willingness to admit that Newton's views were different at different periods in his career, and that even Newton may be guilty of modes of expression that are not free from obscurity. Moreover, Robins speaks in general with greater sincerity than his opponent. But Jurin proves himself the superior of Robins in adhering to a broader and more comprehensive conception of variables and limits.

Pemberton enters the Debate

137. At this stage a new party enters the debate —Henry Pemberton, who had studied medicine and mathematics at Leyden and Paris, had been a friend of Newton, and had edited the third edition of the *Principia*. In an article following the one of Robins in the "Appendix" (August and September 1736),

Pemberton says: "I . . . am fully satisfied, that Mr. Robins has expressed Sir Isaac Newton's real meaning." Pemberton quotes from Newton's Introduction to the *Quadrature of Curves* about prime and ultimate ratios (see our §§ 33, 42), and then remarks; "Here Sir Isaac Newton expressly calls the quantitates *nascentes* and *evanescentes*, whose prime and ultimate ratios he investigates, by the appellation of finite. Now I desire Philalethes to reconcile this passage with his notion of a 'nascent quantity being a quantity not yet arrived at any assignable magnitude how small soever.' And I must farther ask Philalethes, whether he has not here attempted to define a non-entity."

138. Robins's last article and Pemberton's rash challenge led to another flow of words, covering 77 pages in the "Appendix" to the *Republick of Letters* for November, 1736, in an article by Jurin, entitled *Observations upon some Remarks relating to the Method of Fluxions, published in the Republick of Letters for August last, and in the Appendix to that for September.*

Jurin insists that "the method of fluxions, as it is drawn up by Sir Isaac Newton, could not possibly be *formed before* the method of first and last ratio's *was invented*" (p. (6)).

Robins "takes no notice of the letter *o* being used in the book of *Quadratures*, in the very same sense as in the *Analysis*" (p. (8)). "That symbol never denotes any quantity, but what, by a continual decrease, becomes infinitely little, *i.e.* less

than any quantity, and at last vanishes into nothing" (p. (8)).

"He is grossly mistaken in thinking, that quantities, which, before the end of a finite time, come nearer together than to have any assignable difference, will therefore become equal before the end of that time" (p. (12)). "I have clearly proved in November and January last, that Sir Isaac Newton designed no quantities or ratio's to be comprehended within the sense of this lemma, which do not become actually equal" (p. (13)).

"Has then Mr. Robins, . . . offered to shew, that any quantities or ratio's incapable of an actual equality are compared in this lemma? I think not" (p. (22)). In January, "I use the following words, 'This determinate proportion of the finite quantities *a* and *e*, is what I understand by the proportion of the evanescent augments.' This, I say, ought to have been attended to, before this charge against me was renewed" (p. (24)). As regards the ratio between the inscribed and circumscribed figures, "have not I truly expressed it? If my expression be too complex, let these great Geometers shew me a simpler, if they can, and I will make use of that" (p. (34)). Robins's argument about the last form of parallelograms differing from the limiting curve is defective in the minor of the syllogism: "Things which are equal are distinct from each other." "Is it," says Jurin, "the part of a candid and ingenious adversary, to insist always upon the word *equal*, when a more proper expression, as that of *coinciding*, has

been used by his antagonist?" If his argument is sound, "it will hold against my expression, that the figures inscribed and circumscribed do at last *coincide* with the curvilinear figure." Jurin claims "that if Mr. Robins's interpretation of the first *Lemma* be admitted, Sir Isaac's demonstrations, as they now stand, will not be accurate, nor geometrically rigorous," for, "as they now stand, the examples he has given in the several *Lemmata* of the first Section, are of such quantities and ratio's only, as do actually arrive at their respective limits" (pp. (42) and (43)). "Mr. Robins and I have been disputing some time, whether Sir Isaac Newton used indivisibles. That Gentleman maintains that he used them; and grounds his charge upon the term *infinitely little*, which is sometimes to be met with in Sir Isaac Newton's writings: but he does not explain the meaning of that term, when used either by Sir Isaac, or by the writers of indivisibles. I, on the contrary, distinctly explain what I apprehend to be meant by it, both when used by Sir Isaac Newton, and when used by the writers of indivisibles. . . . I supposed the writers upon indivisibles, by an infinitely little quantity, to mean a quantity actually existing, fixed, determinate, invariable, indivisible, less than any finite quantity whatsoever" (p. (73)). Robins quotes Pascal and Barrow as using the term *indefinite* in place of *infinite*, but the writers I quoted use *infinite* and *infinitely little*. There is difference of usage among followers of Cavalieri. "It is not denied,

but that Sir Isaac Newton, by the term *infinitely little*, meant a quantity variable, divisible, that, by a constant diminution, is conceived to become less than any finite quantity whatsoever, and at last to vanish into nothing. By which meaning all that is faulty in the method of indivisibles, is entirely avoided; and that being allowed, the rest is only a dispute about a word" (p. (74)).

Jurin declares in a "Postscript" that "to carry on two controversies at once is more than I have leisure for"; later "I intend to accept of Dr. Pemberton's invitation"; meanwhile Jurin inserts an attestation of "his learned friend *Philelcutherus Oxoniensis*" to the effect that this friend is "fully satisfied, Mr. Philalethes has expressed Sir Isaac Newton's real meaning." The language of this attestation follows exactly the language of Pemberton, except that Philalethes, and not Robins, is now declared the correct interpreter of Newton.

139. In the December issue, 1736, of the *Republick of Letters*, Robins says in an "Advertisement" that "since Philalethes has given loose to passion," he "cannot think anything farther necessary for the satisfaction of impartial readers" (p. 492), and now takes "leave of Philalethes," but cannot resist a few parting shots. Nor could Philalethes resist making reply to this "Advertisement" in an "Appendix" to the December number, 1736, of the *Republick of Letters*, in which he expresses regret "that so long a correspondence should end in discontent or ill humour." Jurin justifies the practice

he exercised in this controversy of offering poetry (usually in Latin) for the sake of readers who are under necessity "of exercising their faith, rather than their reason in this dispute," for "A verse may catch him, who a sermon flies," and for the sake of enlivening the subject for others, "who are judges of the dispute."

140. In this December "Appendix" Jurin then contributes *A Reply to Dr. Pemberton's Postscript*, which takes up 31 pages. Referring to Newton's *Lemma* 1, Jurin says that in his former expression, the quantities "come nearer to equality than to have any assignable difference between them," it never was his intention to assert "that during the time of the approach, the difference between the quantities is not always assignable"; he meant "that, though they shall always have a difference during the finite time, yet, before the end of that time, their difference shall become less than any quantity that can be assigned. And if my words are taken in this sense, the Dr.'s objection immediately falls to the ground" (p. (24)). Mr. Jurin then gives a "demonstration" of the following proposition: "If two lines (1) tend constantly to equality with each other, (2) during any finite time, as, for instance, an hour; (3) and thereby, their difference become less than any quantity that can be assigned, (4) before the end of the hour; then, at the end of that finite time, or at the end of the hour, the lines will be equal." As to Dr. Pemberton's charge that Jurin misinterprets Newton's

nascent and evanescent increments, Jurin says that he discussed this question with Robins. Newton's words in the *Quadratura Curvarum*, viz. *finitarum nascentium vel evanescentium*, may mean "(1) finite nascent or evanescent quantities, or (2) finite quantities when they begin to be, or when they vanish. But the former sense contradicts the second *Lemma* of the second Book of the *Principia*, where Sir Isaac Newton says, *cave intellexeris particulas finitas* . . . and indeed it is contrary to the whole tenor of his doctrine." The second interpretation is "perfectly conformable to all the rest of Sir Isaac Newton's works" (p. (32)). Jurin repeats that a nascent increment is "an increment not yet arrived at any assignable magnitude, how small soever." To Dr. Pemberton's query, whether Jurin "has not here attempted to define a non-entity," Jurin replies that it "ought not to be called simply a *non-entity*, nor simply an *entity*. It is a *non-entity* passing into entity, or entity arising from non-entity, a beginning entity, something arising out of nothing" (p. (37)).

141. The discussion is carried on from this time in a journal called *The Works of the Learned*, into which the *Republick of Letters* and another journal had merged. In the February, 1737, issue Dr. Pemberton appears with *Some Observations on the Appendix to the Present State of the Republick of Letters for December*, 1736, which enjoys the merit of brevity, being limited to only two pages. Pemberton declares that in Newton's passage in the

Quadratura Curvarum, "Philalethes cannot remove my objection by straining one or two of the words to fit his sense"; Newton meant there that vanishing quantities should not be "otherwise than finite quantities" (p. 157). Moreover, "what kind of nothings they must be, which with any propriety can be said to pass into somethings, and for this reason can be capable of bearing proportions, before they are become anything, certainly requires explanation."

A reply by Jurin in *The Works of the Learned* for March, 1637, is kept within the very moderate compass of 10 pages. The title of his contribution is *The Contents of Dr. Pemberton's Observations published the last month.* Nothing here is of interest in the interpretation of Newton.

Dr. Pemberton's reply in the April issue refers to Jurin's phrase, "they come nearer to equality than to have any assignable difference between them": "My objection to the interpretation of Philalethes [in the *Minute Mathematician*, p. 88] is, that these words, which compose the third article of that interpretation, in conjunction with the fourth article can have no other signification, than that the quantities come nearer to equality than to have any difference between them before that point of time, wherein they are supposed by the second article to become equal; all which amounts to this inconsistency, that there is a time, when the quantities have no difference, and yet are not equal" (p. 306). Dr. Pemberton again gives his endorsement of Robins's interpretation of Newton.

Jurin appears with a 12-page article in the May, 1737, number of *The Works of the Learned*, saying: "He still ascribes to my words a meaning, which I have again and again utterly disavowed; not only so, but he changes the words themselves, putting *any difference* instead of *any assignable difference*" (p. 388). As to the Introduction to Newton's *Quadratura Curvarum*, "in that very *Introduction* Sir Isaac Newton has made use of infinitely little quantities, in the sense I understand them, that is, quantities which being at first finite, do by a gradual diminution at last vanish into nothing and consequently must, during their diminution, become less than any quantity that can be assigned" (p. 389). As to evanescent quantities being entities or non-entities, "If this page were divided from top to bottom into two equal parts, one black, and the other white,[1] and Dr. Pemberton were to ask me, whether the middle line, which divides the two parts, were black or white, I apprehend it would be a direct answer to say, it is neither; it cannot properly be called either a black line, or a white line; it is the end of the white and beginning of the black, or the end of the black and beginning of the white" (p. 389). "I was apprised that Mr. Robins had all along expressed the sentiments of Dr. Pemberton" (p. 393). Dr. Pemberton still refuses to give his interpretation of Newton's

[1] As far as I know, Jurin is the first to use colour devices to illustrate subtle points in evanescent quantity or in number. Jules Tannery, in his *Leçons d'Algèbre et d'Analyse*, Paris, 1906, p. 14, uses colour imagery to illustrate the discussion of irrational numbers.

Lemma. "Every body will be satisfied that the true reason of his backwardness, is the fear he is under, that I shall make good my promise, in shewing, that his explanation is either a false one, or, in case it be true, is to all intents and purposes the very same with mine" (p. 396).

In June, 1737, Dr. Pemberton replies again, by repeating his previous assertion against Philalethes's explanation of Newton's *Lemma*, given in the *Minute Mathematician*, but does not permit himself to be drawn into giving an explanation of his own of Newton's *Lemma*.

In Jurin's article in the July issue, 1737, we read: "I did indeed take notice of the prudence Dr. Pemberton used, in passing by my second interpretation, which was so clear and plain, and was so fully illustrated by examples, that there was no possibility of perverting the sense of it" (p. 70). "But since this dispute, which began upon matters of science, . . . unless Dr. Pemberton shall see fit to revive it by giving his so long demanded explication, I shall not judge it worth while to take notice of what he may hereafter write."

Dr. Pemberton followed with some *Observations* in the August, 1737, number, while in the September number there appears "the last reply of Philalethes," and in the October number the final answer by Pemberton. Thus ended a dispute which had for some time ceased to contain much of scientific and historic value.

Debate over Robins's Review of Treatises written by Leonhard Euler, Robert Smith, and Jurin

142. Being in a somewhat combative mood, Robins made attacks upon Euler's treatise on motion, Dr. Robert Smith's optics, and Jurin's essay on vision.[1]

Robins's criticisms of Euler concern mainly the philosophy of the Calculus. Robins quotes Euler's third proposition, "That in any unequal motion the least element of the space described may be conceived to be passed over with an uniform motion," and then says, this "is not universally true," as, for instance, "when those spaces are compared together, which a body accelerated by any force described in the beginning of its motion; for the ultimate proportion of the first of two contiguous spaces, thus described in equal times to the second, is not that of equality, but the ratio of 1 to 3, as is well known to every one acquainted with the common theory of falling bodies" (p. 2). In another place (p. 4) Robins argues that the path assigned by Euler to a certain body "is false even on the confused principles of indivisibles." Some passages in Robins involve the Leibnizian notation in the calculus, and look quite odd in an eighteenth-century publication prepared by a Briton in Great Britain. Robins concludes that most of Euler's errors "are owing to so strong an attachment to the principles, he had imbibed under that inelegant

[1] *Remarks on Mr. Euler's Treatise of Motion, Dr. Smith's Compleat System of Opticks, and Dr. Jurin's Essay upon Distinct and Indistinct Vision.* By Benjamin Robins, London, 1739.

computist, who was his instructor, that he was afraid to trust his own understanding even in cases, where the maxims, he had learnt, seemed to him contradictory to common sense" (p. 30). This master was John Bernoulli.

143. Never losing an opportunity to engage in controversy, Jurin wrote a treatise in reply.[1] We refer only to such parts of this pamphlet, and the ones which followed it, as bear on fluxions or the parties engaged in the discussions on fluxions.

In the preface Jurin says: "I, it seems, am the Reputed Author of the late dissertations under the name of Philalethes Cantabrigiensis, and the other Gentleman [Dr. Robert Smith] is . . . suspected of being my associate. . . . If Dr. Smith were to tell Mr. Robins, what he has often professed to other persons, that he had no hand in those papers; if to confirm this he were to remind him, that Philalethes has declared more than once, he wrote alone and unassisted; if I—But what signifies pleading, when the execution is over? Mr. Robins has already vented his Resentment to the utmost. . . ."

144. Not without interest is the following reference to young Euler in St. Petersburg, whose scientific achievements have been so very extraordinary. Jurin says that to make no reply to Robins's criticisms "might be such a discouragement to the hopeful young writer, whose name is prefixed

[1] *A Reply to Mr. Robins's Remarks on the Essay upon Distinct and Indistinct Vision Published at the End of Dr. Smith's Compleat System of Opticks.* By James Jurin, M.D., London, MDCCXXXIX.

to their common labours, and who possibly, when he comes to study *suo Marte*, and to see with his own eyes, or to meet with abler instructors, may make some figure in the Learned World, that pure humanity induces me to oblige them with this one Reply" (p. 54).

145. Of course, Robins wrote a tract in reply,[1] but only the preface of this tract demands our attention. In answer to the charge made by Jurin, that he (Robins) had conducted the controversy "with passion and abuse," Robins proceeds to explain their past relations to each other.

"About six years since a pamphlet was publish'd under the title of the *Analyst*; in which the author endeavors to shew, that the doctrine of fluxions invented by Sir Isaac Newton is founded on fallacious suppositions. As that writer had a false idea of this doctrine, . . . I thought the most effectual method of obviating his objections would be to explain . . . what Sir Isaac Newton himself had delivered with his usual brevity. . . . And with this view I published a Discourse on Sir Isaac Newton's method of fluxions, and of prime and ultimate ratios. But in the mean time a controversy was carrying on between the author of the *Analyst* and another, who under the name of Philalethes Cantabrigiensis had undertaken the defence of Sir Isaac Newton: and as I at last perceived, both by the concessions

[1] *A Full Confutation of Dr. Jurin's Reply to the Remarks on his Essay upon Distinct and Indistinct Vision.* By Benjamin Robins, London, 1740.

of Philalethes, and the avowed opinions of others, that the erroneous conceptions of the writer of the *Analyst* on this head were more prevalent even amongst those, who approved of the method of fluxions, than I had at first believed; I thought, it might be no unacceptable task more particularly to shew those, who were thus misled, how irreconcileable their opinions were with the tenets of Sir Isaac Newton, and how impossible it would be to defend the accuracy of his doctrine on these their mistaken suppositions; and it was with this intention, that in an account of my book inserted in the *Present State of the Republick of Letters*, some of the errors contained in the writings of Philalethes Cantabrigiensis were endeavoured to be obviated.

"But tho' this discourse was written with great caution, and only mentioned the principles objected to without so much as naming or even insinuating the treatises, from whence they were taken; yet, as Dr. Jurin, who was generally reputed the author of them, was one, that I often conversed with; at my request, before this paper was printed, a common friend carried to him the manuscript, and, without pretending to suppose, whether he was, or was not Philalethes, desired him to read it, and asked him if he thought, Philalethes could be displeased with any thing contained in it; he was also told at the same time, that if he believed any part of it could give offence to that gentleman, whoever he were, it should be struck out, or that I would even let the whole design fall, if he desired it.

"My friend brought me the Doctor's answer importing, that he could not believe, my paper would displease any one, since, if the tenets, I excepted to, were really erroneous, it was reasonable, they should be exposed; and if otherwise, it was the business of Philalethes to defend them . . . it was however added, that I had in two places censured doctrines, which, if I supposed them to be the opinions of Philalethes, I must have misapprehended him. Now . . . I immediately expunged them, and published the remaining part in the *Republick of Letters* for October 1735, as an account of my book on Sir Isaac Newton's method of fluxions, and of prime and ultimate ratios.

"To this Philalethes answered in the following month, and I again replied, till five papers were successively written in this controversy, that is, three by me, and two by him. And all this time so very desirous was I on my part of avoiding irritating circumstances . . . that I thought even the most intimate friend . . . could not be offended with it. . . . But alas . . . Philalethes in his reply, part of which was published in the July following, and the rest in the succeeding month, runs out into the most extravagant heats of passion . . . charging me with dishonestly writing against the convictions of my own judgment. . . . After so gross and unprovoked an abuse, . . . I should surely have been acquitted of any breach of decency, if . . . I had sharply exposed his ignorance in the subjects,

he had attempted. But I chose, if possible, to avoid the ridicule of quarreling on a matter of mere speculation . . . I again requested my friend to speak to Dr. Jurin, and to represent to him the inconveniencies, that would arise from the perseverance of Philalethes in his rash and groundless calumny. My friend accordingly went to Dr. Jurin, and carried with him an answer to so much of Philalethes's paper, as was then published, and told the Doctor, that he came to propose to him a method, that might prevent the controversy betwixt me and Philalethes from degenerating into a passionate personal altercation . . . that therefore, if Dr. Jurin thought it expedient, my paper should be given to a certain gentleman, to whose impartiality and knowledge of the subject in debate no exception could be taken on either side; and that if, when that gentleman had perused it, he should believe, I had in any instance changed my opinion from my first entering into this dispute, I did then promise to submit patiently and without reply to any censures of unfairness and dishonesty, that Philalethes . . . should hereafter think proper, . . . [otherwise] it would then be but common justice, that Philalethes should moderate the remaining part of his performance. . . . But this proposal was rejected. . . . It was immediately given out, that my friends had confessed me to have been foiled in the argument; and were now only sollicitous to support me from the charge of unfairness. . . . The reader will not wonder, if I resolved for the future

to treat him with that freedom, which his unskilfulness authorised. . . ."

146. The above preface constitutes what we may call Robins's *apologia pro vita sua.* It seems only fitting that Jurin should appear with a similar document. This he did in a long *Letter.*[1]

We make the following quotations from Jurin (p. 8):

"About five years ago some passages in a paper of Mr. Robins, were shown to me . . . and a question was put to me, whether I should take it ill, if those passages were printed, it being intimated, that Philalethes, against whom they were designed, might possibly be some friend of mine: and indeed, several persons were then guessed at, all of which happened to be my friends. To this . . . I gave answer, that I should not at all take it ill. But I added, that as I had read the controversy between Philalethes and the Author of the *Analyst*, with some attention, it seemed to me that in one or two passages Mr. Robins imputed opinions to Philalethes, which . . . that gentleman did not hold. . . . Also, I took notice, that Mr. Robins did not rightly explain Sir Isaac Newton's first Lemma. . . . But when I desired to talk with Mr. Robins about the Lemma, before the papers went to the press, as imagining I could convince him that he was in the wrong, answer was made, that the question was

[1] *A Letter to . . . Esquire, In Answer to Mr. Robins's Full Confutation of the Reply to his Remarks on the Essay upon distinct and indistinct Vision.* By James Jurin, M.D., London, 1741.

not whether I thought him in the right or in the wrong, but only whether I should take anything amiss; to which I replied as before. Upon talking with another friend of Mr. Robins a day or two after, I repeated my desire to talk with Mr. Robins about his explanation of the Lemma, before his papers went to the press: but was told that could not be, for that the part of the papers where the Lemma was spoke of, was to go to the press that afternoon. . . . I do not remember, that any offer was made to me of 'letting the whole design fall, if I desired it.' Had any such offer been made, I had at that time so much regard for Mr. Robins, that I think I should at least have desired him to stop the design, till he and I had examined the Lemma together, in order to prevent his exposing himself in the manner he has since done. As to the second application made to me near a year after, it may easily be judged, that I, who gave these gentlemen no reason to think I had any influence over Philalethes, or so much as knew who he was, could neither comply with nor reject their proposal" (p. 9).

Remarks

147. The debate between Jurin and Robins is the most thorough discussion of the theory of limits carried on in England during the eighteenth century. It constitutes a refinement of previous conceptions.

Jurin possessed the more general conception of

a limit in insisting that there are variables which reach their limits. His interpretation of Newton on this point appears to us more nearly correct than that of Robins; Jurin's geometric illustrations of limit-reaching variable, intended to aid the imagination, though as he admits incapable of exhibiting the process "all the way," are nevertheless interesting (see our §§ 124, 132, 133). The imagination is subject to limitations where the reason is still free to act.

Robins, and after him Pemberton, deserve credit in clearly, openly, and completely breaking away from infinitely little quantities, and from prime and ultimate ratios. Robins's conception of a limit was narrow, but this narrowness had certain pedagogical advantages, since it did not involve a mode of advance to the limit which altogether transcended the power of the imagination to follow all the way (see our §§ 117, 118, 129, 130).

It is interesting to observe that both Jurin and Robins disavow belief in the possibility of a subdivision of a line into parts so as to reach a point—they assert "that such subdivision can never be actually finished" (see our §§ 126, 132).

Robins discarded the use of Newton's *moments* in developing the theory of fluxions (see our §§ 119, 120).

Toward the end of his long debate with Robins, Jurin begins to disavow infinitely small quantities. He brings out the difference between infinitesimals as variables, and infinitesimals as constants. He

rejects all quantity "fixed, determinate, invariable, indivisible, less than any finite quantity whatsoever," but he usually admits somewhat hazily a quantity "variable, divisible, that, by a constant diminution, is conceived to become less than any finite quantity whatever, and at last to vanish into nothing." (See our §§ 132, 138, 141.)

While Berkeley's *Analyst* and Berkeley's replies to Jurin and Walton involved purely destructive criticism, the present controversy between Jurin and Robins brought forth valuable constructive results. Jurin's papers against Robins are decidedly superior to those he wrote against Berkeley, though here too they contained much that was not pertinent to the subject and was intended merely to amuse the general reader.

CHAPTER V

TEXT-BOOKS IMMEDIATELY FOLLOWING BERKELEY'S ATTACK

148. The *Analyst* was published in 1734; two years later appeared four books on fluxions. Thus, more British text-books on this subject were published in 1736 than in all the thirty years preceding. That the *Analyst* controversy was largely the cause of this increased productivity there can be no doubt. We proceed to give an account of the books which preceded the publication of Maclaurin's *Treatise of Fluxions*, 1742.

John Colson, 1736

149. Newton's *Method of Fluxions*,[1] said to have been written in 1671, was translated and first published in 1736 by John Colson. Colson had been a student at Christ Church, Oxford, which he left without taking a degree. He was appointed

[1] *The Method of Fluxions and Infinite Series; with its Application to the Geometry of Curve-Lines. By the Inventor, Sir Isaac Newton, Kt., Late President of the Royal Society. Translated from the Author's Latin Original not yet made publick. To which is subjoined, A Perpetual Comment upon the Whole Work, . . .* By John Colson, M.A. and F.R.S., Master of Sir Joseph Williamson's free Mathematical-School at Rochester. London, M.DCC.XXXVI. This book was reissued in 1758.

master of a new mathematical school founded at Rochester, and, in 1739, Lucasian professor of mathematics at Cambridge, in succession to Nicholas Saunderson. Colson was a man of great industry but only ordinary ability.

In his preface, Colson refers to the controversies on fluxions, and says that the defenders as well as their opponents were little acquainted with Newton's own exposition, that this book now published for the first time is "the only genuine and original Fountain of this kind of knowledge. For what has been elsewhere deliver'd by our Author, concerning this Method, was only accidental and occasional" (p. x). Colson accompanies Newton's book "with an ample Commentary" and "particularly with an Eye to the fore-mention'd Controversy" (p. x). Colson in this preface represents Newton as holding the principle "that Quantity is infinitely divisible, or that it may (mentally at least) so far continually diminish, as at last, before it is totally extinguish'd, to arrive at Quantities that may be call'd vanishing Quantities, or which are infinitely little, and less than any assignable Quantity. Or it supposes that we may form a Notion, not indeed of absolute, but of relative and comparative infinity" (p. xi). Colson opposes "indivisibles," as also the "infinitesimal method" and "infinitely little Quantities and infinite orders and gradations of these, not relatively but absolutely such" (p. xii). He argues against "imaginary Systems of infinitely great and infinitely little Quantities, and their

several orders and properties, which, to all sober Inquirers into mathematical Truths, must certainly appear very notional and visionary" (p. xii), for "Absolute Infinity, as such, can hardly be the object either of our Conceptions or Calculations, but relative Infinity may, under a proper regulation" (p. xii). Newton "observes this distinction very strictly, and introduces none but infinitely little Quantities that are relatively so." Colson answers Berkeley's criticism in the *Analyst* of Lemma 2, Book II, in the *Principia* in the following manner:—

"Let X and Y be two variable Lines. . . . Let there be three periods of time, at which X becomes $A-\frac{1}{2}a$, A, $A+\frac{1}{2}a$; and Y becomes $B-\frac{1}{2}b$, B, and $B+\frac{1}{2}b$ . . . Then . . . the Rectangle XY will become . . . $AB-\frac{1}{2}aB-\frac{1}{2}bA+\frac{1}{4}ab$, AB, and $AB+\frac{1}{2}aB+\frac{1}{2}bA+\frac{1}{4}ab$. Now in the interval from the first period of time to the second . . . its whole Increment during that interval is $\frac{1}{2}aB+\frac{1}{2}bA-\frac{1}{4}ab$. And in the interval from the second period of time to the third, . . . its whole Increment during that interval is $\frac{1}{2}aB+\frac{1}{2}bA+\frac{1}{4}ab$. Add these two Increments together, and we shall have $aB+bA$ for the compleat Increment of the Product XY" (p. xiii), called the "Moment of the Rectangle" when a and b are infinitely little.

Another mode of procedure is this: "the Fluxions or Velocities of increase, are always proportional to the contemporary Moments." "When the Increments become Moments, that is, when a and b are

so far diminish'd, as to become infinitely less than A and B; at the same time ab will become infinitely less than either aB or bA (for[1] aB . ab : : B . b, and bA . ab : : A . a), and therefore it will vanish in respect of them. In which case the Moment of the Product or Rectangle will be aB + bA, as before" (p. xv). Newton, however, prefers the more direct way previously explained.

Proceeding to Newton himself, we find (on p. 24) the following definition: "The Moments of flowing Quantities (that is, their indefinitely small Parts, by the accession of which, in infinitely small portions of Time, they are continually increased) are as the Velocities of their Flowing or Increasing. Wherefore if the Moment of any one, as x, be represented by the Product of its Celerity $\dot{x}$ into an indefinitely small Quantity o (that is, by $\dot{x}\,o$), the Moments of the others v, y, z, will be represented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; because $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$." On p. 25 terms containing o as a factor "will be nothing in respect of the rest. Therefore I reject them."

150. Colson appended extensive annotations to Newton's treatise. In these annotations, p. 250, Colson speaks of "smallest particles," but the term "smallest" does not occur in Newton's definition. However, Colson says that he does not mean "atoms" nor "definite and determinate magnitude, as in the Method of Indivisibles," but things "indefinitely small; or continually decreasing, till

[1] Here aB . ab : : B . b means aB : ab : : B : b.

they are less than any assignable quantities, and yet may then retain all possible varieties of proportion to one another. Becoming still more deeply involved in the metaphysics of the subject, Colson adds "that these Moments are not chimerical, visionary, or merely imaginary things, but have an existence *sui generis*, at least Mathematically and in the Understanding, is a necessary consequence from the infinite Divisibility of Quantity, which I think hardly anybody now contests" (p. 251). This he qualifies, "perhaps the ingenious Author of . . . *The Analyst* must be excepted, who is pleased to ask, in his fifth Query, whether it be not unnecessary, as well as absurd, to suppose that finite Extension is infinitely divisible" (p. 251). By *ultimate ratio* Colson means the ratio when the arguments "become Moments" (p. 255). Fearing that *moments*, *infinitely little quantities*, and the like, "may furnish most matter of objection," he says (p. 336) that the symbol *o* at first represents a finite quantity, which then diminishes continually till "it is quite exhausted, and terminates in mere nothing." But "it cannot pass from being an assignable quantity to nothing at once; that were to proceed *per saltum*, and not continually"; hence "it must be less than any assignable quantity whatsoever, that is, it must be a vanishing quantity. Therefore the conception of a Moment, or vanishing quantity, must be admitted as a rational Notion" (p. 336). Again: "The Impossibility of Conception may arise from the narrowness and imperfection

of our Faculties, and not from any inconsistency in the nature of the thing"; these quantities "escape our imagination." Referring to imaginaries, $a\sqrt{-1}$ in the solution of cubic equations, Colson says (pp. 338–9): "These impossible quantities . . . are so far from infecting or destroying the truth of these Conclusions, that they are the necessary means and helps of discovering it. And why may we not conclude the same of that other species of impossible quantities, if they must needs be thought and call'd so? . . . Therefore the admitting and retaining these Quantities . . . 'tis enlarging the number of general Principles and Methods, which will always greatly contribute to the Advancement of true Science. In short, it will enable us to make a much greater progress and proficience, than we otherwise can do, in cultivating and improving what I have elsewhere call'd The *Philosophy of Quantity.*"

151. A review[1] of this book contains the following historical exposition. Sir Isaac Newton, 1665, "found the Proportions of the Increments of indeterminate Quantities. These Increments or *Augmenta Momentanea* he called *Moments*, which others called Particles, infinitely small Parts, and Indivisibles; and the Velocities by which the Quantities increased he called Motions, Velocities of Increase, and Fluxions. He considered Quantities not as composed of Indivisibles, but as generated by local Motion, after the manner of the Ancients . . . and represented such Moments [of Time] by the Letter *o*,

[1] *Republick of Letters*, Art. XI, pp. 223–235, 1736.

or by any other Mark drawn into an Unit" (p. 228). "Fluxions are not Moments, but finite Quantities of another kind." "When Mr. Newton is demonstrating any Proposition, he considers the Moments of Time in the Sense of the Vulgar, as indefinitely small, but not infinitely so; and by that means performs the whole work, in finite Figures, by the Geometry of *Euclid* and *Apollonius*, exactly without any Approximation: and when he has brought the work to an Equation, and reduced the Equation to the simplest Form, he supposes the Moments to decrease and vanish; and from the terms which remain he deduces the Demonstration. But when he is only investigating any Truth, or the Solution of any Problem, he supposes the Moment of Time to be infinitely little, in the Sense of Philosophers, and works in Figures infinitely small."

James Hodgson, 1736

152. James Hodgson, a mathematical teacher and writer, and a fellow of the Royal Society of London, is the author of a book, *The Doctrine of Fluxions.*[1]

Hodgson says in his Introduction that "it is now some years since the greatest Part of this Book was prepared for the Press." There is no direct reference in the book to the *Analyst* controversy, but the declaration is made that the principles upon which fluxions rest need "fear no Opposition."

[1] *The Doctrine of Fluxions, founded on Sir Isaac Newton's Method, Published by Himself in his Tract upon the Quadrature of Curves.* By James Hodgson, London, MDCCXXXVI.

Hodgson also says in his Introduction that most books on fluxions that have hitherto appeared proceeded on the same principles as the Differential Calculus, so that "by calling a *Differential* a *Fluxion*, and a second *Differential* a second Fluxion, etc., they have . . . confusedly jumbled the Methods together," although the principles are really "very different." "The *Differential* Method teaches us to consider Magnitudes as made up of an infinite Number of very small constituent Parts put together; whereas the *Fluxionary* Method teaches us to consider Magnitudes as generated by Motion . . . ; so that to call a *Differential* a *Fluxion*, or a *Fluxion* a *Differential* is an Abuse of Terms." In the method of fluxions, "Quantities are rejected, because they really vanish"; in the differential method they are rejected "because they are infinitely small." Hodgson adds that he always used the differential method "'till I became acquainted with the *Fluxionary* Method." He considers fluxions of quantities (p. 50) "in the first Ratio of their *nascent Augments*, or in the last ratio of their *evanescent Decrements*," and gives an able and faithful exposition of Newton's ideas as found in his *Quadrature of Curves*. He cannot think "there is any more difficulty in conceiving or forming an adequate Notion of a nascent or evanescent Quantity, than there is of a Mathematical Point" (p. xi). In explaining the derivation of the fluxion of the product $xy=z$ he apparently permits (p. xv) the small quantity o to "vanish," and thereupon divides

both sides of the equation $x\dot{y}o+\dot{y}xo=\dot{z}o$ by o. However, in the exposition given on p. 50 he is more careful and divides by o while o is an increment, and obtains $y\dot{x}+\dot{x}y+\dot{y}\dot{z}o=\dot{v}$. Then he says: "Imagine the Quantity o to be infinitely diminished, or, which is the same thing, the Quantity xy to return back again into its arising State; then the Quantity $\dot{x}\dot{y}o$, in this Case, into which o is multiplied, will vanish; whence we shall have $x\dot{y}+y\dot{x}=\dot{v}$ for the Fluxion of the Quantity proposed." Hodgson follows Newton closely and permits the variable to reach its limit.

Thomas Bayes, 1736

153. An anonymous pamphlet of 50 pages, on the *Doctrine of Fluxions*,[1] has been ascribed to Rev. Thomas Bayes. This author contributed in 1763 to the *Philosophical Transactions* a meritorious article on the doctrine of chances.

The pamphlet of 1736 represents a careful effort to present an unobjectionable foundation of fluxions. "The fluxion of a flowing quantity is its rate or swiftness of increase or decrease." Let a, b, x, and y be flowing quantities, let A and B be permanent quantities; if $a:b=A\mp x : B\mp y$, during any time T, and at the end of that Time, a, b, x, y all vanish; then . . . the ratio of A to B is the last ratio of the vanishing quantities a and b (p. 13). This definition is "in effect the same" as that given by Newton.

[1] *Introduction to Doctrine of Fluxions and Defence of the Mathematicians against . . . the Analyst*, 1736.

The author speaks of "that most accurate definition of the ultimate ratio's of vanishing quantities; which we have at the latter end of *Sch. Lemma* XI *Princip.* [see our §§ 10–15], and which is so plain, that I wonder how our author [Berkeley] could help understanding it; which had he done, I am apt to think that all his *Analyst* says concerning the proportion of quantities vanishing with the quantities themselves, had never been heard: For according to this definition, we are not obliged to consider the last ratio as ever subsisting between the vanishing quantities themselves. But between other quantities it may subsist, not only after the vanishing quantities are quite destroyed, but before when they are as large as you please. And the reason why we consider quantities as decreasing continually till they vanish, is not in order to make, but to find out, this last ratio. Sir Isaac Newton does not indeed say that this last ratio is the ratio with which the quantities themselves vanish; but whether he herein speaks with the utmost propriety or not, is a mere nicety on which nothing at all depends" (p. 16, note).

Velocity "signifies the degree of quickness with which a body changes its situation in respect to space"; the fluxion of a quantity "signifies the degree of quickness with which the quantity changes its magnitude." "And when our author asserts, that in order to conceive of a second fluxion, we must conceive of a velocity of velocity, and that this is nonsense; he plainly appeals to the sound and

not the sense of words . . . if . . . you make it synonymous to the word *Fluxion*, then the velocity of velocity . . . is nothing but plain common sense" (p. 19). Moments are not used by the author. The author says that, were he to write a treatise on fluxions, "in order to understand equations where Fluxions of different orders are jumbled together; it would be convenient to represent all Fluxions not as before, but as quantities of the same kind with their Fluents. . . . The Fluxion of a quantity anyhow flowing at any given instant is a quantity found out by taking it to the Fluxion of an uniformly flowing quantity in the ultimate proportion of those synchronal changes which then vanish" (pp. 34, 35). The variables x and x^n have the synchronal augments o and $nox^{n-1}+\frac{(n^2-n)}{2}o^2x^{n-2}+$, etc., which are to one another as $1 : nx^{n-1}+\frac{(n^2-n)}{2}ox^{n-2}+$, etc. "Let now these arguments vanish, and their last ratio will be $1 : nx^{n-1}$." "This our author says is no fair and conclusive reasoning, because when we suppose the 'increments to vanish, we must suppose their proportions, their expressions, and everything else derived from the supposition of their existence to vanish with them.' To this I answer, that our author himself must needs know thus much, viz. That the lesser the increment o is taken, the nearer the proportion of the increments of x and x^n will arrive to that of 1 to nx^{n-1}, and that by supposing

the increment *o* continually to decrease, the ratio of these synchronal increments may be made to approach to it nearer than by any assignable difference, and can never come up with it before the time when the increments themselves vanish. . . . For tho', strictly speaking, it should be allowed that there is no last proportion of vanishing quantities, yet on this account no fair and candid reader would find fault with Sir Isaac Newton, for he has so plainly described the proportion he calls by this name, as sufficiently to distinguish it from any other whatsoever: So that the amount of all objections against the justice of this method in finding out the last proportion of vanishing quantities can arise to little more than this, that he has no right to call the proportions he finds out according to this method by that name, which sure must be egregious trifling. However, as on this head our author seems to talk with more than usual confidence of the advantage he has over his opponents, and gives us what he says is the amount of Sir Isaac's reasoning, in a truly ridiculous light, it will be proper to see on whom the laugh ought to fall, for I am sure somebody must here appear strangely ridiculous, . . . I readily allow whatever consequence he is pleased to draw from it, if it appears that Sir Isaac, in order to find the last ratios proposed was obliged to make two inconsistent suppositions. To confute which nothing more need be said than barely to relate the suppositions he did make.

"1. Then he supposes that x by increasing becomes $x+o$, and from hence he deduces the relation of the increment of x and x^n.

"2. Again, in order to find the last ratio of the increments vanishing, he supposes o to decrease till it vanishes, or becomes equal to nothing. . . . These are evidently no more inconsistent and contradictory, than to suppose a man should first go up stairs, and then come down again. To suppose the increment to be something and nothing at the same time, is contradictory; but to suppose them first to exist, and then to vanish, is perfectly consistent; nor will the consequences drawn from the supposition of their prior existence, if just, be any ways affected by the supposition of their subsequent vanishing, because the truth of the latter supposition no ways would have been an inconsistency; but to suppose them first unequal, and afterwards to become equal, has not the shadow of difficulty in it. . . . must confess there seems to be some objection against considering quantities as generated from moments. What moments, what the *principia jamjam nascentia finitarum quantitatum*, are in themselves, I own, I don't understand. I can't, I am sure, easily conceive what a quantity is before it comes to be of some bigness or other; and therefore moments considered as parts of the quantities whose moments they are, or as really fixed and determinate quantities of any kind, are beyond my comprehension, nor do I indeed think that Sir Isaac Newton himself did thus consider them" (pp. 35–41).

John Muller, 1736

154. John Muller, a German by birth, dates his *Mathematical Treatise*,[1] 1736, from the Tower of London, and dedicates it to the master-general of the ordnance. He was appointed in 1741 head-master of the Royal Military Academy, Woolwich. He was "the scholastic father of all the great engineers this country employed for forty years."

The author's method of explaining fluxions is somewhat unique. "I make no use of infinitely small quantities nor of nascent or evanescent velocities; and yet I think to have explained those Principles, so that any Person of a moderate capacity . . . may be fully convinced of the Truth thereof" (Preface). He begins his conic sections with the postulate: "Grant that two infinite quantities, differing from each other by a finite quantity, may be esteemed equal." He then explains that this postulate "is here of use only to shew the connection of the Conic-Sections," and hastens to assure the reader that "whenever we make use of it in the demonstration of any Proposition, we shall give always another Demonstration independent on it."

In the *Republick of Letters*, June, 1736, occurs the following comment:

"He introduces this [Conic-Sections] by a Postulatum that sounds very absurdly to those that are

[1] *A Mathematical Treatise: Containing a System of Conic-Sections; with the Doctrine of Fluxions and Fluents, Applied to various Subjects.* By John Muller. London, 1736.

not vers'd in mathematical Speculations. 'Grant,' says he, 'that two infinite Quantities, differing from one another by a finite Quantity, may be esteemed equal.' Such would imagine that there could not be two infinite Quantities; or that if there could, they must necessarily be absolutely and not only reputedly equal. But however Hobbes or Berkeley may talk of geometrical Fallacies, or these unexperienced People think, the Adepts in this Science very well know, that more infinite Quantities than two are possible, and that one Quantity may be infinitely greater than an infinite one, and yet be itself infinitely less than a third. But enough of these *Ludibria* Scientiæ, that I may inform the Publick of the more useful Theorems . . ." (pp. 422, 423).

Muller considers in his text a curve generated by a point "urged by two powers acting in two different directions, the one parallel to the Abscisses and the other parallel to the Ordinates. I prove from thence, that if this point (when arrived at a given place) did continue to move with the velocity it has there, it would proceed in a right line touching the Curve in that place . . . So that the three Directions being known in each place, the proportion between the velocities of the urging powers is likewise known." Fluxions are defined as velocities. To find the fluxion of y^2, he puts $y^2 = x$; the subtangent of the parabola is $2y^2$ Since the subtangent is to the ordinate as the velocities along the abscissa and ordinate, he has $2y^2 : y : : \dot{x} : \dot{y}$, or $\dot{x} = 2y\dot{y}$, and $2y\dot{y}$ is the required fluxion. Similarly, to find the

fluxion of y^3, let $x=y^3$. Take $u=z^3$, the $u-x=\overline{z-y} \times zz+zy+yy$, or $z-y : u-x = 1 : zz+zy+yy$. If now y and z approach continually until they coincide with an intermediate ordinate, then $z=y$ and the chord through the extremities of the ordinates y and z will likewise coincide with the tangent. Therefore, the ordinate is to the subtangent as 1 is to $3yy$. Hence the proportion $1 : 3yy = \dot{y} : \dot{x}$, or $\dot{x}=3yy\dot{y}$, the fluxion required. The same argument is applied to y^m. In these demonstrations appeal is made to a geometric figure, and no attention is directed to the ratio $z-y : u-x$ for the difficult case when $y=z$. The author remarks that "though we commonly say that . . . $m\dot{y}y^{m-1}$ is the Fluxion of y^m; yet that expression is not sufficiently accurate: Therefore, the sense in which we desire to be understood is, that $1 : my^{m-1} :: \dot{y} : m\dot{y}y^{m-1}$, that is, unity is to my^{m-1}, or $\dot{y}$ is to $m\dot{y}y^{m-1}$, as the fluxion or velocity with which y is generated, is to the fluxion, or contemporary velocity with which y^m is generated, and so for the rest" (p. 79). Thus, the emphasis is placed upon the *ratios* of velocities.

Anonymous Translation[1] *of Newton's "Method of Fluxions," 1737*

154*a*. Colson's translation from the Latin of Newton's *Method of Fluxions*, published in 1736, was followed in 1737 by a second translation, which

[1] *A Treatise of the Method of Fluxions and Infinite Series, With its Application to the Geometry of Curve Lines. By Sir Isaac Newton, Kt. Translated from the Latin Original not yet published. Designed by the Author for the Use of Learners.* London, MDCCXXXVII.

was anonymous. In it no mention is made of Colson's edition. The anonymous translator says in the preface: "We have reason to believe that what is here delivered, is wrought up to that Perfection in which Sir Isaac himself had once intended to give it to the Publick. The ingenious Dr. Pemberton has acquainted us that he had once prevailed upon him to complete his Design and let it come abroad. But as Sir Isaac's Death unhappily put a stop to that Undertaking, I shall esteem it none of the least Advantages of the present Publication, if it may prove a means of exciting that Honourable Gentleman, who is possessed of his Papers, to think of communicating them to some able Hand; that so the Piece may at last come out perfect and entire." As remarked by G. J. Gray,[1] the two translations were made "from copies of the same manuscript," and differ from each other only "in the mode of expressing the work in English."

James Smith, 1737

155. In his *New Treatise of Fluxions*,[2] Smith says (Preface): "What I call here the New Method, and the Six Propositions immediately following,

[1] *A Bibliography of the Works of Sir Isaac Newton.* By George J. Gray. Second edition, Cambridge, 1907, p. 47.

[2] *A New Treatise of Fluxions, containing, I. The Elements of Fluxions, demonstrated in Two easy Propositions, without first or last Ratios, II. A Treatise of Nascent and Evanescent Quantities, first and last Ratios, III. Sir Isaac Newton's Demonstration of the Fluxions enlarged and illustrated: IV. Answers to the Principal Objections in the Analyst.* By James Smith, A.M., London, 1737.

are entirely New . . . Our common Definition of Motion, *translatio corporis de loco in locum* is certainly imperfect, and I am inclined to think, that Aristotle's old exploded Definition of Motion will, some time or other, come into Vogue again. *Actus entis in potentia, quatenus in potentia est.* Motion is an Effect, and every Effect has a coinstaneous Existence with the Action by which it is produced."

The definitions with which Smith starts out are not very reassuring. "The fluxion of a surface is the Velocity of the generating Line." "The velocity of a generating Line is the Sum of the Velocities of all the Points of that Line, whether these Points move with equal, or unequal Velocities." The rectangle xy "flows or increases by the flowing of both its contiguous Sides" together; but it "flows into Length" by the velocity $y\dot{x}$, and "it flows into Breadth at the very same Instant of Time" by the velocity $x\dot{y}$. "Therefore the Velocity with which it flows into Length and Breadth is the Sum of the synchronic Velocities," $x\dot{y}+y\dot{x}$.

Nor is the second topic displayed with illumination. "A nascent Quantity is a Quantity in the Instant of its commencing to exist." Similar to this is the definition of "evanescent Quantity," as are also the definitions of first and last ratios. Interesting is the following proof that if "two Quantities begin and cease to exist in any finite Time T, . . . they have a first and a last Ratio,"

for, "if they have not a first Ratio, they have not a second nor a third Ratio, etc. Therefore they have no Ratio in the Time T; but in the Time T they are Quantities," and "two quantities of the same kind, as soon or so long as they have any Quantity, Being or Existence [*i.e.* are not absolutely nothing], have a Ratio the one to the other," that is, "they have a Ratio and they have not a Ratio in the Time T, which is absurd." Smith argues also that since two quantities "cannot be in their first Ratio, neither before nor after the Beginning of the Time T, they must have been in their first Ratio at the very Beginning of the Time T, just as they began to exist." Near the close of this part of his book, Smith reveals some of the subtleties of his topic by stating an "Objection" and the "Answer" to it. The Objection: "Nascent and evanescent Quantities are Something or Nothing; for, *Inter ens et non-ens non datur medium.* If Something, then the Ratio of evanescent Quantities is the same with the Ratio before they were evanescent, or when they had any finite Magnitude. . . . If they are mere Nothing, or Non-quanta; then $B\dot{b} / E\dot{e} = 0 / 0 = 0$; . . . which is absurd." In the "Answer" Smith says: "Evanescent Quantities are really nothing, or Non-quanta; for it is evident . . . that upon *b*'s coinciding with B, and *e*'s coinciding with E, the Increments $B\dot{b}$ and $E\dot{e}$ are annihilated, and evanescent Quantities are never accurately evanescent, but upon this or the like Coincidence. And yet it does not follow that their

last Ratio, or the Ratio they nihilesce with, is Nothing. For B*b* / E*e* is neither B*b* nor E*e*, nor B*b* and E*e*, but a Mark or Expression of their Ratio, which may be expressed as well by any other Character. . . . The Increments are indeed annihilated and gone, but their last Ratio remains, and is as real as any Ratio they ever had"; . . . they have as real a Ratio at the last Instant of their Existence; that is, when they are ceasing to be Something, and commencing to be Nothing, as they had at any instant preceding the last Instant of their Existence." . . . "There is, sometimes, something very strange in the Nature of these evanescing Augments, and it is literally true of them, what Juvenal figuratively says of Man.

—Mors sola fatetur,
Quantula sunt hominum corpuscula—

We know nothing of them till they be dead and gone."

Of Part III, in which Smith "demonstrates" Newton's Method of Fluxions, we quote only the last sentence: "I have made use of infinitely little Quantities, and of a second Point as being next to a first Point; but this was only for Illustration sake. There is not the least Occasion for any of these Notions in the Demonstration."

In the last part of Smith's book, Berkeley's contention, "No just Conclusion can be drawn from two contrary Suppositions," is answered by the statement, "This is certainly true, *in sensu composito*, but *in sensu diviso* is intirely false."

We are tempted to make the remark that in 1737 Smith left the subject even more mysterious than he found it.

Thomas Simpson, 1737

156. Thomas Simpson, the son of a weaver, was a self-taught mathematician, and acquired a knowledge of fluxions through Stone's translation of De L'Hospital's *Analyse des infiniment petits*. Simpson was a mathematician of marked power, and influenced considerably the teaching of mathematics in England. In 1737 he brought out his *New Treatise of Fluxions*,[1] which contains some novel features.

"The Fluxions of variable Quantities are always measured by their Relation to each other; and are ever expressed by the finite Spaces that would be uniformly described in equal Times, with the Velocities by which those Quantities are generated."

He finds it easy to show that the fluxion of a rectangular area of constant height and uniformly variable base is as the height drawn into the velocity with which the base changes; also that the fluxion of a curvilinear area generated by an abscissa moving with uniform velocity is at a given point, as the ordinary y for this point, multiplied by that velocity. This last result is applied to finding the fluxion of xy.

Avoiding infinitely small quantities, Simpson finds

[1] *A New Treatise of Fluxions: wherein the direct and inverse Method are demonstrated after a new, clear and concise Manner, with their Application to Physics and Astronomy.* By Thomas Simpson, London, 1737.

the ratio of the fluxions of x and x^2 thus: Let the points m and n move so that the distance h described by n shall always equal the square of the distance g described by m in the same time. Then $(AR)^2 = CS$, $(AR - Rr)^2 = Cs$, and $sS = 2AR \times Rr - (Rr)^2$. But sS is described with accelerated velocity when m moves uniformly, hence sS will be "less than that which would be uniformly described in the same time with the Velocity at the point S, and greater than that which would be described with the Velocity at the point s; and therefore must be equal to the Distance that would be uniformly described with the Velocity at another point e posited somewhere between S and s, in the same Time that the other point m is moving over the Distance rR; therefore rR : $2AR \times Rr - (Rr)^2$: : $g : g(2AR - Rr)$, the Distance that would be described with the Velocity of n, at the point e, in the same Time that m is moving over the Distance g: Now therefore when the points r and s coincide with R and S, then will e coincide with S; . . . and consequently $(2AR - Rr)g$ will then . . . become $2AR \times g$, equal to h the required Distance." The critical part of this proof is "when the points r and s coincide with R and S, then will e coincide with S." A modification of this proof is applied to x^n.

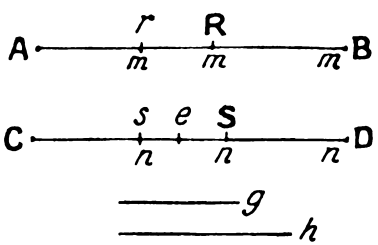


FIG. 7.

Simpson's text marks a departure from Newton

in the definition of fluxion. Newton makes it a velocity, Simpson makes it a finite distance. On the necessity and wisdom of this change there can readily be difference of opinion. But there can be no denying that Simpson developed his theory of fluxions in a manner almost, though not entirely, free from the objections against fluxions that had been advanced by Berkeley; infinitely small quantities are nowhere used. A short but appreciative review of this text appeared in *The Works of the Learned* for July, 1737.

Benjamin Martin, 1739, 1759

157. Benjamin Martin was a mathematician, an optical instrument maker, and a general compiler. He was a self-educated man, and at one time taught reading, writing, and arithmetic. His exposition of fluxions, as found in his *Elements of all Geometry*[1] and in a later work, is below the standard usually reached by him in mathematical writing.

This book, intended as an introduction to modern mathematics, contains in an Appendix an epitome of the doctrine of fluxions. "Since Fluxions are the very small *Increments* and *Decrements* of the Flowing Quantities, or the Velocities of the Motions whereby they increase or decrease, 'tis plain that those Fluxions, or Velocities, themselves may be consider'd as Flowing Quantities, and their Fluxions are call'd Second Fluxions . . ." It would seem

[1] ΠΑΝΓΕΩΜΕΤΡΙΑ; *or the Elements of all Geometry*. By B. Martin, London, M.DCC.XXXIX.

that in this statement a fluxion is "very small" and at the same time a "velocity." A little later the author refers to fluxions as "in the first Ratio of *Augmenta nascentia.*" Evidently, in this Appendix, covering twelve pages, the author has not succeeded in presenting a consistent theory of fluxions.

A fuller exposition was given twenty years later in the *System of Mathematical Institutions, agreeable to the Present State of the Newtonian Mathesis*, by Benjamin Martin, vol. i, London, MDCCLIX. The theory is still confusing. "Indefinitely small Spaces" (p. 362) are represented by $\dot{x}$ and $\dot{y}$, which are called the fluxions of x and y, and said to represent the velocities of moving points. Newton is reported to have at first delivered the idea of what Martin calls a fluxion, under the name of *momentum*, "a Term used in Mechanics to denote the Quantity of Motion generated by a given Quantity of Matter (A), and the Velocity (a) with which it moved conjointly. This *Momentum* therefore was properly represented by (Aa). . . . But instead of this mechanical Notation, we now use $x\dot{x}$ and $y\dot{y}$ for the *Momenta*, or Fluxions. . . ." It is seldom that one encounters a more grotesque conglomeration of unrelated ideas than is presented here. Martin gives John Rowe's mode of deriving the fluxions of xy and xyz.

An Anonymous Text, 1741

158. *An Explanation of Fluxions in a Short Essay on the Theory.* London: Printed for W. Innys, at the

West-End of St. Paul's, MDCCXLI. This anonymous publication of 16 pages was reprinted in 1809 in the fourth edition of John Rowe's *Doctrine of Fluxions*; it constitutes a real contribution to the logic of fluxions. The pamphlet is offered "as an Explanation of the Doctrine itself, and not of Sir Isaac's Manner of delivering it." "About that," he says, "I don't mean, nor pretend to take a Part in any Controversy." He defines fluxions thus: "The word Fluxion properly apply'd always supposes the Generation of some Quantity (term'd Fluent or Flowing Quantity) with an equable, accelerated, or retarded Velocity, and is itself the Quantity which might be uniformly generated, in a constant Portion of Time, with the Amount or Remainder of that Velocity, at the Instant of finding such Fluxion." "Hence, it will appear that the first Fluxions of Quantities are as the Velocities with which those Quantities are increas'd; that second Fluxions are as the Increase or Decrease of such Velocities; and that by second, third, fourth, etc., Fluxions are meant Fluxions, whose Fluents are themselves Fluxions to other proposed Quantities; and the manner of considering and determining them is the very same as tho' they were first Fluxions, they being actually so to the Quantities from which they are immediately derived" (p. 7).

Then follows the lemma:

"The Fluxion of the Area ABC, whether triangular or curvilinear, is the Rectangle $\dot{x}y$."

Suppose B to move along AF while the ordinate

y terminates in the curve AC; "And, at any proposed Position BC, conceive y to become constant," while B "moves uniformly any constant Time, mn, with the Velocity at B, over the Distance $\dot{x}$ or BD; for then will y in the Time mn uniformly generate the Rectangle $\dot{x}y$, which Rectangle is plainly the Fluxion of ABC in this Position (*per Definit.*)." Then follows the illuminating scholium: "It has been commonly objected to the Accuracy of Fluxions, that the Trapezium or curvilinear Space BC*de*D, not the Rectangle $\dot{x}y$, is the Fluxion geometrically exact. But, this Objection is built, I apprehend, upon a false Idea of the Thing. It supposes a Fluxion a complete Part of a flowing Quantity, and an Infinity of Fluxions to constitute the flowing Quantity, which are Mistakes (*per* Definition and Lemma) . . . if $\dot{x}$ be imagined infinitely little, an Infinity of Increments may constitute the Area ABC. But, in Fluxions, our Reasoning is quite different: a Fluxion can no more be called a Part of the Fluent, than an Effect a Part of the Cause. For Instance; from the Fluxion given we know the Fluent, and *vice versa*, just as when a Cause is known to produce a certain Effect, we can infer the one from a Knowledge of the other."

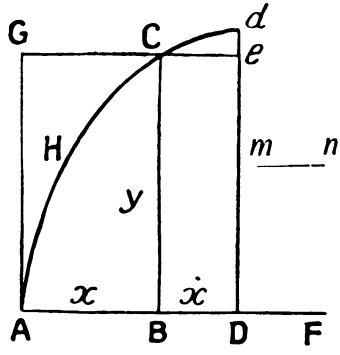


FIG. 8.

We shall find that later this reference to cause and

effect figured in a controversy carried on against Simpson.

As regards the lemma given above, we shall see that the same idea is elaborated in detail by Maclaurin in his work and that a short and even more convincing statement than the one given here is found in the later, revised, text of John Rowe.

From the above lemma, the derivation of the fluxion of xy becomes easy by considering the rectangle ABCG as made up of two parts AHCB and AHCG, and applying the lemma to each part.

John Rowe, 1741, 1757, 1767

159. The first edition (1741) of John Rowe's *Doctrine of Fluxions*[1] appeared anonymously. A copy in the British Museum has the following added by hand after the preface: "This is the first edition of John Rowe's Fluxions. The second came out with his name in 1757 with alterations and additions, and the third came out in 1767 much improved."

In the first edition Rowe begins by stating his programme: "To render the Doctrine of Fluxions plain and easy" by explaining their nature "as deliver'd both by Sir Isaac Newton and by Leibniz." According to Newton, "Fluxion is the same as velocity."

"Definition II [Foreigners Definition]. Quantities are here suppos'd to be generated by a continual Increase, as before; and the indefinitely small Particles

[1] *An Introduction to the Doctrine of Fluxions. Revised by several Gentlemen well skill'd in the Mathematics.* Felicibus inde Ingeniis aperitur Iter—*Claudian.* London, M.DCC.XLI.

whereby they are continually increas'd, are call'd the *Fluxions* of these Quantities" (p. 3). "This is the Notion of Fluxions as deliver'd by Leibnitz and his Followers. But these Fluxions, we shall, in the following Sheets, call by the Names of *Moments*, *Increments* and *Decrements*; that is, *Moments* or *Increments* when the variable Quantities are increasing, and *Decrements* when they are decreasing" (p. 4).

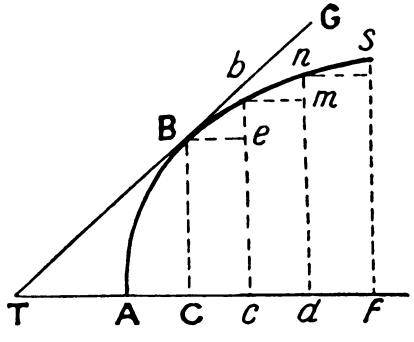


FIG. 9.

"As the Point *b* is continually nearer to a Coincidence with the Tangent TBG the nearer it approaches the Point of Contact B; so if we conceive the Ordinate *cb* to be moved on till it concides with CB; the very first moment before its Coincidence, the Curve B*b*, and Right line BG will be infinitely, or rather indefinitely near a Coincidence with each other; and consequently, in that Case, the Increments B*e*, and *eb* will come indefinitely near to measure the Ratio of the Fluxions of the Absciss and Ordinate AC, and CB, or the Velocities with which they flow at the Point B . . . and therefore (because when any Quantity is increas'd or decreas'd, but by only an infinitely or indefinitely small Particle, that Quantity may be consider'd as remaining the same as it was before;) these Increments may be taken as Proportional to, or for the Fluxions in all Opera-

tions; and, on the contrary, the Fluxion for the Increment" (pp. 5, 6). Accordingly, he deduces the rules of operation by the use of increments, and in the result substitutes the fluxion for the increment.

In finding the fluxion of xy he lets x' and y' be the increments, then the "increase in its nascent state" is such that $x'y'$ "bears no assignable Ratio to either $x'y$ or xy' (for as $x'y' : x'y :: y' : y$ and y' by Supposition is infinitely less than y," and can be "expunged or rejected."

160. The third edition (1767) was commented upon by J. Stubbs, Fellow of Queen's College, Oxford, as follows: "I received your valuable present, and was much surprised to find it so prodigiously improved. Indeed, it so much resembles a New Work, when compared with the First Edition, that I almost wish you had made no mention of its being the Third; but left the two former to be forgotten."

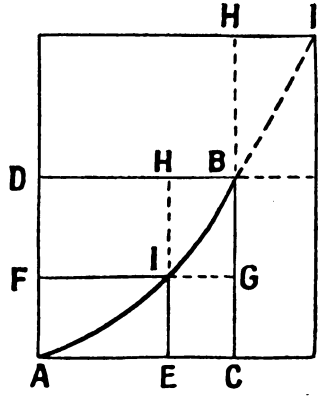


FIG. 10.

The fluxion of xy is now deduced thus: "The fluxion of the curvilinear space AEI is less than the fluxion of the rectangle (of constant altitude) AH before EH reaches BC, and greater after EH passes BC; hence at BC the two fluxions are alike and equal to $y\dot{x}$. Similarly, it follows that the rectangle AG (of constant base) has the same fluxion $x\dot{y}$ at DB as has the curvilinear space AFI. Hence the rectangle

of variable base and altitude AEIF with the vertex I moving along the curve through B has the fluxion $x\dot{y}+y\dot{x}$."

In a footnote Rowe expressed the belief that this mode of deriving the rule is not open to criticism as was the method of using increments which in 1734 was "smartly attacked by the late acute Dr. Berkeley."

Rowe proves by a geometrical method similar to the above that the fluxion of a pyramid of fixed vertex and slant edges, whose variable base xy moves parallel to itself and whose variable altitude is z, is $xy\dot{z}$. Taking a parallelopipedon as equal to three pyramids, he finds the fluxion of xyz to be $\dot{x}yz+x\dot{y}z+xy\dot{z}$. This new way of deriving the fluxion of xyz was copied by "his friend" Benjamin Martin in the *Mathematical Institutions*.

At the end of the third edition of Rowe's *Fluxions* is a bibliography of English works on this subject, and he "particularly refers to the Works of his two celebrated Friends, Mr. Emerson and the late Mr. Simpson."

Berkeley Ten Years After

161. Berkeley, in his *Siris*[1] of 1744, expressed himself as follows: "Concerning absolute space, that phantom of the mechanic and geometrical philosophers (§ 250), it may suffice to observe that it is neither perceived by any sense, nor proved by any reason, and was accordingly treated by the greatest of the ancients as a thing merely visionary.

[1] George Berkeley's *Works*. Edition by A. C. Fraser, vol. ii, Oxford, 1871, p. 468 and note.

From the notion of absolute space springs that of absolute motion . . . " He continues in a footnote: "Our judgment in these matters is not to be overborne by a presumed evidence of mathematical notions and reasonings, since it is plain the mathematicians of this age embrace obscure notions, and uncertain opinions, and are puzzled about them, contradicting each other and disputing like other men: witness their doctrine of Fluxions, about which, within these ten years, I have seen published about twenty tracts and dissertations, whose authors being utterly at variance, and inconsistent with each other, instruct by-standers what to think of their pretensions to evidence."

Remarks

162. In these publications no reference is made to the Jurin-Robins controversy, though Berkeley's *Analyst* is frequently discussed. Excepting only in Benjamin Martin, the definition of a fluxion as a "differential" nowhere appears. Therein we see a step in advance.

The influence of Newton's *Quadrature of Curves* (1704) is evident almost everywhere. An improvement in the mode of deriving the fluxion of a "product" appears in the anonymous *Explanation of Fluxions* and in the revised text by John Rowe (our §§ 158, 160).

Noteworthy is Thomas Simpson's new definition of fluxions; this new definition plays an important rôle during the rest of the century.

163. We quote Sir William Rowan Hamilton's remarks on the lemma of the anonymous *Explanation of Fluxions* (1741) and the derivation of the fluxion of xy, based upon it. Hamilton knew this proof as it is given in a later edition of Simpson's fluxions. Says Hamilton:[1] "I notice that Thomas Simpson treats fluxions as *finite* . . . Thomas Simpson's conceptions appear to have been very clear and distinct, and I do not venture to *say* that the geometrical investigation which he gives of the *fluxion* of a rectangle, avowedly supplied to him by a young but unnamed friend, is *insufficient* in itself, but it fails to convince me, perhaps because I was not early accustomed to fluxions. Certainly there is no *neglecting* of ab, or $\dot{x}\dot{y}$, as *small*; for in fact that *rectangle of the fluxions* is not represented at all in his Figure . . . He conceives the *varying rectangle* xy to be the *sum of two mixtilinear triangles*, of which the *two separate fluxions* are $y\dot{x}$ and $x\dot{y}$. This is very ingenious, but I do not feel sure to what degree I could *rely* on it and build upon it any superstructure, if I were now coming, for the first time, as a *learner*, to the subject. However, I suppose that a pupil, if reasonably modest or even prudent, will take, *for a while*, his teacher's statements upon trust; reserving to himself to *return* upon them, and to examine closely their truth and logic when he shall have acquired some degree of familiarity with the subject taught."

[1] *Life of Sir William Rowan Hamilton*, by R. P. Graves, vol. iii, p. 571.

CHAPTER VI

MACLAURIN'S TREATISE OF FLUXIONS, 1742

164. Colin Maclaurin was educated at the University of Glasgow, and through the influence of Newton was elected professor at the University of Edinburgh. Maclaurin's book on fluxions has been considered the ablest and most rigorous text of the eighteenth century. It was pronounced by Lagrange "le chef d'œuvre de géométrie qu'on peut comparer à tout ce qu' Archimède nous a laissé de plus beau et de plus ingénieux."[1]

In the preface to his *Treatise of Fluxions*[2] Maclaurin says: "A Letter published in the Year 1734, under the Title of the *Analyst*, first gave Occasion to the ensuing Treatise. . . . In the mean Time the Defence of the Method of Fluxions, and of the great Inventor, was not neglected. Besides an Answer to the *Analyst* that appeared very early under the Name of Philalethes Cantabrigiensis . . . a second by the same Hand in Defence of the first, a Discourse by Mr. Robins, a

[1] *Mém. de l'Acad. de Berlin*, 1773; quoted in the art. "Maclaurin" in Sidney Lee's *Dict. of National Biography*.

[2] *A Treatise of Fluxions in Two Books.* By Colin MacLaurin, A.M., Professor of Mathematics in the University of Edinburgh, and Fellow of the Royal Society. Edinburgh, MDCCXLII.

Treatise of Sir Isaac Newton's with a Commentary by Mr. Colson, and several other Pieces were published on this Subject. After I saw that so much had been written upon it to so good Purpose; I was the rather induced to delay the Publication of this Treatise, till I could finish my Design. . . . The greatest Part of the first Book was printed in 1737; But it could not have been so useful to the Reader without the second. . . . In explaining the Notion of a Fluxion, I have followed Sir Isaac Newton in the first Book . . .; nor do I think that I have departed from his Sense in the second Book; and in both I have endeavoured to avoid several Expressions, which, though convenient, might be liable to Exceptions, and, perhaps, occasion Disputes. I have always represented Fluxions of all Orders by finite Quantities, the Supposition of an infinitely little Magnitude being too bold a Postulatum for such a Science as Geometry. But, because the Method of Infinitesimals is much in use, and is valued for its Conciseness, I thought it was requisite to account explicitly for the Truth, and perfect Accuracy, of the Conclusions that are derived from it . . ."

165. In the Introduction to his *Fluxions* Maclaurin says: ". . . When the certainty of any part of geometry is brought into question, the most effectual way to set the truth in a full light, and to prevent disputes, is to deduce it from axioms or first principles of unexceptionable evidence, by demonstrations of the strictest kind, after the manner of the

[*To face page* 182.

COLIN MACLAURIN, A. M.

antient geometricians. This is our design in the following treatise; wherein we do not propose to alter Sir Isaac Newton's notion of a fluxion, but to explain and demonstrate his method, by deducing it at length from a few self-evident truths, in that strict manner: and, in treating of it, to abstract from all principles and postulates that may require the imagining any other quantities but such as may be easily conceived to have a real existence. We shall not consider any part of space or time as indivisible, or infinitely little; but we shall consider a point as a term or limit of a line, and a moment as a term or limit of time . . . [p. 41]. If we are able to join infinity to any supposed idea of a determinate quantity, and to reason concerning magnitude actually infinite, it is not surely with that perspicuity that is required in geometry. In the same manner, no magnitude can be conceived so small, but a less than it may be supposed; but we are not therefore able to conceive a quantity infinitely small . . ."

166. In the posthumous work, *An Account of Sir Isaac Newton's Philosophical Discoveries*, by Colin Maclaurin, 2nd ed., London, 1750, there is printed a life of Maclaurin, from which we glean the following (pp. viii, ix, and xviii) relating to Berkeley's attack in the *Analyst*:

"Mr. Maclaurin found it necessary to vindicate his favourite study, and repel an accusation in which he was most unjustly included. He began an answer to the bishop's book; but as he proceeded,

so many discoveries, so many new theories and problems occurred to him, that, instead of a vindicatory pamphlet, his work came out a complete system of fluxions, with their application to the most considerable problems in geometry and natural philosophy. This work was published at Edinburgh in 1742. . . . His demonstrations had been, several years before, communicated to Dr. Berkeley, and Mr. Maclaurin had treated him with the greatest personal respect and civility: notwithstanding which, in his pamphlet on tar-water,[1] he renews the charge, as if nothing had been done; for this excellent reason, that different persons had conceived and expressed the same thing in different ways. . . . Mr. Maclaurin found it necessary, in demonstrating the principles of fluxions, to reject altogether those exceptionable terms [*infinite* and *infinitesimal*], and to suppose no other than finite determinable quantities, such as Euclid treats of in his geometry."

167. In Chapter I, p. 57, Maclaurin defines a fluxion: "The velocity with which a quantity flows, at any term of the time while it is supposed to be generated, is called its *Fluxion* which is therefore always measured by the increment or decrement that would be generated in a given time by this motion, if it was continued uniformly from that term without any acceleration or retardation: or it may be measured by the quantity that is generated in a given time by an uniform motion which is equal to the generating motion at that term."

[1] In the second edition Berkeley gave the article the name of *Siris*.

The term velocity had been under dispute, particularly in the controversy between Berkeley and Walton. Maclaurin evidently perceived the difficulty in arguing that variable velocity is a physical fact; he says (p. 55), "the velocity of a variable motion at any given term of time is not to be measured by the space that is actually described after that term in a given time, but by the space that would have been described if the motion had continued uniformly from that term. If the action of a variable power, or the velocity of a variable motion, may not be measured in this manner, they must not be susceptible of any mensuration at all"—an argument not likely to convince or silence hostile critics. He quotes Barrow's definition of velocity—"the power by which a certain space may be described in a certain time." In discussing "power" Maclaurin brings in the consideration of "cause" and "effect" in a way that sounds odd in a work laying the foundations to the abstract doctrine of fluxions. Maclaurin uses the word "limit," without giving it a formal definition. Theorem XII reads: "The velocity of a motion that is accelerated or retarded perpetually, is, at any term of the time, to the velocity of an uniform motion, in a ratio that is always a limit between the ratio of the spaces described by these motions in any equal times before that term, and the ratio of the spaces described by them in any equal times after it."

168. In the *Philosophical Transactions*, vol. xlii, for the years 1742–43, London, 1744, Maclaurin

gives an account of his *Treatise of Fluxions*. On p. 330 of these *Transactions* it is pointed out that "the Theory of Motion is rendered applicable to this Doctrine with the greatest Evidence, without supposing Quantities infinitely little or having recourse to prime or ultimate Ratios." Again (p. 336): "There is, however, no Necessity for considering Magnitude as made up of an infinite Number of small Parts; it is sufficient, that no Quantity can be supposed to be so small, but it may be conceived to be diminished further; and it is obvious, that we are not to estimate the Number of Parts that may be conceived in a given Magnitude, by those which in particular determinate Circumstances may be actually perceived in it by Sense; since a greater Number of Parts become visible in it by varying the Circumstances in which it is perceived." Of importance is the following (p. 336): "We shall therefore observe only, that after giving some plain and obvious Instances, wherein a Quantity is always increasing, and yet never amounts to a certain finite Magnitude (as, while the Tangent increases the Arc increases but never amounts to a Quadrant)." That a variable need not reach its limit is also emphasised in other passages, as for instance (pp. 337, 338): "In like manner a curvilineal Area . . . may increase, while the base is produced, and approach continually to a certain finite Space, but never amount to it. . . . A Spiral may in like manner approach to a Point continually, and yet in any Number of Revolutions

never arrive at it. . . . The Author insists on these Subjects, the rather that they are commonly described in very mysterious Terms, and have the most fertile of Paradoxes of any Parts of the higher Geometry."

The ideal of mathematical rigour, as entertained by eighteenth-century writers, was reached by the Greek geometricians, Euclid and Archimedes. To derive the rules of fluxions by the rigorous methods of the ancients was the ambition of Maclaurin. Barring some obvious slips that are easily remedied, Maclaurin certainly reached the ideal he had set. Nor is this so very strange. Fluxions involve questions concerning limits; the ancients overcame the difficulties of such questions by their *method of exhaustion*. It was a rigorous method, but dreadfully tedious. Maclaurin secured his aim at a tremendous sacrifice. His work on fluxions consists of 763 good-sized pages; the first 590 pages do not contain the *notation* of fluxions at all; they deal with the derivation of the fluxions of different geometric figures, of logarithms, of trigonometric functions, also with the discussions of maxima and minima, asymptotes, curvature, and mechanics, in a manner that the ancients might have followed, and with the verbosity of which the ancients are guilty. The consequence was that the work was not attractive reading.

Maclaurin was fully aware of the value of a good notation and ease of operation, for he says of the doctrine (p. 575): "The improvements that have been made by it, either in geometry or in philo-

sophy, are in a great measure owing to the facility, conciseness and great extent of the method of computation, or algebraic part. It is for the sake of these advantages that so many symbols are employed in algebra." But to Maclaurin it seemed "worth while to demonstrate the chief propositions of this method in as clear and compleat a manner as possible, if by this means we can preserve this science from disputes" (p. 102). We shall see that Maclaurin's book did not stop disputes. Had the book been read more, it might have been more effective in this respect. Our studies have led us to the conclusion that Maclaurin was not widely read. A second edition of his *Fluxions* did not appear until 1801. His work was praised highly, but seldom used and digested. We might say of Maclaurin what has been said of the German poet Klopstock :—

> "Wer wird nicht einen Klopstock loben?
> Doch wird ihn jeder lesen?—Nein.
> Wir wollen weniger erhoben,
> Und fleissiger gelesen seyn."

Remarks [1]

169. To what extent, if any, Maclaurin may have been influenced by Robins in the mode of treating

[1] In 1745 there appeared an anonymous publication on fluxions which we have not had the opportunity to examine; it was entitled, *The Harmony of the Ancient and Modern Geometry asserted.* In A. C. Fraser's edition of Berkeley's *Works*, vol. iii, Oxford, 1871, p. 301, it is referred to as follows: "This last and forgotten tract consists of papers given in to the Royal Society in 1742, and treats fluxions as a particular branch of an alleged more general reasoning, called the doctrine of maximinority and minimajority."

fluxions it is difficult to say. Later we shall see what James Wilson states on this point. Certain it is that Maclaurin's views agree much more closely with those of Robins than those of Jurin. Maclaurin stood for the narrower view of limits—limits none of which are reached by the variable. However, the conception of limits does not receive as great a degree of emphasis with Maclaurin as it does in the *Discourse* of Robins.

Of Maclaurin's *Fluxions*, Professor Kelland has remarked: "The Analyst did good service to science, if in no other way, at least by giving occasion to this last work. The principles of the method had been previously exhibited in a concise and obscure manner; Maclaurin developed them after the manner of ancient geometers."

In 1749, Maclaurin's *Treatise of Fluxions* was translated into French by Esprit Pezenas, director of the observatory at Avignon.

As we look back, we see that the eight years immediately following Berkeley's *Analyst* were eight great years, during which Jurin, and especially Robins and Maclaurin, made wonderful progress in the banishment of infinitely small quantities and the development of the concept of a limit. Both before and after that eight-year period there were published books in Great Britain containing a mixture of Continental and British conceptions of the new calculus, a superposition of British symbols and phraseology upon the older Continental concepts.

CHAPTER VII

TEXT-BOOKS OF THE MIDDLE OF THE CENTURY

John Stewart, 1745

170. John Stewart, professor of mathematics at Marischal College in Aberdeen, is known as the translator into English, with commentaries, of Newton's *Quadrature of Curves* and *Analysis by Equations of an Infinite Number of Terms*.[1]

The translator spares no pains in the endeavour to remove any obscurities which the ordinary reader might encounter therein. Newton's Quadrature of Curves takes up 33 pages in John Stewart's volume; Stewart's explanations thereof fill 287 pages. Referring to the controversy between Berkeley and Jurin, Stewart says that "because the Doctrine of prime and ultimate Ratios has been so much controverted of late, I shall here enquire whether we have any distinct Idea thereof." He quotes Newton's *Lemma* 1 in Book I, Section 1 of the *Principia*, also the proof of it, and then argues that the limit *is reached*, for "a Difference less than

[1] *Sir Isaac Newton's Two Treatises of the Quadrature of Curves, and Analysis by Equations of an Infinite Number of Terms, explained.* By John Stewart. London, 1745.

any Thing assignable, is the same Thing as no Difference at all: for repeat it as often as you please, it can never be equal to any finite Quantity: and therefore can bear no Ratio to it, by Def. 4, Bk. 5 [of Euclid's] Elements" (p. 37). Stewart gives definitions of ultimate ratio of quantities and of evanescent quantities, also definitions of prime ratio of quantities and of nascent quantities. The following is a specimen: "The ultimate Ratio of evanescent Quantities is the Limit to which the Ratio of variable Quantities diminishing without Bound, continually approaches, to come nearer to it than by any given Difference; but which never goes beyond; yet no sooner attains to, than the Quantities being diminished infinitely, vanish." The following additional statement follows closely the language of Newton (p. 39): "If any one should object that there can be no ultimate Ratio of continually diminishing and at last evanescent Quantities: because before they vanish it is not the last; and after they vanish, they have no Ratio. The Answer is, that the ultimate Ratio is neither the Ratio of them before they vanish; nor after they vanish; but the Ratio wherewith they vanish, or the Limit to which their varying Ratio no sooner arrives, than they vanish; . . . that Ratio they have that very Instant they vanish. . . . It signifies nothing to say ultimate Quantities cannot be assigned, in regard Quantity is divisible without End: for it is not the Quantities themselves that are hereby determined, but only their Ratio:

which is capable of being determined." This matter, says Stewart, has been so clearly explained by Newton, "that the great Dust which has been raised of late about the Whole of this Doctrine, must be owing to Weakness, or some worse Principle" (p. 40).

William Emerson, 1743 (?), 1757, 1768

171. William Emerson was a self-taught mathematician; he wrote many mathematical texts which indicate a good grasp of existing knowledge, but not great originality. His *Doctrine of Fluxions* appeared at London in 1743 (?). We have before us the third edition, 1768. From it we quote as follows:

"The Velocity of the Increase of any generated Quantity, or the Degree of Quickness (or Slowness) wherewith the new Parts of it continually arise, is called its *Fluxion*."

"The indefinitely small Portions of the Fluent which are generated in any indefinitely small Portions of Time are called *Moments* or *Increments*."

". . . The Moments and Fluxions ought not to be confounded together, since the Moments (being generated by Fluxions) are as different from the Fluxions, as any Effect is different from its Cause."

The following is given as an axiom:

"Quantities, which in any finite Time continually converge to Equality, and *before* the End of that Time, approach nearer to one another than by any given Difference, do at last become equal."

"If any should think this not clear enough to

pass for an Axiom, he may consider it thus; let D be their ultimate Difference, therefore they cannot approach nearer to equality, than by that given Difference D, contrary to the Hypothesis; which Supposition is absurd in all Cases except when D is nothing."

To find the fluxion of $bx'''y'''$, he lets $o\dot{x}$, $o\dot{y}$ be moments, expands the powers of $x+o\dot{x}$ and $y+o\dot{y}$, and finds the increments. Then he divides "by the indefinite Quantity o." "But since the (Velocity or) Fluxion is required wherewith that Moment first arises, in this Case the Moments $o\dot{x}$ and $o\dot{y}$ will also be just arising and therefore nothing, and consequently o will be nothing, and therefore all the Terms wherein it is found will be nothing." The final result then follows. In his Preface Emerson claims that "Velocity must be looked upon as the proper efficient Cause of the Space described; and the Space described the adequate Effect of that Cause." . . . "No increment can be taken so small, but it is still further divisible ad infinitum; and since the Velocity is by Supposition continually variable, it is plain, there can be no two Points of the Increment in both of which the Velocity is accurately the same. It is therefore most manifest, that the Velocity here enquired after is peculiar to one only indivisible Point; . . . that the Velocity in any given Point of the Line described . . . has a certain, fixed, determinate Value. . . . Here a metaphysical Disputant may demand, how it comes to pass, that

any Velocity which continues for no Time at all, can possibly describe any Space at all; or whether its Effect be absolutely nothing, or an infinitely small Quantity, or what it is. Here then it is, that our Reason is at a Stand, and the human Faculties are quite confounded, lost, and bewildered. . . . Now whether such subtile Questions will be ever determined, or not, yet there is one Refuge for us, viz. that it is nothing at all to our Purpose what they are: . . . The Method of Fluxions has no Dependence on these mysterious Disquisitions. What I apprehend the Method of Fluxions to be concerned in, is . . . what a . . . variable Velocity can produce in the whole. And here I think no Reason can be assigned, why a variable Cause should not produce a variable Effect, . . . though we have no Ideas at all of the perpetually arising Increments, or their Magnitude in their nascent or evanescent State, that have so much, and to so little Purpose, confounded and puzzled the mathematical World."

Thomas Simpson, 1750

172. Simpson's *Treatise of Fluxions* of 1737 has already been noticed (our § 156). His text of 1750, *The Doctrine and Application of Fluxions*, London, is new, not only in the title, but to some extent also in the mode of exposition. He says in his Preface (1750) that he has used a tract entitled *An Explanation of Fluxions in a Short Essay on the Theory*, printed by W. Innys and written by one of

his friends who was too modest to put his name to it. (See our §§ 158, 160, 163.) Simpson used his friend's manner of determining the fluxion of a rectangle and of illustrating fluxions of higher order. Simpson defines a fluxion as follows:

"The Magnitude by which any Flowing Quantity would be uniformly increased, in a given Portion of Time, with the generating Celerity at any proposed Position, or Instant (was it from thence to continue invariable), is the Fluxion of the said Quantity at that Position, or Instant."

The derivation of the fluxion of xy is explained after the manner adopted by John Rowe, both authors being indebted for it to the author of *An Explanation of Fluxions in a Short Essay on the Theory.* The same definitions and explanations of the fundamentals are given by Thomas Simpson in the last part of his *Select Exercises for Young Proficients in the Mathematicks*, 1752. In the preface to his *Fluxions* of 1750, Simpson touches some points of philosophic interest. He says:

"By taking Fluxions as *meer Velocities*, the Imagination is confin'd, as it were, to a Point, and without proper Care insensibly involv'd in metaphysical Difficulties: But according to our Method of conceiving and explaining the Matter, less Caution in the Learner is necessary, and the higher Orders of Fluxions are render'd much more easy and intelligible—Besides, tho' Sir Isaac Newton defines Fluxions to be *the Velocities of Motions*, yet he hath Recourse to the Increments, or

Moments, generated in equal Particles of Time, in order to determine those Velocities ; which he afterwards teaches us to expound by finite Magnitudes of other Kinds : Without which (as is already hinted above) we could have but very obscure Ideas of higher Orders of Fluxions : For if Motion in (or at) a Point be so difficult to conceive, that *Some* have, even, gone so far as to dispute the very Existence of Motion, how much more perplexing must it be to form a Conception, not only, of the Velocity of a Motion, but also infinite Changes and Affections of It, in one and the same Point, where all the Orders of Fluxions are to be considered.

" Seeing the Notion of a Fluxion, according to our Manner of defining It, supposes an Uniform Motion, it may, perhaps, seem a Matter of Difficulty, at first View, how the Fluxions of Quantities, generated by Means of accelerated and retarded Motions, can be rightly assigned ; since not any, the least, Time can be taken during which the generating Celerity continues the same : Here, indeed, we cannot express the Fluxion by any Increment or Space, *actually* generated in a given Time (as in uniform Motion). But, then, we can easily determine, what the contemporary Increment, or generated Space *would be*, if the Acceleration, or Retardation, was to cease at the proposed Position in which the Fluxion is to be found : Whence the true Fluxion, itself, will be obtained, without the Assistance of infinitely small Quantities, or any metaphysical Considerations."

Nicholas Saunderson, 1756

173. At the age of twelve months Saunderson lost his eyesight by small-pox; nevertheless, he rose to prominence. He studied at Christ's College, Cambridge, and in 1711 succeeded Whiston as Lucasian professor of mathematics at Cambridge. His *Fluxions*[1] is a posthumous work.

We read (p. 1): "Let AB represent any Moment of Time, whether finite or infinitely small it matters not, terminated by the two Instants A and B. Let x be the Value of any flowing or growing Quantity at any Instant A, whose Velocity at that Instant is such, that if it was to flow during the whole Moment AB with this Velocity, it would gain a certain Increment represented by $\dot{x}$; then is this Quantity $\dot{x}$ called the Fluxion of x at the Instant A, when the Value of the flowing Quantity was x." In the scholium which follows, it is explained that if the velocity is variable, then the increment of the velocity "gained in the time AB will not be the same with its Fluxion above defined, . . . but if the Time AB be infinitely small, then though the Velocity of x at the Instant B be not the same, mathematically speaking, with the Velocity at the Instant A; yet the Difference being infinitely small in Respect of the whole Velocity, it may safely be neglected, where the finite Ratios of Fluxions are only considered; and so this Increment

[1] *The Method of Fluxions Applied to a select Number of Useful Problems. . . .* By Nicholas Saunderson, Late Professor of Mathematics in the University of Cambridge. London, MDCCLVI.

and the Fluxion above defined may be taken for one another, i.e. the Quantity x for so small a Time, may be looked upon as flowing uniformly" (p. 2). Later we read (p. 4) that if the times are infinitely small, the quantity $\dot{v}\dot{x}$ will be "infinitely less" than $v\dot{x}$ or $x\dot{v}$. Here the fluxions $\dot{x}$, $v\dot{x}$, are looked upon as infinitely small.

In the account of the life of Nicholas Saunderson, printed in the first volume of his *Elements of Algebra*, Cambridge, 1740, p. xv, we read: "Our Professor would not be induced by the Desires and Expectations of any, to engage in the war that was lately waged among Mathematicians, with no small Degree of Heat, concerning the *Algorithm or Principles of Fluxions*. Yet he wanted not the greatest Respect for the Memory of Sir Isaac Newton, and thought the whole Doctrine entirely defensible by the strictest Rules of geometry. He owned indeed that the great Inventor, never expecting to have it canvassed with so much trifling Subtility and Cavil, had not thought it necessary to be guarded every where by Expressions so cautious as he might have otherwise used."

John Rowning, 1756

174. A graduate of Magdalene College, Cambridge, and a Fellow there, Rowning interested himself chiefly in natural philosophy, but wrote also *A Preliminary Discourse*[1] on fluxions, with the intention of writing

[1] *A Preliminary Discourse to an intended Treatise on the Fluxionary Method.* By John Rowning, M.A. London, 1756.

a full treatise. But the treatise in question never appeared. After a popular exposition of the ideas of fluxion and fluent, and of Leibniz's infinitely little quantities and their summation, showing how these methods yield important results in natural philosophy, he refers to Berkeley's attacks and the defence made by Philalethes Cantabrigiensis, Walton, and Robins, also Maclaurin, who "declined entering the Combat," but endeavoured to treat the subject "in a Manner less exceptionable." "But no Body, that I know of," continues Rowning, "has explained it in so easy and familiar a Way as I apprehend the Subject capable of." Moreover, Jurin and Walton "carry things . . . no farther than Sir Isaac had done before. They leave them, as to the Objections made by the *Analyst*, exactly as they found them." The difficulties do not lie in the idea of a first fluxion—a velocity. "In this there is Nothing either infinitely great or infinitely little: Nothing obscure." As to higher fluxions, "these Things indeed elude our Senses; but they do not surpass the Understanding" (p. 85). Berkeley's objection to "infinitely small Quantities" is not fatal, "because finite Measures might have been made use of." His other objection, that "such Quantities are in some Cases retained and made use of for a while, and afterwards, to use his own Expression, like Scaffolds to a Building, are rejected as of no Significancy," may be met by the proof that those quantities "are always such as ought by no means to be retained." In further explanation of his

position Rowning says (p. 88), "that the Velocity of any Body is the same at any one Point, or at any one Time, whether the Body moves with an uniform, accelerated, or with a retarded Motion at that Point or Time." This is elucidated by reference to geometric figures, and amounts, in the main, to the explanation given by Rowe in finding the fluxion of xy. One objection to such explanations, which had been raised by Berkeley, was that one could not speak of the velocity a body had *at a point* of space. That such a phraseology is admissible is tacitly assumed by Rowning. What the latter emphasises is that no use is made of the concept of the "infinitely little." As to Berkeley's second objection, that the supposition which is made at the beginning of the process is later displaced by its contrary, as when the symbol o is at first made an actual increment and later in the same process taken as no increment, Rowning argues that terms involving factors oo, ooo, etc., "do arise in consequence of the Acceleration wherewith the Power of x flows, when x itself flows uniformly; and consequently that they arise from the second and higher Fluxions of that Power; and that, therefore, when the first Fluxion of that power is only inquired after . . . they are to be left out and rejected, as appertaining to another Account." It can hardly be claimed that Rowning made a contribution to the theory of fluxions. However, he has a pleasant way of expressing himself. His book was favourably reviewed in the *Monthly Review* (vol. xiv, p. 286).

Israel Lyons, 1758

175. Lyons was a mathematician and botanist. His *Treatise of Fluxions*, London, 1758, is dedicated to Robert Smith, Master of Trinity College, Cambridge, "being the first Essay of a young and unpractised Writer" which "owed its first rude Beginning to the early Encouragement" received from the Master, as the author modestly states. His treatment is geometric. He says: "I reject no Quantities as infinitely smaller than the rest, nor suppose different Orders of Infinitesimals and infinitely great Quantities. But consider the Ratio of the Fluxions as the same as that of the contemporaneous Increments, and take Part of the Increment before and Part after the Fluent is arrived at the Term, where we want the Fluxion, since it is not the Increment after, or the Increment before that we want, but at the very instant, which can no otherwise be found but by considering Part of the Increment before and Part after" (Preface). Fluxions are defined as velocities. "The moments of quantities are the indefinitely small parts, by the addition or subtraction of which, in equal particles of time, they are continually increased or diminished." The author proves the proposition: "The indefinitely small spaces described in equal indefinitely small times are as the velocities," since, "when the time is diminished *ad infinitum*, the difference of the velocities at the beginning and ending of that time will vanish." If two flowing quantities

x and y are to each other in a given ratio, then in $xy=z$ it is argued that $2y=$ incr. of $z \div$ incr. of $x=\dot{z}\div\dot{x}$; hence $\dot{z}=2y\dot{x}$. When $y=x$, this becomes $\dot{z}=2x\dot{x}$; one has also, fluxion $\overline{x+y}^2 = 2\overline{x+v}\times\overline{\dot{x}+\dot{y}}$ $=2\times\overline{x\dot{x}+y\dot{x}+x\dot{y}+y\dot{y}}$.

From this is derived the fluxion of any rectangle xy, thus: The fluxion of $\overline{xy}^2$, or $x^2+2xy+y^2$, is also equal to $2x\dot{x}+2y\dot{y}+$ fluxion of $2xy$. Hence fluxion of $2xy=2\dot{x}y+2x\dot{y}$.

"In the same manner as the quantities x, y, z, are conceived to flow, and to have their fluxions, so may the quantities $\dot{x}$, $\dot{y}$, $\dot{z}$, be supposed to be variable, and therefore have their fluxions, which are thus represented $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, and are called the second fluxions of x, y, z" (p. 11). "The fluent of any quantity as $x'''\dot{x}$ is represented thus $|\overline{x'''\dot{x}}|$."

William West, 1762

176. William West's *Mathematics*[1] is a posthumous work; the author died in 1760. Fluxions are treated from the earlier Newtonian standpoint, infinitely little quantities being used. Some novelty is claimed for this text in the treatment of maxima and minima.

James Wilson, 1761

177. In 1761 Wilson collected some of Benjamin Robins's mathematical tracts in a two-volume book,

[1] *Mathematics.* By the late Rev. Mr. Wm. West of Exeter. Revised by John Rowe, London, 1762. There appeared a second, corrected, edition in 1763.

entitled *Mathematical Tracts of the late Benjamin Robins*. In an Appendix, Wilson inserts some matters of historical interest regarding certain manuscripts of Newton; Wilson also defends Robins against criticisms passed by a French writer, and states his views of Maclaurin's indebtedness to Robins.

Newton's *Method of Fluxions* (see our § 149) was brought out in Paris in 1740 by George Louis Le Clerk, Comte de Buffon, under the title, *La méthode des Fluxions, et des suites infinies*. Buffon prepared a historical Preface, in which he criticised severely Berkeley and Robins for presuming to take exception to anything Newton had written on fluxions or to modify Newton's mode of exposition. Buffon praises Jurin, and then speaks of Robins thus (pp. xxvii–xxix):

". . . il commence par le censurer & par désaprouver sa maniere trop bréve de présenter les choses; ensuite il donne des explications de sa façon, & ne craint pas de substituer ses notions incomplettes aux Démonstrations de ce grand homme. Il avouë que la Géometrie de l'Infini est une science certaine, fondée sur des principes d'une vérité sûre, mais enveloppée, & qui *selon lui n'a jamais été bien connuë*; Newton n'a pas bien lû les Anciens Géometres, son Lemme de la Méthode des Fluxions est obscur & mal exprimé . . .: malheursement les Mathematiciens ont été plus incrédules que jamais, il n'y a pas eu moyen de leur faire croire un seul mot de tout cela, de sorte que

Philalethes comme défenseur de la vérité, s'est chargé de lui signifier qu'on n'en croyoit rien, qu'on entendoit fort bien Newton sans Robin, que les pensées & les expressions de ce grand Philosophes sont justes & très-claires . . ., ce sont des piéces d'une mauvaise critique. . . ."

Buffon presents no argument against the views expressed by Robins, but abuses him for presuming to think independently. This doting attitude toward Newton is justly attacked by James Wilson, in his Appendix to the *Mathematical Tracts of the late Benjamin Robins*, vol. ii, London, 1761, pp. 325–327. Wilson rightly says that if it was a crime for Robins to make mention of the great brevity with which Sir Isaac Newton wrote, Robins was followed in it by Maclaurin and Saunderson. "The truth is," says Wilson, "Sir Isaac Newton at first made the same use of indivisibles, others had done: in his *Analysis per æquationes numero terminorum infinitas*, he expressly says, 'Nec vereor loqui de unitate in punctis, sive lineis infinite parvis[1];' and in his *Lectiones Opticæ* he demonstrated by indivisibles." Wilson contends furthermore that Buffon is wrong in claiming that the mathematicians paid no regard to what Robins had said, that in fact "the best writers soon after trod in Mr. Robins's steps." In fairness to Buffon it should be said, however, that he printed his Preface in 1740, and that Maclaurin, Saunderson, de Bougainville, and d'Alembert, whom

[1] *Comm. Epist*, p. 85.

Wilson mentions as following Robins, wrote at a later date.

178. James Wilson claims[1] that Maclaurin in his *Fluxions* "conformed himself entirely to Mr. Robins's sentiments in regard to Sir Isaac Newton's doctrine," and "has even expressly followed his plan in treating the subject." Jurin had contended (says Wilson) "that Sir Isaac Newton's method, by proving the varying quantities came up to their limits, was more perfect than that of the ancients. Whereas Sir Isaac Newton never claimed such superiority; . . . The coincidence contended for, and thus highly praised by Philalethes, is the very essence of indivisibles." Wilson rightly insists that Buffon's criticisms of Robins are unfair. "When he talks of the obscurity of Mr. Robins's ideas, the insignificancy of his phrases, and the unintelligibleness of his style; he gives the most certain proof, that he had never carefully read his writings, . . . for Mr. Robins is much admired here for the contrary excellencies, on whatever subjects he has employed his pen."

179. Wilson represents Philalethes (Jurin) as championing the use of the infinitely little and of indivisibles. This is putting the case too strongly. In his papers against Berkeley, Jurin uses quantities infinitely little. But toward the end of his debate with Robins he begins to disavow them. Never did Jurin use indivisibles. Few eighteenth-century

[1] *Mathematical Tracts of the late Benjamin Robins*, vol. ii, London, 1761, pp. 312, 315, 320.

writers have brought out as distinctly and clearly as has Jurin the difference between infinitesimals as variables, and indivisibles; Jurin disavowed all quantity "fixed, determinate, invariable, indivisible, less than any finite quantity whatsoever," but he usually did admit somewhat hazily a quantity "variable, divisible, that, by a constant diminution, is conceived to become less than any finite quantity whatever, and at last to vanish into nothing."

Remarks

180. None of the works mentioned in this chapter are great works. Those of William Emerson and Thomas Simpson were the best and the ones most widely used. The first edition of Simpson is of earlier date (1737).

CHAPTER VIII

ROBERT HEATH AND FRIENDS OF EMERSON IN CONTROVERSY WITH JOHN TURNER AND FRIENDS OF SIMPSON

181. The principals, Simpson and Emerson, do not themselves appear in this controversy. During the period of this debate, Robert Heath was editor of *The Ladies' Diary*, which appeared once every year as an almanac. We begin with one of his articles.

Robert Heath, 1746

182. In an article, *Of the Idea, and Nature of Fluxions*,[1] Heath says:

"The Distinction betwixt the *Increments* and Fluxions of Magnitudes, has been this; that the *former* approach in Ratio infinitely near the latter, so that their Difference is unassignable. . . . What we call the *Fluxions*, or *Velocities* of Magnitudes, are only the Fluxions in Chief, or in Part, with which they are born; the Part neglected in the Ratio exactly corresponding with what is rejected in the finite Ratio of the infinitely small Increments, which is therefore the same as the Ratio of our

[1] *The Ladies' Diary: or, the Woman's Almanack, for* 1746.

Fluxions. And hence, whether we call those finite Ratios, Fluxions, or Increments, their Idea, Nature, and Original appear to be the very same thing. For all Things are relative. . . ."

He argues that while we consider a line or plane, generating an area or solid, as of no thickness in the mind, in our notation we represent them as of unit thickness, "and consequently each Line or Plane should be express'd by $o \times L$, and $o \times P$, to denote them as they are in the Mind. But $L \times o$ to o, and $P \times o$ to o, are in the same Ratio with L to 1, and P to 1, by equal Division by o; and those again in the same *Ratio* with $L\dot{x}$ to $\dot{x}$, and $P\dot{x}$ to $\dot{x}$, by equal Multiplication by $\dot{x}$, for the Ratio of Fluxions. But, this finite Notation of *Line* or *Plane*, which we consider of no *Breadth*, or *Thickness*, and yet denote by Unity, each, at the same Time, makes the *Practice* and our *Comprehension* disagree. . . . So that it will be an Error to conclude that the Ratio of the Fluxions of Quantities generated by the Motion of *Lines*, or *Planes*, is arrived at this Way, without the previous Consideration of an Increment; for the very *Lines* and *Planes* must be *Increments*, or *Somethings* next to *Nothings* themselves, before they were what we finitely express them by Notation, or Quantities could never increase or be generated thereby: For to carry a *Line* or *Plane* of no Breadth or Thickness forward, is the same in Terms as to carry Nothing forward. And therefore the Distinction between the Ratio of Increments, and that of Fluxions, is only what the Conception of the

Thing differs from that of its Notation in Practice. . . . Those who desire further Satisfaction as to the Nature of Fluxions, of their noble Use and transcendant Excellence, may consult Mr. Emerson's *Doctrine* of the *whole Art*, which is . . . the best of any. . . . Those writers will find themselves mistaken, who pretended to derive the finite Ratios of Motion, or Fluxions producing Magnitudes, without the previous Consideration of Increments, which include the very Notion of what a Fluxion is. This *some* have attempted by multiplying Quantities into their Velocity, and some by other Means, the Result of which originally depends on *incremental Principles*, if they would consider the Matter as far as it will go." The paper is brought to a finish in the *Ladies' Diary* for 1747.

Main Articles in the Controversy

183. Over the pseudonym of "Cantabrigiensis" there appeared in 1750 an unfriendly review of Simpson's *Doctrine and Application of Fluxions*.[1] The reviewer contended that the definition of a fluxion as the "magnitude by which any flowing quantity would be uniformly increased" (see our § 172) is very "odd"; for, "in quantities uniformly generated, the fluxion must be the fluent itself, or else a part of it." Simpson's endeavour to exclude "velocity" "cannot be made intelligible without introducing velocity into it." "Again, he

[1] *Monthly Review; or, New Literary Journal*, vol. iv, London, 1750, pp. 129–131.

mistakes the effect for the cause; for the thing generated must owe its existence to something, and this can only be the velocity of its motion; but it can never be the cause of itself, as his definition would erroneously suggest." Moreover, it is strange that Simpson "should still stick in the mud and run himself into the old exploded method used by foreigners; and which is subject to all the cavils that have ever been raised against that science."

184. This criticism originated a small tempest. In a journal called *Mathematical Exercises*,[1] its editor, John Turner, makes certain "Observations on certain invidious Aspersions on Mr. Simpson's *Doctrine and Application of Fluxions*, published in the *Monthly Review* for December last, by Cantabrigiensis." Mr. Simpson is there charged as having "mistaken the Effect for the Cause"; Mr. Simpson, says Turner, "builds upon his own Definition;

[1] *Mathematical Exercises* No. III (1751), p. 34. Six numbers of this journal appeared in London in 1750–1752. No. V bears the date 1752; No. VI has no date. Readers are invited "to send their Performances (whether new Problems, Paradoxes, Solutions, etc.) Post paid, to be left with Mr. James Morgan, at the Three-Cranes, in Thames-street . . ." In this connection a statement made by Charles Hutton, in his Memoirs of the Life and Writings of the Author [Thomas Simpson], printed in Thomas Simpson's *Select Exercises in the Mathematics*, new edition, London, 1792, p. xviii, is of interest:

"It has also been commonly supposed that he [Thomas Simpson] was the real editor of, or had a principal share in, two other periodical works of a miscellaneous mathematical nature; viz. the *Mathematician*, and Turner's *Mathematical Exercises*, two volumes, in 8vo, which came out in periodical numbers, in the years 1750 and 1751, etc. The latter of these seems especially to have been set on foot to afford a proper place for exposing the errors and absurdities of Mr. Robert Heath, the then conductor of the *Ladies' Diary* and the *Palladium*; and which controversy between them ended in the disgrace of Mr. Heath, and expulsion from his office of editor to the *Ladies' Diary*, and the substitution of Mr. Simpson in his stead, in the year 1753."

which, he tells us, himself, is not exactly the same as that of Sir Isaac Newton." Mr. Simpson is also charged with plagiarism from Cotes's *Estimatio Errorum.* John Turner says:

"Here his Remarks on the Author's Definition of a Fluxion first demand our Consideration: Mr. Simpson makes it to be, 'the Magnitude by which a flowing Quantity *would be* uniformly increased in a given Time.' This Definition the Critic represents as a very old one; and with regard thereto advances the two following, extraordinary, Positions:

"1. That, in Quantities uniformly generated, the Fluxion must (according to the said Definition) be the Fluent itself, or else a Part of it.

"2. And that, in other Quantities generated by a variable Law, the Fluxion will not be a real, but an imaginary Thing.

"To the first of these Objections I answer, that the Fluxion is neither the Fluent itself nor a Part of it: it is a Quantity of the same Kind with the Fluent; but the Fluent being the Quantity already produced by the generating Point, Line or Surface, supposed still in Motion, and the Fluxion what will arise, hereafter, from the Continuation of that Motion; the latter can no more be denominated a Part of the former than the ensuing Hour a Part of the Time past.

"But his second Observation is a still more glaring Instance of his Disingenuity, and Want of Judgment. Does it follow, because a Body, really, moves over a certain Distance, in a given Time,

with an accelerated, or a retarded Velocity, that there is no Distance over which it might pass in the same Time, with its first Velocity uniformly continued. The Space over which a Body would uniformly move with such, or such, a proposed Velocity, is no less real because no Part of it is actually described with that Velocity" (pp. 36, 37).

185. Then follows an article reprinted from the *Daily Gazetteer* for December 4, last [1750], in which one who signs himself "Honestus" (said to have been John Turner himself) charges that the compiler of the *Ladies' Diary* (Robert Heath) is also the compiler of the *Palladium*, and the best material designed by contributors for the *Diary* are reserved by him for the *Palladium*; that the latter publication is owned by the compiler, while the former is not. Robert Heath wrote a reply in the *Daily Gazetteer* of December 6; four letters follow on this subject.

186. John Turner's defence of Simpson led to the publication of what Turner called a "scurrilous Pamphlet." This pamphlet is without doubt the *Truth Triumphant: or Fluxions for the Ladies*,[1] London, 1752, or else those parts in that pamphlet which appear over the pseudonyms "Ẋ Primus"

[1] The fuller title of the pamphlet is thus: *Truth Triumphant: or, Fluxions for the Ladies. Shewing the Cause to be before the Effect, and different from it; That Space is not Speed, nor Magnitude Motion. With a Philosophic Vision, Most humbly dedicated to his Illustrious High and Serene Excellence, the Sun. For the Information of the Public, by Ẋ, Ẏ, and Ż, who are not of the Family of x, y, z, but near Relations of x', y', and z'.* . . . London. Printed for W. Owen, M.DCC.LII.

and "Ẏ Secundus." These documents evidently emanated from the pen of Robert Heath, assisted possibly by some other adherents of William Emerson. At the risk, perhaps, of not observing strict chronological sequence, we proceed to the consideration of all parts of *Truth Triumphant.* In the dedication "to the Sun," it is stated that "the Family of the Wou'd-be's in this Island is become very numerous, by *uniformly* continuing in their Errors." Thus, both the title-page and the dedication play on Simpson's definition of fluxions and its alleged defects. In the Preface one reads: "Fluxions, then, Ladies, that have so puzzled our wise Mathematicians to define, are the respective Degrees of Motion, at any Instant of Time, of any two things or Bodies that continually flow, or move on, over Space." Four pages are devoted to the explanation of fluxions.

187. Then follow the two criticisms of John Turner's defence of Simpson, signed "Ẋ Primus" and "Ẏ Secundus," to which we have alluded above. In the former of these articles John Turner is treated with contempt. "Who this John Turner is, whether he is Mr. Simpson's Clerk, or his Pupil, or some Dependant on him; or whether he be Mr. Simpson himself, is not very material to the Reader . . ." Turner is continually referred to as "John." To Turner's reply to the first criticism on Simpson's text, "Ẋ Primus" makes rejoinder: "*John* says, the Fluent being the Quantity already produced—Pray how was this Quantity produced,

by some *magic* Art, without any Fluxion? I believe not. . . . For my Part, I know of no Body that ever said, that the Parts of the Fluent that went before were generated by the Fluxion that is to come after, but every Part by its proper Fluxion. . . ." To Turner's reply to the second criticism, "Ẋ Primus" makes rejoinder:

"If there be no Magnitude by which the flowing Quantity is really increased, such a Magnitude is not real, but an imaginary Thing only . . . But *John* thinks, that every Thing that exists in his *Imagination*, really exists in *Nature* . . . Sir Isaac Newton defines Fluxions by the Velocities of the Motions. But Mr. Simpson declares against this, and likewise tells us, that by taking Fluxions for mere Velocities, the Imagination is confin'd, as it were, to a Point. How *his* Imagination is confin'd I don't know; but Sir Isaac Newton chused to define it thus, as very well knowing, that this is the only *solid* Foundation upon which it could be defended against all the impertinent Cavils of ignorant or weak Pretenders." The parting shot by "Ẋ Primus" is—your Great Master will not "think you a fit *Champion* to engage in his Cause for the future; so, good Night, John."

188. The reply made by "Ẏ Secundus" is to the effect that the defender of Simpson is "equally in the Dark" with Simpson himself, "otherwise he would not have gone about to defend so defenceless a Cause, as to vindicate an Absurdity, by representing a Fluxion *to be of the same Kind with the*

Fluent, uniformly generated; when the one is a Quantity of *actual Velocity*, and the other a Quantity of *Space*, described by that Velocity, which can be only proportional to it."

189. After some poetry "To the Family of the the Wou'd-be's," follow "Animadversions on Mr. Simpson's Fluxions," "By Ż Tertius," who quotes a criticism of Simpson from the pen of J. Landen. Where Landen's review first appeared we do not know. As quoted here, Landen objects to the definition of fluxions "as *faulty*, by the Author's different *Idea* given of them to that by the Inventor"; Landen disapproves of "denoting all Quantities whatsoever by Lines, to bring them to one Denomination, and those Lines, to be described by Bodies in Motion." In criticism of fluxions in general, Landen says that the finding, from the velocities, the spaces passed over, and *vice versa*, "may be managed by common *Algebra*, without the least Obscurity. The Business had always been better considered in that Light, without ever making Use of the Term *Fluxions*, as if a new Kind of *Analysis*, tho', in *Fact*, only the *Doctrine of Motion* improved, and applied to Purposes before unthought of."

190. The next article in *Truth Triumphant* is a reprint of the first criticism of Simpson, contributed in 1750 by Cantabrigiensis to the *Monthly Review*. Eight more articles concerning motion, fluxions, and mechanics bring the pamphlet to a close; they make no reference to Simpson. "Heliocentricus"

explains higher fluxions in a way that cannot be called illuminating.

191. Then "Amicus" speaks "of the Use of the Algebraic Cypher, in finding the Fluxions of Algebraic Quantities," letting x increase or decrease, and become $x \pm o$, where the "Increments or Decrements are seen to be $\pm o$," and "dividing by algebraic o," thereupon "algebraically considering o of insensible Value, as before it was consider'd of real sensible Value." Taking a reminiscent mental attitude, "Amicus" says:

"This *algebraic Ratio* of the Fluxions of Quantities, to which the diminishing Value of the *algebraic* Increments or Decrements, from their limited State or Value, tend together, to their geometrical vanishing (by *supposing the variable Value less and less*) has been misconceived, as vanishing together with the *real* geometrical Increments or Decrements they are the *Value of*; whence o has been denominated a *departed*, instead of an *algebraic* Quantity, by a famous $B - p$, tho' it's Reality and Presence still existed before his Eyes; but if o, the Cypher-Value, or *algebraic* Quantity, call'd also *Nothing*, be made to signify *Nothing*, because it is so call'd, the Word *Nothing* with as much Propriety, may be called *no Word*, be allowed to have no Signification among other Words, and be deem'd a mere Blank, as *no* Subject capable of Consideration." Further on in the pamphlet, the query whether there can be "*real* Motion in *no* Time," for "any *one* Point of intermediate Space gone over; especially since an infinite

Number of Points can never actually constitute real Magnitude," and whether "Motion, or Fluxion, can *actually* exist, and be known, but by the next Increment of Space gone over, in some real and next Moment of Time"? These are fundamental problems indeed. Zeno is not mentioned in the pamphlet, but the query involves Zeno's subtle paradox of the "arrow." Nor is the answer given devoid of interest. "But Time, and Motion, flowing over Space, . . . (*since no Quantity can be assign'd, or imagin'd so small, but there will still be smaller*) the respective Degree of Velocity of Motion, or Fluxion (i.e. instantaneous velocities) of that Flowing at any Instant of Time, and Point of Space gone over, will be everywhere assignable by the immediate Increments, as Effects of those preceding Velocities, as has been shewn. Whence it will follow, that certain Degrees of Motion, Fluxion, or Velocity, exist at every instant of Time taken, and Point of Space respectively described; contrary to the *differential* Notion that Foreigners have of this Matter." The weak spot here resides in the words "immediate Increments"; do immediate increments exist in view of the statement in the above parenthesis? The lack of a satisfactory arithmetical continuum comes to view more fully in the antagonism between geometric increases and algebraic increases exhibited in the following passage taken from the pamphlet:

"All the Values of the geometrical Increases flow'd over, in finite Time, can never be algebraically

express'd in infinite Time; in which Sense the *algebraic Increases* being again diminish'd, are said never to converge to the Limits of their geometrical Magnitudes in Motion, but will still have sensible Value; yet supposing the geometric Increases, and their algebraic Values to flow and decrease alike, to something determinate, then o, and o, and it's Powers into a Variable Quantity, and it's Powers, will accurately express the Limits of variable Quantities, or Beginnings of their Increases; which *Limits*, or *Beginnings of Increases* of Quantities, are accurately, as the Fluxions of those Quantities in general."

"Visionarius" closes with a philosophic vision in which four candidates for honors appear before the Goddess of Science. Rejected are the first three, viz. the author of *Mathematical Exercises* (John Turner), the one holding to the motto "A cypher is no Algebraic Quantity" (Bishop Berkeley), and a Grand Magnifier of Fluxions (Thomas Simpson); crowned is the author of the incomparable treatises of *Fluxions* and *Trigonometry* (William Emerson), in whose interests *Truth Triumphant* appears to be mainly written.

192. The probability is that the "scurrilous pamphlet" to which John Turner made reply in an issue of his *Mathematical Exercises* was only a part of what is given in *Truth Triumphant.* The latter is probably a later and enlarged publication. In that reply Turner argues that "it must appear to everyone that, what Mr. Simpson defines as Fluxions,

are exactly such finite Quantities, proportional to the Velocities as Sir Isaac Newton here [1] speaks of; since it is well known that the Quantities produced, or the Distances described, in any given Time, by Motions uniformly continued, are, accurately, as the Velocities of the said Motions."

193. In No. V (1752) of the *Mathematical Exercises* appears another article in the controversy, written by John Turner. There is little in it requiring our attention. It is a reply to two pamphlets, the *Lady's Philosopher* and a new *Palladium*, both publications from the pen of Robert Heath.

Ladies' Diary, 1751, 1752

194. The *Ladies' Diary* for 1751 has an article on *The Nature and Use of the Algebraic Cypher, or Quantity* 0, "by Fluxioniensis"; 0 / 0 is proved to signify "any Value at Pleasure by considering $(a^n - x^n) \div (a - x)$ for $n = 1, 2, 3, 4$, etc., when $x = a$." This "confutes the Notion of some Mathematicians" that 0 / 0 expresses "a Ratio of Equality." Next it is argued that $0^0 = 1$. "Hence," says a second anonymous critic, "all Cypher-Paradoxes, and Mysteries of Ultimate Ratios, or Ratios of Least Increments or Decrements of Quantities, vanish and Day appears. . . ."

"Waltoniensis, making a Distinction between *o* signifying *some* Quantity, and *o* signifying *no* Quantity, or absolute Nothing, says that when x

[1] *Principia*, Bk. II, *Lemma* 2. See our §§ 16–19.

converges to o, in the *ultimate* State before it vanishes, $x^x = 1$; but says, when x entirely vanishes, or becomes absolutely of no Value, that then $x^x = o^0 = o$: But o being supposed no Quantity is contradicted by *Algebraic* Computation, which is general and retains o, in a Mathematical Sense, for a Quantity in the Scale, as much as any other Figure or Literal by which Quantity is denoted and compared. . . . Waltoniensis farther observes that *Fluxions* are the Limits to which the *Ratios* of the Increments or Decrements of Quantities converge, and are assignable from the Principles of Motion only (*uniform*, *accelerated*, and *retarded*); and thinks the Doctrine has nothing to do with infinitely small Quantities, First and Last *Ratios*; and that only finite Quantities need be introduced—'to avoid Disputes, and the dark *Mists* spread over the Process, different to the demonstrative Lights of the Antients.' But Motion refers to the Spaces passed over, by which it is comprehended, measured, and compared: And tho' Mr. *Simpson* has pretended to deduce the *Ratios* of Fluxions of Quantities without the use of *indefinitely small Quantities* (see his *New Doctrine and Application of Fluxions*) yet the Motion of his *Points* along the *Lines* answers to them by the indefinitely small Spaces described together, and are to the same Effect as Quantities taken indefinitely small; which Sir Isaac Newton himself introduced to illustrate the Quantity of relative Motion by. *Fluxions*, as instantaneous Velocities, are also as the Increases or Quantities of Space

passed over together by those *instantaneous Velocities*, uniformly continued; but are not the Spaces or Quantities themselves that *would be* described by them according to Mr. *Simpson's* new Theory, and Application. See *Emerson's* Doctrine and Application of Fluxions. Price 6s. only."

195. In the *Ladies' Diary* for 1752 the reader is amused by satirical remarks on mathematicians. There is also a continuation of the discussion "Of the Cypher-Value and Office of the Algebraic Quantity *o*." "Nihil Maximus says that ·9999, etc. *ad inf.* will never converge to 1, nor yet 1 / (10,000, etc. *ad inf.*) to *o*; because any Quantity infinitely increased or diminished will be still greater or less, and never numerically arrive at Infinity, and *o* Value. . . . That a Distinction should be carefully made between what are called *infinite* and *indefinite* great and small Quantities (*the former of which being impossible*); for what is of *indefinite* Value has Equality, tho' it may be sometimes unassignable; while what is *infinite* is *never* determinable, and has *never* Equality. Hence the numerical Value of *o* and 1 / (1000, etc.), will be for ever different; one being a Quantity of *no* sensible Value, but yet significant, and the other of indefinite small Value. . . . *Infinite* Quantity, or infinite numerical Value, expressed by Authors, is neither practicable nor comprehensible. . . . An infinite *Series* can never precisely converge to a finite or determinate Value; because it *for ever runs on*. The finite Value, taken for that of an *infinite*

Series, is only the Value from whence that *infinite* Series is or may be derived. Mr. Landen thinks that the Value *o* is no algebraic Quantity; but calls it a *mere Blank*, or absolute Nothing . . .; he says, that its peculiar Office is only in arithmetic Notation; while we see it applied to other Use and Office in Treatises of Algebra and Fluxions, as also by himself, for an algebraic Character or Quantity *o*, its own Value."

This discussion of *o* is continued at great length. Confusion arises from the double use of the symbol and from the difficulties surrounding *o* as the limit of variables or sequences. Reference is made twice in this *Ladies' Diary* (1752) to the pamphlet *Truth Triumphant, or Fluxions for the Ladies*, where the nature and office of *o* are discussed, and exception is taken to Landen's views on *o*. "Fluxioniensis" says: "And therefore I should not stick to rank this excellent Reasoner with the great Master of Reason he mentions, the B—p of Cloyne, as he clearly appears to be of the same class."

Popular Impression of the Nature of Fluxions

196. A reviewer of Richard Jack's *Euclid's Data Restored*[1] quotes from Jack's preface what appears to be the opinion of a non-specialist:

"Others, who claim the honour of extending their principles, treat of what they call *Fluxions*, *calculus differentialis*, *infiniment petit's*, *extreme and*

[1] *Monthly Review*, vol. xvi, London, 1757.

ultimate ratios, etc., and with so much obscurity, that no distinct idea of the thing treated, is communicated to the mind of the Reader. From their want of that precision and perspecuity which the Ancients carefully observed in all their writings, the mind becomes clouded with confusion, begins to doubt, which terminates in a disbelief of their principles; for which reason they have been often called upon to demonstrate them: but no demonstration has appeared."

To this the anonymous reviewer of Jack's book replies:

"That the principles of Fluxions stand in need of demonstration, especially since the publication of Maclaurin's works, is certainly a mere pretence, made only to cover the ignorance of the objector . . ."

Remarks

197. In this chapter we have given views held by writers representing the rank and file of mathematical workers. In several passages the need of an adequate theory of a linear continuum makes itself strongly felt.

Some curiosity attaches to the following contemporaneous opinion of *Truth Triumphant*:[1]

"This is an odd assemblage of controversial scraps, chiefly relating to some disputes concerning Mr Emerson's treatise on fluctions, and Mr. Simpson's on the same subject. This most unimportant

[1] *Monthly Review*, vol. v, London, 1751, p. 462.

controversy was first occasioned by the few observations on Mr Simpson's book published in the *Monthly Review*. . . . The author writes in a manner that can do little honour to any party or opinion. And why he chose to give this strange and insignificant production so odd a title, is a mystery that none but himself can clear up."

CHAPTER IX

ABORTIVE ATTEMPTS AT ARITHMETISATION

John Kirkby, 1748

198. In the preface to his *Doctrine of Ultimators* [1] the author states that his doctrine "depends upon scarce any Thing else but a due Application of the Cypher *o* to the analogous Office in Universal Arithmetic, which it is always known to occupy in Common Arithmetic." He argues that "the superlative impropriety of the Word Fluxion, when applied to this Purpose, will fully appear; when we come to consider, that it is put to express an Idea, which arises from the Contemplation of Quantities purely as Quantities: that is, in the same abstract Manner, as they are the proper Subject of Algebra, exclusive of every other Consideration; and con-

[1] *The Doctrine of Ultimators. Containing a new Acquisition to Mathematical Literature, naturally resulting from the Consideration of an Equation, as reducible from its variable to its ultimate State: Or, a Discovery of the true and genuine Foundation of what has hitherto mistakenly prevailed under the improper Names of Fluxions and the Differential Calculus. By means of which we now have that Apex of all Mathematical Science entirely rescued from the blind and ungeometrical Method of Deduction which it has hitherto laboured under; and made to depend upon Principles as strictly demonstrable, as the most self-evident Proposition in the first Elements of Geometry.* By the Reverend Mr. John Kirkby, Vicar of Waldershare in Kent. London, MDCCXLVIII. Pp. 144.

sequently have not the least Regard to Time or Motion, which are necessarily implied in a Fluxion. And the essential Property of a Fluxion is certainly excluded, after the most singular Manner, in the Idea of Quantity considered at its *Ne plus ultra*: that is, in other Terms, when it is in a State, where all Possibility of such imaginary Flux is taken from it. So that the Term Fluxion, when used to this Purpose, if it have any Meaning at all, is as contrary to the true, as Darkness is to Light." He takes an algebraic equation $Aa^0 \pm Ba' \pm Ca^2 \pm Da^3 \pm \ldots \pm Za^n = 0$, assumes the coefficients Y and Z of the two highest terms as fixed, and declares (without proof) that the absolute term A is a maximum when the n roots of the equation are equal. When such an equality exists, the equation is reduced "to its ultimate." When the roots are equal he represents them by $+c$ or $-c$. To reduce the trinomial $A \pm Ba \pm Za = 0$ to its ultimate, "we must make $B / Z = \pm nc^{n-1}$ in the n Power of $c \pm a = 0$. That is (because $c = a$) $B / Z = na^{n-1}$. Therefore the Ultimate required is $B \pm nZa^{n-1} = 0$, or $Ba^0 \pm nZa^{n-1}a^0 = 0$." To be observed here is that Kirkby connects, though only in an obscure way, his ultimate with the coefficient of a in the second term of the binomial expansion of $(c \pm a)^n$. He then pretends to prove "that the Ultimate of the Sum of never so many Equations is the same with the Sum of their respective Ultimates"; hence, the Ultimate of the above general equation is $o \pm Ba^0 \pm 2Caa^0 \pm 3Da^2a^0 \pm \ldots \pm nZa^{n-1}a^0 = 0$. He gives

the rule for finding the "ultimate" or "ultimator" of a^n; this ultimator is $na^{n-1}a^0$; he also gives the rule for writing down "the subject of every Ultimator"; the subject of the ultimator $na^{n-1}a^0$ being a^n+c. He applies these rules when the exponents are fractional.

The "ultimator" of the product of two variables, ae, is found thus. "Put $ae=bee$, and $ae=caa$. Whence $a=be$, and $e=ca$, and $ae=\frac{1}{2}\,bee+\frac{1}{2}\,caa$. The Ultimator of which last is . . . bee^0+caa^0, and consequently is equal to the Ultimator of ae. But $be=a$, and $ca=e$. Therefore these substituted for their Equals in that Ultimator give ae^0+ea^0 for the Ultimator of ae" (p. 43). It will, of course, be noticed that special limitations are placed upon the variables a and e, when the coefficients b and c are tacitly assumed to be constants. Kirkby proceeds to the derivation of the ultimators of fractions and logarithms. He explains the necessity of retaining in the Ultimator each variant (variable) under its o Power. "Without this we cou'd have no Means from the Nature of the thing itself, whereby to distinguish an Ultimator from a Subject." The functions of a^0, e^0 are more than simply to represent unity; just what they are is not very clear, although to the author "it is evident then, as often as any Subject consists of different Variants Ex gr. x, y, z, that the Expressions x^0, y^0, z^0, in the Ultimator have the same Difference in Power with the same Variants under any other common Exponent x^n, y^n, z^n. . . . Therefore the Expression x^0, y^0, z^0,

I conceive may be each fitly called the *Peculiar Unit* of its respective Scale of Powers. Hence every *Ultimator* may be defined to be, *The proper Reference of each Subject in a given Equation to the Peculiar Units of the Powers of all its Variants, in Order to discover the Ratios of those Variants to one another in their Ultimate State.* Which I take to be the true Definition of what has been hitherto most improperly and unintelligibly called a *Fluxion* by some, and a *Differential* by others" (page 49).

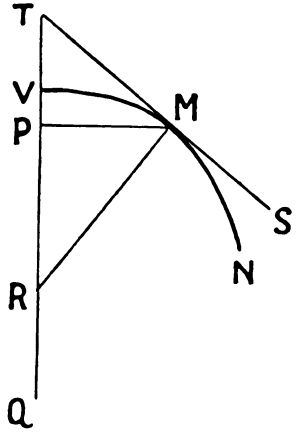


FIG. 11.

199. Kirkby's doctrine may perhaps become plainer by the study of one of his applications. In any curve with the concave side to TQ, the greater abscissa VP (or v) has always the greater "semi-ordinate" PM (or s), "and each are the greatest that they possibly can be to the same Arch VM, or to the same intercepted axis VR. Therefore the Subnormal PR (or $r-v$), and consequently the Normal MR ($=c$) are each the least that they possibly can be to the same Arch VM, or the same intercepted Axis VR (or r). Therefore, if in the last Equation [$r^2-c^2=2rv-v^2-s^2$], c and r be invariable, we have r^2-c^2 an *Ultimum.* Consequently, the Ultimate of that Equation . . . is $2rv^\circ-2vv^\circ-2ss^\circ=0$, or (dividing by 2) $ss^\circ=\overline{r-v}\times v^\circ$. Whence $v^\circ : s^\circ = s : r-v$. That is in all Curves, as

the Ultimator of the Abscissa is to the Ultimator of the Semiordinate; so is the Semiordinate itself to the Subnormal" (p. 51).

The author has occasion to use second and third ultimators and to consider ultimators as variable or invariable. He lets (p. 60) $\dot{x}^\circ$ be the invariable of the first ultimator x°, $\dot{\dot{x}}$ the invariable of the second ultimator $\dot{\dot{x}}$, etc., and warns the reader that his dot does not mean a fluxion. In the more involved applications to curves he lets an infinitely small arch equal $x^\circ = \dot{x}^\circ$. Our impression of the book is that the author's intentions were good when he attempted an arithmetisation. But there is a total lack of clear and rigorous exposition.

200. The *Ladies' Diary*, London, 1750, p. 45, contains a hostile criticism of the *Doctrine of Ultimators* by an anonymous writer (probably the editor, Robert Heath), in which the author of this doctrine is said to declare that fluxions, as explained by Newton, are "absurd and unintelligible," and to place confidence in "the Authority of a certain Irish B—p, a Mathematician as wise as himself. For you must know that this pious B—p (the sagacious Author of the *Analyst*, as he stiles him) out of his religious Zeal against Mathematical Learning, had been engaged in the same senseless Attempt with himself, of degrading the noblest Science. . . . Having thus, as he [the author of "Ultimators"] thinks, overturn'd the Doctrine of Fluxions . . . he has given us instead of it . . . a new Science of

his own, whose Foundation, it seems, depends on *Cyphers*, and Nought Powers full of conceited Expressions. . . . He expresses his Ultimators by the Help of x^0, y^0, z^0, etc., calls them peculiar Units, and of different Values, all of which is absurd. . . . I pass over . . . his using x^0, y^0, z^0, for the same End as others use $\dot{x}$, $\dot{y}$, $\dot{z}$."

John Petvin, 1750

201. In a *Sketch of Universal Arithmetic*,[1] brought out as a posthumous booklet, we encounter a curiosity. Its philosophy of mathematics and of fluxions in particular is set off by the following quotations:

(Page 156) "I do not then consider it [mathematical quantity] as generated or produced, but as that which is. Time and Motion produce nothing of the Kind, and have no Place here. Nor do I consider it as continuous, nor as consisting of very small or infinitely little Parts, but as consisting of Parts in general. These Parts therefore I consider as discrete: And by x, y, z, etc., I understand Multitude. The Ones or Monads, of which x is many, I call x; . . . Nor do I consider x, y, z, etc., barely as many; but as a certain many. So that x, y, z, etc., are Wholes; $\dot{x}$, $\dot{y}$, $\dot{z}$, etc., their respective Parts. These Parts may be considered again as Wholes, consisting of another Order of

[1] *Letters concerning Mind. To which is added, a Sketch of Universal Arithmetic; comprehending the Differential Calculus, and the Doctrine of Fluxions.* By the late Reverend Mr. John Petvin, A.M., Vicar of Ilsington in Devon, London, 1750.

Parts," designated as $\dot{x}$, etc. "Such Things as an Instant, a Point, a Fluxion, she [arithmetic] has nothing to do with. . . . I have joined Fluxion with Point and Instant, because Fluxion seems to be to Motion, as an Instant is to Time; which I suppose to be as a Point is to a Line. Motion cannot be conceived without Time and Space; and when the former runs into an Instant, and the latter into a Point, then it is (as I understand it) that Motion becomes Fluxion. . . . In this Sense Fluxion is no more a Part of Motion than a Point is a Part of a Line." His "parts" are finite increments. The part of xy is $\dot{x}y+\dot{y}x+\dot{x}\dot{y}$. "This Doctrine of Wholes and Parts proceeds upwards from Parts to Wholes, as well as downwards from Wholes to Parts universally" (p. 159). "The Ordinate therefore being x''', when $x'''\dot{x}$ expresses the Fluxion of it, the only Meaning I have for $\dot{x}$ is, that it is the Proportion of a Point to an Instant. And to my Apprehension, a Point may as well be called a last Line, as this called a Velocity." "I have lately deduced some arithmetical Theorems from arithmetical Principles, which other Mathematicians have drawn from Fluxions of Fluxions, etc., and these Theorems fell in with my Design." Just how these deductions were made is not explained by the author.

John Landen, 1758

202. John Landen was a self-educated mathematician of real mathematical power. Had he had the benefits of University training he might have

occupied a much higher rank as a mathematician. Foreigners place him high among his English contemporaries. He wrote *Mathematical Lucubrations*, 1755, and *Residual Analysis*, 1764. We shall consider only his *Discourse concerning Residual Analysis*,[1] 1758. From it we quote as follows:

"Yet, notwithstanding the method of fluxions is so greatly applauded, I am induced to think, it is not the most natural method. . . . The operations therein being chiefly performed with algebraic quantities, it is, in fact, a branch of the algebraic art, or an improvement thereof, made by the help of some peculiar principles borrowed from the doctrine of motion. . . . We may indeed very naturally conceive a line to be generated by motion; but there are quantities . . . which we cannot conceive to be so generated. It is only in a figurative sense, that an algebraic quantity can be said to increase or decrease with some velocity. Fluxions therefore are not immediately applicable to algebraic quantities. . . . It therefore, to me, seems more proper, in the investigation of propositions by algebra, to proceed upon the *anciently-received* principles of that art. . . . That the borrowing principles from the doctrine of motion, with a view to improve the analytic art, was done, not only without any necessity, but even without any peculiar advantage, will appear by showing, that whatever can be done by the method of computa-

[1] *A Discourse Concerning Residual Analysis: A new Branch of the Algebraic Art.* By John Landen. London, 1758.

tion, which is founded on those borrowed principles, may be done as well, by another method founded entirely on the *anciently-received* principles of algebra. . . . It is by means of the following theorem [p. 5], viz.

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x - v} = x^{\frac{m}{n}-1} \times \frac{1 + \frac{v}{x} + \left(\frac{v}{x}\right)^2 + \left(\frac{v}{x}\right)^3 \qquad (m)}{1 + \left(\frac{v}{x}\right)^{\frac{m}{n}} + \left(\frac{v}{x}\right)^{\frac{2m}{n}} + \left(\frac{v}{x}\right)^{\frac{3m}{n}} \qquad (n)},$$

(where m and n are any integers) that we are enabled to perform all the principal operations in our said Analysis."

His Residual Analysis is a method involving vanishing fractions and therefore not free from controversial questions. That the fluxion of x^3 is $3x^2$ is explained according to the Residual Analysis by the consideration that $(y^3 - x^3) \div (y - x) = x^2 + xy + y^2$, which is equal to $3x^2$ when $y = x$. We proceed to give an application in Landen's own words:

203. (Page 5) "Fluxionists, in determining the limit of the ratio of the increments of x and $x^{\frac{m}{n}}$, commonly have recourse to the binomial theorem (which is much more difficult to investigate than the limit they are seeking): But how easily may that limit be found, without the help of that theorem, by the equation exhibited in page 5! Thus, the increment of x being denoted by x', the increment of $x^{\frac{m}{n}}$ is $\overline{x + x'}^{\frac{m}{n}} - x^{\frac{m}{n}}$, and the ratio of those increments is

$$\frac{\overline{x+x'}^{\frac{m}{n}}-x^{\frac{m}{n}}}{x'}=\frac{\overline{x+x'}^{\frac{m}{n}}-x^{\frac{m}{n}}}{\overline{x+x'}-x}=\overline{x+x'}^{\frac{m}{n}-1}$$

$$\times\frac{1+\frac{x}{x+x'}+\overline{\frac{x}{x+x'}}\Big|^{2}+\overline{\frac{x}{x+x'}}\Big|^{3}}{1+\frac{x}{x+x'}\Big|^{\frac{m}{n}}+\frac{x}{x+x'}\Big|^{\frac{2m}{n}}+\frac{x}{x+x'}\Big|^{\frac{3m}{n}}},\qquad \begin{matrix}(m)\\(n)\end{matrix}$$

which, when x' vanishes, is manifestly equal to $\frac{m}{n}x^{\frac{m}{n}-1}$, the limit of the said ratio."

The explanation of the method of drawing tangents is too long for quotation, and we shall limit ourselves to the following outline of it, as given by Landen:

"I consider the curve as already described, without any regard to its generation, and find the value of a certain line (terminated by the curve and its tangent), in algebraic terms involving (s) the subtangent with other quantities; which algebraic expression I observe, from an obvious property of the line it is found to denote, must have a certain property with respect to being positive or negative in certain cases. I therefore assume that expression equal to another which is known to have that very property; and from thence, by means of the theorem mentioned in page 5, readily find the required value of s" (p. 10).

204. Landen's *Discourse* was attacked by an anonymous writer in the *Monthly Review* for June 1759, who claims that the Residual Analysis "is no

other than Sir Isaac Newton's method of differences; and it is well known, that if the differences are diminished so as to vanish, their vanishing ratio becomes that of fluxions" (p. 560), "that his pretended Residual Analysis renders the investigations more tedious and obscure than any other." Landen wrote a reply in the July number, from which we quote only the part relating to the word "function." Says Landen: "He objects to *prime number, function*, etc., as terms never heard before.—Alas! how egregiously does he betray his ignorance!"

James Glenie, 1793

205. James Glenie graduated from the University of St. Andrews, and became a military engineer. He was a prominent Fellow of the Royal Society of London. In his *Antecedental Calculus*,[1] 1793, he begins with the statement, "Having, in a Paper, read before the Royal Society, the 6th of March, 1777, and published in the Philosophical Transactions of that Year, promised to deliver, without any consideration of Motion or Velocity, a Geometrical Method of Reasoning applicable to every purpose, to which the much celebrated Doctrine of Fluxions of the illustrious Newton has been or can

[1] *The Antecedental Calculus, or a Geometrical Method of Reasoning, without any Consideration of Motion or Velocity applicable to every Purpose, to which Fluxions have been or can be applied.* By James Glenie, Esq., M.A. and F.R.S. London, 1793. According to G. Vivanti (see M. Cantor's *Vorlesungen über Geschichte der Mathematik*, vol. iv, Leipzig, 1908, p. 667), James Glenie (1750–1817) was an artillery officer in the war of the American Revolution, later professor of mathematics in the military school of the East India Company.

be, applied; and having taken notice of the same Method, in a small Performance, written in Latin, and printed the 16th of July, 1776, I now proceed to fulfil my promise with as much conciseness as perspicuity and precision will admit of." In his *Antecedental Calculus*, p. 10, he says of Newton: 'I am perfectly satisfied, that had this great Man, discovered the possibility of investigating a general Geometrical Method of reasoning, without introducing the ideas of Motion and Time, . . . he would have greatly preferred it, since Time and Motion have no natural or inseparable connection with pure Mathematics. The fluxionary and differential Caculi are only branches of general arithmetical proportion."

Glenie speaks (p. 3) of "the excess of the magnitude, which has to B a ratio having to the ratio of $A+N$ to B the ratio of R to Q (when R has to Q any given ratio whatever), above the magnitude, which has to B a ratio having to the ratio of A to B the same ratio of R to Q, is geometrically expressed by" a complicated fraction whose denominator is $B^{(R-Q)/Q}$, and whose numerator is the result of expanding by the binomial theorem $(A+N)^{R/Q}$ and then subtracting $A^{R/Q}$ therefrom.

A similar expression is given for the case in which $A-N$ takes the place of $A+N$: "The excess of the magnitude, which has to B a ratio, having to the ratio of A to B the ratio of R to Q, above the magnitude, which has to B a ratio, having to the ratio of $A-N$ to B the ratio of R to Q, is geometrically

expressed by" a fraction whose denominator is $B^{(R-Q)/Q}$, and whose numerator is obtained by expanding and simplifying $A^{R/Q}-(A-N)^{R/Q}$. "But if $A+N$ and $A-N$ stand to B in relations nearer to that of equality than by any given or assigned magnitude of the same Kind, these general expressions become $R/Q \cdot A^{(R-Q)/Q} \cdot N \div B^{(R-Q)/Q}$. This I call the antecedental of the magnitude which has to B such a ratio as has to the ratio of A to B the ratio of R to Q. Now if N the antecedental of A be denoted by $\dot{A}$ or $\overset{a}{A}$ . . . [and] if $Q=1$ and $R=2, 3, 4, 5$, etc., this expression gives $\frac{2A\dot{A}}{B}$, $\frac{3A^2\dot{A}}{B^2}$ . . . respectively." For the "antecedent" $A\frac{A}{B}$ he finds the "antecedental" $\frac{2A\dot{A}}{B}$ or $2\dot{A} + \frac{2M\dot{A}}{B}$ (putting M for $A-B$). Glenie shows that at a point of a curve the antecedentals of the abscissa, ordinate and curve, are as the sub-tangent, the ordinate and the tangent, respectively.

Glenie's calculus involves extremely complicated identities of ratios and examines the antecedents of ratios having given consequents. The style of exposition is poor. In deriving the antecedentals, Glenie quietly drops out all the terms in the numerator that involve powers of N higher than the first power. As this calculus plays no part in the later history of fluxions, we shall give only one more quotation; it relates to the Binomial Theorem

(not used by him in the development of his fundamental formulas). He says (p. 11): "It may not perhaps be improper to add, that, if to the expressions delivered above for the excess of the magnitude, which has to B a ratio, having to the ratio of A + N to B, the ratio of R to Q, above the magnitude, which has to B a ratio, having to the Ratio of A to B the same ratio of R to Q; and for the excess of the magnitude, which has to B a ratio, having to the ratio of A to B the ratio of R to Q, above the magnitude, which has to B a ratio, having to the ratio of A − N to B the ratio of R to Q, be prefixed the magnitude, which has to B a ratio, having to the ratio of A to B the ratio of R to Q, we get a geometrical Binomial, of which, when it is supposed to become numerical, the famous Binomial Theorem of Sir Isaac Newton is only a particular case."

Remarks

206. The classic treatment of fluxions in Great Britain, during the eighteenth century, rests primarily on geometrical and mechanical conceptions. Attempts to found the calculus upon more purely arithmetical and algebraical processes are described in this chapter. All these attempts are either a complete failure or so complicated as to be prohibitive. Easily the ablest among these authors was John Landen. De Morgan says of his Analysis[1]: "It is the limit of D'Alembert supposed to be *attained*,

[1] *Penny Cyclopædia*, Art. "Differential Calculus."

instead of being a *terminus* which can be attained as near as we please. A little difference of algebraical suppositions makes a fallacious difference of form: and though the residual analysis draws less upon the disputable part of algebra than the method of Lagrange, the sole reason of this is that the former does not go so far into the subject as the latter."

In the same article De Morgan speaks of Kirkby's Ultimators thus:

"A something between Landen and D'Alembert, as to principle, published in 1748, was called the 'Doctrine of Ultimators, containing a new Acquisition, etc., or a Discovery of the true and genuine Foundation of what has hitherto mistakenly prevailed under the improper names of Fluxions and the Differential Calculus.'"

CHAPTER X

LATER BOOKS AND ARTICLES ON FLUXIONS

Encyclopædia Britannica, 1771, 1779, 1797

207. The article "Fluxions" in the first edition of the *Encyclopædia Britannica*, Edinburgh, 1771, gives this definition: "The fluxion of any magnitude at any point is the increment that it would receive in any given time, supposing it to increase uniformly from that point; and as the measure will be the same, whatever the time be, we are at liberty to suppose it less than any assigned time." The fluxion of a rectangle is the increment, with the small rectangle at the corner omitted; the latter "is owing to the additional velocity wherewith the parallelogram flows during that time and therefore is no part of the measure of the fluxion." "The increment a quantity receives by flowing for any given time, contains measures of all the different orders of fluxions; for if it increases uniformly, the whole increment is the first fluxion; and it has no second fluxion. If it increases with a motion uniformly accelerated, the part of the increment occasioned by the first motion measures the first fluxion, and the

part occasioned by the acceleration measures the second fluxion. . . . "

The same article is reprinted in the second edition (1779) and the third edition (1797).

Robert Thorp, 1777

208. Thorp made a translation of part of Newton's *Principia*.[1]

In the "advertisement" we read: "The doctrine of prime and ultimate ratios . . . is established, so as to remove the various objections which have been raised against it, since it was first published. To the relations of finite quantities alone the reasoning on this subject is confined." The translation of *quantitates quam minimæ, evanescentes, ultimæ, infinité magnæ*, and the like, has not been literal, yet they are "explained in that sense under which the author cautions his readers to understand them. This is the more necessary, as the terms *infinite, infinitesimal, least possible*, and the like, have been grossly misapplied and abused."

209. In the Commentary to Lemma 1 in Sect. 1 of Bk. I in the *Principia*, Thorp says: "The prime and ultimate ratios of magnitudes . . . are investigated by observing their finite increments or decrements, and thence finding the limits of the ratios of those *variable* magnitudes; not the ratios to which the magnitudes ever actually arrive (for

[1] *Mathematical Principles of Natural Philosophy. By Sir Isaac Newton, Knight. Translated into English, and illustrated with a Commentary*, by Robert Thorp, M.A., vol. i, London, 1777.

they are never, strictly speaking, either prime or ultimate in fact), but those limits to which the ratios of magnitudes perpetually approach; which they can never reach, nor pass beyond; but to which they appear nearer than by any assignable difference." . . . "We now proceed to explain this Lemma more particularly than perhaps might seem necessary, had it not been much controverted, misrepresented, and misunderstood." As one of the conditions of the proposition, Thorp states, is "that quantities and the ratios of quantities must *continually* tend to equality. The one must never become equal to, nor pass beyond the other: their difference must never either vanish to nothing, or become negative." In this restriction Thorp goes even further than had Robins. The following passage from Thorp's commentary is thoroughly in the spirit of Robins: ". . . That we may not be led, from the expression *ultimately* equal, to suppose, that there is an *ultimate* state, in which they are actually equal, we are cautioned in the scholium at the end of this Section [of *Principia*, Bk. I, Sect. 1] in these words, *The ultimate ratios, in which quantities vanish, are not in reality the ratios of ultimate quantities; but the limits to which the ratios of quantities continually decreasing always approach; which they never can pass beyond, nor arrive at, unless the quantities are continually and indefinitely diminished.* According to Thorp, the inscribed or circumscribed polygon can never arrive at the curve. He quotes from Saunderson's Fluxions. By the

doctrine of indivisibles there "has been introduced into mathematical reasoning all that absurd jargon concerning quantities infinitely great, and infinitely little, which has been so much objected to by mathematicians. And, though it has often been elegantly applied by some able geometers to the demonstration of many noble theorems; yet in the hands of less accurate reasoners, it has often led to false conclusions" (p. 71).

F. Holliday, 1777

210. In a somewhat lengthy preface to his *Introduction to Fluxions*[1] the author tells that, when in 1745 he was in London, in company with W. Jones and De Moivre, they expressed great approbation of Emerson's *Fluxions*, with regard to the method of treatment, but thought his book too high for beginners. The author tries to be more diffuse in the laying down of first principles. He derives the fundamental results in two ways: first, by the aid of nascent or evanescent quantities, as suggested by Newton's *Principia*; second, "without using any infinitely small quantities, or vanescent Parallelograms, which perhaps will be more acceptable to many of my Readers." Holliday explains at great length the Scholium (see our §§ 10–15) on prime and ultimate ratios, and gives a short account of the invention of fluxions as given in the review of

[1] *An Introduction to Fluxions, Designed for the Use, and Adapted to the Capacities of Beginners.* By the Reverend F. Holliday, Vicar of West Markham and Bothamsall, Nott's. London, 1777.

Collins's *Commercium Epistolicum* in the *Philosophical Transactions*, 1717. Though following Newton closely, variations were bound to arise. Thus, Holliday says (p. 73), "Fluxions are not magnitudes but the *velocities* with which magnitudes, varying by a continual motion, increase or decrease." It cannot be claimed that Holliday made any contribution to the philosophy of fluxions, nor even that he profited as much as he might by the refinements in the logic which had been made by English writers since the time of Newton.

Charles Hutton, 1796, 1798

211. In his *Mathematical Dictionary*, London, 1796, Charles Hutton makes reference to the advantage of Simpson's definition of a fluxion as a magnitude uniformly generated in a finite time, the imagination being now no longer confined to a single point and to the velocity at that point; moreover, "higher orders of Fluxions are rendered much more easy and intelligible."

212. From the part on fluxions in Hutton's *Course of Mathematics*[1] we take the following:

"The rate or proportion according to which any flowing quantity increases, at any position or instant, is the Fluxion of the said quantity, at that position or instant: and it is proportional to the magnitude by which the flowing quantity would be uniformly increased, in a given time with the

[1] *A Course of Mathematics.* By Charles Hutton. London, 4th ed., 1803–1804, vol. ii, p. 279. [First ed., 1798–1801.]

generating celerity uniformly continued during that time."

". . . If the motion of increase be accelerated, the increment so generated, in a given finite time, will exeeed the fluxion: . . . But if the time be indefinitely small, so that the motion be considered as uniform for that instant; then these nascent increments will always be proportional, or equal, to the fluxions, and may be substituted instead of them in any calculation."

The fluxion of xy is derived in two ways: the first by the method of considering the rectangle composed of two parts, as previously expounded by Rowe.

The second method finds algebraically the increment $xy'+yx'+x'y'$, "of which the last term $x'y'$ is nothing, or indefinitely small, in respect of the other two terms, because x' and y' are indefinitely small in respect of x and y. . . . Hence, by substituting $\dot{x}$ and $\dot{y}$ for x' and y', to which they are proportional, there arises $x\dot{y}+y\dot{x}$ for the true value of the fluxion of xy."

S. Vince, 1795, 1805

213. Vince's *Principles of Fluxions* appeared in 1795 as the second volume of the *Principles of Mathematics and Natural Philosophy in Four Volumes*,[1] which were brought out under the general editorship of James Wood. A second

[1] *The Principles of Mathematics and Natural Philosophy in Four Volumes. Vol. II, The Principles of Fluxions: Designed for the Use of Students in the University.* By the Rev. S. Vince, A.M., F.R.S., Cambridge, 1795.

edition of Vince was printed in 1805. From this second edition we quote:

P. 1: "The velocities with which flowing quantities increase or decrease at any point of time, are called the *fluxions* of those quantities at that instant.

"As the velocities are in proportion to the increments or decrements *uniformly* generated in a given time, such increments or decrements will represent the fluxions."[1]

Vince also quotes Newton on the generation of quantities by motion: "Sir I. Newton, in the Introduction to his *Quadrature of Curves*, observes that 'these geneses really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner, the ancients, by drawing moveable right lines along immoveable right lines, taught the geneses of rectangles.'"

Vince gives no formal definition of a *limit*; but his philosophy of this subject is disclosed by the two following quotations (pp. 4 and 5): "By keeping the ratio of the vanishing quantities thus expressed by finite quantities, it removes the obscurity which may arise when we consider the quantities themselves; this is agreeable to the reasoning of Sir I. Newton in his *Principia*, Lib. I, Sect. 1, Lem. 7, 8, 9."

"It has been said, that when the increments are

[1] "This is agreeable to Sir I. Newton's ideas on the subject. He says: 'I sought a method of determining quantities from the velocities of the motions or increments with which they are generated; and calling these velocities of the motions or increments, *fluxions*, and the generated quantities *fluents*, I fell by degrees upon the method of fluxions.'—Introd. to *Quad. Curves*."

actually vanished, it is absurd to talk of any ratio between them. It is true; but we speak not here of any ratio then existing between the quantities, but of that ratio to which they have approached as their *limit*; and that ratio still remains. Thus, let the increments of two quantities be denoted by ax^2+mx and bx^2+nx; then the *limit* of their ratio, when $x=0$, is $m:n$; for in every state of these quantities, $ax^2+mx : bx^2+nx :: ax+m : bx+n ::$ (when $x=0$) $m:n$. As the quantities therefore approach to nothing, the ratio approaches to that of $m:n$ as it's limit. We must therefore be careful to distinguish between the ratio of two evanescent quantities, and the *limit* of their ratio; the former ratio never arriving at the latter, as the quantities vanish at the instant that such a circumstance is about to take place."

By aid of the binomial theorem, Vince finds the fluxion of x^n, when the fluxion of x is given; he then finds the fluxion of xy by considering $(x+y)^2=x^2+2xy+y^2$, by which the fluxion of $2xy$ can be found in terms of the fluxions $(x+y)^2$, x^2 and y^2.

Agnesi—Colson—Hellins, 1801

214. The *Analytical Institutions*[1] is the first calculus that was written by a woman. The authoress

[1] *Analytical Institutions, in four books: Originally written in Italian, by Donna Maria Gætana Agnesi, Professor of the Mathematicks and Philosophy in the University of Bologna. Translated into English by the late Rev. John Colson, M.A., F.R.S., and Lucasian Professor of the Mathematicks in the University of Cambridge. Now first printed, from the Translators Manuscript, under the inspection of the Rev. John Hellins, B.D., F.R.S.* Vols. i and ii. London, 1801.

is the noted Maria Gætana Agnesi, of the University of Bologna.

The Italian original was first published at Milan in 1748. The two volumes of the translation were printed at the expense of Baron Maseres. In an introduction, Hellins points out that Colson hoped to interest the ladies of England in the study of fluxions by his translation of the work of the great Italian lady, "And, in order to render it more easy and useful to the Ladies of this country, . . . he [Colson] had designed and begun a popular account of this work, under the title of *The Plan of the Lady's System of Analyticks*; explaining, article by article, what was contained in it. But this he did not live long enough to finish."

215. Colson dealt with Agnesi's work somewhat as Stone had dealt with that of De L'Hospital, inasmuch as both translators substituted the notation of Newton in place of that of Leibniz. The word fluxions ("flussioni") occurs in the original Italian of Agnesi's masterly work. How Colson's conscience may have troubled him, when a fluxion stood out in his translation as something "infinitely little," may be judged when we consider that in 1736 he brought out an English translation, with an extensive comment, of Newton's *Method of Fluxions*. With Newton a fluxion always meant a *velocity*.

We quote a few passages from Colson's Agnesi (vol. ii, pp. 1, 2):

"The Analysis of infinitely small Quantities,

which is otherwise called the *Differential Calculus*, or the *Method of Fluxions*, is that which is conversant about the differences of variable quantities, of whatever order those differences may be."

"Any infinitely little portion of a variable quantity is called it's *Difference* or *Fluxion*; when it is so small, as that it has to the variable itself a less proportion than any that can be assigned; and by which the same variable being either increased or diminished, it may still be conceived the same as at first."

On p. 3 we read that certain lines in a figure "will be quantities less than any that can be given, and therefore will be *inassignable*, or *differentials*, or *infinitesimals*, or, finally, *fluxions*. Thus, by the common Geometry alone, we are assured that not only these infinitely little quantities, but infinite others of inferior orders, really enter the composition of geometrical extension."

"These propositions," says a reviewer[1] of the translation, "may appear exceptionable, in point of language, to the *rigorists* in geometry; but they are nevertheless founded on good principles, and furnish rules for the comparison of evanescent quantities, which will prove safe guides in investigation. The demonstrations appear to us to be perfectly sound (if the word *infinite* be taken in its true sense, as denoting merely the absence of any limit), with the exception, perhaps, of the first theorem, which (as is not a little curious to remark) is liable to the

[1] *Edinburgh Review*, vol. iii, 1805, p. 405.

same objection that has been made of the lemma of Newton's Principia. In both instances, also, the error is rather apparent than real." The first theorem in question states that the two intersecting perpendiculars to a curve drawn at the ends of "an infinitely little portion of it of the first order," "may be assumed as equal to each other." We wonder what Robins and Maclaurin would have thought, had they been alive in 1801 and 1805, and read these definitions and comments! What horrible visions would these ghosts of departed quantities have brought to Bishop Berkeley, had he been alive! But the nineteenth century was destined to bring back to British soil still greater accentuations of infinitesimals.

T. Newton, 1805

216. The Rev. T. Newton says in the preface of his *Illustrations of Sir Isaac Newton's Method*:[1]

"Every Mathematician now considers the whole doctrine of Prime and Ultimate Ratios in no other light, than as a Doctrine of Limits." Young readers of Sir Isaac Newton's *Principia* encounter difficulties because commentators have made "use of the terms of Indivisibles, in their explanations; . . . Newton expressly says, that by the ultimate ratios of quantities he means the ratios of their limits.[2] And when he wants to infer the

[1] *An Illustration of Sir Isaac Newton's Method of Reasoning. By Prime and Ultimate Ratios.* By the Rev. T. Newton, Rector of Tewin, Herts; late Fellow of Jesus College, Cambridge. Leeds, 1805.

[2] See our §§ 12, 15.

equality of inequality of those limits from some relation of the variable quantities, which are never supposed absolutely to reach their limits, it certainly requires something more than a definition to shew this. . . . It is not my intention to detain the reader, with answering the objections of the *Analyst* and his followers, because it has been already done by others in a satisfactory manner. . . . Notwithstanding the assertions of some modern writers, the method of ultimate ratios is extremely perspicuous, strictly logical, and more concise than any other of modern invention; . . . it neither involves the strange notion, that a straight line may be a part of a curve, and a plane superficies a part of a concave or convex one; nor the unintelligible idea of adding and subtracting indivisibles, or inconceivably small magnitudes. Whatever magnitudes are compared, according to this method, they are always supposed to be finite."

T. Newton begins with the following two definitions (p. 1):

"If a variable quantity, either increasing or decreasing, approaches to a fixed quantity, the difference between them being continually diminished, so as at length to become less than any assignable quantity; the fixed quantity is called the Limit of the variable quantity."

"If the ratio of two variable quantities continually approaches to a fixed ratio, so as to come nearer to it than by any assignable difference; the

fixed ratio is called the Limiting Ratio of the variable quantities."

William Dealtry, 1810, 1816

217. In the preface of Dealtry's *Principles of Fluxions*[1] (1816) we read:

"The method of Fluxions rests upon a principle purely analytical; namely, the theory of limiting ratios; and the subject may therefore be considered as one of pure mathematics, without any regard to ideas of time and velocity. But the usual manner of treating it is to employ considerations resulting from the theory of motion. This was the plan of Sir Isaac Newton in first delivering the principles of the method; and it is adopted in the following Work, from the belief, that it is well adapted for illustration."

Dealtry defines a "fluxion of a quantity at any point of time" as "its increment or decrement, taken proportional to the velocity with which the quantity flows at that time." . . .

"When a quantity increases with a velocity which continually varies, the quantity, which measures the fluxion, is a limit between the preceding and succeeding increments, and is ultimately equal to either of them." He explains that "the word ultimately is intended to denote that particular instant, when the time is diminished *sine limite*,"

[1] *The Principles of Fluxions: Designed for the Use of Students in the Universities.* By William Dealtry, B.D., F.R.S., late Fellow of Trinity College, Cambridge. 2nd ed., Cambridge, 1816.

and quotes Newton's Scholium, Sect. 1, in the *Principia*. He points out, also, that if x increases uniformly, x^2 increases with accelerated velocity, and the part of the increment x'^2 is the effect of the acceleration, and therefore, by his definition of fluxion, to be "omitted in taking the fluxions" (p. 8).

New Editions, 1801–1809

218. William Davis, who was a bookseller in London and editor of the *Companion to the Gentleman's Diary*, appears also as the editor of new editions of three different texts on fluxions. In 1801 he saw through the press the second edition of Maclaurin's *Treatise of Fluxions*; in 1805 the third edition of Thomas Simpson's *Doctrine and Application of Fluxions*. In 1809 appeared the fourth edition of John Rowe's *Doctrine of Fluxions*, revised "by the late William Davis."

Remarks

219. Among some of the authors of this period there is less concern than among writers of former years about the attainment of the rigour of the ancients. Perhaps the effects of the revival of the ideals of Euclid and Archimedes which followed the publication of the *Analyst* were gradually subsiding. It would not be fair to this age to judge its mathematical status altogether by the authors which we have selected. There was a movement under way

at this time which is reflected in the literature that will be under consideration in the next chapter.

Both before the time of Berkeley's *Analyst* and after the time of Maclaurin's *Fluxions* there appeared in Great Britain texts which superposed British symbols and phraseology upon the older Continental concepts. The result was a system, destitute of scientific interest. Newton's notation was poor and Leibniz's philosophy of the calculus was poor. That result represents the temporary survival of the least fit of both systems. The more recent international course of events has been in a diametrically opposite direction, namely, not to superpose Newtonian symbols and phraseology upon Leibnizian concepts, but, on the contrary, to superpose the Leibnizian notation and phraseology upon the limit-concept, as developed by Newton, Jurin, Robins, Maclaurin, D'Alembert, and later writers.

CHAPTER XI

CRITICISMS OF FLUXIONS BY BRITISH WRITERS UNDER THE INFLUENCE OF D'ALEMBERT, LAGRANGE, AND LACROIX

Review of Lagrange's "Fonctions analytiques," 1799

220. Important is a review[1] of Lagrange's *Théorie des fonctions analytiques*, which, as is well known, is an attempt to deduce the principles of the calculus, diverted of all reference to infinitely small or evanescent quantities, limits or fluxions, and reduced to the algebra of finite quantities. The reviewer gives a general criticism of the methods of fluxions and the differential calculus. He discusses the principle of motion: "It will not be denied that this principle is introduced purely for the purpose of illustration, . . . on the ground of convenience. . . . The mathematical principle, on which the doctrine of fluxions depends, is a definition . . . and fluxions were defined to be velocities. . . . Now velocity is nothing real, but is only the relation between the space described and the time of describing it;—of which relation we have a clear idea when the motion is uniform." The reviewer continues: "In variable

[1] *Monthly Review*, London, vol. xxviii, 1799, Appendix.

motion, however, we inquire what velocity is; and here it is defined to be the relation between the space which *would* be described were the motion continued uniform from any point, and the time. Still difficulties remained; this definition might convey to the mind a general idea of the nature of velocity, but was of no *mathematical* use, since the *space which would be described* could not be immediately ascertained and determined. Another step was therefore to be made, and which was made by establishing this proportion; if V be the velocity, S the space, which would be described, and T the time, S′ the space really described, and T′ the corresponding time; then $V = \frac{S}{T}$ = ultimate ratio of $\frac{S'}{T'}$, when S′ and T′ are indefinitely diminished."

Again he says:

"On the ground of perspicuity and evidence, the understanding is not much assisted by being directed to consider all quantity as generated by motion; . . . when such quantities as weight, density, force, resistance, etc., become the object of inquiry . . . then the true end of the figurative mode of speech, *illustration*, is lost. . . . That which happened to Aristotle has happened to Newton; his followers have bowed so implicitly to his authority, that they have not exercised their reason. The method of fluxions had never so acute, so learned, and so judicious a defender as Maclaurin:—yet whoever consults it . . . finds the author speaking of

'causes and effects,' of 'the springs and principles of things,' and proposing to deduce the 'relation of quantities by comparing the powers which are conceived to generate them';—will be convinced that this could only happen from so able a mathematician having failed to seize the right principles." "If English mathematicians first adopted Newton's method from veneration to him, . . . they have since persevered in it (we may almost say) against conviction." The reviewer claims that the criticisms of D'Alembert, Torelli, and Landen have shown that the use of motion is unnecessary and unreal. We have given citations from Landen in an earlier chapter (see our §§ 202, 203). D'Alembert is quoted as saying fifty years previous:

"Introduire ici le mouvement, c'est y introduire une idée étrangère, et qui n'est point nécessaire à la démonstration : d'ailleurs on n'a pas d'idée bien nette de ce que c'est que la vîtesse d'un corps à chaque instant, lorsque cette vîtesse est variable. La vîtesse n'est rien de réel ; . . . c'est le rapport de l'espace au tems, lorsque la vîtesse est uniforme ; . . . Mais lorsque le mouvement est variable, ce n'est plus le rapport de l'espace au tems, c'est le rapport de la différentielle de l'espace à celle du tems ; rapport dont on ne peut donner d'idée nette, que par celle des *limites*. Ainsi, il faut nécessairement en revenir à cette dernière idée, pour donner une idée nette des *fluxions*."[1]

[1] Art. "Fluxion" in *Encyclopédie, ou Dictionnaire raisonné des sciences*, etc., t. 6, Paris, 1756.

221. The reviewer states that foreign mathematicians have written treatises in which motion is entirely excluded, "and in some of these treatises, the principles of the doctrine in question have been laid down with a considerable degree of evidence and exactness." The Residual Analysis of Landen rests on "a process purely algebraical: but the want of simplicity . . . is a very great objection to it." The reviewer is of the opinion that Euler and D'Alembert give "the most clear and precise notions of the principles on which the differential calculus is established." He refers to Euler's *Institutiones calculi differentialis*, 1755. D'Alembert, says the reviewer, "observes that the method is really founded on that of prime and ultimate ratios, or of limits, which latter method is only an algebraical translation of the former; that, in fact, there are no such things as infinitely small quantities; and that, when such quantities are mentioned, it is by the adoption of a concise mode of speech for the purpose of simplifying and abridging the reasoning;—that the true object of consideration is the limit of the ratio of the finite differences of quantities."

The reviewer continues: "The explanations given by Euler and D'Alembert, beyond all doubt, deserve much consideration, yet their method of considering the doctrine of fluxions is not completely satisfactory, but is objectionable on two grounds: first, that we have no clear and precise notion of the ratio of quantities, when those quantities are in their vanishing state, or cease to be quantities;

secondly, the connection and natural order of the sciences are interrupted, if we give a distinct and independent origin to that which in fact, is a branch of analysis derived from the same common stock, whence all the other branches are deduced." Then follows a sympathetic account of the foundations for the calculus laid by Lagrange in his *Théorie des fonctions analytiques*, 1798. In passing, the reviewer remarks that "Emerson, Stone, Simpson, Waring, etc., have published treatises on fluxions; in none of which, however, are the principles clearly laid down."

Review of a Memoir of Stockler, 1799

222. In the same journal[1] there is a review of a memoir on fluxions written by the Portuguese mathematician, Garçao Stockler, who modifies the explanation of fundamentals by the introduction of a "hypothetical fluxion" (a uniform velocity that generates a quantity equal to the real increment generated during the actually variable motion), which is always contained between the proper fluxions at the first and second instant under consideration. By diminishing the interval of time, the hypothetical fluxion approaches the true fluxion more nearly than by any assignable quantity. Here also, the real object of consideration is the *limit*. The reviewer argues that the fundamental principles are not new, and that the objections to Newton's fluxions apply equally to those of Stockler. In a reply to

[1] *Monthly Review*, vol. xxviii, London, 1799, p. 571.

the *Monthly Review*, Stockler denies the reviewer's allegation that he [Stockler] supposed quantity to be generated by motion. "The idea of motion, and the idea of velocity, are too particular to be admitted into a general theory of fluent quantities."[1]

Review of Lacroix's "Calcul différentiel," 1800

223. A review of S. F. La Croix's *Traité du calcul différentiel*[2] served as the occasion of further comments and criticisms of fundamental concepts:

"Who would direct his ridicule against the refinements, subtleties, and trifling of the schoolmen, if he read what has been written by some men who were presumed to be the greatest masters of reason, and whose employment and peculiar privilege consisted in deducing truth by the justest inferences from the most evident principles? The history of the differential calculus, indeed, shows that even mathematicians sometimes bend to authority and a name, are influenced by other motives than a love of truth, and occasionally use (like other men) false metaphysics and false logic. No one can doubt this, who reads the controversial writings to which the invention of fluxions gave rise: he will there find most exquisite reasonings concerning quantities which survived their grave, and, when they ceased to exist, did not cease to operate; concerning an infinite derivation of velocities,—and a progeny of

[1] *Monthly Review*, vol. xxxii, p. 497.
[2] *Monthly Review*, vol. xxxi, London, 1800, p. 493.

infinitesimals smaller than the 'moonshine's wat'ry beams,' and more numerous than

> 'Autumnal leaves that strow the brooks
> In Vallombrosa.' (Milton, *Par. Lost*, i, 302.)

"The contemporaries and partizans of Newton were men infinitely inferior to him in genius, but they had zeal, and were resolved to defend his opinions and judgments. Hence they undertook the vindication of fluxions, according to the principles and method of its author; although it may be fairly inferred, from the different explanations given of that doctrine by Newton in different parts of his works, that Newton himself was not perfectly satisfied of the stability of the grounds on which he had established it."

The reviewer quotes (p. 497) from Lacroix's preface:

"These notions [velocities, motions], although rigorous, are foreign to geometry, and their application is difficult. . . . Properly speaking, fluxions were to him [Newton] only a means of giving a sensible existence to the quantities on which he operated. The advantage of the method of fluxions over the differential calculus in point of metaphysics, consists in this; that, fluxions being finite quantities, their moments are only infinitely small quantities of the first order, and their fluxions are finite; by these means, the consideration of infinitely small quantities of superior orders is avoided. . . . I can only mention a method which Landen gave in 1758, to avoid consideration of infinity of

motions, or of fluxions, since it rests on a very elegant algebraic theorem which cannot be given in a work of this nature. The freedom with which Landen divests himself of national prejudice stamps a remarkable character on his work; he is perhaps the only English mathematician, who has acknowledged the inconvenience of the method of fluxions." . . .

"We can always descend from the function to the differential coefficient or from the primitive function to the derived function: but, generally speaking, the reverse step is attended with the greatest difficulty."

"The rivals of Newton thought and invented for themselves; had they been influenced by his authority, and devoted their talents to the perfection of synthesis, science must have been considerably retarded. To the improvement of the algebraical analysis, is to be attributed the amazing advances of physical astronomy."[1]

Review of Carnot's "Réflexions," 1801

224. In the *Monthly Review*[2] (London) for 1801 there is a short and unimportant account of Lazare N. M. Carnot's new book, *Réflexions sur la métaphysique du calcul infinitésimal*, 1797. Carnot explains the correctness of results obtained by the infinitesimal calculus of Leibniz on the theory of compensation of errors—a theory which had been

[1] *Monthly Review*, vol. xxxii, p. 491.
[2] *Monthly Review*, vol. xxxiv, 1801, p. 463.

advanced much earlier by Berkeley in his *Analyst.* Mr Philip E. B. Jourdain has found clear indications of this theory in Maclaurin's *Fluxions* and in Lagrange's *Théorie des fonctions analytiques.* The method of limits is explained by Carnot in the manner of D'Alembert. "Of fluxions, indeed," says the reviewer, "as founded on the strange basis of velocity, there is no account."

Robert Woodhouse, 1803

225. In 1803, Robert Woodhouse published his *Principles of Analytical Calculation.*[1] Woodhouse had graduated B.A. at Caius College, Cambridge, in 1795, as senior wrangler. He then held a scholarship and a fellowship at Caius College, devoting himself to mathematics. He has the distinction of being the first to strongly encourage the study in England of the mathematical analysis which had been created on the Continent by Swiss and French mathematicians. In his *Principles of Analytical Calculation* he discussed the methods of infinitesimals and limits, and Lagrange's theory of function, pointing out the merits and defects of each. "By thus exposing the unsoundness of some of the Continental methods, he rendered his general support of the system far more weighty than if he had appeared to embrace it as a blind partisan."[2]

226. The ideas set forth in this book are, on the

[1] *The Principles of Analytical Calculation*, by Robert Woodhouse, A.M., F.R.S. Cambridge, 1803.

[2] Art. "Woodhouse, Robert," in Sidney Lee's *Dictionary of National Biography.*

whole, in such close agreement with those advanced in the preceding reviews, that the query naturally arises, whether Woodhouse is not the author of those reviews. We have reached no final decision on this point.

In the preface Woodhouse passes in review the different methods of establishing the foundations of the calculus. He criticises the use of motion in the proof of the binomial and other related theorems. "It required no great sagacity to perceive, that a principle of motion, introduced to regulate processes purely algebraical, was a foreign principle." If the binomial theorem and related theorems for the development of a function be established by algebra, independently of motion, then "from the second term of this expansion, the fluxion or differential of a quantity may be immediately deduced, and in a particular application, it appears to represent the velocity of a body in a motion. The fluxionists pursue a method totally the reverse; they lay down a principle of motion as the basis of their calculus, thence deduce some of the first processes, and establish the binomial theorem, by which it is said, the extraction of roots may be effected. . . . The project of extracting the square and cube roots of algebraical quantities by a principle of motion, is surely revolting to the common sense."

"Of his own method, Newton left no satisfactory explanation: those who attempted to explain it, according to what they thought the notions of its author, and . . . by reasoning which fairly may be

called tedious and prolix. Of the commentators on the method of fluxions, Maclaurin is to be esteemed most acute and judicious, but his Introduction exhibits rather the exertions of a great genius struggling with difficulties, than a clear and distinct account of the subject he was discussing." To remove this prolixity, it was proposed, conformably to the notions of Newton, to call in the doctrine of prime and ultimate ratios or of limits. Euler and D'Alembert, on the other hand, rejected motion, but retained limiting ratios, failing, however, in supplying a satisfactory explanation therefor. Woodhouse is the earliest English mathematician who speaks in respectful and appreciative terms of the services to mathematics rendered by Bishop Berkeley. In fact, Woodhouse admits as valid some of Berkeley's objections which had been declared invalid. The methods of treating the calculus "all are equally liable to the objection of Berkeley, concerning the fallacia suppositionis, or the *shifting of the hypothesis.*" Thus, in fluxions and the method of limits, x is increased by i, and, in the case of x^m, the increment of the function, divided by i, is $mx^{m-1}+\frac{m(m-1)}{z}x^{m-2}i+$, etc.; then, putting, $i=0$, there results mx^{m-1}. But since the expansion of $(x+i)^m$ was effected "on the express supposition, that i is some quantity, if you take $i=0$, the hypothesis is, as Berkeley says, shifted, and there is a manifest sophism in the process" (p. xii).

227. As another objection to limits, or prime and ultimate ratios, Woodhouse declares that "the method is not perspicuous, inasmuch as it considers quantities in the state, in which they cease to be quantities."

Moreover, "the definition of a limit, is neither simple nor concise" (p. xvii). "The name of Berkeley has occurred more than once in the preceding pages: and I cannot quit this part of my subject without commending the *Analyst* and the subsequent pieces, as forming the most satisfactory controversial discussion in pure science, that ever yet appeared: into what perfection of perspicuity and of logical precision, the doctrine of fluxions may be advanced, is no subject of consideration: But, view the doctrine as Berkeley found it, and its defects in metaphysics and logic are clearly made out. If, for the purpose of habituating the mind to just reasoning . . . I were to recommend a book, it should be the *Analyst.*" "The most diffuse and celebrated antagonists of Berkeley, are Maclaurin and Robins, men of great knowledge and sagacity: but the prolixity of their reasonings confirms the notion, that the method they defend is an incommodious one."

"Landen, I believe, first considered and proposed to treat the fluxionary calculus merely as a branch of Algebra: After him, M. Lagrange, a name ever to be celebrated, in the Berlin Acts for 1772, laid down its analytical principles; and subsequently in his *Théorie des fonctions analytiques*, 1796, he has resumed the subject: in this treatise, the author

expressly proposes, to lay down the principles of the differential calculus, independently of all consideration of infinitely small, or vanishing quantities, of limits, or of fluxions" (p. xviii). While Woodhouse considers Lagrange's discussion as very valuable, he does not find it free from logical faults.

William Hales, 1804

228. As a protest against the new movement and a vindication of Newton from the attacks upon fluxions in the *Monthly Review*, William Hales prepared a book, the *Analysis Fluxionum*, which was published in Maseres' *Scriptores Logarithmici*, vol. v, London, 1804. Hales endeavours to show that the doctrine of prime and ultimate ratios is really the same as the doctrine of the limits of the ratios. Hales's fundamental definitions are:

"Rationes ultimæ sunt limites, ad quos quantitatum sine fine decrescentium rationes, 1, semper appropinquant; et, 2, quas propiùs assequi possunt quàm pro datâ quâvis differentiâ; 3, nunquam verò transgredi; 4, nec priùs attingere, quam quantitates ipsæ diminuuntur in infinitum."

"Momentum est fluentis augmentum aut *decrementum momentaneum*; id est, tempore quam minimo genitum. Estque fluxioni proportionale."

After Hales's work had gone to press, he became acquainted with Benjamin Robins's *Discourse* of 1735, and published in appendices[1] numerous extracts from

[1] Maseres, *Scriptores Logarithmici*, vol. v, pp. 848, 854, 856.

it. Says Hales: "It is far superior indeed to the subsequent explanations of professed commentators; and it is a high gratification to myself to find, that the mode of explanation, which I adopted of the Doctrine of Limits, is precisely the same as Robins's; long before I had seen his admirable treatise, which did not fall into my hands until lately, a considerable time after the publication of the *Analysis Fluxionum.*" Maseres calls the *Discourse* of Robins "the ablest tract that has ever been published on the subject." Hales's text and the appendices to it contain considerable historical material, consisting mainly of references to and quotations from earlier writings. In view of the testimony of Laplace, Legendre, and Lacroix on the superiority of the method of fluxions, "how was it possible," asks Hales, that the eyes of the Monthly Reviewers "could still be so holden . . . as still to assert, that Newton himself was not perfectly satisfied of the stability of the ground on which he had established his Method of Fluxions!" Hales's motive in opposing Continental ideas was probably partly theological. D'Alembert, considered by him a hostile critic of Newton, is called "a philosophizing infidel," one "of the original conspirators against Christianity," "at once the glory and disgrace of the French Academy of Sciences," whose last words were "a terrific contrast to the death of the Christian Philosopher," Colin Maclaurin.[1]

The publication of Hales's Fluxions in large

[1] Maseres, *Scriptores Logarithmici*, vol. v, pp. 176-182.

quarto form and in the Latin language, the inclusion in the Appendix of matters foreign to the subject of the book, together with the attempt to maintain a system of notation and mode of exposition that was beginning to be considered provincial, caused the book to "fall still-born from the press."

Encyclopædia Britannica, 1810

229. In the fourth edition of the *Encyclopædia Britannica*, Edinburgh, 1810, the article "Fluxions" is wholly rewritten, and is much more extensive than the article in former editions. There is a lengthy historical introduction, and emphasis is placed upon work done on the Continent. It observes "that there is no work in the English language that exhibits a complete view of the theory of fluxions, with all the improvements that have been made upon it to the present time." Mention is made then of "several excellent works in the French language," mentioning Cousin, Bossut, La Croix, L'Huilier.

Letting u be "any function" of x, the *limit* of the ratio $(u' - u) / h$ is defined as "a quantity to which the ratio may approach nearer than by any assignable difference, but to which it cannot be considered as becoming absolutely equal." The article asserts that the method of fluxions "rests upon a principle purely analytical, namely the theory of limiting ratios; and this being the case, the subject may be treated as a branch of pure mathematics,

without having occasion to introduce any ideas foreign to geometry. Sir Isaac Newton, however, in first delivering the principles of the method, thought proper to employ considerations drawn from the theory of motion. But he appears to have done this chiefly for the purpose of illustration, for he immediately has recourse to the theory of limiting ratios, and it has been the opinion of several mathematicians of great eminence (such as Lagrange, Cousin, La Croix, etc., abroad, and Landen in this country) that the consideration of motion was introduced into the method of fluxions at first without necessity, and that succeeding writers on the subject ought to have established the theory upon principles purely mathematical, independent of the ideas of time and velocity, both of which seem foreign to investigations relating to abstract quantity." "By the fluxions then of two variable quantities having any assigned relation to each other, we are in the following treatise always to be understood to mean *any indefinite quantities which have to each other the limiting ratio of their simultaneous increments* (we . . . mean the ratio of the *numerical* values of the increments, which may always be compared with each other, whether the variable quantities be of the same kind, as both lines, or both surfaces, etc., or of different kinds, as the one a line, and the other a surface). The Newtonian notation is used in the article exclusively."

Lacroix's "Elementary Treatise," 1816

230. The translation of Lacroix's *Elementary Treatise on the Differential and Integral Calculus*[1] in 1816 marks an important period of transition.

From the "Advertisement" we quote:

This work of Lacroix "may be considered as an abridgement of his great work on the Differential and Integral Calculus, although in the demonstration of the first principles, he has substituted the method of limits of D'Alembert, in the place of the more correct and natural method of Lagrange, which was adopted in the former. The first part of this Treatise, which is devoted to the exposition of the principles of the Differential Calculus, was translated by Mr. Babbage. The translation of the second part, which treats of the Integral Calculus, was executed by Mr. G. Peacock, of Trinity College, and by Mr. Herschel, of St. John's College, in nearly equal proportions."

On p. 2 the process of differentiation of $u = ax^2$ is explained, so that $2ax$ "is the limit" of the ratio $(u' - u) / h$, or it is "the value towards which this ratio tends in proportion as the quantity h diminishes, and to which it may approach as near as we choose to make it."

Thus Lacroix's definition, like D'Alembert's, does not prohibit the limit to be reached. In Note A, added by the translators, we read:

[1] *An Elementary Treatise on the Differential and Integral Calculus.* By S. F. Lacroix. Translated from the French. Cambridge. 1816.

"A limit, according to the notions of the ancients, is some fixed quantity, to which another of variable magnitude can never become equal, though in the course of its variation it may approach nearer to it than any difference that can be assigned." Thus, the method of limits is here ascribed by the translators to the ancients, which is an act of reading into the ancient expositions a theory not actually there. The ancient "Method of Exhaustions" is merely a prelude to the theory of limits. Peacock gives in Note A a history of the theory of limits, in which researches on the Continent are dwelled upon and the contribution made by Newton is explained, but no reference is made to Jurin, Robins, and Maclaurin. In Note B Peacock states that the method used by Lacroix in this treatise "was first given by D'Alembert, in the *Encyclopédie*" article "Différentiel." Evidently Peacock was not altogether friendly toward this method, for in Note B he proceeds "directly to show in what manner this calculus may be established upon principles which are entirely independent of infinitesimals or limits," and then informs the reader "that we are indebted for the principal part of the contents of this note, to the Calcul des Fonctions of Lagrange and the large treatise by our author, on the Differential and Integral Calculus." Peacock proceeds to give an account of Lagrange's calculus of functions and of the method of fluxions. Attention is called to "the difficulty of denoting the operations of finding the different orders of

fluxions" according to the Newtonian notation, "when for u we put the function itself, which it represents."

231. The attitude of some British mathematicians of the early part of the nineteenth century toward the discussions of the fundamental concepts of the calculus carried on during the eighteenth century is exhibited in the following passage from John Leslie's *Dissertation* on the progress of mathematical and physical science:[1]

"The notion of flowing quantities, . . . appears on the whole, very clear and satisfactory; nor should the metaphysical objection of introducing ideas of motion into Geometry have much weight. Maclaurin was induced, however, by such cavelling, to devote half a volume to an able but superfluous discussion of this question. As a refinement on the ancient process of Exhaustions, the noted method of Prime and Ultimate Ratios . . . deserves the highest praise for accuracy of conception. It has been justly commended by D'Alembert, who expounded it copiously, and adapted it as the basis of the Higher Calculus. The same doctrine was likewise elucidated by our acute countryman Robins; . . . Landen, one of those men so frequent in England whose talents surmount their narrow education, produced in 1758, a new form of the Fluxionary Calculus, under the title of Residual Analysis, which, though framed with little elegance,

[1] Dissertation Fourth, in the *Encyclopædia Britannica*, 7th ed., vol. i, 1842, pp. 600, 601.

may be deemed, on the whole, an improvement on the method of ultimate ratios."

Remarks

232. The first part of the nineteenth century marks a turning-point in the study and teaching of mathematics in Great Britain. Attention has been directed to the efforts of Woodhouse to introduce the higher analysis of the Bernoullis, Euler, Clairaut, and Lagrange. His efforts were strongly and ably seconded by three other young men at Cambridge, John Frederick William Herschel, Charles Babbage, and George Peacock, who used to breakfast together on Sunday mornings, and in 1812 founded the "Analytical Society at Cambridge," for the promotion, as Babbage humorously expressed it, of "the principles of pure D-ism in opposition to the *Dot*-age of the University." The translation into English of Lacroix's *Elementary Treatise* and the publication, in 1820, of *Examples* with their solutions, brought the more perfect notation of Leibniz and the refined analytical methods to the attention of young students of mathematics in England.[1]

[1] Before the nineteenth century, the use in England of the Leibnizian notation *dz* and *∫ydz* is exceedingly rare. In our § 54 we saw that about the beginning of the eighteenth century these symbols were used by John Craig in articles published in the London *Philosophical Transactions*. When criticising Euler, Benjamin Robins once used the Leibnizian notation; see our § 142. Mr. Philip E. B. Jourdain has brought to my attention the fact that the sign of integration ∫ occurs also in a book, entitled, *Second Volume of the Instructions given in the Drawing School established by the Dublin Society. . . . Under the Direction of Joseph Fenn, heretofore Professor of Philosophy in the University of Nantes. Dublin*, MDCCLXXII. De Morgan refers to this work in a letter to Hamilton. See Graves' *Life of Sir William Rowan Hamilton*, vol. iii, p. 488. See also our *Addenda*, p. 289.

As usually happens in reformations, so here, some meritorious features were discarded along with what was antiquated. William Hales, in 1804, referred to the much neglected *Discourse* of Benjamin Robins (1735), with its full and complete disavowal of infinitesimals and clear-cut, though narrow, conception of a limit. By a curious turn in the process of events, Robins was quite forgotten in England, and D'Alembert's definition was recommended and widely used in England. Now Robins and D'Alembert had the same conception of a limit; both held to the view that variables cannot reach their limits. However, there was one difference between the two men: Robins embodied this restriction in his definition of a limit; D'Alembert omitted it from his definition, but referred to it in his explanatory remarks. D'Alembert says:[1]

"On dit qu'une grandeur est la *limite* d'une autre grandeur, quand la seconde peut approcher de la première plus près que d'une grandeur donnée, si petite qu'on là puisse supposer, sans pourtant que la grandeur qui approche, puisse jamais surpasser la grandeur dont elle approche; ensorte que la différence d'une pareille quantité à la *limite* est absolument inassignable." Further on in the same article we read: "A proprement parler, la *limite* ne coïncide jamais, ou ne devient jamais égale à la quantité dont elle est la *limite*; mais celle-ci s'en

[1] Art. "Limite" in the *Encyclopédie, ou dictionnaire raissonné des Sciences des arts et des métiers, publié par M. Diderot, et M. D'Alembert.* Paris, 1754. See also the later edition of Geneva, 1772.

approche toujours de plus en plus, & peut en différer aussi peu qu'on voudra. . . . On dit que la somme d'une progression géométrique décroissante dont le premier terme est a & le second b, est $(a-b) / (aa)$; cette valeur n'est poit proprement la somme de la progression, c'est la *limite* de cette somme, c'est-à-dire la quantité dont elle peut approcher si près qu'on voudra, sans jamais y arriver exactement."

233. That even the best expositions of limits and the calculus that the Continent had to offer at that time were recognised in England to be imperfect, is shown by a passage in a letter which William Rowan Hamilton wrote in 1828 to his friend John T. Graves :[1]

"I have always been greatly dissatisfied with the phrases, if not the reasonings, of even very eminent analysts, on a variety of subjects. . . . An algebraist who should thus clear away the metaphysical stumbling-blocks that beset the entrance of analysis, without sacrificing those concise and powerful methods which constitute its essence and its value, would perform a useful work and deserve well of Science."

[1] *Life of Sir William Rowan Hamilton*, by Robert P. Graves, vol. i, 1882, p. 304.

CHAPTER XII

MERITS AND DEFECTS OF THE EIGHTEENTH-CENTURY BRITISH FLUXIONAL CONCEPTIONS

Merits

234. There are, perhaps, no intuitional conceptions available in the study of the calculus which are clearer and sharper than motion and velocity. There is, therefore, a certain advantage in approaching the first study of the differential calculus or of fluxions by the consideration of motion and velocity. Even in modern teaching of the elements to beginners, we cannot afford to ignore this advantage offered by the eighteenth-century British mode of treating the calculus.

A second point of merit lies in the abandonment of the use of infinitely little quantities. Not all English authors of the eighteenth century broke away from infinitesimals, but those who did were among the leaders: Robins, Maclaurin, Simpson, Vince, and a few others. The existence of infinitesimals (defined as infinitely small constants) was looked upon by philosophers and by many mathematicians as doubtful. Their subjective existence was hardly more probable than their objective existence. These mystic creations occupied a hypo-

thetical twilight zone between finite quantity and no quantity. Their abandonment added to the clearness and logical rigour of mathematics. From the standpoint of rigour, the British treatment of the calculus was far in advance of the Continental. It is certainly remarkable that in Great Britain there was achieved in the eighteenth century, in the geometrical treatment of fluxions, that which was not achieved in the algebraical treatment until the nineteenth century; it was not until after the time of Weierstrass that infinitesimals were cast aside by many mathematical writers on the Continent.

235. There is a perversity in historic events exhibited in the fact that after infinitesimals had been largely expelled in the eighteenth century from Great Britain as undesirable, unreal, and mischief-making, they should in the nineteenth century be permitted to return again and to flourish for a time as never before. About 1816 the Leibnizian notation of the calculus and the vast treasures of mathematical analysis due to the Bernoullis, Euler, D'Alembert, Clairaut, Lagrange, Laplace, Legendre, and others, which were all expressed in that notation, found their way into England. This influx led to enrichment and advancement of mathematics in England, but also to a recrudescence—this return of the infinitely small. How thoroughly the infinitesimal invaded certain parts of British territory is seen in Price's large work on the *Infinitesimal Calculus*, a work which in many ways is most admirable and useful.

236. After the development of the theory of limits by the English mathematicians and by such Continental writers as D'Alembert and Lacroix, it would hardly seem necessary even for the sake of brevity to reintroduce the old-time infinitesimal which could be "dropped" whenever it was very small, yet stood in the way. But at all times, and particularly in the eighteenth and beginning of the nineteenth centuries, there have been mathematicians who cared little for the logical foundations of their science. Fascinated by the ease with which the calculus enabled them to dispose of difficult problems in the theory of curves, ordinary mechanics, and celestial motions, they felt more like poets, and held the sentiments toward logic that a distinguished bard entertained toward pure intellectualism when he contemplated the beauties of the rainbow:

> "Triumphal arch that fill'st the sky,
> When storms prepare to part,
> I ask not proud philosophy
> To teach me what thou art."

Defects

237. All the eighteenth-century expositions of the foundations of the calculus—even the British—are defective. Without attempting an historical treatment or a logical exposition of later developments, we desire to point out briefly what some of these defects were.

In the first place, the doctrine of fluxions was so closely associated with geometry, to the neglect of

analysis, that, apparently, certain British writers held the view that fluxions were a branch of geometry. In the preface to the *Gentleman's Diary* of London, the new editor, Mr Wildbore, said at the commencement of his editorship in 1781, "the doctrine of fluxions depends on principles purely geometrical, as is very satisfactorily demonstrated by that incomparable geometer, the late Dr Robert Simson of Glasgow in his Opera posthuma."

In the second place, as pointed out by Landen and Woodhouse, there was an unnaturalness in founding the calculus upon the notions of motion and velocity. In a real way, these notions seem to apply only to a limited field in the applications of the calculus, namely, to dynamics. In other fields, motion and velocity are wholly foreign concepts which, if applicable at all, are so only in a figurative sense.

238. Newtonian writers lay great stress upon such conceptions as a line generated by the motion of a point, a surface generated by the motion of a line, and a solid generated by the motion of a surface. We have already referred to the pedagogical advantages of this view, in teaching beginners. But as a final logical foundation this view is inadequate. Not all continuous curves can be conceived as traceable by the motion of a point. An example frequently quoted, in discussions of this sort, is the curve

$$y = 0 \text{ for } x = 0,\ y = x \sin \frac{1}{x} \text{ for } x \neq 0.$$

Let us try to trace this curve by the motion of a point starting from the origin of co-ordinates. In which direction must the point move from the origin? To answer this question we differentiate, and find $dy / dx = \sin(1/x) - (1/x)\cos(1/x)$. At the origin we have $x = 0$ and $y = 0$. No value can be assigned to dy / dx, because $1/x$ has no meaning when $x = 0$; moreover, the equation $y = x \sin(1/x)$ is expressly stated above to apply only when x is not zero. There is, therefore, no way of ascertaining the direction in which the point must depart from the origin. Perhaps we can do better if the moving point is started at another part of the curve. An attempt to plot the curve reveals the fact that it lies between two right lines, of which one makes with the x-axis an angle of $45°$, the other an angle of $-45°$. As the point moves along the curve toward the origin, the curve is found to oscillate with ever-increasing rapidity. When we try to determine the direction by which it jumps into the origin, we encounter the same difficulty as before. As long as x is finite, the direction of motion is determinable. But as soon as we try $x = 0$, the determination is impossible. This conclusion must be accepted, in spite of the fact that the curve is continuous in all its parts, including the origin. This example illustrates the inadequacy of motion as a fundamental concept.

239. Difficulties are encountered in the notion of velocity. Is variable velocity an objective reality? Take a body falling from rest. We say that its

velocity is $ds / dt = gt$. At the end of the first second, the velocity is g. If we ask ourselves, How far does the body move with the velocity g? we must admit that no distance can be assigned. We cannot say that the body moves from a certain point to the point immediately beneath; there is no such point immediately beneath. For, as soon as we try to locate such a point, it occurs to us that we can imagine at least one point located *between* the two points under consideration. This intermediate point serves our purposes no better, for a fourth point located between it and the initial point is easily detected, and so on, without end. Thus it is seen that no distance, however small, can be assigned through which a body falls with a given velocity. We are thus compelled to reject variable velocity as a physical fact. What, then, is $ds / dt = gt$? Clearly it is merely a limit, a mathematical concept, useful in mathematical analysis, but without physical reality. To say that ds / dt represents the distance a body *would* fall in unit time after the instant indicated by t, is to assign it merely hypothetical meaning, destitute of concreteness. While these considerations in themselves may not debar the use of velocity as a mathematical concept upon which to build the calculus, they show that the concept is not as simple as it would seem to be at first approach.

The reader will have observed that in all discussion of limits during the eighteenth century the question of the *existence of a limit* of a convergent sequence was never raised; no proof was ever given

that a limit actually exists. In this respect the treatment was purely intuitive.

240. Another defect in the British exposition of fluxions was in the use of the word "quantity." No definition of it was given, yet quantities were added, subtracted, multiplied, and divided. It is possible to treat *quantities* or *magnitudes* without the use of number. The fifth and tenth books of Euclid's *Elements* contain such treatment. We may speak of the ratio of one *magnitude* to another *magnitude*, or we may speak of the ratio of one *number* to another *number*. Which was meant in the treatment of fluxions? Straight lines were drawn and the ratios of parts of these lines were written down. What were these the ratios of? Were they the ratios of the line-segments themselves, or the ratios of the numbers measuring the lengths of these line-segments? No explicit answer to this was given. Our understanding of authors like Maclaurin, Rowe, and others is that in initial discussions such phrases as "fluxions of curvilineal figures," "fluxion of a rectangle," are used in a non-arithmetical sense; the idea is purely geometrical. When later the finding of the fluxions of terms in the equations of curves is taken up, the arithmetical or algebraical conception is predominant. Rarely does a writer speak of the difference between the two. Perhaps

> "His notions fitted things so well,
> That which was which he could not tell."

241. Analytical geometry practically identified geometry with arithmetic. It was tacitly assumed

that to every distance corresponds a number and to every number there corresponds a distance. Number was thus given a geometrical basis. This situation continued into the nineteenth century. This metrical view involved the entire theory of measurement, which assumed greater difficulties with the advent of the non-Euclidean geometries. The geometrical theory of number became less and less satisfactory as a logical foundation. Hence the attempts to construct purely arithmetical theories.[1]

A good share of those difficulties arose from irrational numbers, which could not be avoided in analytical geometry. This occurrence is not merely occasional; irrational ratios are at least as frequent as rational ones. What is an irrational number? How do we operate with irrational numbers? What constitutes the sum, difference, product or quotient, when irrational numbers are involved? No explicit answer was given to these questions. It was tacitly assumed without fear, that it is safe to operate with irrational numbers as if they were rational. But such assumptions are dangerous. They might lead to absurdities. Even if they do not, this matter demands attention when mathematical rigour is the aim.

242. Perhaps it may be worth while to recall to the reader's mind illustrations of the danger resulting from taking operations known to yield consistent

[1] For a historical account of the number concept and the founding of the theory of transfinite numbers during the nineteenth century, read Philip E. B. Jourdain's "Introduction" to Cantor's *Transfinite Numbers*, The Open Court Publishing Co., 1915.

results when a certain limited class of numbers is involved, and applying them to numbers of a more general class. Suppose a and b to be rational, positive numbers, not zero; we find, let us agree, consistent results in the operation $a+b$, $a-b$ when $a>b$, and $a\times b$, and $a\div b$. Let us now consider the class composed of rational numbers, both positive and negative; suppose, moreover, that we introduce 0 in order to give interpretation to the operation $a-a$. If in this extended class of numbers we admit the four operations $a+b$, $a-b$, $a\times b$, $a\div b$, trouble arises even after due consideration has been given to the negative numbers. There may arise the following well-known paradox. Let $a=b=1$, then $a^2-b^2=a-b$. Divide both sides of the last equation by $a-b$, and we have $a+b=1$, or $2=1$. Where is the difficulty? The answer is known to every schoolboy: We have used $a-b$, or 0, as a divisor; we have extended the operation of division to the larger class of numbers, and to zero, without first assuring ourselves that such an extension is possible in every case; division by zero is inadmissible.

243. A less familiar example is the following. Let us suppose that, for real exponents, it is established that $(A^x)^y=A^{xy}$. When we apply this process to imaginary exponents, trouble arises. Take the equation $e^{2m\pi i}=e^{2n\pi i}$, where m and n are distinct integers, $i=\sqrt{-1}$, $\pi=3\cdot14159\ldots$, and $e=2\cdot718\ldots$ That this equation holds is evident, for $e^{2m\pi i}=\cos 2m\pi+i\sin 2m\pi=\cos 2n\pi+i\sin 2n\pi=e^{2n\pi i}$. If

both sides of $e^{2m\pi i} = e^{2 \cdot n\pi i}$ are raised to the power i / 2, we obtain $e^{-m\pi} = e^{-n\pi}$. Here all the letters stand for real numbers; since m and n are not equal to each other, this last equation is an absurdity. The assumption that a rule of operation valid for real exponents was valid also for imaginary exponents, has led to papable error.

Examples of this sort emphasise the need of caution when operations, known to be valid for a certain class of numbers, are applied to numbers belonging to a larger class. Special examination is necessary. These remarks are pertinent when operations applicable to rational numbers are extended to a class which embraces both rational and irrational numbers. What are the numbers called irrational? It is hardly sufficient to say that an irrational number is one which cannot be expressed as the ratio of two rational numbers. A negative definition of this sort does not even establish the existence of irrational numbers. Considerable attention has been paid to the definition of irrational numbers as limits of certain sequences of rational numbers. Thus, $\sqrt{2}$ may be looked upon as the limit of the sequence of rational fractions obtained by the ordinary process of root-extraction, namely, the sequence, 1, 1·4, 1·41, 1·414, 1·4142, . . . This attempt to establish a logical foundation for irrational numbers was not successful. We endeavour, in what follows, to make this matter plainer.

244. Let us agree that in building up an arith-

metical theory we have reached a development of rational numbers (integers and rational fractions). We wish, next, to define *limit* and also *irrational number.* An early nineteenth-century definition of limit was: "When the successive values attributed to a variable approach a fixed value indefinitely so as to end by differing from it as little as is wished, this fixed value is called the limit of all the others." Since, according to our supposition, we are still in the field of rational numbers, this limit, unless it happens to involve only rational numbers and to be itself only a rational number, is, in our case, non-existent and fictitious.

If now, as stated above, an irrational number is defined as the limit of certain sequences of rational fractions, trouble arises. The existence of such a limit is often far from evident. But aside from that general consideration, the difficulty of the situation in our case is apparent: Irrational numbers are limits, but limits themselves are non-existent or fictitious, unless they are rational numbers. To avoid this breakdown in the logical development, it was found desirable to define irrational number without using limits.

245. With the view of avoiding the use of limits in the definition of irrational number, and at the same time avoid inelegant and difficult assumptions, involving complicated considerations relating to the nature of space,[1] devices were invented by several

[1] On this point consult the article "Geometry" in the *Encyclopædia Britannica*, 11th edition, the part on Congruence and Measurement.

logicians independently, which freed the number concept from magnitude and established number theory on the concept of order. Chief among the workers in this field were Méray, Weierstrass, Dedekind, and Georg Cantor. It is to them that we owe representations of number, both rational and irrational, which have yielded a much more satisfactory theory of limits, and in that way have vastly improved the logical exposition of the differential calculus. These theories have brought about the last stages of what is called the *arithmetisation* of mathematics. As now developed in books which aim at extreme rigour, the notion of a limit makes no reference to quantity and is a purely ordinal notion. Of this mode of treatment the eighteenth century had never dreamed.

ADDENDA TO §§ 54, 58, 73

246. ADDITIONAL data on the fundamental conceptions relating to fluxions and on the use of the Newtonian and the Leibnizian notations in England during the lifetime of Newton are contained in George Cheyne's *Philosophical Principles of Religion*, Part II, London, 1716. Part I of this book appeared first in 1705. Like Berkeley's *Analyst*, which was written later, Cheyne's book, Part I and Part II, had for its primary purpose the refutation of atheism. Cheyne says in his preface to the third edition of Part I, "that Atheism, may be eternally confounded, by the most distant Approaches to the true Causes of natural Appearances. And that if the Modern Philosophy demonstrates nothing else, yet it infallibly proves Atheism to be the most gross Ignorance."

Part II of Cheyne's book consists of three chapters and of seven pages of "Additions." He says in his preface to this part that, excepting one short note, the third chapter and the "Additions" are "what the reverend and ingenious Mr. John Craig sent me about seven years ago, when I desired him . . . to write me down his Thoughts on, correct or alter,

what I had formerly published on this Head in the first Edition of this Work."

Cheyne uses in Part II, p. 20, the notation $\dot{x}$ to denote a distance B*b* when he supposes "*b* infinitely near to B." In § 58 we pointed out that in 1704 Cheyne wrote once $\dot{x} = 1$, but nowhere in the present book does $\dot{x}$ denote a finite quantity. He argues that $1 / \dot{x} = \infty\ 1$, that $1 / 0 = \infty$; hence that $\dot{x} = o$, or "relative nothing," which is "the least Part of the Finite, to which it is related or compared." On p. 21 he calls $\dot{x}$ "an infinitely little Part of x." On p. 12 he speaks of the "absolute infinite" as "admitting of neither Increase, nor Diminution, or of any Operation that mathematical Quantity is subjected to," while (p. 13) "absolute nothing" is "neither capable of increasing nor diminishing, nor of any wise altering any Mathematical Quantity to which it is apply'd, but stands in full opposition to absolute Infinite." On the other hand, "indefinite" or "relative infinite" quantities (p. 29) "are not properly either Finite or Infinite, but between both." The "relative nothing" (p. 8) "is an infinitely little Quantity, as it stands related to a given Finite, by the perpetual Subtraction of which from it self it is generated. Let *o* stand for relative nothing. Thus *o*1 is a *relative infinitely little* Quantity, as it stands related to Unity, by the perpetual Subtraction of which from it self, it is generated; that is $o1 = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1$ ec., and *oa* is an infinitely little Quantity, as it stands related to the given Finite *a*, by the perpetual Subtraction

of which from it self, it is generated; that is $oa = a - a + a - a + a - a$ ec." In the "Advertizement" following p. 190 this is further explained thus: "Relative Nothing is said here to be generated by a perpetual Subtraction, tho' the Signs by alternately + and −. For these Reasons, because relative Infinite, was said to be generated by a perpetual Addition, and because that after the first Term, every two succeeding ones in relative Nothing 1 is equivalent to $o1$ thus $1 - 1 + 1 - 1 + 1 - 1$, &c. . . . $= 1 - o1 - o1 - o1$ &c."

247. These explanations are intended by Cheyne merely as introductions to the later chapters, particularly that by John Craig, who (p. 167) declares that o cannot be an absolute nothing, "for an infinite Number of absolute Nothings cannot make 1, but by o is understood an infinitely small part, as in the *calc. diff.* dx is an infinitely small part of x, so that dx is as o to x: Not that dx is absolutely nothing, for it is divisible into an infinite Number of Parts, each of which is ddx." To make the point still plainer, John Craig continues (p. 168): "But then it may be inquir'd what is the Quotient that arises from the Division of 1 by absolute Nothing. I say there is no Quotient because there is no Division: Therefore it is a Mistake to say the Quotient is 1 or Unity undivided, which is demonstrably false, neither is the Quotient $= o$. For properly speaking there is no Quotient, and therefore it is an Error to assign any. In like manner, it is an Error to say, that $o \times a$ makes the Product

o; for properly speaking there is no Product. It is true, this of Multiplication has no influence upon Practice, but that of Division has. From hence it appears, that a Curve is said to meet with its Asymptote, when the Ordinate is infinitely little." Then follows a startling view which had been held about sixty years before by John Wallis in his *Arithmetica Infinitorum*, 1655,[1] but Craig makes no reference to him. Craig argues (p. 169): "This same Notion does explain how it comes to pass that 1 divided by a negative Number gives a Quotient greater than Infinite." Curiously, he represents the logarithmic curve $y = \log x$ as crossing the y-axis at $y = -\infty$, for since the curve approaches the y-axis infinitely near when positive x approaches zero, "we may conceive the Logarithmic Curve continued as intersecting" the y-axis, so as to form "one continued Curve." Accordingly negative numbers have logarithms that are real and negative. His further argument amounts to this: For values of x that are equal to 1 divided by a negative number, y in $y = \log x$ is negative and is less than its value $-\infty$ arising when $x = o$ (presumably in the sense that $-2 < -1$). "Ergo x is a Number greater than infinite." Considering the approach of the logarithmic curve towards its asymptote, Craig says (p. 170) that "here it is observable, that there are affirmative Numbers less than nothing denoted by the several Powers of dx, as dx^2, dx^3, ec., or by the second, third, ec. Differences, and these Numbers may be aptly

[1] Wallis, *Opera*, I, p. 409, Prop. CIV.

represented by the Ordinates of the logarithmic Curve," continued from $y=-\infty$ away from the origin when dx^n is affirmative, or towards the origin when dx^n is negative. In the "Additions," p. 185, Craig devotes six pages to "An Answer to Mr. Varignon's Reflections upon Spaces greater than infinite," in which Craig uses the Leibnizian symbol $\int$ five times, as in $\int: x^{-e}dx=\frac{x^{1-e}}{1-e}$. Nowhere in the book under consideration does Craig use the notation of Newton. The "Additions" are dated "September 23d, 1713."

248. George Cheyne was a pupil of the Scotch physician, Archibald Pitcairne (1652–1713), who is the author of two books on fluxions (which we have not seen), viz. *Fluxionum Methodus inversa; Sive Quantitatum Fluentium Leges generaliores. Ad celeberrimum virum, Archibaldum Pitcarnium, Medicum Edinburgensem*; and *Rudimentorum Methodi Fluxionum inversæ Specimina adversus Abr. de Moivre.* Pitcairne's mathematical bent more or less influenced his medical theories. He liked to ridicule others, and was himself ridiculed in a publication, *Apollo Mathematicus; or, the Art of curing Diseases by the Mathematicks, according to the Principles of Dr. Pitcairne*, 1695.

INDEX

Absolute motion, 86, 179.
  nothing, 219, 291.
  space, 178, 179.
"Achilles," 97, 125.
Agnesi, M. G., 247–250.
Alembert, d', *see* D'Alembert,
Analyst (Berkeley's), 34, 57–95, 101, 117, 128, 141, 142, 145, 148, 149, 151, 153, 155, 157, 158, 165, 179, 181, 183, 189, 199, 229, 251, 253, 254, 263, 266, 289.
Analytical Society at Cambridge, 274.
Angle of contact, 42.
Antecedental calculus, 235–239.
Apollonius, 31, 155.
Archimedes, 181, 187, 253.
Aristotle, 166, 256.
Arithmetisation, 225–239, 288.

Babbage, Charles, 271, 274.
Barrow, Isaac, 1, 48, 52, 65, 132, 185; his definition of velocity, 185.
Bayes, Thomas, 157.
Berkeley, George (Bishop), 2, 34, 56–95, 87–91, 94–96, 101, 112, 123, 148, 149, 151, 158, 163, 168, 171, 178, 183, 184, 188, 190, 199, 200, 203, 205, 216, 218, 222, 229, 250, 263, 265, 266; his first reply to Jurin and Walton, 72–78; his second reply to Walton, 85–87; Berkeley's *lemma*, 59, 71, 84, 93, 159, 160, 161, 168, 200, 265; Berkeley ten years after, 178, 179.
Bernoulli, D., 41, 274, 278.
  John, 31, 32, 44, 114, 140, 274, 278.
Binomial theorem, 60, 233, 237, 247, 264.
Bossut, Ch., 269.
Bougainville, L. A. de, 204.
Brewster, David, 26.
Buffon, Comte de, 203–205.
Buonaparte, 90.

Calculus, pre-history of, 1; controversy on invention of, 38, 39, 47, 49; *see* Differential calculus.
Cantabrigiensis, 209, 210, 215.
Cantor, Georg, 284, 288.
  Moritz, 235.
Carnot, L. N. M., 62, 262, 263.
Carré, L., 41.
Cause and effect, 79, 166, 174, 185, 192–194, 210, 212, 257.
Cavalieri, 1, 26, 100, 116, 132.
Cheyne, George, 40, 41, 289–290, 293.
Clairaut, A. C., 274, 278.
Clarke, John, 47.
Clerk, G. L. Le, *see* Buffon, Comte de.
Collins, John, 26, 47, 48, 112, 244.
Colour device in numbers, 137.
Colson, John, 149–154, 164, 165, 182, 247, 248.
Commercium Epistolicum, 26–29, 36, 47–49, 52, 55, 112, 204, 244.
Compensation of errors, 62, 63, 68, 262.
Cotes, Roger, 31, 39, 55, 211.
Cournot, A. A., 94.

Cousin, J. A. J., 269, 270.
Craig, John, 37–39, 55, 274, 289, 291–293.

Daily Gazetteer, 212.
D'Alembert, 204, 238, 239, 254, 255, 257, 263, 265, 268, 271–273, 275, 278, 279.
Davis, William, 253.
Dealtry, William, 252.
Dedekind, R., 288.
De l'Hospital, 41, 50, 51, 53, 54, 66, 84, 169, 248.
De Moivre, A., 39–41, 243.
De Morgan, A., 26, 29, 32, 34, 38, 39, 90, 91, 94, 238, 239, 274.
"Dichotomy," 119.
Diderot, D., 275.
Differential calculus, 45, 48, 52, 58, 61, 62, 114, 118, 139, 156, 222, 225, 239, 249, 258, 260, 261, 271, 277, 288.
method of Leibniz, 28, 29, 37, 38, 49, 217; *see* Differential calculus.
Ditton, H., 40, 41, 44, 46, 47, 50, 53, 55, 56.
Duhamel, J. M. C., 94.
Duillier, Fatio de, 39, 55.

Edinburgh Review, 249.
Edleston, J., 31, 32, 40.
Effect, *see* Cause and effect.
Emerson, William, 178, 192–194, 206, 209, 213, 218, 221, 223, 243, 259.
Encyclopædia Britannica, 240, 269, 273, 287.
Enestrom, G., 30.
Errors, not to be neglected, 19, 23, 28, 32, 34; *see* Infinitesimals, Infinitely little quantities.
Euclid, 7, 10, 31, 119, 155, 184, 187, 191, 222, 253.
Euler, L., 139, 140, 258, 265, 274.
Evanescent quantities, 6, 8–10, 19, 20, 23, 24, 26, 35, 62, 63, 68, 70, 71, 73, 76, 77, 79, 81, 105, 106, 116, 131, 135, 137, 156, 162, 165–168, 191, 247, 255.
Exhaustion, method of, 1, 76, 100, 107, 128, 187, 272, 273.

Fatio de Duillier, 39, 55.
Fenn, John, 274.
Flowing quantities; *see* Fluents.
Fluents, 15, 16, 18, 21, 22, 25, 27–29, 43, 49, 55, 70, 89, 112, 113, 159, 162, 173, 174, 192, 195, 199, 201, 202, 209, 211, 213–215, 246, 273.
Fluxions, criticisms of, 58, 72, 90, 225, 229, 232, 235, 236, 255–276; definition of—by Emerson, 192, in *Encyclopædia Britannica*, 240, by Maclaurin, 184, by I. Newton, 11, 13–16, 18, 21, 22, by Simpson, 169, 179, 195, 211, 213, 219, 244; erroneously defined, 39–56, 71, 156, 171, 172, 198, 240, 248; notation of, 15, 16, 28–33, 38–40, 43, 46, 48, 49, 54, 186, 202, 248, 270, 273, 289–293; pre-history of, 1; second, third, etc., fluxions, 39, 42, 47, 55, 56, 63, 74, 77, 80, 82, 89, 90, 173, 182, 196, 199, 241, 272.
Fraser, A. C., 178, 188.
Function, 235, 269.

Galileo, 1.
Gentleman's Diary, 280.
"Ghosts of departed quantities," 63, 68, 73, 85, 216, 250
Gibson, G. A., 93, 96.
Glenie, James, 235–239.
Graves, J. T., 276.
R. P., 38, 90, 91, 94, 180, 274, 276.
Gray, G. J., 165.
Gregory, David, 39.
James, 48.
St. Vincent, 1, 99.

Hales, William, 267–269, 275.
Halley, E., 39, 57.
Hamilton, W. R., 38, 90, 91, 94, 180, 274, 276.
Harris, John, 40, 41, 55.
Hayes, Ch., 41–43, 53, 55, 56.
Heath, Robert, 207, 210, 212, 213, 229.
Hellins, John, 247–250.

Herschel, J. F. W., 271, 274.
Hobbes, Thomas, 163.
Hodgson, James, 155–157.
Holliday, F., 243, 244.
Hospital, De l', 41, 50, 51, 53, 54, 66, 84, 169, 248.
Hutton, Charles, 210, 244, 245.

Imaginary quantities, 154.
Imagination, straining of, 120–123, 125, 147, 154, 195, 214, 217.
Incommensurables, 7, 10.
Indivisibles, method of, 5, 7, 8, 24, 29, 100, 106, 114, 116, 117, 127, 128, 132, 133, 139, 150, 152, 154, 183, 204, 205, 243.
Infinite divisibility, 5, 7, 10, 61, 89, 153, 193.
Infinite series, method of, 20, 24, 164, 221.
Infinitely little quantities, 15–17, 20, 21, 24, 26–28, 30–36, 39, 40–43, 45, 47, 48, 50, 51, 53, 54, 56, 58, 61, 76–78, 80, 86, 94, 114, 117, 118, 123, 132, 133, 137, 147, 150–156, 162, 168, 169, 174, 182, 183, 196, 197, 199–202, 220, 243, 248, 258, 261, 277, 290–291; avoiding their use, 142, 162, 169, 171, 183, 184, 186, 189, 275; *see* Infinitesimals.
Infinitesimals, 29, 32, 35, 36, 46, 50, 56, 58, 59, 72, 74–76, 83, 89, 94, 111, 114, 118, 147, 182, 184, 201, 206. 207, 241, 250, 263, 275, 277–279.
Infinity, 162, 221, 249; absolute, 151, 163, 292; relative, 151, 162, 163, 183, 184, 290.
Irrational numbers, 284, 286-288.

Jack, Richard, 222, 223.
Jones, William, 17, 43, 243.
Jourdain, Philip E. B., 29, 263, 274, 284.
Jurin, James, 57, 64, 72, 91, 93, 139, 140. 189, 190, 199, 203, 205, 206, 254. 272; Jurin's first reply to Berkeley, 64–69; second reply, 80–85; controversy with Robins, 96–148, 179; controversy with Pemberton, 129–138; explains history of controversy with Robins, 145–146.
Juvenal, 168.

Keill, John, 31, 48.
Kelland, Philip, 189.
Kirkby, John, 225–230.
Klopstock, 188.
Kowalewski, G., 17.

Lacroix, S. F., 255, 260–262, 268–275, 279.
Ladies' Diary, 207, 209, 210, 212, 219, 221, 222, 229.
Lady's Philosopher, 219.
Lagrange, J. L., 94, 181, 239, 255, 259, 266, 270–272, 274, 278; his *Fonctions analytiques*, 255, 259, 263, 266, 272.
Landen, J., 215, 222, 231–235, 239, 257, 258, 261, 262, 266, 270, 273, 280.
Laplace, P. S., 268, 278.
Last ratio; *see* Ultimate ratio.
Lee, Sidney, 181, 263.
Legendre, A. M., 268, 278.
Leibniz, G. W., 1, 28, 32, 33, 37, 38, 41, 47–49, 52, 53, 55, 94, 106, 112–114, 118, 175, 176, 199, 248, 254, 262, 274; Differential method of, *see* Differential calculus.
Leibnizian notation, 32, 37, 38, 48, 49, 53, 54, 139, 254, 274, 278, 289, 291–293.
Leslie, John, 273.
L'Huilier (Lhuilier), S. A. J., 269.
Limit, 1, 3–5, 7–10, 25, 33, 35, 36, 62, 70, 75–78, 83, 89, 92, 93, 97, 99, 101, 104, 106, 116, 119, 129, 146, 147, 183, 185, 189, 191, 218, 220, 238, 241, 242, 247, 250, 251, 254, 255, 258, 259, 263, 265–269, 271, 272, 275, 276, 279, 282, 286–288; limit not reached, 97, 102, 107, 116, 119, 131, 147, 186, 189,

221, 241, 242, 275; limit is reached, 97, 98, 101, 103, 119, 123, 124, 147, 157, 190, 271.
Locke, John, 69.
Lyons, Israel, 201.

Maclaurin, Colin, 181, 199, 203–205, 250, 253, 254, 256, 263, 266, 268, 272, 273, 277, 283; on infinitely small magnitudes, 182; life of, 183.
Maclaurin's *Treatise of Fluxions*, 149, 175, 181–189, 223, 254, 263; French translation, 189.
Martin, Benjamin, 171, 172, 178, 179.
Maseres, Baron, 248, 267, 268.
Mathematical Exercises (Turner's), 210, 218, 219.
Mathematician, 210.
"Maximinority," 188.
Méray, H. C. R., 288.
"Minimajority," 188.
Minute Mathematician, 80, 136, 138.
Moivre, De, 39.
Moments, 11–14, 16, 25–28, 30, 31, 33, 39, 40, 42, 46, 47, 49, 58, 59, 66, 67, 70, 71, 73, 74, 76–78, 89, 91, 92, 99, 100, 105, 106, 116, 118, 123, 128, 147, 151–155, 159, 161, 172, 176, 183, 192, 193, 196, 197, 201; notation for, 25, 27, 28, 31; origin of term, 26.
Moment of a rectangle, 12, 13, 58, 66, 71, 74, 75, 79, 82, 83, 91, 92, 151.
Monthly Review, 200, 209, 210, 215, 222, 224, 234, 255, 259–262, 267, 268.
Morgan, De; *see* De Morgan, A.
Morgan, James, 210.
Motte, A., translator of *Principia*, 4, 12, 103.
Muller, John, 162–164.

Napier, J., 1.
Nascent magnitudes, 11, 13, 22, 43, 44, 58, 59, 62, 66, 70, 73, 74, 77, 81, 84, 106, 113, 116, 118, 126, 135, 156, 161, 162, 165–167, 172, 177, 191, 194, 243, 245.
Navier, L. M. H., 94.
Newton, Isaac, 2–36, 38–41, 43, 49, 52, 53, 55, 56, 61, 64, 65, 67–70, 75, 78, 81–86, 88, 91, 92, 96, 100, 102, 104, 111–114, 118, 122, 123, 127 129, 130, 133, 137, 141–143, 147, 152, 154, 155, 157, 158, 160, 161, 164, 165, 168, 170–173, 175, 181–183, 191, 192, 195, 198, 203–205, 211, 214, 219, 220, 229, 235, 241, 244, 248, 252, 254, 256, 257, 261, 264, 268, 270, 272.
Newton's *Analysis per æquationes*, 17, 36, 48, 190, 204.
lemmas, 3, 43, 75–77, 83, 91, 101–103, 105, 107, 109, 110, 115, 118, 124, 128, 131, 132, 134, 135, 137, 138, 145, 146, 151, 158, 189, 190, 203, 241, 242, 246, 250, 254.
manuscripts, 29.
Method of Fluxions, 149, 164, 248.
notation for fluxions; *see* Fluxions, notation of.
Opticks, 17, 127, 204.
Principia, quotations from, 2–14, 43; *see Principia* (Newton's).
Quadratura Curvarum; see *Quadratura Curvarum*.
use of infinitely little quantities, 29, 30, 32, 33–36, 155.
Newton, T., 250, 251.
Nieuwentiit, B., 41, 45, 46, 102.
Notation for fluxions, *see* Fluxions, notation of; for calculus, *see* Leibnizian notation.
Nothing, absolute, 219, 290–291; relative, 290–291.

Oldenburgh, 26.
Ostwald's *Klassiker*, 17.

Palladium, 210, 212, 219.
Parent, A., 114.
Pascal, B., 132.

Peacock, George, 271, 272, 274.
Pemberton, H., 96, 125, 129–138, 147, 165.
Petvin, John, 230.
Pezenas, Esprit, 189.
"Philalethes Cantabrigiensis," 64, 72, 82, 101, 106, 108–110, 117, 122, 124–128, 133, 136–138, 140–144, 181, 199, 204, 205; *see* Jurin, James.
"Phileleutherus Oxoniensis," 133.
Philosophical Transactions, 26, 38–41, 112, 118, 157, 185, 235, 244, 274.
Pitcairne, Archibald, 293.
Poisson, S. D., 94.
Portsmouth collection, 30.
Present State of the Republick of Letters, 99, 101, 106, 109, 111, 117, 123–125, 129, 130, 133, 135, 142, 143, 154, 162.
Price, B., 278.
Prime and ultimate ratios, 5, 6, 8, 19, 20, 22, 24, 29, 33, 35, 36, 62, 71, 80, 81, 86, 88, 89, 96, 100, 109, 113, 114, 117, 118, 122, 124, 126, 128, 130, 141, 143, 147, 166, 186, 190, 241, 243, 250, 258, 265, 267, 273.
Principia (Newton's), 28, 30, 31–33, 36, 53, 67–69, 74–77, 86, 101–104, 107, 110, 126, 128, 129, 135, 158, 190, 219, 241–243, 253; quotations from, 2–14, 43.

Quadratura Curvarum (Newton's), 17–26, 28, 31–34, 36, 43, 50, 53, 67, 69, 71, 74, 76, 77, 112, 124, 130, 135–137, 156, 179, 190, 246.
Quadrature of curves; see *Quadratura Curvarum*.

Raphson, Joseph, 49, 55
Ratios, first and last; *see* Prime and ultimate ratios.
Relative infinite; *see* Infinity, relative.
Relative motion, 86.
Relative nothing; *see* Nothing, relative.
Republick of Letters; see *Present State of the Republick of Letters*.
Residual analysis (Landen's), 232–235, 239, 258, 273.
Rigaud, S. P., 14, 29, 30.
Robartes, F., 41.
Roberval, 1.
Robins, B., 96–148, 179, 181, 188, 189, 199, 242, 250, 254, 266, 267, 272–275; Robins's *Mathematical Tracts*, 202–206; Robins's *Discourse*, 96–100, 115, 141, 142, 267, 268, 275; explains history of controversy with Jurin, 141–145.
Rowe, John, 172, 173, 175, 178, 179, 195, 202, 253, 283.
Rowning, John, 198–200.

Saunderson, Nicholas, 150, 197, 198, 204, 242.
Simpson, Thomas, 169–171, 175, 178–180, 194–196, 206, 209, 210–215, 218, 220, 221, 223, 224, 244, 253, 259, 277; his definition of fluxions, 244.
Simson, Robert, 280.
Sloane, H., 48.
Smith, James, 165–169.
Smith, Robert, 139, 140, 201.
Solidus, use of, 12.
Stevin, S., 99.
Stewart, John, 17, 21, 25, 190–192.
Stirling, James, 50, 55.
Stockler, G., 259.
Stone, E., 50, 51, 53–56, 169, 248, 259.
St. Vincent, Gregory, 1, 99.
Stubbs, J., 177.

Tacquet, Andrew, 99.
Tannery, Jules, 137.
Taylor, Brook, 50.
Thorp, R., 241–243; translator of *Principia*, 3, 8.
Torelli, G., 257.
Truth Triumphant, 212, 213, 215, 218, 222, 223.
Turner, John, 210–214, 218, 219.

Ultimate ratios, 6, 7, 9, 10, 19, 22–24, 35, 68, 70, 78, 86, 98, 99, 103, 115, 153, 158–161, 168, 191, 219, 223, 242.
Ultimate velocity, 7, 10; *see* Velocity.
Ultimators, doctrine of, 225.

Valerius, Lucas, 99.
Varignon, P., 293.
Velocity, 11, 13–18, 21, 28–30, 32, 33, 36, 43–45, 49, 50, 58, 62, 63, 66, 70, 71, 74, 76–80, 83, 85, 87, 88, 93, 106, 112, 123, 151, 152, 158, 163, 164, 166, 169–171, 173, 175, 184, 185, 193–197, 199, 200, 201, 207, 209, 212, 214, 215, 217, 219, 220, 231, 235, 244, 246, 252, 261, 263, 270, 277, 280; velocity, ultimate, 6, 9, 36; velocity criticised as a fundamental concept, 255, 264, 280–282.
Vince, S., 245–247, 277.
Vivanti, G., 235.

Wallis, John, 41, 100, 292; Wallis's *Algebra*, 14, 30, 32, 33, 39.
Walton, John, 57, 69, 77, 85, 86, 91–96, 148, 185, 199; his first reply to Berkeley, 69–72; his second reply, 78–80; second edition of second reply, 87.
Warburton, Bishop William, 73.
Waring, E., 259.
Weierstrass, K., 288.
Weissenborn, H., 92.
West, William, 202.
Whately, 90.
Whiston, William, 197.
Wildbore, 280.
Wilson, James, 96, 99, 189, 202–206.
Witting, A., 30.
Wood, James, 245.
Woodhouse, R., 93, 263–267, 274, 280.
Works of the Learned, 135–137, 171.

Zeno (of Elea), 90, 119, 125, 217.

PRINTED IN GREAT BRITAIN BY NEILL AND CO., LTD., EDINBURGH

Berkeley
THE UNIVERSITY OF CALIFORNIA · LIBRARY OF THE UNIVERSITY OF CALIFORNIA · LIBRARY OF THE UNIVERSITY OF CALIFORNIA